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On Trim Loss Optimization for Circular Objects under Certain Constraints

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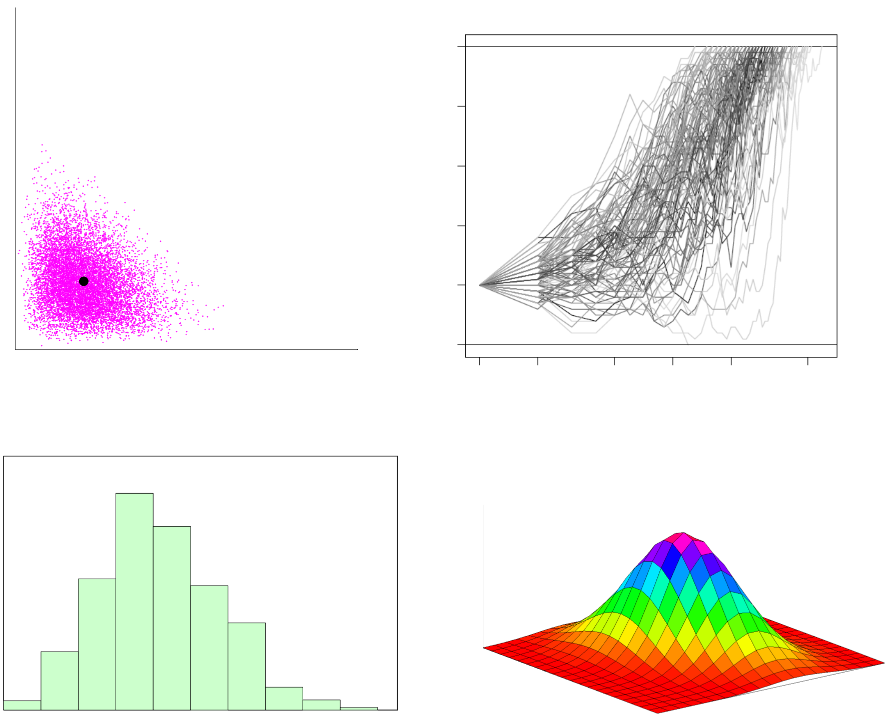

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On Trim Loss Optimization for Circular Objects under Certain Constraints[1]

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Abstract

An enterprise is cutting circular objects, so called rounds, out of so called coils, i.e. steel strips wound into rolls. The enterprise is interested in cutting the desired number of rounds with minimal trim loss where some restrictions due to the production process have to be taken into consideration. In the past the steel producer delivered coils in different widths ordered by the enterprise whereas they are produced in an international standard width of 1250 mm. Therefore the steel producer wants to deliver only coils in international standard widths or pieces subdivided from standard coils to avoid unusable coil remainders. The producer offers to compensate for the increasing amount of trim loss at the enterprise by a price reduction. This paper describes a method in order to give an upper bound for the increase in trim loss for a given production program, if the offer will be accepted.

Zusammenfassung

In einem stahlverarbeitenden Unternehmen werden aus Stahlblechbahnen Kreise gestanzt, wobei einige produktionstechnische Restriktionen zu beachten sind. Bislang wurden vom Stahlerzeuger die Stahlblechbahnen in beliebigen Breiten geliefert. Da diese Bahnen aber in der internationalen Standardbreite von 1250 mm produziert werden, musste der Produzent bislang die ungenutzten Teilbahnen irgendwie verwerten. Daher möchte der Stahlerzeuger nur noch Bahnen in der Standardbreite liefern, die er auch in Teilbahnen zerschneiden würde. Zum Ausgleich würde er sich durch eine Preissenkung am Zusatzverschnitt beim verarbeitenden Unternehmen beteiligen. In der Arbeit wird eine Methode beschrieben, mit der eine obere Schranke für die dadurch bedingte Zunahme des Verschnitts zu einem vorgegebenen Produktionsprogramm bestimmt werden kann.

KEYWORDS: Trim loss, cutting stock, optimization, Kepler's conjecture, sphere packings

MSC2010 Subject Classification: 52C26, 52C15, 11H31, 05B40, 90C90

1 Introduction

An enterprise E is cutting circular objects, so called rounds, out of coils, i.e. steel strips wound into rolls. The enterprise is interested in cutting the needed number of rounds with minimal trim loss

[1]Talks on this topic were given at the Joint Statistical Seminare Marburg–Wroclaw in Wroclaw, 27. 9. 2005, and at the conference in honour of the 80^{th} birthday of Prof. Dr. Dietrich Kölzow in Erlangen, 22. 7. 2011.

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where some constraints due to the production process have to be taken into consideration. In the past the steel producer P delivered coils in arbitrary widths according to E's demand. Since the coils are always produced in an international standard width of 1250 mm, P has to care about the remains.

P therefore offers E the following deal:
If E would be willing to order only standard coils or subdivided standard coils P would be willing to compensate for the increasing amount of trim loss by a certain price reduction. For its decision on accepting the deal or not, E has to estimate the increase in trim loss for his production program.

This paper describes a method to calculate the trim loss for a given production program first for arbitrary coil widths without further restrictions and additionally for coils subdivided from standard coils.

In practice some further restrictions due to the cutting process still have to be taken into account. Additionally, in the second optimization one further restriction occurs, since the twin coils should be used in nearly the same lengths, as we will see.

About 400 years ago Johannes Kepler [7] considered the problem of the densest packing of spheres. For which arrangement of identical spheres do we have a minimal amount of space between the spheres? A short survey through the history of this and similar problems is given by Wills [16]. Leppmeier [9] presents a nicely readable introduction and gives an overview on sphere packings and related topics. An introduction from a more mathematical point of view on sphere packings, a lot of related problems and their connection are also given by Conway & Sloane [1].

Obviously the problem of densest packings of spheres and other objects is related to packing and cutting stock problems. For which arrangement of objects a maximal number of such objects can be packed into a given container of fixed size? In two dimensions we could regard it as a barbecue problem: By which arrangement of (identical) steaks a maximal number of such steaks can be placed on a (limited) grill?

In this manner of speaking the paper deals with the arrangement of circular steaks on long rectangular grills only.

In the literature this concrete practical problem together with the relevant restrictions is not yet considered. Most papers treat cutting stock problems for rectangular pieces.

Dyckhoff et al. [3] mention only three papers dealing with real cutting stock problems with pieces of other than rectangular shapes until 1984. Lampl & Stahl [8] consider cutting stock problems of spheres out of certain rectangles, where spheres of different sizes can be cut simultaneously. Niederhausen [11] just describes a software using a Monte-Carlo-procedure for the determination of optimal cutting plans of arbitrarily shaped objects out of given rectangles of fixed size. At last Ellinger et al. [4] treat cutting of different pieces of clothes in the textile industry, which usually are not in the shape of a circle. Furthermore they use the simplex method of linear optimization in order to determine an optimal pattern of different objects to satisfy one customer's orders but not the overall minimization of the trim loss by arranging the pieces of clothes.

Subsequent papers either consider concrete rectangular cutting problems or sphere packings without concrete applications in cutting stock or packing problems. The last ones mainly deal with the sausage conjecture of L. Fejes Tóth [13] in higher dimensions or deduce bounds for the possible number of (unit-)circles, which can be arranged in a convex area (see Segre, Mahler [12], Groemer [5], Wegner [14], Wegner [15]).

According to Hinxman [6] the problem treated in this paper can be classified as a $1\frac{1}{2}$ dimensional problem, since the length of the rectangular coil is not fixed but has to be minimized under consideration of some further restrictions.

2 Overview

First we will describe restrictions due to the production process as well as the handling of these constraints during the optimization process.

Afterwards we scrutinize possible optimal arrangements of rounds of the same size on a coil of fixed width. We will find just a few possible arrangements which contain the optimal one. For all these arrangements trim loss can be calculated simply. This enables us to compute the amount of trim loss, the proportion of trim loss and the required amount of the coil for every round size and every coil width under consideration.

Using simple optimization methods we are able to find the optimal coil widths as well as the minimal trim loss proportion for a fixed number of different coils (which is determined by the stock capacity or specific requirements of the entrepreneur). Optimization in this paper is realized by the simplex method of Nelder & Mead [10].

In order to find optimal coil widths for coils which unite to standard coils the same optimization methods can be used. But then another problem comes up. Optimization will lead to different parts of a standard coil which are required in very different lengths usually, which means we will have remainings of some coils which will increase permanently. Hence a modification is described which ensures a similar usage of twin coils but can unfortunately not guarantee to find the optimum. But at least it will give an upper bound for the minimal trim loss proportion which is available in an reasonable amount of time and will be useful for the decision of the enterprise E in the introductory problem.

The paper concludes with some exemplary computational results.

3 Consideration of restrictions due to the production process

In the present case some restrictions regarding the production of the rounds have to be taken into account.

1. Rounds must be arranged in a lattice structure, i.e. rounds in each line have to be arranged with equal space between them horizontally. The rounds of the second line can be shifted to the left or right compared to the rounds in the first row. Then the rounds in the third row are arranged in the same way as the rounds in the first row, the rounds in the fourth row are arranged like the rounds in the second row and so on.

2. For rounds of different size different cutting heads have to be used. In each change of the cutting head the coil has to be taken out of the machine where usually the coil will be scratched a bit. Thus always a few metres of the coil length (the exact amount cannot be specified in advance) are cut off and are loss, too. Hence a small number of changes of the cutting head will be advantageous which means that all rounds of one size should be cut at the same time with one cutting head before it is changed.
   Moreover cutting of rounds of different sizes from the same part of a coil at the same time will not be possible. Hence arrangements of different rounds as they are considered in the paper of Lampl & Stahl [8] are not allowed in our case.

3. Between any two rounds there has to be some space, in german called **Innensteg** Is (usually about 3-5 mm).
   This restriction can be kept by arranging rounds of fictional radius $r_e = r + \frac{1}{2}\text{Is}$ instead of the real radius r. If two rounds with radius r_e touch each other then the distance between the borders of the real rounds is precisely Is.

4. Some space at the border of the coil is used for the guidance through the machine. This distance Rs, in german called **Randsteg**, to the border (usually about 3-4 mm) cannot be used for rounds, too.
   As distance to the border the maximum of Rs and $\frac{1}{2}$ Is must be respected. Since Is is already taken into account by using r_e the distance to the border must at least be $\text{Rs}_e = \text{Rs} - \frac{1}{2}\,\text{Is}$, which in practice will always be positive.
   We can respect this restriction by calculating with an effectively usable coil width of $b - 2\text{Rs}_e$ instead of the real coil width b.

5. The cutting head can move at most a certain distance a (for the used machine of the manufacturer it holds $a = 425$ mm) from the middle of the coil to the left or to the right. Hence the middle of the furthest round (at the left and also at the right side) can be placed at most in a distance of a from the middle of the coil.
   Thus the maximal coil width which can be used for the arrangement of rounds of radius r_e due to the movement restriction of the cutting head will be

$$2\left(a + r + \frac{1}{2}\,\text{Is}\right) = 2(a + r_e)\,.$$

   Wider coils can be guided through the machine but some space at the border will be unusable.

   Observe that this restriction together with the last one can be considered by computing with an effective coil width of

$$b_e = \min\left\{b - 2\text{Rs}_e; 2(a + r_e)\right\}\ .$$

   Furthermore observe that the effective coil width depends on the size of the round.

6. Coils of a given thickness are not substitutable by coils of other thicknesses. Hence optimizing the total trim loss means minimizing trim loss for each thickness separately.

Consideration of restrictions 3 - 5 is illustrated in figure 1.

Remark: Innensteg, Randsteg and a probably not usable part of the coil at the border (see restrictions 3 - 5) is treated as trim loss.

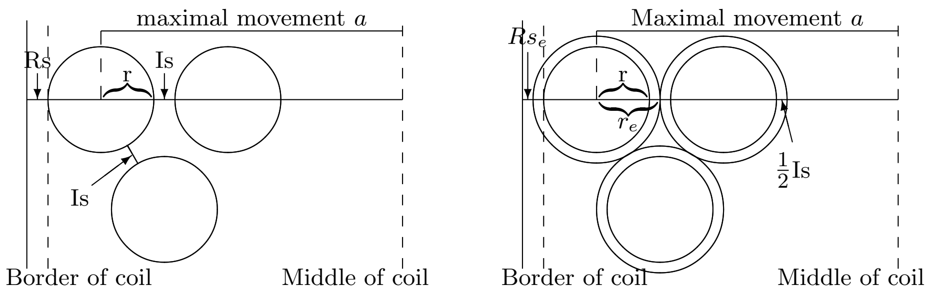


Figure 1: Illustration of restrictions 3 - 5

Since the enterprise E has to store all coils of different widths and different thicknesses it usually wants to use only a small number of different coils, in our case about 4 - 6 coils of each thickness.

4 Some basic relations

The computation of the amount of trim loss uses some simple basic geometric relations, which are regarded now.

Because of the placement of the rounds in a lattice structure according to restriction 1, the arrangements of the first two rows will repeat. Therefore, we just focus on a 'window', which is determined by the (vertical) borders of the coils and the (horizontal) lines of the centres of the rounds in the first and the third row, as illustrated in figure 2. The trim loss proportion of the produced number of rounds will be the trim loss proportion of the window without considering the first and the last produced row which is not important according to restriction 2.

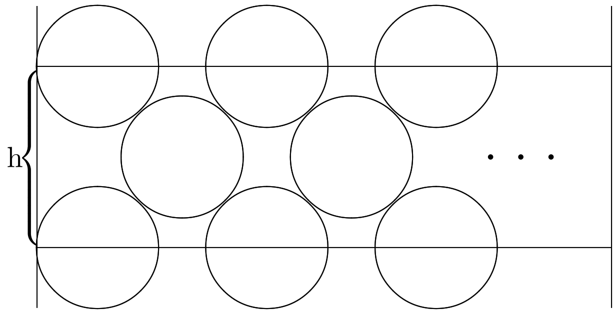


Figure 2: Window

If the window's height is called h and m denotes the number of rounds in the first two rows (and simultaneously the number of rounds in the window) we get

$$\text{Exploitation proportion:} \quad N = \frac{100mr^2\pi}{hb}\ \% \tag{1}$$

$$\text{Trim loss proportion:} \quad V = 100\left(1 - \frac{mr^2\pi}{hb}\right)\% \tag{2}$$

In order to compute the exploitation and the trim loss proportion the window's height h is needed. It can be found by elementary geometric reflections. In figure 3 we consider certain arrangements which involve the relevant cases developed later on.

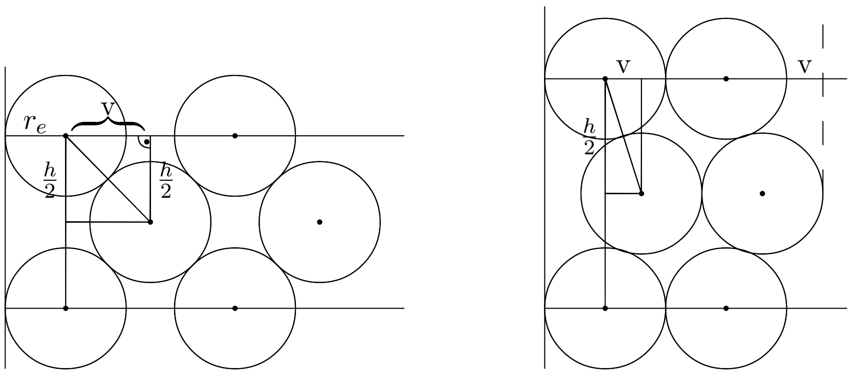


Figure 3: Computation of the window's height

Let v denote the horizontal shift of the rounds of the second row compared to the rounds of the first row. For both arrangements we can conclude

$$\frac{h}{2} = \sqrt{(2r_e)^2 - v^2} = \sqrt{4r_e^2 - v^2}$$

$$h = 2\sqrt{4r_e^2 - v^2} \tag{3}$$

5 Optimal arrangements of rounds of fixed size on a coil of fixed width

Now we will look for optimal arrangements of rounds on a coil of fixed width. First of all let us assume that n rounds have to be placed into the first row and develop possibly optimal arrangements of the rounds. Later on we will be able to conclude which numbers n are really possible or even useful in the first row.

Thus, let us assume that n rounds have to be put into the first row. Without loss of generality we can also assume that at most n rounds are placed in the second row (otherwise change rows one and two).

Because of the lattice structure of the rounds there must be either n or $n-1$ rounds in the second row. Henceforth we will denote these kinds of arrangements by (n,n) and (n,n-1) respectively.

In both cases trim loss will be minimal, if the window containing this fixed number $m = 2n$ or $m = 2n - 1$ of rounds respectively will have minimal height (see (1)).

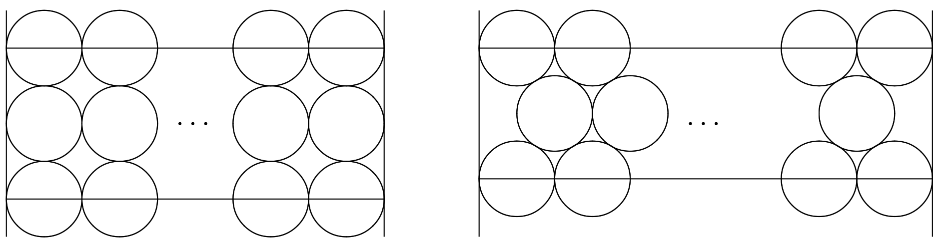
Obviously it must hold $b_e \geq 2nr_e$, otherwise the effective coil width is to small for n rounds in one row.

Starting with $b_e = 2nr_e$ we will develop favourable arrangements and then consider how to find a minimal window height for the rounds if the effective coil width increases.

The results are summarized in table 1 together with important characteristics.

First of all, let us assume $b_e = 2nr_e$ which means that the n rounds in the first row have to lie directly side by side without leaving any space between them.
Then for an (n,n)-arrangement the second row has to look like the first row as illustrated in figure 4a), because any slight shift to the right requires a slightly wider coil.

For an (n,n-1)-arrangement the window's height is going to be minimal, if the $n-1$ rounds of the second row are positioned as far upwards as possible as shown in figure 4b).



a) b)

Figure 4: Possible arrangements for $b_e = 2nr_e$

Obviously, for other arrangements the window height will increase in both cases leading to a higher amount of trim loss.

A simple comparison of the usable coil proportion shows that the (n,n)-arrangement in figure 4a) will be better than the (n,n-1)-arrangement in figure 4b) if and only if $n < 2 + \sqrt{3}$, i.e. $n < 4$.

What happens, if the effective coil width is slightly increasing?

Let us consider (n,n)-arrangements first. In this case the rounds in the second row can be shifted to the right into the gap between the rounds of the first row. Thereby the window's height is decreasing compared to the placement as in figure 4a) with unused coil space in the first and third row at the right border. An arrangement as shown in figure 5 arises.[3]

[3]Those arrangements are not considered by the enterprise but will be considered in this paper. A simple command can exclude this case in the software from further consideration if desired.

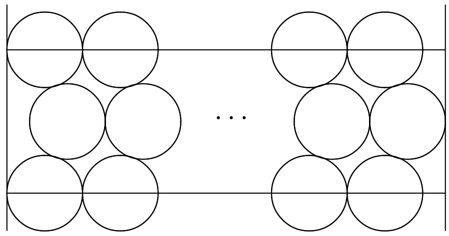


Figure 5: (n,n)-arrangement for $2nr_e \leq b_e \leq (2n+1)r_e$

If we would shift apart the rounds in the first row in case of $b_e \leq (2n+1)r_e$ then the rounds in the second and third row have to move up thereby increasing the window's height as well as the trim loss.

Such an arrangement where each round in the second row touches exactly one round of the first row is possible only if $b_e - 2nr_e < r_e$. For $b_e - 2nr_e = r_e$ we obtain the arrangement in figure 6.

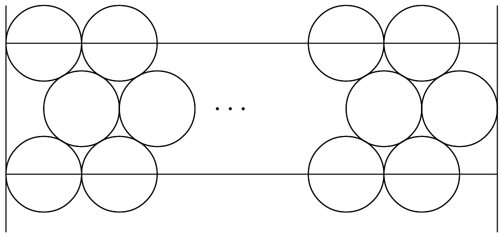


Figure 6: (n,n)-arrangement for $b_e = (2n+1)r_e$

Hence, in case of $2nr_e \leq b_e \leq (2n+1)r_e$ the arrangement in figure 5 is possible which contains the arrangements in figures 4a) and 6 as special cases for $b_e = 2nr_e$ and $b_e = (2n+1)r_e$, respectively.

If we increase the effective coil width in figure 6 or in figure 4b) then the window's height would decrease by shifting the rounds apart in each row as shown in figure 7. The more the coil width

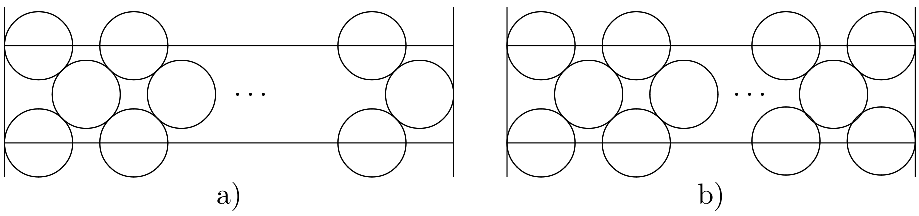


Figure 7: Possible arrangements with some space between rounds

increases the more the window's height decreases, because the rounds of the second row would move downwards. The minimal height of the window is obtained, if the rounds of the third row touch the rounds of the first row as shown in figure 8. For a wider coil the additional space cannot be used to reduce the window's height, if we hold on to (n,n)- or (n,n-1)-arrangements.

The (n,n)-arrangement in figure 8a) requires $b_e \geq (2n-1)\sqrt{3}r_e + 2r_e$ and the (n,n-1)-arrangement in figure 8b) $b_e \geq 2(n-1)\sqrt{3}r_e + 2r_e$.

Hence the (n,n)-arrangement in figure 7a) is possible only if $(2n+1)r_e \leq b_e \leq 2r_e + (2n-1)\sqrt{3}r_e$. As specific cases the arrangements in figure 6 (left bound) and figure 8a) (right bound) are contained. Also, the (n,n-1)-arrangement in figure 7b) is possible only if $2nr_e \leq b_e \leq 2r_e + 2(n-1)\sqrt{3}r_e$ and contains the arrangements in figure 4b) (left bound) and in figure 8b) (right bound) as special cases,

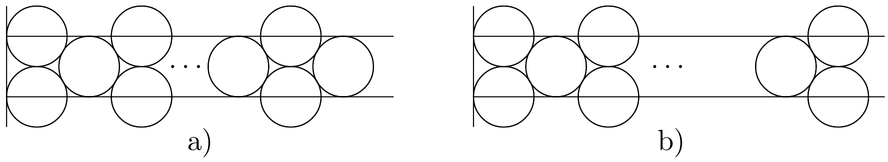


Figure 8: Arrangements with maximal space between the rounds

too.
The assignments in figures 8a) and 8b) are optimal for (n,n)- and (n,n-1)-arrangements respectively, where the additional space at the right border cannot be used to decrease the trim loss for fixed n. Observe, that we only consider (n,n)- and (n,n-1)-arrangements for fixed n so far. Trim loss might be lower for (n+1,n+1)- or (n+1,n)-arrangements .

Table 1 summarizes which arrangements have to be checked for fixed possible and useful n in order to find the optimal arrangement (n,n)- or (n,n-1)-arrangement which minimizes the trim loss for fixed coil width b and fixed radius r of a round. The horizontal shift v and the window's height h is computed according to the elementary geometric considerations in section 4 and formula (3).

Table 1: Summary of possible cases

Arrangement	Fig.	Condition	v	h
(n,n)	5	$2nr_e \le b_e \le (2n+1)r_e$	$b_e - 2nr_e$	$2\sqrt{4r_e^2 - v^2}$
(n,n)	7a	$(2n+1)r_e < b_e \le [(2n-1)\sqrt{3}+2]r_e$	$\frac{b_e - 2r_e}{2n-1}$	$2\sqrt{4r_e^2 - v^2}$
(n,n)	8a	$b_e > [(2n-1)\sqrt{3}+2]r_e$	$\sqrt{3}r_e$	$2r_e$
(n,n-1)	7b	$2nr_e \le b_e \le 2r_e[(n-1)\sqrt{3}+1]$	$\frac{b_e - 2r_e}{2(n-1)}$	$2\sqrt{4r_e^2 - v^2}$
(n,n-1)	8b	$b_e > 2r_e[(n-1)\sqrt{3}+1]$	$\sqrt{3}r_e$	$2r_e$

Furthermore, we are now able to determine which numbers n of rounds in the first row are possible and useful. First of all it must hold $2nr_e \le b_e$, consequently $n \le \frac{b_e}{2r_e}$ that means $n \le n_{\max} = \left[\frac{b_e}{2r_e}\right]$, where $[x]$ denotes the largest integer less or equal to x.
The smallest useful n is given, if the largest possible distance between the rounds in the first row as shown in figure 8b) occurs. Hence, $n_{\min}$ is just the maximal number n for which we have

$$b_e \quad \ge \quad (2n-1)\sqrt{3}r_e + 2r_e$$

that means

$$n \quad \le \quad \frac{b_e - 2r_e}{2\sqrt{3}r_e} + \frac{1}{2}$$

and thus

$$n_{\min} \quad = \quad \left[\frac{b_e - 2r_e}{2\sqrt{3}r_e} + \frac{1}{2}\right] .$$

Summarizing these results we can see that for a given effective coil width b_e only numbers n from the following interval (4) are possible respectively useful as the number of rounds in the first row:

$$\left[\frac{b_e - 2r_e}{2\sqrt{3}r_e} + \frac{1}{2}\right] \quad \le n \le \quad \left[\frac{b_e}{2r_e}\right] . \qquad (4)$$

Hence, for given coil width b and radius r we can calculate b_e and r_e as well as the minimal and maximal number for n according to the interval (4). Then we can check the trim loss for each n out of the interval (4) and the accompanying (n,n)- and (n,n-1)-arrangement according to table 1. From this finite and small number of cases we can easily find the optimal arrangement, that means the one with minimal trim loss.

6 Optimizing the coil widths for arbitrary choice of the width

Above we described how to find the optimal arrangement together with the amount and the proportion of trim loss for a given size of the round, a given coil width and the desired number of rounds of this size to be produced.

Now let $\{(r_1, m_1), \ldots, (r_l, m_l)\}$ be the desired production program where r_i denotes the radius and m_i the demand of round i.

For a given number k of different coils with fixed widths $b_1, \ldots, b_k$ we can place each round on each coil in order to find the optimal coil, the minimal trim loss proportion and the total amount of trim loss for the whole production program by summing over all different rounds. This means that for a fixed production program the trim loss proportion for a given vector $(b_1, \ldots, b_k)$ of coil widths is the value $f(b_1, \ldots, b_k)$ of a function f. Our target is to determine the global minimum of this function f.

This minimum can be found by usual optimization methods. The author cannot recommend gradient methods in this case, since the target function (dependent on the production program) often is very flat with several local minima and might be not differentiable at some points. As an example illustrations of the target function for one and for two coils are shown in figures 9 and 10. Additionally for two coils the contour lines of the target function are given in figure 11. In this picture the stars mark the global minima at about 1090 and 919 mm and by symmetry at (919,1090), too. It should be mentioned that all these plots were made using the software R [2].

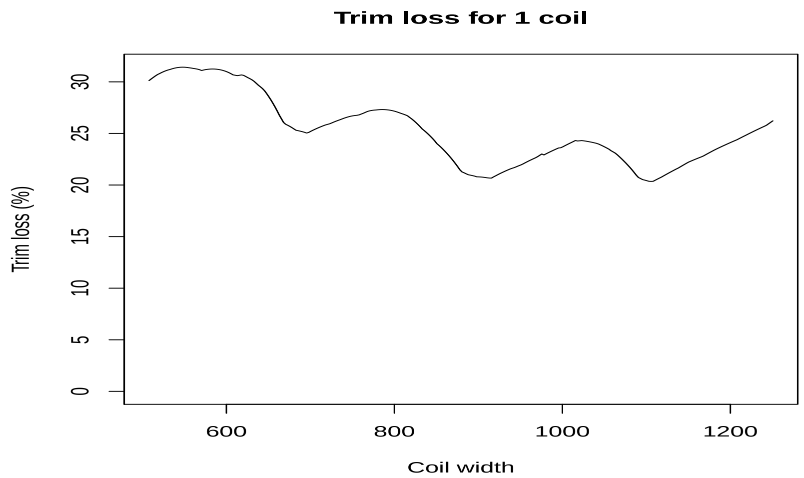


Figure 9: Target function for one coil

Computations were carried out using the simplex method of Nelder & Mead [10]. For unfeasible coil widths, e.g., $b_i < 0$ or $b_i > 1250$ the value of the target function was set to 100%.

Since the simplex method may stop in a local minimum several computations were run using randomly generated starting values.

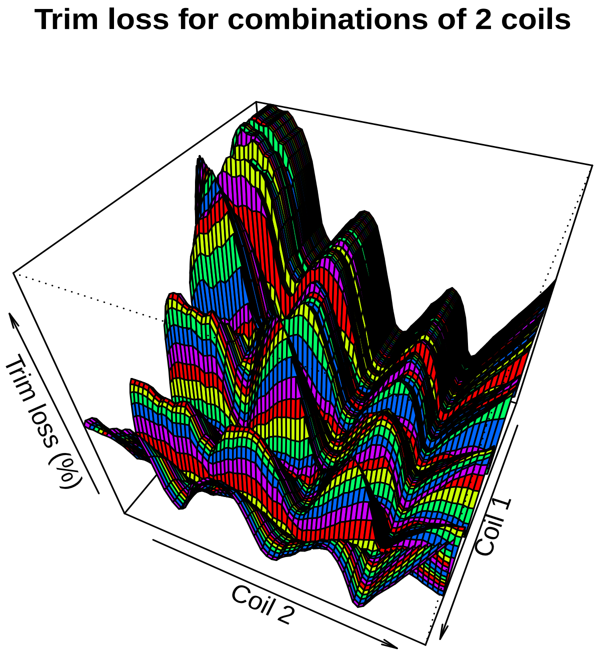


Figure 10: Target function for two coils

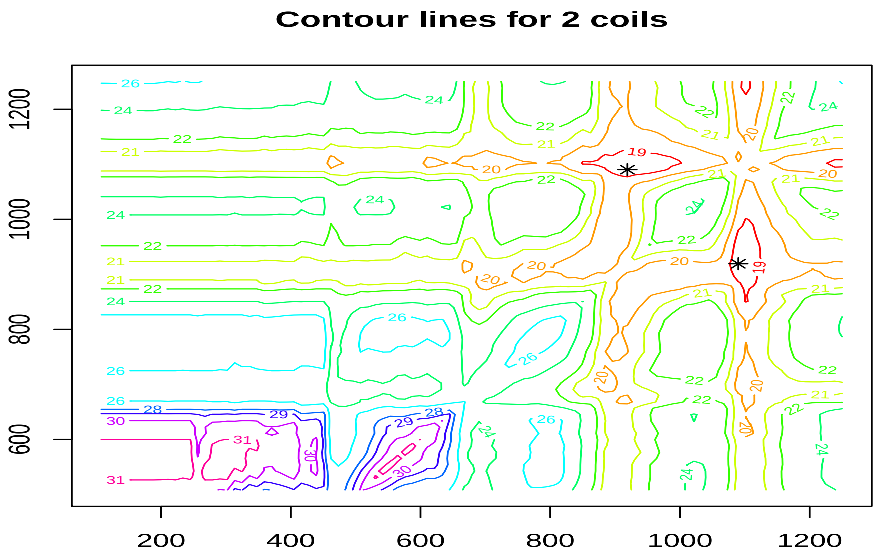


Figure 11: Contour lines of the target function for two coils

The optimal coil widths are found within a few seconds with sufficient precision on a usual personal computer for the considered production program. Observe that in this case the producer delivers arbitrary coil widths and takes care about the second part of each standard coil. Observe, that trim loss in this case means the unneeded amount or proportion only for the delivered coil.

7 Optimizing coil widths for standard coils

In the preceding section a method is described to determine optimal coil widths without other restrictions than $b_i \leq 1250$ on the widths for a given production program. Now the aim is to find the amount of increase of the trim loss proportion, if only standard coils are chosen. Subdividing a standard coil in more than two parts small coils results in small coils which are only usable for a few small sizes of rounds. Hence we only consider the partition of a standard coil in two parts. A generalization for a partition in three or more coils is straightforward.
Below we will speak of twin coils if they are both parts of a standard coil.

For $k = 1$ we just have one standard coil where trim loss can be calculated easily. For an even number k always two coils must complete to a standard coil, whereas for an odd number k an additional standard coil is used.

If at most k coils of different widths are desired then (up to renumbering) it holds

$$b_i + b_{i+[k/2]} = 1250 \quad i = 1, \ldots, \left[\frac{k}{2}\right],$$

where b_j denotes the width of coil j $(j = 1, \ldots, k)$.
For odd k additionally $b_k = 1250$ must hold.
Hence we can use the first $[k/2]$ coil widths as variables, while the others are determined according to the restrictions above.

It seems simple to find the optimal coil width in the same manner as before, e.g. using the simplex method of Nelder & Mead or other optimization methods. If the restrictions $0 \leq b_i \leq 1250$ $(i = 1, \ldots, [k/2])$ do not hold, the trim loss proportion is set to 100%. Specific cases, e.g. if for an additional coil the trim loss proportion cannot be reduced, should result e.g. in a solution $b_i = 1250$ and $b_{i+[k/2]} = 0$ for some i or a solution with two twin coils which are equal to other twin coils.

The straight line in figure 12 represents the points (b_1, b_2) with $b_1 + b_2 = 1250$. We are looking for an optimum on this straight line. The optimal value is located near the point (345, 905), marked by a star.

Unfortunately the optimum on this line is not really what we are looking for, because an important implicit restriction is not taken into account so far. To come upon the problem let us assume as an example we find the optimal coil widths $b_1 = 500$, $b_2 = 750$ of a twin coil. The value of the target function f just gives the trim loss proportion, if every round is cut out of 'its best coil', that means out of that coil, where the least trim loss remains. But if we look at the needed lengths of both coils we may find that we need 10000 metres of the first part of the coil but 20000 metres of the second one. Hence we will have an increasing amount of remaining coil parts which we do not use.
This means we are actually looking for a solution where twin coils are used in almost identical lengths.

In order to achieve a rather similar use of twin coils the following procedure is suggested:
First, look for the optimal assignment of the rounds to the coils without regarding this restriction as described above. Then compute the used lengths for each coil according to the production program. The greater length of twin coils determines the order length hence the unused area of the shorter twin coil is treated as trim loss (10000 metres for the coil of width 500 mm in the example above) and thus increases the trim loss proportion.

If twin coils are used in very different lengths it might be better not to cut some rounds out of their 'best' coil. The rearrangement of rounds from coils with larger length needed than its twin to other coils might decrease the total amount of trim loss. Hence, in order to find the minimal trim loss every

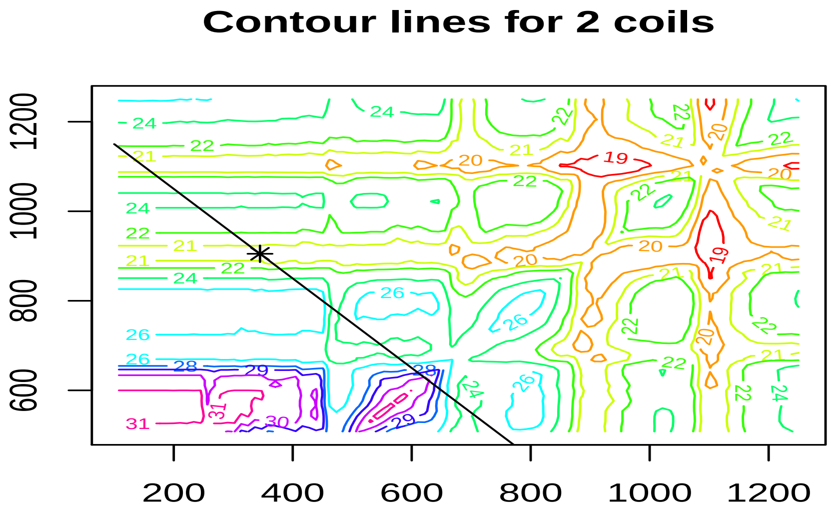


Figure 12: Contour lines of the target function for two coils

possible assignment of the rounds to the coils might have to be taken into account. For l different sizes of rounds and k coils k^l assignments are possible, a number which is increasing very quickly with increasing k or l.
Observe, that for our first procedure, which looks for the best coil for each round, just $k \cdot l$ assignments have to be checked. For $l = 20$ different rounds and $k = 5$ coils we have $5^{20} \approx 10^{14}$ possibilities instead of $5 \cdot 20 = 100 = 10^2$ according to our first procedure.
Since the total amount of trim loss can be calculated only at the end it is not possible to decide in advance which assignments might be advantageous.
Also observe that all of those assignments must be carried out within every optimization step.

Hence we use the following modification to find a better solution:
We use our first procedure, where every round is assigned to its best coil, as a starting point. Then we rearrange every round from one coil to another coil in order to decrease the amount (or proportion) of trim loss. We try one round after the other and start again with the first round unless no rearrangement of a round to another coil leads to a reduction of the trim loss proportion. Even if this solution must not be a minimum it certainly gives an upper bound for the trim loss proportion for this production program when only standard coils are ordered.

8 Some Examples

Let us assume the desired production program in table 2 with $l = 20$ different rounds. We computed the optimal coil widths and the amount of trim loss for $k = 1, 2, \ldots, 5$ coils according to the above mentioned procedure (optimal widths rounded to integer values of milimeters).

'Case A' denotes the case, where coil widths can be chosen arbitrarily, whereas 'case B' indicates the case of twin coils. Always 20 vectors of randomly selected starting values for the coil widths were selected for the calculations. Coils are ordered in increasing lengths.

For $k = 1$ a coil of width 1106 mm would be optimal. If (case B) a standard coil would be used the trim loss percentage would increase by about 6 percentage points. For the assumed production

Table 2: Production program

Round No i	Radius [mm]	Needed amount [pieces]	Round No i	Radius [mm]	Needed amount [pieces]
1	50	12000	11	125	65000
2	55	4500	12	130	13500
3	60	6000	13	140	4000
4	70	6500	14	150	24000
5	75	10000	15	160	14000
6	90	2500	16	175	10000
7	100	12000	17	180	2500
8	110	14000	18	200	8000
9	120	110000	19	225	10000
10	122.5	35000	20	250	2000

Table 3: Results for one coil

$k = 1$	Case A Coil width [mm]	Case A Coil area [m^2]	Case A Coil length [m]	Case B Coil width [mm]	Case B Coil area [m^2]	Case B Coil length [m]
Coil 1	1106	24181.68	21864.09	1250	26096.92	20877.54
V		20.35 %			26.19 %	

program in case B the needed area (and hence the amount of trim loss) would be about 2000 m^2 larger.

Table 4: Results for two coils

$k = 2$	Case A Coil width [mm]	Case A Coil area [m^2]	Case A Coil length [m]	Case B Coil width [mm]	Case B Coil area [m^2]	Case B Coil length [m]
Coil 1	915	11717.63	12806.15	308	5987.18	19438.89
Coil 2	1095	11768.83	10747.79	942	19270.09	20456.58
V		17.99 %			24.67 %	

Using two coils instead of one would allow to reduce the amount of trim loss by more than two percentage points if no restrictions on the coil widths are made. If a standard coil divided into two twins of widths 308 mm and 942 mm has to be ordered the trim loss would be about 6.7 percentage points higher than in the unrestricted case. Nevertheless about 1000 m more are needed from the wider coil than from the smaller one. But compared to the one-coil case more than 400 m less are needed in the two-coil case.

Using three coils instead of two in case A the trim loss proportion would be less than 17%, whereas for two standard coils, one subdivided into twins of widths 308 mm and 942 mm the proportion of trim loss would increase by nearly 6 percentage points. The used lengths of the twin parts coincide rather perfectly.

If four coils are used compared to the three coils solution the trim loss proportion could further be reduced by about 0.7 percentage points. In case B a solution is found where one standard coil should

Table 5: Results for three coils

$k=3$	Case A Coil width [mm]	Case A Coil area [m^2]	Case A Coil length [m]	Case B Coil width [mm]	Case B Coil area [m^2]	Case B Coil length [m]
Coil 1	915	9936.40	10859.45	308	4558.56	14800.53
Coil 2	970	3861.95	3981.18	942	13936.57	14794.66
Coil 3	1095	9386.91	8572.52	1250	6389.11	5111.29
V		16.92 %			22.61 %	

Table 6: Results for four coils

$k=4$	Case A Coil width [mm]	Case A Coil area [m^2]	Case A Coil length [m]	Case B Coil width [mm]	Case B Coil area [m^2]	Case B Coil length [m]
Coil 1	915	7864.01	8594.54	150	278.20	1854.67
Coil 2	975	3862.49	3961.53	307	2089.59	6806.47
Coil 3	1095	9386.91	8572.52	943	6475.22	6866.61
Coil 4	1243	1876.93	1510.00	1100	15427.45	14024.96
V		16.22 %			26.24 %	

be subdivided into parts with widths 150 mm and 1100 mm and the second one into twins of widths 307 mm and 943 mm. From the smallest coil almost nothing is used whereas very much is used from the widest coil. The difference in lengths of about 12 km is treated as trim loss again. The proportion of trim loss would be about ten percentage points higher than in case A and surprisingly would also exceed our solution in the three coil case by about 3.6 percentage points. This observation is due to unfavourable starting values and to the suboptimality of the procedure, too.
For rounds which are assigned to rarely used coils every reassignment of them could not decrease the amount of trim loss further.
A rearrangement of rounds will not lead to the real optimum in general.
Even if an increase in the number of simulations can help we will describe a simple correction in the software program to avoid this case in the next section.

Table 7: Results for five coils

$k=5$	Case A Coil width [mm]	Case A Coil area [m^2]	Case A Coil length [m]	Case B Coil width [mm]	Case B Coil area [m^2]	Case B Coil length [m]
Coil 1	850	1929.80	2270.35	154	498.64	3237.90
Coil 2	915	7864.01	8594.54	336	1897.04	5645.95
Coil 3	979	2883.29	2945.14	914	5157.82	5643.13
Coil 4	1000	1236.40	1236.40	1096	3551.21	3240.15
Coil 5	1095	9077.02	8289.52	1250	14072.66	11258.13
V		16.22 %			23.50 %	

Using five instead of four coils can not reduce trim loss for arbitrary coil widths in our example. In case of twin coil widths trim loss proportion can be decreased by nearly 3 percentage points, but still is more than 7 percentage points higher than in the unrestricted case. Used lengths of twin coils are nearly identical. But it should be noticed that the result is still worse than in the three coil case, which again indicates the suboptimality of the procedure.

9 Some final comments

In order to assess the actually computed trim loss proportion it should be noticed that the minimum value for an arrangement of circles in the plane would be $1 - \frac{\pi}{2\sqrt{3}} \approx 9.31\%$ (see [9], S. 46). But such a small value is only possible in practice for very wide coils, very small rounds and very small values of Rs and Is. For the rounds of the above production program it was never observed a smaller amount of trim loss than 14.8%.
Even if computations can stop in a local minimum the results will give at least an upper bound of the increasing amount of trim loss.

Only for an even number of coils we often found that the smallest and the largest twin coils in case B are used in very different lengths. Hence we would recommend an odd number of coils.

Even if the procedure described above usually shows stable and plausible results, it has to be mentioned that especially in the restricted case sometimes only suboptimal solutions have been found as mentioned above. Therefore, the number of simulations should not be choosen too small.
Furthermore, we recommend calculations for coil numbers 1, 2, ... up to a maximum number of coils. In this manner we can recognize also, for which number of coils the proportion of trim loss is decreasing only a little by using an additional coil but for the enterprise it requires to store this additional coil.

Some simple modifications are recommended for practical use:

To avoid worse results for a greater number of coils, as discovered in the example, it might be possible – besides the increase of the number of starting values – to use the optimal coil widths for the actual coil number k as starting values for calculations with the increased coil number $k + 1$. In case A the starting value for the additional coil could be 1250 mm. In case B for even k we can also use an additional coil with width 1250 mm. For odd k the additional coil $k + 1$ should be used as twin of the coil with initial width 1250 mm, what means that its starting width should be set to 0 mm.

Instead of deterministic rearranging of the rounds, where the achievable optimum depends on the order of the rounds it is possible to rearrange the rounds randomly. This would increase the number of steps but will provide a more or less high probability of finding the global optimum up to a given accuracy.

Now let us reconsider our starting question for the enterprise:
Should it accept the offer of the steal producer or not?
For our production program we would conclude that the offer should be accepted if the price reduction would be at least about 7%.

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