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# Studying Spillovers in a New Keynesian Continuous Time Framework with Financial Markets

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## Abstract

The recent financial and real economic crises have made it clear that macroeconomists need to better account for the influence of financial markets. This paper explores the consequences of treating the interaction between different financial markets, monetary policy, and the real economy seriously by developing a fully dynamic theoretical model in continuous time. Starting from a standard New Keynesian framework, we reformulate and extend the model by means of stochastic differential equations so as to analyse spillover effects and steady-state properties. We solve the nonlinear model using stochastic differential equations rather than third-order perturbation. Applying Bayesian estimation methods, we estimate the model for the US economy and analyse the financial markets influence on business cycles and monetary policy.

**JEL Classification Numbers:** C02, C63, E44, E47, E52, F41

**Keywords:** New Keynesian Model, Phillips Curve, Taylor Rule, Stochastic Differential Equations

## 1 Introduction

The recent economic crisis emphasises the relevance of financial markets to macroeconomic stability. This should not be a surprise: financial markets played an important role in previous major crises too, such as the Great Depression and the Asian financial crisis. Arguably, these crises were associated with financial market turmoil and each required a different stabilisation policy to avoid further financial contagion. The Asian crisis was caused by speculative attacks on the Thai Baht and the Great Depression followed a major stock market crash. The ongoing European sovereign debt crisis was triggered by speculative attacks on government bonds following an increase in public debt, which debt was at least partly incurred as a result of government intervention in the banking sector in the aftermath of the recent financial crisis.

In light of this situation, we believe it is crucial to achieve a better understanding of the role different financial markets play in a macroeconomy. Furthermore, observing not only financial market actions but also financial market reactions to different policies could be helpful in designing appropriate policies. Put differently, policymaker responses to shocks could likely be improved by the availability of a thorough theoretical framework. However, the common macroeconomic approaches to monetary policy taken by central banks, both analytically and empirically, tend to downplay the role the financial system plays in the conduct of monetary policy and rarely, if ever, take into consideration different financial markets. Thus, we argue

that academic research can assist central banks by placing greater emphasis on financial markets and moving beyond common macroeconomic models. Therefore, the purpose of this paper is to explore the consequences of treating seriously the interaction between financial markets, monetary policy and the real economy by developing a fully dynamic theoretical modelling framework. We are particularly interested in the relationship between financial markets and monetary policy, which is characterised by a substantial degree of simultaneity. We use stochastic differential equations to cope with the likely nonlinearity of macroeconomic relationships. We thus avoid third-order perturbation methods and can more easily estimate the model. Furthermore, Yu (2013) emphasises that there are compelling reasons why continuous-time models should appeal to both economists and financial specialists, one of these being that 'the economy does not cease to exist in between observations' (Bartlett 1946). Moreover, at an aggregate level, economic decision-making almost always involves many agents and is typically conducted over the course of a month and thus continuous-time models may provide a good approximation of the actual dynamics of economic behaviour. Another important advantage of continuous-time models is that they provide a convenient mathematical framework for the development of financial economic theory, enabling relatively simple and often analytically tractable ways to price financial assets. Continuous-time models also permit treating stock and flow variables separately. Finally, by using this class of models, a well-defined mathematical analysis of model stability can be applied (Thygesen 1997).

In mainstream macroeconomic research, the New Keynesian (NK) model (Blinder (1997), Clarida, Gali, and Gertler (1999), Romer (2000), and Woodford (1999)) is a frequent starting point for analysing monetary policy. We follow this strand of literature and adopt the NK model as our baseline approach. However, as our main question of interest is the interaction between monetary policy and financial markets, we need to extend the NK model. Our starting point is that monetary policy reacts to financial markets and financial markets react to monetary policy and that this relationship is characterised by a notable degree of simultaneity. This is not a new idea; other studies have taken such simultaneity into account. However, most of this work is empirical in nature, for example, Bjornland and Leitemo (2009), Rigobon (2003), and Rigobon and Sack (2003). Rigobon (2003) and Rigobon and Sack (2003) observe the effects of bonds and stocks on monetary policy and vice versa by applying a novel approach (identification through heteroscedasticity) to circumvent simultaneity issues. Bjornland and Leitemo (2009) adopt a VAR approach to study how financial markets affect monetary policy. Technically, the authors

deal with the simultaneity issue by imposing *a priori* short-run and long-run restrictions. Both studies find evidence of monetary policy reaction to financial market developments.

Less formally, Hildebrand (2006) argues that financial markets are the link between monetary policy and the real economy and are an important part of the transmission mechanism for monetary policy. Moreover, he argues that financial markets reflect expectations about future inflation and output and, therefore, are also affected by monetary policy. Christiano et al. (2008) propose a more formal empirical approach. When considering the problem of boom-bust cycles in the economy, they find evidence that it might be more expedient for monetary policymakers to target credit growth instead of inflation.

Our paper makes several contributions to the literature. First, we combine and extend the models of Smets and Wouters (2002) and Bekaert, Cho, and Moreno (2010) and compute a closed-economy New Keynesian model that includes a financial market sector. Derivation of the monetary policy rule is in line with Smets and Wouters (2002). We follow Bekaert, Cho, and Moreno (2010), Brunnermeier and Sannikov (2012), and Christiano, Motto, and Rostagno (2014) for our modelling of a financial market sector.

Second, we undertake a formal mathematical analysis of the extended model in a continuous-time framework. This requires transforming the NK model in stochastic differential equations, using numerical algorithms to derive stable solutions, and studying the evolution of various variables over time. Our reading of the literature is that this combination of applying Taylor rules to financial markets and extending the NK model with two dynamic financial market equations is unique.

Third, we estimate model parameters for the United States using Bayesian estimation techniques and compare the estimated impulse-response functions with those from the theoretical simulations.

Our approach is similar to that taken by Asada et al. (2006) and Chen et al. (2006a) and to Fernández-Villaverde, Posch, and Rubio-Ramírez (2012). These authors transform the Keynesian AS/AD model into a disequilibrium model with a wage-price spiral and include two Phillips curves, one targeting wages and the other targeting prices. The model is transformed into five differential equations-explaining real wages, real money balances, investment climate, labour intensity, and inflationary climate-and its dynamics are analysed extensively. Malikane and Semmler (2008b) extend this framework by including the exchange rate and Malikane and Semmler (2008a) consider asset prices. However, none of these studies include both stock and

bond markets. Since our aim is to close the gap between classic macroeconomic research and finance, we employ a discrete-time model as a starting point.

In the next section, we develop the theoretical baseline model. Section 3 contains the derivation of the continuous-time model. In Section 4, we compare discrete- and continuous-time results of the model. Section 5 contains an analysis employing empirically estimated parameters for the United States. Section 6 concludes.

## 2 The Baseline Model

Our model is based on the New Keynesian model of Smets and Wouters (2002, 2007) (respectively Allsopp and Vines (2000), Blanchard (2007, 2008), Clarida, Gali, and Gertler (1999), and Gali and Monacelli (2005)).

### 2.1 Households

We assume that the economy is inhabited by a continuum of infinitely-living consumers  $i \in [0, 1]$ . First, we consider a consumption index, such as that of Dixit and Stiglitz (1977),  $C_t$  ( $P_t$ ) which consists of goods  $c_t^j$  produced by firm  $j$ .  $\eta_c$  is the elasticity of demand. Similarly, we define a production price index  $P_t$ , using  $p_t^j$ .

$$C_t = \left[ \int_0^1 (C_t^j)^{\frac{\eta_c-1}{\eta_c}} dj \right]^{\frac{\eta_c}{\eta_c-1}} \quad (1)$$

Accordingly, the aggregate price index (the CPI) is given by

$$P_t = \left[ \int_0^1 (P_t^j)^{1-\eta_c} dj \right]^{\frac{1}{1-\eta_c}} \quad (2)$$

Consumption is maximised subject to  $\int_0^1 (P_t^j C_t^j) dj = Z_t$ , where  $Z_t$  are expenditures.

$$C_t^j = \left( \frac{P_t^j}{P_t} \right)^{-\eta_c} C_t \quad (3)$$

Our discrete time model is based on Smets and Wouters (2002, 2007). We extend it, following Paoli, Scott, and Weeken (2010a), by including assets in the household's budget constraint.

Each household  $i$  provides a different type of labour. Households seek to maximise the discounted sum of expected utilities with regard to consumption  $C_t$  and labour  $N_t$  subject to a period-by-period budget constraint. Using a constant relative risk aversion utility function, the representative household's lifetime utility can be written as

$$E_0 \sum_{t=0}^{\infty} \beta^t u_t^i (C_t^i, N_t^i) \quad (4)$$

where  $\beta$  is the discount factor. Specifically, it is

$$u_t^i = \epsilon_t^U \left( \frac{1}{1 - \sigma_c} (C_t^i - hC_{t-1}^i)^{1 - \sigma_c} - \frac{\epsilon_t^L}{1 + \sigma_l} (N_t^i)^{1 + \sigma_l} \right) \quad (5)$$

where  $h$  represents an external habit formation,  $\epsilon_t^U$  is a general shock to preferences,  $\epsilon_t^L$  is a specific shock to labour and  $\sigma_c$ , and  $\sigma_l$  are elasticities of consumption, money, and labour. Households maximise their utility due to the intertemporal budget constraint

$$\begin{aligned} & \frac{W_t^i}{P_t} N_t^i + R_t^k Z_t^i K_{t-1}^i - a(Z_t^i) K_{t-1}^i - C_t - I_t^i - A_t^i - \frac{B_t^i R_t^{-1} - B_{t-1}^i}{P_t} + \frac{Div_t}{P_t} Equ_{t-1}^i \\ & - \sum_{j=0}^J \left( \frac{V_{t,t+m}^B}{P_t} B_{t,j}^i - \frac{V_{t-1,t+m-1}^B}{P_t} B_{t-1,t+m}^i \right) - \frac{(V_t^E Equ_t^i - V_{t-1}^E Equ_{t-1}^i)}{P_t} = 0 \end{aligned} \quad (6)$$

where  $W_t$  is the nominal wage,  $Div_t$  are dividends, and  $(R_t^k Z_t^i - a(Z_t^i)) K_{t-1}^i$  is the return on the real capital stock minus capital utilisation costs. Furthermore,  $B_t^i$  denotes a one-period bond;  $B_{t,j}^i$  denotes an  $n$ -period bond with the respective  $V_t^C$  its price.  $Equ_t^i$  is a share in an equity index with  $V_t^E$  its respective value and  $A_t^i$  are state-contingent claims, and  $I_t^i$  are investments in capital.

Capital is accumulated according to

$$K_t^i = (1 - \delta) K_{t-1}^i + \left( 1 - V \left( \frac{I_t^i}{I_{t-1}^i} \right) \right) I_t^i \quad (7)$$

where  $\delta$  is the depreciation rate and  $V(\cdot)$  is similarly defined as in Smets and Wouters (2002).

## 2.2 Firms

### 2.2.1 Domestic Firms

Final goods are supplied by monopolistically competitive firms using a CES function

$$Y_t = \left( \int_0^1 (Y_t^j)^{\frac{1}{\mu_t}} dj \right)^{\mu_t} \quad (8)$$

where  $Y_t^j$  is the input of the intermediate good  $Y_t^j$  and  $\mu_t$  is a price elasticity. Final good producers maximise their profits subject to the production function

$$\max \left( P_t Y_t - \int_0^1 P_t^j Y_t^j \right) \quad (9)$$

### 2.3 Intermediate Firms

Intermediate goods  $Y_t^j$  are produced using a Cobb-Douglas production function

$$Y_t^j = \epsilon_t^F (\tilde{K}_t^j)^\alpha (N_t^j)^{1-\alpha} \quad (10)$$

where  $\epsilon_t^F$  is a technology shock,  $\tilde{K}_t$  are capital services, and  $N_t$  is the labour input. Firm profits are immediately paid out as dividends

$$\frac{Equ_{t-1}^j Div_t^j}{P_t} = \frac{P_t^j}{P_t} Y_t^j - \frac{W_t}{P_t} N_t^j - R_t^k \tilde{K}_t^j \quad (11)$$

Firms minimise their costs with respect to the production technology. Nominal profits of firm  $j$  are given by

$$\pi_t^j = \left( \frac{P_t^j}{P_t} - MC_t \right) Y_t^j = \left( \frac{P_t^j}{P_t} - MC_t \right) \left( \frac{P_t}{P_t^j} \right)^{\frac{-\mu_t}{\mu_t-1}} Y_t \quad (12)$$

The pricing kernel is derived from the first-order conditions (FOCs) of the households. This gives the pricing kernel for the discount rate  $\frac{1}{1+R_t}$ .

Each period, a fraction of the firms  $(1 - \theta)$  are able to adjust prices, while the remainder follow a rule of thumb. We denote  $\pi_t = \frac{P_t}{P_{t-1}}$  and  $\pi$  is the steady state inflation.

We obtain

$$\begin{aligned}
P_t &= \left[ \theta (P_{t-1} \pi_{t-1}^l \pi^{1-l})^{\frac{1}{1-\mu_t}} + (1-\theta) \overline{P}_t^{\frac{1}{1-\mu_t}} \right]^{1-\mu_t} \\
\Leftrightarrow 1 &= \left[ \theta (\pi_{t-1} \pi_{t-1}^l \pi^{1-l})^{\frac{1}{1-\mu_t}} + (1-\theta) (\overline{P}_t^*)^{\frac{1}{1-\mu_t}} \right]^{1-\mu_t}
\end{aligned} \tag{13}$$

## 2.4 Wage Setting

Each household supplies labour based on the following labour-bundling function

$$N_t = \left( \int_0^1 (N_t^i)^{\frac{1}{\gamma_n}} \right)^{\gamma_n} \tag{14}$$

where  $\gamma_n$  is wage elasticity and  $1 \leq \gamma_n < \infty$ . Similarly, the demand for labour is given by

$$N_t^i = \left( \frac{W_t^i}{W_t} \right)^{\frac{\gamma_n}{1-\gamma_n}} N_t \tag{15}$$

Reflecting the specification of the firms' optimisation problem, they face a random probability  $(1 - \theta_h)$  of changing the nominal wage. The reoptimised wage of household  $i^{th}$  is  $\overline{W}_t^i$ , whereas the unchanged wage is adjusted by  $W_{t+1}^i = W_t^i \pi_t^{l_h} \pi^{1-l_h}$ .

Households then maximise their optimal wage subject to the demand for labour and the budget constraint.

## 2.5 The Financial Sector

The literature contains several approaches for including a financial sector in a macroeconomic model. Gertler, Kiyotaki, and Queralto (2012) apply a micro-based model and incorporate a banking sector and financial frictions. However, as our focus here is on spillovers from the asset markets to the real economy, intermediaries are of limited interest to us. Brunnermeier and Sannikov (2012) construct a macroeconomic model with an emphasis on variations in risk preferences and the degree of information across households and financial experts, but since they do not model real economic effects, their framework is not suitable for our study of financial and macroeconomic spillovers. Therefore, we follow Bekaert, Cho, and Moreno (2010) and Paoli, Scott, and Weeken (2010b) and model bond and asset yields first in discrete time, after which we switch to continuous time. Moreover, we extend the NK framework of Paoli, Scott, and Weeken (2010b) by defining different term structures and rigidities.

As long as there are no frictions, our model exhibits the classic equity and term premia puzzle. As demonstrated by Campbell and Cochrane (1999) in the context of endowment economies, consumption habits can be used to solve this puzzle. When switching off capital adjustment costs, we confirm the previous result of Boldrin, Christiano, and Fisher (2001) that, in a production economy, consumption habits by themselves are not sufficient solve the puzzle. In fact, consumer-investors can eat into capital and, thereby, change production plans. In other words, we need to ensure that households do not merely dislike consumption volatility, they have to be prevented from doing something about it; capital adjustment costs are a useful modelling device for achieving this.

The presence of state-contingent claims implies that we can price all financial assets in the economy based on no-arbitrage arguments. The price of a zero-coupon bond is therefore given by the FOCs. We follow Binsbergen et al. (2012) and Paoli, Scott, and Weenen (2010b) and model the term structure recursively

$$\frac{1}{R_t} = \beta E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} \right] \quad (16)$$

Tobin's Q is given by  $q_t^i = \frac{\varphi_t^i}{\lambda_t^i}$  in

$$q_t^i = \beta E_t \left( \frac{\lambda_{t+1}}{\lambda_t} (R_t^k Z_t^i - a(Z_t^i)) + q_{t+1}^i (1 - \delta) \right) \quad (17)$$

A real zero-coupon bond returns one unit of consumption at maturity, and the remaining time is  $j$ . For  $j = 1$ , it is

$$-\lambda_t \frac{V_{t,1}^B}{P_t} = E_t \left[ \beta \lambda_{t+1} \frac{V_{t+1,0}}{P_{t+1}} \right] \quad (18)$$

$$\Leftrightarrow V_{t,1}^B = E_t \left[ -\beta V_{t+1,0} \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} \right] \quad (19)$$

$V_{t+1,1}$  is the price of a real bond of original maturity  $m = 2$  with one period left. Assuming no arbitrage, this price equals the price of a  $m = 1$  bond issued next period. Bond prices thus can be defined recursively (using  $SDF_t = \beta \frac{\lambda_{t+1}}{\lambda_t}$ )

$$V_{t,t+m}^B = E_t \left[ -\beta \frac{\lambda_{t+1}}{\lambda_t} V_{t+1,t+m-1}^B \frac{P_t}{P_{t+1}} \right] \quad (20)$$

$$= E_t(SDF_{t+1} \pi_{t+1} V_{t+1,t+m-1}^B) \quad (21)$$

Assuming  $V_{t,1} = 1$  in terms of one unit of consumption and applying recursion, we obtain

$$V_{t,t+m}^B = E_t((SDF_{t+1} \pi_{t+1})^j) \quad (22)$$

Real yields are then given by

$$R_{t+1,t+m}^B = (V_{t,t+m}^B)^{-\frac{1}{j}} \quad (23)$$

Regarding the stocks, we derive

$$1 = E_t \left[ -\beta \frac{P_t}{P_{t+1}} \frac{\lambda_{t+1}}{\lambda_t} \frac{V_t^E + Div_{t+1}}{V_t^E} \right]$$

implying a real return of

$$R_{t+1}^E = \frac{V_t^E + Div_t}{V_t^E} \frac{P_t}{P_{t+1}}$$

The model reconciles the non-stationary behaviour of consumption found in the macro literature with the assumption of stationary interest rates in the finance literature. Andreasen (2012), Andreasen (2010), and Wu (2006) show that this specification equals standard finance models such as those of Dai and Singleton (2000), Duffie and Kan (1996), and Duffie, Pan, and Singleton (2000) .

## 2.6 The Monetary Reaction Function

In line with Ball (1998), Justiniano, Primiceri, and Tambalotti (2010), Leitemo and Soderstrom (2005), Lubik and Schorfheide (2007), Lubik and Smets (2005), Smets and Wouters

(2007), Svensson (2000), and Taylor (1993), we apply a standard Taylor rule.

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_R} \left[ \left(\frac{\pi_t \pi}{\pi_{t-1}}\right)^{\psi_\pi} \left(\frac{Y_t}{Y_{t-1}}\right)^{\psi_Y} \right]^{1-\rho_{E^*}} \eta_{mp,t} \quad (24)$$

where  $\eta_{mp,t}$  is a monetary policy shock.

$$\log(\eta_{mp,t}) = \rho_{mp,t} \log(\eta_{mp,t-1}) + \epsilon_{mp,t} \quad (25)$$

where  $\epsilon_{mp,t}$  is *i.i.d.*  $N(0, \sigma_{mp}^2)$

## 2.7 Market Clearing

We treat equities and bonds as financial instruments and assign them to a new sector, the financial market. A shortcoming of this sector is that it has no purpose other than an allocative one. However, in building a bridge between finance and macroeconomic research, we model financial instruments stochastically and thus the solely allocational purpose of the financial market sector is appropriate for our analysis. In principle, introducing a banking sector or entrepreneurs would easily overcome this shortcoming. However, for the sake of simplicity, we adhere to the solely allocative purpose. Thus, the market clearing condition is

$$Y_t = C_t + I_t + FM + a(Z_t)K_{t-1} \quad (26)$$

where  $FM$  is the financial market sector.

## 3 Linearisation and Continuous Time

### 3.1 Linearisation

Equities and bonds must be linearised at least up to the second order. Indeed, Andreasen (2012) propose linearising to the third order so as to capture the time-varying effects of the term structure. Normal linearisation would yield risk-neutral market participants, implying similar prices for all assets. We overcome this dilemma and follow Paoli, Scott, and Weeken (2010b) and Wu (2006) by assuming  $\ln(SDF_{t+1}\pi) = \alpha_0 + \alpha_1 s_t + \alpha_2 \epsilon_{t+1}$ , where  $s_t$  are the state variables.  $\frac{P_t}{P_{t+1}} = \pi_{t+1}$  is the inflation rate. We then apply a third-order approximation for the

relevant yields. We linearise the economy using perturbation.

Following Wu (2006), we use the approach of Jermann (1998) and log-linearise around the non-stochastic steady state and solve the resulting system of linear difference equations. The issue of non-stationarity is tackled by rescaling all variables. We assume that the model solution is given by variables  $z_t$  such that

$$z_t = \mu_z + \Sigma_z s_t \quad (27)$$

where  $s_t$  is an AR(1) process. In line with Binsbergen et al. (2012), the bond equation has the following form

$$\begin{aligned} \ln(VB_{t,m}) &= E_t(\ln(SDF_{t+1}) + \ln(VB_{t+1,m-1})) \\ &+ \frac{1}{2} \text{var}(\ln(SDF_{t+1}) + \ln(VB_{t+1,m-1})) \\ &+ \frac{1}{6} \text{skew}(\ln(SDF_{t+1}) + \ln(VB_{t+1,m-1})) \end{aligned} \quad (28)$$

Equation (28) holds exactly when the conditional distribution of the bond price  $VB$  and the stochastic discount factor  $SDF$  are joint log-normal, but it still holds approximately if this is not the case (Wu 2006). Our model falls within the class of affine term structure models, since the logarithm of the bond prices is a system of linear (or affine) functions of the state variables. Our model is somewhat similar to standard finance models such as Dai and Singleton (2000), Duffie and Kan (1996), and Duffie, Pan, and Singleton (2000), but it includes a macroeconomic base and therefore bridges the gap between finance and macroeconomic research.

## 3.2 Continuous-Time Modelling

One disadvantage of NK models is the absence of an explicit solution for the baseline model. A standard response to this problem is to linearise the model around the steady state. However, this causes a loss of information. Term structure models must be linearised at least up to the second order, which aggravates the loss of information and, furthermore, there is no longer an explicit solution to the model.

We propose a different approach. Reflecting work by **and chen06b**; Asada et al. (2006) and Chen et al. (2006a), we switch to a continuous-time framework by using difference equations transformed to differential equations. NK models are based on difference equations such

as, for example, the consumption equation, which depends on the actual and prior values. Using time-scale calculus, we can then transform these equations into differential equations. To explicitly account for shocks, we propose stochastic differential equations. In contrast to Fernández-Villaverde, Posch, and Rubio-Ramírez (2012), who works only within the continuous-time domain, we first derive the discrete model. This allows us to directly link our approach to standard macroeconomic theory and compare core results derived using the continuous-time models with those from the discrete-time model. Time-scale calculus makes it possible to switch from discrete time to continuous time. Time-scale calculus is a formal unification of the theory of difference equations with that of differential equations. A general overview is provided by Agarwal et al. (2002) and Turnovsky (2000); the theoretical basis for applying this concept to stochastic differential equations is given by Sanyal (2008).

We augment the standard representation of stocks and bonds (see Merton (1969)) to reflect our assumptions of simultaneity and highly interacted markets by incorporating the monetary policy rate and the log-linearised stock price into the bond yield equation. Similarly, we add the bond yield and the monetary policy rate to the stock price equation. Since we consider stock prices to be an important link between the real and the monetary economy, we add the output gap to the stock market equation. In the price equations for stocks and bonds, we account for the future stream of dividend payments by adding the inflation rate positively and the (nominal) interest rate negatively. This approach accounts for the Fisher effect and, hence, stock prices rise with increasing inflation. Including the output gap in the stock market equation is in line with Cooper and Priestley (2009) and Vivian and Wohar (2013). A general approach to accounting for macroeconomic factors in stock returns is provided by Pesaran and Timmermann (1995). We make no explicit assumptions about how bond and stock markets influence each other. When bond yields are inside a small interest rate band (of realistic size), the correlation between bonds yields and stock yields (and equity prices) is mainly positive. However, when yields are outside this band, the correlation is mainly negative. In line with the relevant finance literature (Barndorff-Nielsen and Shephard 2001; Fama 1965; Malkiel and Fama 1970), we model stock prices as geometric Brownian motion processes and bond yields as Ornstein-Uhlenbeck processes. Since disturbances are specified as AR(1) processes, we can interpret the main macroeconomic variables as standard Brownian motions.

$$dS = (S_t((r - \lambda\mu) + \rho_b b_t + \rho_y y_t + \rho_\pi \pi_t))dt + \sqrt{V_t}dW_S(t) + \sum_{i=1}^{dN_t} J(Q_i) \quad (29)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_V(t)$$

$$db_t = (\gamma_b b_t + \gamma_S S_t + \gamma_y y_t - \gamma_i(R_t - \pi_t))dt + \sigma_b dW_b(t) \quad (30)$$

where  $J(Q)$  is the Poisson jump amplitude,  $Q$  is an underlying Poisson amplitude mark process ( $Q = \ln(J(Q) + 1)$ ), and  $N(t)$  is the standard Poisson jump-counting process with jump density  $\lambda$  and  $E(dN(t)) = \lambda dt = Var(dN(t))$ .  $dW_s$  and  $dW_v$  denote Brownian motions.  $\beta$  is the long-term mean level.

Stability is an important aspect of differential equations. We employ Lyapunov techniques to analyse the stability of our model. For a thorough discussion of these techniques, see (Khasminskii 2012). We apply the following Lyapunov function:

$$V(x) = \|x\|_2^2 = \left( \sqrt{\sum |x_i|^2} \right)^2 \quad (31)$$

where  $\|\cdot\|_2$  denotes the Euclidean norm. Since the zero solution is only locally stable, there is no global stable rest point. However, even though there are only parameter-dependent partial solutions, these are 'almost' stable. In the following section, we analyse stability for each set of parameters we derive.

Because our focus is on business cycles, we concentrate our dynamic analysis on short-run adjustments. Within this timeframe, there is no guarantee that the variables will actually return to their starting values.

## 4 Comparison of Discrete- and Continuous-Time Results in the Standard Model

To show that our continuous-time model encompasses standard macroeconomic results derived in discrete-time models, we compare both types of specification. As there is no explicit analytical solution, we compare the respective impulse response functions. For the parameters, we choose standard values from the literature (see column 2 of Table 1).

Calibration of the household and firm side is standard. Elasticities of substitution regarding

investments and consumption  $(\eta_d^c, \eta_d^i, \eta_f^c, \eta_f^i)$  vary between 1.30 and 1.50 (Fernández-Villaverde 2010). The household's utility function is similar to the one employed by Smets and Wouters (2007). The elasticity for substitution of consumption  $\sigma$  is 1.20; the elasticity of substitution for labour  $\sigma_l$  is 1.25. On the supply side, we assume standard Calvo-pricing parameters as in Smets and Wouters (ibid.). The Calvo parameters for prices  $\theta$  and wages  $\theta_h$  are 0.75. We use monetary policy parameters similar to those of Adolfson et al. (2011) and Lindé (2005). The inflation parameter  $\psi_\pi$  is 1.20, reflecting our assumption that a central bank's chief goal is disinflation stability. Further details can be found in the cited literature.

We analyse monetary policy shocks and shocks to the financial market instruments (bonds and stocks). In general, we find that the main difference between continuous- and discrete-time adjustment is the high number of small fluctuations within the continuous-time framework. This reveals that the adjustment process to the steady state is somewhat different from what even third-order approximation in discrete time can capture. Our main goal is to analyse monetary policy and financial market behaviour and we therefore concentrate on shocks to monetary policy and financial variables.

Figure 1 shows the impulse response functions after a contractionary monetary policy shock. For both specifications, the shock causes a decrease in output and inflation. However, the additional nonlinearities and the better-specified interaction with the financial market reveal more fluctuations in the continuous-time compared to the discrete-time scenario. On the one hand, the discrete-time scenario looks like a text-book economy, where monetary policy is able to steer the adjustment in a way such that there is no overshooting. The exception is investment, which shows no tendency to return to the baseline within our observation window. In the continuous-time case, on the other hand, the recession is followed by a boom and only then does the economy return to the baseline. Stabilisation policy as modelled by the Taylor rule leads to several interventions by the central bank, as financial market fluctuations which were triggered by monetary policy, feed back into the real economy and contribute to the resulting business cycle. Reflecting these adjustments, bond prices, inflation, and the stock market are also subject to more fluctuation. Overall, we find a much more realistic picture of the effects of a monetary policy shock, i.e., one that is more in line with the stylised facts derived from VAR models (see Bernanke, Boivin, and Elias (2005) and Christiano, Eichenbaum, and Evans (2005)).

Figure 2 studies the response after a negative bond price shock. A negative bond price

shock can be interpreted as an increase in market participants' risk aversion. In both models, financial and real variables show a cyclical adjustment pattern back to equilibrium. There are also clear spillovers between the two financial markets and to the real economy. However, the continuous-time model is characterised by much richer dynamics than is the discrete-time model, for instance, the decline in GDP is smooth rather than abrupt as in the discrete-time scenario. Thus, we can actually generate realistic-looking business cycles in the real economy via financial market shocks. This is a very clear illustration of how important financial market shocks are to the macroeconomy.

Figure 3 shows the results of a negative stock price shock. Here, the differences between the models are even more marked. Whereas in the discrete-time model, the stock market crash causes almost no decline in output or investments and recovers fast, the shock causes implausible dynamic adjustments of the other variables in the system, e.g. increasing GDP and investments, shock causes a recession followed by a boom in continuous time due to increased investment opportunities. Furthermore, the negative stock price shock leads to a cyclical bond market adjustment in the continuous-time model. So again the continuous-time model provides plausible interactions between the real economy and the financial markets. We argue that the more realistic dynamics illustrate the effectiveness of using findings from financial literature in macroeconomic models. As financial markets are better simulated and integrated into the system of equations, a stock market shock is transmitted more realistically to other parts of the economy

To summarise, all simulations suggest that the continuous-time model captures business cycles better than do standard DSGE models. The improvement is especially noteworthy in the case of the financial variables, where we observe a full adjustment process instead of immediate reactions.

## 5 Employing Empirically Estimated Parameters: The United States

We now test our model with real-world data using Markov Chain Monte Carlo (MCMC) methods, a Bayesian estimation technique (approximate Bayesian computation; see Beaumont, Zhang, and Balding (2002)), particularly suited to estimating stochastic differential equations. In principle, the technique allows estimating even more complex models, such as multi-country

models with financial markets. Here, reflecting the assumption of a closed economy made in the theoretical analysis, we estimate the model using US data. We then compare the theoretical with the empirical impulse response functions. Since estimating a nonlinear model is difficult, two inputs are crucial for obtaining plausible results through MCMC estimations: first, the choice of priors and, second, the choice of initial values. Our choice of prior distributions for New Keynesian models is similar to that of, among others, Smets and Wouters (2007), Negro et al. (2007) or Lindé (2005). We follow Kimmel (2007) and Jones (2003) in choosing normal distributions for the variance of bond and stock price shocks. An overview of the priors is given in columns 3 to 5 of Table 1.

We apply the MCMC estimation to the United States. Data are from the Federal Reserve Bank of St. Louis and the US Bureau of Labor Statistics. We employ quarterly data from 1957:Q1 to 2013:Q4. We use a bandpass filter (as suggested by Christiano and Fitzgerald (1999)) to eliminate noise from the variables and employ a Hodrick Prescott filter as a robustness check. The inflation series is constructed by the GDP deflator. We use the S&P 500 stock market index to capture stock prices.

Table 1 contains the results of the MCMC estimations. The posterior values for the means and variances are set out in columns 6 and 8. Note that these parameters are in line with previous findings, such as those of Smets and Wouters (2002, 2007). This suggests that our approach is a tractable way of solving and estimating nonlinear DSGE models. Figure 4 shows impulse response functions following an contractionary monetary policy shock. The results are similar to those found in the extant literature (e.g. Justiniano, Primiceri, and Tambalotti (2010)). Our findings are also in line with the calibrated continuous-time model outlined above. However, we observe only negligible reactions by the investments in the estimation if the policy rate is cut, but stronger reactions if the policy rate is later increased.

## 6 Conclusions

In this paper, we study the relationship between macroeconomic and financial variables in the framework of a continuous-time New Keynesian DSGE model. Hence, method-wise, we provide another way of analysing nonlinear New Keynesian models without linearising, a particularly important advantage when it comes to DSGE models with term structure as linearisation can seriously distort the results. Usually, therefore, third-order perturbation methods are called for

but solutions are difficult to find with these methods and estimations are even more difficult to implement. In contrast, stochastic differential equations can be used to both implement and estimate these models. Moreover, the continuous-time approach reveals greater dynamic fluctuations compared to the standard linearised models. By including financial market term structure in continuous time, we can implement bonds, stocks, call prices, and the like in our model, for instance, in the form of the Vasicek model, jump-diffusion processes, or Black Scholes calls. We thus combine classic research from the field of finance with macroeconomic models. In addition, we can interact the financial markets and account for financial market integration. Model stability is proven by applying a Lyapunov function. Thus, in our analysis, we combine New Keynesian macroeconomic analysis, classic finance research, and standard mathematical procedures.

Our main research quest is to understand the effects of financial market shocks and monetary policy in a theoretical framework that allows for feedback between financial markets and the real economy. In line with economic theory and empirical evidence, we begin with a steady-state solution. We simulate financial market reaction to monetary policy and monetary policy reaction to financial turmoil in the context of a standard Taylor rule. For our simulations, the model parameters are based on empirical findings from work on the Taylor rule, the New Keynesian Phillips curve, and the IS curve; our financial equations rely on findings by Merton (1970) and Black and Scholes (1973). However, we extend all financial equations by accounting for spillover effects from monetary variables to real variables and vice versa. We analyse monetary shocks as well as financial market shocks and follow their transmission to the real economy.

In general, we find that the scenarios based on continuous-time models exhibit much more realistic adjustment patterns than those based on discrete-time models. First, a contractionary monetary policy shock has negative effects on output and inflation as well as repercussions for both financial markets. The financial market fluctuations cause spillovers to the real economy and induce further adjustments in monetary policy. As a result, we find that monetary policy shocks cause adjustments in real and financial variables. Second, the responses after a negative bond price shock reveal a cyclical adjustment pattern back to equilibrium. Hence, using financial market shocks, we can generate realistic-looking business cycles in output and inflation. The continuous-time models also provide highly satisfactory results in the case of a simulated stock market crash; the observed adjustments are far more realistic than those seen

using discrete-time models.

We then test our theoretical model with real-world data from the United States. Employing quarterly data over the period 1957:Q1 to 2013:Q4, we use Bayesian estimation techniques to derive the model's parameters. The simulation results support those from the purely theoretically parameterised continuous-time model. We find spillover effects from monetary policy to financial markets and from financial markets to monetary policy.

Our study discovers some useful ways of implementing very complicated nonlinear DSGE models that will make it easier to study financial markets in both closed and open economies and facilitate the estimation of nonlinear models using different Bayesian methods. However, we analysed only a closed economy and did not account for international spillovers. Hence, there are various ways of productively extending our analysis. First, looking at an open economy with monetary spillover effects might provide additional insight. The model also permits comparing different monetary policy reaction functions; for example, we could include a financial market indicator in the Taylor rule. Furthermore, we did not account for a fiscal authority. Thus, extending the model by including government fiscal policy targets could be of interest, especially since we observe that bond markets hardly react to monetary policy. The hypothesis that bond markets are relatively more driven by fiscal policy could be tested with a differently specified model. Moreover, the model could be expanded to account for an even greater number of financial markets. The inclusion of the Black-Scholes formula or advanced option-pricing techniques would permit analysing an almost complete model of the financial system. Finally, in light of the recent financial and sovereign debt crises, a specific analysis of crises could be interesting. As outlined above, the model is capable of measuring the impact of financial market bubbles in a two-economy framework. Including jump-diffusion processes would add another source of financial market volatility and could help explain times of crisis.

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## 7 Technical Appendix

### The Baseline Model

#### Households

We assume that the economy is inhabited by a continuum of consumers  $i \in [0, 1]$ . First, we consider a consumption index, such as that of Dixit and Stiglitz (1977),  $C_t$  ( $P_t$ ) which consists of goods  $c_t^j$  produced by firm  $j$ .  $\eta_c$  is the elasticity of demand. Similarly, we define a production price index  $P_t$ , using  $p_t^j$ .

$$C_t = \left[ \int_0^1 (C_t^j)^{\frac{\eta_c-1}{\eta_c}} dj \right]^{\frac{\eta_c}{\eta_c-1}} \quad (32)$$

Accordingly, the aggregate price index (the CPI) is given by

$$P_t = \left[ \int_0^1 (P_t^j)^{1-\eta_c} dj \right]^{\frac{1}{1-\eta_c}} \quad (33)$$

Consumption is maximised subject to  $\int_0^1 (P_t^j C_t^j) dj = Z_t$ , where  $Z_t$  are expenditures.

$$C_t^j = \left( \frac{P_t^j}{P_t} \right)^{-\eta_c} C_t \quad (34)$$

There is a continuum of households  $i$  which live infinitely. Households seek to maximise the discounted sum of expected utilities with regard to consumption  $C_t$  and labour  $N_t$  subject to a period-by-period budget constraint. Using a constant relative risk aversion utility function (CRRA), the representative household’s lifetime utility can be written as

$$E_0 \sum_{t=0}^{\infty} \beta^t u_t^i(C_t^i, N_t^i) \quad (35)$$

where  $\beta$  is the discount factor. In particular it is

$$u_t^i = \epsilon_t^U \left( \frac{1}{1 - \sigma_c} (C_t^i - hC_{t-1}^i)^{1 - \sigma_c} - \frac{\epsilon_t^L}{1 + \sigma_l} (N_t^i)^{1 + \sigma_l} \right) \quad (36)$$

where  $h$  represents an external habit formation,  $\epsilon_t^U$  is a general shock to preferences,  $\epsilon_t^L$  is a specific shocks to labour and  $\sigma_c$ , and  $\sigma_l$  are elasticities of consumption, money and labour. Households maximise their utility due to the intertemporal budget constraint

$$\begin{aligned} & \frac{W_t^i}{P_t} N_t^i + R_t^k Z_t^i K_{t-1}^i - a(Z_t^i) K_{t-1}^i \\ & - \frac{B_t^i R_t^{-1} - B_{t-1}^i}{P_t} - \sum_{j=0}^J \left( \frac{V_{t,t+m}^B}{P_t} B_{t,j}^i - \frac{V_{t-1,t+m-1}^B}{P_t} B_{t-1,t+m}^i \right) \\ & - \frac{(V_t^E Equ_t^i - V_{t-1}^E Equ_{t-1}^i)}{P_t} + \frac{Div_t}{P_t} Equ_{t-1}^i - C_t - I_t^i - A_t^i = 0 \end{aligned} \quad (37)$$

where  $W_t$  is the nominal wage,  $Div_t$  are dividends,  $(R_t^k Z_t^i - a(Z_t^i)) K_{t-1}^i$  is the return on the real capital stock minus capital utilisation costs. Furthermore,  $B_t^i$  denote a one-period bonds,  $B_{t,j}^i$  denotes a  $n$ -period bond with respective  $V_t^C$  its price.  $Equ_t^i$  is a share in an equity index and with respective value  $V_t^E$  and  $A_t^i$  are stage-contingent claims,  $I_t^i$  are investments.

Furthermore, households are subject to capital accumulation

$$K_t^i = (1 - \delta) K_{t-1}^i + \left( 1 - V \left( \frac{I_t^i}{I_{t-1}^i} \right) \right) I_t^i \quad (38)$$

We obtain the first order conditions

$$C_t^i : \quad \epsilon_t^U (C_t^i - h_t C_{t-1}^i)^{-\sigma_c} - \beta h E_t [\epsilon_{t+1}^U (C_{t+1}^i - h C_t^i)^{-\sigma_c}] - \lambda_t = 0 \quad (39)$$

$$N_t^i : \quad \epsilon_t^U \epsilon_t^L (N_t^i)^{\sigma_L} - \lambda_t \frac{W_t^i}{P_t} = 0 \quad (40)$$

$$B_t^i : \quad -\frac{\lambda_t}{R_t P_t} + \beta E_t \left[ \frac{\lambda_{t+1}}{P_{t+1}} \right] = 0 \quad (41)$$

$$Z_t^i : \quad R_t^k - a'(Z_t^i) = 0 \quad (42)$$

$$K_t^i : \quad \beta E_t [\lambda_{t+1} (R_t^k Z_t^i - a(Z_t^i))] - \varphi_t + \beta E_t [\varphi_{t+1} (1 - \delta)] = 0 \quad (43)$$

$$I_t^i : \quad -\lambda_t + \varphi_t \left( 1 - V \left( \frac{I_t^i}{I_{t-1}^i} \right) \right) - \varphi_t \left( \frac{I_t^i}{I_{t-1}^i} V' \left( \frac{I_t^i}{I_{t-1}^i} \right) \right) + \beta E_t \left( \varphi_{t+1} \left( \frac{I_{t+1}^i}{I_t^i} \right)^2 V' \left( \frac{I_{t+1}^i}{I_t^i} \right) \right) = 0 \quad (44)$$

$$Equ_t^i : \quad -\frac{\lambda_t V_t^E}{P_t} + \beta \left( E \left( \lambda_{t+1} \frac{V_{t+1}^E + Div_{t+1}^i}{P_{t+1}} \right) \right) = 0 \quad (45)$$

$$B_{t,t+m}^i : \quad -\lambda_t \frac{V_{t,t+m}^B}{P_t} + E_t \left[ \beta \lambda_{t+1} \frac{V_{t+1,t+m-1}^B}{P_{t+1}} \right] = 0 \quad (46)$$

Following Fernández-Villaverde (2010) we assume the capital adjustment costs  $a$  to be like

$$a(u) := \gamma_1(u - 1) + \gamma_2(u - 1)^2$$

the investment adjustment cost function is

$$S \left( \frac{x_t}{x_{t-1}} \right) = \frac{\kappa}{2} \left( \frac{x_t}{x_{t-1}} - \Lambda_x \right)^2$$

in our case  $x_t = I_t^i$  and  $\Lambda_x$  is the growth rate of investment  $\delta$ .

## Firms

### Domestic Firms

Final goods are derived by monopolistic competition using a CES function

$$Y_t = \left( \int_0^1 (Y_t^j)^{\frac{1}{\mu_t}} dj \right)^{\mu_t} \quad (47)$$

where  $Y_t^j$  is the input of the intermediate good  $Y_t^j$  and  $\mu_t$  is a price elasticity. Final good producers maximise their profits subject to the production function

$$\max \left( P_t Y_t - \int_0^1 P_t^j Y_t^j \right) \quad (48)$$

The first order conditions are given by

$$0 = -P_t^j + P_t \mu_t \left( \int_0^1 (Y_t^j)^{\frac{1}{\mu_t}} dj \right)^{\mu_t - 1} \left( \frac{1}{\mu_t} (Y_t^j)^{\frac{1 - \mu_t}{\mu_t}} \right) \quad (49)$$

$$\Leftrightarrow 0 = -P_t^j + P_t \left( \left( \int_0^1 (Y_t^j)^{\frac{1}{\mu_t}} dj \right)^{\mu_t} \right)^{\frac{\mu_t - 1}{\mu_t}} (Y_t^j)^{\frac{1 - \mu_t}{\mu_t}} \quad (50)$$

$$\Leftrightarrow \frac{P_t^j}{P_t} = Y_t^{\frac{\mu_t - 1}{\mu_t}} (Y_t^j)^{-\frac{\mu_t - 1}{\mu_t}} \quad (51)$$

$$\Leftrightarrow Y_t^j = \left( \frac{P_t}{P_t^j} \right)^{\frac{\mu_t}{\mu_t - 1}} Y_t \quad (52)$$

Integrating 52 into 48 we obtain

$$Y_t = \left( \int_0^1 \left( \left( \frac{P_t}{P_t^j} \right)^{\frac{\mu_t}{\mu_t-1}} Y_t \right)^{\frac{1}{\mu_t}} dj \right)^{\mu_t} \quad (53)$$

$$\Leftrightarrow Y_t = \left( \int_0^1 \left( \frac{P_t}{P_t^j} \right)^{\frac{1}{\mu_t-1}} Y_t^{\frac{1}{\mu_t}} dj \right)^{\mu_t} \quad (54)$$

$$\Leftrightarrow Y_t = Y_t \left( \int_0^1 \left( \frac{P_t}{P_t^j} \right)^{\frac{1}{\mu_t}} dj \right)^{\mu_t} \quad (55)$$

$$\Leftrightarrow 1 = P_t^{\frac{\mu_t}{\mu_t-1}} \left( \int_0^1 (P_t^j)^{-\frac{1}{\mu_t-1}} dj \right)^{\mu_t} \quad (56)$$

$$\Leftrightarrow P_t^{-\frac{\mu_t}{\mu_t-1}} = \left( \int_0^1 (P_t^j)^{\frac{1}{1-\mu_t}} dj \right)^{\mu_t} \quad (57)$$

$$\Leftrightarrow P_t = \left( \int_0^1 (P_t^j)^{\frac{1}{1-\mu_t}} dj \right)^{1-\mu_t} \quad (58)$$

## Intermediate Firms

The intermediate good  $Y_t^j$  is produced using a Cobb-Douglas production function

$$Y_t^j = z_t^{1-\alpha} \epsilon_t^F \Phi_t (\tilde{K}_t^j)^\alpha (N_t^j)^{1-\alpha} \quad (59)$$

where  $\Phi_t$  is the total factor productivity,  $\epsilon_t^F$  is a technology shock,  $\tilde{K}_t$  are capital services,  $N_t$  is the labour input. Firms profits are immediately paid out as dividends

$$\frac{Equ_{t-1}^j Div_t^j}{P_t} = \frac{P_t^j}{P_t} Y_t^j - \frac{W_t}{P_t} N_t^j - R_t^k \tilde{K}_t^j \quad (60)$$

Firms minimise their costs with respect to the production technology

$$\tilde{K}_t^j : \quad R_t^k - \Gamma_t \alpha z_t^{1-\alpha} \epsilon_t^F \Phi_t (\tilde{K}_t^j)^{\alpha-1} (N_t^j)^{1-\alpha} \quad (61)$$

$$\Leftrightarrow \frac{R_t^k}{\alpha z_t^{1-\alpha} \epsilon_t^F \Phi_t (\tilde{K}_t^j)^{\alpha-1} (N_t^j)^{1-\alpha}} = \Gamma_t \quad (62)$$

$$N_t^j : \quad \frac{W_t}{P_t} - \Gamma(1-\alpha) z_t^{1-\alpha} \epsilon_t^F \Phi_t (\tilde{K}_t^j)^\alpha (N_t^j)^{-\alpha} = 0 \quad (63)$$

$$\Leftrightarrow \frac{W_t}{P_t(1-\alpha) z_t^{1-\alpha} \epsilon_t^F \Phi_t (\tilde{K}_t^j)^\alpha (N_t^j)^{-\alpha}} = \Gamma_t \quad (64)$$

This implies

$$\frac{R_t^k}{\alpha z_t^{1-\alpha} \epsilon_t^F \Phi_t(\tilde{K}_t^j)^{\alpha-1} (N_t^j)^{1-\alpha}} = \frac{W_t}{P_t(1-\alpha) z_t^{1-\alpha} \epsilon_t^F \Phi_t(\tilde{K}_t^j)^\alpha (N_t^j)^{-\alpha}} \quad (65)$$

$$\Leftrightarrow \frac{R_t^k}{W_t} = \frac{\alpha z_t^{1-\alpha} \epsilon_t^F \Phi_t(\tilde{K}_t^j)^{\alpha-1} (N_t^j)^{1-\alpha}}{P_t(1-\alpha) z_t^{1-\alpha} \epsilon_t^F \Phi_t(\tilde{K}_t^j)^\alpha (N_t^j)^{-\alpha}} \quad (66)$$

$$\Leftrightarrow \frac{W_t}{R_t^k} = \frac{P_t(1-\alpha) \tilde{K}_t^j}{\alpha N_t^j} \quad (67)$$

$$\Leftrightarrow \tilde{K}_t^j = \frac{\alpha}{1-\alpha} \frac{W_t}{P_t R_t^k} N_t^j \quad (68)$$

We interpret the Lagrangian parameters as marginal costs

$$\frac{R_t^k}{\alpha z_t^{1-\alpha} \epsilon_t^F \Phi_t\left(\frac{\alpha}{1-\alpha} \frac{W_t}{P_t R_t^k} N_t^j\right)^{\alpha-1} (N_t^j)^{1-\alpha}} = MC_t \quad (69)$$

$$\Leftrightarrow \frac{(R_t^k)^\alpha \left(\frac{W_t}{P_t}\right)^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha} z_t^{1-\alpha} \epsilon_t^F \Phi_t} = MC_t \quad (70)$$

Nominal profits for firm  $j$  are therefore given by

$$\pi_t^j = \left(\frac{P_t^j}{P_t} - MC_t\right) Y_t^j = \left(\frac{P_t^j}{P_t} - MC_t\right) \left(\frac{P_t}{P_t^j}\right)^{\frac{-\mu_t}{\mu_t-1}} Y_t \quad (71)$$

The pricing kernel is derived from the FOCs of the households

$$\frac{\lambda_t}{P_t} = \beta E_t \left( \frac{(1+R_t)\lambda_{t+1}}{P_{t+1}} \right) \quad (72)$$

$$\Leftrightarrow \beta E_t(1+R_t) = E_t \frac{P_{t+1} \lambda_t}{\lambda_{t+1} P_t} \quad (73)$$

This gives the pricing kernel for the discount rate  $\frac{1}{1+R_t}$ .

Each period a fraction of firms  $(1-\theta)$  is able to adjust prices, the remaining fraction follows a rule of thumb. We denote  $\pi_t = \frac{P_t}{P_{t-1}}$  and  $\pi$  is the steady state inflation.

$$E_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\lambda_{t+s}}{\lambda_t} \left( \frac{\overline{P}_t^j}{P_{t+s}} \prod_{l=1}^s (\pi_{t+l-1}^l \pi^{1-l}) - MC_{t+s} \right) Y_{t+s}^j \quad (74)$$

$$s.t. \left( \frac{\overline{P}_t^j \prod_{l=1}^s (\pi_{t+l-1}^l \pi^{1-l})}{P_{t+s}} \right)^{-\frac{\mu_{t+s}}{\mu_{t+s}-1}} Y_{t+s} = Y_{t+s}^j \quad (75)$$

it is  $-\frac{\mu_{t+s}}{\mu_{t+s}-1} = \delta$

$$E_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\lambda_{t+s}}{\lambda_t} \left( \left( \frac{1}{P_{t+s}} \overline{P}_t^j \prod_{l=1}^s (\pi_{t+l-1}^l \pi^{1-l}) \right)^{\delta+1} \frac{Y_{t+s}}{P_{t+s}^{\delta}} - MC_{t+s} \left( \overline{P}_t^j \prod_{l=1}^s (\pi_{t+l-1}^l \pi^{1-l}) \right)^{\delta} \frac{Y_{t+s}}{P_{t+s}^{\delta}} \right) \quad (76)$$

it is

$$E_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\lambda_{t+s}}{\lambda_t} \left( \left( \frac{1}{P_{t+s}} \right)^{\delta} \overline{P}_t^{j\delta} (\delta+1) \left( \prod_{l=1}^s (\pi_{t+l-1}^l \pi^{1-l}) \right)^{\delta+1} \frac{Y_{t+s}}{P_{t+s}^{\delta}} \right. \\ \left. - E_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\lambda_{t+s}}{\lambda_t} \left( MC_{t+s} \delta \overline{P}_t^{j\delta-1} \left( \prod_{l=1}^s (\pi_{t+l-1}^l \pi^{1-l}) \right)^{\delta} \frac{Y_{t+s}}{P_{t+s}^{\delta}} \right) \right) = 0 \quad (77)$$

$$E_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\lambda_{t+s}}{\lambda_t} \left( \left( \overline{P}_t^{j*} \right)^{\delta} \left( \prod_{l=1}^s \frac{1}{\pi_{t+l}} \right)^{1+\delta} (1+\delta) \left( \prod_{l=1}^s (\pi_{t+l-1}^l \pi^{1-l}) \right)^{\delta+1} Y_{t+s} \right) \\ - E_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\lambda_{t+s}}{\lambda_t} \left( \delta MC_{t+s} \left( \overline{P}_t^{j*} \right)^{\delta-1} \left( \prod_{l=1}^s \frac{1}{\pi_{t+l}} \right)^{\delta} \left( \prod_{l=1}^s (\pi_{t+l-1}^l \pi^{1-l}) \right)^{\delta} Y_{t+s} \right) = 0 \quad (78)$$

From 58 we get

$$P_t = \left[ \theta (P_{t-1} \pi_{t-1}^l \pi^{1-l})^{\frac{1}{1-\mu_t}} + (1-\theta) \overline{P}_t^{\frac{1}{1-\mu_t}} \right]^{1-\mu_t} \quad (79)$$

$$\Leftrightarrow 1 = \left[ \theta (\pi_{t-1} \pi_{t-1}^l \pi^{1-l})^{\frac{1}{1-\mu_t}} + (1-\theta) (\overline{P}_t^*)^{\frac{1}{1-\mu_t}} \right]^{1-\mu_t} \quad (80)$$

## Wage Setting

Each household sells his labour due to the labour bundling function

$$N_t = \left( \int_0^1 (N_t^i)^{\frac{1}{\gamma_n}} \right)^{\gamma_n} \quad (81)$$

where  $\gamma_n$  is the wage elasticity and  $1 \leq \gamma_n < \infty$ . Similarly, the demand for labour is given by

$$N_t^i = \left( \frac{W_t^i}{\bar{W}_t} \right)^{\frac{\gamma_n}{1-\gamma_n}} N_t \quad (82)$$

Similarly to the firm's problem households face a random probability  $1 - \theta_h$  of changing nominal wage. The  $i^{th}$  households reoptimised wage is  $\bar{W}_t^i$ , whereas the unchanged wage is given by  $W_{t+1}^i = W_t^i \pi_t^{\iota_h} \pi^{1-\iota_h}$ , Households then maximise their optimal wage subject to the demand for labour and the budget constraint.

$$N_{t+s}^i = \left( \frac{\bar{W}_t^i \prod_{l=1}^s (\pi_{t+l-1}^{\iota_h} \pi^{1-\iota_h})}{W_{t+s}} \right)^{\frac{\gamma_n}{1-\gamma_n}} N_{t+s} \quad (83)$$

The Langrangian function is

$$E_t \sum_{s=0}^{\infty} (\beta \theta_h)^s \left( -\frac{\epsilon_{t+s}^L}{1 + \sigma_l} (N_{t+s}^i)^{1+\sigma_l} + \lambda_{t+s} \left( \frac{\bar{W}_t^i \prod_{l=1}^s (\pi_{t+l-1}^{\iota_h} \pi^{1-\iota_h})}{P_{t+s}} N_{t+s} \right) \right) \quad (84)$$

$$+ E_t \sum_{s=0}^{\infty} (\beta \theta_h)^s \left( \lambda_{t+s}^b \left( N_t^i - \left( \frac{\bar{W}_t^i \prod_{l=1}^s (\pi_{t+l-1}^{\iota_h} \pi^{1-\iota_h})}{W_{t+s}} \right)^{\frac{\gamma_n}{1-\gamma_n}} N_{t+s} \right) \right) \quad (85)$$

FOCs

$$\bar{W}_t^i : E_t \sum_{s=0}^{\infty} (\beta \theta_h)^s \left( \lambda_{t+s} \frac{\prod_{l=1}^s (\pi_{t+l-1}^{\iota_h} \pi^{1-\iota_h})}{P_{t+s}} N_{t+s} \right) \quad (86)$$

$$- E_t \sum_{s=0}^{\infty} (\beta \theta_h)^s \left( \epsilon_{t+s}^L \left( \frac{\bar{W}_t^i \prod_{l=1}^s (\pi_{t+l-1}^{\iota_h} \pi^{1-\iota_h})}{W_{t+s}} \right)^{\frac{\gamma_n}{1-\gamma_n}} N_{t+s} \right)^{\sigma_l+1} \frac{\gamma_n}{1-\gamma_n} \frac{1}{\bar{W}_t^i} \right) \quad (87)$$

Wages therefore evolve as

$$W_t = \left[ \theta_h (W_{t-1} \pi_{t-1}^{\iota_h} \pi^{1-\iota_h})^{\frac{1}{1-\gamma_n}} + (1 - \theta_h) \bar{W}_t^{\frac{1}{1-\gamma_n}} \right]^{1-\gamma_n} \quad (88)$$

## The Financial Sector

$$\frac{1}{R_t} = \beta E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} \right] \quad (89)$$

$$\frac{1}{R_t^*} = \beta E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} \frac{S_{t+1}}{S_t} \right] \quad (90)$$

Remember that

$$\lambda_t = \epsilon_t^U (C_t^i - h_t C_t - 1^i)^{-\sigma_c} - \beta h E_t [\epsilon_{t+1}^U (C_{t+1}^i - h C_t^i)^{-\sigma_c}]$$

The UIP condition is similarly given by the households FOCs

$$\frac{1}{R_t} = \frac{1}{R_t^*} E_t \left[ \frac{S_{t+1}}{S_t} \right] \quad (91)$$

Tobin's Q is given by  $q_t^i = \frac{\varphi_t^i}{\lambda_t^i}$  in

$$q_t^i = \beta E_t \left( \frac{\lambda_{t+1}}{\lambda_t} (R_t^k Z_t^i - a(Z_t^i)) + q_{t+1}^i (1 - \delta) \right) \quad (92)$$

More generally, the FOCs are given as

$$Equ_t^i : \quad -\frac{\lambda_t V_t^E}{P_t} + \beta \left( E \left( \lambda_{t+1} \frac{V_t^E + Div_{t+1}^i}{P_{t+1}} \right) \right) = 0 \quad (93)$$

$$B_{t,t+m}^i : \quad -\lambda_t \frac{V_{t,t+m}^B}{P_t} + E_t \left[ \beta \lambda_{t+1} \frac{V_{t+1,t+m-1}^B}{P_{t+1}} \right] = 0 \quad (94)$$

A real zero coupon bond returns one unit of consumption at maturity. For  $j = 1$  it is

$$-\lambda_t \frac{V_{t,1}^B}{P_t} = E_t \left[ \beta \lambda_{t+1} \frac{V_{t+1,0}}{P_{t+1}} \right] \quad (95)$$

$$\Leftrightarrow V_{t,1}^B = E_t \left[ -\beta V_{t+1,0} \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} \right] \quad (96)$$

For  $j = 2$  it is

$$-\lambda_t \frac{V_{t,2}^B}{P_t} = E_t \left[ \beta \lambda_{t+1} \frac{V_{t+1,1}}{P_{t+1}} \right] \quad (98)$$

$$\Leftrightarrow V_{t,2}^B = E_t \left[ -\beta V_{t+1,1} \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} \right] \quad (99)$$

$V_{t+1,1}$  is the price of a real bond of original maturity  $m = 2$  with one period left. Assuming no arbitrage, this price equals the price of a  $m = 1$  bond issued next period. Bond prices can thus be defined recursively (using  $SDF_t = \beta \frac{\lambda_{t+1}}{\lambda_t}$ )

$$V_{t,t+m}^B = E_t \left[ -\beta \frac{\lambda_{t+1}}{\lambda_t} V_{t+1,t+m-1}^B \frac{P_t}{P_{t+1}} \right] \quad (100)$$

$$= E_t(SDF_{t+1} \pi_{t+1} V_{t+1,t+m-1}^B) \quad (101)$$

Assuming  $V_{1,t} = 1$  in terms of one unit of consumption we apply recursion and get

$$V_{t,t+m}^B = E_t((SDF_{t+1} \pi_{t+1})^j) \quad (102)$$

Real yields are then given by

$$R_{t+1,t+m}^B = (V_{t,t+m}^B)^{-\frac{1}{j}} \quad (103)$$

Regarding the assets we derive

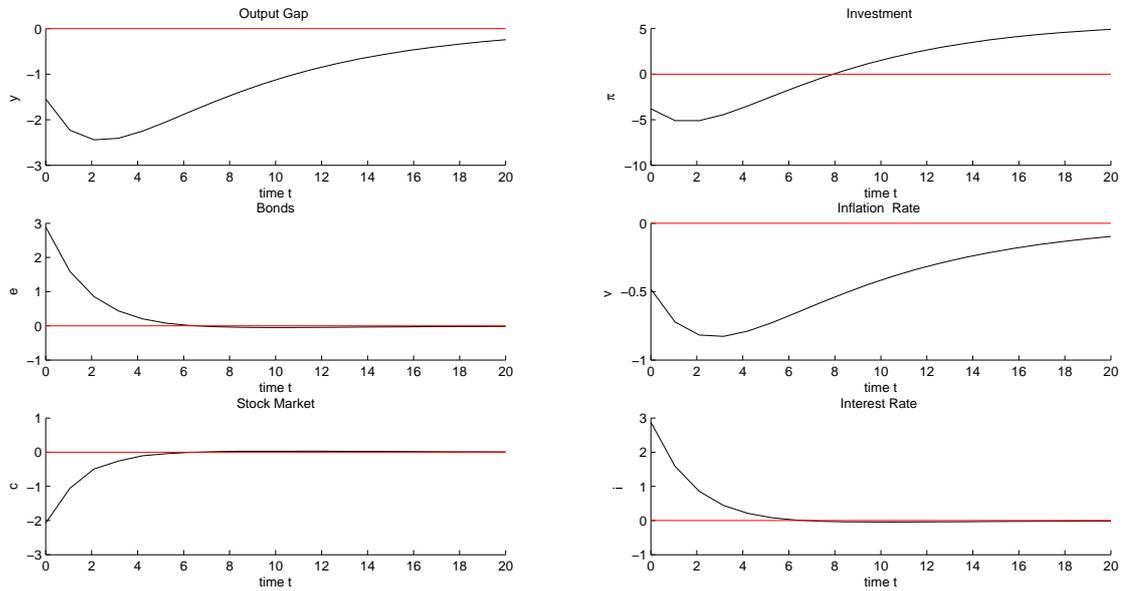
$$1 = E_t \left[ -\beta \frac{P_t}{P_{t+1}} \frac{\lambda_{t+1}}{\lambda_t} \frac{V_t^E + Div_{t+1}}{V_t^E} \right]$$

with real return

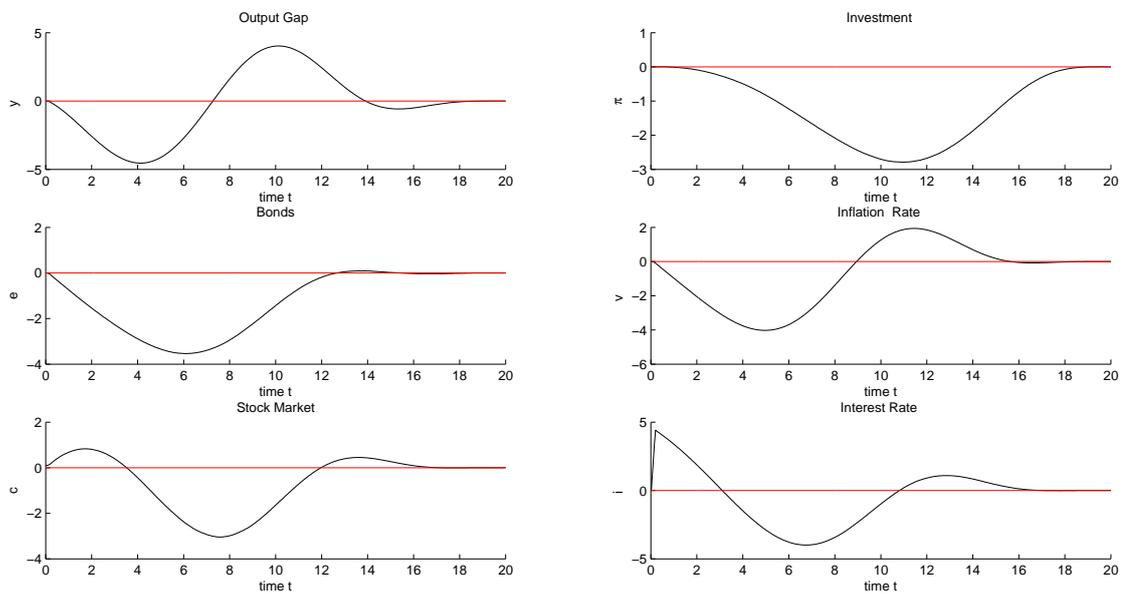
$$R_{t+1}^E = \frac{V_t^E + Div_t}{V_t^E} \frac{P_t}{P_{t+1}}$$

# 8 Figures and Tables

Figure 1: Contractionary Monetary Policy Shock

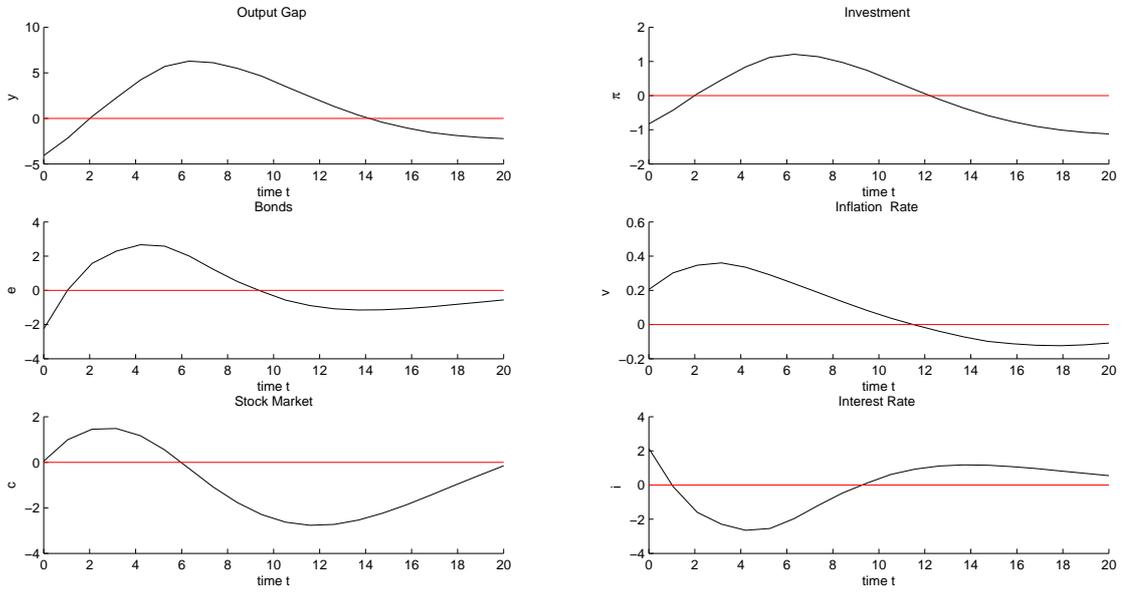


## Discrete Time

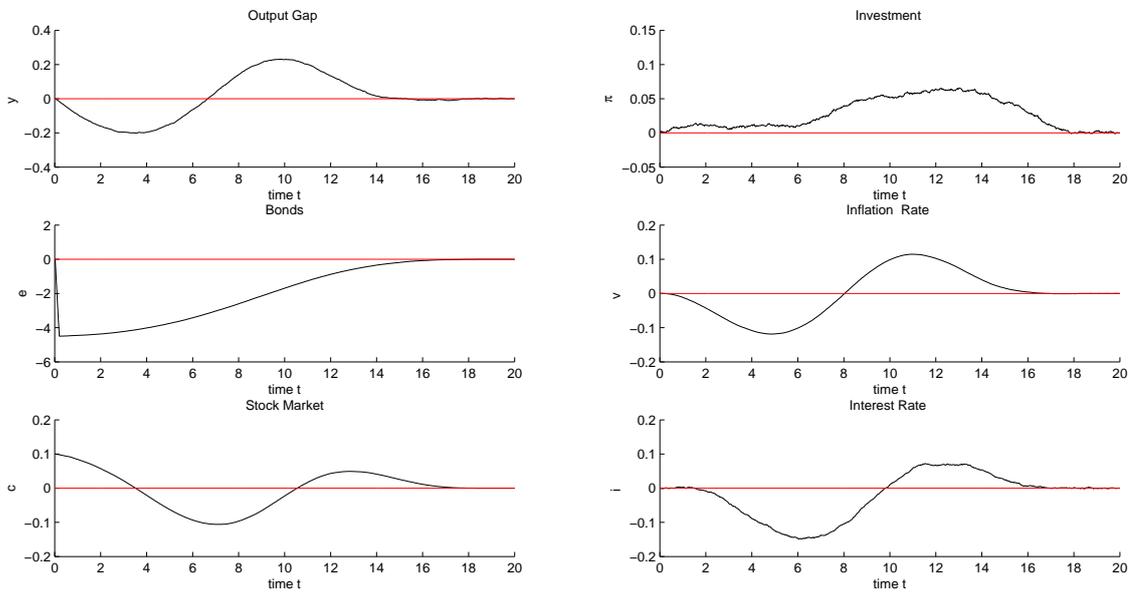


## Continuous Time

Figure 2: Negative Bond Market Shock

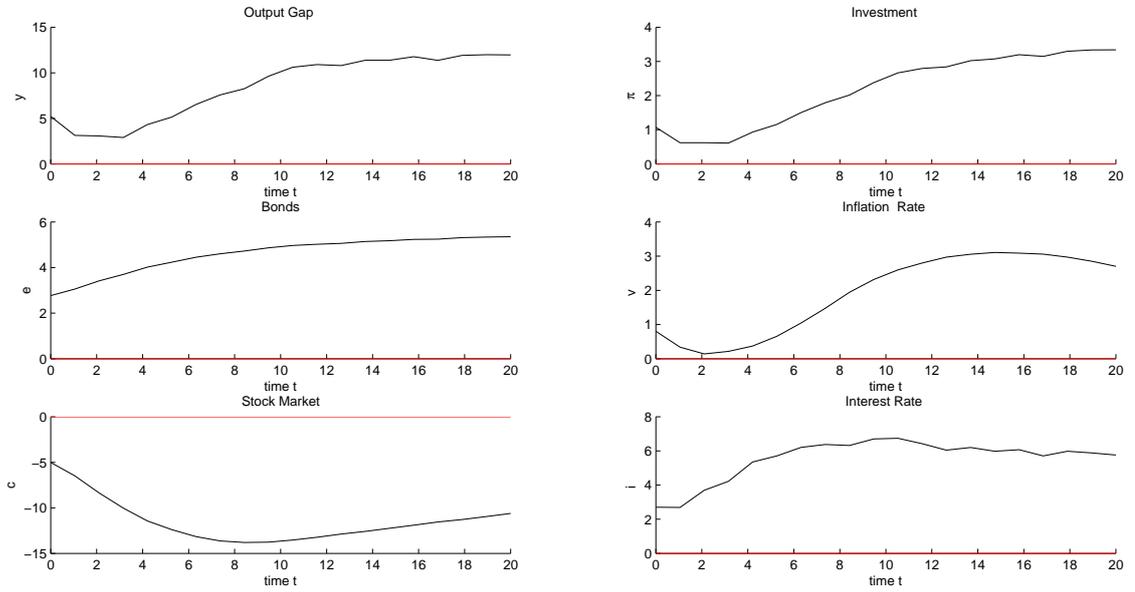


Discrete Time

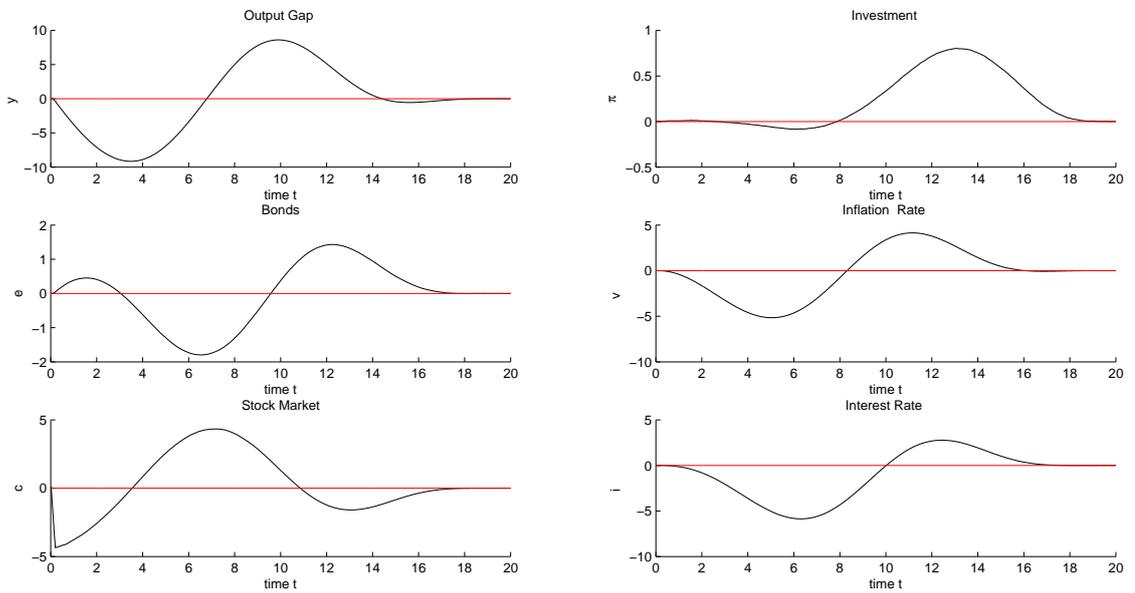


Continuous Time

Figure 3: Negative Stock Market Shock

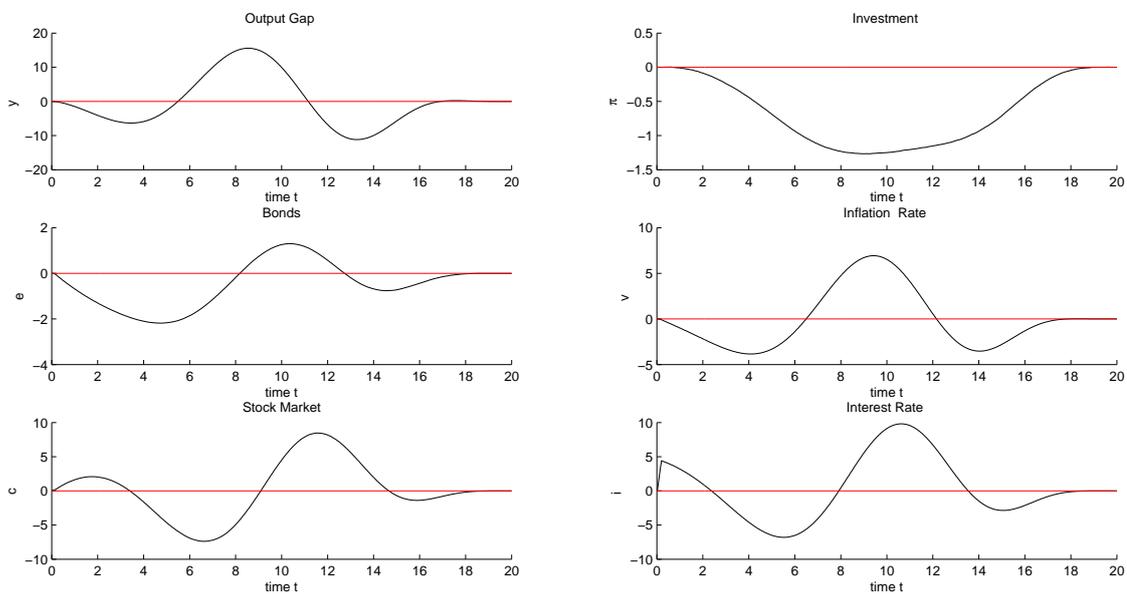


Discrete Time



Continuous Time

Figure 4: Impulse Response Function: USA



Contractionary Monetary Policy Shock

Table 1: Parameters, Priors and Posteriors

param	Value	distrib	prior		posterior	
			mean	variance	mean	variance
$h$	0.970	Beta	0.7	0.10	0.78	0.25
$\sigma_c$	1.000	N	1.5	0.40	0.98	0.98
$\sigma_l$	1.200	N	1.5	0.38	0.88	1.14
$\kappa$	9.510	N	4.0	1.50	10.1	4.23
$\alpha$	0.210	N	0.3	0.05	0.35	0.19
$\theta$	0.820	Beta	0.5	0.20	0.47	0.29
$\mu$	0.630	Beta	0.5	0.20	0.61	0.29
$\theta_w$	0.680	Beta	0.5	0.20	0.43	0.25
$\gamma_n$	0.620	Beta	0.5	0.20	0.62	0.27
$\rho_R$	0.770	Beta	0.5	0.20	0.90	0.26
$\psi_y$	0.190	Beta	0.5	0.20	0.57	0.27
$\psi_\pi$	1.290	N	0.0	2.0	5.01	6.35
$\rho_b$	0.200	N	0.0	2.0	1.77	2.74
$\rho_y$	0.200	N	0.0	2.0	0.13	0.12
$\rho_\pi$	0.200	N	0.0	2.0	2.14	1.56
$\gamma_S$	0.200	N	0.0	2.0	0.08	0.64
$\gamma_i$	0.200	N	0.0	2.0	0.46	1.41
$\gamma_y$	0.200	N	0.0	2.0	0.59	2.10
$\sigma_b$	0.500	-	-	-	-	-
$\sigma$	0.500	-	-	-	-	-
$\mu$	0.400	-	-	-	-	-
$\kappa$	1.500	-	-	-	-	-
$\beta$	0.980	-	-	-	-	-
$\eta_c$	0.600	-	-	-	-	-
$\gamma_1$	0.050	-	-	-	-	-
$\gamma_2$	0.001	-	-	-	-	-
$\iota$	0.710	-	-	-	-	-
$\iota_w$	0.680	-	-	-	-	-
$\delta$	0.025	-	-	-	-	-