Refined Parameterizations for Computing Colored Cuts in Edge-Colored Graphs*

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Abstract. In the COLORED (s,t)-CUT problem, the input is a graph $G = (V, E)$ together with an edge-coloring $\ell : E \to C$, two vertices s and t, and a number k. The question is whether there is a set $S \subseteq C$ of at most k colors, such that deleting every edge with a color from S destroys all paths between s and t in G. We continue the study of the parameterized complexity of COLORED (s,t)-CUT. First, we consider parameters related to the structure of G. For example, we study parameterization by the number ξ_i of edge deletions that are needed to transform G into a graph with maximum degree i. We show that COLORED (s,t)-CUT is W[2]-hard when parameterized by ξ_3, but fixed-parameter tractable when parameterized by ξ_2. Second, we consider parameters related to the coloring ℓ. We show fixed-parameter tractability for three parameters that are potentially smaller than the total number of colors $|C|$ and provide a linear-size problem kernel for a parameter related to the number of edges with a rare edge color.

1 Introduction

The design of networks that are robust against failure of network components is an important step in the quest for secure communication systems [9]. Since current communication networks are in fact multilayer networks, it is important to consider *multiple failure* scenarios where a failure of a single layer may affect direct connections between many different nodes at once—even if these nodes are spread widely throughout the network [1,5]. Thus, it has been proposed to use edge-colored graphs consisting of a graph $G = (V, E)$, a color set C, and an edge-coloring $\ell : E \to C$ to model the layers. If a network layer fails, then all edges with the corresponding color become unavailable for communication. In other words, we may think of these edges as being removed from the graph. One measure for the vulnerability of a network in this model is the number of layers that have to fail in order to disconnect two given important nodes s and t. To compute this vulnerability measure, one needs to solve the following computational problem [1,5].

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Colored (s,t)-cut
Input: An edge-colored graph $(G = (V, E), C, \ell)$, two vertices s and t, and a positive integer k.
Question: Is there a subset of colors $S \subseteq C$ with $|S| \leq k$ such that s and t are not in the same connected component in $G' := (V, E \setminus E_S)$, where $E_S := \{e \in E \mid \ell(e) \in S\}$?

Colored (s,t)-cut is NP-hard [1,5]. Motivated by this hardness, we study the parameterized complexity of the problem.

Known Results and Related Work. To our knowledge, Colored (s,t)-cut was first introduced in a directed version in the context of the analysis of directed attack graphs [7,10]. It was shown, by a reduction from Hitting Set, that in this setting computing (s,t)-cuts with few colors is NP-hard [7,10]. While the graph is directed in this case, the reduction can be easily adapted to show NP-hardness of the undirected case by discarding all edge directions in the constructed graph G. Moreover, this reduction also implies that Colored (s,t)-cut is W[2]-hard when parameterized by k. In later work, this reduction from Hitting Set and the above-mentioned hardness results were also discovered directly for Colored (s,t)-cut [6,8,11,12]. When the same reduction is from Vertex Cover, the special case of Hitting Set where every hyperedge has size two, then the resulting instance of Colored (s,t)-cut has a vertex cover of size two [12] making the problem NP-hard even in this very restricted case. Moreover, Colored (s,t)-cut is NP-hard even if G is a complete graph [11].

On the positive side, by considering all possible choices for choosing the k colors that shall be removed, Colored (s,t)-cut can be solved in $n^{\mathcal{O}(k)}$ time. This implies an $n^{\mathcal{O}(\Delta)}$-time algorithm, where Δ is the maximum degree of G, since instances with $\Delta \leq k$ are trivial yes-instances. Moreover, Colored (s,t)-cut can be solved in $\mathcal{O}(2^c \cdot (n+m))$ time, where $c := |C|$ is the number of colors. Colored (s,t)-cut can be solved in polynomial time when each edge color appears in at most two (s,t)-paths [8,11] and if every edge color has span one. Herein, the *span* of a color is the number of connected components in the subgraph of G that contains only the edges of this color and their endpoints [1]. The latter result was later extended to an algorithm with running time $2^{c_{\text{span}}} \cdot n^{\mathcal{O}(1)}$ where c_{span} is the number of edge colors that have span at least two [2,8,11]. Colored (s,t)-cut is fixed-parameter tractable (FPT) with respect to the combination of $p_{\max}$ and k where $p_{\max}$ is the number of edges of a longest simple path between s and t [13]. Finally, Colored (s,t)-cut is FPT with respect to the number of (s,t)-paths in G [8]. For all known nontrivial parameters that lead to FPT algorithms, that is, for c, $p_{\max} + k$, c_{span}, and for the number of (s,t)-paths, Colored (s,t)-cut does presumably not admit a polynomial problem kernel [8,11].

Our Results. We study new parameterizations for Colored (s,t)-cut. Recall that it is known that Colored (s,t)-cut is W[2]-hard for the budget parameter k and that Colored (s,t)-cut is NP-hard even when G has a vertex cover of size two. The latter result excludes tractability for most standard parameterizations that are related to the structure of G, for example for the treewidth of G, the vertex deletion distance to forests (known as feedback vertex set number), or

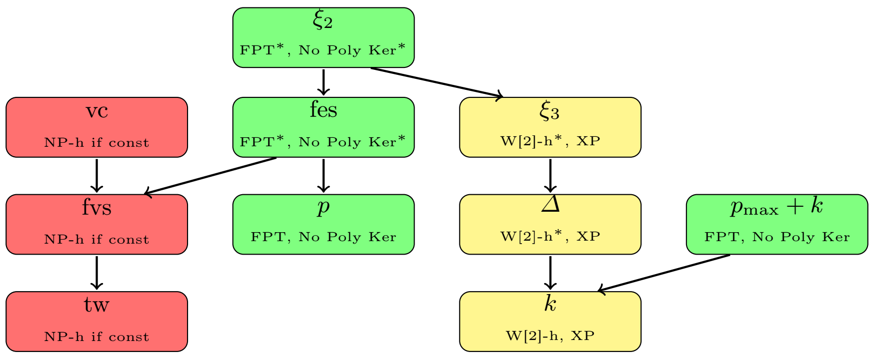

Fig. 1: The parameterized complexity of COLORED (s, t)-CUT for structural graph parameters; vc, fes, fvs, and tw denote the vertex cover number, feedback edge set number, feedback vertex set number, and treewidth, respectively. New results are marked by an asterisk. An arrow $a \to b$ symbolizes that $a \geq g(b)$ for some function g in all graphs. Note that $\xi_2 \to$ fes holds only for connected graphs; for COLORED (s, t)-CUT we assume that G is connected.

the vertex deletion distance to graphs with maximum degree i: the corresponding parameters are never larger than the size of a smallest vertex cover of G. Thus, we first consider parameters that are related to the *edge deletion distance* to tractable cases of COLORED (s, t)-CUT. Our results are shown in Fig. 1.

Since COLORED (s, t)-CUT can be solved in polynomial time on graphs with constant maximum degree Δ, we consider parameterization by ξ_i, the number of edges that need to be deleted in order to transform G into a graph with maximum degree i. We show that for all $i \geq 3$, COLORED (s, t)-CUT is W[2]-hard for ξ_i. This also implies W[2]-hardness for the parameter Δ: For a vertex of degree $\Delta \geq i$, at least ξ_i incident edges have to be deleted to decrease its degree to i. Hence, $\Delta \leq \xi_i + i$. Consequently, our result strengthens the W[2]-hardness for the parameter k, as $k \leq \Delta$ in all non-trivial instances. Hence, the known $n^{\mathcal{O}(\Delta)}$-time algorithm for graphs with constant maximum degree cannot be improved to an algorithm with running time $f(\Delta) \cdot n^{\mathcal{O}(1)}$. We then show an FPT algorithm for parameterization by ξ_2. This algorithm is obtained via the FPT algorithm for the parameter "number p of simple (s, t)-paths in G". The latter algorithm also gives an FPT algorithm for parameterization by the feedback edge set number of G, the number of edges that need to be removed to transform G into a forest. We also observe that COLORED (s, t)-CUT does not admit a polynomial kernel for ξ_2 and for the feedback edge set number of G.

We then study parameterizations that are related to the edge-coloring ℓ of G; our results are shown in Fig. 2. Assume that $C = \{\alpha_1, \ldots, \alpha_c\}$ and there are at least as many edges with color α_i as with color α_{i+1} for all $i < c$. For any number q, we let the parameter $m_{>q} := |\{e \in E \mid \ell(e) = \alpha_j \text{ for } j > q\}|$ denote the number of edges with a color that is not among the q most frequent colors.

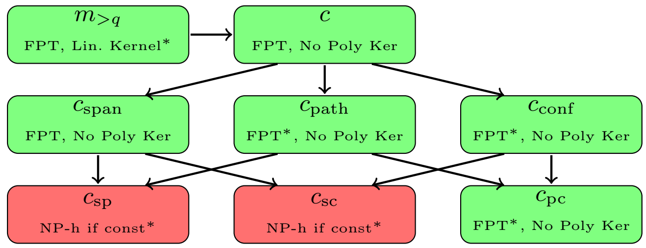

Fig. 2: An overview of the parameterized complexity of Colored (s,t)-cut for color-related parameters as analyzed in Sections 3 and 4. New results are marked by an asterisk. An arrow $a \to b$ between two parameters a and b, symbolizes that $a \geq g(b)$ for some function g in all instances.

Note that $c \leq m_{>q} + q$ and $m_{>q} \leq m$. Hence, for constant q, the parameter $m_{>q}$ is an intermediate parameter between c and m. We show that for all constant q, Colored (s,t)-cut admits a problem kernel of size $\mathcal{O}(m_{>q})$.

We then provide a general framework to obtain FPT algorithms for parameters that are potentially smaller than c, the number of colors. To formulate the framework, we identify certain properties of color sets in the input instances that directly give an FPT algorithm for the parameterization by the size of this color set. We then provide four applications of this framework. The first application is for c_{span}, the number of colors with span at least two. For this parameterization, an FPT algorithm is already known [2,8,11], and an algorithm with the same running time can be obtained by applying our framework. The second application is for parameterization by the number c_{path} of colors that appear in at least three (s,t)-paths. Using our framework, we extend the known polynomial-time algorithm for $c_{\text{path}} = 0$ to an FPT algorithm with running time $2^{c_{\text{path}}} \cdot n^{\mathcal{O}(1)}$. The third application is for the parameterization by c_{conf} which we define as follows. Two colors i and j are in *conflict* if G contains some (s,t)-path containing i and j. Then, c_{conf} is the number of colors i that are in conflicts with at least three other colors. We show by applying our framework, that Colored (s,t)-cut can be solved in $2^{c_{\text{conf}}} \cdot n^{\mathcal{O}(1)}$ time. Finally, we strengthen the latter two results by showing an FPT algorithm for the parameter c_{pc} counting the number of colors which are in at least three paths and in at least three conflicts. The parameter c_{pc} can be seen as an "intersection" of c_{path} and c_{conf}. We also show that Colored (s,t)-cut is NP-hard even when every color has span one or occurs in at most two paths, and NP-hard even when every color has span one or occurs in at most two conflicts. Thus, an FPT algorithm is unlikely for the intersection of c_{span} with c_{path} or c_{conf}, denoted by c_{sp} and c_{sc}, respectively.

Preliminaries. An *edge-colored graph* is a triple $\mathcal{H} = (G = (V,E), C, \ell : E \to C)$ where G is an undirected graph, C is a set of colors and $\ell : E \to C$ is an *edge coloring*. We extend the definition of ℓ to edge sets $E' \subseteq E$ by defining $\ell(E') := \{\ell(e) \mid e \in E'\}$. We let n and m denote the number of vertices and edges in G, respectively, and c the size of the color set C. We call $|I| := m + n$ the *size* of an instance I. We assume $k < m$ and that all input graphs are connected, since connected components containing neither s nor t may be removed.

In a graph $G = (V, E)$, we call a sequence of vertices $P = (v_1, \ldots, v_k) \in V^k, k \geq 1$, a *path* of length $k-1$ if $\{v_i, v_{i+1}\} \in E$ for all $1 \leq i < k$. If $v_i \neq v_j$ for all $1 \leq i < j \leq k$, then we call P *vertex-simple*. If not mentioned otherwise, we only talk about vertex-simple paths. Furthermore, we say that a path $(v_1, \ldots, v_k)$ is a (v_1, v_k)-*path*. We denote with $V(P) := \{v_i \mid 1 \leq i \leq k\}$ the vertices of P and with $E(P) := \{\{v_i, v_{i+1}\} \mid 1 \leq i < k\}$ the edges of P. Hence, $\ell(E(P))$ denotes the set of colors of a path P in a colored graph $(G = (V, E), C, \ell)$. Given two paths $P_1 = (v_1, \ldots, v_k)$ and $P_2 = (w_1, \ldots, w_r)$ in G, we define the *concatenation* as $P_1 \cdot P_2 := (v_1, \ldots, v_k, w_1, \ldots, w_r)$. Note that $P_1 \cdot P_2$ is a path if $\{v_k, w_1\} \in E$. Let $\mathcal{H} = (G, C, \ell)$ be a colored graph and let $s, t \in V$ be two vertices in G. We say that $\tilde{C} \subseteq C$ is a *colored* (s,t)-*cut* in G if for every (s,t)-path P in G, $\ell(E(P)) \cap \tilde{C} \neq \emptyset$. We denote by $\mathcal{C}(\mathcal{H}) := \{\ell(E(P)) \mid P \text{ is an } (s,t)\text{-path in } G\}$ the collection of sets of colors of vertex simple (s,t)-paths in G. Note that $\tilde{C} \subseteq C$ is a colored (s,t)-cut in G if and only if $\tilde{C} \cap C' \neq \emptyset$ for all $C' \in \mathcal{C}(\mathcal{H})$. The following lemma implies that we can efficiently compute an "or"-composition of many COLORED (s,t)-CUT instances.

Lemma 1 ([11]). *Let $I_1, I_2, \ldots, I_i$ be a set of* COLORED (s,t)-CUT *instances with the same budget k. Then, we can compute in linear time an instance I' with budget k such that I' is a yes-instance if and only if I_j is a yes-instance for at least one $j \in \{1, \ldots, i\}$ and $|I'| \leq \sum_{j=1}^{i} |I_i|$.*

For the standard notions on parameterized complexity, refer to [3,4]. Due to lack of space, several proofs are deferred to the full version of this paper.

2 Structural Graph Parameters

As discussed above, COLORED (s,t)-CUT is unlikely to be FPT for vertex deletion parameters. We thus consider edge deletion parameters.

Definition 1. *Let $G = (V, E)$ be a graph and $i \geq 0$ be an integer. Further, let $\xi_i := \min\{|E'| \mid E' \subseteq E, G - E' \text{ has a maximum degree of } i\}$ be the* edge deletion distance to a maximum degree of *i.*

Since COLORED (s,t)-CUT can be solved in polynomial time for constant Δ, the parameter ξ_i measures the distance to a trivial case. Since $\Delta \leq \xi_i + i$, COLORED (s,t)-CUT parameterized by ξ_i is in XP when i is constant. The larger i, the smaller the parameter value ξ_i will be in most instances. We now show that even for small i, namely for $i = 3$, an FPT algorithm for ξ_i is unlikely.

Theorem 1. COLORED (s,t)-CUT *parameterized by ξ_3 is* W[2]-*hard even on planar graphs.*

We now show that this result is tight by showing an FPT algorithm for ξ_2 which is obtained via an FPT algorithm for p, the number of (s,t)-paths in G.

Proposition 1 ([8]). Colored (s,t)-cut *is* FPT *parameterized by* p *and does not admit a polynomial kernel unless* NP $\subseteq$ coNP/poly.

For a graph $G = (V, E)$, we call $F \subseteq E$ a *feedback edge set* if $G - F$ is a forest. We define with $\text{fes} := \min\{|F| \mid F \text{ is a feedback edge set}\}$ the *feedback edge set number*. The following can be obtained by applying Proposition 1.

Proposition 2. Colored (s,t)-cut *is* FPT *parameterized by* fes *or* ξ_2 *does not admit a polynomial kernel for* $\text{fes} + \xi_2$*, unless* NP $\subseteq$ coNP/poly.

3 A Kernel for the Number of Edges with Rare Colors

In this section, we give a linear problem kernel for Colored (s,t)-cut parameterized by the number of edges whose color is not among the top-q most frequent colors. More precisely, we define a family of parameters $m_{>q}$ for every $q \in \mathbb{N}$ as follows. For a Colored (s,t)-cut-instance I with color set C, let $(\alpha_1, \alpha_2, \ldots, \alpha_c)$ be an ordering of the colors in C such that the number of edges with color α_i is not smaller than the number of edges with color α_{i+1} for all $i \in \{1, \ldots, c-1\}$. For a given constant q, let $\tilde{C} \subseteq C$ be the set of the q most frequent colors. We then define $m_{>q}$ as the number of edges that are not assigned to a color in $\tilde{C}$. In the following, we show a linear problem kernel for $m_{>q}$ for every q.

Informally, the kernel is based on the following idea: Since q is a constant, we may try all possible partitions of $\{\alpha_1, \ldots, \alpha_q\}$ into a set of colors C_r that we want to remove and a set of colors C_m that we want to keep. Fix one partition (C_r, C_m). Under the assumption posed by this partition, we can simplify the instance as follows. The edges of C_r can be deleted. Moreover, all vertices that are connected by a path P in G, such that $\ell(E(P)) \subseteq C_m$ cannot be separated anymore under this assumption. Thus, all vertices of P can be merged into one vertex. To formalize this merging, we give the following definition. For a colored graph $(G = (V, E), \ell)$ and a set $C_m \subseteq C$, we define $[v]_{C_m} := \{u \in V \mid \exists P = (v, \ldots, u) \text{ in G} : \ell(E(P)) \subseteq C_m\}$ as the set of vertices that are connected to v by a path only colored in C_m. If C_m is clear from the context, we may only write $[v]$. The instance that can be built for specific sets C_r and C_m is defined as follows.

Definition 2. *Let* $I = (G, C, \ell, s, t, k)$ *be a* Colored (s,t)-cut *instance and let* $C_r, C_m \subseteq C$ *with* $C_r \cap C_m = \emptyset$*. The* remove-merge-instance *of* I *with respect to* (C_r, C_m) *is* $\text{rmi}(I, C_r, C_m) := (G' = (V', E'), C', \ell', [s], [t], k - |C_r|)$*, where* $C' := C \setminus (C_r \cup C_m)$*,* $V' := V'_1 \cup V'_2$*, and*

$$\begin{aligned}
V'_1 &:= \{[v] \mid v \in V\},\\
V'_2 &:= \{v^{\alpha}_{\{[u],[w]\}} \mid [u], [w] \in V'_1, \alpha \in C', [u] \neq [w],\\
&\qquad \exists u' \in [u], w' \in [w] : \{u', w'\} \in E, \ell(\{u', w'\}) = \alpha\},\\
E' &:= \{\{[w], v^{\alpha}_{\{[u],[w]\}}\}, \{[u], v^{\alpha}_{\{[u],[w]\}}\} \mid v^{\alpha}_{\{[u],[w]\}} \in V'_2\}, \textit{and}\\
\ell'(\{[u], v^{\alpha}_{\{[u],[w]\}}\}) &:= \alpha.
\end{aligned}$$

The vertices of V_2' only exist to prevent G' from having parallel edges. We first show that a remove-merge-instance can be computed efficiently.

Proposition 3. *Let $I = (G = (V, E), C, \ell, s, t, k)$ be a* Colored (s,t)-cut *instance, and let $I' = \mathrm{rmi}(I, C_r, C_m)$ be the remove-merge-instance of I for some $C_r, C_m \subseteq C$ such that $C_r \cap C_M = \emptyset$. Then, $|I'| \in \mathcal{O}(|I|)$ and I' can be computed in $\mathcal{O}((|C_r| + |C_m|) \cdot m)$ time.*

We now show that for any $\tilde{C} \subseteq C$, we can solve the original instance by creating and solving all possible remove-merge-instances for subsets of $\tilde{C}$.

Lemma 2. *Let $I := (G = (V, E), C, \ell, s, t, k)$ be a* Colored (s,t)-cut *instance and let $\tilde{C} \subseteq C$, then I is a yes-instance if and only if there is a subset $C_r \subseteq \tilde{C}$ such that the remove-merge-instance $I' := \mathrm{rmi}(I, C_r, \tilde{C} \setminus C_r)$ is a yes-instance.*

Theorem 2. *For every constant $q \in \mathbb{N}$,* Colored (s,t)-cut *admits a problem kernel of size $\mathcal{O}(m_{>q})$ that can be computed in $\mathcal{O}(|I|)$ time.*

Proof. Let $I = (G = (V, E), C, \ell, s, t, k)$ be an instance of Colored (s,t)-cut and let $\tilde{C} = \{\alpha_1, \alpha_2, \dots, \alpha_q\} \subseteq C$ be the set of the q most-frequent colors. We first describe how to compute an equivalent instance I' from I in linear time and afterwards we show that $|I'| \in \mathcal{O}(m_{>q})$.

Construction of I'. We start by computing the set $\mathcal{I} = \{\mathrm{rmi}(I, C_r, \tilde{C} \setminus C_r) \mid C_r \subseteq \tilde{C}\}$ containing for every $C_r \subseteq \tilde{C}$, the remove-merge instances of I with respect to $(C_r, \tilde{C} \setminus C_r)$. Note that $|\mathcal{I}| = 2^q \in \mathcal{O}(1)$. We write $\mathcal{I} = \{I_1, I_2, \dots, I_{2^q}\}$ and let $I_i =: (G_i = (V_i, E_i), C_i, \ell_i, [s]_i, [t]_i, k_i)$ denote each instance $I_i \in \mathcal{I}$. By Proposition 3 we can compute every $I_i \in \mathcal{I}$ in $\mathcal{O}(q \cdot |I|) = \mathcal{O}(|I|)$ time. Therefore, we can compute $\mathcal{I}$ in $\mathcal{O}(|I|)$ time. Note that $\max_{i \in \{1,\dots,2^q\}} k_i = k$ and that $C_i = C \setminus \tilde{C}$ for every $i \in \{1, \dots, 2^q\}$.

Next, we apply the algorithm of Lemma 1 on all instances of $\mathcal{I}$. Note that the budgets k_i of the instances $I_i \in \mathcal{I}$ might not be equal. Thus, in order to apply Lemma 1 we transform every instance $I_i \in \mathcal{I}$ into an instance I_i^* by adding auxiliary vertices $v_1, \dots, v_{k-k_i}$ to V_i and auxiliary edges $\{[s]_i, v_j\}$ and $\{[t]_i, v_j\}$ for every $j \in \{1, \dots, k-k_i\}$ to E_i. Let V_i^* and E_i^* be the resulting sets. Finally, we set $k_i^* = k$ and $\ell_i^*(e) = \ell_i(e)$ if $e \in E_i$ and $\ell_i^*(\{[s]_i, v_j\}) = \ell_i^*(\{[t]_i, v_j\}) = \alpha_j$ for every $j \in \{1, \dots, k-k_i\}$. Note that we added at most $k-k_i$ vertices and $2(k-k_i$ edges to every instance I_i and that $k-k_i \leq q$. Since q is a constant, $|I_i^*| \in \mathcal{O}(|I_i|)$ and I_i^* can be computed from I_i in $\mathcal{O}(|I_i|)$ time.

Let $\mathcal{I}^* = \{I_1^*, \dots, I_{2^q}^*\}$ be the resulting set of instances. Note that the budget is k in all instances in $\mathcal{I}^*$. Therefore, we can apply Lemma 1 on the 2^q instances in $\mathcal{I}^*$ and compute an instance I' in $\mathcal{O}(|I|)$ time, such that I' is a yes-instance if and only if there exists some $i \in \{1, \dots, 2^q\}$ such that I_i^* is a yes-instance. We defer the proof of the equivalence of I and I'.

Size of I'. It remains to give a bound for the size of I'. By Definition 2 of remove-merge-instances, every $I_i \in \mathcal{I}$ contains no edges with a color in $\tilde{C}$, and subdivides every other edge of I. Therefore, every $I_i \in \mathcal{I}$ contains at most $2m_{>q}$ edges. Since $|I_i^*| \in \mathcal{O}(|I_i|)$ we conclude $|I_i^*| \in \mathcal{O}(m_{>q})$ for every $I_i^* \in \mathcal{I}$. Finally, by Lemma 1 it holds that $|I'| \leq \sum_{i=1}^{2^q} |I_i^*| \in \mathcal{O}(m_{>q})$, since $2^q \in \mathcal{O}(1)$. □

4 Parameterization by Color Subsets

In this section we present a general framework for color parameterizations of Colored (s,t)-cut leading to an FPT algorithm. To apply our framework, one has to check two properties of the parameterization.

Definition 3. *A function π that maps every instance $I = (G, C, \ell, s, t, k)$ of* Colored (s,t)-cut *to a subset $\pi(I) \subseteq C$ of the colors of I is called a* color parameterization. *If for every* Colored (s,t)-cut *instance I, $\pi(I)$ can be computed in polynomial time and I can be solved in polynomial time if $\pi(I) = \emptyset$, then π is called a* polynomial color parameterization.

In the following, we will only deal with polynomial color parameterizations. Next, we will use remove-merge-instances to transform an instance I of Colored (s,t)-cut to a set $\mathcal{I}$ of remove-merge-instances of Colored (s,t)-cut such that $\pi(I') = \emptyset$ for each $I' \in \mathcal{I}$ and $\mathcal{I}$ has size $f(\pi(I))$ for some computable function f. Each I' can be solved in polynomial-time since π is polynomial and $\pi(I') = \emptyset$. This leads to an FPT algorithm.

Definition 4. *A color parameterization π has the* strong remove-merge property *if for every* Colored (s,t)-cut *instance I, every $\tilde{C}$ and every $C_r \subseteq \tilde{C}$ it holds that $\pi(I') \subseteq \pi(I)$ where $I' := \mathrm{rmi}(I, C_r, \tilde{C} \setminus C_r)$. Further, π has the* weak remove-merge property *if for every* Colored (s,t)-cut *instance I and every $C_r \subseteq \pi(I)$ it holds that $\pi(I') = \emptyset$ where $I' := \mathrm{rmi}(I, C_r, \pi(I) \setminus C_r)$.*

Lemma 3. *If π has the strong remove-merge property, π also has the weak remove-merge property.*

Lemma 4. *Let π be a polynomial color parameterization with the weak remove-merge property. Then, any instance I of* Colored (s,t)-cut *can be solved in $2^{|\pi(I)|}|I|^{\mathcal{O}(1)}$ time and* Colored (s,t)-cut *does not admit a polynomial kernel for $|\pi(I)|$, unless $\mathrm{NP} \subseteq \mathrm{coNP/poly}$.*

Proof. First, we present an FPT algorithm with the claimed running time. Let I be an instance of Colored (s,t)-cut. We compute $\pi(I)$ and the set $\mathcal{I}$ of all remove-merge-instances for G with respect to $\pi(I)$ and answer yes if and only if there is some $I' \in \mathcal{I}$ such that I' is a yes-instance. This algorithm is correct due to Lemma 2. Since π is a polynomial color parameterization, we can compute $\pi(I)$ in polynomial time. Since $|\mathcal{I}| = 2^{|\pi(I)|}$, we can compute $\mathcal{I}$ in $2^{|\pi(I)|}|I|^{\mathcal{O}(1)}$ time. Since π is a polynomial color parameterization that has the weak remove-merge property, we can solve each $I' \in \mathcal{I}$ in $|I|^{\mathcal{O}(1)}$ time. Hence, this algorithm runs in $2^{|\pi(I)|}|I|^{\mathcal{O}(1)}$ time. The kernel lower bound follows from the fact that in every instance I of Colored (s,t)-cut it holds that $|\pi(I)| \leq c$ and Colored (s,t)-cut admits no kernel when parameterized by c, unless $\mathrm{NP} \subseteq \mathrm{coNP/poly}$. □

Next, we apply Lemma 4 to three polynomial color parameterizations. The proof for the parameterization by c_{span} is deferred to the full version.

4.1 Number of Path-Frequent Colors

This parameter counts the number of colors occurring on many (s,t)-paths.

Definition 5. *Let $I = (G = (V,E), C, \ell, s, t, k)$ be a* Colored (s,t)-cut *instance. A color $\alpha \in C$ is called* path-frequent *if there exist at least three vertex-simple (s,t)-paths such that at least one edge on each path has color α.*

By C_{path} we denote the function that maps each Colored (s,t)-cut instance I to the set of path-frequent colors of I. Further, for a fixed instance I, let $c_{\text{path}} := |C_{\text{path}}(I)|$. For a fixed color α one can test in polynomial time whether α is path-frequent [11]. Further, an instance I of Colored (s,t)-cut can be solved in polynomial time if $C_{\text{path}}(I) = \emptyset$. [11]. Thus, the following holds.

Lemma 5. *The function C_{path} is a polynomial color parameterization. Moreover, for every α that is contained in at most two (s,t)-paths we can compute all these (s,t)-paths in polynomial time.*

Lemma 6. *The function C_{path} has the strong remove-merge property.*

Proof. Let $I = (G, C, \ell, s, t, k)$ be an instance of Colored (s,t)-cut, let $\tilde{C} \subseteq C$, let $C_r \subseteq \tilde{C}$ be the colors which will be removed and let $I' = (G', C', \ell', [s], [t], k - |C_r|) := \text{rmi}(I, C_r, \tilde{C} \setminus C_r)$ be the resulting remove-merge-instance. We show that $C_{\text{path}}(I') \subseteq C_{\text{path}}(I)$. Assume towards a contradiction that there is a color $\alpha \in C_{\text{path}}(I') \setminus C_{\text{path}}(I)$. Thus, there are three vertex-simple $([s],[t])$-paths P_i for $i = \{1,2,3\}$ in G' such that $\ell'(E(P_i)) \subseteq C \setminus C_{\text{path}}(I)$ and each path contains an edge of color α. By construction of G', we can assume without loss of generality that $P_i = ([v_1], v^{\alpha_1}_{[v_1],[v_2]}, [v_2], \ldots, [v_{i_r}])$ for some $i_r \in \mathbb{N}$ where $s \in [v_1]$ and $t \in [v_{i_r}]$. By definition of G', it follows that there exists some $v_i^{j_{\text{in}}} \in [v_i]$ and some $v_i^{j_{\text{out}}} \in [v_{i+1}]$ such that $e_i^j := \{v_i^{j_{\text{in}}}, v_i^{j_{\text{out}}}\} \in E$ with $\ell(e_i^j) = \alpha_i$ for each j, $1 \leq j < i_r$, where $\alpha_i \in C \setminus \tilde{C}$. Further, we set $v_i^{1_{\text{in}}} = s$ and $v_{i_r}^{j_{\text{out}}} = t$, and since $v_i^{j_{\text{in}}}, v_i^{j_{\text{out}}} \in [v_i]$, we can conclude that there is a path P_i^j from $v_i^{j_{\text{in}}}$ to $v_i^{j_{\text{out}}}$ in G such that $\ell(E(P_i^j)) \subseteq \tilde{C} \setminus C_r$. Then $P^i := P_i^1 \cdot P_i^2 \cdot \ldots \cdot P_i^{i_r}$ is a vertex-simple (s,t)-path in G such that $\ell(E(P^i)) \subseteq C \setminus C_r$. Hence, there exist at least three paths from s to t such that at least one edge has color α, a contradiction. ◻

Lemmas 4, 5, and 6 now give an FPT algorithm.

Theorem 3. Colored (s,t)-cut *can be solved in $\mathcal{O}(2^{c_{\text{path}}}|I^{O(1)}|)$ time.*

4.2 Number of Colors in at least Three Conflicts

The next parameter concerns colors which occur on vertex-simple (s,t)-paths with many different colors.

Definition 6. *Let $I = (G = (V,E), C, \ell, s, t, k)$ be a* Colored (s,t)-cut *instance. Two colors $\alpha, \beta \in C$ form a* conflict *if there exists an (s,t)-path such that at least one edge on this path has color α and at least one edge has color β.*

By C_{conf} we denote the function that maps an instance I of Colored (s,t)-cut to the set of colors of I which are in conflict with at least three different colors. Further, for a fixed instance I, let $c_{\text{conf}} := |C_{\text{conf}}(I)|$.

Lemma 7. *Let $D \subseteq C$ be a color set of size at most three, then we can determine in polynomial time if there is an (s,t)-path P on G such that $D \subseteq \ell(E(P))$.*

Lemma 8. *The function C_{conf} is a polynomial color parameterization.*

Lemma 9. *The function C_{conf} has the strong remove-merge property.*

Proof. Let $I = (G, C, \ell, s, t, k)$ be an instance of Colored (s,t)-cut, let $\tilde{C} \subseteq C$, let $C_r \subseteq \tilde{C}$ be the colors which will be removed and let $I' = (G', C', \ell', [s], [t], k - |C_r|) := \text{rmi}(I, C_r, C_{\text{conf}} \setminus C_r)$ be the resulting remove-merge-instance. We show that $C_{\text{conf}}(I') \subseteq C_{\text{conf}}(I)$. Assume towards a contradiction that there exist a color $\alpha \in C_{\text{conf}}(I')$ such that $\alpha \notin C_{\text{conf}}(I)$ and α forms conflicts with colors $\beta_1, \beta_2, \beta_3$. Let $P = ([v_1], v^{\alpha_1}_{[v_1],[v_2]}, [v_2], \ldots, [v_x])$ for some $x \in \mathbb{N}$ be a vertex-simple (s,t)-path in G' containing at least one edge of color α and at least one edge of color β_i for some $i \in \{1,2,3\}$, where $s \in [v_1]$ and $t \in [v_x]$. By definition of G' it follows that there exist some $v^{\text{in}}_j \in [v_j]$ and some $v^{\text{out}}_j \in [v_{i+1}]$ such that $e_j := \{v^{\text{in}}_j, v^{\text{out}}_j\} \in E$ with $\ell(e_j) = \alpha_j$ for each $1 \leq j < x$ where $\alpha_i \in C \setminus \tilde{C}$. Further, we set $v^{\text{in}}_1 = s$ and $v^{\text{out}}_x = t$. Since $v^{\text{in}}_j, v^{\text{out}}_j \in [v_j]$ we can conclude that there is a path P_j from v^{in}_j to v^{out}_j in G such that $\ell(E(P_j)) \subseteq \tilde{C} \setminus C_r$. Then $P^* := P_1 \cdot P_2 \cdot \ldots \cdot P_x$ is a vertex-simple (s,t)-path in G such that P^* contains at least one edge of color α and at least one edge of color β_i. Hence, color α forms conflicts with each β_i, a contradiction. □

Lemmas 4, 8, and 9 now give an FPT algorithm.

Theorem 4. Colored (s,t)-cut *can be solved in $\mathcal{O}(2^{c_{\text{conf}}}|I|^{O(1)})$ time.*

4.3 Parameter Intersections

In the following we study Colored (s,t)-cut parameterized by the pairwise intersection of all three parameters of the previous sections.

Theorem 5. *Let I be an instance of* Colored (s,t)-cut *and let π, ϕ be color parameterizations with the strong remove-merge property. Then the intersected parameter $\rho(I) := \pi(I) \cap \phi(I)$ also has the strong remove-merge property.*

Proof. Fix a set $\tilde{C} \subseteq C$, fix a set $C_r \subseteq \tilde{C}$ and let $I' = \text{rmi}(I, C_r, \tilde{C} \setminus C)$ be the resulting remove-merge-instance. We have to show that $\rho(I') \subseteq \rho(I)$. By definition, $\rho(I') = \pi(I') \cap \phi(I')$. Since π and ϕ are strong, we have $\pi(I') \subseteq \pi(I)$ and $\phi(I') \subseteq \phi(I)$. Hence, $\rho(I') \subseteq \pi(I) \cap \phi(I) = \rho(I)$. □

We now study the pairwise intersection of color parameterizations.

Definition 7. *Let* $C_{\text{pc}}(I) := C_{\text{path}}(I) \cap C_{\text{conf}}(I)$ *denote the function that maps an instance* I *of* Colored (s,t)-cut *to the set of colors of* I *which are path-frequent and contained in at least three conflicts. Further, let* $c_{\text{pc}} := |C_{\text{pc}}(I)|$.

Theorem 6. Colored (s,t)-cut *can be solved in* $\mathcal{O}(2^{c_{\text{pc}}}|I|^{O(1)})$ *time.*

Proof. We will prove this theorem by applying Lemma 4. First, we observe that C_{pc} has the weak remove-merge property: Since C_{path} and C_{conf} both have the strong remove-merge property, C_{pc} also has the strong remove-merge property due to Theorem 5.

Second, we show that C_{pc} is polynomial. According to Lemmas 5 and 8 it can be determined in polynomial time whether a color α is in $C_{\text{path}}(I)$ or in $C_{\text{conf}}(I)$. Thus, $C_{\text{pc}}(I)$ can be computed in polynomial time.

It remains to show that an instance $I = (G = (V,E), C, \ell, s, t, k)$ can be solved in polynomial time if $C_{\text{pc}}(I) = \emptyset$. Recall that $\mathcal{C}(I) := \{\ell(E(P)) \mid P$ is a vertex-simple (s,t)-path in $G\}$. Without loss of generality we can assume that each set $D \in \mathcal{C}(I)$ has size at least two. We first show that $\mathcal{C}(I)$ can be computed in polynomial time when $C_{\text{pc}}(I) = \emptyset$. Let $\alpha \in C \setminus C_{\text{path}}(I)$, then there exist at most two paths containing an edge with color α. Both paths can be computed in polynomial time according to Lemma 5. Let $\alpha \in C \setminus C_{\text{conf}}(I)$. In other words, α forms conflicts with at most two other colors β and γ. The colors β and γ can be computed according to Lemma 8. Hence, $\mathcal{C}(I)$ contains at most three sets containing α. Each subset $D \in \mathcal{C}(I)$ can be computed as follows: If color α forms a conflict only with one other color β, then $\{\alpha, \beta\}$ is the unique set in $\mathcal{C}(I)$ containing α. This set can be computed in polynomial time. Now, assume color α forms conflicts with colors β and γ. Next, test if $T := \{\alpha, \beta_1, \beta_2\} \in \mathcal{C}(I)$. This can be done in polynomial time due to Lemma 7. If $T \notin \mathcal{C}(I)$, $\{\alpha, \beta_1\}, \{\alpha, \beta_2\} \in \mathcal{C}(I)$ and there is no other set $D \in \mathcal{C}(I)$ such that $\alpha \in D$. If $T \in \mathcal{C}(I)$, then test for each $i \in \{1,2\}$ whether s and t are connected in $G[\ell^{-1}(\{\alpha, \beta_i\})]$. If yes, the set $\{\alpha, \beta_i\}$ is contained in $\mathcal{C}(I)$.

From $\mathcal{C}(I)$, we now construct an instance $\mathcal{I} := (\mathcal{G} = (\mathcal{V}, \mathcal{E}), C, \ell', s, t, k)$ of Colored (s,t)-cut as follows: For each $D \in \mathcal{C}(I)$ create an (s,t)-path P with $\ell'(P) = D$. Note that S is a colored (s,t)-cut for G if and only if S is a colored (s,t)-cut for $\mathcal{G}$.

Now, we show that a colored (s,t)-cut S with $|S| \leq k$ can be computed in polynomial time for $\mathcal{I}$. Let $\alpha \in C_{\text{path}}(\mathcal{I})$. Hence, $\alpha \in C \setminus C_{\text{conf}}(\mathcal{I})$. Hence, $\mathcal{C}(I)$ contains exactly three sets $T_1 = \{\alpha, \beta_1, \beta_2\}, T_2 = \{\alpha, \beta_1\}$ and $T_3 = \{\alpha, \beta_2\}$ containing color α. Note that if there is a fourth set $D \in \mathcal{C}(I)$ such that $\beta_j \in D$ and $D \setminus T_1 \neq \emptyset$ for some $j \in \{1,2\}$, then $\beta_j \in C_{\text{path}}(I) \cap C_{\text{conf}}(I)$, that is, β_j is in at least four paths in G and β_j forms conflicts with at least three different colors. This contradicts the assumption $C_{\text{pc}} = \emptyset$. Hence, such a set $D \in \mathcal{C}(I)$ does not exist. In other words, there is no color γ such that γ forms a conflict with β_j for $j \in \{1,2\}$. The only possible further set containing β_1 or β_2 can be $T_4 := \{\beta_1, \beta_2\}$. First, assume $T_4 \in \mathcal{C}(I)$. Then each colored (s,t)-cut S of G contains at least two of α, β_1, and β_2. Without loss of generality, add α and β_1 to S. Second, if $T_4 \notin \mathcal{C}(I)$, adding α to S covers each T_i for $i \in \{1,2,3\}$.

Afterwards, for each color α we have $\alpha \notin C_{\text{path}}(I')$ and we can apply Lemma 5. Hence, if $C_{\text{pc}}(I) = \emptyset$, I can be solved in polynomial time. □

As in Definition 7, one can define $C_{\text{ps}}(I) := C_{\text{path}}(I) \cap C_{\text{span}}(I)$ and $C_{\text{sc}}(I) := C_{\text{span}}(I) \cap C_{\text{conf}}(I)$. We show that both of them are not polynomial.

Proposition 4. Colored (s,t)-cut *is* NP*-hard even for instances* I *where* $C_{\text{ps}}(I) = \emptyset$ *and* $C_{\text{sc}}(I) = \emptyset$.

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