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Learning-by-doing in Two Sectors, Production Structure, Leisure and Optimal Endogenous Growth

*by Matthias Göcke **

Abstract: *A model with two different production sectors and endogenous growth based on the accumulation of sector-specific human capital due to learning-by-doing is presented. Accumulation of experience is measured by means of sectoral production output aggregated over time. Growth is controlled by a dynamic optimisation of the use of time for working in the different sectors or for leisure. Transitional dynamics of production growth, especially of structural change towards a 'new' sector (with relatively scarce experience), of the optimal sectoral distribution of working time and of leisure as well as the corresponding steady state levels are derived and a numerical simulation is performed.*

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Learning-by-doing in Two Sectors, Production Structure, Leisure and Optimal Endogenous Growth

1. Introduction

While performing a production activity, specific skills (i.e. 'experience') are acquired as a *joint* or *by-product* (Rosen, 1972). This is different from real physical capital accumulation based on investment or from human capital created by schooling, since in these cases the respective type of capital is the main product of an accumulation process – investment or education decisions – which can be controlled independently. However, in growth theory the prevalent method of modelling the process of experience accumulation is not done by combining learning directly with production activity, but to connect the accumulation of experience to the accumulation of real physical capital ("learning-by-investing": Arrow, 1962, and e.g. Romer, 1986, pp. 1018 ff., or Greiner, 1996, 2003). Of course, learning is positively influenced by investment, since new experience is especially arising by using newly installed machines. However, directly binding investment and learning has the disadvantage of a lack of separation, since only a 'composite capital stock' is modelled, implicitly comprising real capital and experience. As a consequence, only a single costate of this composite capital stock is available. In models with investment as one control variable, and dynamically optimal leisure (working) time as another independent control, mismatch problems result (like multiplicity of optimal growth paths and steady states, see de Hek, 1998, and Ladrón-de-Guevara/Ortigueira/Santos, 1997).

In this paper we follow an opposite approach: Human capital accumulation due to learning-by-doing is directly modelled as a by-product of production, while investment and real capital accumulation is disregarded for reasons of simplicity.[1] Based on the concept of the experience curve the stock of experience of a sector is described based on the sectoral production output which is aggregated over time (Lucas, 1993, pp. 259 ff.). If real investment is disconnected from learning, it can be disregarded for reasons of concentration on learning-by-doing. This has the advantage of reducing the model's complexity. Assuming consumption of goods and leisure as the sources of utility – and disregarding investment – one only has to determine leisure time plus the sectoral distribution of working time, and not additionally to optimise consumption vs. saving.

[1] See Göcke (2010) for a one-sector model with an explicitly separated modelling of dynamically optimal real capital accumulation via investing and experience accumulation via working.

Moreover, the mismatch problem outlined above is avoided by separating learning-by-working from investment.

In our model the sector-specific learning is controlled by the usage of labour (working time) for producing in two different sectors of an economy. However, in both sectors learning occurs as a by-product. This is a difference to two-sector models in the tradition of Uzawa (1965) and Lucas (1988), where 'educational human capital' is the main output of a separate schooling sector.[2] From a technical point of view, we duplicate a one-sector model of Göcke (2002) by a second production sector. Compared to Göcke (2002) the dimensions are doubled, since our extended model has two types of sector specific experience capital stocks, and correspondingly the sector-specific working times as the two controls. The inclusion of two different types of sector-specific experience capital has two advantages. First, we can explicitly explain sustained endogenous growth based on spill-over effects of experience between both sectors, though sector specific experiences partially show diminishing returns each. This is an advantage compared to one-sector AK-type models – like the Göcke (2002) model – with only one capital stock, since in these models a linear influence of this single accumulated factor is required for sustained endogenous growth. Secondly, modelling sector-specific experience allows to analyse changes of the composition of the experience stocks. During the transition towards a steady state the relative size of sector specific experience changes, and the transitional dynamics are determined by changes of this proportion. Thus, compared to a one sector model our sectoral decomposition gives a better explanation of the transitional dynamics. Summarising, our two-sector model is able to illustrate sectoral change and inter-sectoral learning spill-over effects if a 'new' sector with a relatively low level of sector-specific experience is introduced, which is competing with an 'old' experience-rich sector. These transitional dynamics may serve as an explanation of growth rates above the steady state level, if a new technology becomes widespread and if experience in this modern sector is growing rapidly, and if positive productivity spill-overs to other traditional sectors are observed (as e.g. in the case of the modern IT-sector).

The outline of the paper is as follows: Dynamic optimisation based on a general formulation of the model is presented in section 2. In section 3 the optimisation results of the general model are applied to a simple example with Cobb-Douglas type production functions and a logarithmic utility function. Dynamics for the case of a 'new'

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2 Actually, since time utilisation between working or learning at school is rival, in Lucas-Uzawa-type models "learning–***or***–doing" (Chamley, 1993) is modelled. See Göcke (2004) for a Lucas-Uzawa-type model which combines learning-by-doing in a production sector with learning-by-schooling in an educational sector.

sector with relatively low experience stock and the consequences for structural change and transitional economic growth are illustrated by a numerical simulation in section 4. Section 5 concludes.

2. Dynamic optimisation in a general formulation

Population size is neglected for reasons of simplicity, and a generalised formulation of the model with two production sectors is presented in per capita terms. Per capita production output x_t in sector 1 is based on real 'experience' ξ_t in producing in sector 1 (the p.c. human capital accumulated in sector 1), sector 2 production 'experience' ψ_t, due to a spill-over of experience between both sectors, and the share q_t of the time potential which is spent on working in sector 1 ($0 \leq q_t \leq 1$).

(1) $$x_t = x(\xi_t, \psi_t, q_t) \qquad \text{with: } \frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \psi}, \frac{\partial x}{\partial q} > 0 \quad \text{(production function, sector 1)}$$

with:
- x : sector 1 output per capita (equal to consumption of sector 1 goods)
- ξ : per capita human capital based on learning-by-doing in production sector 1
- ψ : p.c. experience based on learning-by-doing in production sector 2
- q : share of time potential spent on working in sector 1 (with $0 \leq q \leq 1$)
- t : index of time

Per capita output y_t in sector 2 is primary based on p.c. “experience” ψ_t in sector 2 production, spill-over of “experience” ξ_t in sector 1, and the share ω_t of the time potential which is spent on working in sector 2.

(2) $$y_t = y(\psi_t, \xi_t, \omega_t) \qquad \text{with: } \frac{\partial y}{\partial \psi}, \frac{\partial y}{\partial \xi}, \frac{\partial y}{\partial \omega} > 0 \quad \text{(production function, sector 2)}$$

with:
- y : sector 2 output per capita (equal to consumption of sector 2 goods)
- ω : share of time spent on working in sector 2 [with $0 \leq \omega_t \leq 1$ and $0 \leq (q_t + \omega_t) \leq 1$]

Experience ξ_t in sector 1 is accumulated based on ‘new learning’ $a[x_t(.)]$ due to production activity x_t during working time q_t, minus a depreciation (i.e. via forgetting) with a constant rate μ. Analogously, sector 2 experience ψ_t is accumulated by new learning $b[y_t(.)]$ based on production y_t during working time ω_t in sector 2. For simplicity, depreciation rate μ on human capital is the same for both sectors:

(3) $$\mathring{\xi}_t \equiv \frac{d\xi}{dt} = a[x(\xi_t, \psi_t, q_t)] - \mu \cdot \xi_t \qquad \text{with: } \frac{\partial a}{\partial x} > 0 \quad \text{(experience accum. in sect. 1)}$$

(4) $$\mathring{\psi}_t \equiv \frac{d\omega}{dt} = b[y(\psi_t, \xi_t, \omega_t)] - \mu \cdot \psi_t \qquad \text{with: } \frac{\partial b}{\partial y} > 0 \quad \text{(experience accum. in sect. 2)}$$

with a[.] : learning per capita in sector 1 as a function of output x
b[.] : learning per capita in sector 2 as a function of output y
μ : depreciation rate (i.e. unlearning) $(0 \leq \mu \leq 1)$
° : derivative with respect to time [i.e. (d(.)/dt)]

Dynamic optimisation is done via determining the time path of the working time in both sectors, q_t and ω_t, based on a representative individual's time separable utility function. Utility comes from the p.c. consumption of the goods produced in both sectors and from leisure. Since leisure time is the residual of working time the share of leisure time is $(1-q_t-\omega_t)$. Overall utility U is the intertemporal aggregation of instantaneous utility at time t (u_t) applying the rate of time preference ρ as the discount rate:

$$U = \int_0^\infty u_t[\, x_t\,,\, y_t\,,\, (q_t+\omega_t)\,] \cdot e^{-\rho\cdot t}\, dt \qquad \text{with: } \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} > 0\,,\ \frac{\partial u}{\partial (q_t+\omega_t)} < 0 \tag{5}$$

The (present-value) Hamiltonian (6), the first order optimality conditions (7) and (8), the motion of the costates, (9) and (10), i.e. of the shadow prices, λ_1 for experience ξ_t in sector 1, and λ_2 for ψ_t of sector 2, and the transversality conditions (11) are:

$$H = u_t \cdot e^{-\rho\cdot t} + \lambda_1(t)\cdot \overset{\circ}{\xi}_t + \lambda_2(t)\cdot \overset{\circ}{\psi}_t \quad \Rightarrow \tag{6}$$

$$H = u_t[x(\xi_t,\psi_t,q_t),\, y(\psi_t,\xi_t,\omega_t),\, (q_t+\omega_t)]\cdot e^{-\rho\cdot t}$$

$$+\, \lambda_1(t)\cdot\{\, a[x(\xi_t,\psi_t,q_t)] - \mu\cdot\xi_t \,\} \;+\; \lambda_2(t)\cdot\{\, b[y(\psi_t,\xi_t,\omega_t)] - \mu\cdot\psi_t \,\}$$

$$\frac{\partial H}{\partial q} = 0: \qquad \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial q} + \frac{\partial u}{\partial q}\right)\cdot e^{-\rho\cdot t} + \lambda_1\cdot\left(\frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial q}\right) = 0 \tag{7}$$

$$\Rightarrow \; (7') \quad \lambda_1 = e^{-\rho\cdot t}\cdot\frac{-du_q}{da_q} \qquad \text{with: } du_q \equiv \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial q} + \frac{\partial u}{\partial q} \quad \text{and} \quad da_q \equiv \frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial q}$$

$$\frac{\partial H}{\partial \omega} = 0: \qquad \left(\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \omega} + \frac{\partial u}{\partial \omega}\right)\cdot e^{-\rho\cdot t} + \lambda_2\cdot\left(\frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial \omega}\right) = 0 \quad \Rightarrow \tag{8}$$

$$\Rightarrow \; (8') \quad \lambda_2 = e^{-\rho\cdot t}\cdot\frac{-du_\omega}{db_\omega} \qquad \text{with: } du_\omega \equiv \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \omega} + \frac{\partial u}{\partial \omega} \quad \text{and} \quad db_\omega \equiv \frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial \omega}$$

$$\frac{\partial H}{\partial \xi} = -\overset{\circ}{\lambda}_1 \qquad = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}\cdot e^{-\rho\cdot t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi}\cdot e^{-\rho\cdot t} + \lambda_1\cdot\left(\frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial \xi} - \mu\right) + \lambda_2\cdot\left(\frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial \xi}\right) \tag{9}$$

$$\frac{\partial H}{\partial \psi} = -\overset{\circ}{\lambda}_2 \qquad = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \psi}\cdot e^{-\rho\cdot t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \psi}\cdot e^{-\rho\cdot t} + \lambda_1\cdot\left(\frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial \psi}\right) + \lambda_2\cdot\left(\frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial \psi} - \mu\right) \tag{10}$$

(11) $$\lim_{t\to\infty}(\lambda_1(t)\cdot\xi_t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty}(\lambda_2(t)\cdot\psi_t) = 0$$

With an interpretation of the mathematical terms in eqs. (7), (7'), (8), and (8'):

$\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial q} > 0$:	marginal instantaneous utility from an increase in sector 1 production/consumption due to more working time
$\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \omega} > 0$:	marginal instantaneous utility from an increase in sector 2 production/consumption due to more working time
$\frac{\partial u}{\partial q} < 0$:	dis-utility of less leisure time (due to more working in sector 1)
$\frac{\partial u}{\partial \omega} < 0$:	dis-utility of less leisure time (due to more working in sector 2)
$e^{-\rho\cdot t} > 0$:	discounting future utility (↔ 'present value')
$\lambda_1 > 0$:	shadow price of sector 1 experience (with implicit discounting of future utility)
$\lambda_2 > 0$:	shadow price (= costate) of sector 2 experience
$\frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial q} > 0$:	marginal sector 1 learning due to additional working time
$\frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial \omega} > 0$:	marginal sector 2 learning due to additional working time

The effects of an increase in working time (of q or ω) on learning are positive: $da_q > 0$ and $db_\omega > 0$. Due to this positive intertemporal externality of working & learning, a negative momentary marginal utility of working is dynamically optimal ($du_q < 0$ and $du_\omega < 0$). In contrast, a purely static optimisation of leisure would imply marginal utility at every moment instantaneously to be balanced, i.e. $du_q = du_\omega = 0$.

Using eqs. (7') and (8'), the movement of the costates (9) and (10) can be reformulated as growth rates of the shadow prices ($\hat{\lambda}_1$, $\hat{\lambda}_2$) in order to reveal the net internal marginal rates of return to both human capital stocks, i.e. the rate of return to sector 1 experience (r_ξ) and to sector 2 experience (r_ψ). (A hat '^' indicates a variable's growth rate.)

(12) $$r_\xi = -\hat{\lambda}_1 \equiv \frac{-\mathring{\lambda}_1}{\lambda_1} = \frac{e^{-\rho\cdot t}}{\lambda_1}\cdot\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi}\right) + \frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\lambda_2}{\lambda_1}\cdot\frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial \xi} - \mu$$

$$\Rightarrow \quad r_\xi = \frac{du_\xi\cdot da_q}{-du_q} + \frac{du_\omega\cdot db_\xi\cdot da_q}{db_\omega\cdot du_q} + da_\xi - \mu$$

$$\text{with: } du_\xi \equiv \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi}\right), \quad da_\xi \equiv \frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial \xi} \quad \text{and} \quad db_\xi \equiv \frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial \xi}$$

$$(13)\qquad r_\psi = -\hat{\lambda}_2 \equiv \frac{-\mathring{\lambda}_2}{\lambda_2} = \frac{e^{-\rho\cdot t}}{\lambda_2}\cdot\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\psi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\psi}\right)+\frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial\psi}+\frac{\lambda_1}{\lambda_2}\cdot\frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial\psi}-\mu$$

$$\Rightarrow\quad r_\psi = \frac{du_\psi\cdot db_\omega}{-du_\omega}+\frac{du_q\cdot da_\psi\cdot db_\omega}{da_q\cdot du_\omega}+db_\psi-\mu$$

$$\text{with: } du_\psi \equiv \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\psi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\psi}\right),\quad da_\psi \equiv \frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial\psi}\ \text{ and }\ db_\psi \equiv \frac{\partial b}{\partial y}\cdot\frac{\partial y}{\partial\psi}$$

The derivative of the optimality conditions (7) and (8) with respect to time (using $\mathring{\lambda}_1$, $\mathring{\lambda}_2$, $\mathring{\xi}$, and $\mathring{\psi}$) leads to differential equations for the two control variables, i.e. the working time in both sectors as: $(dq/dt) \equiv \mathring{q} = \mathring{q}(\xi,\psi,q,\omega)$ and $(d\omega/dt) \equiv \mathring{\omega} = \mathring{\omega}(\xi,\psi,q,\omega)$. In combination with the motions of the state variables ($\mathring{\xi}$ and $\mathring{\psi}$), the dynamics of the economy are determined by a system of four differential equations.

The first steps of this procedure are as follows: The derivative of (7') and (8') with respect to time lead to eqs. (14) and (15). E.g. the growth rate $\hat{\lambda}_1$ of the shadow price of experience in sector 1 is determined by the growth rate of marginal utility related to changes in sector 1 working time q minus the growth of marginal learning due to changes of q (i.e. $d\hat{u}_q$ – $d\hat{a}_q$). The growth rate $\hat{\lambda}_2$ of the shadow price of sector 2 experience is analogously determined based on sector 2 working time ω:

$$(14)\qquad \hat{\lambda}_1 = \frac{d\mathring{u}_q}{du_q}-\frac{d\mathring{a}_q}{da_q}-\rho = d\hat{u}_q - d\hat{a}_q - \rho$$

$$(15)\qquad \hat{\lambda}_2 = \frac{d\mathring{u}_\omega}{du_\omega}-\frac{d\mathring{b}_\omega}{db_\omega}-\rho = d\hat{u}_\omega - d\hat{b}_\omega - \rho$$

Combining (12) and (14) leads to a Ramsey rule analogy of dynamically optimal working in sector 1: an optimal decision on sector 1 working time q implies that the net return to experience ξ [LHS of (16)] has to cover the sum of the discount rate and the 'shrinking rate' (i.e. negative growth rate) of marginal utility of working time [RHS of (16)]. The decrease of marginal utility consists of the decrease of momentary marginal 'utility' of working (– $d\hat{u}_q$) extended by the growth rate of the marginal effects on learning ($d\hat{a}_q$). The net rate of return to experience (r_ξ) is based on three valuable effects: <1> on learning in sector 1 related to the marginal sector 1 productivity of experience (da_ξ), <2> on a “cross-sectional” learning spill-over related to marginal ξ-productivity in the other sector (db_ξ), which has to be converted by the relation (λ_2/λ_1), and <3> on a direct marginal utility effect of experience (du_ξ) due to more production/consumption based on marginal ξ-productivity in both sectors, corrected by a discounting effect ($e^{-\rho\cdot t}$) in order to determine the present value of du_ξ, and related by the shadow price of sector 1 experience (λ_1):

$$(16)\qquad da_\xi + \frac{\lambda_2}{\lambda_1}\cdot db_\xi + \frac{e^{-\rho\cdot t}\cdot du_\xi}{\lambda_1} - \mu \;=\; \rho - d\hat{u}_q + d\hat{a}_q \quad = \; r_\xi$$

Combining (13) and (15) leads to an analogous 'Ramsey rule' of dynamically optimal working ω in sector 2:

$$(17)\qquad db_\psi + \frac{\lambda_1}{\lambda_2}\cdot da_\psi + \frac{e^{-\rho\cdot t}\cdot du_\psi}{\lambda_2} - \mu \;=\; \rho - d\hat{u}_\omega + d\hat{b}_\omega \quad = \; r_\psi$$

3. Cobb-Douglas production, logarithmic utility and dynamic optimisation

As a simple example, a Cobb-Douglas type version of per capita production and log utility is presented. Both sectors are symmetrically modelled concerning production elasticities and learning efficiency (time index "t" is omitted):

(18) $x = q\cdot\xi^{\alpha}\cdot\psi^{(1-\alpha)}$ (C.D.-type sector 1 p.c. production function)

(19) $y = \omega\cdot\xi^{(1-\alpha)}\cdot\psi^{\alpha}$ (C.D.-type sector 2 per capita production)

(20) $\mathring{\xi} = \phi\cdot x - \mu\cdot\xi$ (p.c. learning by-doing in sector 1)

(21) $\mathring{\psi} = \phi\cdot y - \mu\cdot\psi$ (change of sector 2 experience p.c.)

(22) $\theta \equiv \dfrac{\xi}{\psi}$ (sector1-to-sector2-experience ratio, "1-to-2-intensity")

(23) $\Rightarrow\; \mathring{\theta} = \phi\cdot[q\cdot\theta^{\alpha} - \omega\cdot\theta^{(2-\alpha)}]$ (change of 1-to-2-intensity)

with: α : production elasticity of experience in the same sector ($0.5 \le \alpha \le 1$)
$(1-\alpha)$: cross-sectoral production elasticity of experience (as spill-over to the other sector)
ϕ : productivity parameter in the learning function ($\phi > 0$)

The sum of elasticities of both accumulated factors is assumed to be exactly one $[\alpha + (1-\alpha) = 1]$. This ensures non-diminishing returns to both human capital stocks as a necessary condition for sustained endogenous growth.

A logarithmic instantaneous utility u_t, concerning the *momentary* choice between leisure time$(1-q_t-\omega_t)$ and consumption of both types of goods (x_t, y_t) and at *each* point t in time is assumed:[3]

$$(24)\qquad U = \int_0^\infty u_t \cdot e^{-\rho\cdot t}\, dt$$

$$\text{with}\quad u_t = \frac{b}{2}\cdot \ln(x_t) + \frac{b}{2}\cdot \ln(y_t) + (1-b)\cdot \ln(1-q_t-\omega_t) \quad \text{and} \quad 0 < b \leq 1$$

The Hamiltonian and the FOCs of our problem are:

$$(25)\qquad H = [\, \frac{b}{2}\cdot \ln(q\cdot\xi^{\alpha}\cdot\psi^{(1-\alpha)}) + \frac{b}{2}\cdot \ln(\omega\cdot\xi^{(1-\alpha)}\cdot\psi^{\alpha}) + (1-b)\cdot \ln(1-q_t-\omega_t)\,]\cdot e^{-\rho\cdot t}$$
$$+ \lambda_1(t)\cdot [\, \phi\cdot q\cdot\xi^{\alpha}\cdot\psi^{(1-\alpha)} - \mu\cdot\xi\,]$$
$$+ \lambda_2(t)\cdot [\, \phi\cdot \omega\cdot\xi^{(1-\alpha)}\cdot\psi^{\alpha} - \mu\cdot\psi\,]$$

$$(26)\qquad \frac{\partial H}{\partial q} = 0 \;=\; e^{-\rho\cdot t}\cdot\left(\frac{b-1}{1-q-\omega} + \frac{b}{2q}\right) + \lambda_1\cdot\frac{\phi\cdot x}{q} \quad \Rightarrow$$

$$(26')\qquad \lambda_1 = \left(\frac{b}{2} - \frac{q\cdot(1-b)}{1-q-\omega}\right)\cdot\frac{e^{-\rho\cdot t}}{\phi\cdot x}$$

$$(27)\qquad \frac{\partial H}{\partial \omega} = 0 \;=\; e^{-\rho\cdot t}\cdot\left(\frac{b-1}{1-q-\omega} + \frac{b}{2\omega}\right) + \lambda_2\cdot\frac{\phi\cdot y}{\omega} \quad \Rightarrow$$

$$(27')\qquad \lambda_2 = \left(\frac{b}{2} - \frac{\omega\cdot(1-b)}{1-q-\omega}\right)\cdot\frac{e^{-\rho\cdot t}}{\phi\cdot y}$$

The motions of the costates in the C.-D.-production / log-utility example are:

$$(28)\qquad \frac{\partial H}{\partial \xi} = -\mathring{\lambda}_1 \;=\; \frac{b\cdot e^{-\rho\cdot t}}{2\xi} + \lambda_1\cdot\left(\frac{\alpha\cdot\phi\cdot x}{\xi} - \mu\right) + \lambda_2\cdot\frac{(1-\alpha)\cdot\phi\cdot y}{\xi}$$

$$(29)\qquad \frac{\partial H}{\partial \psi} = -\mathring{\lambda}_2 \;=\; \frac{b\cdot e^{-\rho\cdot t}}{2\psi} + \lambda_1\cdot\frac{(1-\alpha)\cdot\phi\cdot x}{\psi} + \lambda_2\cdot\left(\frac{\alpha\cdot\phi\cdot y}{\psi} - \mu\right)$$

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3 Assuming a log-utility function avoids consumption saturation, where productivity increases are mainly utilised for an expansion of leisure time. Thus, for log-utility endogenous growth is not limited by the demand side. For a discussion of this problem of growth models allowing the choice between leisure and consumption see Baldassarri/DeSantis/Moscarini (1994) and Göcke (2002).

Via the costate motions (28) and (29) the marginal net rate of return to both sector's human capital (r_ξ and r_ψ) can be calculated, and reformulated using the 1-to-2-intensity $\theta \equiv (\xi/\psi)$:

$$r_\xi = -\hat{\lambda}_1 = \frac{2\cdot\phi\cdot x\cdot(1-b)\cdot[(1-\alpha)\cdot\omega+\alpha\cdot q]}{\xi\cdot(2q-b\cdot q-b+b\cdot\omega)} - \mu = \tag{30}$$

$$= \frac{2\cdot\phi\cdot q\cdot(1-b)\cdot[(1-\alpha)\cdot\omega+\alpha\cdot q]}{\theta^{(1-\alpha)}\cdot(2q-b\cdot q-b+b\cdot\omega)} - \mu$$

$$r_\psi = -\hat{\lambda}_2 = \frac{2\cdot\phi\cdot y\cdot(1-b)\cdot[(1-\alpha)\cdot q+\alpha\cdot\omega]}{\psi\cdot(2\omega-b\cdot\omega-b+b\cdot q)} - \mu \tag{31}$$

$$= \frac{2\cdot\phi\cdot\omega\cdot\theta^{(1-\alpha)}\cdot(1-b)\cdot[(1-\alpha)\cdot q+\alpha\cdot\omega]}{2\omega-b\cdot\omega-b+b\cdot q} - \mu$$

The higher the rate of return to experience, the higher is the production in a sector, related to experience in the same sector's production (x/ξ and y/ψ) and the lower is the level of experience in a sector in relation to the level of experience of the other sector. Thus, a higher 1-to-2-intesinty θ results c.p. in a lower r_ξ and a higher r_ψ. Note that the rates of return are determined by the intensity θ and not by the levels of ξ and ψ.

Taking the derivative with respect to time of the costates (λ_1 and λ_2) as calculated based on the optimality conditions (26') and (27') lead to a second expression for the motion of each shadow price and for both rates of return:

$$\frac{d\lambda_1}{dt}\cdot\frac{-1}{\lambda_1} = \frac{-\mathring{\lambda}_1}{\lambda_1} = -\hat{\lambda}_1 = r_\xi \tag{32}$$

$$= \rho-\mu+\frac{(b-2)\cdot\mathring{q}-b\cdot\mathring{\omega}}{2q-b\cdot q-b+b\cdot\omega}+\theta^{(\alpha-1)}\cdot\alpha\cdot\phi\cdot q+\theta^{(1-\alpha)}\cdot(1-\alpha)\cdot\phi\cdot\omega+\frac{\mathring{q}}{q}-\frac{\mathring{q}+\mathring{\omega}}{1-q-\omega}$$

$$\frac{d\lambda_2}{dt}\cdot\frac{-1}{\lambda_2} = \frac{-\mathring{\lambda}_2}{\lambda_2} = -\hat{\lambda}_1 = r_\psi \tag{33}$$

$$= \rho-\mu+\frac{(b-2)\cdot\mathring{\omega}-b\cdot\mathring{q}}{2\omega-b\cdot\omega-b+b\cdot q}+\theta^{(1-\alpha)}\cdot\alpha\cdot\phi\cdot\omega+\theta^{(\alpha-1)}\cdot(1-\alpha)\cdot\phi\cdot q+\frac{\mathring{\omega}}{\omega}-\frac{\mathring{q}+\mathring{\omega}}{1-q-\omega}$$

Combining (30) with (32) and (31) with (33) – and some algebra – leads to 3 differential equations which completely determine the dynamics of the system. The dynamics of state of the system are summarised by the intensity $\mathring{\theta}(\theta_t,q_t,\omega_t)$, and the optimal motion of both working times as the control variables, $\mathring{q}(\theta_t,q_t,\omega_t)$ and $\mathring{\omega}(\theta_t,q_t,\omega_t)$. Since the explicit results are too extensive, a representation of $\mathring{q}$ and $\mathring{\omega}$ is skipped (see the Appendix).

The steady state under endogenous growth is characterised by a constant (non-zero) growth rate of both stocks ξ and ψ, by a constant intensity, and by constant control ratios. Thus, the conditions of a steady state are: a constant 1-to-2-intensity (θ_{st} with $\mathring{\theta}=0$), and constant working times (q_{st} with $\mathring{q}=0$, and ω_{st} with $\mathring{\omega}=0$). The use of these conditions leads to a steady state 1-to-2-intensity of exactly one and to an identical steady state working time in both sectors (as a consequence of the symmetric model structure):

$$(34)\quad \mathring{\theta}=0 \;\wedge\; \mathring{q}=0 \;\wedge\; \mathring{\omega}=0$$

$$\Rightarrow\quad \theta_{st}=1 \quad \text{and} \quad q_{st}=\omega_{st}=\frac{1}{4}-\frac{\rho}{2\cdot b\cdot\phi}+\frac{1}{2}\cdot\sqrt{\frac{1}{4}-\frac{\rho}{b\cdot\phi}+\frac{\rho^2}{b^2\cdot\phi^2}+\frac{2\rho}{\phi}}$$

From (20) and (21) the growth rate of the experience stock in sector 1 and 2 is:

$$(35)\quad \hat{\xi}\equiv\frac{\mathring{\xi}}{\xi} = \phi\cdot\frac{x}{\xi}-\mu = \phi\cdot q\cdot\xi^{(\alpha-1)}\cdot\psi^{(1-\alpha)}-\mu = \phi\cdot q\cdot\theta^{(\alpha-1)}-\mu$$

$$(36)\quad \hat{\psi}\equiv\frac{\mathring{\psi}}{\psi} = \phi\cdot\frac{y}{\psi}-\mu = \phi\cdot\omega\cdot\xi^{(1-\alpha)}\cdot\psi^{(\alpha-1)}-\mu = \phi\cdot\omega\cdot\theta^{(1-\alpha)}-\mu$$

From (18) and (19) the growth rate of the production flow in both sectors is:

$$(37)\quad \begin{aligned}\hat{x} &= \hat{q}+\alpha\cdot\hat{\xi}+(1-\alpha)\cdot\hat{\psi}\\ &= \hat{q}+\alpha\cdot\phi\cdot q\cdot\theta^{(\alpha-1)}+(1-\alpha)\cdot\phi\cdot\omega\cdot\theta^{(1-\alpha)}-\mu\end{aligned}$$

$$(38)\quad \begin{aligned}\hat{y} &= \hat{\omega}+(1-\alpha)\cdot\hat{\xi}+\alpha\cdot\hat{\psi}\\ &= \hat{\omega}+(1-\alpha)\cdot\phi\cdot q\cdot\theta^{(\alpha-1)}+\alpha\cdot\phi\cdot\omega\cdot\theta^{(1-\alpha)}-\mu\end{aligned}$$

With $q_{st}=\omega_{st}$, $\theta_{st}=1$, $\hat{q}_{st}=\hat{\omega}_{st}=0$ from (35), (36), (37) and (38) a common steady state growth rate of production flows and experience capital stock results in the long-run:

$$(39)\quad \begin{aligned}&\hat{x}_{st}=\alpha\cdot\phi\cdot q_{st}+(1-\alpha)\cdot\phi\cdot\omega_{st}-\mu = \phi\cdot q_{st}-\mu=\phi\cdot\omega_{st}-\mu = \hat{y}_{st}=\hat{\xi}_{st}=\hat{\psi}_{st}\\ &\Rightarrow\quad \hat{x}_{st},\hat{y}_{st}>0 \quad \text{if} \quad q_{st},\omega_{st}>\frac{\mu}{\phi}\end{aligned}$$

A positive endogenous growth with $\hat{x}_{st}=\hat{y}_{st}=\hat{\xi}_{st}=\hat{\psi}_{s}>0$ results if the net rate of return to both human capital stocks in the long-run stays above the (discount) rate of time preference ρ, since in this case the positive incentive to work in order to accumulate experience remains. From (32) and using $r_{\xi,st}>\rho$ a condition equivalent to (39) follows:

(40) $\quad r_{\xi,st} = \rho - \mu + \phi \cdot q_{st} > \rho \quad \Leftrightarrow \quad q_{st}, \omega_{st} > \frac{\mu}{\phi} \quad \Leftrightarrow \quad \hat{x}_{st} = \hat{y}_{st} = \hat{\xi}_{st} = \hat{\psi}_s > 0$

By applying $r_{\xi,st} > \rho$ to (30) this leads to:

(41) $\quad r_{\xi,st} = r_{\psi,st} = \frac{2 \cdot \phi \cdot (1-b) \cdot q_{st}^2}{(2 q_{st} - b)} - \mu > \rho$

A combination of (40) and (41) leads to the following condition for positive long-run growth:

(42) $\quad \phi > \phi_{crit} = \frac{2\mu \cdot (\rho + \mu \cdot b)}{b \cdot (\rho + \mu)}$

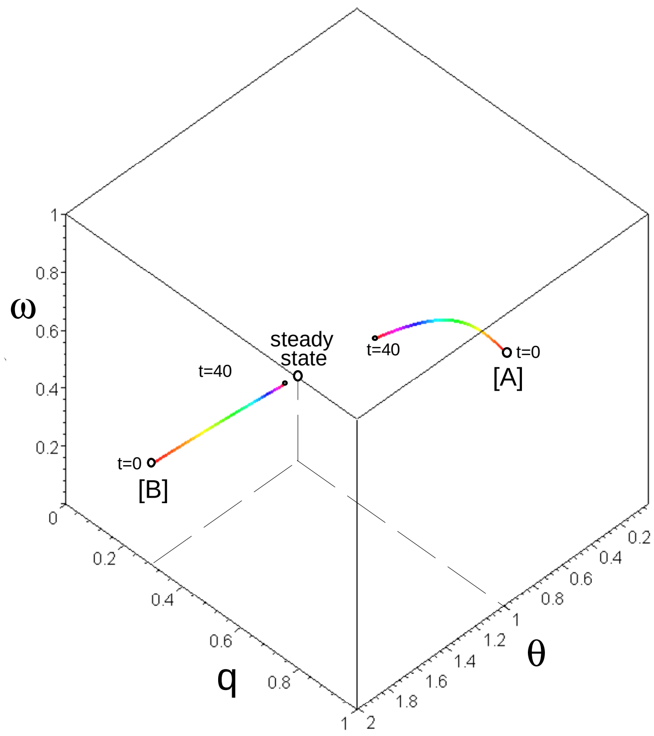
Thus, a high learning-productivity parameter $\phi > \phi_{crit}$ ensures in the long-run a net rate of return to experience above the discounting rate ρ, and thus an incentive to work long in order to accumulate more experience as a prerequisite for positive growth. This necessary/critical level of the learning productivity parameter ϕ_{crit} is the higher, the higher is the depreciation μ on experience, the higher is the rate of time preference ρ, and the lower is the weighting b of consumption goods, resp. the higher is the weight of leisure $(1-b)$, in the utility function.

Local stability surrounding the steady state of the entire system, set up by the three differential equations $\mathring{q}$, $\mathring{\omega}$ and $\mathring{\theta}$, can be analysed by means of the characteristic roots of the coefficient matrix of a first order Taylor expansion around the steady state. If adequate parameter values are chosen (e.g. excluding unbounded utility), the system shows one eigenvalue with a negative real part (corresponding to the predetermined variable θ) and two eigenvalues with a positive real part (related to both jump variables/controls q and ω). Since a typical optimal control problem with infinite horizon is analysed we observe saddle path stability.

4. A numerical simulation

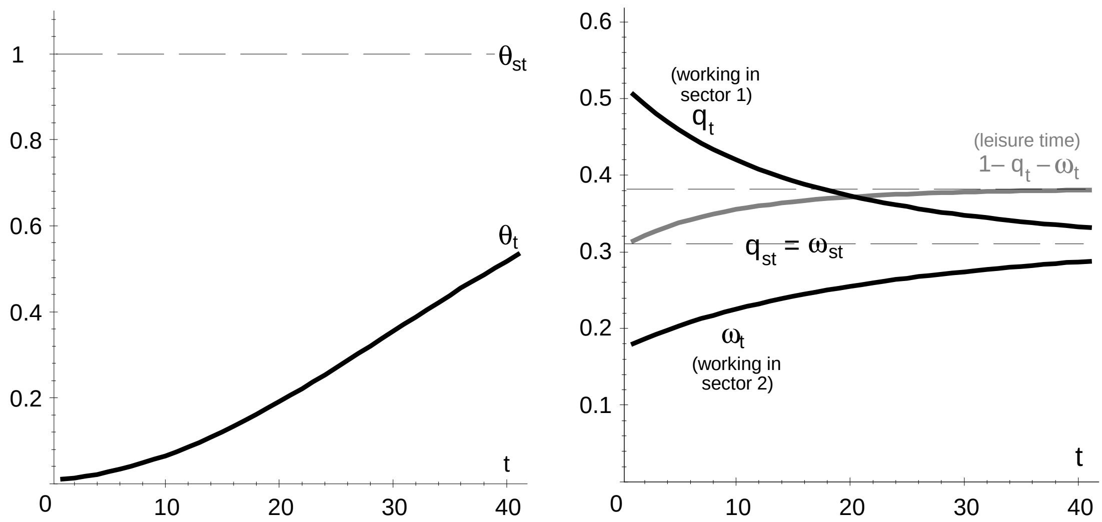
The parameters for the numerical example are $\phi = 1/5$; $b = 1/2$; $\rho = 1/10$; $\mu = 1/20$, $\alpha = 3/4$. The simulation was calculated with two alternative initial points on the saddle path: trajectory [A], starting with $\theta(t=0) = 0.01$ at a 1 percent level of the steady state intensity $\theta_{st} = 1$; and trajectory [B], with $\theta(t=0) = 1.99$ starting from above the steady state intensity. Finding initial points on the saddle path was done by using the time-elimination-method (see Barro & Sala-i-Martin, 1995, pp. 490 f.); for simulation the software MAPLE was used. The stable saddle path is depicted in Fig. 1 as a trajectory for 40 periods (t=0,..,40) in the (θ, q, ω)-space converging to the steady state point.

Fig. 1: Stable saddle path



Steady state: θ_{st} = 1 and $q_{st} = \omega_{st}$ = 0.30902 (⇒ $1–q_{st}–\omega_{st}$=0.38197);
trajectory [A]: θ(t=0) = 0.01 ⇒ q(t=0) = 0.50548 , ω(t=0) = 0.18044 ⇒ 1–q–ω(t=0) = 0.31408;
trajectory [B]: ψ(t=0) = 1.99 ⇒ q(t=0) = 0.28548 , ω(t=0) = 0.33438 ⇒ 1–q–ω(t=0) = 0.38014.

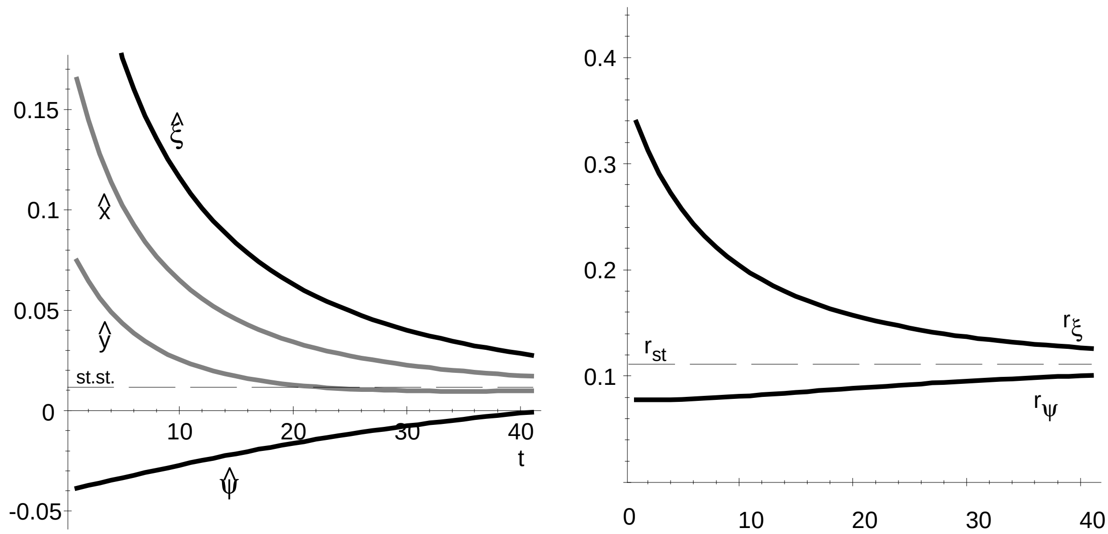
Fig. 2: Time paths of 1-to-2-intensity θ, of working time in sector 1 and 2, q and ω, and of leisure time (1–q–ω) for trajectory [A]



The dynamics are in Fig. 2 and Fig. 3 explicitly illustrated for trajectory [A] of the stable saddle path, with a relatively low level of experience in sector 1: θ(t=0) = 0.01. I.e.

sector 1 can be interpreted as an emerging ('new') sector with relatively scarce sectoral experience, and sector 2 is a traditional ('old') sector with 'a lot of' experience. The dynamics of the ratio $\theta \equiv (\xi/\psi)$ dominates the dynamics of the entire system. As eqs. (30), (31), (32), and (33) demonstrate, the marginal rate of return to both sector's experience capital is determined by the 1-to-2-intensity θ. For a low ratio $\theta < 1$ the rate of return r_ξ to (relatively scarce) experience in sector 1 is high (and above the steady state rate of return r_{st}), while the rate of return to experience in sector 2 production r_ψ is low (and below r_{st}). In the long run, a process of arbitrage is induced by dynamically optimal working time decisions, leading to a higher growth in sector 1 production compared to sector 2, which implies an increasing experience intensity θ, and leading to a convergence of both rates of return to the same level r_{st} (as an 'interest parity') of the two types of experience capital. Since even in the long run, the rate of return to both accumulated factors is above utility discounting via time preference ($r_{st} = 0.1180$ > $\rho = 0.1$), an incentive to accumulate experience via working long remains and, thus, an endogenous growth with a positive common steady state growth rate ($\hat{x}_{st} = \hat{y}_{st} = \hat{\xi}_{st} = \hat{\psi}_{st} = 0.0118$) results. The dynamics of the sectoral working times can be explained based on both rates of return. Since the return on experience r_ξ in sector 1 is initially high (due to a low ratio $\theta \equiv \xi/\psi$) working in sector 1 is more profitable, due to the implicit accumulation of the relatively scarce sector 1 experience ξ. Because of the high growth rate of production in sector 1, the scarcity of ξ is reduced and r_ξ converges to r_{st}. The very high growth rate $\hat{\xi}$ of experience in the emerging/'new' sector 1 is the source of a high growth $\hat{x}$ in this sector, however, due to a spill-over of this experience growth to the traditional/'old' sector during the period of rapid sector 1 growth, even sector 2 growth is above the common steady state growth rate $\hat{y}_{st}$. The initially high sector 1 return r_ξ leads to a high share of overall working time $(q+\omega)$ and, consequently, to a low level of leisure time $(1-q-\omega)$, which later increases during the convergence towards steady state.

Fig. 3: Time paths of growth rate of production, of sector specific experience growth, and of the rates of return in both sectors



Steady state: $\hat{x}_{st} = \hat{y}_{st} = \hat{\xi}_{st} = \hat{\psi}_{st} = 0.01180$; $r_{st} = 0.11180$.

In our simple setting there is no constraint of shifting labour time to the 'new' sector 1 with its high learning potential due to the low level of ξ-experience. This is of course an extreme assumption about the intersectoral flexibility in the employment of labour. Thus, limitations of the number of working places in sector 1 or the costs of changing the intersectoral production structure, e.g. due to a limited endowment with sector specific machines are neglected.

5. Conclusion

A macroeconomic growth model based on two production sectors with sectoral learning-by-doing is presented. An experience curve analogy is applied, i.e. accumulation of sector specific human capital is measured by means of a sector's output aggregated over time. The growth of the economy is controlled by the dynamically optimal decision on working versus leisure time, and on the distribution of working time between both production sectors. Utility is based on consumption of the goods produced in both sectors and on leisure time. Including the future effects of higher productivity due to learning-by-doing, i.e. accumulation of experience due to working, c.p. increases the dynamically optimal working time on the costs of leisure. The long-run steady state growth as well as the transitional dynamics are driven by the marginal rates of return to human capital (experience) in the two sectors. These sector-specific rates of return are determined by the relative size of both experience stocks, i.e. by the ratio θ of experience in sector 1 relative to experience in sector 2. If e.g. the sector 1 is an emerging ('new') sector with a relatively low experience stock compared to a traditional ('old') sector 2, the partial marginal return to experience in sector 1 is above its long run steady state level. This high marginal rate of return reflects an extra incentive to shift labour and to increase production in the 'new' sector, in order to learn and accumulate the relatively scare (and thus productive) type of experience of the emerging sector. Consequently, time usage is shifted in favour of working in the 'modern' sector (while labour dedicated to the 'old' sector and leisure time is reduced). During this transitional process the 'new' sector's output growth rate is above the long run steady state growth level. Due to a spill-over of the rapid human capital accumulation in the 'new' sector to the other sector, even the 'old/traditional' sector may experience growth rates above the steady state level. These processes may represent situations after important technological innovations were broadly disseminated, founding new/emerging sectors in the economy (e.g., as the starting point of industrialisation, the introduction of the steam machine, or as a modern example, the increasing prevalence of digital communication and information technology). In these circumstances a (rapid) shift of the workforce towards the new sector and a rapid learning of the new technology are the base of an exceptionally high transitional output growth of the whole economy. Of course, the simple set up of our model with (sectoral) learning as the sole source of growth is unrealistic in order to describe the entire growth dynamics. Especially the omission of real capital accumulation via saving and investment in the 'new/modern' sector is a shortcoming. An explicit inclusion of a creation of workplaces with a sufficient endowment of sector specific real capital/machines as a prerequisite of structural change will result in a setting where economic growth and the speed of change towards 'new/modern' sectors may be slowed down compared to our model, where there are no limits of shifting resources (working time) between sectors.

References

Arrow, K.J. (1962), The Economic Implications of Learning by Doing, Review of Economic Studies, 29, 155-173.

Baldassarri, M., DeSantis, P., Moscarini, G. (1994), Allocation of Time, Human Capital and Endogenous Growth, Baldassarri, M., Paganetto, L., Phelps, E.S. (eds.), International Differences in Growth Rates, Basingstoke, Hampshire, 95-110.

Barro, R.J., Sala-i-Martin, X. (1995), Economic Growth, New York u.a.

Chamley, C. (1993), Externalities and Dynamics in Models of 'Learning or Doing' , International Economic Review, 34/3, 583-609.

de Hek, P.A. (1998), An Aggregative Model of Capital Accumulation with Leisure-dependent Utility., Journal of Economic Dynamics and Control 23: 255-276.

Göcke, M. (2002), Leisure versus Learning-by-doing – Saturation Effects and Utility-side Limits to Endogenous Growth, Economic Modelling, 19, 585-609.

Göcke, M. (2004), Learning-by-doing versus Schooling in a Model of Endogenous Growth, Journal of Economics, 83/1, 47-69.

Göcke, M. (2010), A One-Sector Model with Learning-by-doing, Investment, Leisure, and Optimal Growth, MAGKS – Joint Discussion Paper Series in Economics by the Universities of Marburg, Aachen, Gießen, Göttingen, Kassel & Siegen, No. 36-2010.

Greiner, A. (1996), Endogeneous Growth Cycles: Arrow's Learning by Doing Reconsidered, Journal of Macroeconomics, 18/4, 587-604.

Greiner, A. (2003), On the Dynamics of an Endogenous Growth Model with Learning by Doing, Economic theory, 21/1, 205-214

Ladrón-de-Guevara, A., Ortigueira, S., Santos, M.S. (1997), Equilibrium Dynamics in Two-Sector Models of Endogenous Growth, Journal of Economic Dynamics and Control. , 21, 115-143.

Lucas, R.E. (1988), On the Mechanics of Economic Development, Journal of Monetary Economics, 22/1, 3-42.

Lucas, R.E. (1993), Making a Miracle, Econometrica, 61, 251-273.

Romer, P. M. (1986), Increasing Returns and Long-Run Growth, Journal of Political Economy, 94/5, 1002-1037.

Rosen, S. (1972), Learning by Experience as Joint Production, Quartely Journal of Economics, 86, 366-382.

Uzawa, H. (1965), Optimum Technical Change in an Aggregate Model of Economic Growth, International Economic Review, 6, 18-31.

Appendix:

MAPLE-Output for the optimal motion of both working times as the control variables, $\mathring{q}(\theta_t,q_t,\omega_t)$ and $\mathring{\omega}(\theta_t,q_t,\omega_t)$.

$$\frac{d(q)}{dt} = q\,(-2\,\omega\,\theta^{(-1+\alpha)}\,b\,\phi\,q - 3\,\omega^2\,\theta^{(1-\alpha)}\,b^2\,\alpha\,\phi + 6\,q^2\,b\,\theta^{(-1+\alpha)}\,\omega\,\phi + 2\,q^2\,b\,\theta^{(-1+\alpha)}\,\omega^2\,\phi + 3\,\omega^2\,\theta^{(-1+\alpha)}\,b^2\,\alpha\,\phi\,q +$$

$$- b^2\,\theta^{(1-\alpha)}\,\phi\,\omega^4 - 4\,q\,\theta^{(1-\alpha)}\,\omega^3\,\alpha\,\phi - 4\,q^2\,\theta^{(1-\alpha)}\,\omega^2\,\alpha\,\phi + 4\,\omega^2\,\theta^{(-1+\alpha)}\,b\,\phi\,q + 2\,q^3\,b\,\theta^{(1-\alpha)}\,\phi\,\omega - q\,b^2\,\theta^{(1-\alpha)}\,\omega$$

$$+ q\,b^2\,\theta^{(1-\alpha)}\,\omega\,\phi - \omega\,\theta^{(1-\alpha)}\,b^2\,q^2\,\alpha\,\phi - 2\,q\,b\,\theta^{(1-\alpha)}\,\omega\,\alpha\,\phi + q^2\,b^2\,\theta^{(1-\alpha)}\,\omega\,\phi - q^2\,b^2\,\theta^{(-1+\alpha)}\,\omega^2\,\alpha\,\phi - 4\,q^2\,b\,\theta$$

$$+ \omega\,\theta^{(1-\alpha)}\,b^2\,\alpha\,\phi - 4\,q\,b\,\theta^{(1-\alpha)}\,\phi\,\omega^3 - q^3\,b^2\,\theta^{(1-\alpha)}\,\phi\,\omega + 3\,q^2\,b^2\,\theta^{(-1+\alpha)}\,\alpha\,\phi + q^3\,b^2\,\theta^{(1-\alpha)}\,\alpha\,\omega\,\phi - \theta^{(-1+\alpha)}\,b^2\,\alpha$$

$$- 2\,\omega^3\,\theta^{(-1+\alpha)}\,b^2\,\phi\,q + \omega^3\,\theta^{(-1+\alpha)}\,b^2\,\alpha\,\phi\,q + 2\,q\,b\,\theta^{(1-\alpha)}\,\omega\,\phi - q\,b^2\,\theta^{(1-\alpha)}\,\omega\,\alpha\,\phi - 6\,q^2\,b\,\theta^{(-1+\alpha)}\,\omega\,\alpha\,\phi + 2\,\omega\,\theta$$

$$+ 6\,\omega^3\,\theta^{(-1+\alpha)}\,b\,\phi\,q + 4\,q^3\,b^2\,\theta^{(-1+\alpha)}\,\omega\,\phi + 4\,q\,\theta^{(-1+\alpha)}\,\omega^3\,\alpha\,\phi + 2\,q^2\,\theta^{(1-\alpha)}\,\alpha\,\phi\,\omega^2\,b + 2\,b\,\theta^{(1-\alpha)}\,\omega^3\,\alpha\,\phi + 2\,q\,\theta$$

$$- \omega\,\theta^{(-1+\alpha)}\,b^2\,\alpha\,q^3\,\phi + 4\,\omega\,\theta^{(-1+\alpha)}\,b\,\alpha\,q^3\,\phi + 4\,q^2\,\theta^{(-1+\alpha)}\,\alpha\,\omega^2\,\phi + 6\,q\,b\,\theta^{(1-\alpha)}\,\omega^2\,\alpha\,\phi + \omega^3\,\theta^{(1-\alpha)}\,b^2\,\alpha\,\phi - 6\,q^2\,b$$

$$- 4\,q^2\,b\,\theta^{(1-\alpha)}\,\omega^2\,\phi + 3\,b^2\,\theta^{(1-\alpha)}\,\omega^2\,\phi - b^2\,\theta^{(1-\alpha)}\,\phi\,\omega^3 - 2\,b\,\theta^{(1-\alpha)}\,\phi\,\omega^3 + 2\,b\,\theta^{(1-\alpha)}\,\phi\,\omega^4 + 4\,q\,\theta^{(1-\alpha)}\,\omega^3\,\phi - 4$$

$$+ 2\,q\,b\,\rho + q\,b^2\,\rho - 4\,q\,b^2\,\rho\,\omega - 4\,\omega^2\,\theta^{(-1+\alpha)}\,b^2\,\phi\,q + 3\,b^2\,\rho\,\omega - 2\,b\,\rho\,\omega^2 - b^2\,\rho - 2\,q\,b\,\rho\,\omega - b^2\,\rho\,\omega^3 - b^2\,\rho\,\omega^2 - 4\,q^3\,b\,\theta$$

$$+ 4\,q^2\,\theta^{(1-\alpha)}\,\omega^2\,\phi - 4\,q^2\,\theta^{(-1+\alpha)}\,\omega^2\,\phi + 2\,b\,\rho\,\omega^3 + 2\,q^3\,b\,\rho - 4\,q^2\,b\,\rho + q\,b^2\,\rho\,\omega^2 + q^2\,b^2\,\rho\,\omega - q^3\,b^2\,\rho + \omega\,\theta^{(-1+\alpha)}\,b^2$$

$$+ 3\,q\,b^2\,\theta^{(1-\alpha)}\,\phi\,\omega^3 + q^2\,b^2\,\rho + 2\,q^2\,b\,\rho\,\omega) \Big/ (b\,(-1 + q + \omega)\,(-2\,q^2 + b\,q^2 + 2\,b\,q - 2\,b\,q\,\omega - b - 2\,\omega^2 + b\,\omega^2 + 2\,b\,\omega))$$

$$\frac{d(\omega)}{dt} = \omega\,(-2\,q^4\,b\,\theta^{(-1+\alpha)}\,\alpha\,\phi + 2\,\theta^{(-1+\alpha)}\,b\,\alpha\,q^3\,\phi - 6\,\omega\,\theta^{(1-\alpha)}\,b\,q^2\,\alpha\,\phi - q\,b^2\,\theta^{(1-\alpha)}\,\omega^3\,\alpha\,\phi + 2\,\omega\,b\,\rho - 2\,\omega\,\theta^{(-1+\alpha)}\,b$$

$$+ 4\,\omega^2\,\theta^{(-1+\alpha)}\,b\,\alpha\,\phi\,q - q^2\,b^2\,\theta^{(-1+\alpha)}\,\omega^2\,\alpha\,\phi + 3\,q^3\,b^2\,\theta^{(-1+\alpha)}\,\omega\,\phi + 4\,q\,b^2\,\theta^{(1-\alpha)}\,\phi\,\omega^3 - 6\,q\,b^2\,\theta^{(1-\alpha)}\,\omega^2\,\phi - \omega^3$$

$$- \omega\,\theta^{(-1+\alpha)}\,b^2\,\alpha\,q\,\phi + 2\,q^2\,b\,\theta^{(1-\alpha)}\,\omega^2\,\phi - \omega\,\theta^{(1-\alpha)}\,b^2\,\alpha\,\phi - 4\,q^3\,\theta^{(-1+\alpha)}\,\omega\,\alpha\,\phi - 4\,q^2\,b\,\theta^{(-1+\alpha)}\,\omega^2\,\phi - 3\,\omega^3\,\theta$$

$$+ 4\,q\,\theta^{(1-\alpha)}\,\omega^3\,\alpha\,\phi\,b - 4\,q^3\,b\,\theta^{(1-\alpha)}\,\alpha\,\omega\,\phi + 2\,q^2\,b^2\,\theta^{(1-\alpha)}\,\omega^2\,\phi - \omega^2\,\theta^{(-1+\alpha)}\,b^2\,\alpha\,\phi\,q + 6\,q^3\,b\,\theta^{(1-\alpha)}\,\phi\,\omega + 4\,q^2\,b\,\theta$$

$$+ \omega\,\theta^{(-1+\alpha)}\,b^2\,\phi\,q - 2\,q\,b\,\theta^{(1-\alpha)}\,\omega\,\phi + \omega^3\,\theta^{(-1+\alpha)}\,b^2\,\alpha\,\phi\,q - 4\,q^3\,b\,\theta^{(-1+\alpha)}\,\omega\,\phi - 4\,q\,b\,\theta^{(1-\alpha)}\,\phi\,\omega^3 + 2\,\omega^3\,\theta$$

$$+ 2\,\omega\,\theta^{(-1+\alpha)}\,b\,\phi\,q + \omega^2\,\theta^{(-1+\alpha)}\,b^2\,\phi\,q - 2\,\omega^3\,\theta^{(-1+\alpha)}\,b\,\alpha\,\phi\,q + 3\,\omega^2\,\theta^{(1-\alpha)}\,b^2\,\alpha\,\phi + 6\,q^2\,b\,\theta^{(-1+\alpha)}\,\omega\,\alpha\,\phi + 4\,q^2\,\theta$$

$$- 4\,\omega^2\,\theta^{(-1+\alpha)}\,b\,\phi\,q - 4\,q^2\,b\,\theta^{(-1+\alpha)}\,\omega\,\phi - 6\,q\,b\,\theta^{(1-\alpha)}\,\omega^2\,\alpha\,\phi - 2\,q^2\,b^2\,\theta^{(-1+\alpha)}\,\omega\,\phi - \omega^2\,\theta^{(1-\alpha)}\,b^2\,q^2\,\alpha\,\phi + \theta$$

$$- 2\,q^3\,b^2\,\theta^{(1-\alpha)}\,\phi\,\omega + 2\,q\,b\,\theta^{(1-\alpha)}\,\omega\,\alpha\,\phi + 3\,\omega\,\theta^{(1-\alpha)}\,b^2\,q^2\,\alpha\,\phi - 3\,q^2\,b^2\,\theta^{(-1+\alpha)}\,\alpha\,\phi - 4\,q^2\,b^2\,\theta^{(1-\alpha)}\,\omega\,\phi + q^4\,b^2\,\theta$$

$$+ 4\,q^3\,\theta^{(1-\alpha)}\,\alpha\,\phi\,\omega + 2\,q\,b\,\rho\,\omega^2 - q^4\,b^2\,\theta^{(-1+\alpha)}\,\phi - 4\,q^2\,\theta^{(1-\alpha)}\,\omega^2\,\phi + 4\,q^3\,\theta^{(-1+\alpha)}\,\omega\,\phi + 2\,q^4\,b\,\theta^{(-1+\alpha)}\,\phi - q^3\,b^2\,\theta$$

$$- 4\,q^3\,\theta^{(1-\alpha)}\,\phi\,\omega - 2\,q^3\,b\,\theta^{(-1+\alpha)}\,\phi + 3\,q^2\,b^2\,\theta^{(-1+\alpha)}\,\phi + 3\,q\,b^2\,\rho - 4\,q\,b^2\,\rho\,\omega + 2\,q^2\,\theta^{(-1+\alpha)}\,\alpha\,\omega^2\,\phi\,b + 3\,q^2\,b^2\,\theta$$

$$- b^2\,\rho\,\omega^3 + b^2\,\rho\,\omega^2 + 2\,b\,\rho\,\omega^3 + 2\,q^3\,b\,\rho - 2\,q^2\,b\,\rho + q\,b^2\,\rho\,\omega^2 + q^2\,b^2\,\rho\,\omega - q^3\,b^2\,\rho - q^2\,b^2\,\rho + 2\,q^2\,b\,\rho\,\omega) \Big/ (b\,(-1 + q + \omega$$

$$(-2\,q^2 + b\,q^2 + 2\,b\,q - 2\,b\,q\,\omega - b - 2\,\omega^2 + b\,\omega^2 + 2\,b\,\omega))$$