MAGKS

Aachen Siegen Marburg Gießen Kassel Göttingen

Joint Discussion Paper
Series in Economics

by the Universities of

Aachen · Gießen · Göttingen
Kassel · Marburg · Siegen

ISSN 1867-3678

No. 49-2009

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Innovation, Income Distribution, and Product Variety

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Innovation, Income Distribution, and Product Variety

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December 2009

Abstract

On the basis of a modification of K. Lancaster's characteristics approach and a special class of non-homothetic utility functions individual demand functions are derived. Individual demand is determined in a complex way by the income as well as the product qualities and the unit costs of the offered products. It becomes clear that product innovations (changes in product quality), process innovations (changes of unit costs) and changes in personal income distribution (e. g. due to income taxation and redistribution) all influence product variety in a very different way.

JEL-Classifications

L0, D1, O0

Key words

innovation, income distribution, product variety, Lancaster's characteristics approach

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1 Introduction

Product variety is an interesting phenomenon which consumers have to deal with. For consumer decisions there is usually not only one product to choose from but several or even very many, differing in prices and qualities. Which products are chosen, in the last instance depends on the income of the consumer, the product prices and product qualities and of course on the consumer's preferences.

If one considers real purchase decisions some essential statements can be made:

- Consumers do usually not demand all offered products, even when there is no lack of information.
- Consumers with a low income tend to demand low-priced products and thus usually products with a lower quality, while households with a high income will demand high-quality products, which tend to be more expensive.
- If a consumer's income increases, the demand for the previously demanded products usually does not increase at the same percentage as the income increase. Rather, with an increased income the consumer will demand products with higher quality (and thus usually at a higher price).

These observations allow the conclusion that the income distribution in an economy is highly important for the product variety that can be observed. It can be assumed that in an egalitarian society the product variety is smaller than in a society with differences in income.

It is the aim of this contribution to investigate the connection between income distribution and product variety. Furthermore, it will be shown how product and process innovation influence product variety. Apparently, an innovative firm can, with a product innovation, which may show in a completely new product or a quality improvement, take market shares from another firm or even drive it out of the market entirely so that then the product variety is reduced. A process innovation may trigger the same effects if the process innovation is reflected in the supply price via a reduction of the unit costs.

In order to be able to treat these questions, a model will be developed which in particular will answer the question how in a heterogeneous market with $\tilde{m}$ suppliers k ($k \in \tilde{\mathcal{M}} := \{1, \ldots, \tilde{m}\}$) the prices of the offered products p_k, the qualities a_k and the income b_h of a household h (respectively, the income distribution of the H households h ($h \in \mathcal{H} := \{1, \ldots, H\}$) determine the individual (respectively, aggregate) demand and thus also the product variety. The developed model will initially provide individual demand functions x_{hk}, with

$$x_{hk} = x_{hk}(a, p, b_h),$$

of a household h of product k, where $a = (a_1, \ldots, a_{\tilde{m}})^T$ represents the vector of the product qualities and $p = (p_1, \ldots, p_{\tilde{m}})^T$ the vector of the product prices.

Therefore this contribution is connected to questions of consumption theory. But it deviates from traditional consumption theory in essential issues. First of all, this contribution dispenses with homothetic preferences. Although homothetic preferences are widely assumed in economic theory as a rule empirical research shows no evidence for homothetic preferences (cf. e.g. *Deaton/Muellbauer* (1980) and *Zweimüller* (2000a, b)).

If homothetic utility functions are assumed linear income expansion paths and linear Engel curves will result (cf. *Mas-Colell/Whinston/Green* (1995)). A major consequence is further that changes in income distribution have no influence on the structure of the demand (given by demand shares), even when the entire budget of the households changes. It is precisely this result that does (usually) not hold if non-homothetic utility functions are assumed. In this case non-linear Engel curves and non-linear income expansion paths result. Another consequence is that (as a rule) income taxation and redistribution lead to changes in the demand structure.

Another point concerns the integration of product quality and product variety.[1] If one is to assume neither that a household always demands only one unit of one single product nor that all products are demanded per se, the majority of consumption theory approaches does not lead any further. However, one route that can an will be taken here is taking up the works of K. Lancaster (cf. in particular *Lancaster* (1966, 1971, 1991)).

The basic idea of this contribution consists in the combination of non-homothetic utility function and a modification of Lancaster's characteristics approach.[2] This approach facilitates the representation of the facts described above with the help of a simple model. In addition, this approach shows the importance of the complex »neighborhood relations« of the offered products in a market for the demand structure.

The contribution has the following structure. After some brief theoretical remarks in section 2 the modification of the Lancaster approach will be described and some essential aspects of non-homothetic utility functions will be treated in section 3. In section 4 individual demand functions will be derived. Section 5 is concerned with the derivation of the aggregate demand functions and the product variety. A short resume in section 6 concludes the contribution.

[1] For modeling of product qualities see *Payson* (1994), *Swan/Taghavi* (1992) and *Wadman* (2000).

[2] The basic idea was given with simple numeric examples in *Voßkamp* (1996).

2 The Basics

2.1 Product variety

Although real economies in many ways are characterized by heterogeneous subjects and objects, economic theory in general pays only very little attention to heterogeneity, even though many authors point out the importance of heterogeneity.[3] This also applies to the heterogeneity of products and thus product variety.

The aspect of product variety may be taken up in very different ways. In contrast to the empirical surveys, which – similar to approaches in botany and zoology – attempt to record products systematically by classifying the characteristics of the products according to technological conditions and types of usage,[4] this survey chooses a more economic approach which is motivated exclusively by a consumer perspective. The products offered in a market are substitutional from the consumers' point of view. However, the products differ in their technological characteristics.[5] In order to keep the present analysis of manageable size, it is assumed that every product can be characterized by the value of a one-dimensional variable which is to represent the quality a_k of the product k. Thus product variety in this survey is essentially based on the different product varieties which are characterized by different product qualities. This – highly simplified – presentation of a one-dimensional product variety facilitates very helpful references to the theory of heterogeneous oligopolies.[6]

The aspect of product variety is connected to essential economic problems. *Lancaster* (1990: 190) mentions the following questions which are of importance with regard to the consideration of product variety:[7]

> 1. »*The individual consumer.* How many of the available variants within a single product group will the individual choose? What determines the choice?
> 2. *The individual firm.* What degree of product variety is most profitable for the firm to offer in a given competitive situation?
> 3. *Market equilibrium.* What degree of product variety will result from the operation of the market within a particular competitive structure?

[3] Cf. *Kirman* (1992) and *Stoker* (1993).

[4] Cf. *Saviotti* (1996) and the contributions quoted *Bils/Klenow* (2001).

[5] Cf. *Lancaster* (1966). The approach chosen here can also be interpreted a a radical simplification of the approach by *Saviotti* (1996).

[6] It is problematic to reduce the quality of a product to one dimension. It is known, for example, that consumers actually do perceive numerous characteristics when purchasing consumer goods. However, since it is very difficult to measure characteristics, in particular with regard to quality variables, here, as is usual in the economic literature, a one-dimensional quality factor is assumed (cf. *Payson* (1994) und *Wadman* (2000)).

[7] Italics are taken over from the original.

4. *The social optimum.* What degree of product variety is optimal for society on some criterion? How is this related to the market equilibrium?«

This contribution is concerned with some of these questions in the subsequent sections. First, the determinants of product variety will be determined. Then it will be clarified what impact innovations and the income distribution have on product variety.

2.2 Income distribution and product variety

The relationship between income distribution and product variety is one-sided: While product variety has no direct impact on the income distribution, the other way round there is an elementary impact since the income distribution represents one, if not even the central, determinant of product variety. In an economy where the incomes of the households are identical only few products of similar quality will be able to persist. Therefore it can be assumed that product variety in this case will be relatively small. In an economy where the incomes are distributed very unevenly, a much higher product variety will develop because on the one hand there will be households which, due to a high income, will demand high-quality products which as a rule will also be expensive, while on the other hand there are also households which, due to a low budget, will demand only low-priced products which as a rule will be of lesser quality.

The traditional demand theory has provided numerous contributions about consumer behavior (cf. e. g.*Deaton/Muellbauer* (1980)), which, however, can treat the mentioned aspects only in parts. One reason for these inadequacies is the dominant usage of homothetic utility functions, which leads to results that are not compatible with the phenomena described above.

If utility-maximizing consumers and homothetic utility functions are assumed, linear income expansion paths and Engel curves are accepted (cf. *Mas-Colell/Whinston/Green* (1995)). The following also applies: If homothetic utility functions are assumed, changes in income distribution apparently have no influence on the demand structure. Even more: With an income-neutral redistribution of the incomes of the households the aggregate demand volumes for the various goods (or products) will not change. But this does not agree with empirical observations. In particular, if there is a change in income, households will determine their expenditure shares depending on whether they are demanding essential goods or luxury goods.

Traditional consumption theory has many answers to individual aspects of this issue, but it is not possible to extract a consumption theory from the known literature which is consistent for the questions posed in this survey. Therefore an alternative demand model will be developed in the following, based on the Lancaster approach (*Lancaster* (1966)) and non-homothetic utility functions, which can be used to answer the highlighted questions.

2.3 Innovation and product variety

The connection between product innovation and product variety is canonical. In the course of the Schumpeterian competition product innovations result from the R&D activities of the firms which generate new varieties and thus contribute to an (c.p.) increase of the product variety. This result occurs in the case of new products as well as in the case of product innovations which result in quality improvements.

However, if a firm can improve the quality of its product it will win market shares or even be able to push out some competitors from the market, assuming that oligopolistic competition works. In the latter case, (c. p.) product variety apparently is reduced.

Process innovations have quite the same impact. If process innovations result in cost reductions which are reflected in the prices, an innovating firm will also be able to strengthen its position in the market. As in the case of product innovations there is an effect which is reducing product variety and an effect which is increasing product variety.

3 The modification of the Lancaster approach

3.1 Lancaster's characteristics approach

Following *Lancaster* (1966) a market with $\tilde{m}$ single-product-firms is considered, which offer a heterogeneous good which displays s_0 characteristics. It is assumed that one unit of a good k ($k \in \tilde{\mathcal{M}} := \{1, \ldots, \tilde{m}\}$) contains $\lambda_{sk} \geq 0$ units of the characteristic s ($s \in \mathcal{S} := \{1, \ldots, s_0\}$). Furthermore it is assumed that z_{hsk} represents the total quantity of units of the characteristic s, which are contained by the quantity x_{hk}, which a household h ($h \in \mathcal{H} := \{1, \ldots, H\}$) consumes. Therefore the following applies:

$$\lambda_{sk} = \frac{z_{hsk}}{x_{hk}}$$

Thus it is assumed that the households' assessments of the products' features do not differ.

If household h consumes the bundle of goods represented by $x_h := (x_{h1}, \ldots, x_{h\tilde{m}})^T$, this is connected to the consumption of the characteristic s in the amount of z_{hs}. Apparently the following applies:

$$z_{hs} = \sum_{k \in \tilde{\mathcal{M}}} z_{hsk} = \sum_{k \in \tilde{\mathcal{M}}} \lambda_{sk} x_{hk} \tag{1}$$

It is furthermore assumed that the income of household h is given by b_h and the prices of products are given by the vector $p := (p_1, \ldots, p_{\tilde{m}})^T$. Following the Lancaster approach, a household h will not maximize the utility in immediate dependence of the bundle of goods but rather in dependence of the vector of the characteristics quantities $z_h := (z_{h1}, \ldots, z_{hs_0})^T$. Further, $U_h(z_h)$ shall be a strictly concave utility function. Then the utility maximization problem of the household will be given by

$$\begin{aligned}
&\max_{z \in \mathrm{I\!R}_+^{s_0}} U_h(z_h) \\
&\text{s. t.} \quad z_h = \Lambda x_h \\
&\qquad\quad b_h \geq p^T x_h \\
&\qquad x_h, z_h \geq 0
\end{aligned} \tag{2}$$

where

$$\Lambda := \begin{pmatrix} \lambda_{11} & \ldots & \lambda_{1\tilde{m}} \\ \vdots & & \vdots \\ \lambda_{s_0 1} & \ldots & \lambda_{s_0 \tilde{m}} \end{pmatrix} \in \mathrm{I\!R}_+^{s_0 \times \tilde{m}}$$

represents the matrix of the characteristics coefficients λ_{sk}.

In the case of $s_0 = 2$ – where the characteristics r and s will be considered – the optimization problem can be represented in a two-dimensional (z_{hr}, z_{hs})-diagram. If the household spends its entire income b_h to purchase the good l, the demand will be given by:

$$x_{hk}^{max} = \begin{cases} 0 & k \neq l \\ b_h/p_l & k = l. \end{cases}$$

The following characteristics quantities will result from this:

$$z_{hrl}^{max} = \lambda_{rl} x_{hl}^{max} \tag{3}$$

$$z_{hsl}^{max} = \lambda_{sl} x_{hl}^{max} \tag{4}$$

Thus a point $\mathcal{A}_{hl}$ with the coordinates

$$\mathcal{A}_{hl} = (z_{hrl}^{max}, z_{hsl}^{max}) = (\lambda_{rl} b_h/p_l, \lambda_{sl} b_h/p_l) \in \mathrm{I\!R}_+^2 \tag{5}$$

can be determined for a given budget b_h for every product $l \in \tilde{\mathcal{M}}$. Figure 1 – which will be explained in more detail in the following – shows an example for $\tilde{m} = 3$ products.

It will further be assumed that the household can spend its entire disposable income (totally or partially) to purchase one or several products. Then the possible characteristics quantities z_{hr} und z_{hs} which the household h can consume can be characterized by a tuple which is in the convex hull which is formed by the points $\mathcal{A}_{h1}$ $(k \in \tilde{\mathcal{M}})$ and the point $\mathcal{O} = (0, 0)$. Figure 1 shows a typical possibility set for the case $\tilde{m} = 3$.

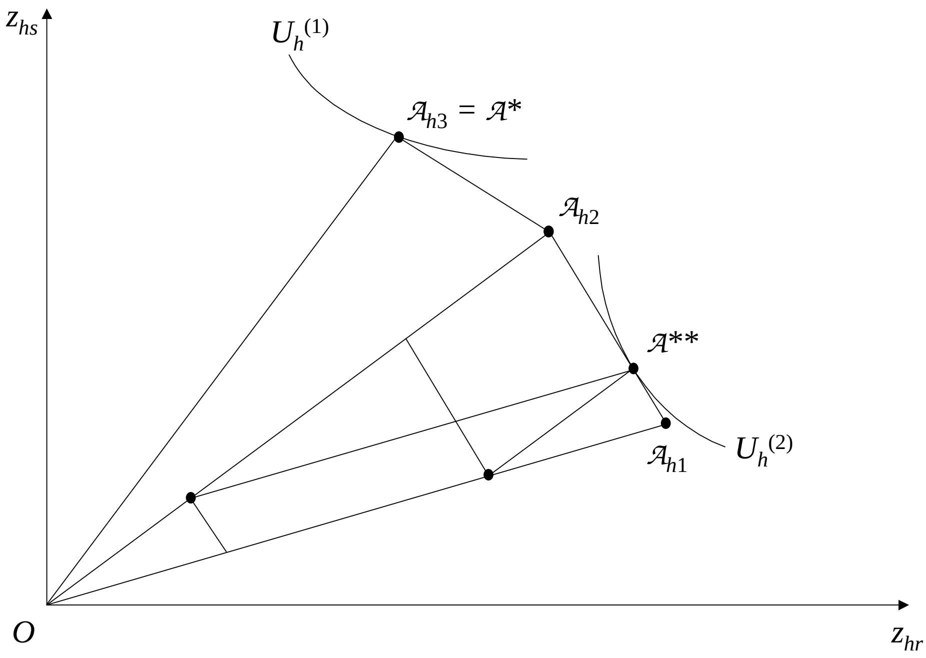

Figure 1: The Lancaster approach

If there is no lack of information and the behaviour is rational a household will choose a tuple which is characterized by a point $\mathcal{A}^* = (z^*_{hr}, z^*_{hs})$ or $\mathcal{A}^{**} = (z^{**}_{hr}, z^{**}_{hs})$ on the edge of the possibility set. If strictly concave utility functions are assumed the optimal point will lie on the north-eastern edge of the possibility set. Thus products which are characterized by points lying inside the set are obviously not in demand.[8]

Two solutions are conceivable in principle. On the one hand the household h will – depending on its utility function – often only demand one product. In this case a vertex solution will result (cf. $\mathcal{A}^*$ and $U_h^{(1)}$ in figure 1). On the other hand edge solutions are also possible (cf. $\mathcal{A}^{**}$ und $U_h^{(2)}$ in figure 1). In that case the household is demanding two »adjacent« products. The division of the budget is then determined by the corresponding linear combination of $\mathcal{A}_{hl}$ and $\mathcal{A}_{hk}$:

$$\mathcal{A}^{**} = (z^{**}_{hr}, z^{**}_{hs}) = \eta_h \mathcal{A}_{hl} + (1 - \eta_h)\mathcal{A}_{hk} \tag{6}$$

where η_h determines the share of the income that is spent on product l

[8] For the modified Lancaster approach these aspects will be discussed in details (cf. subsection 3.3).

3.2 The modification

For the issue of this paper the Lancaster approach will be modified. In particular, it will be assumed that only two characteristics r and s will be considered. Furthermore the following shall apply for the matrix of the characteristics coefficient Λ:

$$\Lambda = \begin{pmatrix} \lambda_{r1} & \dots & \lambda_{r\tilde{m}} \\ \lambda_{s1} & \dots & \lambda_{s\tilde{m}} \end{pmatrix} = \begin{pmatrix} 1 & \dots & 1 \\ a_1 & \dots & a_{\tilde{m}} \end{pmatrix} \tag{7}$$

The specific structure of Λ can be motivated as follows: We will consider a market with $\tilde{m}$ suppliers which offer a heterogeneous good. Thus the products of the supplier are in some respects similar or even identical, because otherwise they would not be offered in one market. This is the justification for considering the characteristics coefficients relating to the first characteristic r to be identical. For reason of standardization we will additionally assume $\lambda_{rk} = 1$ for all $k \in \tilde{\mathcal{M}}$. The second row represents the product qualities of the respective products. λ_{sk} thus represents the product quality of product k.

Due to the equation (7) the equation (1) can be simplified. For the two characteristics r and s the following applies:

$$\begin{aligned} z_{hr} &= \sum_{k\in\tilde{\mathcal{M}}} z_{hrk} = \sum_{k\in\tilde{\mathcal{M}}} \lambda_{rk} x_{hk} = \sum_{k\in\tilde{\mathcal{M}}} x_{hk} \\ z_{hs} &= \sum_{k\in\tilde{\mathcal{M}}} z_{hsk} = \sum_{k\in\tilde{\mathcal{M}}} \lambda_{sk} x_{hk} = \sum_{k\in\tilde{\mathcal{M}}} a_k x_{hk} \end{aligned}$$

Now z_{hr} represents the total quantity consumed by the household h. z_{hs} in contrast shows how many quality units are consumed by the household h. The utility maximization problem is simplified in the same way (cf. the optimization problem (2)) to be:

$$\max_{(z_{hr},z_{hs})^T\in\mathbb{R}^2_+} U_h(z_{hr}, z_{hs})$$

$$\text{s. t.} \quad \begin{aligned} z_{hr} &= \sum_{k\in\tilde{\mathcal{M}}} x_{hk} \\ z_{hs} &= \sum_{k\in\tilde{\mathcal{M}}} a_k x_{hk} \\ b_h &\geq p^T x_h \\ x_h, z_h &\geq 0 \end{aligned}$$

Due to the special form of the matrix Λ every product l can be marked in a rather simple way in a (z_{hr}, z_{hs})-diagram by a point with the coordinates z_{hrl}^{max} and z_{hsl}^{max}. With (5) and (7) the following applies:

$$z_{hrl}^{max} = \lambda_{rl} x_{hl}^{max} = (1/p_l) b_h \tag{8}$$

$$z_{hsl}^{max} = \lambda_{sl} x_{hl}^{max} = (a_l/p_l) b_h \tag{9}$$

Consequently the points

$$\mathcal{A}_{hl} = ((1/p_l)b_h, (a_l/p_l)b_h) \tag{10}$$

represent the individual products $l \in \tilde{\mathcal{M}}$.

Thus every product is characterized uniquely by the point $\mathcal{A}_{hl}$ since both components are obviously linear in b_h. As a consequence every product l is characterized by its price p_l and its quality a_l.

In this contribution it will further be assumed that the firms produce their products at constant unit costs and offer them according to the price equals marginal cost rule so that $p_l = c_l$ applies. These simple assumptions allow a simple representation of the effects of innovations. Process innovations reduce the costs and therefore the price of the respective product. Product innovations can be interpreted as quality improvements. Consequently the points $\mathcal{A}_{hl}$ are shifted by innovations.

3.3 Competitive products

For the further course of the analysis the term *competitiveness of products* is important.

Definition 1 *Assume rational households and perfect information. A product k is competitive if there is a strictly concave utility function so that the product k is in demand.*

The chosen definition results in a product being called competitive even when there is only a single household with a strictly concave utility function which has not to be specified further. This definition seems to be very weak. However, not every product is competitive. The subset $\mathcal{M} \subseteq \tilde{\mathcal{M}}$ of the competitive products can be determined with the proposition 1.

Proposition 1 *Assume $\tilde{m}$ products. Every product $k \in \tilde{\mathcal{M}}$ shall be characterized by its quality a_k and its price p_k. Moreover, $a_k \neq a_l$ und $p_k \neq p_l$ shall apply for all $k, l \in \tilde{\mathcal{M}}$.*[9] *$\tilde{C}$ shall be the convex set which is formed by the points $\mathcal{A}_{hk}$ ($k \in \tilde{\mathcal{M}}$) and the $\mathcal{O} = (0,0)$. Further, k^{max} shall be the index of the product with highest quality-price index and k^{min} the index of the product with the lowest price:*

$$\begin{aligned} k^{max} &= \arg\max_{k \in \tilde{\mathcal{M}}} a_k/p_k \\ k^{min} &= \arg\min_{k \in \tilde{\mathcal{M}}} p_k. \end{aligned}$$

Then, a product k is competitive if and only if the following conditions hold:

[9] This restriction saves the discussion of special cases that are not of interest. In particular, the question is eliminated how the demand is divided between to products which are identical with regard to product quality and product price.

1. *$\mathcal{A}_{hk}$ is a point at the margin of $\tilde{\mathcal{C}}$.*

2. $p_k < p_{k^{max}}$

3. $a_k/p_k > a_{k^{min}}/p_{k^{min}}$

If $\mathcal{C} \subseteq \tilde{\mathcal{C}}$ describes the convex set which is formed by the points $\mathcal{A}_{hk}$ $(k \in \mathcal{M})$ and $\mathcal{O} = (0,0)$ then the competitive products apparently are also marked by marginal points of the set $\mathcal{C}$.

The proof of this proposition is rather simple. First, it can be noted that the considerations are not dependent on the budget b_h of the household because both components of $\mathcal{A}_{hl}$ are linear in b_h.

It is obvious that a product is competitive if the three mentioned conditions are fulfilled. The other way round, these three conditions must hold. The first condition excludes products being considered competitive which represent inner points of the convex set $\tilde{C}$. The inner points characterize non-competitive products since a higher utility level can be achieved with the same budget.

The other two conditions obviously make only points on the north-eastern margin relevant. Other marginal points of the quantity $\tilde{C}$ may characterize non-competitive products. In these cases the consumer can achieve a higher utility with the exclusive demand of the product k^{max} or k^{min}. The background for this is the assumption of strictly concave utility functions.

Figure 2 illustrates the situation. The products 1 to 3 ($\mathcal{A}_{h1}$ to $\mathcal{A}_{h3}$) apparently are competitive. The products 6 to 8 ($\mathcal{A}_{h6}$ to $\mathcal{A}_{h8}$) apparently are not, since they are not marginal points. A point in the convex hull of $\mathcal{O}$, $\mathcal{A}_{hk^{min}}$ (here: $\mathcal{A}_{h1}$) and $\mathcal{J}_1$ (e.g. product 4, marked by $\mathcal{A}_{h4}$) cannot represent a competitive product because in this case the demand of $\mathcal{A}_{hk^{min}}$ would have a higher utility. With analogous reasoning products can be excluded (e.g. product 5, marked by $\mathcal{A}_{h4}$) which can be characterized by a point in the convex hull of $\mathcal{O}$, $\mathcal{A}_{hk^{max}}$ (here: $\mathcal{A}_{h3}$) and $\mathcal{J}_2$.

The question of the influence of the income b_h on the set of the competitive products is answered again by proposition 2:

Proposition 2 *$\tilde{m}$ products shall be given. Then the set of the competitive products is independent of the income b_h of the considered household.*

The result can be immediately understood with the equations (8), (9) and (10): Since both coordinates of the points $\mathcal{A}_{hl}$ are linear in b_h there is only a stretching, which does however not change the structure of the possibility set C. Put differently: The set of competitive products is identical for all households.

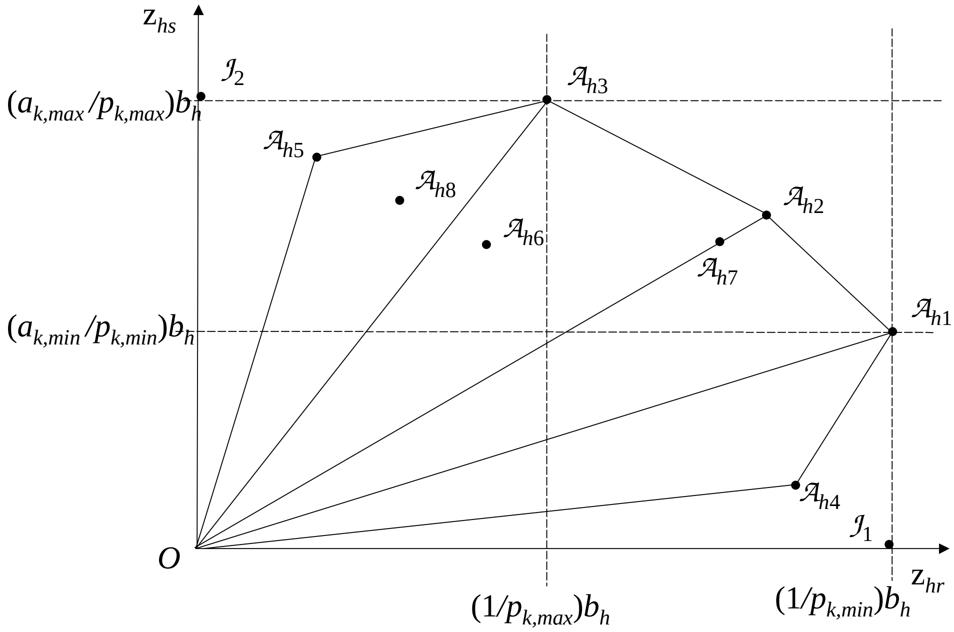

Figure 2: Definition of competitive products

3.4 The canonical order of the competitive products

The special structure of the matrix Λ facilitates a simple characterization of products which have the same quality, the same price and the same quality-price-relation. As shown before, a product l, which is characterized by the product quality a_l and the price p_l, can be represented in the (z_{hr}, z_{hs})-diagram. It is easy to show that the following applies:[10]

1. Products with identical prices lie on a vertical straight line (e.g. the products $\mathcal{A}_{h1}$ and $\mathcal{A}_{h3}$ or $\mathcal{A}_{h4}$ and $\mathcal{A}_{h2}$). The lower the price, the further the products' distance from O. If two products with identical prices are supplied, (c. p.) only the product with the higher product quality is competitive (for the example: $\mathcal{A}_{h3}$ and $\mathcal{A}_{h4}$).

2. Products with identical quality-price ratios lie on a horizontal straight line (e.g. products $\mathcal{A}_{h1}$ and $\mathcal{A}_{h2}$ or $\mathcal{A}_{h3}$, $\mathcal{A}_{h4}$ and $\mathcal{A}_{h5}$). The higher the quality-price relation the further the products' distance from O. If two products with

[10] Cf. also figure 3. For the figure the disposable income was determined by $b_h = 1$. For reasons of linguistic simplification a point $\mathcal{A}_{hl}$ will also be called product l in the following if $\mathcal{A}_{hl}$ represents the product l.

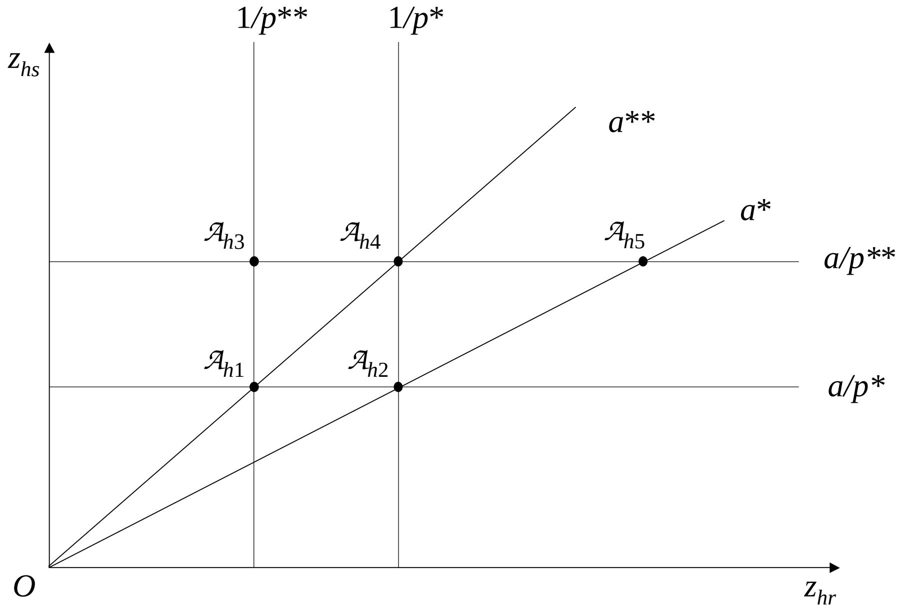

Figure 3: The competitive products in the (z_{hr}, z_{hs})-diagram

identical quality-price relations are supplied, (c.p.) only the one with the lowest price is competitive (for the example: $\mathcal{A}_{h2}$ and $\mathcal{A}_{h4}$).

3. Products with identical product qualities lie on a straight line through O (e.g. products $\mathcal{A}_{h1}$ and $\mathcal{A}_{h4}$ or $\mathcal{A}_{h2}$ and $\mathcal{A}_{h5}$). The higher the product quality the higher the slope of the corresponding line. If two products with identical qualities are supplied, (c.p.) only the product with the lowest price is competitive (for the example: $\mathcal{A}_{h4}$ and $\mathcal{A}_{h5}$).

Apparently the following applies for the example in figure 3:

$$\begin{aligned} a_1 = a_4 = a^{**} &> a^* = a_2 = a_5 \\ p_1 = p_3 = p^{**} &> p^* = p_2 = p_4 \\ a_3/p_3 = a_4/p_4 = a_5/p_5 = (a/p)^{**} &> (a/p)^* = a_1/p_1 = a_2/p_2. \end{aligned}$$

From these considerations an unambiguous enumeration can be found for the products which is identical with regard to the prices, product qualities and price-quality relations. Generally it can be shown that with the restriction to competitive products the products can always be ordered canonically:

Proposition 3 *Assume m competitive products k with product quality a_k and product price p_k ($k \in \mathcal{M}$). $a_k \neq a_l$ and $p_k \neq p_l$ shall apply for all $k, l \in \mathcal{M}$. With the assumptions mentioned above a permutation π[11] can be found so that the following applies:*

$$\begin{aligned} a_{\pi(1)} < a_{\pi(2)} < \quad \ldots \quad &< a_{\pi(m-1)} < a_{\pi(m)} \\ p_{\pi(1)} < p_{\pi(2)} < \quad \ldots \quad &< p_{\pi(m-1)} < p_{\pi(m)} \\ a_{\pi(1)}/p_{\pi(1)} < a_{\pi(2)}/p_{\pi(2)} < \quad \ldots \quad &< a_{\pi(m-1)}/p_{\pi(m-1)} < a_{\pi(m)}/p_{\pi(m)}. \end{aligned}$$

The permutation is unique.

The proof of Proposition 3 is simple, but technically. For that reason we omit it.

In the following we assume that the products are ordered as in Proposition 3 presented.

3.5 Implications of the approach

The Lancaster approach as well as the modified Lancaster approach point to an important problem: A household with a certain income will always demand only one product l or two adjacent products l and $l+1$. Now, if h_0 households are considered, this situation will not change if homothetic utility functions are assumed. This implication is rather unsatisfactory. There are two solutions:

1. We make use of the assumption of homothetic but not of the assumption of identical utility functions.

2. We make use of the assumption of non-homothetic and of the assumption of identical utility functions.

The first option is not particularly attractive. If different preferences for the individuals in an economy are assumed, virtually every result is conceivable. The analysis becomes arbitrary. Therefore only the second variant remains, which, however, leads to much more serious technical difficulties, as can be seen from the following sections where non-homothetic utility functions are combined with the modified Lancaster approach. In order to restrict the complexity of the approach, a very special class of utility functions will be used, which will be introduced in the next subsection.

3.6 Non-homothetic utility functions

Very different classes of non-homothetic utility functions are treated in the literature.[12] Non-homothetic utility functions which do not immediately lead to great

[11] In mathematical terms a permutation – simply speaking – a enumeration of the products. In the mathematical sense an arrangement is a permutation on the set $\mathcal{M}$.

[12] For an introduction to non-homothetic preferences and utility functions cf. *Mas-Colell/Whinston/Green* (1995) as well as *Dow/da Costa Werlang* (1992) and *Bosi* (1998).

difficulties in the technical handling are (amongst others) Stone-Geary utility functions, the generalized CES utility functions (cf. *Sato* (1974,1975)) or the utility functions classified in *Lewbel* (1987).

In this contribution, non-homothetic utility functions are chosen which result as special cases from the »extended addilog utility functions« (cf. e.g. *Atkeson/Ogaki* (1996)), which can be generally described by (with $\mathcal{M} := \{1, \ldots, m\}$):

$$U_h(x_{h1}, \ldots, x_{hm}) \;=\; \sum_{k \in \mathcal{M}} \frac{\zeta_k}{1 - J_k}[(x_{hk} - F_k)^{1-J_k} - 1],$$

where x_{hk} ($k \in \mathcal{M}$) represents the quantity that the household h consumes of the good X_k.[13] For the parameters the following is assumed:[14].

$$F_k \geq 0, \quad \zeta_k \geq 0, \quad J_k > 0.$$

If the price of the good k is given by p_k and the income of the household h is given by b_h, the household will – under the usual assumptions of household theory – maximize its utility while taking the budget restriction into account

$$b_h \;=\; \sum_{k \in \mathcal{M}} p_k x_{hk}.$$

This utility maximization problem of the household leads immediately to the following Lagrange-function:[15]

$$L(x_{h1}, \ldots, x_{hm}; \lambda) \;=\; \sum_{k \in \mathcal{M}} \frac{\zeta_k}{1 - J_k}[(x_{hk} - F_k)^{(1-J_k)} - 1] + \lambda(b_h - \sum_{k \in \mathcal{M}} p_k x_{hk}).$$

For the partial derivations the following applies:

$$\frac{\partial L}{\partial x_{hk}} \;=\; \frac{\zeta_k}{(1 - J_k)}(1 - J_k)(x_{hk} - F_k)^{-J_k} - \lambda p_k = \zeta_k (x_{hk} - F_k)^{-J_k} - \lambda p_k \quad (11)$$

and

$$\frac{\partial L}{\partial \lambda} \;=\; b_h - \sum_{k \in \mathcal{M}} p_k x_{hk}.$$

For the ratio of the derivations to x_{hk} and x_{hl} one gets – after re-ordering and setting to zero – the following from the corresponding equations from (11)):[16]

$$\frac{\zeta_k}{\zeta_l} \frac{(x_{hk} - F_k)^{-J_k}}{(x_{hl} - F_l)^{-J_l}} \;=\; \frac{p_k}{p_l}.$$

[13]In this subsection non-homothetic utility functions are primarily discussed in the traditional context. I.e. the utility depends on the consumed bundle of goods. We will apply results of this subsection in the next sections when utility depends on the consumed characteristics units.

[14]We will come to some special parameter constellations in a minute. The cases $J_k = 1$ have not be excluded from the parameter space due to the existence of continuities

[15]λ here represents the Lagrange-multiplier.

[16]A labeling of the optimal values with * was dispensed with.

Here some features of the utility function can be seen.[17] For $J_1 = \ldots = J_m$ homothetic preferences result. For the special case $J_1 = \ldots = J_m = 1$ a linear expenditure system results. If the subsistence parameters are dispensed with ($F_k = 0$ for all $k \in \mathcal{M}$), the utility functions introduced by H. S. Houthakker result.

For this survey, a very special parameter constellation is selected. Now, if $m = 2$, $F_k = F_l = 0$, $J_k = 2$ and $J_l = 1$ is selected, it follows that[18]

$$\frac{\zeta_k}{\zeta_l} \frac{x_{hk}^{-2}}{x_{hl}^{-1}} = \frac{p_k}{p_l}.$$

With $\zeta := \zeta_l/\zeta_k$ the following holds:

$$x_{hl} = \zeta \frac{p_k}{p_l} x_{hk}^2. \tag{12}$$

The equation (12) apparently describes the income expansion paths which are given by parables. Therefore the utility functions from the class of the »extended addilog utility functions«, which show the feature (12) should be called *parabolic utility functions*.

The advantage of this specific subclass of utility functions will become clear in many places in the course. This formulation allows the explicit determination of demand functions, which is often difficult with many non-homothetic utility functions.

For illustration purposes we will take a brief look at figure 4. First of all, for a certain price ratios p_k/p_l straight budget lines for different budgets b_h appear in the (x_{hk}, x_{hl})-diagram. The income expansion path is given by a parable which is determined by the equation (12). Of course this parable represents the combinations which represent a utility maximum with the different budgets. Put geometrically, this parable represents all points in which the indifference curves have the slope of the budget lines. It becomes clear that with an increasing income b_h the demand for both products k and l increases, as does the relation of x_{hl} to x_{hk}. The latter can easily be seen from the corresponding angles in the triangles which are given by $\mathcal{O}$, $\mathcal{A}^*$ and $\mathcal{B}^*$ or $\mathcal{O}$, $\mathcal{A}^{**}$ and $\mathcal{B}^{**}$ (cf. figure 4).

The features of the parabolic utility functions as compared to normal homothetic utility functions become clear: If (due to the income) the demand for X_k doubles, with parabolic utility functions the demand for X_l will quadruple. For homothetic utility functions the following applies: If a doubling of the demand for good X_k occurs due to changes in income, the demand for good X_l will double as well.

For utility functions which under the assumption of utility maximising households generate the condition (12) the expenditures $b_{hk} = p_k x_{hk}$ und $b_{hl} = p_l x_{hl}$ can be

[17] Cf. again *Atkeson/Ogaki* (1996).

[18] In order to keep the notation clear, the two goods are not, as would be usual, marked by the indices 1 and 2, but by the indices k and l. Therefore the following applies: $\mathcal{M} = \{k, l\}$.

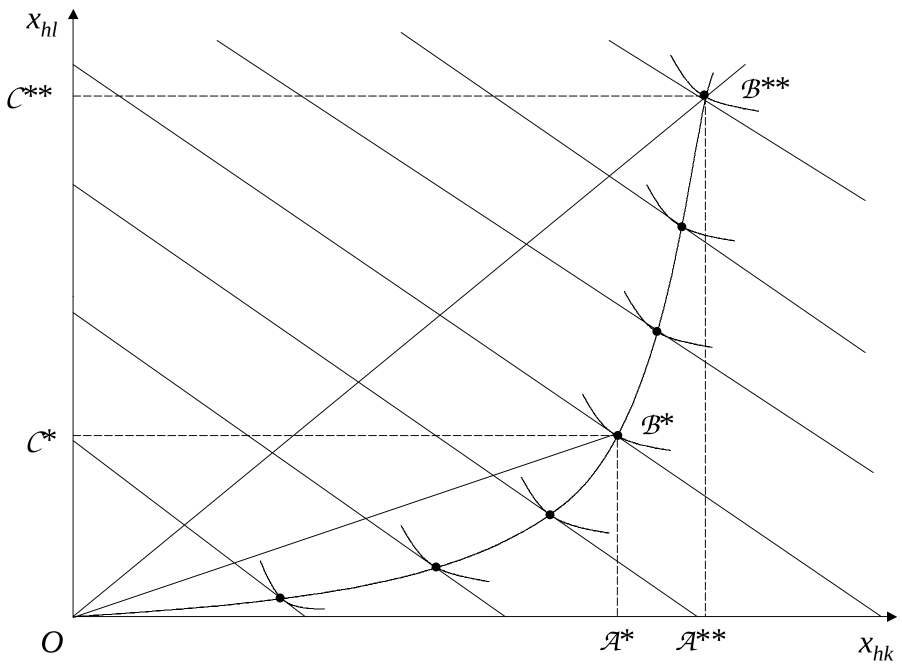

Figure 4: Utility maximization in the case of »parabolic« utility functions (goods area)

determined in a simple way[19] with the application of the budget restriction

$$p_k x_{hk} + p_l x_{hl} \quad = \quad b_h$$

of a household with income b_h. With (12) the following results

$$b_h = b_{hk} + b_{hl} = b_{hk} + p_l x_{hl} = b_{hk} + \zeta p_k x_{hk}^2 = b_{hk} + \frac{\zeta}{p_k} b_{hk}^2$$

and thus the following quadratic equation:

$$b_{hk}^2 + \frac{p_k}{\zeta} b_{hk} - \frac{p_k}{\zeta} b_h \quad = \quad 0. \qquad (13)$$

As one solution we get:

$$\begin{aligned} b_{hk} &= -\frac{p_k}{2\zeta} + \sqrt{\left(\frac{p_k}{2\zeta}\right)^2 + \frac{p_k}{\zeta} b_h} \\ &= -\frac{p_k}{2\zeta} + \sqrt{\frac{p_k^2}{4\zeta^2} + \frac{p_k}{\zeta} b_h}. \qquad (14) \end{aligned}$$

[19] A utility maximization problem under side conditions, which is solved with the help a Lagrange approach does not have to be applied here any longer. This is the essential advantage of not assuming certain utility functions, but features with regard to the income expansion path (here in particular the feature given by equation (12)).

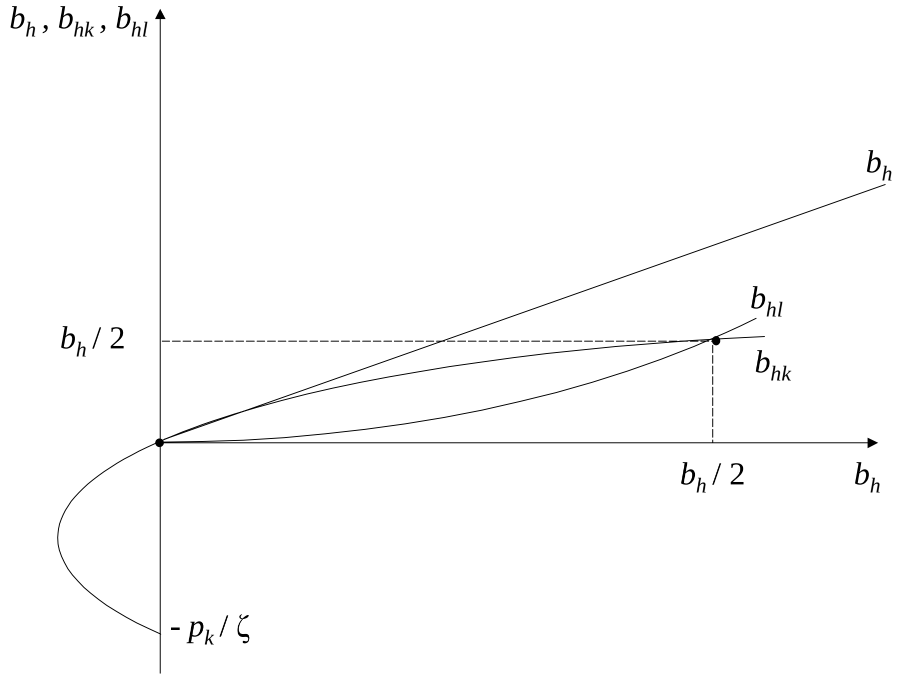

Figure 5: The expenditures in the case of »parabolic« utility functions

The other solution of the equation (13) is always negative and thus economically not meaningful and will therefore not be considered further.

b_{hl} then is given by:

$$\begin{aligned} b_{hl} &= b_h - b_{hk} \\ &= b_h + \frac{p_k}{2\zeta} - \sqrt{\frac{p_k^2}{4\zeta^2} + \frac{p_k}{\zeta} b_h}. \end{aligned} \tag{15}$$

Figure 5 shows the expenditures (qualitatively) for the selected »parabolic« utility functions. The demand functions now can be determined with the equations (14) and (15):

$$\begin{aligned} x_{hk} &= \frac{b_{hk}}{p_k} = -\frac{1}{2\zeta} + \sqrt{\frac{1}{4\zeta^2} + \frac{b_h}{\zeta p_k}} \\ x_{hl} &= \frac{b_{hl}}{p_l} = \frac{b_h}{p_l} + \frac{p_k}{2\zeta p_l} - \sqrt{\frac{p_k^2}{4\zeta^2 p_l^2} + \frac{p_k}{\zeta p_l^2} b_h}. \end{aligned}$$

4 Individual demand

4.1 Basic assumptions

For the subsequent considerations we make use of the following assumptions:

(A1) Assume $m \geq 2$ competitive products k with product qualities a_k and product prices p_k $(k \in \mathcal{M})$. $a_k \neq a_l$ and $p_k \neq p_l$ shall again apply for all $k, l \in \mathcal{M}$.[20] The canonical enumeration of the products is chosen as it was introduced on proposition 3.

(A2) The disposable income of a household h is given by b_h.

(A3) The households' behavior is given as in the modified Lancaster approach (cf. particularly the optimization problem (2)).

(A4) The preferences of the individuals are given by non-homothetic utility functions. In particular, »parabolic« utility functions are assumed.

On the basis of these assumptions individual demand functions are derived. For this, the optimization problem (2) must be solved. Due to the special features of the utility function $U_h(z_{hr}, z_{hs})$ there is the elegant possibility to use the results from the previous subsection 3.6 if the income expansion paths are parameterized in the form

$$z_{hs} = -\upsilon s_k z_{hr}^2$$

where

$$s_k = \frac{a_{k+1}/p_{k+1} - a_k/p_k}{1/p_{k+1} - 1/p_k}$$

for $k \in \mathcal{M}_{-m}$ represents the slope of the section of the line $\overline{A_{hk}A_{h,k+1}}$ (cf. equation (12)).[21]

With this approach s_k is the equivalent of p_k/p_l and υ is the equivalent of ζ. Figure 6 illustrates once more the situation for the case $m = 2$. First, six households $(h = 1, \ldots, 6)$ with different budgets can be observed. The possibility sets for the households apparently are represented by the sets which in the figure are unambiguously determined by the triangles with the corners $\mathcal{A}_{h1}$, $\mathcal{A}_{h2}$ and $\mathcal{O}$ $(h \in \{1, \ldots, 6\})$. From the set which is relevant for a given household h this household then chooses the combination (z_{hr}^*, z_{hs}^*), which generates the utility maximum of the household h.

It becomes clear that a tangential solution only results for certain budgets. This is the case for budgets b_h which lie between b_3 and b_5. For low budgets $b_h \leq b_3$

[20] The case $m = 1$ is excepted because in this case only one product $k = 1$ exists on which all demand is placed. In order to avoid an extensive distinction of cases, $m = 1$ will as a rule not be considered in the following.

[21] s_k is obviously independent of the budget b_h and therefore does not carry an index h.

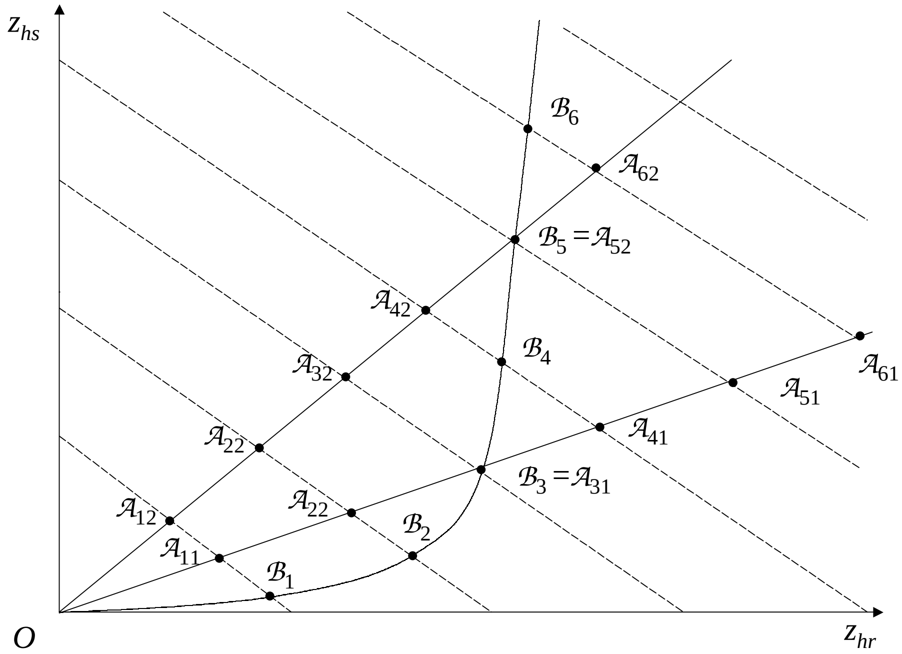

Figure 6: Utility maximization in the case of »parabolic« utility functions (characteristics area)

a vertex solution results where only the lower-priced product 1 is demanded. For high incomes $b_h \geq b_5$ only the higher quality product 2 is demanded. The relevant income limits obviously are determined by the parable and the marked rays and thus by the prices p_k, the product qualities a_k and the budget b_h.

If a third product was to be considered now, with a product quality $a_3 > a_2$ and a price $p_3 > p_2$, apparently a second parable would have to be considered, if the other two products do not lose their competitiveness with the addition of the third product. This parable would then generate the income limits for the products 2 and 3.

4.2 The deviation of individual demand curves

In this subsection it will first be shown how the demand for the individual products depends on the income of a household.

Proposition 4 *Assume (A1) to (A4). If* $k \in \mathcal{M} \setminus \{1, m\} = \{2, \ldots, m-1\}$*, then unique income limits* b_k^l*,* b_k^L*,* b_k^H *and* b_k^h *with* $0 < b_k^l < b_k^L < b_k^H < b_k^h$ *exist, so that the* *expenditures $b_{hk}(b_h)$ of the household are given by:*

$$b_{hk}(b_h) = \begin{cases} 0 & for \quad b_h \in [0, b_k^l] \\ (1-\eta_{k-1}(b_h))b_h & for \quad b_h \in]b_k^l, b_k^L[\\ b_h & for \quad b_h \in [b_k^L, b_k^H] \\ \eta_k(b_h)b_h & for \quad b_h \in]b_k^H, b_k^h[\\ 0 & for \quad b_h \in [b_k^h, \infty[\end{cases} . \tag{16}$$

If $k = 1$, then unique income limits b_k^H and b_k^h with $0 \leq b_k^H < b_k^h$ exist, so that the expenditures $b_{hk}(b_h)$ of the household are given by:

$$b_{hk}(b_h) = \begin{cases} b_h & for \quad b_h \in [0, b_k^H] \\ \eta_k(b_h)b_h & for \quad b_h \in]b_k^H, b_k^h[\\ 0 & for \quad b_h \in [b_k^h, \infty[\end{cases} .$$

If $k = m$, then unique income limits b_k^l and b_k^L with $0 < b_k^l < b_k^L$ exist, so that the expenditures $b_{hk}(b_h)$ of the household are given by:

$$b_{hk}(b_h) = \begin{cases} 0 & for \quad b_h \in [0, b_k^l] \\ (1-\eta_{k-1}(b_h))b_h & for \quad b_h \in]b_k^l, b_k^L[\\ b_h & for \quad b_h \in [b_k^L, \infty] \end{cases} .$$

Furthermore the following yields for $k \in \mathcal{M}_{-1} := \mathcal{M} \setminus \{1\}$:

$$\eta_k(b_h) = \frac{-1/(2\upsilon) - b_h/p_{k+1} + \sqrt{1/(4\upsilon^2) - b_h(a_k - s_k)/(\upsilon p_k s_k)}}{b_h/p_k - b_h/p_{k+1}}$$

with

$$s_k = \frac{a_{k+1}/p_{k+1} - a_k/p_k}{1/p_{k+1} - 1/p_k}$$

and

$$\begin{aligned} b_k^l &= b_{k-1}^H \\ b_k^L &= b_{k-1}^h. \end{aligned}$$

The system of individual demand curves resulting from this approach, which are given by

$$x_{hk} = b_{h_k}/p_k \tag{17}$$

actually shows interesting features. In figure 7 the typical characteristics of the expenditure shares $\tilde{b}_{hk} = b_{hk}/b_h$ for the case $m = 5$ are shown.

Although the proof of the proposition 4 is intuitive, extensive technical considerations are needed. The starting point for these considerations is the result that the utility maximization of the household always leads to a tangential or a vertex solution. In the case of competitive products, $2m - 1$ »different« cases must be distinguished:

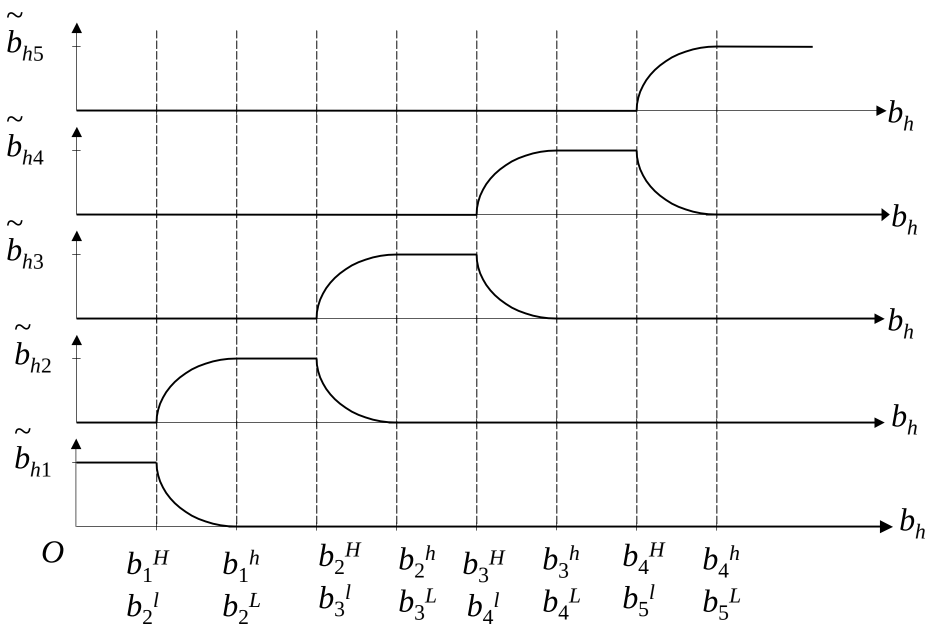

Figure 7: Typical individual expenditure shares

- m »different« vertex solutions, i.e. the household only demands one product k ($k \in \mathcal{M}$).
- $m-1$ »different« tangential solutions, i.e. the household demands two adjacent products k and $k+1$ ($k \in \mathcal{M}_{-m}$).

The demand for product k is now essentially determined by four different income limits. Without taking further notice of some particularities for $k = 1$ and $k = m$, b_k^l, b_k^L, b_k^H and b_k^h for $k = 2, \ldots, m-1$ can be determined, which generate five intervals:

$$b_h \in \left\{ \begin{array}{ccl} [0, b_k^l] & : & \text{Case (0,l)} \\ ]b_k^l, b_k^L[& : & \text{Case (l,L)} \\ [b_k^L, b_k^H] & : & \text{Case (L,H)} \\ ]b_k^H, b_k^h[& : & \text{Case (H,h)} \\ [b_k^h, \infty[& : & \text{Case (h,}\infty\text{)} \end{array} \right. .$$

- Case (L,H): In this case the household will spend the entire income on the purchase of the product k. For $b_h = b_k^L$ the vertex solution lies on an indifference curve whose slope in this point corresponds exactly to the slope of the line $\overline{\mathcal{A}_{h,k-1}\mathcal{A}_{hk}}$. If the income increases an indifference curve is reached which represents a higher utility level. However, the slope of the relevant indifference curve decreases. For the income $b_h = b_k^H$ a vertex solution is only just realized. The slope of the indifference curve then corresponds to the slope of the line $\overline{\mathcal{A}_{kh}\mathcal{A}_{h,k+1}}$.

- Case (H,h): For $b_h \in]b_k^H, b_k^h[$ the products k and $k+1$ are demanded. For this case the slope of the indifference curve always corresponds to the slope if the line $\overline{\mathcal{A}_{hk}\mathcal{A}_{h,k+1}}$. The expenditure share for product k decreases with increasing income b_h.

- Case (h,∞): If the income b_h exceeds the limit b_k^h, the expenditure share for k is reduced to 0.

- Case (l,L): This case is similar to the case (H,h): The slope of the line $\overline{\mathcal{A}_{h,k-1}\mathcal{A}_{hk}}$ is always of importance here. Starting from $b_h = b_k^L$ a reduction of the income b_h leads to a reduction of the expenditure share for k. For $b_h = b_k^l$ this share obviously becomes 0.

- Case (0,l): For an income below b_k^l the household will not demand the product any more.

To determine the income limits b_k^i ($i \in \{l, L, H, h\}$, $k \in \mathcal{M} \setminus \{1, m\}$) the slopes s_k of the line $\overline{\mathcal{A}_{hk}\mathcal{A}_{h,k+1}}$ have to be determined. With (3) and (4) the following applies (for $k \in \mathcal{M}_{-m}$):

$$
\begin{aligned}
s_k &:= \frac{z_{hs,k+1}^{max} - z_{hs,k}^{max}}{z_{hr,k+1}^{max} - z_{hr,k}^{max}} \\
&= \frac{(a_{k+1}/p_{k+1})b_h - (a_k/p_k)b_h}{b_h/p_{k+1} - b_h/p_k} \\
&= \frac{a_{k+1}/p_{k+1} - a_k/p_k}{1/p_{k+1} - 1/p_k}.
\end{aligned}
$$

s_k of course does not depend on b_h ab.
The straight line containing the line $\overline{\mathcal{A}_{hk}\mathcal{A}_{h,k+1}}$, is given by (for $k \in \mathcal{M}_{-m}$):

$$
\mathcal{L}_k := \left\{ (z_{hr}, z_{hs}) \in \mathbb{R}_+^2 : \; s_k = \frac{z_{hs} - (a_k/p_k)b_h}{z_{hr} - b_h/p_k} \right\}.
$$

Transformations result in:

$$
\mathcal{L}_k = \{ (z_{hr}, z_{hs}) \in \mathbb{R}_+^2 : \; z_{hs} = s_k z_{hr} + b_h(a_k - s_k)/p_k \}.
$$

Furthermore, the following parables are needed, which are important because of the assumption (A4) (for $k \in \mathcal{M}_{-m}$):

$$
\mathcal{P}_k := \{ (z_{hr}, z_{hs}) \in \mathbb{R}_+^2 : \; z_{hs} = -\upsilon s_k z_{hr}^2 \}.
$$

Finally, the straight line through the point Oand the points $\mathcal{A}_{hk}$ (for $k \in \mathcal{M}$) are needed:

$$\mathcal{R}_k \;:=\; \{(z_{hr}, z_{hs}) \in \mathrm{IR}_+^2 : \; z_{hs} = a_k z_{hr}\}.$$

The points of intersection of the source straight lines $\mathcal{R}_k$with the parables $\mathcal{P}_k$ allow the determination of the income limits b_k^i. Apparently $\mathcal{R}_k \cap \mathcal{P}_k$ generates the income limit $b_k^H = b_{k+1}^l$. $\mathcal{R}_{k+1} \cap \mathcal{P}_k$ results in $b_k^h = b_{k+1}^L$, where $\mathcal{O}$ can always be disregarded. For the first case $z_{hr}^2 = a_k z_{hr}$ and $z_{hr} = b_k^H/p_k$ will result from $-\upsilon s_k$:

$$b_k^H \;=\; a_k p_k/(\upsilon s_k).$$

Analogously we can derive:

$$b_k^h \;=\; a_{k+1} p_{k+1}/(\upsilon s_k).$$

After the determination of the income limits b_k^i it remains to be shown that for those cases where two products are demanded the expenditure shares are determined as in proposition 4. For the cases (l,L) and (H,h) therefore the expenditure shares $\eta_k(b_h)$ have to be determined.[22]

The tangential solution which results in these cases is determined by the point of intersection of the parable $\mathcal{P}_k$ and the straight line $\mathcal{L}_k$. From the corresponding equation

$$s_k z_{hr} + b_h(a_k - s_k)/p_k \;=\; -\upsilon s_k z_{hr}^2$$

the following solutions result

$$z_{hr}^{1,2} = -\frac{1}{2\upsilon} \pm \sqrt{\frac{1}{4\upsilon^2} - \frac{a_k - s_k}{\upsilon p_k s_k} b_h}$$

For economic reasons only the positive solution is meaningful:

$$z_{hr}^* = -\frac{1}{2\upsilon} + \sqrt{\frac{1}{4\upsilon^2} - \frac{a_k - s_k}{\upsilon p_k s_k} b_h}.$$

The solution $z_h^* = (z_{hr}^*, z_{hs}^*)$ represents a mix of the two products k and $k+1$. For the determination of the expenditure shares equation (6) will be used:

$$z_h^* \;=\; \eta_k \mathcal{A}_{hk} + (1 - \eta_k)\mathcal{A}_{h,k+1}.$$

The following applies:

$$\begin{pmatrix} z_{hr}^* \\ z_{hs}^* \end{pmatrix} \;=\; \eta_k \begin{pmatrix} b_h/p_k \\ (a_k/p_k) b_h \end{pmatrix} + (1 - \eta_k) \begin{pmatrix} b_h/p_{k+1} \\ (a_{k+1}/p_{k+1}) b_h \end{pmatrix}. \tag{18}$$

[22] Cf. also the results from subsection 3.6.

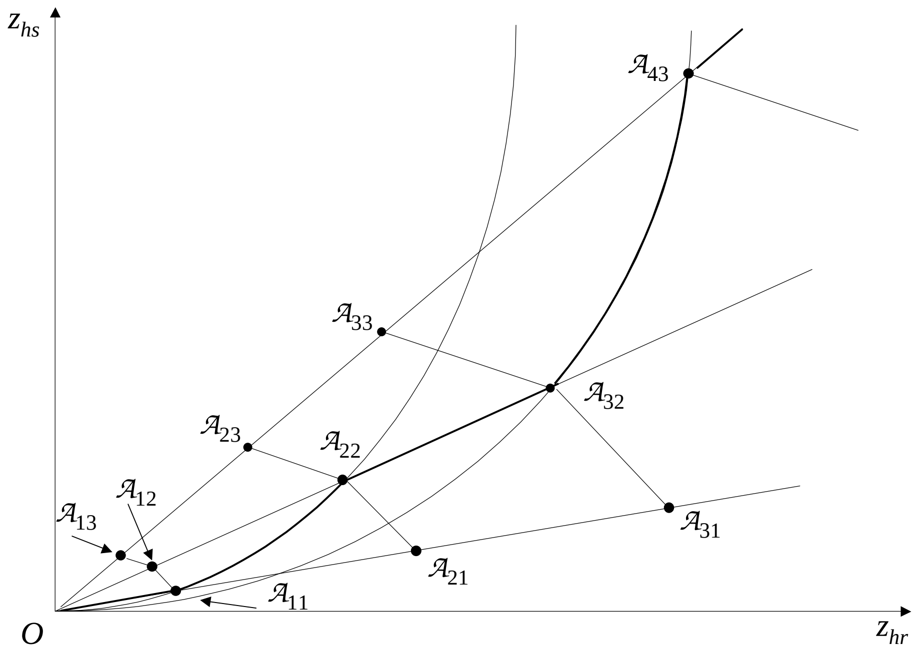

Figure 8: A typical income expansion path

From the first row of equation (18) we can now determine $\eta_k(b_h)$:

$$\begin{aligned} \eta_k(b_h) &= \frac{z^*_{hr} - b_h/p_{k+1}}{b_h/p_k - b_h/p_{k+1}} \\ &= \frac{-1/(2\upsilon) - b_h/p_{k+1} + \sqrt{1/(4\upsilon^2) - b_h(a_k - s_k)/(\upsilon p_k s_k)}}{b_h/p_k - b_h/p_{k+1}}. \end{aligned} \tag{19}$$

Thus proposition 4 is proven.

A second proposition relates to the structure of the income expansion paths:

Proposition 5 *Assume (A1) to (A4). Then the income expansion path consists of $m-1$ parts of the rays $\mathcal{R}_k$ ($k \in \mathcal{M}_{-1}$), $m-1$ parable sections of the parables $\mathcal{P}_k$ ($k \in \mathcal{M}_{-1}$) and a ray which represents a subset of $\mathcal{R}_m$. The ends of sections of the lines and parable sections are given canonically by the income limits b^i_k ($i = l, L, H, h$).*

The proof of the proposition 5 is based on the proof of the proposition 4. Figure 8 shows a typical income expansion path for the case $m = 3$. Since the income expansion path results piece by piece it is not surprising that it is continuous but not differentiable.

4.3 The impact of process innovations on individual demand

For the analysis of the influence of process innovations we focus on incremental innovations which result in marginal price reductions. In order to grasp the the impact of process innovations we have to investigate the cases introduced in proposition 4. The following considerations apply to products $k = 2, \ldots, m-1$.[23] By considering marginal price changes we will first exclude the option that a transition from one of the five cases considered in the previous subsection to another takes place. Consequently, the five cases can be processed successively when we now consider the price dependence of the demand x_{hk}.

- Case (L,H): In this case only one product k is demanded. Consequently the following applies: $x_{hk} = b_h/p_k$. For the price elasticity apparently the following applies:

$$\mathrm{E}\left(x_{hk}, p_k\right) = -1.$$

  The reason is clear: If a household spends its budget completely on one product the demand will decrease by one percent if the price is increased by one percent.

- Case: (H,h): In this case two adjacent products k and $k+1$ are demanded. With the equations (16) and (17) it becomes clear that

$$x_{hk} = x_{hk}(a_k, a_{k+1}, p_k, p_{k+1}, b_h)$$

  applies. With proposition 4 and especially equation (19) and some transformations one gets:

$$\begin{aligned} \mathrm{E}\left(x_{hk}, p_k\right) &< 0 \\ \mathrm{E}\left(x_{hk}, p_{k+1}\right) &> 0. \end{aligned}$$

  The usual features of oligopolistic competition show: a firm will loose demand if it increases the product price. If the competitor increases its price, the demand will rise.

- Case (l,L): The results are – for obvious reasons – similar to the results of the case (H,h). The following applies:

$$\begin{aligned} \mathrm{E}\left(x_{hk}, p_k\right) &< 0 \\ \mathrm{E}\left(x_{hk}, p_{k-1}\right) &> 0. \end{aligned}$$

  Compared to the case discussed before now the product with the next lowest price $k-1$ is the product with which the product k has to compete.

[23] For $k = 1$ and $k = m$ similar results can be derived. However, it should be noted that in these cases the results for the case (L,H) will appear for the intervals (0,l) and (l,L) or (H,h) and (h,∞) respectively.

- Case (0,l) and case (h,∞): In these cases the demand for k is always 0. Thus marginal price changes have no influence on the demand for product k.

On the whole the results are not surprising but still remarkable: Every firm competes – if it competes at all – only with those firms which offer adjacent products according to the canonical enumeration. Thus, if we have $m = 5$ competitive products a firm with a product of medium product quality ($k = 3$) is not in immediate competition with the firms which offer the product with the highest quality ($k = 5$) and the lowest-priced product ($k = 1$).

4.4 The impact of product innovations on individual demand

As with the the process innovation case, the five known cases must be distinguished for the analysis of the effects of marginal changes of the product qualities which are a result of product innovation:

- Case (L,H): In this case only one product k is demanded. Consequently the following applies: $x_{hk} = b_h/p_k$. For the quality elasticity the following apparently applies:

$$\mathrm{E}\left(x_{hk}, a_k\right) = 0.$$

- Case: (H,h): In this case two adjacent products k and $k+1$ are demanded. The following applies:

$$\begin{aligned} \mathrm{E}\left(x_{hk}, a_k\right) &> 0 \\ \mathrm{E}\left(x_{hk}, a_{k+1}\right) &< 0. \end{aligned}$$

  Thus a firm k can achieve demand increases by improving the product quality a_k. Improvements of the product quality a_{k+1} reduce the demand.

- Case (l,L): The results are symmetrical to the results of the case (H,h). The following applies:

$$\begin{aligned} \mathrm{E}\left(x_{hk}, a_k\right) &> 0 \\ \mathrm{E}\left(x_{hk}, a_{k-1}\right) &< 0. \end{aligned}$$

- Case (0,l) and case (h,∞): In these cases the demand for k is always 0. Thus neither marginal price changes nor marginal product quality changes have an influence on the demand.

Thus the results for the quality competition are almost symmetrical to the results for the price competition. However, there is remarkable difference. In the case (L,H) the firm has no incentive to innovate with respect to a product with a higher quality. In that case a higher product quality does not generate a higher demand for the innovator's product.

4.5 Implications of the approach

The approach provides a foundation for oligopolistic and particularly also for local competition. It can be seen that here firms are in price and quality competition, initially depending on the income of an individual household. However, the approach also shows that there may be situations where an individual firm has a monopolistic market position. If it is in competition to other products then always only with one of the immediately adjacent products. These statements however only apply to marginal price and quality changes.

5 Aggregate demand and product variety

5.1 The impact of income distribution

The aggregate demand is determined by explicit aggregation. If the expenditures of a household h ($h \in \mathcal{H}$) for the product k are given by b_{hk}, then the following will apply for the aggregate expenditures b_k:

$$b_k = \sum_{h \in \mathcal{H}} b_{hk}.$$

The aggregate demand x_k is then given by:

$$x_k = b_k / p_k.$$

The explicit determination of the aggregate demand depends on the special form of the income distribution. The following proposition 6 shows that the range of products which are demanded by the households at all is essentially determined by the income distribution.

Proposition 6 *Assume (A1) to (A4). Furthermore, assume for the disposable income b_h of the households $b_h \in [b^{min}, b^{max}]$. Then the indexes k_0^l and k_0^h ($k_0^l, k_0^h \in \mathcal{M}$) exist, so that the following applies:*

$$x_k \begin{cases} = 0 & \textit{for all} \quad k < k_0^l \\ > 0 & \textit{for} \quad k = k_0^l \\ \geq 0 & \textit{for all} \quad k = k_0^l + 1, \ldots, k_0^h - 1 \\ > 0 & \textit{for} \quad k = k_0^h \\ = 0 & \textit{for all} \quad k > k_0^h, \end{cases} \tag{20}$$

with

$$k_0^l = \max(\{1\} \cup \{k \in \mathcal{M}_{-1} : b^{min} > b_k^h\}) \tag{21}$$
$$k_0^h = \min(\{m\} \cup \{k \in \mathcal{M}_{-m} : b^{max} < b_k^l\}). \tag{22}$$

The proposition shows that there may be competitive products which are not demanded because in the final instance their product quality is too low. At the same time, there may be products with a very high product quality which are not demanded because they are too expensive. Apparently the income distribution and in particular the range of incomes essentially determines the product variety that can persist in a market.[24]

Proposition 6 assumes a finite number of households. Assuming many households which is certainly justified with regard to real economies with many millions of households it can be expected that all products with indices between k_0^l and k_0^h will be demanded. Then Equation 20 can be reduced to:

$$x_k \begin{cases} = & 0 & \text{for all} & k < k_0^l \\ > & 0 & \text{for all} & k = k_0^l, \ldots, k_0^h \\ = & 0 & \text{for all} & k > k_0^h, \end{cases} \tag{23}$$

However, if the number of households h_0 is very small (if for example $h_0 < m/2$ applies), the case may occur that not all products are demanded although e.g. the products $k = 1$ and $k = m$ are demanded.

5.2 The impact of income taxation

The influence of income taxation on the product variety becomes clear most of all when considering the influence of income taxation. Therefore we will consider three tax regimes in particular. With the gross income y_h and the disposable income b_h of a household h three tax regimes τ, T and β shall be characterized as follows:

- Tax regime τ:

  A proportional income tax with a tax rate τ which is identical for all households is levied. Then the following applies:

  $$b_h = (1 - \tau) y_h.$$

- Tax regime T:

  A poll tax of T/h_0 is levied, which burdens all households $h \in \mathcal{H}$ in the same way:

  $$b_h = y_h - T/h_0.$$

- Tax regime β:

[24] The proof of the proposition immediately follows from the considerations of the previous section. It may seem surprising that in the equations (21) and (22) the indexes $k = 1$ and $k = m$ are especially listed. The background for this is that the variables b_1^l and b_m^h have not been defined.

In this case it is assumed that a part of the gross income is redistributed neutrally. With

$$\bar{y} = \sum_{h \in \mathcal{H}} y_h / h_0$$

the following applies in this case:

$$b_h = (1-\beta) y_h + \beta \bar{y},$$

where β $(0 \leq \beta \leq 1)$ determines the degree of redistribution: In the case $\beta = 0$ there is no redistribution. For $\beta = 1$ identical disposable incomes result.

It will be shown now that the introduced tax regimes τ, T and β have an essential influence on the product variety. In order to keep the analysis transparent, we will make use of one additional assumption:

(A5) If products k_l and k_h are demanded, then also all products $k = k_l + 1, \ldots, k_h$ are demanded.

The following proposition, which can easily be proven, summarizes some results:

Proposition 7 *Assume (A1) to (A5) and $y_h \in [(1-\alpha)\bar{y}, (1+\alpha)\bar{y}]$. Then the introduction of the tax regimes τ, T and β may lead to changes in product variety, given by $k_0^l, \ldots, k_0^h$. In particular, the tax regimes induce different sequences $k_i^l, \ldots, k_i^h$ (with $i \in \{\tau, T, \beta\}$) of products which are demanded. The following applies:*

- *Tax regime τ:*

$$\begin{aligned} k_\tau^l &= \max(\{1\} \cup \{k \in \mathcal{M}_{-1} : (1-\tau)(1-\alpha)\bar{y} > b_k^h\}) \leq k_0^l \\ k_\tau^h &= \min(\{m\} \cup \{k \in \mathcal{M}_{-m} : (1-\tau)(1+\alpha)\bar{y} < b_k^l\}) \leq k_0^h; \end{aligned}$$

- *Tax regime T:*

$$\begin{aligned} k_T^l &:= \max(\{1\} \cup \{k \in \mathcal{M}_{-1} : (1-\alpha)\bar{y} - T/h_0 > b_k^h\}) \leq k_0^l \\ k_T^h &:= \min(\{m\} \cup \{k \in \mathcal{M}_{-m} : (1+\alpha)\bar{y} - T/h_0 < b_k^l\}) \leq k_0^h; \end{aligned}$$

- *Tax regime β:*

$$\begin{aligned} k_\beta^l &:= \max(\{1\} \cup \{k \in \mathcal{M}_{-1} : ((1-\alpha) + \beta\alpha)\bar{y} > b_k^h\}) \\ k_\beta^h &:= \min(\{m\} \cup \{k \in \mathcal{M}_{-m} : ((1+\alpha) - \beta\alpha)\bar{y} < b_k^l\}). \end{aligned}$$

The results for the tax regimes τ and T follow immediately from (21) and (22) by inserting the values for the minimum and maximum income (after taxation or redistribution) respectively. The same applies for the regime β. However, here some reformations must be carried out first: For the minimum income b_β^{min} the following applies after redistribution:[25]

$$\begin{aligned} b_\beta^{min} &= (1-\beta)(1-\alpha)\bar{y} + \beta\bar{y} \\ &= ((1-\alpha) - (1-\alpha)\beta + \beta)\bar{y} \\ &= ((1-\alpha) + \beta\alpha)\bar{y}. \end{aligned}$$

The proposition shows an interesting result: With taxation the range of the demanded products shifts towards the lower-price products. Therefore the case may occur where an increase of the tax rate τ or the poll tax T/h_0 leads to high quality products not being demanded any longer because no households show a corresponding income. The other way round, suppliers with low-price products have an opportunity to sell their products.

With the help of the revenue-neutral redistribution the range of the demanded products can also be changed. If the incomes are equalized (this is the case with $\beta = 1$), then at most two products will be able to stay in the market. For values of $\beta < 0$ the income distribution apparently will be spread out even further. Thus the product variety can be extended in both directions – towards low-price and towards high quality products.

5.3 The impact of process innovation

In this section marginal and drastic price changes will be investigated. Following the results from subsection 4.3 we will first discuss the consequences of marginal price changes. To simplify the analysis we will assume that all products are demanded, i.e. the following will apply:

(A6) $k_0^l = 1$ und $k_0^h = m$.

Then the following applies:

Proposition 8 *Assume (A1) to (A6). Then a marginal price change due to a process innovation of p_k ($k = 2, \ldots, m-1$) only has an influence on the demand of the products $k-1$, k and $k+1$. For $k = 1$ (or $k = m$) only the demand of the products $k = 1$ and $k = 2$ (or $k = m-1$ and $k = m$). In particular the following applies for the corresponding price elasticities:*

- *for $k = 1$:*

$$\mathrm{E}\,(x_l, p_1) \quad \begin{cases} < 0 & \textit{for} \quad l = 1 \\ > 0 & \textit{for} \quad l = 2 \\ = 0 & \textit{for} \quad l > 2; \end{cases}$$

[25] For b_β^{max} an analogous calculation applies.

- *for* $k = 2, \ldots, m-1$*:*

$$\mathrm{E}(x_l, p_k) \quad \begin{cases} = 0 & \textit{for} \quad l < k-1 \\ > 0 & \textit{for} \quad l = k-1 \\ < 0 & \textit{for} \quad l = k \\ > 0 & \textit{for} \quad l = k+1 \\ = 0 & \textit{for} \quad l > k+1; \end{cases}$$

- *for* $k = m$*:*

$$\mathrm{E}(x_l, p_m) \quad \begin{cases} = 0 & \textit{for} \quad l < m-1 \\ > 0 & \textit{for} \quad l = m-1 \\ < 0 & \textit{for} \quad l = m. \end{cases}$$

The results can be immediately derived from the results of the section 4.3. It can be seen here also that for incremental changes of individual prices and product qualities changes only result in changes of demand for the concerned firm or its »neighbors«.

5.4 The impact of product innovations

The analysis of the quality increases due to product innovation is very similar to the analysis of process innovation. Particularly for the marginal quality changes the results are very similar:

Proposition 9 *Assume (A1) to (A6). Then a marginal quality change of* a_k *(*$k = 2, \ldots, m-1$*) only has an influence on the demand of the products* $k-1$*,* k *and* $k+1$*. For* $k = 1$ *(or* $k = m$*) only the demand of the products* $k = 1$ *and* $k = 2$ *(or* $k = m-1$ *and* $k = m$*) changes. In particular, the following applies:*

- *for* $k = 1$*:*

$$\mathrm{E}(x_l, a_1) \quad \begin{cases} > 0 & \textit{for} \quad l = 1 \\ < 0 & \textit{for} \quad l = 2 \\ = 0 & \textit{for} \quad l > 2; \end{cases}$$

- *for* $k = 2, \ldots, m-1$*:*

$$\mathrm{E}(x_l, a_k) \quad \begin{cases} = 0 & \textit{for} \quad l < k-1 \\ < 0 & \textit{for} \quad l = k-1 \\ > 0 & \textit{for} \quad l = k \\ < 0 & \textit{for} \quad l = k+1 \\ = 0 & \textit{for} \quad l > k+1; \end{cases}$$

- *for* $k = m$*:*

$$\mathrm{E}(x_l, a_m) \quad \begin{cases} = 0 & \textit{for} \quad l < m-1 \\ < 0 & \textit{for} \quad l = m-1 \\ > 0 & \textit{for} \quad l = m. \end{cases}$$

6 Conclusion

In this contribution a complex demand model was presented, which – based on non-homothetic utility functions and a modified Lancaster approach – allows the dependence of individual and aggregate demand functions on product qualities, product prices and the disposable incomes so that particularly the determinants of product variety can be determined.

Furthermore the approach offers a foundation of local oligopolistic price and quality competition where on the whole very similar results were found for price and quality competition.

The model was also able to show what product variety can be maintained in the market from the perspective of the demand. The decisive factor for this is the income distribution and consequently the income taxation and redistribution.

The theoretical approach points to empirical surveys. For individual markets which are characterized by price and quality competition empirical studies are conceivable. This applies in particular to the automobile market. The model can supply explanations as to why individual manufacturers of high quality automobiles are better able to maintain their position than others.

The presented approach also offers interesting possibilities for growth theoretical issues. The present model was introduced into a growth model in *Voßkamp* (2005) as a module for the representation of the demand side in order to investigate the importance of product variety for the growth of an economy.

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