On Explaining Integer Vectors by Few Homogeneous Segments☆

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Abstract

We extend previous studies on "explaining" a nonnegative integer vector by sums of few homogeneous segments, that is, vectors where all nonzero entries are equal and consecutive. We study two NP-complete variants which are motivated by radiation therapy and database applications. In Vector Positive Explanation, the segments may have only positive integer entries; in Vector Explanation, the segments may have arbitrary integer entries. Considering several natural parameterizations such as the maximum vector entry γ and the maximum difference δ between consecutive vector entries, we obtain a refined picture of the computational (in-)tractability of these problems. For example, we show that Vector Explanation is fixed-parameter tractable with respect to δ, and that, unless NP $\subseteq$ coNP/poly, there is no polynomial kernelization for Vector Positive Explanation with respect to the parameter γ. We also identify relevant special cases where Vector Positive Explanation is algorithmically harder than Vector Explanation.

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1. Introduction

In this work we study two variants of a "mathematically fundamental" [4], NP-complete combinatorial problem motivated by cancer radiation therapy planning [12] and database and data warehousing applications [1, 22]:

VECTOR (POSITIVE) EXPLANATION
Input: A vector $\mathcal{A} \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0$.
Question: Can $\mathcal{A}$ be explained by at most k (positive) segments?

Herein, a *segment* is a vector in $\{0, a\}^n$ for some $a \in \mathbb{Z} \setminus \{0\}$ where all a-entries occur consecutively, and a segment is positive if a is positive. An *explanation* is a set of segments that component-wise sum up to the input vector. For example, in case of VECTOR EXPLANATION (VE for short) the vector $(4, 3, 3, 4)$ can be explained by the segments $(4, 4, 4, 4)$ and $(0, -1, -1, 0)$, and in case of VECTOR POSITIVE EXPLANATION (VPE for short) it can be explained by $(3, 3, 3, 3)$, $(1, 0, 0, 0)$, and $(0, 0, 0, 1)$. Throughout the article an *entry*[4] of a vector refers to a pair consisting of a position (that is, an index) and the value of the vector at this position. Both problems have a simple well-known geometric interpretation (see Figure 1.1).

VE occurs in the database context and VPE occurs in the radiation therapy context. Motivated by previous work providing polynomial-time solvable special cases [1, 4], polynomial-time approximation [5, 26] and fixed-parameter tractability results [6, 9] (approximation and fixed-parameter algorithms both exploit problem-specific structural parameters), we head for a systematic parameterized and multivariate complexity analysis [15, 23, 28] of both problems; see Table 1 for a survey of parameterized complexity results (the parameters therein are formally defined in Definition 1.1).

[4]Naturally, being "consecutive", "first", "last", or "next" always refers to the position of entries and being "equal", "positive", "negative", or the "difference between two entries" refers to the value of the entries.

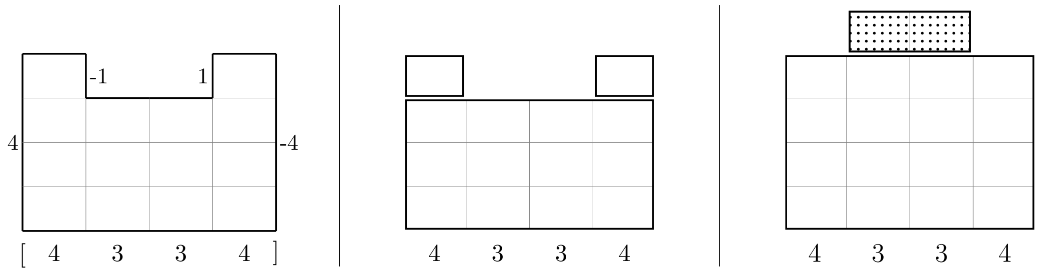

Figure 1.1: Illustration of the geometric interpretation of an input vector $\mathcal{A} = (4, 3, 3, 4)$ (left-hand side), an explanation of it using only positive segments (middle), and an explanation with one negative segment (dotted pattern on the right-hand side). Vector $\mathcal{A}$ is represented by a tower of blocks where each position i on the x-axis has $\mathcal{A}[i]$ many blocks. Each segment $I \in \{0, a\}^n$ is represented by a height-a rectangle starting and ending in the corresponding first and last a-entry of I (their different positions on the y-axis are only to draw them in a non-overlapping fashion). A set of segments explains $\mathcal{A}$ if for each i the sum of the heights of the rectangles intersecting a position i on the x-axis is $\mathcal{A}[i]$.

Previous work. Agarwal et al. [1] studied a polynomial-time solvable variant ("tree-ordered") of VE relevant in data warehousing. Karloff et al. [22] initiated a study of the two-dimensional ("matrix") case of VE and provided NP-completeness results as well as polynomial-time constant-factor approximations. Parameterized complexity aspects of VE and its two-dimensional variant seem unstudied so far.

The literature on VPE is richer. For a detailed description of the motivation from radiation therapy refer to the survey of Ehrgott et al. [12]. Concerning computational complexity, VPE is known to be strongly NP-complete [3] and APX-hard [4]. A significant amount of work has been done to achieve polynomial-time approximation algorithms for minimizing the number of segments. For instance, Bansal et al. [4] provide a 24/13-approximation which improves on the straightforward factor of two [26] (see also Biedl et al. [5]).

Improving a previous fixed-parameter tractability result for the parameter "maximum value γ of a vector entry" by Cambazard et al. [9], Biedl et al. [6] developed a fixed-parameter algorithm solving VPE in $2^{O(\sqrt{\gamma})} \cdot \gamma n$ time with n being the number of entries in the input vector. Moreover, the parameter "maximum difference between two consecutive vector entries" has been exploited for developing polynomial-time approximation algorithms [5, 26]. Finally, we remark that most of the previous studies also looked at the two-dimensional ("matrix") case, whereas we focus on the one-dimensional ("vector") case.

Table 1: An overview of previous and new results. ILP-FPT refers to the fact that the result is proven by an integer linear programming formulation and the exploitation of a result of Lenstra [25].

Parameters	Vector Explanation	Vector Pos. Explanation
max. value γ	ILP-FPT (Cor. 3.2)	$2^{O(\sqrt{\gamma})} \cdot \gamma n$ [6] no poly. kernel (Thm. 3.4)
max. difference δ of two consecutive entries		$O(n^{\delta} \cdot e^{\pi\sqrt{2\delta/3}})$ (Thm. 3.3)
# of peaks p and δ		ILP-FPT (Thm. 3.1)
number k of segments	$k! \cdot k + n$ (Thm. 4.1) $(2k-1)$-entry kernel (Thm. 4.1)	
$\kappa = 2k - n$	$\kappa^{O(\kappa)} + n^{O(1)}$ (Thm. 4.5(ii)) 3κ-entry kernel (Thm. 4.5(ii))	$k^{O(\kappa)} + n^{O(1)}$ (Thm. 4.5(i)) W[1]-hard (Thm. 4.6)
$n - k$	NP-complete for $(n-k) = 1$ (Thm. 4.7)	
max. segment length ξ	$\xi \geq 3$: NP-complete (Thm. 4.7) $\xi \leq 2$: $O(n^2)$ (Thm. 4.8)	
max. number ϕ of segments overlapping at some position	$\phi = 1$: trivial $\phi = 2$ (and $\xi = 3$ and $n - k = 1$): NP-complete (Thm. 4.7)	

Parameters under Study. We observe that the combinatorial structure of the considered problems is extremely rich, opening the way to a more thorough study of the computational complexity landscape under the perspective of problem parameterization. We take a closer look at these parameterization aspects. This helps in better understanding and exploiting problem-specific properties. To start with, note that previous work [6, 9], motivated by the application in radiation therapy, studied the parameterization by the maximum vector entry γ. They showed fixed-parameter tractability for VPE parameterized by γ, which we complement by showing the nonexistence (under a standard complexity-theoretic assumption) of a corresponding polynomial-size problem kernel. Using an integer linear program formulation, we also show fixed-parameter tractability for VE parameterized by γ. Moreover, for the perhaps most obvious parameter, the number k of explaining segments, we show fixed-parameter tractability for both problems.

Before providing a formal and comprehensive list of parameters that are studied in this work, we introduce the following known data reduction rule [4].

Reduction Rule 1.1. *If the input vector $\mathcal{A}$ has two consecutive equal entries, then remove one of them.*

The correctness of Reduction Rule 1.1 is obvious, as there is always a minimum-size explanation such that for each segment S it holds that $S[i] = S[i+1]$ in case of $\mathcal{A}[i] = \mathcal{A}[i+1]$. For notational convenience, we use $\mathcal{A}[0] = \mathcal{A}[n+1] = 0$ and thus in case that $\mathcal{A}[0] = \mathcal{A}[1] = 0$ or $\mathcal{A}[n] = \mathcal{A}[n+1] = 0$ we also apply Reduction Rule 1.1 to them. We emphasize that we consider neither $\mathcal{A}[0]$ nor $\mathcal{A}[n+1]$ as part of the input vector $\mathcal{A} \in \mathbb{N}_0^n$. It is easy to observe that Reduction Rule 1.1 can be exhaustively applied (applying it once more would not change the outcome) in $O(n)$ time to an input vector and we call the resulting vector *reduced*. A central consequence is that in each explanation of a reduced input vector $\mathcal{A}$ it holds that for each position $i \in \{0, \ldots, n\}$ there is at least one segment S such that $S[i] \neq S[i+1]$. This implies that if $n + 1 > 2k$, then the instance is a trivial no-instance. Moreover, $k \geq n$ would allow to use one segment for each input vector entry and thus the instance would be a trivial yes-instance. Hence, we may assume throughout the rest of the paper that $\mathcal{A}[i] \neq \mathcal{A}[i+1]$ for all $0 \leq i \leq n$ and that $k < n < 2k$. We now have the ingredients to provide a formal definition of all parameters considered in this work.

Definition 1.1. *For an input vector $\mathcal{A} \in \mathbb{N}_0^n$ define:*

- *the maximum difference δ between two consecutive vector entries, that is, $\delta := \max_{0 \leq i \leq n} |\mathcal{A}[i] - \mathcal{A}[i+1]|$;*
- *the number p of peaks where a position $i \in \{1, \ldots, n\}$ is a peak if $\mathcal{A}[i-1] < \mathcal{A}[i] > \mathcal{A}[i+1]$;*
- *maximum value $\gamma := \max_{1 \leq i \leq n} \mathcal{A}[i]$;*
- *number k of allowed segments in an explanation;*
- *"distance from triviality"-parameters $n - k$ and $\kappa := 2k - n$;*
- *maximum segment length ξ (number of nonzero entries) in an explanation;*
- *maximum number ϕ of segments overlapping at some position, that is, the maximum number of segments in an explanation which have a nonzero entry at a particular vector position.*

Our Contributions. Table 1 summarizes our and previous results with respect to the above parameters. Note that, since we assume by the above discussion that $k < n < 2k$, the parameters $n - k$ and κ are well-defined. Indeed, both can be interpreted as "distance from triviality" parameterizations [8, 20, 28]. We prove that, somewhat surprisingly, VE and VPE are already NP-hard for $n - k = 1$. Furthermore, we show that VE and VPE are polynomial-time solvable when κ is a constant, motivating a thorough study of the parameter κ. Interestingly, while we show that VPE is W[1]-hard for parameter κ, we prove that VE is fixed-parameter tractable for κ. Finally, we show NP-completeness for VE and VPE when $\xi = 3$ and $\phi = 2$.

Organization. In Section 2, we present a number of combinatorial properties of vector explanation problems which may be of independent interest and which are used throughout our work. In Section 3, we study the "smoothness of input vector"-parameters γ, δ, and p. In Section 4, we present results for further parameters as discussed above, and we conclude in Section 5 with some challenges for future research.

Parameterized Complexity Preliminaries. A parameterized problem is *fixed-parameter tractable* and belongs to the corresponding parameterized complexity class FPT if each instance (I, ρ), consisting of the "classical" problem instance I and the parameter ρ, can be solved in $f(\rho) \cdot |I|^{O(1)}$ time for some computable function f solely depending on ρ. A *kernelization* algorithm is a polynomial-time algorithm that transforms each instance (I, ρ) of a problem L into an instance (I', ρ') of L such that $(I, \rho) \in L \Leftrightarrow (I', \rho') \in L$ (equivalence) and $\rho', |I'| \leq g(\rho)$ for some function g [19, 24]. The instance (I', ρ') is called a (problem) *kernel* of size $g(\rho)$ and in case of g being a polynomial it is a *polynomial kernel*. A kernelization algorithm is often described by a set of *data reduction rules* whose exhaustive application leads to a kernel. Formally, a data reduction rule transforms an instance (I, ρ) of a parameterized problem L into another instance (I', ρ') of L such that $(I, \rho) \in L \Leftrightarrow (I', \rho') \in L$. An instance is called *reduced* with respect to a data reduction rule if one further application of the rule has no effect on the instance.

If a parameterized problem can be solved in polynomial running time where the degree of the polynomial depends on ρ (such as $|I|^{f(\rho)}$), then, for parameter ρ, the problem is said to lie in the—strictly larger [10]—class XP. Note that containment in XP ensures polynomial-time solvability for a constant parameter ρ whereas FPT additionally ensures that the degree of the corresponding polynomial is independent of the parameter ρ.

The basic class of (presumable) parameterized intractability is $W[1]$. A problem that is shown to be *W[1]-hard* by means of a *parameterized reduction* from a W[1]-hard problem is not FPT, unless FPT = W[1]. A parameterized reduction maps an instance (I, ρ) in $f(\rho) \cdot |I|^{O(1)}$ time to an equivalent instance (I', ρ') with $\rho' \leq g(\rho)$ for some computable functions f and g. See the monographs [10, 16, 27] for a more detailed introduction.

We use the unit-cost RAM model where arithmetic operations on numbers count as a single computation step.

2. Further Notation and Combinatorial Properties

We say that a segment $I \in \{0, a\}^n$ for some $a \in \mathbb{Z} \setminus \{0\}$ is of *weight* a and it *starts* at position ℓ and *ends* at positions r if $I[i] = a$ for all $1 \leq \ell \leq i < r \leq n$ and all other entries are zero. We will briefly write $[\ell, r]$ for such a segment and we say that it *covers* position i whenever $\ell \leq i < r$.[5] Because this notation suppresses the weight of the segment, we will associate a weight function $\omega : \mathcal{I} \to \mathbb{Z} \setminus \{0\}$ with a set $\mathcal{I}$ of segments that relates each segment to its weight. A set $\mathcal{I}$ of segments with a corresponding weight function ω forms an *explanation* for $\mathcal{A} \in \mathbb{N}_0^n$ if for each $1 \leq i \leq n$ the total sum of weights of all segments covering position i is equal to $\mathcal{A}[i]$. We also say that $(\mathcal{I}, \omega)$ *explains* $\mathcal{A}$ and refer to $|\mathcal{I}|$ as *solution size*. We call segments with positive weight *positive segments*, and otherwise *negative segments*. Hence, any explanation for a VPE-instance is allowed to contain only positive segments.

Since we assume that in a preprocessing phase Reduction Rule 1.1 is exhaustively applied, without loss of generality it holds that $\mathcal{A}[i] \neq \mathcal{A}[i+1]$ for all $0 \leq i \leq n$. Hence, the difference between two consecutive entries in $\mathcal{A}$ is *never* zero. It will turn out that the difference between consecutive entries in $\mathcal{A}$ is an important quantity.

Definition 2.1. *For an input vector $\mathcal{A} \in \mathbb{N}_0^n$ where any two consecutive entries are different from each other, the* tick vector $T \in \mathbb{N}^{n+1}$ *is defined to be $T[i] = \mathcal{A}[i] - \mathcal{A}[i-1]$ for all $i \in \{1, \ldots, n+1\}$. A position $i \in \{1, \ldots, n+1\}$ is an* uptick *if $T[i] > 0$ and otherwise it is a* downtick*. The* size *of the corresponding up- or downtick is $|T[i]|$.*

Given a tick vector T, the corresponding input vector $\mathcal{A}$ is uniquely determined as $\mathcal{A}[i] = \sum_{j=1}^{i} T[j]$. Thus, we will call an explanation for $\mathcal{A}$ also

[5] Note that $[\ell, r]$ does not cover position r, but it covers position ℓ.

an explanation for its tick vector T. Observe that the parameter *maximum difference δ between consecutive entries* is the maximum absolute value in T.

We next define a structure for an explanation and subsequently prove that there is always a minimum-size explanation of this structure.

Definition 2.2. *An explanation is called* regular *if each positive segment starts at an uptick and ends at a downtick, and each negative segment starts at a downtick and ends at an uptick.*

By the following theorem we can assume that each input vector admits a *regular* explanation. For VPE it corresponds to Bansal et al. [4, Lemma 1].

Theorem 2.1. *Let $(\mathcal{I}, \omega)$ be a size-k explanation of an input vector $\mathcal{A}$. Then, there is a regular size-k explanation $(\mathcal{I}', \omega')$ for $\mathcal{A}$. Furthermore, if $(\mathcal{I}, \omega)$ contains only positive segments, then $(\mathcal{I}', \omega')$ also does so.*

Proof. Let $\mathcal{A}$ be an input vector and let $(\mathcal{I}, \omega)$ be a non-regular explanation of $\mathcal{A}$. We say that a segment $I \in \mathcal{I}$ has a *starts wrongly* if I is positive and starts at a downtick or if I is negative and starts at an uptick. We denote such a starting position as a wrong start. Otherwise, we say the start is *correct*. We define *wrong and correct ends* analogously. Correspondingly, we call a position a *wrong start* (*wrong end*) if there is segment starting (ending) wrongly at this position.

Let $\overline{\mathcal{A}} = (\mathcal{A}[n], \mathcal{A}[n-1], \ldots, \mathcal{A}[1])$ be the vector formed by reversing $\mathcal{A}$ and let $\overline{\mathcal{I}}$ be the set of segments formed by reversing each segment in $\mathcal{I}$. Clearly, $(\overline{\mathcal{I}}, \omega)$ is an explanation for $\overline{\mathcal{A}}$. Hence, we may assume that in $\mathcal{A}$ there is a segment with a wrong start, as we otherwise consider $\overline{\mathcal{A}}$. We will provide a restructuring procedure whose application to $\mathcal{I}$ does not decrease the smallest (leftmost) wrong start, the sum of the absolute weights of the segments starting at the smallest wrong start of I strictly decreases, and it does not increase the number of wrong ends. Thus by iteratively applying this restructuring one can "replace" from left to right all segments that start wrongly with segments that start correctly. Then the reversal vector $\overline{\mathcal{A}}$ does not have any wrongly ending segments and thus by applying the same procedure again to $\overline{\mathcal{A}}$ one removes all wrongly ending segments in $\mathcal{A}$ without introducing any new wrongly starting segments.

We now describe the restructuring procedure. Let $I = [\ell_i, r_i] \in \mathcal{I}$ be a segment starting at the smallest wrong start ℓ_i. Since I is a wrongly starting segment there is a segment $J = [\ell_j, r_j]$ with $\ell_j \leq \ell_i$ such that either the sign

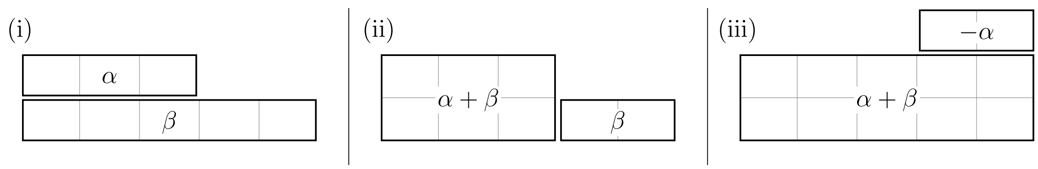

Figure 2.1: Illustration of the three configurations where two segments are *messy overlapping*. That is, in Configuration (i), one segment with weight α and one with weight β, start at the same position. In Configuration (ii), the first segment ends at the same position where the second starts. In Configuration (iii), both segments end at the same position. Given an explanation where two messy overlapping segments are in one of the three specific configurations, one can transform them into two new segments with any of the remaining configurations, obtaining a new explanation. Note that if the two segments in Configuration (i) have opposite weight sign and the same absolute weight, that is, $\alpha+\beta=0$, or if the two segments in Configuration (ii) have the same weight, that is, $\alpha=0$, then after the transformation, one would introduce a segment with zero weight which will then be removed.

of the weights of I and J are equal and J ends at ℓ_i (Case 1), or J has an opposite weight sign and starts at ℓ_i (Case 2). Clearly, Case 2 occurs only if explanation $\mathcal{I}$ contains negative segments. In Case 1, our restructuring procedure only introduces segments of the same weight sign as I. (This ensures that the restructured explanation contains negative segments only if $(\mathcal{I},\omega)$ does so.) In either case, we say that I and J are in a *messy overlapping configuration*, that is, either they start at the same position (configuration (i)) or one segment ends where the other starts (configuration ii)). We will also call the configuration where two segments end at the same position messy overlapping (configuration (iii)). See Figure 2.1 how to transform the configurations into each other while preserving an explanation.

Case 1: $\omega(I)$ and $\omega(J)$ have the same sign, and J ends at ℓ_i (correctly). Thus, segments I and J are in Configuration (ii), implying that $\omega(J)=\alpha+\beta$ and $\omega(I)=\beta$ in the terminology of Figure 2.1. If α and β have the same sign which implies that $\alpha+\beta$ and α have the same sign, then "transform" I, J into the two segments in Configuration (i). Otherwise, transform I, J into the two segments in configuration (iii). We remark here that if $\alpha=0$, then the segment I' will be ignored. One can verify that the constructed set of segments is a size-k explanation which has a smaller absolute weight sum of wrongly starting segments starting at ℓ_i than the original one $(\mathcal{I},w)$, and which does not contain a wrongly starting segment starting at a position prior to ℓ_i. Additionally, observe that we neither introduced new wrong ends nor, if $\omega(I)$ and $\omega(J)$ both are positive, created negative segments.

Case 2: $\omega(I)$ and $\omega(J)$ have different signs, and J starts at ℓ_i.
Thus, the segments I and J are in configuration (i), implying that, when using the terminology of Figure 2.1, α and β have different signs. If $|\beta| \geq |\alpha|$, then we transform I, J into the two segments in Configuration (iii). If $\alpha + \beta = 0$, then we just remove the segment with weight $\alpha + \beta$. One can verify that the constructed set of segments is a size-k explanation that, due to α and β having different signs, has a smaller absolute weight sum of wrongly starting segments at position ℓ_i than the original explanation $(\mathcal{I}, \omega)$. Furthermore, as $|\beta| \geq |\alpha|$ and thus the signs of $\alpha + \beta$ (if it is not zero) and $-\alpha$ are the same as of β, we neither introduce new wrong ends nor new wrongly starting segments starting at a position prior to ℓ_i. If $|\alpha| > |\beta|$, then transform I, J into configuration (ii). Again, since α and β have different signs, the thus-constructed set of segments is a size-k explanation that has a smaller absolute weight sum of wrongly starting segments starting in ℓ_i than the original explanation $(\mathcal{I}, \omega)$. Further, because $|\alpha| > |\beta|$, the two segments with weights $\alpha + \beta$ and β have the same sign as α and β, respectively. Hence, this transformation does not introduce new wrong ends. □

Remark: Theorem 2.1 implies containment of VE in NP as it upper-bounds the segment weights in an explanation in the numbers occurring in the instance. For VPE this directly follows from the problem definition.

The following corollary summarizes the consequences of Theorem 2.1. To state them, we introduce the following terminology.

Definition 2.3. *An input vector is* single-peaked *if it contains only one peak. A* single-peaked instance *is an instance with a single-peaked vector.*

Corollary 2.2. *(i) For any* Vector Positive Explanation *or* Vector Explanation *instance there is a minimum-size explanation such that there is only one segment that covers the first position, it is positive, and it ends at a downtick. Symmetrically, there is a minimum-size explanation such that there is only one segment that covers the last position and it is positive and starts at an uptick.*

(ii) Any single-peaked Vector Explanation *instance $(\mathcal{A}, k)$ is an equivalent* Vector Positive Explanation *instance.*

Proof. (i): Since position 1 is an uptick and position $n+1$ is a downtick, by Theorem 2.1 it directly follows that in a regular explanation all segments covering the first or last position are positive and thus start in upticks and end in downticks. Moreover, if there are two positive segments covering the first position, then they are messy overlapping as they are in Configuration (i) (Figure 2.1). Hence, transforming them into the two segments in Configuration (ii) results in an explanation where one segment less covers the first position. Analogously, two segments covering the last position are in Configuration (iii) and can be transformed into the two segments in Configuration (ii).

(ii): By Theorem 2.1 if there is any size-k explanation, then there is also a regular size-k explanation which starts negative segments in downticks and ends them in upticks. However, in single-peaked instances all upticks precede the first downtick. □

The following theorem states that for VE one can arbitrarily permute the entries of a tick vector without changing the solution size for the corresponding input vectors.

Theorem 2.3. *Let $T \in \mathbb{N}^{n+1}$ be an arbitrary tick vector and let $T' \in \mathbb{N}^{n+1}$ be a tick vector that results from T by arbitrarily permuting the entries in T. For* Vector Explanation *it holds that there is a size-k explanation for T if and only if there is a size-k explanation for T'.*

Proof. We prove Theorem 2.3 for two tick vectors T and T' where, for some i, $T'[i] = T[i+1]$, $T'[i+1] = T[i]$, and $T[j] = T'[j]$ for all other entries j. It is clear that one can arbitrarily permute the entries in T by applying these "flips" to consecutive entries. Let $\mathcal{A}$ be the input vector corresponding to T and let $\mathcal{A}'$ be the input vector corresponding to T'. It follows that $\mathcal{A}'[j] = \mathcal{A}[j]$ for every $j \neq i$ and $\mathcal{A}'[i] = \mathcal{A}[i-1] + \mathcal{A}[i+1] - \mathcal{A}[i]$. For any k, we prove that $(\mathcal{A}', k)$ is a yes-instance if and only if $(\mathcal{A}, k)$ is a yes-instance. However, as "flipping" $T'[i]$ and $T'[i+1]$ in T' results in T, the equivalence is symmetric and it is thus sufficient to prove that if $(\mathcal{A}, k)$ is a yes-instance, then so is $(\mathcal{A}', k)$.

Let $(\mathcal{I}, \omega)$ be an explanation for $\mathcal{A}$. We construct $(\mathcal{I}', \omega')$ by replacing some segments in $\mathcal{I}$. The general idea is that if a segment started or ended at position i, then it is modified such that it starts or ends at $i+1$ and vice versa. The only exception are the segments which start at i and end at $i+1$, for which we swap the endpoints and negate the weight. Formally, $\mathcal{I}'$ is defined as follows:

$$\begin{aligned}
\mathcal{I}' &= \mathcal{I}'_0 \cup' \mathcal{I}'_1 \cup \mathcal{I}'_2 \cup \mathcal{I}'_3 \cup \mathcal{I}'_4 \cup \mathcal{I}'_5 \cup \mathcal{I}'_6, \text{where}\\
\mathcal{I}'_0 &= \{[a,b] \in \mathcal{I} \mid a,b < i \vee a,b > i+1\},\\
\mathcal{I}'_1 &= \{[a,b] \in \mathcal{I} \mid a < i \wedge b > i+1\},\\
\mathcal{I}'_2 &= \{[a,i+1] \mid [a,i] \in \mathcal{I}\},\\
\mathcal{I}'_3 &= \{[a,i] \mid [a,i+1] \in \mathcal{I} \wedge a < i\},\\
\mathcal{I}'_4 &= \{[i+1,b] \mid [i,b] \in \mathcal{I} \wedge b > i+1\},\\
\mathcal{I}'_5 &= \{[i,b] \mid [i+1,b] \in \mathcal{I}\},\\
\mathcal{I}'_6 &= \{[i,i+1]\} \cap \mathcal{I}.
\end{aligned}$$

Let $\omega'([i,i+1]) = -\,\omega([i,i+1])$ if $[i,i+1] \in \mathcal{I}$, and for the other segments of $\mathcal{I}'$ set the weight ω' to be equal to the weight of the corresponding segment in $\mathcal{I}$.

Obviously, $|\mathcal{I}'| = |\mathcal{I}|$ and, hence, it remains to show that $(\mathcal{I}', \omega')$ explains $\mathcal{A}'$. As a segment of $\mathcal{I}'$ covers a position $j \neq i$ if and only if the corresponding segment in $\mathcal{I}$ of the same weight covers j, it is clear that $(\mathcal{I}', w')$ explains every position $\mathcal{A}'[j] = \mathcal{A}[j]$ with $j \neq i$. To prove that it also explains position i, let $s_x = \sum_{I \in \mathcal{I}'_x} \omega'(I)$ for all $x \in \{1, \ldots, 6\}$. Since $(\mathcal{I}, \omega)$ explains $\mathcal{A}$ and the weight of the segments (except those potentially in $\mathcal{I}'_6$) are equal, it holds that

$$\begin{aligned}
\mathcal{A}[i-1] &= s_1 + s_2 + s_3,\\
\mathcal{A}[i] &= s_1 + s_3 + s_4 - s_6, \text{ and}\\
\mathcal{A}[i+1] &= s_1 + s_4 + s_5.
\end{aligned}$$

The sum of the weights of segments covering $\mathcal{A}'[i]$ is $s_1 + s_2 + s_5 + s_6$ and thus, together with $\mathcal{A}'[i] = \mathcal{A}[i-1] + \mathcal{A}[i+1] - \mathcal{A}[i]$, the equations above prove that $(\mathcal{I}', w')$ also explains $\mathcal{A}'[i]$. □

The following corollary summarizes combinatorial properties of VE which can be directly deduced from Theorem 2.3 as it allows to arbitrarily order the entries of a tick vector. They are used throughout the paper and may be of independent interest for future studies.

Corollary 2.4. *Let* $(\mathcal{A}, k)$ *be an instance of* Vector Explanation. *Then, the following holds.*

(i) The instance $(\mathcal{A}, k)$ can be transformed in $O(n)$ time into an equivalent single-peaked Vector Explanation*-instance $(\mathcal{A}', k)$ such that the maximum difference between two consecutive entries is the same in $\mathcal{A}$ and $\mathcal{A}'$.*

(ii) The instance $(\mathcal{A}, k)$ can be transformed in $O(n)$ time into an equivalent Vector Explanation*-instance $(\mathcal{A}', k)$ such that the maximum value in $\mathcal{A}'$ is less than two times the maximum difference between consecutive entries in $\mathcal{A}$.*

Proof. (i): By Theorem 2.3, permuting the entries of the tick vector of $\mathcal{A}$ such that all upticks precede all downticks, that is, the new input vector is single-peaked, results in an equivalent instance. Clearly, this can be done in $O(n)$ time.

(ii): Let δ be the maximum difference between any two consecutive entries in $\mathcal{A}$, or equivalently, the maximum absolute value in the tick vector T of $\mathcal{A}$. Start creating a permuted tick vector T' of T by assigning to $T'[1]$ an arbitrary positive entry from T. Next, whenever $T'[1], \ldots, T'[i-1]$ are already assigned, if $\sum_{j=1}^{i-1} T'[j] < \delta$ and there is a positive entry in T that is not yet assigned to one of $T'[1], \ldots, T'[i-1]$, then assign $T'[i]$ to be this entry. In this way, we ensure that every entry of the input vector corresponding to T' is less than twice of δ. Otherwise set it to one of the remaining negative entries of T that is not yet assigned. Clearly, a partition of T's entries in positive and negative entries can be computed in $O(n)$ time, and using it one can easily achieve the above assignment. □

3. Parameterization by Input Smoothness

In this section, we examine how the computational complexity of VE and VPE is influenced by parameters that measure how "smooth" the input vector $\mathcal{A} \in \mathbb{N}_0^n$ is. We assume that $\mathcal{A}$ is reduced with respect to Reduction Rule 1.1 and thus all consecutive entries in $\mathcal{A}$ have different values. We consider the following three measurements:

- the maximum difference δ between two consecutive entries in $\mathcal{A}$,
- the number p of peaks, and
- the maximum value γ occurring in $\mathcal{A}$.

Our main results are fixed-parameter algorithms for the combined parameter (p, δ) in case of VPE and for the parameter δ in case of VE. For the parameter maximum value γ, we show that VPE does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly.

Next, relying on a deep result by Lenstra [25] and providing an integer linear programming formulation where the number of variables is upper-bounded in the number of peaks p and the maximum difference δ, we prove fixed-parameter tractability with respect to (p, δ).

Theorem 3.1. Vector Positive Explanation *is fixed-parameter tractable with respect to the combined parameter number p of peaks and maximum difference δ.*

Proof. We provide an integer linear program (ILP) formulation for VPE where the number of variables is a function of p and δ. This ILP decides whether there is a regular size-k explanation (restricting to regular explanations is sufficient by Theorem 2.1). In a regular explanation the multiset of weights of segments that start at an uptick sum up to the uptick size. Analogously, this holds for segments ending at a downtick. Motivated by this fact, we introduce the following notion: For a positive integer x, we say that a multiset $X = \{x_1, x_2, \ldots, x_r\}$ of positive integers *partitions* x if $x = \sum_{i=1}^{r} x_i$. Similarly, we say that X *partitions* an uptick (downtick) i of size x if X partitions x. Let $\mathcal{P}(x)$ denote the set of all multisets that partition x.

In the ILP formulation for VPE, we describe a solution by "fixing" for each position i a multiset X_i of positive integers which partitions the uptick (downtick) at i. The crucial observation for our ILP is that if a set of consecutive upticks contains more than one uptick of size x, then it is sufficient to fix how many of these upticks were partitioned in which way. In other words, one does not need to know the partition for each position; instead one can distribute freely the partitions of x onto the upticks of size x. This also holds for consecutive downticks. Since each peak is preceded by consecutive upticks and succeeded by consecutive downticks, and since we introduce variables in the ILP formulation to "model" how many upticks (downticks) exist between two consecutive peaks, the number of variables in the formulation is upper-bounded by a function of p and δ. We now give the details of the formulation. Herein, we assume that the peaks are ordered from left to right; we refer to the i-th peak in this order as *peak i*.

For an integer $x \in \{1, \ldots, \delta\}$, let $\operatorname{occ}(x, i)$ denote the number of upticks of size x that directly precede peak i, that is, the number of upticks succeeding peak $i-1$ and preceding peak i. Similarly, let $\operatorname{occ}(-x, i)$ denote the number of downticks of size x that directly succeed i. For two positive integers y and x

with $y \leq x$ and a multiset $P \in \mathcal{P}(x)$ let $\text{mult}(y, P)$ denote how often y appears in P. We use $\text{mult}(y, P)$ to "model" how many segments of weight y start (end) at some uptick (downtick) that is partitioned by P.

To formulate the ILP, we introduce for each peak i, each $x \in \{1, \ldots, \delta\}$, and each $P \in \mathcal{P}(x)$ two nonnegative variables $\text{var}_{x,P,i}$ and $\text{var}_{-x,P,i}$. The variables respectively correspond to the number of upticks directly preceding peak i and downticks directly succeeding peak i of size x that are partitioned by P in a possible explanation of $\mathcal{A}$. To enforce that a particular assignment to these variables corresponds to a valid explanation, we introduce the following constraints.

First, for each peak i and each $1 \leq x \leq \delta$ we ensure that the number of directly preceding size-x upticks (succeeding size-x downticks) that are partitioned by some $P \in \mathcal{P}(x)$ is equal to the number of directly preceding size-x upticks (succeeding size-x downticks):

$$\forall i \in \{1, \ldots, p\},\ \forall x \in \{-\delta, \ldots, \delta\} \setminus \{0\} : \sum_{P \in \mathcal{P}(x)} \text{var}_{x,P,i} = \text{occ}(x, i). \qquad (1)$$

Second, we ensure that for each peak i and each value $y \in \{1, \ldots, \delta\}$ the number of segments of weight y that end directly after peak i is at most the number of segments of weight y that start at positions (not necessarily directly) preceding peak i minus the number of segments of weight y that end at positions succeeding some peak $j < i$. Informally, this means that we only "use" the available number of segments of weight y. To enforce this property, for each peak $1 \leq i \leq p$ and each possible segment weight $1 \leq y \leq \delta$ we add:

$$\sum_{j=1}^{i} \sum_{x=y}^{\delta} \sum_{P \in \mathcal{P}(x)} \Big(\underbrace{\text{mult}(y, P) \cdot \text{var}_{x,P,j}}_{\#\text{ of started weight-}y\text{ segments}} - \underbrace{\text{mult}(y, P) \cdot \text{var}_{-x,P,j}}_{\#\text{ of ended weight-}y\text{ segments}} \Big) \geq 0 \qquad (2)$$

Finally, we ensure that the total number of segments is at most k:

$$\sum_{i=1}^{p} \sum_{x=1}^{\delta} \sum_{P \in \mathcal{P}(x)} \sum_{y=1}^{x} \text{mult}(y, P) \cdot \text{var}_{x,P,i} \leq k. \qquad (3)$$

Correctness: The equivalence of the ILP instance and $(\mathcal{A}, k)$ can be seen as follows. Assume that there is a size-at-most-k explanation $(\mathcal{I}, \omega)$ for $(\mathcal{A}, k)$, where the segments start at upticks and end at downticks. Recall that by definition of $\mathcal{P}(x)$, for any uptick i of size x there is a partition in $\mathcal{P}(x)$ that corresponds to the weights of the segments starting in i. For each peak i, for any value $1 \leq x \leq \delta$ and each $P \in \mathcal{P}(x)$, count how many upticks of size x that directly precede peak i are explained by segments in $\mathcal{I}$ (segments that start in this uptick) whose weights correspond to $\mathcal{P}(x)$ and set $\mathrm{var}_{x,P,i}$ to this value. Symmetrically, do the same for the downticks succeeding peak i and set $\mathrm{var}_{-x,P,i}$ accordingly. It is straightforward to verify that Constraint set (1) and Constraint set (2) and constraint (3) hold.

Conversely, assume that there is an assignment to the variables such that Constraint sets (1) and (2) and Constraint (3) are fulfilled. We form an explanation $(\mathcal{I}, \omega)$ as follows: For any peak i and any value $1 \leq x \leq \delta$ with $\mathrm{occ}(x,i) > 0$ let $\mathcal{P}_{i,x}$ be the multiset of elements from $\mathcal{P}(x)$ that contains each $P \in \mathcal{P}(x)$ exactly $\mathrm{var}_{x,P,i}$ times. By Constraint set (1), $|\mathcal{P}_{i,x}| = \mathrm{occ}(x,i)$. For an arbitrary ordering of $\mathcal{P}_{i,x}$ and the upticks of size x directly preceding peak i, add to $\mathcal{I}$ for the jth element $\mathcal{P}_j$ of $\mathcal{P}_{i,x}$ exactly $|\mathcal{P}_j|$ segments with weight corresponding to $\mathcal{P}_j$ and let them start at the jth uptick with size x that directly precedes peak i. By Constraint (3) we added at most k segments. It remains to specify the end of the segments. Symmetrically to the upticks, for each downtick directly succeeding peak i of size x let $\mathcal{P}_{i,x}$ be the multiset of elements from $\mathcal{P}(x)$ containing each $P \in \mathcal{P}(x)$ exactly $\mathrm{var}_{-x,P,i}$ times. For the jth element $\mathcal{P}_j$ of $\mathcal{P}_{i,x}$ and the jth downtick directly succeeding peak i (again both with respect to any ordering) and for each $\alpha \in \mathcal{P}_j$ pick any weight-α segment from $\mathcal{I}$ (so far without end) and let it at the jth downtick. Observe that the existence of this segment is ensured by Constraint set (2). Finally, it remains to argue that the end of each segment in $\mathcal{I}$ is determined. This follows from the fact that Constraint set (1) and Constraint set (2) together imply for each $1 \leq y \leq \delta$ that

$$\sum_{i=1}^{p} \sum_{x=y}^{\delta} \sum_{P \in \mathcal{P}(x)} (\mathrm{mult}(y,P) \cdot \mathrm{var}_{x,P,i} - \mathrm{mult}(y,P) \cdot \mathrm{var}_{-x,P,i}) = 0,$$

and thus the total number of opened weight-y segments is equal to the number of ended weight-y segments.

Running time: The number of variables in the constructed ILP instance is

$$p \cdot \sum_{x \in \{-\delta,\ldots,\delta\} \setminus \{0\}} |\mathcal{P}(|x|)| = 2p \sum_{x=1}^{\delta} |\mathcal{P}(x)| \leq 2\delta p \cdot |\mathcal{P}(\delta)| \leq 2\delta p \cdot e^{\pi\sqrt{\frac{2}{3}\delta}} =: f(\delta, p),$$

where the last inequality is due to de Azevedo Pribitkin [2]. Then, due to a deep result in combinatorial optimization, the feasibility of the ILP can be decided in $O(f(\delta,p)^{2.5f(\delta,p)+o(f(\delta,p))} \cdot |L|)$ time, where $|L|$ is the size of the instance [17, 21, 25]. Moreover, as we have $O(\delta p)$ constraints, we also have $|L| = O(\delta^2 p^2 \cdot e^{\pi\sqrt{\frac{2}{3}\delta}})$. □

Observe that Theorem 3.1 implies that VE is fixed-parameter tractable with respect to the maximum difference δ: By Corollary 2.2(ii) & Corollary 2.4(i), in linear time one can transform input instances of VE into equivalent single-peaked instances of VPE without increasing the maximum difference δ.

Corollary 3.2. Vector Explanation *parameterized by the maximum difference* δ *is fixed-parameter tractable.*

It remains open whether VPE is fixed-parameter tractable with respect to δ. Note that the argumentation for VE (Corollary 3.2) cannot be transferred, since there may be more than one peak in an instance. However,the following theorem shows that VPE is in XP for the parameter maximum difference δ.

Theorem 3.3. Vector Positive Explanation *is solvable in* $O(n^\delta \cdot e^{\pi\sqrt{2\delta/3}})$ *time.*

Proof. We describe a dynamic programming algorithm that finds a regular minimum-size explanation. Every explanation for a size-n vector $\mathcal{A}$ can be interpreted as an extension of an explanation for the same vector without the last entry, where some segments that originally only covered position $n-1$ may be stretched to also cover position n and some new length-one segment may start at position n.

Our algorithm uses the above relation between explanations for the vector $\mathcal{A}[1,\dots,n]$ and explanations for the vector $\mathcal{A}[1,\dots,n-1]$. Due to Theorem 2.1, it only considers regular explanations, implying that each segment starts at an uptick and ends at a downtick. Since all upticks and downticks have size at most δ, the algorithm furthermore only considers solutions in which all segments have weight at most δ.

We fill a table T which has entries of type $T(i, d_1, \dots, d_j, \dots, d_\delta)$ where $0 \le i \le n$ and $0 \le d_j \le k$ with $1 \le j \le \delta$. An entry $T(i, d_1, \dots, d_j, \dots, d_\delta)$ contains the minimum number of segments explaining vector $\mathcal{A}[1,\dots,i]$ such that d_j segments of weight j cover position i. If no such explanation exists, then the entry is set to ∞. By definition of the table entries, there is a solution for VPE if and only if

$$\min_{(d_1,\ldots,d_\delta)\in\{0,\ldots,k\}^\delta} T(n, d_1, \ldots, d_\delta) \leq k.$$

In the following, we show how to fill the table. As initialization, set $T(0, d_1, \ldots, d_\delta) \leftarrow \infty$ if there is some $d_j > 0$ and set $T(0, 0, \ldots, 0) \leftarrow 0$.

For increasing $i \leq n$, compute the table for each $(d_1, \ldots, d_\delta) \in \{0, \ldots, k\}^\delta$ as follows. If $\mathcal{A}[i] = \sum_{j=1}^{\delta} d_j \cdot j$ and $\mathcal{A}[i] > \mathcal{A}[i-1]$, then set

$$T(i, d_1, \ldots, d_\delta) \leftarrow \min_{d'_1 \leq d_1, \ldots, d'_\delta \leq d_\delta} \left(T(i-1, d'_1, \ldots, d'_\delta) + \sum_{j=1}^{\delta} (d_j - d'_j) \right). \quad (4)$$

If $\mathcal{A}[i] = \sum_{j=1}^{\delta} d_j \cdot j$ and $\mathcal{A}[i] < \mathcal{A}[i-1]$, then set

$$T(i, d_1, \ldots, d_\delta) \leftarrow \min_{d'_1 \geq d_1, \ldots, d'_\delta \geq d_\delta} T(i-1, d'_1, \ldots, d'_\delta). \quad (5)$$

Otherwise, set

$$T(i, d_1, \ldots, d_\delta) \leftarrow \infty. \quad (6)$$

The correctness of the initialization follows directly from the table definition. For the remaining computation we can thus assume that there is some i such that all entries $T(i', d_1, \ldots, d_\delta)$ with $(d_1, \ldots, d_\delta) \in \{0, \ldots, k\}^\delta$ and $i' < i$ were computed correctly.

As discussed above, we interpret an explanation of $\mathcal{A}[1, \ldots, i]$ as extension of an explanation for $\mathcal{A}[1, \ldots, i-1]$. There are exactly two groups of segments covering position i: those also covering position $i-1$ and those starting at position i. Let the set of segments covering position i be described by $(d_1, \ldots, d_\delta)$ such that $\mathcal{A}[i] = \sum_{j=1}^{\delta} d_j \cdot j$ and $\mathcal{A}[i] > \mathcal{A}[i-1]$. Due to Theorem 2.1, no segment ends at position i, but since $\mathcal{A}[i] > \mathcal{A}[i-1]$ at least one new segment has to start at position i. By setting $(d'_1, \ldots, d'_\delta)$ such that $d'_j \leq d_j, 1 \leq j \leq \delta$, one considers all possible extensions for explanations of $\mathcal{A}[i-1]$ such that no segment ends at position i. Clearly, $\sum_{j=1}^{\delta} (d_j - d'_j)$ further segments have to start at position i to explain $\mathcal{A}[i]$. Hence, Assignment (4) is correct.

Now, describe the set of segments covering position i by $(d_1, \ldots, d_\delta)$ such that $\mathcal{A}[i] = \sum_{j=1}^{\delta} d_j \cdot j$ and $\mathcal{A}[i] < \mathcal{A}[i-1]$. By Theorem 2.1 no new segment starts at position i. The algorithm considers all possible explanations where some segments end at position i and the other segments survive to explain $\mathcal{A}[i]$. Thus, Assignment (5) is correct.

For a given $(d_1, \ldots, d_\delta) \in \{0, \ldots, k\}^\delta$, to find an explanation for $\mathcal{A}[1, \ldots, i]$ such that $\mathcal{A}[i] \neq \sum_{j=1}^{\delta} d_j \cdot j$ is impossible because such an explanation does not explain position i. Thus Assignment (6) is correct.

The size of the table is upper-bounded by n^δ since we only have to consider table entries $T[i, d_1, \ldots, d_\delta]$ with $\mathcal{A}[i] = \sum_{j=1}^{\delta}(d_j \cdot j)$. The trivial upper bound of $O(n^\delta)$ for computing each table entry already leads to a running time of $O(n^{2\delta})$. However, the number of entries that have to be considered is smaller. For Assignment (4), one only has to consider those entries of Table T that do not have value ∞. Hence, $\sum_{j=1}^{\delta} |d_j - d'_j| \leq |\mathcal{A}[i] - \mathcal{A}[i-1]| \leq \delta$. This implies that for each table entry the number of previous entries that have to be considered in the minimization is upper-bounded by the number of different multisets that sum up to δ and thus is upper-bounded by $O(e^{\pi\sqrt{\frac{2}{3}\delta}})$ [2]. A similar argument applies for Assignment (5). The overall running time follows. □

VPE is known to be fixed-parameter tractable when parameterized by the maximum value γ [6]. We complement this result by showing a lower bound on the kernel size, and thus demonstrate limitations on the power of polynomial-time preprocessing.

Theorem 3.4. *Unless NP $\subseteq$ coNP/poly, there is no polynomial kernel for* VECTOR POSITIVE EXPLANATION *parameterized by the maximum value* γ.

Proof. We provide an *AND-cross-composition* [7, 11] from the 3-PARTITION problem [18, SP15]. This is a polynomial-time algorithm that gets as input a set of 3-PARTITION-instances and computes an instance $(\mathcal{A}, k)$ of VPE such that the maximum value γ occurring in $(\mathcal{A}, k)$ is polynomially bounded in the maximum of sizes of the input 3-PARTITION instances and $(\mathcal{A}, k)$ is a yes-instance if and only if all given 3-PARTITION instances are yes-instances.

> 3-PARTITION
> **Input:** A multiset $S = \{a_1, \ldots, a_{3m}\}$ of positive integers and an integer bound B with $m \cdot B = \sum_{i=1}^{3m} a_i$ and $B/4 < a_i < B/2$ for every $i \in \{1, \ldots, 3m\}$.
> **Question:** Is there a partition of S into m subsets $P_1, \ldots, P_m$ with $|P_j| = 3$ and $\sum_{a_i \in P_j} a_i = B$ for every $j \in \{1, \ldots, m\}$?

3-Partition is NP-complete even if B (and thus all a_i's) is bounded by a polynomial in m [18]. We show that this variant of 3-Partition AND-cross-composes to VPE parameterized by the maximum value γ. Then, results of Bodlaender et al. [7] and Drucker [11] imply that VPE does not have a polynomial kernel with respect to parameter γ, unless NP $\subseteq$ coNP/poly.

First, let (S, B) be a single instance of 3-Partition. We show that it reduces to an instance $(\mathcal{A}', 3m)$ of VPE. This reduction is similar to a previous NP-hardness reduction for VPE due to Bansal et al. [4]. We define $\mathcal{A}'$ as length-$(4m-1)$ vector:

$$\left(a_1, a_1 + a_2, \ldots, \sum_{i=1}^{j} a_i, \ldots, \sum_{i=1}^{3m} a_i = mB, (m-1)B, (m-2)B, \ldots, B\right).$$

If a partition $P_1, \ldots, P_m$ of S forms a solution, then the set of segments $\{[i, 3m+j] \mid a_i \in P_j\}$ each with weight $w([i, 3m+j]) = a_i$ is an explanation for the vector $\mathcal{A}'$. Conversely, let $(\mathcal{I}, \omega)$ be a regular explanation for $(\mathcal{A}', 3m)$. Since every segment starts at an uptick and ends at a downtick, $\mathcal{I}$ contains $3m$ segments and the segment starting at position i has weight a_i. Since $B/4 < a_i < B/2$ for each integer $a_i \in S$, exactly three segments end at a downtick whose size is exactly B. Thus, grouping the segments according to the position they end at, we get the desired partition of S, solving the instance of 3-Partition.

Now let $(S_1, B_1), \ldots, (S_t, B_t)$ be instances of 3-Partition such that $S_r = \{a_1^r, \ldots, a_{3m_r}^r\}$ and $B_r \le {m_r}^c$ for every $r \in \{1, \ldots, t\}$ and some constant c. We build an instance $(\mathcal{A}, k)$ of VPE by first using the above reduction for each (S_r, B_r) separately to produce a vector $\mathcal{A}'_r$, and then concatenating the vectors $\mathcal{A}'_r$ one after another, leaving a single position of value 0 in between. The total length of the vector $\mathcal{A}$ is $4(\sum_{r=1}^{t} m_r) - 1$ and we set $k = 3\sum_{r=1}^{t} m_r$.

Due to the argumentation for the single instance case, on the one hand, if each of the instances is a yes-instance, then there is an explanation using $3m_r$ segments per instance (S_r, B_r), that is $3\sum_{r=1}^{t} m_r$ segments in total. Conversely, we need at least $3m_r$ segments to explain $\mathcal{A}'_r$ and there is an explanation with $3m_r$ segments if and only if (S_r, B_r) is a yes-instance. Since all segments are positive and the subvectors $\mathcal{A}'_r$ are separated by an entry with value zero, no segment can span over two subvectors. In other words, no segment can be used to explain more than one of the $\mathcal{A}'_r$'s. Therefore, an explanation for $\mathcal{A}$ with $3\sum_{r=1}^{t} m_r$ segments implies that (S_r, B_r) is a yes-instance for every $r \in \{1, \ldots, t\}$.

Finally, observe that the maximum value γ in the vector $\mathcal{A}$ is equal to $\max_{r=1}^{t} m_r B_r \leq \max_{r=1}^{t} {m_r}^{c+1}$ and, thus, it is polynomially bounded in $\max_{r=1}^{t} |S_r|$. Hence, 3-PARTITION AND-cross-composes to VPE parameterized by the maximum value γ, and there is no polynomial kernel for this problem unless NP $\subseteq$ coNP/poly. □

4. Parameterizations of the Size and the Structure of Solutions

We now provide fixed-parameter tractability and (parameterized) hardness results for further natural parameters. Specifically, we consider the number k of segments in the solution, so-called "above-guarantee" and "below-guarantee" parameterizations (which are smaller than k), the maximum segment length ξ, and the maximum number ϕ of segments covering a position.

For the parameter k we develop a search tree algorithm for VPE and VE where the depth and the branching degree of the search tree are bounded by the solution size k. This is achieved by combining Reduction Rule 1.1, Corollary 2.2, and Corollary 2.4. The first part of Theorem 4.1 follows directly from exhaustively applying Reduction Rule 1.1.

Theorem 4.1. *Any instance of* VECTOR POSITIVE EXPLANATION *or* VECTOR EXPLANATION *can be reduced in $O(n)$ time to an equivalent one with at most $(2k-1)$ entries. Furthermore,* VECTOR POSITIVE EXPLANATION *and* VECTOR EXPLANATION *can be solved in $O(k! \cdot k + n)$ time.*

Proof. We start with the algorithm for VPE which works as follows. After exhaustive application of Reduction Rule 1.1 branch over all possible segments covering the last entry. Due to Corollary 2.2(i), it suffices to search for exactly one segment starting at one of the upticks and ending at the last entry. For each branch assign the value $\mathcal{A}[n]$ as weight to the segment and solve the instance consisting of the remaining entries recursively. To this end, decrease each of the entries covered by the segment by $\mathcal{A}[n]$, and decrease k by one. If some entry becomes negative or if $k < 0$, then discard the branch.

The exhaustive application of Reduction Rule 1.1 can be performed in $O(n)$ time, afterwards $n \leq 2k - 1$ (this also implies the first part of the theorem). The search tree produced by the branching algorithm has depth at most k. In the i-th level of the search tree, one branches over at most $k + 1 - i$ upticks. The steps performed in each search tree node take $O(k)$ time since $n \leq 2k - 1$. The overall running time thus is $O(k! \cdot k + n)$.

For VE we first apply Corollary 2.4(i) to transform our instance into a single-peaked instance (this is necessary to avoid negative entries). The rest works analogously to VPE. □

The first part of Theorem 4.1 implies that for a reduced instance every explanation needs at least $\lfloor n/2 \rfloor + 1$ segments. Hence, it is interesting to study parameters that measure how far we have to exceed this lower bound for the solution size: such above-guarantee parameters can be significantly smaller than k. For this reason, we study a parameter that measures $k - (\lfloor n/2 \rfloor + 1)$. For ease of presentation, we define this parameter as $\kappa := 2k - n$. The concepts of "clean" and "messy" positions, which are defined as follows, are crucial for the design of our algorithms.

Definition 4.1. *Let $(\mathcal{A}, k)$ be an instance of* Vector Explanation *or* Vector Positive Explanation *and let $\mathcal{I}$ be an explanation for $\mathcal{A}$. A segment $I = [i, j] \in \mathcal{I}$ is* clean *if all other segments start and end at positions different from i and j. A position i is* clean with respect to $\mathcal{I}$ *if it is the start or endpoint of a clean segment in $\mathcal{I}$. A position or segment that is not clean is called* messy.

Remark: Note that a segment is messy if and only if it is in one of the messy overlapping configurations shown in Figure 2.1.

Messy positions and segments have the following useful relation to the parameter κ: if κ is small, then there are only few messy positions and segments.

Lemma 4.2. *Let $(\mathcal{A}, k)$ be a yes-instance of* Vector Positive Explanation *that is reduced with respect to Reduction Rule 1.1. Then, every explanation of $(\mathcal{A}, k)$ of size at most k has at most 2κ messy segments and at most 3κ messy positions.*

Proof. Let x denote the number of messy segments in some arbitrary explanation for $(\mathcal{A}, k)$. Since $(\mathcal{A}, k)$ is reduced with respect to Reduction Rule 1.1, every position of $\mathcal{A}$ is the starting point or endpoint of some segment. In particular, every messy segment shares at least one endpoint with another messy segment. Hence, there are at most $3x/2$ messy positions in the explanation. Furthermore, since there are at most $k - x$ clean segments there are at most $2(k-x)$ clean positions. Thus, $n \leq 2(k-x)+3x/2$ which implies $x \leq 2\kappa$, and the number of messy positions is at most $(3x/2) \leq 3\kappa$. □

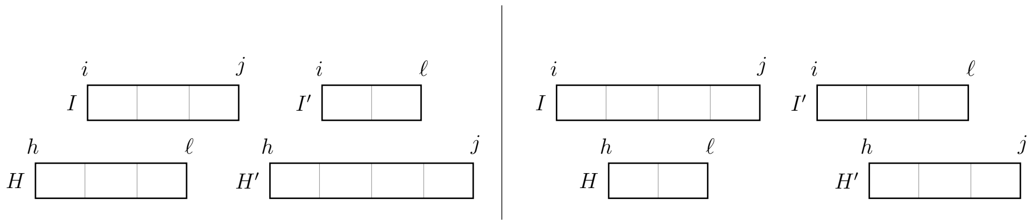

Figure 4.1: Illustration of the proof of Lemma 4.3. The two equal-weight segments $H = [h, \ell]$ and $I = [i, j]$ with $i < \ell < j$ are replaced by $I' = [i, \ell]$ and $H' = [h, j]$ without changing the weight. The case $h < i$ is shown on the left, and the case $h > i$ is shown on the right.

In order to exploit a small value of κ algorithmically, the main challenge is to identify clean segments as their number is not bounded in the parameter κ. For VPE, we can identify clean segments if the set of clean positions is known: In this case, one may greedily add for any clean uptick a clean segment that starts at this uptick and ends at the next downtick of the same size. The following lemma gives a formal proof of this claim.

Lemma 4.3. *Let $(\mathcal{A}, k)$ be a* Vector Positive Explanation *instance and let $\mathcal{I}$ be an explanation for it. Furthermore, let $[i, j] \in \mathcal{I}$ be a clean segment and let ℓ be a clean position such that $i < \ell < j$ and ℓ and j are downticks of the same size. Then, there is an explanation $\mathcal{I}'$ for $(\mathcal{A}, k)$ containing the clean segment $[i, \ell]$ such that $|\mathcal{I}'| = |\mathcal{I}|$.*

Proof. Since ℓ is clean, there is a clean segment $H = [h, \ell]$ whose weight equals the downtick size of ℓ and hence the weight of $I = [i, j]$. Consider a segment set $\mathcal{I}'$ obtained from $\mathcal{I}$ by replacing I with $I' = [i, \ell]$ and H with $H' = [h, j]$. The weights remain unchanged. In particular, the weights of I' and H' are the same as the weights of I and H. It is straightforward to verify, by a case distinction whether $h > i$ or not, that $\mathcal{I}'$ with the adjusted weight function is still an explanation. See Figure 4.1 for an illustration of the two cases. □

For single-peaked instances of Vector Positive Explanation, the situation is even more favorable: we can directly identify clean positions and the segments starting and ending at these clean positions.

Lemma 4.4. *Let $(\mathcal{A}, k)$ be a single-peaked instance of* Vector Positive Explanation*. If vector $\mathcal{A}$ has an uptick i and a downtick j of the same sizes, then there is a minimum-size explanation for $(\mathcal{A}, k)$ containing the segment $[i, j]$ with weight equal to the size of the uptick i.*

Proof. Let $(\mathcal{A}, k)$ be a single-peaked VPE instance and let T be the tick vector of $\mathcal{A}$. By Corollary 2.2(ii) $(\mathcal{A}, k)$ is an equivalent VE instance and thus by Theorem 2.3 we may assume that $j = i + 1$. Furthermore, let $(\mathcal{I}, \omega)$ be a regular minimum-size explanation of $(\mathcal{A}, k)$. Let I_s be all segments in $\mathcal{I}$ starting in i and let I_e be all segments in $\mathcal{I}$ ending in $i + 1$ and, additionally, do not start in i. Hence $I_s \cap I_e = \emptyset$.

Let T' be a copy of T and "subtract" the segments in $\mathcal{I} \setminus (I_s \cup I_e)$: For each segment $[\ell, r] \in \mathcal{I} \setminus (I_s \cup I_e)$ of weight a decrease $T'[\ell]$ by a and increase $T'[r]$ by a. Additionally, subtract $[i, i + 1]$ with weight $T[i]$, meaning that we set $T'[i]$ and $T'[i + 1]$ from $\pm T[i]$ to zero. Observe that if there is a size-$(|I_s| + |I_e| - 1)$ explanation for the input vector corresponding to T', then combining this explanation with the segments from $\mathcal{I} \setminus (I_s \cup I_e)$ and the segment $[i, i + 1]$ of weight $T[i]$ gives a minimum-size explanation for $(\mathcal{A}, k)$. Thus, it remains to show that there is a size-$(|I_s| + |I_e| - 1)$ explanation for the input vector corresponding to T'. Since we subtracted all segments in $\mathcal{I} \setminus (I_s \cup I_e)$, all positions in T' that have nonzero entries are the start or end of a segment in $I_s \cup I_e$. Moreover, since we additionally subtracted $[i, i + 1]$ with weight $T[i]$, there are at most $|I_e|$ upticks and at most $|I_s|$ downticks in T'. Hence, the corresponding input vector of T' has, after an application of Reduction Rule 1.1, at most $|I_e| + |I_s| - 1$ entries. Therefore, it has a trivial explanation of size at most $|I_e| + |I_s| - 1$. □

We now have all ingredients to provide our two tractability results with respect to the parameter κ. More precisely, we show membership in XP for VPE and fixed-parameter tractability for VE and single-peaked VPE. The main approach of the algorithms is as follows: For VPE, we guess all messy positions, then we greedily identify the clean segments (using Lemma 4.3), and then we solve the remaining instance (which now has size bounded in κ). For single-peaked VPE and VE, we can directly reduce the instance to one that has only messy positions (using Lemma 4.4).

Theorem 4.5. *(i)* Vector Positive Explanation *can be solved in* $O((2k)^{3\kappa} \cdot (2\kappa)! \cdot \kappa \cdot k + k \log k + n)$ *time.*

(ii) Any single-peaked instance of Vector Positive Explanation *and any instance of* Vector Explanation *can be reduced in* $O(n + k \log k)$ *time to an equivalent instance with at most* 3κ *entries. Moreover,* Vector Explanation *and single-peaked* Vector Positive Explanation *are solvable in* $O((2\kappa)! \cdot \kappa + k \log k + n)$ *time.*

Proof of Theorem 4.5(i). We prove that VPE can be solved in $O((2k)^{3\kappa} \cdot (2\kappa)! \cdot \kappa \cdot k + k \log k + n)$ time. Let $(\mathcal{A}, k)$ be an instance of VPE and let T be the tick vector corresponding to $\mathcal{A}$. We may assume via a preprocessing step running in $O(n)$ time that Reduction Rule 1.1 has been exhaustively applied and thus the number of positions is at most $2k$.

The algorithm works as follows. Let U (D) be the set of all upticks (downticks) in T. Sort the values in U and D in ascending order according to their absolute sizes and use their position in T as a tie-breaker (smaller positions come first). This can be done in $O(k \log k)$ time. Next, branch into the at most $(2k)^{3\kappa}$ possibilities for choosing all of the at most 3κ messy positions (Lemma 4.2). If the guess was correct, then for each clean uptick there is a clean downtick of equal size.

By Lemma 4.3 there is a minimum-size explanation that contains a segment starting in any clean uptick position i and ending at the first clean downtick position $j > i$ with the same size. We next find and remove these segments: Initialize $\tilde{k}$ by the value of k and also T' by T. Iterate over all clean upticks in the order of U and find for each of them the first clean downtick in D which starts to its right. Delete the up- and downtick from T' and decrease parameter $\tilde{k}$ by one. Clearly, by using two pointers, one for U and one for D, iterating over U and finding the downtick in D can be done in $O(k)$ time as by the order of U and D one has to move the pointers only to the right. Moreover, if at some point of the iteration we do not find any "matching" downtick or at the end there remain some clean downticks in D, then we abort this branch as the guess of clean positions was incorrect. Let $\mathcal{A}'$ be the input vector corresponding to the final T'. Note that since all positions in $\mathcal{A}'$ are messy, by Lemma 4.2 it follows that $|\mathcal{A}'| \leq 2\kappa$ and $\tilde{k} \leq 3\kappa$. Hence, Theorem 4.1 solves the remaining instance $(\mathcal{A}', \tilde{k})$ in $O((2\kappa)! \cdot \kappa)$ time. The overall running time is $O((2k)^{3\kappa} \cdot (2\kappa)! \cdot \kappa \cdot k + k \log k + n)$.

Proof of Theorem 4.5(ii): Due to Corollary 2.4(i), we can assume that the given instance of VE is single-peaked. Also, because of Corollary 2.2(ii), we only need to investigate whether the given single-peaked instance is a yes-instance for VPE. We first apply Reduction Rule 1.1 exhaustively. After that, if there is an uptick and a downtick of the same size, then by Lemma 4.4 there is an optimal solution containing a segment starting at the uptick and ending at the downtick of weight equal to the size of the uptick. Hence, by applying a similar procedure as in the proof of Theorem 4.5(i) (sort up- and downticks by their size) one finds and eliminates all these segments in $O(k \log k)$ time. Note that by removing such a segment from the input vector the length of the vector is reduced by two, while k is reduced by one, so κ stays the same.

In the remaining instance all positions are messy and thus by Lemma 4.2 there are at most 3κ messy positions and 2κ messy segments explaining them. Thus, one ends up with a problem kernel having at most 3κ positions. By Theorem 4.1, this kernel can be solved in $O((2\kappa)! \cdot 2\kappa + 3\kappa)$ time. □

Theorem 4.5(ii) implies that single-peaked VPE and VE are fixed-parameter tractable with respect to κ. For VPE, we obtained a polynomial-time algorithm for every fixed value of κ but not a fixed-parameter algorithm for κ. As we show in the following, such an algorithm is unlikely.

Theorem 4.6. Vector Positive Explanation *is W[1]-hard with respect to* κ.

Proof. We present a parameterized reduction from the Subset Sum problem.

> Subset Sum [18, SP13]
> **Input:** A multiset $X = \{x_1, \ldots, x_\ell\}$ of positive integers and two positive integers y and Φ.
> **Question:** Is there a size-Φ subset X' of X such that $\sum_{x_i \in X'} x_i = y$?

Subset Sum is W[1]-hard with respect to the solution size Φ [13]. In the following, we use $t := \sum_{1 \le i \le \ell} x_i$ to denote the total sum of the integers in X. Note that by modifying the x_i's we can assume that for every size-$(\Phi - 1)$ subset X' the sum $\sum_{x_i \in X'} x_i$ is *less than* y: adding t to each input integer, and $\Phi \cdot t$ to y results in an instance for which this holds. Next, we describe the parameterized reduction.

The input vector $\mathcal{A}$ has length $2\ell + 1$. For $i \le \ell$, we set $\mathcal{A}[i] := \sum_{j=1}^{i} x_j$. Let $\mathcal{A}[\ell+1] = t - y$. For $i \ge \ell + 2$, we set $\mathcal{A}[i] = \mathcal{A}[2\ell+2-i]$. The number of allowed segments is set to $\ell + \Phi$. Consequently, $\kappa = 2(\ell+\Phi) - (2\ell+1) = 2\Phi - 1$.

We complete the proof by showing that for this construction the following equivalence holds.

> (X, y, Φ) is a yes-instance of Subset Sum $\Leftrightarrow$ $(\mathcal{A}, \ell + \Phi)$ is a yes-instance of VPE.

"$\Rightarrow$": Let X' be a size-Φ subset of X whose values sum up to y. Then, consider the following set $\mathcal{I}$ of segments.

For each $x_i \notin X'$, add the segment $J_i = [i, 2\ell + 3 - i]$. There are $\ell - \Phi$ such segments. For each $x_i \in X'$, add two segments $I_i = [i, \ell + 1]$

and $I'_i = [\ell+2, 2\ell+3-i]$. For each of these two types of segments there are Φ of them. Hence, $|\mathcal{I}| = \ell + \Phi$. For each $1 \le i \le \ell$ set the weights of the segments J_i, I_i and I'_i to x_i. Now, $\mathcal{I}$ explains $\mathcal{A}$: First, for each $i \le \ell$, $\mathcal{A}[i] = \sum_{j \le i} x_j$ is explained by $\{J_j \mid j \le i \wedge x_j \notin X'\} \cup \{I_j \mid j \le i \wedge x_j \in X'\}$. Second, $\mathcal{A}[\ell+1] = t - y$ is explained by exactly the segments J_j with $x_j \notin X'$. Finally, for $i > \ell + 1$, $\mathcal{A}[i] = \mathcal{A}[2\ell+2-i] = \sum_{j \le 2\ell+2-i} x_j$ is explained by $\{J_j \mid j \le 2\ell+2-i \wedge x_j \notin X'\} \cup \{I'_j \mid j \le 2\ell+2-i \wedge x_j \in X'\}$.

"$\Leftarrow$": Let $\mathcal{I}$ be a set of $\ell + \Phi$ segments that explain $\mathcal{A}$. By Theorem 2.1 we can assume that $\mathcal{I}$ is regular. First, note that for each position $i \le \ell$, there is at least one segment that starts at i. Also, each of these segments has a weight of at most the maximum value in X. Since for any X' with $|X'| < \Phi$ it holds that $\sum_{x_i \in X'} x_i < y$ and the size of downtick $\ell + 1$ is y, at least Φ segments end at $\ell + 1$. Similarly, for each $i \ge \ell + 3$ there is at least one segment that ends at position i, and each of these segments has a weight of at most x_j for some $x_j \in X$. Further, since the size of uptick $\ell + 2$ is y, at least Φ segments start at $\ell + 2$. This implies that there are *exactly* ℓ segments starting in the first ℓ positions and *exactly* Φ segments ending at position $\ell+1$. Therefore, for each $i \le \ell$ there is exactly one segment starting at i which has weight x_i. Since Φ of these segments end at position $\ell + 1$, they correspond to a size-Φ set $X' \subseteq X$. Finally, since $\mathcal{A}[\ell] = t$ and $\mathcal{A}[\ell+1] = t - y$ the sum $\sum_{x_i \in X'} x_i$ of the integers in this set is exactly y. □

Parameter κ used in Theorems 4.5 and 4.6 measures to what extent the solution exceeds the lower bound $\lfloor n/2 \rfloor + 1$. Another bound on the solution size is n: If $k = n$, then any instance of VPE or VE is a trivial yes-instance. Hence, it is interesting to consider the parameter $n - k$. Furthermore, it is natural to consider explanations with restricted segment length ξ or the maximum number ϕ of segments overlapping at some position. The following theorem shows that VPE and VE are already NP-complete even if $k = n - 1$, $\xi \ge 3$, and $\phi = 2$. To this end, we reduce from the NP-complete Partition problem [18, SP12]. In terms of parameterized complexity this implies that, unless P=NP, VPE is not fixed-parameter tractable with respect to the "maximum segment length ξ", the "maximum number ϕ of segments overlapping at some position", and the "below guarantee parameter" $n - k$.

Theorem 4.7. Vector Positive Explanation *and* Vector Explanation *are weakly NP-complete even if $k = n - 1$ and every yes-instance has an explanation of at most k segments where each position is covered by at most two segments and each segment has length at most three.*

Proof. We reduce from the weakly NP-complete PARTITION problem.

PARTITION [18, SP12]
Input: A multiset of positive integers $S = \{a_1, \ldots, a_t\}$.
Question: Is there a subset $S' \subseteq S$ such that $\sum_{a_i \in S'} a_i = \sum_{a_i \in S \setminus S'} a_i$?

Given an instance $S = \{a_1, \ldots, a_t\}$ of PARTITION, we create an input instance $(\mathcal{A}, k)$, where $\mathcal{A}$ is a vector of length $3t + 1$ and $k = 3t$. More specifically, $\mathcal{A}^T$ is the vector

$$\begin{pmatrix} 1 & & \\ 2 & & \\ 2 & + & (t+1) \cdot a_1 \\ 3 & + & (t+1) \cdot a_1 \\ 4 & + & (t+1) \cdot a_1 \\ 4 & + & (t+1) \cdot (a_1 + a_2) \\ & \vdots & \\ 2j-1 & + & (t+1) \cdot \sum_{i=1}^{j-1} a_i \\ 2j & + & (t+1) \cdot \sum_{i=1}^{j-1} a_i \\ 2j & + & (t+1) \cdot \sum_{i=1}^{j} a_i \\ & \vdots & \\ 2t-1 & + & (t+1) \cdot \sum_{i=1}^{t-1} a_i \\ 2t & + & (t+1) \cdot \sum_{i=1}^{t-1} a_i \\ 2t & + & (t+1) \cdot \sum_{i=1}^{t} a_i \\ t & + & \frac{1}{2}(t+1) \cdot \sum_{i=1}^{t} a_i \end{pmatrix}$$

Obviously, the reduction runs in polynomial time. It remains to show that

> $S = \{a_1, \ldots, a_t\}$ is a yes-instance of PARTITION $\Leftrightarrow$ $(\mathcal{A}, k = 3t)$ is a yes-instance of VPE and VE.

"$\Rightarrow$": Let $S' \subseteq S$ be a solution for the PARTITION instance, meaning that $\sum_{a_i \in S'} a_i = \sum_{a_i \in S \setminus S'} a_i$. Further, let $S'_j := S' \cap \{a_1, \ldots, a_j\}$, $\overline{S}_j := \{a_1, \ldots, a_j\} \setminus S'$, and $S'_0 = \overline{S}_0 := \emptyset$. We construct the set $\mathcal{I}$ of segments consisting of six subsets and their weights as follows (we use the notation $[\ell, r; a]$ for a weight-a segment that starts at ℓ, ends at r and does not include r):

$$\mathcal{I}_1 = \{[3j-2, 3j+1; j + (t+1) \cdot \sum_{a_i \in S'_{j-1}} a_i] \mid a_j \notin S'\},$$

$$\mathcal{I}_2 = \{[3j-1, 3j; j + (t+1) \cdot \sum_{a_i \in \overline{S}_{j-1}} a_i] \mid a_j \notin S'\},$$

$$\mathcal{I}_3 = \{[3j, 3j+2; j + (t+1) \cdot \sum_{a_i \in \overline{S}_j} a_i] \mid a_j \notin S'\},$$

$$\mathcal{I}_4 = \{[3j-1, 3j+2; j + (t+1) \cdot \sum_{a_i \in \overline{S}_{j-1}} a_i] \mid a_j \in S'\},$$

$$\mathcal{I}_5 = \{[3j-2, 3j; j + (t+1) \cdot \sum_{a_i \in S'_{j-1}} a_i] \mid a_j \in S'\},$$

$$\mathcal{I}_6 = \{[3j, 3j+1; j + (t+1) \cdot \sum_{a_i \in S'_j} a_i] \mid a_j \in S'\}.$$

As there are exactly three segments for each a_j, there are $3t$ segments in total. Note that if $a_j \notin S'$, then $S'_{j-1} = S'_j$. Otherwise $a_j \notin \overline{S'}$ and $\overline{S}_{j-1} = \overline{S}_j$.

Now, we show that $\mathcal{I}$ with weight function ω explains vector $\mathcal{A}$. Let $j \in \{1, \ldots, t\}$. At position $3j-2 = 3(j-1)+1$ we have $\mathcal{A}[3j-2] = 2j - 1 + (t+1)\sum_{i=1}^{j-1} a_i$. If $a_j \notin S'$, then segment $[3j-2, 3j+1]$ from $\mathcal{I}_1$ covers $3j-2$ and if $a_j \in S'$, then segment $[3j-2, 3j]$ from $\mathcal{I}_5$ covers $3j-2$. Both segments have weight $j + (t+1)\sum_{a_i \in S'_{j-1}} a_i$. Additionally, if $a_{j-1} \notin S'$, then segment $[3(j-1), 3(j-1)+2]$ from $\mathcal{I}_3$ also covers $3j-2$ and if $a_{j-1} \in S'$, then segment $[3(j-1)-1, 3(j-1)+2]$ from $\mathcal{I}_4$ also covers $3j-2$. In both cases the weight of the segment is $(j-1) + (t+1)\sum_{a_i \in \overline{S}_{j-1}} a_i$. In the former case this holds by definition. In the latter case, since $a_{j-1} \in S'$, it holds that $a_{j-1} \notin \overline{S'}$ and, thus, $\overline{S'}_{j-2} = \overline{S'}_{j-1}$. Summarizing, in each case the weights of the two segments covering position $3j-2$ sum up to

$$\begin{aligned}
&\left(j + (t+1) \cdot \sum_{a_i \in S'_{j-1}} a_i\right) + \left((j-1) + (t+1) \cdot \sum_{a_i \in \overline{S}_{j-1}} a_i\right) \\
&= 2j - 1 + (t+1) \cdot \sum_{i=1}^{j-1} a_i \\
&= \mathcal{A}[3j-2].
\end{aligned}$$

In the same way, at position $3j-1$, we have $\mathcal{A}[3j-2] = 2j+(t+1)\sum_{i=1}^{j-1} a_i$. If $a_j \notin S'$, then only segments $[3j-2, 3j+1]$ from $\mathcal{I}_1$ and $[3j-1, 3j]$ from $\mathcal{I}_2$ cover and explain this position, since

$$\begin{aligned}&\left(j + (t+1) \cdot \sum_{a_i \in S'_{j-1}} a_i\right) + \left(j + (t+1) \cdot \sum_{a_i \in \overline{S}_{j-1}} a_i\right)\\ &= 2j + (t+1) \cdot \sum_{i=1}^{j-1} a_i\\ &= \mathcal{A}[3j-1].\end{aligned}$$

Otherwise, only segments $[3j-1, 3j+2]$ from $\mathcal{I}_4$ and $[3j-2, 3j]$ from $\mathcal{I}_5$ cover and explain this position, since

$$\begin{aligned}&\left(j + (t+1) \cdot \sum_{a_i \in \overline{S}_{j-1}} a_i\right) + \left(j + (t+1) \cdot \sum_{a_i \in S'_{j-1}} a_i\right)\\ &= 2j + (t+1) \cdot \sum_{i=1}^{j-1} a_i\\ &= \mathcal{A}[3j-1].\end{aligned}$$

Also, at position $3j$, we have $\mathcal{A}[3j] = 2j + \sum_{i=1}^{j} a_i$. If $a_j \notin S'$, then only segments $[3j-2, 3j+1]$ from $\mathcal{I}_1$ and $[3j, 3j+2]$ from I_3 cover and explain this position since the sum of their weights equals

$$\begin{aligned}&\left(j + (t+1) \cdot \sum_{a_i \in S'_{j}} a_i\right) + \left(j + (t+1) \cdot \sum_{a_i \in \overline{S}_{j}} a_i\right)\\ &= 2j + (t+1) \cdot \sum_{i=1}^{j} a_i\\ &= \mathcal{A}[3j].\end{aligned}$$

This also holds for the case that $a_j \in S'$. Finally, we have only one segment covering the position $3t+1$ with weight

$$\begin{aligned} t+(t+1)\sum_{a_i\in\overline{S}_t} a_i &= t+(t+1)\cdot\sum_{a_i\in S\setminus S'} a_i \\ &= t+\frac{1}{2}(t+1)\cdot\sum_{i=1}^{t} a_i \\ &= \mathcal{A}[3t+1]. \end{aligned}$$

"$\Leftarrow$": Let $\mathcal{I}$ with weights ω be a regular explanation for vector $\mathcal{A}$ with at most k segments. As all upticks precede all downticks, all segments in $\mathcal{I}$ are positive. More precisely, as there are exactly $k=3t$ upticks, exactly one positive segment starts at every uptick and ends either at position $3t+1$ or $3t+2$.

We denote the segment of $\mathcal{I}$ starting at position $3i$ by I_i. Obviously, $\omega(I_i)=(t+1)\cdot a_i$. Furthermore, there are $2t$ segments of weight one. Now set $S' := \{a_i \mid I_i \text{ ends at position } 3t+2\}$. We show that S' is a solution of the Partition instance S: Let $x\in\{0,\ldots,2t\}$ be the number of segments of weight 1 that cover position $3t+1$. We have $x+(t+1)\sum_{a_i\in S'} a_i = \mathcal{A}[3t+1] = t+\frac{1}{2}(t+1)\cdot\sum_{i=1}^{t} a_i$. As $|t-x|\le t$, we have $\sum_{a_i\in S'} a_i = \frac{1}{2}\sum_{i=1}^{t} a_i$. Hence, S' is a solution for the Partition instance S.

As we can see from the reduction, every yes-instance of Partition is reduced to a yes-instance that can be explained by segments with $\xi=3$ and $\phi=2$ and every no-instance is reduced to an instance that cannot be explained by segments of any size. The statement of Theorem 4.7 follows. □

Next, we show that, in contrast to the NP-completeness for $\xi\ge 3$ (Theorem 4.7), VPE and VE are polynomial-time solvable for $\xi\le 2$.

Theorem 4.8. Vector Explanation *and* Vector Positive Explanation *can be solved in* $O(n^2)$ *time for maximum segment length* $\xi=2$.

Proof. We devise a dynamic programming algorithm for VPE. Afterwards, we show how to extend our algorithm to VE.

Let $(\mathcal{A},k)$ be an input instance, where $\mathcal{A}$ is a vector of length n. Since $\xi=2$ we may assume that the last position is only covered by either one length-two segment or one length-one segment, but not both: otherwise, there are two length-two segments covering the last position which can be transformed into a solution with two length-one segments. Due to this, if we have an optimal solution for a vector of length x, then we can find an optimal solution for a vector of length $x+1$ which contains either an additional length-one segment or a length-two segment covering the last position.

Based on this idea, we use dynamic programming with a table D indexed by $1,\dots,n$: For each $j \leq n$, we store in $D(j)$ the minimum number of segments needed to explain the subvector $\mathcal{A}[1,\dots,j]$. Let $D(0) = 0$ for simplicity. For $j = 1$, we set $D(1) = 1$ which is obviously correct. Now assume that for an index $j \leq n$, $D(i)$ was already computed for each $i < j$ and we now compute $D(j)$. We begin with $i := j$ and $a_i^j := \mathcal{A}[j]$. We set

$$a_{i-1}^j := \mathcal{A}[i-1] - a_i^j \text{ and } i := i - 1$$

as long as

$$a_i^j > 0 \text{ and } i > 1. \tag{*}$$

The idea behind this computation is that we can assume that each chain of q overlapping length-two segments fully explains all positions covered by segments from the chain. In particular, this includes the last covered position j and the first position $j-q$ and the latter implies that $a_{j-q}^j = 0$. If this is not the case, then the explanation uses at least $q+1$ segments to explain $q+1$ positions which could also be achieved with $q+1$ length-one segments. If Condition (*) does not hold, then there are two cases: If $a_i^j = 0$, then let $D(j) \leftarrow \min\{D(j-1)+1, D(i-1)+j-i\}$; otherwise let $D(j) := D(j-1)+1$. Finally, once the table is completed, we answer yes if $D(n) \leq k$, and no otherwise.

As the algorithm obviously works in $O(n^2)$ time, it remains to show that the algorithm fills the table correctly. The proof is by induction on j. Obviously $D(1)$ is computed correctly. For $j \leq n$, assume $D(i)$ is computed correctly for all $i < j$. We show that $D(j)$ is also computed correctly.

We first show that there is an explanation for $\mathcal{A}[1,\dots,j]$ with $D(j)$ segments. We have two cases: If $D(j) = D(j-1)+1$, then we use the explanation for $\mathcal{A}[1,\dots,j-1]$ with $D(j-1)$ segments and add a single length-one segment with weight $\mathcal{A}[j]$ to explain $\mathcal{A}[j]$. Otherwise, there is an $i \in \{1, j-1\}$ such that $D(j) = D(i-1)+j-i$. Let $a_j^j := \mathcal{A}[j]$, and $a_x^j := \mathcal{A}[x] - a_{x+1}^j$ for $i \leq x \leq j-1$. Note that $a_i^j = 0$ because of Condition (*). Then, we use the explanation for $\mathcal{A}[1,\dots,i-1]$ with $D(i-1)$ segments and add a set $\mathcal{I}$ of $j-i$ length-two segments such that for each $z \in \{i,\dots j-1\}$, we have a segment $I_z = [z, z+2]$ with weight a_{z+1}^j. Clearly, positions from 1 to $i-1$ are already explained. Since a_i^j equals zero, we have $\mathcal{A}[i] = a_{i+1}^j$ which is also the weight of I_i. Thus, $\mathcal{I}$ explains $\mathcal{A}[i]$. For $z \in \{i+1, \dots, j-1\}$, we have $\mathcal{A}[z] = a_z^j + a_{z+1}^j$ and $\mathcal{A}[j] = a_j^j$. Hence, the subvector $\mathcal{A}[i+1, \dots, j]$ is also explained by $\mathcal{I}$.

Next, we show that $D(j)$ is minimal. Suppose that there is an explanation $(\mathcal{I}, \omega)$ of $\mathcal{A}[1, \dots, j]$ with r segments. We will show that $r \geq D(j)$. Without loss of generality, we can assume that every length-one segment exclusively covers a position, since otherwise we can either merge two length-one segments or split one length-two segment into two length-one segments and merge one of them with the original length-one segment. We also assume that entry $\mathcal{A}[j]$ is positive as otherwise $D(j) = D(j-1) \leq r$. Let i be the last position such that all segments in $\mathcal{I}$ covering i start at i. If $i = j$, then $\mathcal{I} \setminus \{[j, j+1]\}$ is an explanation for $\mathcal{A}[1, \dots, j-1]$, and $r \geq D(j-1) + 1 \geq D(j)$ as $D(j-1)$ is optimal. If $i < j$, then $\mathcal{I}$ contains a chain of $j - i$ overlapping length-two segments $I_{i+1} = [i, i+2], \dots, I_j = [j-1, j+1]$ starting at i and ending at $j+1$. Since these are the only segments explaining positions $i, \dots, j$, their weights are $\omega(I_j) = \mathcal{A}[j]$ and $\omega(I_z) = \mathcal{A}[z] - \omega(I_{z+1})$, $j - 1 \geq z \geq i + 1$. Position i is only explained by I_{i+1}, so we have $\mathcal{A}[i] = \omega(I_{i+1}) = \mathcal{A}[i+1] - \omega(I_{i+2})$. Hence, it follows from the definition of the dynamic programming algorithm that $a_z^j = \omega(I_z)$, $i + 1 \leq z \leq j$. This means that the algorithm stops at position i with $a_i^j = \omega(I_{i+1}) - a_{i+1}^j = 0$. Thus, $D(j) = \min\{D(j-1)+1, D(i-1) + j - i\} \leq D(i-1) + j - i$. Furthermore, $\mathcal{I} \setminus \{I_z \mid z \in \{i+1, \dots j\}\}$ is an explanation for $\mathcal{A}[1, \dots, i-1]$. Hence, $r \geq D(i-1) + j - i \geq D(j)$ because $D(i-1)$ is optimal.

To solve VE, it is sufficient to change Condition (*) in the loop of the above algorithm to "... as long as $a_i^j \neq 0$ and $i > 1$." The rest of the proof remains the same. □

5. Conclusion and Open Questions

We explored the parameterized complexity of Vector Explanation and Vector Positive Explanation with respect to various parameterizations. By considering the tick vector concept, we gained further combinatorial insights into Vector Explanation and Vector Positive Explanation. In particular, we showed that for Vector Explanation the tick vector can be arbitrarily permuted. Several of our fixed-parameter algorithms for Vector Explanation and Vector Positive Explanation are based on this observation. Furthermore, we found that, surprisingly, Vector

Positive Explanation is presumably harder than Vector Explanation, for example concerning the distance from triviality parameter $\kappa = 2k - n$.

It would be interesting to significantly improve on several of the running time upper bounds of our (theoretical) tractability results (cf. Table 1 for an overview). In particular, obtaining tight lower and upper running time bounds for the parameter number k of segments seems to be a challenging and interesting research task. Moreover, we also left open a number of concrete problems. We conclude with three of them:

- Is Vector Positive Explanation fixed-parameter tractable with respect to the maximum difference δ?
- Does Vector Explanation parameterized by δ or parameterized by γ admit a polynomial kernel?
- Is Vector (Positive) Explanation fixed-parameter tractable with respect to the parameter "number of different values in the input vector $\mathcal{A}$"? This parameter would be a natural version of "parameterization by the number of numbers" [14].

Last but not least, we would like to point to the challenging task to transfer our study to the case of a 2-dimensional ("matrix") input [22].

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