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THE CENTRAL LIMIT THEOREM FOR STATIONARY MARKOV CHAINS UNDER INVARIANT SPLITTINGS

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The central limit theorem (CLT) for stationary ergodic Markov chains is investigated. We give a short survey of related results on the CLT for general (not necessarily Harris recurrent) chains and formulate a new sufficient condition for its validity. Furthermore, Markov operators are considered which admit invariant orthogonal splittings of the space of square-integrable functions. We show how conditions for the CLT can be improved if this additional structure is taken into account. Finally we give examples of this situation, namely endomorphisms of compact Abelian groups and random walks on compact homogeneous spaces.

Keywords: Central limit theorem; homogeneous space; invariant splitting; Markov chain; random walk; stationary process.

1. Introduction

Let $(S, \mathcal{A}, \mu)$ be a probability space and $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary, ergodic Markov chain with state space S, transition operator Q and stationary distribution μ. It is assumed that $(\xi_k)_{k\in\mathbb{Z}}$ is realized on its own sample path space $(\Omega, \mathcal{B}, P_\mu)$. Denote by L_2^0 the subspace of $L_2^{\mathbb{R}}(S, \mathcal{A}, \mu) = L_2$ consisting of functions with $\int f\, d\mu = 0$. Given $f \in L_2^0$ let

$$S_n(f) = f(\xi_1) + \cdots + f(\xi_n).$$

We will be interested in conditions on $f \in L_2^0$ under which $S_n(f)$ satisfies the central limit theorem (CLT)

$$S_n(f)/\sqrt{n} \overset{n\to\infty}{\Longrightarrow} N(0, \sigma^2(f)), \tag{1}$$

with

$$ES_n(f)^2/n \xrightarrow{n\to\infty} \sigma^2(f). \tag{2}$$

Here $N(0, \sigma^2)$ denotes the normal law with mean 0 and variance σ^2, which in the degenerate case $\sigma^2 = 0$ is equal to the Dirac measure at 0, and $\Rightarrow$ denotes weak convergence of distributions.

Definition 1.1. If for a function $f \in L_2^0$, (1) and (2) hold, we say that f is asymptotically normal.

Asymptotic normality of functions $f \in L_2^0$ has been studied extensively in the literature, and we want to review some of the results. A very simple sufficient condition (cf. Gordin & Lifšic [11]) is the existence of a solution $g \in L_2$ to the equation

$$f = g - Qg. \tag{3}$$

If (3) holds, f is asymptotically normal with limit variance given by

$$\sigma^2(f) = |g|_2^2 - |Qg|_2^2, \tag{4}$$

where $|\cdot|_2$ is the L_2-norm. Under the same assumption the functional central limit theorem (FCLT) also holds (see, for instance, Borodin & Ibragimov [1]). Furthermore, we may change P_μ to the probability measure P_s corresponding to the chain started from a point $s \in S$ so that $\xi_0 = s$, and the CLT and FCLT still hold true for almost all $s \in S$ with respect to μ. In general we may conclude (3) from the convergence in L^2 of the series

$$\sum_{n=0}^{\infty} Q^n f. \tag{5}$$

Then we can set $g = \sum_{n=0}^{\infty} Q^n f$. The convergence of (5) implies, in particular, $|Q^n f|_2 \xrightarrow[n\to\infty]{} 0$. An obvious sufficient condition under which the series (5) converges is

$$\sum_{n=0}^{\infty} |Q^n f|_2 < \infty. \tag{6}$$

However, the assumptions (6) and even (3) seem to be too restrictive, at least for some special classes of transition operators. A bounded operator K in a Hilbert space is called *normal* if it commutes with its adjoint: $KK^* = K^*K$. Let D denote the closed unit disc in $\mathbb{C}$. Gordin & Lifšic [12] observed that the condition

$$\int_D \frac{1}{|1-z|} \rho_f(dz) < \infty \tag{7}$$

is sufficient for f to be asymptotically normal, provided the operator Q is normal. Here ρ_f is the spectral measure of f with respect to Q. In this case the formula for the limiting variance $\sigma^2(f)$ reads

$$\sigma^2(f) = \int_D \frac{1-|z|^2}{|1-z|^2} \rho_f(dz). \tag{8}$$

Observe that (7) is equivalent to the solvability in L_2 of the equation

$$f = (I - Q)^{1/2} g. \tag{9}$$

The detailed proofs of these results, along with some generalizations and applications, were published by Gordin & Lifšic in ([1], Chap. 4, Secs. 7–9). See also Derriennic & Lin [3], and for the self-adjoint case, which corresponds to *reversible* chains, Kipnis & Varadhan [13]. It was assumed in the above-mentioned papers that the path space of $(\xi_k)_{k\in\mathbb{Z}}$ is endowed with the stationary measure P_μ. Under slightly stronger conditions the same conclusion was established in [4] for μ-almost all measures $P_s, s \in S$, provided that Q is a normal operator. All the above-mentioned papers used martingale approximation.

A natural question now arises whether it is possible to relax the known conditions for the CLT which make sense for arbitrary transition operators (like (5) or (6)). A related problem is to obtain a formula for the limiting variance valid (as (8)) in some cases when (3) has no solution in L_2. Recently some results in this direction have been obtained. Maxwell & Woodroofe [15] showed that if $f \in L_2^0$ satisfies

$$\sum_{n\geq 1} n^{-3/2} \left| \sum_{k=0}^{n-1} Q^k f \right|_2 < \infty, \tag{10}$$

then f is asymptotically normal. In case $\left| \sum_{k=0}^{n-1} Q^k f \right|_2 = O(n^\alpha)$ for some $\alpha < 1/2$, Derriennic & Lin [6] showed that the FCLT for the Markov chain, started at a point, holds true.

Several questions however remain open. We do not know, for example, whether (9) implies the CLT without normality assumptions. It can be shown using the results in [5] that (9) is equivalent to the convergence in L_2 of $\sum_{n\geq 1} n^{-3/2} \sum_{k=0}^{n-1} Q^k f$ (compare this to (10)).

An important class of Markov operators Q is specified by the condition

$$QQ^* = I. \tag{11}$$

Such operators are sometimes called *coisomteries* because the adjoint operator Q^* is an isometry in view of (11). The corresponding class of Markov chains is useful to produce examples of transition operators with properties very far from normal (remark that only those coisomteries are normal which are unitaries; the latter leads to $\sigma^2(f) = 0$ in (16)).

This paper is organised as follows. In Sec. 2, we derive a new sufficient condition for a function $f \in L_2^0$ to be asymptotically normal. This is applied to certain linear processes. Our main interest is the situation in which there exists an orthogonal splitting

$$L_2^{\mathbb{C}}(S, \mathcal{A}, \mu) = \bigoplus_{i\in\mathcal{I}} H_i$$

invariant under the Markov operator, i.e. $QH_i \subset H_i$ for every i. Here $\mathcal{I}$ is a countable index set. Evidently, this is the case if Q is normal with discrete spectrum and the H_i correspond to eigenspaces of different eigenvalues. However, there are several other interesting examples, even of coisometries, for which such invariant splittings exist. In Sec. 3 we prove two conditions, adapted to invariant splittings, under which f is asymptotically normal. The first is related to the condition derived in Sec. 2, the second to (10) as introduced by Maxwell & Woodroofe [15]. It is shown that these are indeed improvements as compared to the original conditions. Finally, in Sec. 4, we consider two examples, namely endomorphisms of compact Abelian groups and random walks on compact homogeneous spaces. Let us mention that using invariant splittings, Denker and Gordin [2] proved the FCLT for a certain class of transformations of the two-dimensional torus.

In the present paper we consider only discrete splittings. In principle, similar results can be obtained for some splittings with a nontrivial continuous part which is a direct integral of Q-invariant subspaces. For example, this is the case if one considers a random walk on a non-compact homogeneous space of finite invariant measure such as $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,Z)$.

2. A Sufficient Condition for the CLT

In this section we will prove asymptotic normality for functions $f \in L_2^0$ which satisfy

$$|Q^n f|_2 \xrightarrow{n\to\infty} 0 \tag{12}$$

and

$$\sum_{n=0}^{\infty} \left(|Q^n f|_2^2 - |Q^{n+1} f|_2^2\right)^{1/2} < \infty. \tag{13}$$

Our discussion is based on the following well-known result, which is contained implicitly in [1, 9, 18].

Theorem 2.1. *Let $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary ergodic Markov chain. Assume that $f \in L_2^0$ satisfies*

$$\lim_{n\to\infty} \sup_{m\geq 0} \left(\left| \sum_{k=n}^{n+m} Q^k f \right|_2^2 - \left| \sum_{k=n+1}^{n+m+1} Q^k f \right|_2^2 \right) = 0 \tag{14}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} Q^k f \right|_2^2 = 0. \tag{15}$$

Then f is asymptotically normal with limit variance

$$\sigma^2(f) = \lim_{n\to\infty} \left(\left| \sum_{k=0}^{n} Q^k f \right|_2^2 - \left| \sum_{k=1}^{n+1} Q^k f \right|_2^2 \right). \tag{16}$$

This will be used to prove the following:

Theorem 2.2. *Let $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary ergodic Markov chain. Assume that $f \in L_2^0$ satisfies* (12) *and* (13). *Then it also satisfies* (14) *and* (15) *and consequently, f is asymptotically normal with limit variance $\sigma^2(f)$ given in* (16).

First let us prove two lemmas.

Lemma 2.1. *For any $f \in L_2$, $n, m \geq 0$,*

$$\left|\sum_{k=m}^{n+m-1} Q^k f\right|_2^2 - \left|Q \sum_{k=m}^{n+m-1} Q^k f\right|_2^2 \leq \left(\sum_{k=m}^{n+m-1} \left(|Q^k f|_2^2 - |Q^{k+1} f|_2^2\right)^{1/2}\right)^2 . \quad (17)$$

Proof. The map $(f, g) \mapsto \langle f, g\rangle - \langle Qf, Qg\rangle$ is a non-negative, symmetric bilinear form since Q is a contraction, and hence gives rise to the seminorm $(|f|_2^2 - |Qf|_2^2)^{1/2}$. Relation (17) now follows by applying the triangle inequality. □

Lemma 2.2. *Assume that $f \in L_2$ satisfies* (12). *Then for every $n \geq 0$,*

$$\left|\sum_{k=0}^{n-1} Q^k f\right|_2^2 = \sum_{l=0}^{\infty} \left(\left|\sum_{k=0}^{n-1} Q^{k+l} f\right|_2^2 - \left|\sum_{k=0}^{n-1} Q^{k+l+1} f\right|_2^2\right) . \quad (18)$$

Proof. For $N > 0$ we have

$$\left|\sum_{k=0}^{n-1} Q^k f\right|_2^2 = \sum_{l=0}^{N-1} \left(\left|\sum_{k=0}^{n-1} Q^{k+l} f\right|_2^2 - \left|Q\sum_{k=0}^{n-1} Q^{k+l} f\right|_2^2\right) + \left|Q^N \sum_{k=0}^{n-1} Q^k f\right|_2^2 .$$

The remainder term vanishes as $N \to \infty$ due to (12). □

Proof of Theorem 2.2. We want to show that the conditions of Theorem 2.1 hold. From Lemma 2.1 it follows directly that (13) implies (14). In order to show (15), we compute

$$\begin{aligned} n^{-1}\left|\sum_{k=0}^{n-1} Q^k f\right|_2^2 &= n^{-1}\sum_{l=0}^{\infty} \left(\left|\sum_{k=0}^{n-1} Q^{k+l} f\right|_2^2 - \left|Q\sum_{k=0}^{n-1} Q^{k+l} f\right|_2^2\right) \\ &\leq n^{-1}\sum_{l=0}^{\infty}\left(\sum_{k=0}^{n-1}\left(|Q^{k+l} f|_2^2 - |Q^{k+l+1} f|_2^2\right)^{1/2}\right)^2 \\ &\leq n^{-1}\sum_{k=0}^{n-1}\left(\sum_{l=0}^{\infty}\left(|Q^{k+l} f|_2^2 - |Q^{k+l+1} f|_2^2\right)^{1/2}\right)^2 \\ &= n^{-1}\sum_{k=0}^{n-1}\left(\sum_{r=k}^{\infty}\left(|Q^{r} f|_2^2 - |Q^{r+1} f|_2^2\right)^{1/2}\right)^2 . \end{aligned} \quad (19)$$

Since $\sum_{r=k}^{\infty}(|Q^r f|_2^2 - |Q^{r+1} f|_2^2)^{1/2} \to 0$, (15) follows. This proves the theorem. □

Example 2.1. (Linear processes) Let $(\eta_k)_{k\in\mathbb{Z}}$ be a sequence of independent identically distributed random variables with values in $\mathbb{R}$. Let us form a sequence $(\xi_k)_{k\in\mathbb{Z}}$ by setting $\xi_k = (\dots, \eta_{k-1}, \eta_k)$. This is a Markov sequence with transition operator satisfying (11). Assume that the variables η_k have second moments and expectation 0. Set

$$f(\xi_k) = \sum_{j=0}^{\infty} a_j \eta_{k-j},$$

where $(a_j)_{j\geq 0}$ is a real sequence satisfying $\sum_{j=0}^{\infty} a_j^2 < \infty$. Then (6) means that

$$\sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} a_k^2 \right)^{1/2} < \infty,$$

while (13) only requires

$$\sum_{n=0}^{\infty} |a_n| < \infty.$$

3. CLT Under Invariant Splittings

In this section we study the situation in which there is an orthogonal splitting of the L_2-space invariant under the Markov operator. First let us give a general sufficient condition, adapted to an orthogonal splitting, for convergence of a series in a Hilbert space.

Let H be a real or complex Hilbert space with norm denoted by $\|\cdot\|$ and let $H = \bigoplus_{i\in\mathcal{I}} H_i$ be a splitting into closed, orthogonal subspaces, where $\mathcal{I}$ is a countable index set. For $x \in H,\ i \in \mathcal{I}$ let x^i denote the orthogonal projection of x onto H_i, so that $x = \sum_{i\in\mathcal{I}} x^i$.

Lemma 3.1. *If* $(x_n)_{n\geq 1} \subset H$ *satisfies*

$$\sum_{i\in\mathcal{I}} \left(\sum_{n\geq 1} \|x_n^i\| \right)^2 < \infty, \tag{20}$$

then the series $\sum_{n\geq 1} x_n$ *converges.*

This is easily proved using the Cauchy criterion. The next lemma shows that taking into account the orthogonal splitting indeed brings an improvement.

Lemma 3.2.

$$\sum_{i\in\mathcal{I}} \left(\sum_{n\geq 1} \|x_n^i\| \right)^2 \leq \left(\sum_{n\geq 1} \|x_n\| \right)^2.$$

Proof. Since $\|x_n\| = (\sum_{i\in\mathcal{I}} \|x_n^i\|^2)^{1/2}$, after expanding both sides, we have to show that

$$\sum_{n_1,n_2\geq 0} \underbrace{\sum_{i\in\mathcal{I}} \|x_{n_1}^i\| \|x_{n_2}^i\|}_{A} \leq \sum_{n_1,n_2\geq 0} \underbrace{\left(\sum_{i\in\mathcal{I}} \|x_{n_1}^i\|^2 \right)^{1/2} \left(\sum_{i\in\mathcal{I}} \|x_{n_2}^i\| \right)^{1/2}}_{B}.$$

This follows since $A \leq B$ from the Schwarz inequality. □

Corollary 3.1. *If there are two splittings $H = \bigoplus_{i\in\mathcal{I}} H_i = \bigoplus_{i'\in\mathcal{I}'} H_{i'}$ such that for each $i \in \mathcal{I}$ there is a $i' \in \mathcal{I}'$ with $H_i \subset H_{i'}$, then*

$$\sum_{i\in\mathcal{I}} \left(\sum_{n\geq 1} \|x_n^i\| \right)^2 \leq \sum_{i'\in\mathcal{I}'} \left(\sum_{n\geq 1} \|x_n^{i'}\| \right)^2 . \tag{21}$$

Now consider once more a stationary ergodic Markov chain $(\xi_n)_{n\in\mathbb{Z}}$ on $(\Omega, \mathcal{B}, P_\mu)$ with state space $(S, \mathcal{A}, \mu)$, transition operator Q and stationary initial distribution μ. Since we have to deal with complex-valued functions, observe that Q also acts as a contraction on $L_2^{\mathbb{C}}(S, \mathcal{A}, \mu)$ and that $\overline{Qf} = Q\bar{f}$, where "$^-$" denotes complex conjugation. Assume that there is a splitting

$$L_2^{\mathbb{C}}(S, \mathcal{A}, \mu) = \bigoplus_{i\in\mathcal{I}} H_i$$

of $L_2^{\mathbb{C}}(S, \mathcal{A}, \mu)$ into closed orthogonal subspaces H_i that are invariant under Q, i.e. $QH_i \subset H_i$. We denote by Q_i the restriction of Q to H_i and by f_i the orthogonal projection of $f \in L_2^{\mathbb{C}}(S, \mathcal{A}, \mu)$ onto H_i. An application of Lemma 3.1 immediately gives

Proposition 3.1. *If $f \in L_2^0$ satisfies*

$$\sum_{i\in\mathcal{I}} \left(\sum_{n\geq 0} |Q_i^n f_i|_2 \right)^2 < \infty, \tag{22}$$

then the series (5) *converges and consequently, f is asymptotically normal.*

This result will be improved in two ways. Firstly, we give a criterion analogously to (13) modified to the context of invariant splittings. Observe that Lemmas 2.1 and 2.2 remain true for complex valued f.

Theorem 3.1. *Let $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary ergodic Markov chain. Assume that $L_2^{\mathbb{C}}(S, \mathcal{A}, \mu) = \bigoplus_{i\in\mathcal{I}} H_i$ is a splitting into orthogonal closed subspaces H_i that are invariant under the transition operator Q. If $f \in L_2^0$, $f = \sum_{i\in\mathcal{I}} f_i$ with $f_i \in H_i$ satisfies* (12) *and*

$$\sum_{i\in\mathcal{I}} \left(\sum_{n=0}^{\infty} \left(|Q_i^n f_i|_2^2 - |Q_i^{n+1} f_i|_2^2 \right)^{1/2} \right)^2 < \infty, \tag{23}$$

then it also satisfies (14) *and* (15) *and consequently, f is asymptotically normal with limit variance $\sigma^2(f)$ given by* (16).

Remark 3.1. If there exists an invariant splitting, Lemma 3.2 means that (23) is a weaker condition than (13), where this splitting is not taken into account. Furthermore, by (21), the finer the splitting in (23), the weaker the condition. If the splitting is finite (i.e. $\mathcal{I}$ is finite), (23) and (13) are in fact equivalent. Similar comments apply to (24) as compared with (10).

Proof of Theorem 3.1. If (23) holds, then using Lemma 2.1

$$\begin{aligned}
&\sup_{n\geq 1}\left(\left|\sum_{k=m}^{n+m-1} Q^k f\right|_2^2 - \left|Q\sum_{k=m}^{n+m-1} Q^k f\right|_2^2\right)\\
&= \sup_{n\geq 1}\sum_{i\in\mathcal{I}}\left(\left|\sum_{k=m}^{n+m-1} Q_i^k f_i\right|_2^2 - \left|Q_i\sum_{k=m}^{n+m-1} Q_i^k f_i\right|_2^2\right)\\
&\leq \sum_{i\in\mathcal{I}}\sup_{n\geq 1}\left(\left|\sum_{k=m}^{n+m-1} Q_i^k f_i\right|_2^2 - \left|Q_i\sum_{k=m}^{n+m-1} Q_i^k f_i\right|_2^2\right)\\
&\leq \sum_{i\in\mathcal{I}}\left(\sum_{k=m}^{\infty}\left(|Q_i^k f_i|_2^2 - |Q_i^{k+1} f_i|_2^2\right)^{1/2}\right)^2,
\end{aligned}$$

and this tends to 0 as $m\to\infty$. If in addition (12) holds, then from (19) it follows that

$$\begin{aligned}
n^{-1}\left|\sum_{k=0}^{n-1} Q^k f\right|_2^2 &= n^{-1}\sum_{i\in\mathcal{I}}\left|\sum_{k=0}^{n-1} Q_i^k f_i\right|_2^2\\
&\leq n^{-1}\sum_{i\in\mathcal{I}}\sum_{k=0}^{n-1}\left(\sum_{r=k}^{\infty}\left(|Q_i^r f_i|_2^2 - |Q_i^{r+1} f_i|_2^2\right)^{1/2}\right)^2\\
&= n^{-1}\sum_{k=0}^{n-1}\sum_{i\in\mathcal{I}}\left(\sum_{r=k}^{\infty}\left(|Q_i^r f_i|_2^2 - |Q_i^{r+1} f_i|_2^2\right)^{1/2}\right)^2
\end{aligned}$$

which once more tends to 0. This proves the theorem. □

Now we will put (10) into the context of invariant splittings. Consider the subspaces of $L_2^{\mathbb{C}}(\Omega,\mathcal{A},P_\mu)$ defined by

$$H_i' = \{f(X_1) - Qf(X_0),\ f\in H_i,\ i\in\mathcal{I}\}.$$

From

$$E\Big((f(X_1)-(Qf)(X_0))\overline{(g(X_1)-(Qg)(X_0))}\Big) = \langle f,g\rangle - \langle Qf,Qg\rangle,\quad f,g\in L_2^{\mathbb{C}}(S,\mathcal{A},\mu),$$

it follows that for different $i\in\mathcal{I}$ these spaces are orthogonal in $L_2^{\mathbb{C}}(\Omega,\mathcal{A},P_\mu)$, and hence so are their closures. Denote

$$V_n(f) = \sum_{k=0}^{n-1} Q^k f,\qquad f\in L_2^{\mathbb{C}}(S,\mathcal{A},\mu).$$

For the proof of the next theorem we rely heavily on results from Maxwell & Woodroofe [15].

Theorem 3.2. *Let $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary ergodic Markov chain. Assume that $L_2^{\mathbb{C}}(S,\mathcal{A},\mu) = \bigoplus_{i\in\mathcal{I}} H_i$ is a splitting into orthogonal closed subspaces H_i that are* *invariant under the transition operator Q. Assume that $f \in L_2^0$, $f = \sum_{i \in \mathcal{I}} f_i$ with $f_i \in H_i$ satisfies*

$$\sum_{i \in \mathcal{I}} \left(\sum_{n \geq 1} |V_n(f_i)|_2 / n^{3/2} \right)^2 < \infty. \tag{24}$$

Then f is asymptotically normal.

Proof. For the proof we depend on several facts from Maxwell & Woodroofe [15]. Our goal is to obtain a representation $S_n(f) = M_n + A_n, \ n \geq 1$, where (M_n) is a martingale with stationary increments w.r.t. $\mathcal{F}_n = \sigma(\xi_k, \ k \leq n)$, and $EA_n^2/n \to 0$. Asymptotic normality of f is then immediate. Given $\varepsilon > 0$ set $g_\varepsilon = ((1+\varepsilon)I - Q)^{-1} f$, so that $(1+\varepsilon) g_\varepsilon - Q g_\varepsilon = f$. Then

$$S_n(f) = M_{n,\varepsilon} + \varepsilon S_n(g_\varepsilon) + A_{n,\varepsilon}, \tag{25}$$

where

$$\begin{aligned} M_{n,\varepsilon} &= \sum_{k=1}^{n} \big(g_\varepsilon(X_k) - (Q g_\varepsilon)(X_{k-1}) \big), \\ A_{n,\varepsilon} &= (Q g_\varepsilon)(X_0) - (Q g_\varepsilon)(X_n). \end{aligned}$$

We want to show that

$$\varepsilon \cdot S_n(g_\varepsilon) \to 0, \quad M_{n,\varepsilon} \to M_n \quad \text{and} \quad A_{n,\varepsilon} \to A_n, \ \ \varepsilon \to 0 \quad \text{in } L_2(\Omega, \mathcal{A}, P_\mu), \tag{26}$$

where $(M_n)_{n \geq 1}$ and $(A_n)_{n \geq 1}$ have the properties specified above. For this it suffices to show that

(1) $\varepsilon \|g_\varepsilon\|^2 \to 0, \varepsilon \to 0$,
(2) $(M_{1,\delta_n})_{n \geq 1}$, with $\delta_n = 1/2^n$, converges as $n \to \infty$ in $L_2(\Omega, \mathcal{A}, P_\mu)$,

see [15]. To show $\varepsilon |g_\varepsilon|_2^2 \to 0$, observe that

$$g_\varepsilon = \sum_{i \in \mathcal{I}} g_\varepsilon^i, \quad g_\varepsilon^i = ((1+\varepsilon)I - Q)^{-1} f_i \in H_i.$$

In [15] it is shown that with $\delta_k = 1/2^k$,

$$\sum_{k \geq 1} \sqrt{\delta_k} \sup_{\delta_k \leq \varepsilon < \delta_{k-1}} |g_\varepsilon^i|_2 \leq C \sum_{n \geq 1} |V_n f_i|_2 / n^{3/2}, \tag{27}$$

where $C > 0$ from now on denotes a generic constant. In particular, $\varepsilon |g_\varepsilon^i|_2^2 \to 0$, $\varepsilon \to 0$ and

$$\sqrt{\varepsilon} |g_\varepsilon^i|_2 \leq C \sum_{n \geq 1} |V_n f_i|_2 / n^{3/2} \quad \forall \ \varepsilon > 0. \tag{28}$$

Since

$$\varepsilon |g_\varepsilon|_2^2 = \sum_{i \in \mathcal{I}} \varepsilon |g_\varepsilon^i|_2^2, \tag{29}$$

it follows from (28) and (24) that the right of (29) is dominated by a convergent series. Therefore

$$\lim_{\varepsilon\to 0} \varepsilon\,|g_\varepsilon|_2^2 = \sum_{i\in\mathcal{I}} \lim_{\varepsilon\to 0} \varepsilon\,|g_\varepsilon^i|_2^2 \;= 0.$$

Next we want to show that $M_{1,\varepsilon} = g_\varepsilon(X_1) - Qg_\varepsilon(X_0)$ converges to a limit in $L_2(\Omega, \mathcal{A}, P_\mu)$ along the sequence δ_n. To this end notice that

$$M_{1,\varepsilon} = \sum_{i\in\mathcal{I}} M_{1,\varepsilon}^i, \quad M_{1,\varepsilon}^i = g_\varepsilon^i(X_1) - Qg_\varepsilon^i(X_0) \in H_i',$$

is an orthogonal decomposition in $L_2^{\mathbb{C}}(\Omega, \mathcal{A}, P_\mu)$. We have

$$M_{1,\delta_n} = \sum_{k=1}^{n} (M_{1,\delta_k} - M_{1,\delta_{k-1}}) + M_{1,\delta_0}.$$

In [15] it is shown that

$$\sum_{n=1}^{\infty} \|M_{1,\delta_n}^i - M_{1,\delta_{n-1}}^i\|_{L_2^{\mathbb{C}}(\Omega,\mathcal{A},P_\mu)} \le C \sum_{n\ge 1} |V_n(f_i)|_2 / n^{3/2}. \tag{30}$$

From (30) and (24) it follows that

$$\sum_{i\in\mathcal{I}} \left(\sum_{n\ge 1} \|M_{1,\delta_n}^i - M_{1,\delta_{n-1}}^i\|_{L_2^{\mathbb{C}}(\Omega,\mathcal{A},P_\mu)} \right)^2 < \infty,$$

and therefore we can apply Lemma 3.1 to obtain convergence of the series $\sum_{n\ge 1}(M_{1,\delta_n} - M_{1,\delta_{n-1}})$ and hence that of M_{1,δ_n}. This concludes the proof of the theorem. □

The following corollary is easily deduced.

Corollary 3.2. *A function $f \in L_2^0$ satisfies* (24) *if*

$$\sum_{i\in\mathcal{I}} \left(\sum_{n\ge 0} \frac{|Q_i^n f_i|_2}{\sqrt{n+1}} \right)^2 < \infty. \tag{31}$$

Remark 3.2. It can be shown that if (24) holds, the limits $\sigma^2(f_i) = \lim_{n\to\infty} E|M_{1,\delta_n}^i|^2$, $i \in \mathcal{I}$, exist and

$$\sigma^2(f) = \sum_{i\in\mathcal{I}} \sigma^2(f_i).$$

4. Examples

In this section we consider two examples in which the Markov operator admits an invariant splitting in the sense of Sec. 3. In case of a normal operator with discrete spectrum and the canonical decomposition of $L_2^{\mathbb{C}}(S, \mathcal{A}, \mu)$ into eigenspaces of distinct eigenvalues, (23), (24) and (31) are stronger requirements than (7). However, there are other interesting examples of such invariant splittings. Firstly we consider exact endomorphisms of compact Abelian groups. Here the transfer operator, evidently a coisometry, plays the role of the Markov operator Q. Furthermore we study random walks on compact homogeneous spaces.

Recall that a measure-preserving transformation T of a probability space $(S, \mathcal{A}, \mu)$ is called *exact* (cf. Rohlin [16]) if the σ-field $\bigcap_{n=0}^{\infty} T^{-n}\mathcal{A}$ is trivial.

Example 4.1. (Exact group endomorphisms) The central limit theorem for the following class of transformations was studied by Leonov [14] using moment-based arguments. Let T: $G \to G$ be an endomorphism of a compact separable Abelian group G. Denote by μ_G the normalized Haar measure. Let Γ be the dual group consisting of characters, which form an orthonormal basis of $L_2^{\mathbb{C}}(G, \mu_G)$ (cf. Rudin [17]). Denote $T^*\chi = \chi \circ T$, then T^* is a homomorphism of Γ. We have the following relations between T and T^*.

- T surjective $\Leftrightarrow$ T^* injective.
- T exact $\Leftrightarrow \bigcap_{n\geq 1} T^{*^n}\Gamma = \{0\}$.

Indeed, if T is onto, then T^* is evidently injective. On the other hand, T is onto if and only if

$$f \circ T = g \circ T \Rightarrow f = g \quad \forall\ f, g \in L_2^{\mathbb{C}}(G, \mu_G).$$

Since T^* is injective, the equality

$$f \circ T = \sum_{\chi\in\Gamma} \langle f, \chi\rangle T^*\chi = \sum_{\chi\in\Gamma} \langle g, \chi\rangle T^*\chi = g \circ T$$

implies $\langle f, \chi\rangle = \langle g, \chi\rangle$ and hence $f = g$. Furthermore exactness of T is equivalent to

$$E(f|\, T^{-n}\mathcal{B}) \to Ef \quad \text{ in } L_2^{\mathbb{C}}(G, \mu_G),$$

where f is considered as a random variable on $(G, \mathcal{A}, \mu_G)$, and $\mathcal{A}$ denotes the Borel sigma-algebra of G. But if $f = \sum_{\chi\in\Gamma}\langle f, \chi\rangle\chi$ we have that

$$E(f|\, T^{-n}\mathcal{B}) = \sum_{\chi\in T^{*n}\Gamma} \langle f, \chi\rangle\chi,$$

and the second equivalence follows.

From now on we assume that T is onto and exact. The group Γ can be partitioned into *grand orbits* defined by

$$\mathcal{O}(\chi) = \{\gamma \in \Gamma\colon \exists\ n, m \geq 0\colon T^{*^n}\chi = T^{*^m}\gamma\}.$$

We assume that G is infinite, so that Γ is countably infinite. We claim that there exists a countably infinite set $\tilde{\Gamma}$ such that the different grand orbits (except for the trivial orbit $\{1\}$) are given by

$$\{\tilde{\gamma}, T^*\tilde{\gamma}, {T^*}^2\tilde{\gamma}, \ldots\}, \quad \tilde{\gamma} \in \tilde{\Gamma}. \tag{32}$$

In fact, set $\tilde{\Gamma} = \Gamma \setminus T^*\Gamma$ (here $\setminus$ is the set-theoretic difference). Since $T^*\Gamma$ is an infinite subgroup $\neq \Gamma$, $\tilde{\Gamma}$ must also be infinite. Since T^* is injective, the sets in (32) are indeed grand orbits. If χ has no first predecessor, then for every $n \geq 0$ there is a $\gamma \in \Gamma$ such that ${T^*}^n\gamma = \chi$, which would imply $\chi \in \bigcap_{n \geq 0} {T^*}^n\Gamma$, a contradiction to exactness. We obtain a splitting

$$L_2^0 = \bigoplus_{\tilde{\gamma} \in \tilde{\Gamma}} L_2(\tilde{\gamma}), \tag{33}$$

where $L_2(\tilde{\gamma})$ denotes the closure of the subspace generated by $\mathcal{O}(\tilde{\gamma})$. Let

$$U_T \colon L_2^{\mathbb{C}}(G, \mu_G) \to L_2^{\mathbb{C}}(G, \mu_G), \quad U_T f = f \circ T.$$

Evidently, U_T preserves the splitting (33), and hence the same holds for its dual, the transfer operator, denoted by V_T. On the components of the splitting, V_T acts as left shift. More precisely, if

$$f = \sum_{n \geq 0} \langle f, {T^*}^n\tilde{\gamma} \rangle {T^*}^n\tilde{\gamma}$$

for some $\tilde{\gamma} \in \tilde{\Gamma}$, then

$$V_T f = \sum_{n \geq 0} \langle f, {T^*}^{n+1}\tilde{\gamma} \rangle {T^*}^n\tilde{\gamma}.$$

Let $(\xi_n)_{n \in \mathbb{Z}}$ be a Markov chain with stationary distribution μ_G and transition operator V_T. Then $(U_T^n f)_{n \geq 0}$ is a time reversal of $(f(\xi_n))_{n \geq 0}$, i.e.

$$\big(f(\xi_0), \ldots, f(\xi_n)\big) \sim \big(U_T^n f, \ldots, f\big), \quad n \geq 0,$$

where $\sim$ means that the random vectors are equal in distribution. Therefore an application of Theorem 3.1 with $Q = V_T$ gives

Theorem 4.1. (Leonov, [14]) *Suppose that* $f \in L_2^0$ *satisfies*

$$\sum_{\mathcal{O}} \left(\sum_{\chi \in \mathcal{O}} |\langle f, \chi \rangle| \right)^2 < \infty \tag{34}$$

where the first sum is taken over all grand orbits. Then the sequence $n^{-1/2} \sum_{k=0}^{n-1} f \circ T^n$ *is asymptotically normal with variance*

$$\sigma^2(f) = \sum_{\mathcal{O}} \left| \sum_{\chi \in \mathcal{O}} \langle f, \chi \rangle \right|^2.$$

The formula for the limit variance is deduced using (16). Leonov [14] in fact considered general ergodic group endomorphisms. Theorem 4.1 can be derived from his results when restricting to the case of an exact endomorphism. As a particular example, consider $G = \mathbb{T}^1$, the 1-torus, and let $Tx = 2x \bmod 1$. Then the dual group is isomorphic to $\mathbb{Z}$, and grand orbits can be indexed by odd integers.

Example 4.2. (Random walks on compact homogeneous spaces) Let G be a locally compact second countable group and let K be a closed subgroup such that the homogeneous space $X = G/K$ of left cosets with the quotient topology is compact. We assume that K is a unimodular group and that the left-invariant Haar measure on G is also invariant with respect to the right action of K (in particular, this is the case if both G and K are unimodular). Under these assumptions, there exists a unique probability measure μ on X invariant with respect to the natural action of G on X given by $(g, x) \mapsto gx$, $x = hK$. Denote by $\pi = \pi_X$ the unitary representation of G in $L_2^{\mathbb{C}}(X, \mu)$ defined by

$$(\pi(g)f)(x) = f(g^{-1}x), \quad x \in X, \;\; f \in L_2^{\mathbb{C}}(X, \mu).$$

Assume that for every continuous function ϕ on G with compact support, the operator $\int_G \phi(g)\pi(g)dg$ is a compact operator on $L_2^{\mathbb{C}}(X, \mu)$ (the integration over g is performed respective to the left-invariant Haar measure on G). This holds true e.g. for discrete K (cf. [10], p. 21) and for compact G. Then the representation π can be decomposed into a sum of countably many irreducible unitary representations of G where any such representation occurs with finite multiplicity. The proof for the case of discrete K can be found in ([10], Sec. 2.3), however, discreteness is only used to obtain compactness of the operators $\int_G \phi(g)\pi(g)dg$ and therefore the proof also works under our assumptions.

In principle our results apply to this splitting of π into irreducible representations. However, since this splitting is not canonic, our conditions from Sec. 3 for a specific function $f \in L_2^0$ to satisfy the CLT depend on the particular choice of the splitting. A more satisfactory way is to pass to a coarser splitting of π_X given by

$$\pi_X = \bigoplus_{\alpha \in \mathrm{Irr}(\pi_X)} \pi_\alpha. \tag{35}$$

Here $\mathrm{Irr}(\pi_X)$ denotes the set of equivalence classes of irreducible unitary representations of G which realize as subrepresentations of π_X, and $\pi_\alpha = \pi_{X,\alpha}$ is the *primary* (or *isotipic*) subrepresentation of π of type α, that is, π_α is a maximal subrepresentation of π every irreducible subrepresentation of which is equivalent to α. Every π_α can be decomposed into some number m_α (called *multiplicity*) of copies of α. Unlike the decomposition of π into irreducible representations, its decomposition (35) into primary components is canonic. Let H_{π_α} denote the subspace of $L_2^{\mathbb{C}}(X, \mu)$ the representation π_α acts on, and denote by f_α the orthogonal projection of f onto H_{π_α}.

If Q is a Borel probability measure on G, it induces a Markov operator on $L_2^{\mathbb{C}}(X,\mu)$ given by

$$Qf(x) = \int_G f(g^{-1}x)dQ(g), \quad x \in X, \;\; f \in L_2^{\mathbb{C}}(X,\mu).$$

This Markov operator preserves μ, and since the spaces H_{π_α} are invariant under the representation π_X, they are also invariant under Q. Therefore an application of Corollary 3.2 gives

Theorem 4.2. *Let $(\xi_n)_{n\in\mathbb{Z}}$ be a random walk on the compact homogeneous space $X = G/K$ with transition operator Q and stationary distribution μ, as specified above. If Q is ergodic and if $f \in L_2^0$ satisfies*

$$\sum_{\alpha\in\mathrm{Irr}(\pi_X)} \left(\sum_{n\geq 0} \frac{|Q^n f_\alpha|_2}{\sqrt{n+1}} \right)^2 < \infty, \tag{36}$$

then the sequence $\sqrt{n}^{-1}\sum_{k=0}^{n-1} f(\xi_k)$ is asymptotically normal with variance

$$\sigma^2(f) = \lim_{n\to\infty} \frac{1}{n} E\left(\sum_{k=0}^{n-1} f(\xi_k)\right)^2 .$$

Let us consider more explicitly the case of a compact group G. We denote by μ_G the normalized Haar measure on G and by $\hat{G}$ the set of equivalence classes of unitary irreducible representations of G. Since G is compact, each $\alpha \in \hat{G}$ is finite dimensional (cf. [8], p. 74). We let $\alpha \in \hat{G}$ also stand for some representative of its equivalence class, acting on the vector space V_α of dimension n_α. The character χ_α of α is given by $\chi_\alpha(g) = \mathrm{tr}(\alpha(g))$, where tr denotes the trace of the corresponding automorphism of V_α.

Let σ be a subrepresentation of the left regular representation π_G of G (where $\pi_G(g)f(\cdot) = f(g^{-1}\cdot)$, $f \in L_2^{\mathbb{C}}(G,\mu_G)$, $g \in G$). It decomposes canonically into a sum $\sigma = \bigoplus_{\alpha\in\mathrm{Irr}(\sigma)} \sigma_\alpha$ of isotipic subrepresentations, and the orthogonal projection of $f \in H$ to H_{σ_α} is given by $f_\alpha = n_\alpha\, f * \chi_\alpha$.

Now consider a closed subgroup K of G (possibly the trivial subgroup $\{1\}$), and as above let $X = G/K$ be the corresponding compact homogeneous space. The spaces $L_2^{\mathbb{C}}(X,\mu)$ and

$$L_2^K(G,\mu_G) = \big\{ f \in L_2^{\mathbb{C}}(G,\mu_G) \colon f(gk) = f(g) \text{ for all } k \in K \big\},$$

the subspace of $L_2^{\mathbb{C}}(G,\mu_G)$ consisting of functions which are right-invariant under K, are isometric (cf. [8], p. 101). Therefore $L_2^{\mathbb{C}}(X,\mu)$ may be considered as a subspace of $L_2^{\mathbb{C}}(G,\mu_G)$, and π_X as a subrepresentation of π_G. Thus we have the above formula for the orthogonal projections f_α of $f \in L_2^{\mathbb{C}}(X,\mu)$ onto $H_{\pi_{X,\alpha}}$. A representation $\alpha \in \hat{G}$ occurs in $\mathrm{Irr}(\pi_X)$ if and only if $\dim V_\alpha^K > 0$, where

$$V_\alpha^K = \{ v \in V_\alpha \colon \alpha(k)v = v \text{ for all } k \in K \}.$$

In fact, the multiplicity of α in $\pi_{X,\alpha}$ is given by $m_\alpha = \dim V_\alpha^K$. On $L_2^K(G, \mu_G)$, the operator Q is simply the convolution $Qf = Q * f$.

Finally let us mention that for Q having finite support, the CLT for a random walk on a compact homogeneous space was established by Dolgopyat ([7], p. 193).

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