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The strength to share: Adopting durable elite patronage

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The strength to share:

Adopting durable elite patronage

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Abstract

A common expectation is that weak incumbents are the ones most willing to rely on patronage, trading state resources for the support of elites, since that support can help them survive electoral competition. We show that this intuition changes once patronage is durable, developing an infinite-horizon model of electoral competition in which an incumbent can adopt such an exchange with elites once and for all. Because the channel attaches to office rather than to its creator, it can later benefit a rival who wins office after turnover. This creates a tradeoff: patronage raises the chance of staying in office today but builds an asset that may strengthen an opponent tomorrow. We characterize an adoption threshold and show that strong incumbents are more willing to adopt than weak ones, expecting a longer tenure in which to use the channel and a smaller risk of handing it to a rival. Secondary forces work through the same logic. Larger elite contributions widen adoption. Electoral stability generally favors it, though its effect becomes single-peaked for a sufficiently weak and patient incumbent. The effect of patience is single-peaked unless the incumbent is sufficiently strong, in which case adoption rises throughout.

Keywords: political patronage, hybrid regimes, incumbents, elites, elections, dynamic games

JEL: C73, D02, D72, D73

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1 Introduction

Patronage refers to the discretionary allocation of state resources in exchange for political support. In its broadest form, politicians and parties distribute public jobs, contracts, favors, protection, or other state-controlled benefits to those whose support can help them acquire, exercise, or retain power (Eisenstadt & Roniger, 1984; Hicken, 2011; Kitschelt & Wilkinson, 2007; Piattoni, 2001; Roniger & Güneş-Ayata, 1994; Scott, 1969). This paper studies a particular form of this exchange: the incumbent-elite patronage exchange, a *quid pro quo* in which the incumbent transfers state resources to the elites, who in turn commit private resources to strengthen the incumbent's position in office. The mechanism is related to broader accounts of political survival, elite co-optation, and patronal politics (Bueno de Mesquita et al., 2003; Gandhi & Przeworski, 2006; Hale, 2014), but differs in making the two-sided exchange between the incumbent and politically useful elites explicit.[1] This focus distinguishes the exchange we study from clientelism between politicians and individual voters. The relevant source of support is not a large body of isolated voters receiving minor favors, but a small set of elites whose private resources can shift political competition. For the incumbent, elite support raises the probability of remaining in office. For the elites, access to state resources raises the returns to their economic position.

The exchange is most consequential in regimes that combine meaningful electoral competition with weak constraints on executive discretion. Such regimes are described in different ways: competitive authoritarianism, electoral authoritarianism, defective democracy, informational autocracy, and hybrid regimes (Bogaards, 2009; Diamond, 2002; Guriev & Treisman, 2019; Levitsky & Way, 2002; Schedler, 2002). We use the last term throughout the paper. By hybrid regimes we mean political settings in which elections are competitive enough that incumbents face a real risk of losing office, while institutional checks are weak enough that incumbents can use state resources, administrative

[1] Elite support takes a variety of forms, including providing campaign contributions and financing campaign operations (Grossman & Helpman, 1996, 2001; Hillman & Ursprung, 1988), shaping the information environment through favorable media coverage (Besley & Prat, 2006; Guriev & Treisman, 2019; Szeidl & Szűcs, 2021), or mobilizing voters through local networks and organizational infrastructure (Stokes et al., 2013). What unifies these channels is a common logic: the elites commit private resources that lie beyond the incumbent's reach, such as private assets, organizational capacity, social standing, or control over information, while the incumbent compensates them with access to state resources.

authority, or regulatory discretion to advantage political allies (Brownlee, 2007; Levitsky & Way, 2010; Schedler, 2006).[2] More generally, political accountability and the organization of political decision-making shape how openly rents can be created and assigned (Hillman & Ursprung, 2016). Elections create demand for elite support because incumbents must mobilize votes, shape information, and deter defections, while weak constraints create supply because incumbents can reward supportive elites with access to state resources. Office is especially valuable in this setting because it confers both formal authority and access to rents, which the incumbent can appropriate (Ferejohn, 1986) or allocate to sustain political coalitions (Svolik, 2012).

The decision to adopt such an exchange is not random. A common expectation is that vulnerable incumbents are most likely to rely on patronage, since when electoral competition threatens their hold on office even a marginal gain in organized support can determine survival, and patronage converts control over the state into the co-optation of influential elites. Yet this view overlooks a defining feature of incumbent-elite patronage exchanges: once established, they tend to outlive the politicians who created them. Adopting the exchange does not deliver a one-time boost that fades after the election. It builds a durable political infrastructure: stable flows of state resources to the elites, shared expectations between the elites and the state apparatus, and settled routines that resist reversal once in place (Pierson, 2000). This infrastructure survives turnover because the conditions that produced it survive turnover. The elites who hold politically useful private resources remain in place after office changes hands (Acemoglu & Robinson, 2008), and any incoming incumbent faces the same electoral pressures that gave the original incumbent reason to compensate them. Dismantling the channel would do worse than waste an existing asset, since it would leave the elites free to deploy their resources against the new

[2] This definition places hybrid regimes between closed autocracies and consolidated democracies. Closed autocracies are outside our scope because elections do not create a meaningful risk of turnover. Consolidated democracies are not our main focus, since institutional oversight makes a durable incumbent-elite patronage exchange more costly to sustain, though the exchange can still arise where executive discretion remains high, auditing institutions are weak, or party organizations are sufficiently embedded to mediate access to state resources over time (Golden, 2003; Kopecký et al., 2012; Scheiner, 2006; Shefter, 1994; Warner, 2007). Informational autocracy is closely related to our setting because incumbents rely less on overt repression and more on information management, elite co-optation, and controlled political competition (Guriev & Treisman, 2019, 2020, 2022). The concept is not identical to hybrid regime, but it identifies a central mechanism through which elite support helps incumbents survive under formally competitive institutions.

incumbent and force the costly construction of alternative networks. The same logic that led to adoption therefore gives every successor reason to keep the channel running.[3]

This durability is central to the incumbent's decision. The advantage the channel generates accrues to whoever holds office in the future, including a successor who did not adopt it, so the choice extends beyond the next election to a longer-run question about future control of office. This raises the central question of the paper: under what conditions does an incumbent choose to adopt an exchange whose benefits may eventually serve a successor?

To investigate this question, we develop an infinite-horizon model of electoral competition between two office-seeking politicians. In every period, citizens choose between the incumbent and the opposition. Some voters are more inclined toward one politician than the other, while election-level conditions can move the electorate in a common direction. Office is valuable because only the incumbent controls the state's rents and can appropriate them while in power, so holding office is itself the prize the politicians compete for. At the outset, the initial incumbent makes a once-and-for-all decision between no-patronage and patronage that applies in every future period regardless of who holds office. Under no-patronage the full office rent is retained by the

[3] Comparative evidence supports this pattern. Patronage networks have survived leadership transitions across a wide range of settings. Broad studies cover sub-Saharan Africa across the shift from single-party rule to multiparty competition during the third wave of democratization (Van de Walle, 2007), post-Soviet Eurasia under patronal politics (Hale, 2014), the United States across generations of incumbents (Shefter, 1994), Mexico under the Institutional Revolutionary Party (PRI) (Magaloni, 2006), and European democracies across repeated changes of governing party (Kopecký et al., 2012). Three additional examples illustrate the persistence more directly. In Greece, Pappas and Assimakopoulou (2012) document the depth of party patronage, while Eleftheriadis (2014) describes the enduring political role of shipowners, media owners, and other influential actors across governments and electoral cycles. In Italy, patronage practices did not disappear with alternation in office or changes in party configuration: during the Silvio Berlusconi era, political influence rooted in ties among party organization, economic power, and media control remained relevant across successive rounds of competition, and similar structures continued to shape political dynamics afterward (Di Mascio, 2012; Fabbrini, 2013). In Malaysia, under competitive authoritarian rule, the United Malays National Organisation (UMNO) assembled an extensive patronage system with Malay business elites over decades of dominance, and after the coalition lost power in 2018, many of the machine-based, patronage, and clientelist structures formed under the previous regime remained politically relevant (Gomez & Jomo, 1999; Slater, 2010; Weiss, 2020). Taken together, the settings discussed here do not all map perfectly onto the incumbent-elite patronage exchange modeled in this paper. Some involve broader patronage systems, voter-facing party machines, or diffuse elite networks. They nevertheless support the core persistence assumption: once an incumbent-elite patronage exchange, or a closely related patronage channel, becomes embedded in political competition, it can outlast the incumbent or party that first benefited from it.

current incumbent, but no elite support is available. Under patronage, rents are diverted to the elites in every future period in exchange for support that raises the probability of reelection of whoever is then the incumbent. The elites are modeled as a non-strategic source of support: once the channel is activated, they receive a transfer from office rents in each period and provide support to the incumbent. Section 5 returns to additional considerations concerning the role of the incumbent, voters, and elites.

The mechanism that drives the analysis is the durability of the channel. While the incumbent remains in power, it converts rents into elite support and improves reelection prospects, and after turnover the same channel becomes available to the successor. Patronage therefore generates a political asset that can strengthen the incumbent today but may later be used by a successor. The transfer required to activate this channel is exogenous: it is the realized price of elite support under the prevailing institutional environment rather than a choice of the incumbent.[4] The model characterizes an adoption threshold that separates the transfers at which patronage is adopted from those at which it is rejected: patronage is adopted when the required transfer falls below the threshold and rejected when it lies above. This threshold has a clear interpretation: it is the largest share of office rents the incumbent will give up in each period of incumbency to secure elite support, and it is the central object of the analysis.

Electoral outcomes are determined by a probabilistic voting framework in which an aggregate electoral shock creates election-level uncertainty.[5] Two parameters govern the distribution of this shock. Incumbent strength captures how favorable the electoral environment is to the incumbent on average. Electoral stability captures how predictable outcomes are from one election to the next, so that a more stable environment is one in which strength translates more reliably into winning. The central comparative statics ask how the initial

[4]Treating the required transfer as exogenous reflects the idea that political actors operate within institutional environments whose main constraints are not chosen inside the model. North (1990) emphasizes that institutions structure the incentives and feasible actions of political actors. In our setting, the institutional environment determines how costly it is to convert office rents into elite support, while the incumbent decides only whether that cost is worth accepting.

[5]The probabilistic voting structure follows Lindbeck and Weibull (1987) and Persson and Tabellini (2000). The framework is widely used because it captures electoral uncertainty while preserving a tractable link between political advantages and winning probabilities. The shock represents campaign conditions, economic news, scandals, or other election-level forces that affect broad segments of the electorate at the same time.

incumbent's strength and discount factor, the stability of the electoral environment, and the elite contribution shape the willingness to accept the recurring transfer.

We begin with incumbent strength, which constitutes the central result of the model. Because the channel is durable, its value depends on how long the incumbent expects to control office. Strong incumbents are more willing to adopt because a longer expected tenure gives more periods in which to draw on the channel and recover the rent diverted to the elites, and lowers the chance that the channel soon passes to a successor. Weak incumbents face the opposite logic: elite support remains valuable in the short run, but shorter expected tenure leaves fewer periods in which to recover the diverted rent, while a higher risk of turnover raises the chance that the same channel eventually strengthens a successor. The elite contribution works in the expected direction: a larger contribution raises the per-period value of the channel and makes the incumbent willing to accept a higher transfer to obtain it. The model therefore reverses the usual expectation that electoral insecurity should increase reliance on patronage, the common intuition in distributive politics under which targeted state resources matter most when marginal support can decide the outcome (Dixit & Londregan, 1996; Stokes, 2005). The mechanism instead points to political security, organizational control, and a long expected tenure in office as the conditions under which durable patronage becomes easier to sustain.[6]

Electoral stability shapes patronage because it changes how surely today's political advantage carries into tomorrow. When elections are more stable, the support an incumbent buys from elites is less likely to be overturned by chance, so it more reliably secures another term in office and the patronage channel becomes worth more. For most incumbents this makes patronage more attractive as elections grow more stable. The relationship reverses only for an incumbent who is at once weak and patient. There, greater stability eventually

[6]Several findings fit parts of this logic. Dominant parties build their patronage capacity while electorally strong, as the PRI did through its command of public resources (Greene, 2010). Established ruling parties reward loyal politicians with legislative leadership posts (Reuter & Turovsky, 2014), and leaders already holding office widen cabinet patronage to lengthen their tenure (Arriola, 2009). Durable business-elite ties rest on privileged access to the state (Rithmire, 2023; Sinanoglu, 2025). These are not exact instances of our exchange, yet in each the co-opting actor is already strong, a dominant party, an entrenched ruling party, or a leader who controls the state, which is the position of strength from which the exchange is sustained.

works the other way, because it makes the weakness persist rather than fade, so a loss of office becomes more likely and the channel passes to a successor, which makes patronage less worth adopting. The effect of stability on patronage is therefore single-peaked, rising at first and falling once the electoral environment becomes stable enough, and only for an incumbent both weak and patient enough for the later loss of office to outweigh the early advantage.[7]

Patience matters because it changes how much the future weighs against the present, and it has a natural reading in terms of the incumbent's political horizon.[8] Since the patronage channel pays off only over time, its effect on adoption is single-peaked for most incumbents, rising at first and then falling. When patience is low, what counts is the near future, in which an incumbent who adopts is still likely to hold office and to draw on the elite support, so adoption grows more attractive. When patience is high, the later future counts as well, and the incumbent is likely to have lost office by then, so the channel comes to serve a successor instead, which makes adoption less attractive. Only for a sufficiently strong incumbent does the threshold rise throughout, since the prospect of the channel passing to a successor never comes to dominate. This horizon reading also situates the mechanism against work where the effect of political horizons varies across institutional settings: term limits, for instance, can alter policy choices and electoral accountability as incumbents approach their final eligible term (Smart & Sturm, 2013). Here the single-peaked shape arises for a different reason, namely whether the advantage generated by adoption remains with the incumbent or passes to a successor.

The paper contributes to three related bodies of work. The first brings together work on patronage, clientelism, campaign finance, and distributive politics. Much of this literature models patronage as a contingent electoral exchange: politicians allocate targeted benefits to voters, brokers, or organized

[7]This connects to a broader literature on how stability and uncertainty shape intertemporal political choices (Alesina & Tabellini, 1990; Persson & Svensson, 1989; Przeworski, 1991). In that literature the stability of political and economic conditions alters the stakes of redistribution and the value of holding power (Acemoglu & Robinson, 2005; Boix, 2003), and the same reasoning extends to dynamic electoral settings, where a current choice carries consequences into later periods (Besley & Coate, 1998).

[8]Related work shows how sufficiently long political horizons can make current policy responsive to future electoral support. For example, Hillman and Long (2022) show that immigration policy may reflect politicians' incentives to consider immigrants as future voters. More broadly, longer political time horizons can encourage investment in governance, institutions, and state capacity because leaders who value the future internalize more of the returns from current choices (Clague et al., 1996; McGuire & Olson, 1996; Olson, 1993).

constituencies in return for political support, and continued access to those benefits depends on political success (Dixit & Londregan, 1996; Robinson & Verdier, 2013; Stokes, 2005). Related models address why parties use targeted transfers rather than public goods (Lizzeri & Persico, 2001), how brokers mediate the delivery of benefits (Calvo & Murillo, 2004; Nichter, 2008), or how outside contributions shift electoral incentives (Baron, 1994; Grossman & Helpman, 2001). Wong and Marinov (2026) show that campaign contributions can generate policy divergence by raising the turnout of impressionable voters, particularly when only the incumbent can benefit. We share the feature that the advantage attaches asymmetrically to the incumbent, but the margin differs. Their contributions come from outside the state and shift current competition, whereas in our setting the incumbent diverts office rents from within it to sustain a durable channel whose advantage may later accrue to a successor.

The second relates to work on the co-optation of economic elites and the rents, regulation, and corruption that sustain it. Classic accounts of regulation and rent-seeking explain how public authority creates valuable privileges for private actors (Krueger, 1974; Stigler, 1971). Political-support models add that governments may choose protection and other interventions in response to the support generated by gainers and losers, rather than solely to advance social welfare (Hillman, 1982). A separate tradition on corruption and rent extraction treats political control over public decisions as a good that can be priced and exchanged (McChesney, 1987, 1997; Shleifer & Vishny, 1993). These arguments clarify why the elites value access to the state. A further strand on elite-centered political survival examines the reverse side of the exchange. Selectorate theory treats private goods to a winning coalition as central to leader survival (Bueno de Mesquita et al., 2003). A separate line examines how authoritarian institutions and power-sharing arrangements help rulers contain elite defection through controlled access to rents (Boix & Svolik, 2013; Gandhi & Przeworski, 2006; Svolik, 2012). Other studies analyze ethnic favoritism, cabinet allocation, oligarchic property rights, informational control, or business co-optation as mechanisms through which rulers maintain the loyalty of powerful groups (Francois et al., 2015; Guriev & Sonin, 2009; Guriev & Treisman, 2020; Padró i Miquel, 2007; Rithmire, 2023; Sinanoglu, 2025).[9] These accounts treat exchanges between rulers and elites as a feature of the

[9]Further examples include kleptocracy and divide-and-rule strategies (Acemoglu et al., 2004), as well as the institutional foundations of political authority (Myerson, 2008).

political setting and focus on how they are sustained. Our contribution steps back to the prior decision: when does an incumbent choose to adopt such an exchange in the first place?

The third draws on dynamic models of incumbency advantage, institutional persistence, and intertemporal political choice. A related rent-seeking literature studies rents that endure across periods even though rights to them may be re-contested or eliminated by future governments (Aidt & Hillman, 2008). Repeated-election models examine how incumbency advantages arise from credibility, information, or political budgeting, and how such advantages shape electoral competition over time (Alesina, 1988; Bernhardt et al., 2004; Besley & Coate, 1998).[10] A separate strand shows how current political actors may preserve inefficient arrangements, shape property rights, or choose rules that constrain future incumbents (Acemoglu & Robinson, 2008; Lagunoff, 2009). The closest precedent for our dynamic structure is Dunning (2010), where the initial incumbent fixes central-government oil rents high or low once and for all, and a weak incumbent may keep them low because high rents would otherwise strengthen a likely successor. We share the insight that an office-based advantage is less attractive when future control of office is uncertain, so that strength sustains the willingness to commit. Our mechanism extends it: patronage is not a level of rent but a recurring transfer paid in every period of incumbency, so strength matters on two fronts at once. A strong incumbent collects the residual rent over a longer tenure and runs a smaller risk of equipping a successor, while a weak incumbent gains on neither. The single-peaked effects of electoral stability and patience on a weak incumbent's adoption decision follow from this tradeoff.

The remainder of the paper proceeds as follows. Section 2 develops the model. Section 3 presents the results. Section 4 provides the political context for the analysis. Section 5 discusses extensions and implications. Section 6 concludes.

[10] Related contributions provide existence results for Markov equilibria in dynamic elections (Duggan, 2000) and analyze political selection and the persistence of bad governments (Acemoglu et al., 2010).

2 Model

2.1 Setup

We study an infinite-horizon electoral competition between two politicians, denoted by α and β. Time is discrete, $t = 0, 1, 2, \ldots$. In each period, one politician holds office and the other remains outside office. We refer to the politician in office as the *incumbent*, indexed by $i \in \{\alpha, \beta\}$, and to the other politician as the *opposition*, indexed by $o \in \{\alpha, \beta\} \setminus \{i\}$. Elections determine whether these roles persist or switch across periods. Thus, the politician who is the incumbent in period t may become the opposition in period $t+1$, and the value of any advantage tied to office depends on who holds it over time.

There is a continuum of citizens $j \in [0, 1]$ with unit mass. In every period, citizens vote in an election that determines which politician holds office in the next period. We use a probabilistic voting framework in which electoral outcomes depend on voters' individual preferences, an aggregate electoral shock, and any political advantage attached to the incumbent. The feature that matters for the analysis is monotonicity: when the incumbent receives a larger advantage, the probability of reelection increases. This allows us to represent elite support as an electoral benefit that can shift the probability of holding office without modeling every channel through which elites influence voters.

A central element of the model is that this advantage can be generated by the elites. Let the group of elites have measure $e > 0$, fixed over time. The elites matter because they can provide political support, not because their own votes are electorally decisive, so we treat their direct electoral weight as negligible: as the mass of elites tends to zero, their votes have no direct effect on the outcome. This isolates the role of elite support as an organized political contribution rather than as a bloc of votes, as captured by Assumption 1.

Assumption 1. ***Negligible electoral mass of elites***
In every period t, the mass of elites e is treated as vanishingly small, so their own votes have no direct effect on the electoral outcome.

The elites matter through the support they give the incumbent.[11] We represent

[11]This reflects a common asymmetry in political settings where large electorates coexist with a small number of influential actors. The elites are too few to affect electoral outcomes through

this support as a fixed contribution c that improves the incumbent's standing with voters.[12] The contribution can take several forms, such as financing campaign operations, providing favorable media coverage, or mobilizing voters through local networks and organizational infrastructure. The elites hold a resource endowment $\bar{c}$, and the contribution c is the share of it they put to work for the incumbent. This endowment is large enough to cover the contribution, so that $c \leq \bar{c}$. We take the contribution as fixed rather than as a quantity the elites fine-tune from one election to the next, because the resources behind it, an established media outlet, a standing local network, a habitual base of campaign funding, are assets already in place rather than dials reset each time. We model the elites as a non-strategic source of support.[13]

What the incumbent gives in return comes from the rent of office. Holding office gives the incumbent control over an exogenous rent $R > 0$ in each period, representing the material value of office: access to state revenues, discretion over public budgets, control over procurement, and other state-controlled resources. The rent is fixed over time and common knowledge, and part of it can be transferred to the elites in exchange for their support. Patronage therefore raises the probability of holding office while reducing the rent the incumbent retains in each period of incumbency.

This is the exchange the patronage channel captures, a *quid pro quo* in which the incumbent gives up part of the office rent and the elites provide support that improves the incumbent's electoral position. In each period, the institutional environment determines a required transfer

$$x \in [0, R). \tag{1}$$

The transfer x is the price of activating elite support in that period, for example a public contract steered to an allied firm, a license granted on easy terms, cheap state credit, or official protection from scrutiny. As with the contribution, we also treat this price as fixed, because the institutions that set it, the strength of oversight and the going terms for access to the state, move slowly relative to

their own ballots, but they can matter because they control resources and organizational channels that can be deployed on behalf of a politician.

[12]The contribution c can also be interpreted as effort rather than as a material resource.

[13]This abstracts from bargaining, internal coordination problems, and heterogeneity within the elite group, and isolates the incumbent's decision to adopt the patronage channel. In settings where elite coordination is contested, the elite group can be interpreted as a dominant coalition or as the organized segment of the elites whose support is politically relevant.

the electoral choices we study. The incumbent does not choose it, but observes it at the outset and decides whether to accept it. If the incumbent accepts, x is diverted to the elites, the incumbent retains $R - x$, and the elites provide the contribution c. If the incumbent rejects, the full rent R is retained, but no contribution is obtained. Consistent with their non-strategic role, once patronage is chosen the elites accept the transfer and provide the contribution.

Treating x as exogenous reflects the institutional environment in which the exchange takes place: it is the price at which elite support can be activated, not a quantity the incumbent sets. The model therefore does not ask what determines this price. It asks how high a transfer the incumbent is willing to accept, and how the remaining parameters shape this adoption threshold.

The elite contribution is strictly positive whenever the channel is active, since it represents genuine political support supplied to the incumbent. This is captured in Assumption 2.

Assumption 2. ***Positive elite contribution***
For any required transfer $x \in [0, R)$, the induced elite contribution satisfies

$$c > 0. \tag{2}$$

The contribution is positive by construction, since c represents political support supplied to the incumbent, and allowing the elites to undermine the incumbent would require a different object than the patronage channel modeled here. We impose no ordering between the contribution and the transfer, since the two are measured in different units: c is political support and enters the voting rule in the units of the electoral shock and the voter bias, while x is an amount of rent. What ties the contribution to the transfer is its backing rather than its size, as the recurring transfer is what sustains the elites' participation in the exchange.[14] The non-strategic treatment fits the same logic: the resources the elites deploy derive their value from continued access to the state, so support tends to follow whoever grants that access rather than being strategically withheld, which makes a steady exchange of the contribution for the transfer a natural description of how these relationships operate once in place.

A further feature of the exchange is that only the incumbent can operate it. Control of office provides the administrative authority, state resources,

[14] Non-material motives such as ideology or loyalty could also sustain elite support, but we abstract from these and treat the contribution as backed by the recurring transfer.

and discretion over implementation that make the transfer feasible before the election, instruments the opposition lacks while out of office.[15] This incumbency-specific capacity still does not make elections non-competitive: patronage tilts the contest by shifting the electoral environment in the incumbent's favor, but it does not eliminate the possibility of turnover.

The final feature of the exchange is its durability. Once activated, the channel remains available to whoever holds office in later periods, so the initial incumbent's choice extends beyond the current election to every future incumbent operating under it. This is captured in Assumption 3.

Assumption 3. ***Durability of the patronage channel***
At $t = 0$, the initial incumbent chooses $d \in \{0, 1\}$, where $d = 0$ denotes no-patronage *and $d = 1$ denotes* patronage*. The choice of d remains fixed for all subsequent periods.*

Under $d = 0$ the environment remains under no-patronage, and the incumbent in each period retains the full rent but receives no elite contribution. Under $d = 1$ it remains under patronage, and the incumbent in each period transfers rents to the elites and receives their support.[16]

Durability captures a structural feature of the channel: it is institutional, not personal. The instruments through which rents are diverted, the routines that govern their allocation, and the elites who supply support all remain in place when office changes hands, so a new incumbent inherits a functioning channel rather than building one from scratch. Dismantling it would release the elites to deploy their resources against the new incumbent and force the costly construction of alternative networks, which gives every successor reason to keep it running. The model takes this in its sharpest form, with an irreversible choice that makes the dependence of adoption on future control of office transparent.

Together with competition between the same two politicians across periods, this establishes the central mechanism. The channel is durable, the opposition can later hold office, and the residual rent $R - x$ is collected only by whoever

[15] In such settings the incumbent controls state tools to reward allied elites and can limit the opposition's access to similar support, through funding, organization, media exposure, or local networks.

[16] For simplicity, we assume that the channel persists indefinitely, which keeps both the solution and the notation tractable. The results do not depend on this assumption. The same logic operates when the channel persists for a finite number of periods, since the incumbent still trades current rent diversion for a future advantage that may serve a successor.

holds office at the time, so the value of adoption depends on who is expected to control office over time. This dependence is the source of the incentives that govern the adoption decision.

Having defined the patronage channel, we turn to the electoral environment and show how elite support enters electoral outcomes and shapes the incumbent's probability of remaining in office.

2.2 Voting

At the beginning of the game, the initial incumbent makes the adoption decision

$$d \in \{0, 1\}, \tag{3}$$

where $d = 0$ denotes no-patronage and $d = 1$ denotes patronage. Under no-patronage, the incumbent relies only on the ordinary electoral environment. Under patronage, the channel is active and the incumbent receives elite support in exchange for rents transferred to the elites.

In every period t, citizens vote between the incumbent i and the opposition o. Let ω^j denote voter j's individual bias in favor of β relative to α. Thus, $\omega^j > 0$ makes voter j more favorable to β, while $\omega^j < 0$ makes voter j more favorable to α. Let ζ denote an aggregate electoral shock that moves all voters in the same direction. We use the same sign convention: $\zeta > 0$ shifts the electorate toward β, while $\zeta < 0$ shifts it toward α.

The patronage channel affects the election through the elite contribution c. The resulting advantage attached to incumbency is

$$C = d \cdot c. \tag{4}$$

Thus, C is zero under no-patronage and equals the elite contribution under patronage.[17] Since $c > 0$, the patronage channel can only shift the electoral environment in favor of the incumbent.

A voter supports the incumbent whenever the incumbency advantage is at least as large as the combined effect of the electoral shock and the voter's

[17]We take the incumbency advantage to be one-to-one in the elite contribution for simplicity. Nothing of substance changes if instead $C = d \cdot f(c)$ for some strictly increasing function f, since what matters is only that more elite support raises the advantage attached to incumbency.

individual bias. Formally, voter j votes for the incumbent if and only if

$$C \geq \omega^j + \zeta. \tag{5}$$

If inequality (5) fails, voter j votes for the opposition.[18] This rule makes the role of elite support transparent: a higher C relaxes the condition under which the incumbent wins a given voter and therefore increases the probability of reelection.

Equation (5) nests the two political environments. Under no-patronage, the incumbent receives no elite advantage, so voter j's choice is governed only by ω^j and ζ. If α is the incumbent, voter j votes for α if and only if

$$0 \geq \omega^j + \zeta. \tag{6}$$

The analogous condition applies when β is the incumbent, with the roles of α and β reversed.

Under patronage, the incumbent receives the elite advantage c. If α is the incumbent, voter j votes for α if and only if

$$c \geq \omega^j + \zeta. \tag{7}$$

If α is the opposition, then β is the incumbent and receives the advantage. Voter j votes for α if and only if

$$0 \geq c + \omega^j + \zeta. \tag{8}$$

Otherwise, voter j votes for β. Equations (7) and (8) show that the elite contribution shifts the voting threshold toward the incumbent, an advantage the opposition cannot access within the same period.

We now specify the distributions of the individual voter bias ω^j and the aggregate electoral shock ζ. The voter-specific bias is uniformly distributed,

$$\omega^j \sim U\left[-\frac{1}{2\sigma_\kappa}, \frac{1}{2\sigma_\kappa}\right], \qquad \sigma_\kappa > 0, \tag{9}$$

[18] An indifferent voter, for whom $C = \omega^j + \zeta$, is counted for the incumbent, reflecting a mild preference for the status quo. The convention is in any case immaterial, since indifference pins down a single value of ω^j and the bias is continuously distributed, so indifferent voters form a set of measure zero.

independently across voters $j \in [0,1]$. Its support is symmetric around zero, so individual biases create no aggregate advantage for either politician. Since the support width is $1/\sigma_\kappa$, the density is

$$f_\omega(w) = \sigma_\kappa \qquad \text{for } w \in \left[-\frac{1}{2\sigma_\kappa}, \frac{1}{2\sigma_\kappa}\right]. \tag{10}$$

The aggregate electoral shock is uniformly distributed with location parameter $\mu \in \mathbb{R}$,

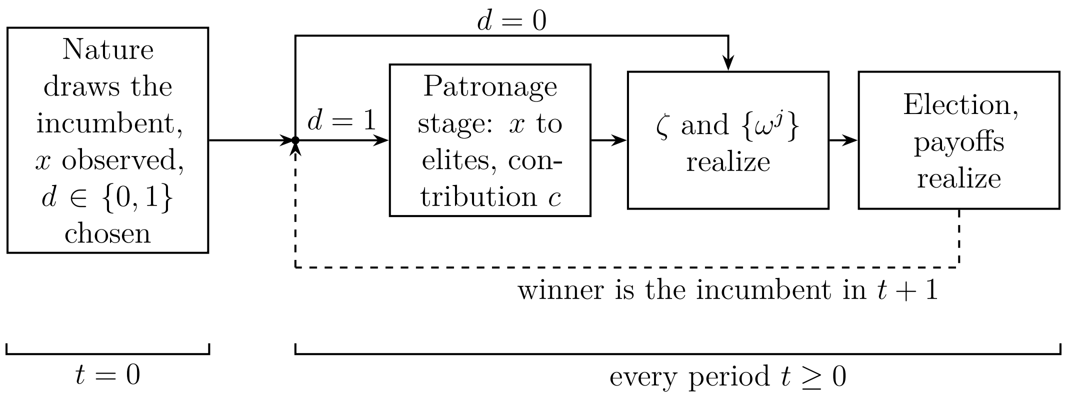
$$\zeta \sim U\left[\mu - \frac{1}{2\sigma_\lambda}, \mu + \frac{1}{2\sigma_\lambda}\right], \qquad \sigma_\lambda > 0. \tag{11}$$

The parameter μ shifts the distribution without changing its dispersion. Since positive values of ζ favor β, a positive μ is an aggregate bias toward β and a negative μ an aggregate bias toward α. Substantively, μ captures the average direction of election-level conditions, such as broad evaluations of competence, economic performance, scandals, security events, or other common factors that move voters toward one politician rather than the other. The support width of ζ is $1/\sigma_\lambda$, so the density is

$$f_\zeta(z) = \sigma_\lambda \qquad \text{for } z \in \left[\mu - \frac{1}{2\sigma_\lambda}, \mu + \frac{1}{2\sigma_\lambda}\right]. \tag{12}$$

The parameter σ_λ governs electoral stability. A larger σ_λ narrows the support of the shock, so election-level conditions vary less from one election to the next and the electoral shock is less able to swing the result. Outcomes then track the incumbent's standing and advantage more closely, so the average electoral environment and the elite advantage C translate more reliably into reelection. A smaller σ_λ widens the support, so the shock is more volatile and can more easily overturn that advantage at the polls.

Figure 1: Timing of the game



Definition 1. ***Strength and electoral stability***

1. *The parameter $\mu \in \mathbb{R}$ measures incumbent strength. We say that α is* strong *when $\mu < 0$ and* weak *when $\mu > 0$.*[19] *The symmetric statement applies to β, who is* strong *when $\mu > 0$ and* weak *when $\mu < 0$. When $\mu = 0$, the electoral environment is* balanced *between α and β.*

2. *The parameter $\sigma_\lambda > 0$ measures electoral stability. We say that the electoral environment is* stable *when σ_λ is high and* volatile *when σ_λ is low.*

Politicians discount future payoffs with a common per-period discount factor $\delta \in (0,1)$, where a higher δ means a more patient incumbent who places more weight on future periods.[20]

Before turning to the no-patronage and patronage cases, in which we derive the winning probabilities of each politician in each role, we impose a single parameter restriction that secures well-defined winning probabilities throughout the analysis.

[19]Other labels could serve equally well, such as *popular* and *unpopular*, since μ captures how favorable the electoral environment is to the incumbent on average.

[20]We use *patience* and *discount factor* interchangeably throughout.

Assumption 4. ***Well-defined winning probabilities***
The model parameters satisfy

$$c + |\mu| \leq \frac{1}{2\sigma_\lambda} \iff \sigma_\lambda \leq \frac{1}{2(c + |\mu|)}. \tag{13}$$

Assumption 4 ensures that every winning probability derived below lies in $[0, 1]$. Intuitively, it bounds the combined effect of the elite contribution and the average electoral environment so that the implied vote shares stay within the range generated by the support of ζ. The role of the elite contribution is transparent. Under no-patronage, c drops out of winning probabilities and the condition reduces to $|\mu| \leq 1/(2\sigma_\lambda)$, which keeps zero within the support of ζ. Under patronage, the elite contribution shifts the relevant threshold by exactly c, so the same logic requires an additional margin of c. The combined condition $c + |\mu| \leq 1/(2\sigma_\lambda)$ covers both cases.

The condition also carries a substantive reading. The right-hand side $1/(2\sigma_\lambda)$ is half the width of the support of ζ, so it shrinks as the environment becomes more stable. The assumption therefore requires that the combined advantage from the elite contribution and incumbent strength remain no greater than the room left by electoral volatility. Away from the boundary of the assumption, neither can settle the election in advance, so the electoral shock retains enough weight to keep turnover possible and the advantage adoption confers remains one that a future election can take away.

2.2.1 Winning probabilities under no-patronage

We begin with the no-patronage case, so $d = 0$ and $C = 0$ in Equation (4). Consider a realized aggregate electoral shock ζ. When α is the incumbent, voter j supports α if and only if

$$0 \geq \omega^j + \zeta. \tag{14}$$

Equivalently, there is a cutoff $\hat{\omega}(\zeta)$ such that voters with $\omega^j \leq \hat{\omega}(\zeta)$ vote for α, while voters with $\omega^j > \hat{\omega}(\zeta)$ vote for β. Solving Equation (14) gives

$$\hat{\omega}(\zeta) = -\zeta. \tag{15}$$

Since $\omega^j \sim U\left[-\frac{1}{2\sigma_\kappa}, \frac{1}{2\sigma_\kappa}\right]$ with density $f_\omega(w) = \sigma_\kappa$ on its support, the vote share of α when α is the incumbent is

$$\nu_\alpha^i(\zeta) = \int_{-\frac{1}{2\sigma_\kappa}}^{-\zeta} \sigma_\kappa \, \mathrm{d}w = \frac{1}{2} - \sigma_\kappa \zeta, \tag{16}$$

where the superscript i denotes that α is the incumbent.[21] In the no-patronage case, the voting rule does not depend on incumbency status once candidate identities are fixed. Thus, when α is the opposition, denoted by the superscript o, the vote share is

$$\nu_\alpha^o(\zeta) = \frac{1}{2} - \sigma_\kappa \zeta. \tag{17}$$

The corresponding vote shares for β follow from binary competition. When β is the incumbent,

$$\nu_\beta^i(\zeta) = 1 - \nu_\alpha^o(\zeta) = \frac{1}{2} + \sigma_\kappa \zeta, \tag{18}$$

and when β is the opposition,

$$\nu_\beta^o(\zeta) = 1 - \nu_\alpha^i(\zeta) = \frac{1}{2} + \sigma_\kappa \zeta. \tag{19}$$

Equations (16)–(19) show that the electoral shock shifts vote shares symmetrically: higher values of ζ increase support for β and reduce support for α by the same amount.

Elections are decided by strict majority. In the no-patronage case, α wins if and only if $\nu_\alpha^r(\zeta) > 1/2$, where $r \in \{i, o\}$. Using Equation (16), the condition for α to win when α is the incumbent is

$$\nu_\alpha^i(\zeta) > \frac{1}{2} \quad \Longleftrightarrow \quad \frac{1}{2} - \sigma_\kappa \zeta > \frac{1}{2} \quad \Longleftrightarrow \quad \zeta < 0. \tag{20}$$

Thus, the probability that α wins when α is the incumbent is

$$p_\alpha^i = \Pr\left(\nu_\alpha^i(\zeta) > \frac{1}{2}\right) = \Pr(\zeta < 0). \tag{21}$$

By Assumption 4, zero lies in the support of ζ. The cumulative distribution function of ζ at zero is therefore

$$F_\zeta(0) = \frac{\frac{1}{2\sigma_\lambda} - \mu}{\frac{1}{\sigma_\lambda}} = \frac{1}{2} - \sigma_\lambda \mu. \tag{22}$$

[21] When the cutoff falls outside the support of ω^j, the vote share is zero or one, so all vote-share expressions are understood as truncated to $[0, 1]$.

Combining Equations (21) and (22) gives

$$p_\alpha^i = \frac{1}{2} - \sigma_\lambda \mu. \tag{23}$$

Because α's vote share is the same in both roles in the no-patronage case, the same winning condition applies whether α is the incumbent or the opposition:

$$p_\alpha^o = \Pr\left(\nu_\alpha^o(\zeta) > \frac{1}{2}\right) = \frac{1}{2} - \sigma_\lambda \mu. \tag{24}$$

Since elections are binary, the corresponding probabilities for β are[22]

$$p_\beta^i = p_\beta^o = \frac{1}{2} + \sigma_\lambda \mu. \tag{25}$$

Equations (16)–(19) show that, without elite support, vote shares move only with the realized electoral shock. A higher ζ shifts votes toward β, while σ_κ determines the size of this shift.[23] Equations (23)–(25) give the corresponding winning probabilities. These depend on incumbent strength, through μ, and on electoral stability, through σ_λ. This is immediate from the distribution of ζ: μ shifts the average electoral environment toward one politician, while higher σ_λ makes this average position matter more by reducing the dispersion of the electoral shock.

2.2.2 Winning probabilities under patronage

We now turn to the patronage case, so $d = 1$ and $C = c$ in Equation (4). Suppose first that α is the incumbent. Voter j supports α when the elite contribution is large enough to offset the realized shock and the voter-specific bias:

$$c \geq \omega^j + \zeta. \tag{26}$$

The corresponding cutoff is

$$\hat{\omega}(\zeta) = c - \zeta. \tag{27}$$

[22] Assumption 4 ensures that p_α^i, p_α^o, p_β^i, and p_β^o in Equations (23)–(25) lie in $[0, 1]$.

[23] In Equation (16), σ_κ scales how the electoral shock moves α's vote share: a higher σ_κ concentrates voter biases around zero, so a given shock moves a larger share of voters. Under strict majority rule, however, winning depends only on whether the vote share exceeds one half, and since $\sigma_\kappa > 0$ scales the majority condition without changing its sign, it drops out of the winning probabilities that follow.

Using $f_\omega(w) = \sigma_\kappa$ on $\left[-\frac{1}{2\sigma_\kappa}, \frac{1}{2\sigma_\kappa}\right]$, the vote share of α as incumbent is

$$\nu_\alpha^i(\zeta) = \int_{-\frac{1}{2\sigma_\kappa}}^{c-\zeta} \sigma_\kappa \, \mathrm{d}w = \frac{1}{2} + \sigma_\kappa(c-\zeta). \tag{28}$$

The vote share of β as opposition is

$$\nu_\beta^o(\zeta) = 1 - \nu_\alpha^i(\zeta) = \frac{1}{2} - \sigma_\kappa(c-\zeta). \tag{29}$$

Now suppose that α is the opposition. The incumbent is then β, who receives the elite contribution. Voter j supports α only when the elite contribution, the shock, and the voter-specific bias do not favor β:

$$0 \geq c + \omega^j + \zeta. \tag{30}$$

The cutoff is therefore

$$\hat{\omega}(\zeta) = -c - \zeta. \tag{31}$$

The vote share of α as opposition is

$$\nu_\alpha^o(\zeta) = \int_{-\frac{1}{2\sigma_\kappa}}^{-c-\zeta} \sigma_\kappa \, \mathrm{d}w = \frac{1}{2} - \sigma_\kappa(c+\zeta). \tag{32}$$

The vote share of β as incumbent is

$$\nu_\beta^i(\zeta) = 1 - \nu_\alpha^o(\zeta) = \frac{1}{2} + \sigma_\kappa(c+\zeta). \tag{33}$$

By Assumption 4, both c and $-c$ lie in the support of ζ, so the winning events under strict majority rule again reduce to threshold conditions on ζ. When α is the incumbent,

$$p_\alpha^i = \Pr\left(\nu_\alpha^i(\zeta) > \frac{1}{2}\right) = \frac{1}{2} + \sigma_\lambda(c-\mu), \tag{34}$$

so the probability that β wins as opposition is

$$p_\beta^o = 1 - p_\alpha^i = \frac{1}{2} - \sigma_\lambda(c-\mu). \tag{35}$$

When α is the opposition,

$$p_\alpha^o = \Pr\left(\nu_\alpha^o(\zeta) > \frac{1}{2}\right) = \frac{1}{2} - \sigma_\lambda(c+\mu), \tag{36}$$

and the corresponding probability that β wins as incumbent is[24]

$$p_\beta^i = 1 - p_\alpha^o = \frac{1}{2} + \sigma_\lambda(c + \mu). \tag{37}$$

Equations (28)–(29) and (32)–(33) show how vote shares change when the patronage channel is active. The elite contribution c shifts votes toward the incumbent by $\sigma_\kappa c$, so the same voter-specific structure now converts elite support into an incumbency advantage.[25] Equations (34)–(37) give the resulting winning probabilities. Relative to no-patronage, the new term is c: it raises the probability that the incumbent remains in office, and its effect is stronger when the electoral environment is more stable.

Having derived the winning probabilities for each politician in each role, under both no-patronage and patronage, we now turn to the solution strategy.

2.3 Solution strategy

Our strategy is to compare the initial incumbent's value of holding office under patronage with its value under no-patronage. Because the adoption decision $d \in \{0, 1\}$ is fixed once at $t = 0$, each choice of d defines a stationary environment and hence a single continuation value for the incumbent. This value accounts for the full future path of play, including the periods in which the incumbent loses office and the patronage channel serves the rival instead. The adoption decision therefore reduces to comparing these two values.

Suppose without loss of generality that α is the initial incumbent. From this point onward, we refer to α as *she* and to β as *he*. Let U_α^i denote α's continuation value when she enters a period as the incumbent, and let U_α^o denote her continuation value when she enters a period as the opposition.

When α is the incumbent, she receives the current rent $R - dx$, which equals R under no-patronage and $R - x$ under patronage. After the election, she remains the incumbent with probability p_α^i and becomes the opposition with probability $1 - p_\alpha^i$. Her continuation value as incumbent therefore satisfies

$$U_\alpha^i = R - dx + \delta \left(p_\alpha^i U_\alpha^i + \left(1 - p_\alpha^i\right) U_\alpha^o \right). \tag{38}$$

The first term is the current rent net of any transfer to the elites. The second

[24] Assumption 4 ensures that p_α^i, p_α^o, p_β^i, and p_β^o in Equations (34)–(37) lie in $[0, 1]$.

[25] As in the no-patronage case, σ_κ scales how the advantage moves vote shares but drops out of the winning probabilities, since strict majority rule reduces the winning event to a threshold condition on ζ.

is the discounted continuation value, weighted by the probabilities of remaining the incumbent and moving to the opposition.

Rearranging (38) and solving for U_α^i yields

$$U_\alpha^i = \frac{R - dx + \delta\left(1 - p_\alpha^i\right) U_\alpha^o}{1 - \delta p_\alpha^i}. \tag{39}$$

The right-hand side still depends on U_α^o. To eliminate this term, we derive a corresponding expression for α's value as the opposition.

When α is the opposition, she receives no current rent. After the election, she wins with probability p_α^o and becomes the incumbent in the next period, or she loses with probability $1 - p_\alpha^o$ and remains the opposition. Her continuation value as opposition therefore satisfies

$$U_\alpha^o = \delta\left(p_\alpha^o U_\alpha^i + \left(1 - p_\alpha^o\right) U_\alpha^o\right). \tag{40}$$

Solving (40) for U_α^o gives

$$U_\alpha^o = \frac{\delta p_\alpha^o}{1 - \delta\left(1 - p_\alpha^o\right)} U_\alpha^i. \tag{41}$$

Substituting Equation (41) into Equation (39) and solving for U_α^i yields

$$U_\alpha^i = \frac{\left(R - dx\right)\left(1 - \delta\left(1 - p_\alpha^o\right)\right)}{\left(1 - \delta\right)\left(1 - \delta\left(p_\alpha^i - p_\alpha^o\right)\right)}. \tag{42}$$

The basic comparative statics of U_α^i follow directly from this expression (see Lemma A.1 in Appendix A). The terms R and x affect the rent retained in each period of incumbency: a higher R increases the value of holding office, while a higher x weakly lowers it when patronage is adopted. The probabilities p_α^i and p_α^o affect the expected fraction of future periods spent in office. A higher p_α^i raises the probability that α remains the incumbent, while a higher p_α^o raises the probability that she returns to office after losing.

The winning probabilities p_α^i and p_α^o depend on the adoption decision d. Under no-patronage, the incumbency advantage is absent. Under patronage, it is present. The mapping from parameters to (p_α^i, p_α^o) therefore differs across the two cases, and the comparative statics with respect to c, μ, σ_λ, and δ must be evaluated separately under each.

Equation (42) depends on the adoption decision d through two channels. First, d enters the current rent through $R - dx$. Second, d enters the continuation value because the electoral environment differs across no-patronage and patronage, which changes the winning probabilities. To make this dependence explicit, we write the incumbent value and the winning probabilities as functions of d:

$$U^i_\alpha(d) = \frac{(R - dx)\left(1 - \delta\left(1 - p^o_\alpha(d)\right)\right)}{(1-\delta)\left(1 - \delta\left(p^i_\alpha(d) - p^o_\alpha(d)\right)\right)}. \tag{43}$$

The probabilities $p^i_\alpha(d)$ and $p^o_\alpha(d)$ follow from Equations (23), (24), (34), and (36). Under no-patronage, there is no incumbency advantage and α's winning probability is the same whether she is the incumbent or the opposition. Under patronage, the incumbency advantage shifts vote shares in favor of the incumbent and this symmetry breaks. Formally,

$$p^i_\alpha(d) = \begin{cases} \frac{1}{2} - \sigma_\lambda \mu, & d = 0, \\ \frac{1}{2} + \sigma_\lambda\left(c - \mu\right), & d = 1, \end{cases} \tag{44}$$

and

$$p^o_\alpha(d) = \begin{cases} \frac{1}{2} - \sigma_\lambda \mu, & d = 0, \\ \frac{1}{2} - \sigma_\lambda\left(c + \mu\right), & d = 1. \end{cases} \tag{45}$$

The initial incumbent chooses $d \in \{0, 1\}$ to maximize her continuation value as incumbent, $U^i_\alpha(d)$. Adoption costs x in every period of incumbency, raises p^i_α by $\sigma_\lambda c$, and lowers p^o_α by $\sigma_\lambda c$, since the advantage accrues to whichever politician is the incumbent in any given period.

2.4 Continuation values

We evaluate Equation (43) under no-patronage and under patronage, using the winning probabilities in Equations (44) and (45), to obtain explicit expressions for $U^i_\alpha(0)$ and $U^i_\alpha(1)$.

2.4.1 No-patronage

Under no-patronage, $p^i_\alpha(0) = p^o_\alpha(0) = \frac{1}{2} - \sigma_\lambda\mu$, so the difference $p^i_\alpha(0) - p^o_\alpha(0)$ in the denominator of Equation (43) vanishes and the numerator factor reduces to $1 - \delta\left(\frac{1}{2} + \sigma_\lambda\mu\right)$. Setting $d = 0$ therefore yields

$$U^i_\alpha(0) = \frac{R\left(1 - \delta\left(\frac{1}{2} + \sigma_\lambda\mu\right)\right)}{1-\delta}. \tag{46}$$

2.4.2 Patronage

Under patronage, $p^i_\alpha(1) = \frac{1}{2} + \sigma_\lambda(c - \mu)$ and $p^o_\alpha(1) = \frac{1}{2} - \sigma_\lambda(c + \mu)$, so $p^i_\alpha(1) - p^o_\alpha(1) = 2\sigma_\lambda c$ and the numerator factor reduces to $1 - \delta\left(\frac{1}{2} + \sigma_\lambda(c + \mu)\right)$. Setting $d = 1$ therefore yields

$$U^i_\alpha(1) = \frac{(R - x)\left(1 - \delta\left(\frac{1}{2} + \sigma_\lambda\left(c + \mu\right)\right)\right)}{\left(1 - \delta\right)\left(1 - 2\delta\sigma_\lambda c\right)}. \tag{47}$$

Equations (46) and (47) are the two continuation values the initial incumbent compares. Comparative statics for both are provided (see Lemmas A.3, A.4, A.5, and A.6 in Appendix A). We now turn to the results, beginning with the adoption decision that this comparison defines.

3 Results

3.1 Patronage adoption

The initial incumbent α chooses $d \in \{0, 1\}$ to maximize the value of holding office at the start of the game. Before this choice is made, the transfer $x \in [0, R)$ required to activate elite support is realized exogenously and observed by α. It is the price of activating the patronage channel under the institutional environment, which α takes as given, deciding only whether to adopt patronage at that transfer. The adoption decision solves

$$d^* \in \underset{d \in \{0,1\}}{\arg\max}\, U^i_\alpha(d), \tag{48}$$

and reduces to a comparison of $U^i_\alpha(1)$ and $U^i_\alpha(0)$. The incumbent weighs the transfer x against the electoral advantage generated by elite support. Because the choice is durable, both sides of the tradeoff recur in every period in which the channel operates, with the elite contribution accruing to whoever holds office at the time.

Lemma 1. ***Adoption threshold***

Let $\tilde{x}$ be the normalized adoption threshold, obtained in the proof:

$$\tilde{x} = \frac{\delta \sigma_\lambda c \left(1 - \delta \left(1 + 2\sigma_\lambda \mu\right)\right)}{1 - \delta \left(\frac{1}{2} + \sigma_\lambda \left(c + \mu\right)\right)}, \tag{49}$$

and let

$$\bar{x} = R\tilde{x}. \tag{50}$$

Then

$$U_\alpha^i(1) > U_\alpha^i(0) \iff x < \bar{x}. \tag{51}$$

Proof: See Appendix A.

Lemma 1 gives a single cutoff $\bar{x} = R\tilde{x}$: the incumbent adopts patronage at every required transfer below $\bar{x}$ and rejects it above.[26] The two thresholds are the same object in different units. The transfer x is an amount of rent, so the cutoff it is compared against, $\bar{x}$, is also an amount of rent: the largest transfer the incumbent will accept. Dividing by the office rent gives $\tilde{x} = \bar{x}/R$, the same cutoff written as a share of that rent, namely the largest fraction of each period's rent the incumbent will divert to the elites in exchange for their support. The share $\tilde{x}$ does not depend on how valuable office is and is the object the comparative statics work with, while $\bar{x} = R\tilde{x}$ restores the scale and so is larger when office is worth more.[27]

The threshold governs the decision only when $\tilde{x}$ lies strictly inside $(0, 1)$, where a higher required transfer can move the incumbent from adoption to rejection. The two boundaries are degenerate. At $\tilde{x} \geq 1$ the incumbent adopts at every feasible transfer, and at $\tilde{x} \leq 0$ she rejects at every feasible transfer. The upper boundary never binds, because Assumption 4 keeps $\tilde{x} < 1$ throughout. Only the lower corner can arise, and only when α is weak and the environment stable enough (see Lemma A.2 in Appendix A).[28] We rule out negative thresholds with the following restriction.

[26]At $x = \bar{x}$ the incumbent is indifferent between the two regimes. We assume patronage is not adopted in this case, consistent with the strict inequality in Lemma 1.

[27]The formula for $\tilde{x}$ contains strength, stability, patience, and the elite contribution, but not R. Office value therefore cannot change the share $\tilde{x}$, it enters only through $\bar{x} = R\tilde{x}$, scaling the share into an amount.

[28]When $\mu \leq 0$ the threshold is already non-negative. When $\mu > 0$ it stays non-negative only if $\sigma_\lambda \leq (1-\delta)/(2\delta\mu)$. At the boundary $\sigma_\lambda = (1-\delta)/(2\delta\mu)$ the threshold is $\tilde{x} = 0$, so $\bar{x} = 0$ and patronage is rejected even at a zero transfer.

Assumption 5. ***Non-negative adoption threshold***

If $\mu > 0$, then

$$\sigma_\lambda \le \frac{1-\delta}{2\delta\mu} \iff \delta \le \frac{1}{1+2\sigma_\lambda\mu}. \tag{52}$$

Assumption 5 secures $\tilde{x} \ge 0$. It is automatic when $\mu \le 0$ and binds only when α is weak, where it bounds how durable that weakness may be. For a weak incumbent, a more stable environment and a higher discount factor make the channel more likely to end up serving a successor than α herself, and weight those later periods more heavily, until the channel is no longer worth adopting at all. The assumption rules out negative thresholds while retaining the knife-edge case $\tilde{x} = 0$, at which patronage is rejected at every transfer, and keeps the weak incumbents for whom the channel still has value. The bound relaxes as σ_λ, μ, or δ falls: a more volatile environment leaves α a greater chance of holding office, less weakness leaves less of it to carry forward, and less patience places less weight on the later periods.

Under Assumptions 4 and 5 the threshold satisfies $\tilde{x} \in [0, 1)$, so the required transfer decides adoption. We now turn to how the adoption threshold responds to incumbent strength, the elite contribution, electoral stability, and the discount factor.

3.2 Comparative statics

We study how the model's parameters shape the adoption threshold on the admissible parameter range, where the adoption decision depends on the realized required transfer x:

$$\alpha \text{ adopts patronage} \iff x < \bar{x} = R\tilde{x}. \tag{53}$$

Shifts in $\tilde{x}$ therefore translate directly into changes in the range of transfers at which adoption is optimal: a higher $\tilde{x}$ widens that range, so patronage is adopted at more prices, and a lower $\tilde{x}$ narrows it. We begin with incumbent strength μ and the elite contribution c, which most directly shape the attractiveness of patronage.

Lemma 2. ***Comparative statics of the adoption threshold: The effect of incumbent strength and the elite contribution***
$\tilde{x}$ in Equation (49) satisfies the following:

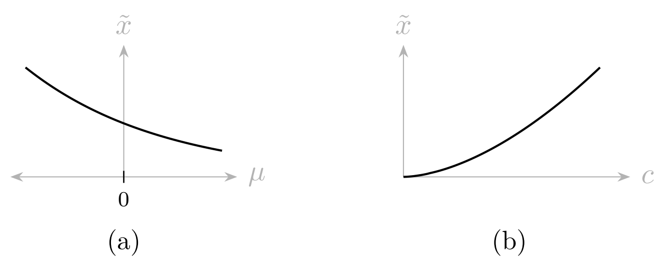
2.1 $\tilde{x}$ is strictly decreasing in μ.

2.2 $\tilde{x}$ is increasing in c. The increase is strict unless $\mu > 0$ and $\sigma_\lambda = \dfrac{1-\delta}{2\delta\mu}$.

Proof: See Appendix A.

Lemma 2 records the two parameters that move the threshold most directly: it falls in incumbent strength μ and rises in the elite contribution c. We take each in turn.

Figure 2: Adoption threshold, incumbent strength, and elite contribution



Note: Panel (a) illustrates Part 2.1, in which the adoption threshold is strictly decreasing in incumbent strength μ. Panel (b) illustrates Part 2.2, in which the threshold is increasing in the elite contribution c. The curves are schematic.

The effect of μ is the central comparative static of the model, shown in panel (a) of Figure 2. As α becomes weaker, or equivalently as μ rises, the threshold falls and patronage is adopted at a narrower range of transfers. The mechanism is the durability of the channel. While α holds office, elite support raises her probability of reelection, but after turnover the same support raises the probability of reelection of β, who is then the incumbent. A weaker α expects to lose office sooner, so she has fewer periods in which to recover the diverted rent and a larger share of future periods in which the channel serves β rather than her. She is therefore willing to divert a smaller share of the office rent, and the threshold falls.

This reverses the common expectation that weak incumbents rely most heavily on patronage. Patronage is a durable investment whose return depends on holding office long enough to use it. A strong incumbent expects to draw on the channel over many periods and accepts a higher transfer, while a weak incumbent expects to lose office and leave the channel to a successor, so adoption is less attractive. Patronage is better understood as a way to protect an existing political position than as a tool of last resort for incumbents about to lose office.

The effect of c, shown in panel (b), is more direct. A larger elite contribution raises $U^i_\alpha(1)$, while $U^i_\alpha(0)$ does not depend on c (see Lemmas A.5 and A.3 in Appendix A). The incumbent is therefore willing to give up a larger share of the office rent to activate elite support, and the threshold rises.[29]

We next examine how the adoption threshold responds to electoral stability σ_λ and the discount factor δ.

The admissible range of σ_λ depends on incumbent strength. Assumption 4 always bounds σ_λ from above, with the bound $1/(2(c+|\mu|))$ equal to $1/(2(c-\mu))$ when $\mu \leq 0$ and to $1/(2(c+\mu))$ when $\mu > 0$. Assumption 5 adds a second upper bound only when $\mu > 0$ and is automatically satisfied when $\mu \leq 0$. The upper bound of the admissible range is therefore

$$\bar{\sigma}_\lambda = \begin{cases} \dfrac{1}{2(c-\mu)}, & \mu \leq 0, \\ \min\left\{ \dfrac{1}{2(c+\mu)}, \ \dfrac{1-\delta}{2\delta\mu} \right\}, & \mu > 0. \end{cases} \tag{54}$$

[29]The increase is strict except at the boundary of Assumption 5.

Lemma 3. ***Comparative statics of the adoption threshold: The effect of electoral stability***

Let σ_λ^* be the location of the interior maximum of the adoption threshold in σ_λ, when it exists, obtained in the proof:

$$\sigma_\lambda^* = \frac{\mu(2-\delta) - \sqrt{\mu(2-\delta)\left(\mu - c(1-\delta)\right)}}{2\delta\mu(\mu+c)}. \tag{55}$$

Let $\hat{\delta} = \dfrac{7-\sqrt{17}}{8}$ *and, for* $\delta > \hat{\delta}$,

$$\hat{\mu} = \frac{c(1-\delta)(2-\delta)}{7\delta - 4\delta^2 - 2}. \tag{56}$$

Then $\tilde{x}$ in Equation (49) satisfies the following on $(0, \bar{\sigma}_\lambda]$:

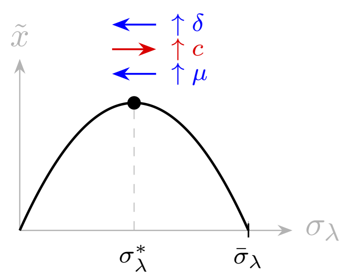
3.1 If $\delta > \hat{\delta}$ and $\mu > \hat{\mu}$, then $\tilde{x}$ is single-peaked in σ_λ, with a unique interior maximum at σ_λ^* satisfying

$$\frac{\partial \sigma_\lambda^*}{\partial \mu} < 0, \qquad \frac{\partial \sigma_\lambda^*}{\partial c} > 0, \qquad \frac{\partial \sigma_\lambda^*}{\partial \delta} < 0. \tag{57}$$

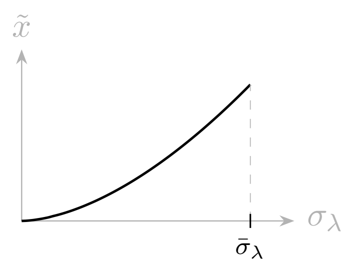
3.2 Otherwise, $\tilde{x}$ is strictly increasing in σ_λ.

Proof: See Appendix A.

Figure 3: Adoption threshold and electoral stability



(a) $\delta > \hat{\delta}$ and $\mu > \hat{\mu}$



(b) otherwise

Note: Panel (a) illustrates Part 3.1, in which the adoption threshold is single-peaked in electoral stability σ_λ, with a unique interior maximum at σ_λ^*. The arrows show the maximum shifting lower with μ and δ, and higher with c. Panel (b) illustrates Part 3.2, in which the threshold is strictly increasing. The curves are schematic, and the position of the interior maximum is illustrative.

Lemma 3 describes how electoral stability affects the adoption threshold, as shown in Figure 3. As a rule, more stable elections raise the threshold, as in Part 3.2. A higher σ_λ narrows the electoral shock, so the elite support carries the incumbent to reelection more surely. The patronage channel is then worth more while she holds office, so she adopts it at a wider range of transfers and the threshold rises across the admissible range.

The exception, in Part 3.1, is an incumbent who is both weak enough and patient enough, for whom the threshold rises and then falls. Stability here pulls in opposite directions through two forces, a gain and a loss. At low stability the shock is dispersed, so her weakness carries little weight in the election, and further stability mainly strengthens the grip that elite support gives her on office, so the gain leads and the threshold rises. At high stability the shock is narrow, so her weakness becomes more determinative and losing office more likely, and further stability mainly entrenches the successor who inherits the channel, so the loss leads and the threshold falls. The reversal appears only if she is weak enough, $\mu > \hat{\mu}$,[30] for the loss to prevail, and patient enough, $\delta > \hat{\delta}$, to value the later periods in which it falls on her.

The peak sits where the gain and the loss balance, and the parameters move it. A weaker or more patient incumbent strengthens the loss, since a weaker incumbent is more likely to be pushed out and a more patient one cares more about the later periods in which the channel serves a successor, so the threshold turns down at a lower level of stability, and $\partial\sigma_\lambda^*/\partial\mu < 0$ and $\partial\sigma_\lambda^*/\partial\delta < 0$. A larger elite contribution strengthens the gain, since more support is collected while she holds office, so the turn comes later and $\partial\sigma_\lambda^*/\partial c > 0$.

The discount factor satisfies $\delta \in (0,1)$ by definition. For $\mu > 0$, Assumption 5 adds the tighter bound $\delta \leq 1/(1+2\sigma_\lambda\mu)$, so the admissible range is bounded above by

$$\bar{\delta} = \min\left\{1,\ \frac{1}{1+2\sigma_\lambda\mu}\right\}, \tag{58}$$

which equals $1/(1+2\sigma_\lambda\mu) < 1$ when $\mu > 0$ and equals 1 when $\mu \leq 0$. Since $\delta < 1$ throughout, the admissible range is $\left(0, \bar{\delta}\right]$ when $\mu > 0$ and $(0,1)$ when

[30] Differentiating Equation (56) for $\delta > \hat{\delta}$, where $7\delta - 4\delta^2 - 2 > 0$, gives $\partial\hat{\mu}/\partial c = (1-\delta)(2-\delta)/(7\delta - 4\delta^2 - 2) > 0$ and $\partial\hat{\mu}/\partial\delta = -c(5\delta^2 - 12\delta + 8)/(7\delta - 4\delta^2 - 2)^2 < 0$, the latter negative because $5\delta^2 - 12\delta + 8$ has no real root and so stays positive. A larger elite contribution makes patronage more valuable while she holds office, so she must be weaker before stability turns against her, and the weakness needed for the reversal rises. A more patient incumbent cares more about the periods after she loses office, so a smaller weakness already triggers the reversal, and that threshold falls.

$\mu \leq 0$.

Lemma 4. ***Comparative statics of the adoption threshold: The effect of the discount factor***
Let δ^* be the location of the interior maximum of the adoption threshold in δ, when it exists, obtained in the proof:

$$\delta^* = \frac{2(1 + 2\sigma_\lambda \mu) - \sqrt{2(1 + 2\sigma_\lambda \mu)\big(1 + 2\sigma_\lambda(\mu - c)\big)}}{(1 + 2\sigma_\lambda \mu)(1 + 2\sigma_\lambda(\mu + c))}. \tag{59}$$

Then $\tilde{x}$ in Equation (49) satisfies the following on the admissible range of δ:

4.1 If $\delta^* < \bar{\delta}$, then $\tilde{x}$ is single-peaked in δ, with a unique interior maximum at δ^* satisfying

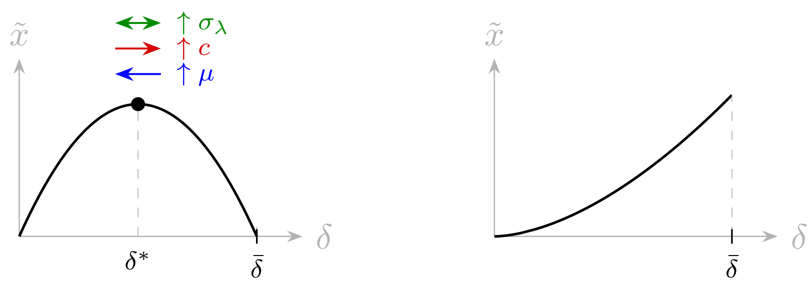
$$\frac{\partial \delta^*}{\partial \mu} < 0, \qquad \frac{\partial \delta^*}{\partial c} > 0, \tag{60}$$

while the effect of σ_λ on δ^* is ambiguous.

4.2 Otherwise, $\tilde{x}$ is strictly increasing in δ.

Proof: See Appendix A.

Figure 4: Adoption threshold and the discount factor



(a) $\delta^* < \bar{\delta}$ (b) otherwise

Note: Panel (a) illustrates Part 4.1, in which the adoption threshold is single-peaked in the discount factor δ, with a unique interior maximum at δ^*. The arrows show the maximum shifting lower with μ and higher with c, while the effect of σ_λ is ambiguous. Panel (b) illustrates Part 4.2, in which the threshold is strictly increasing. The curves are schematic, and the position of the interior maximum is illustrative.

The discount factor shapes the adoption threshold through the same opposition of forces, as shown in Figure 4. Now, though, the single-peaked case is the rule rather than the exception: for most incumbents the threshold rises and then falls in δ, as in Part 4.1. Here the gain and the loss are weighed against each other by patience itself. The patronage channel is durable, so its value comes from future periods. At low patience the incumbent weighs mainly the near future, where she still holds office and the elite support works for her, so the gain leads and the threshold rises. At high patience she also weighs the far future, where she may have lost office and the channel serves a successor, so the loss leads and the threshold falls. The threshold peaks at an interior discount factor δ^* where the two balance.

The threshold rises throughout instead, as in Part 4.2, only for a strong incumbent, and only once she is strong enough (see proof of Lemma 4 in Appendix A). A weak incumbent is likely to lose office, so the loss always overtakes the gain as patience grows and the threshold is single-peaked. A strong incumbent rarely loses office, so the loss is faint, and past a level of strength it never overtakes the gain on the admissible range, so the threshold only rises.

The peak again sits where the gain and the loss balance, and the parameters move it. A weaker incumbent strengthens the loss, since she is more likely to be pushed out and leave the channel to a successor, so the turn comes at a lower discount factor and $\partial\delta^*/\partial\mu < 0$. A larger elite contribution strengthens the gain, since more support is collected while she holds office, so the turn comes later and $\partial\delta^*/\partial c > 0$. Electoral stability strengthens both forces at once, making office more secure while she holds it and the successor's hold more secure once she loses it, so its effect on the peak is ambiguous.

3.3 Predictions

Taken together, the results tie the adoption of patronage to how long the incumbent expects to hold office, not to momentary electoral need. The central result is that adoption follows from strength and a long horizon rather than from vulnerability. The reason is the durability of the channel. While α holds office, the elite support she buys raises her own probability of reelection, so the exchange pays off only if she keeps office long enough to recover the rent she diverts to the elites. After turnover the same support raises the probability of reelection of β, so the channel serves a successor rather than herself. A stronger incumbent expects both more periods in which to draw on the channel and less risk that it soon passes to a rival, so she values it more, and the threshold rises with strength and with a larger elite contribution (see Lemma 2). This reverses the common expectation that the incumbents most at risk of losing office are the ones who reach for patronage.

The same opposition of a gain in office and a loss after turnover shapes the effects of electoral stability and the discount factor. Greater stability makes elite support translate more surely into the incumbent's own reelection, so it generally raises adoption, and its effect turns single-peaked only for an incumbent who is sufficiently weak and patient, for whom further stability makes the channel more likely to serve a successor (see Lemma 3). Greater patience weights the future periods on both sides of turnover at once, so its effect is single-peaked for most incumbents, rising while the periods in office dominate and falling once the periods after turnover weigh more heavily. Only for a sufficiently strong incumbent, who rarely loses office, does patience raise adoption throughout (see Lemma 4).

These results yield a small set of predictions about where patronage is adopted. The first concerns strength: a strong incumbent is more likely to adopt the channel than a weak one. The second turns on electoral stability: a more stable environment raises adoption for such an incumbent. The third involves the discount factor: greater patience does the same once the incumbent is sufficiently strong. A larger elite contribution raises adoption too, since the incumbent secures more elite support in each period she holds office. This effect is very direct, and we do not focus on it in the cases, since wherever elite support is more valuable we naturally expect more patronage of this kind.

The single-peaked reversals are subtler. For a weak incumbent, further stability can lower the value of patronage, and for all but the strongest incumbents further patience eventually does the same, since in each case the relevant force makes the channel more likely to serve a successor, so adoption may fail precisely where it appears most attractive. They are hard to recognize in observed cases, because the absence of adoption is consistent with explanations the model does not address (see Parts 3.1 and 4.1).

These predictions describe a recognizable profile of the adopting incumbent, one who is strong, faces a stable electoral environment, and expects to hold office well into the future. The cases illustrate the predictions rather than test them, and we read them in that spirit.

4 Political context

Three incumbents come close to the profile the model associates with adoption: Vladimir Putin in Russia, Recep Tayyip Erdoğan in Turkey, and Viktor Orbán in Hungary. The relevant periods are the first two terms of Vladimir Putin's presidency through the 2000s, the rise and consolidation of Recep Tayyip Erdoğan's party from 2002 through the 2010s, and Viktor Orbán's tenure since 2010. In each period the regime held multiparty elections that were contested but tilted toward the incumbent, the defining feature of a hybrid regime (Levitsky & Way, 2010; Lührmann et al., 2018). For Russia we have in mind these earlier years rather than the more closed system that came later, when elections no longer carried real uncertainty.

The profile set out above brings together three features: a strong incumbent, a stable electoral environment, and a long political horizon. These three incumbents shared all three, and each commanded the state resources that such an exchange requires (Fazekas & Tóth, 2016; Gürakar, 2016; Guriev & Rachinsky, 2005). In each case a durable patronage channel was built with economic elites once the incumbent's position was secure, not while it was under threat. We take the three in turn, beginning with the most consolidated.

We begin with Russia, where Vladimir Putin built the channel from the most commanding position of the three. Through most of the 2000s Vladimir Putin held approval ratings above sixty percent and brought the legislature, the regions, and the major broadcasters under central control, a concentration of authority with few checks remaining (Hale, 2014; Treisman, 2011). The economic elites that had emerged from the privatizations of the 1990s controlled a large share of the economy and were a natural counterparty for an exchange (Guriev & Rachinsky, 2005). From this position Vladimir Putin reset the terms on which they held what they had. The elites kept their fortunes and gained access to state contracts so long as they backed the leadership and stayed out of opposition, an arrangement Markus (2015) describes as ownership made conditional on political loyalty. The wealthiest among them secured their holdings by aligning with the state rather than against it (Markus & Charnysh, 2017), and public procurement became a standing instrument for directing resources to elites whose support could be relied upon. This is the model's exchange in its clearest institutional form, an incumbent strong enough to set the terms and an elite that supplies political support in return for a recurring flow of state resources. The channel was set in place only after the presidency and the major institutions were under firm control.

Turkey under Recep Tayyip Erdoğan shows the same exchange with unusually direct evidence on its mechanics. Recep Tayyip Erdoğan and his Justice and Development Party (AKP) won successive parliamentary majorities from 2002 and reached a degree of electoral dominance with few precedents in Turkish multiparty politics (Esen & Gümüşçü, 2016). From that base Recep Tayyip Erdoğan assembled a loyal business class through the allocation of state resources, embedding the ties between political authority and allied firms within formal institutions rather than leaving them to informal favor (Bektaş, 2025; Buğra & Savaşkan, 2014). The exchange is clearest in public contracting, the elites delivering political and financial backing and the government delivering the contracts in return. Gürakar (2016) examines tens of thousands of high-value procurement awards and finds that firms connected to Recep Tayyip Erdoğan and the AKP won them more often and on better terms than unconnected firms, while repeated amendments to the procurement law widened the discretion through which the awards were made. The two sides of the model's exchange, the transfer of state resources out and the political support coming back, are visible here as a practice that deepened across a decade of accumulating majorities rather than under any immediate threat.

Hungary under Viktor Orbán shows the same logic in a setting embedded within the European Union. Viktor Orbán and Fidesz won a constitutional supermajority in 2010 and the capacity to reshape the institutional order, renewed by further supermajorities in 2014 and 2018 (Bíró-Nagy, 2017; Kornai, 2015). From that position Viktor Orbán cultivated a class of loyal owners across business and media, rewarding allied firms with public resources and concentrating media ownership in friendly hands, a system in which the concentration of political power and the accumulation of allied wealth advanced together (Magyar, 2016; Scheiring, 2020). The pattern is visible in the contracting data. Fazekas and Tóth (2016) find that political influence over public procurement in Hungary is systematic and far-reaching, and the share of contracts won by firms tied to Viktor Orbán and Fidesz tracked the electoral cycle, rising once the party had secured another term and receding before votes (Dávid-Barrett & Fazekas, 2020). The same favor ran through the media, where government advertising rewarded supportive outlets and was withheld from critical ones (Szeidl & Szűcs, 2021). The channel expanded after electoral victories rather than before them, when Viktor Orbán's position was already secure, which is the pattern the model predicts.

The three cases bear most directly on the central prediction, that patronage is adopted from strength rather than from weakness. In none of them did a cornered incumbent reach for elite support to survive an immediate threat. The channel was built in each case by an incumbent already in a strong and secure position, the reverse of the expectation that the incumbents most at risk are the ones who turn to patronage.

These incumbents also operated in unusually stable electoral environments, the second condition the model attaches to adoption. Because each was already strong, the gain from elite support while in office counted for more than any loss were office to change hands later, so a more stable environment only added to the appeal of the channel. When an incumbent is unlikely to be voted out, elite support buys continued office at little risk, which is what makes it worth paying for. In all three cases the result of national elections had become hard to swing, with the incumbent's lead largely unchanged from one contest to the next (Simpser, 2013). Russia shows this most fully, where party regulation, registration barriers, and control over the major broadcasters narrowed the range of likely results, while the consolidation of United Russia into a dominant party organized elites and voters around the incumbent (Golosov, 2011; Hale, 2014; Reuter, 2017). The Turkish field was tilted in the same direction through control over broadcasting, selective enforcement against opposition media, and the use of state resources in campaigns (Esen & Gümüşçü, 2016). Hungary held its advantage firm across successive votes through concentrated media ownership, redrawn electoral districts, and state advertising directed to allied outlets (Bíró-Nagy, 2017; Szeidl & Szűcs, 2021). Where the result was this stable, the support these incumbents had secured was in little danger of being undone at the next election, which is the case in which the model makes the channel most valuable.

Each also looked far into the future, the third condition. For an incumbent this strong, expecting to keep office rather than lose it, the periods ahead are mostly ones in which office is kept and the channel keeps working for the incumbent rather than passing to a rival, so a longer horizon adds to its worth. The return from patronage accrues only over the years in office, and counts for most to one who expects many of them, and the horizon facing these three was effectively open. Vladimir Putin moved between the presidency and the premiership and later won a constitutional amendment that opened the way to further terms (Hale, 2014). Recep Tayyip Erdoğan recast the office itself, shifting from prime minister to president in a restructuring that reset the limits on his tenure (Esen & Gümüşçü, 2016). Viktor Orbán, prime minister once before between 1998 and 2002 and back in power since 2010, faced no binding term limit and held the means to sustain his lead. None of the three had reason to expect to leave office soon, so the long run of future returns that makes the channel worth building was theirs to count on.

Having presented these cases, we now turn in the next section to discuss some of the model's extensions and implications.

5 Discussion

At the heart of the model is a single decision. At the outset the incumbent chooses, once and for all, whether to adopt a durable patronage channel at a transfer set by the environment, and the analysis asks how high a transfer is worth accepting. Everything else, the conduct of the elites, the behavior of voters, and the value of office, is held fixed so that this choice can be seen clearly.

This section revisits the main simplifications behind that choice, on the side of the elites, the voters, and the incumbent. In each case we explain what a richer treatment would add and why we set it aside. The aim is to show that these choices serve the question we ask rather than the answer we reach.

5.1 The elites

The elites enter the model as a non-strategic source of support. Once the channel is active they receive a transfer in each period and supply support in return, with no choice of their own. This is the strongest simplification in the paper, and it is worth being clear about what it leaves out. A strategic treatment of the elites would give them a payoff, an opportunity cost of political support, perhaps a budget constraint, and it would replace the fixed exchange with a bargaining or participation problem between the incumbent and the elites. That is a substantive extension and a natural one. We do not pursue it because it adds a second optimization on the side of the elites and shifts attention toward how the terms of the exchange are set, whereas our concern is the prior one of whether the incumbent adopts the channel at all.

A related simplification is that the contribution c is a fixed parameter, independent of the transfer. Even without making the elites strategic, one could let the contribution depend on the transfer, writing $c = c(x)$ with

$$\frac{dc}{dx} = c'(x) > 0, \tag{61}$$

so that a larger transfer funds more support. This is appealing, but it changes what the model studies. The adoption condition is

$$R\,\delta\sigma_\lambda c\Big(1 - \delta(1 + 2\sigma_\lambda\mu)\Big) > x\Big(1 - \delta(\tfrac{1}{2} + \sigma_\lambda(c + \mu))\Big) \tag{62}$$

(see proof of Lemma 1 in Appendix A). With c fixed, the left side is constant and the right side rises in x, so the condition holds for transfers below a threshold, and that threshold is the largest transfer worth accepting for a given return. Now let the contribution rise with the transfer in the simplest way, $c(x) = kx$ with $k > 0$. Substituting and moving every term to one side, adoption requires

$$\delta k\sigma_\lambda\, x^2 + \left(R\,\delta k\sigma_\lambda\Big(1 - \delta(1 + 2\sigma_\lambda\mu)\Big) + \delta\sigma_\lambda\mu + \tfrac{\delta}{2} - 1\right)x > 0. \tag{63}$$

The left side is now a parabola in x that opens upward and passes through the origin, since the coefficient on x^2 is $\delta k\sigma_\lambda > 0$. The adoption set is therefore no longer an interval of transfers below a threshold. Over the range where patronage is adopted, a larger transfer does not make the exchange less attractive, because paying more also buys more support. The transfer no longer behaves like a price, and the threshold loses the meaning that is the focus of the analysis. We therefore keep c fixed, which lets the transfer be read as the price of elite support and the threshold as the central object of the paper.

5.2 The voters

Voters in the model are summarized by the distribution of their bias and the aggregate shock, with elite support entering only through its effect on the probability of reelection. A richer treatment would distinguish informed from uninformed voters, in the manner of Baron (1994) and Grossman and Helpman (1996), letting elite support work through channels such as campaign spending and the shaping of information rather than as a single effect on the vote. Suppose a share λ of voters are informed and the rest are not. The uninformed are the ones elite support can move, through advertising, favorable coverage, and local mobilization. The informed see through such efforts and are unmoved, and may even turn against the incumbent, reading the visible reliance on bought support as a diversion of public resources that they penalize at the polls. The two groups then vote under different conditions, so elite support raises the incumbent's vote among the uninformed and may lower it among the informed, and its net effect scales with the size of each group.

Formally, the contribution c that previously entered the winning probabilities now appears scaled by the uninformed share $1-\lambda$. With no patronage there is no contribution to scale, so α's winning probability is the same whether in office or in opposition,

$$p_\alpha^i = p_\alpha^o = \tfrac{1}{2} - \sigma_\lambda \mu, \tag{64}$$

exactly as in the baseline, since with no channel the split makes no difference. Under patronage the channel attaches to whoever holds office and reaches only the uninformed, so the two roles come apart. As incumbent α draws on the contribution through the uninformed share,

$$p_\alpha^i = \tfrac{1}{2} - \sigma_\lambda\Big(\mu - (1-\lambda)c\Big), \tag{65}$$

while as opposition, facing a β incumbent who now draws on it instead,

$$p_\alpha^o = \tfrac{1}{2} - \sigma_\lambda\Big(\mu + (1-\lambda)c\Big). \tag{66}$$

A penalty by the informed would subtract a further term in λ from the probability of whichever side relies on the channel. Carried through the continuation values to the adoption threshold, the effect is straightforward: a larger informed share lowers the electoral return to elite support and so lowers the willingness to adopt, and where the informed penalize visible patronage the threshold falls faster. This sharpens the interpretation without disturbing the mechanism, which requires only that elite support raise the probability of reelection. We leave voters in reduced form for that reason, and note that the extension would change the level of the threshold rather than its comparative statics.

5.3 The incumbent

Three features of the incumbent are held fixed, and each could be relaxed. The first is the office rent R, constant and independent of who holds office. Letting it vary with type, competence, administrative capacity, or fiscal reach would connect adoption to the quality of the incumbent. A natural formulation gives each politician $\phi \in \{\alpha, \beta\}$ a type $\theta_\phi \in \{\ell, h\}$, low or high, high with probability π_ϕ, and a rent that depends on it,

$$R_\phi = \begin{cases} R^\ell, & \theta_\phi = \ell, \\ R^h, & \theta_\phi = h, \end{cases} \qquad 0 < R^\ell < R^h, \tag{67}$$

with expected rent $(1-\pi_\phi)R^\ell + \pi_\phi R^h$. The value of office then depends on the type of the incumbent and of the successor, so the model carries type uncertainty on both sides. This adds complexity and comes at the cost of the tractability that makes the adoption threshold transparent, while speaking to a question, the quality of the office and its holders, outside the scope of the analysis. We keep R fixed so that the adoption decision turns on the durability of the channel and the incumbent's expected tenure rather than on a second source of variation whose effects would be hard to separate from the other variables of the model in general, and from strength in particular.

The second is the transfer x. Letting the incumbent choose it would turn the model into a pricing or bargaining problem, in which the incumbent sets the terms of the exchange rather than responding to them. We instead read x as the institutional price of elite support, set by the environment in which the incumbent operates. This reading matches the settings the paper has in mind, where the cost of securing elite backing is shaped by executive discretion, the strength of oversight, and the control of procurement, conditions an individual incumbent takes as given rather than sets. It also defines what we ask. The adoption threshold is meaningful only against a price the incumbent does not choose, since it measures the largest such price worth accepting. Endogenizing x would replace that with how an incumbent sets the terms of the exchange, moving the paper from a study of adoption to one of bargaining. We therefore treat x as exogenous and ask how high a price the incumbent will accept.

The third concerns electoral stability, which the model treats as a fixed feature of the environment. The cases suggest that in practice strong incumbents do much to produce the stability they enjoy, through, for example, control of the media, the rules of competition, and the resources of the state, so that stability is partly a consequence of their incumbency and not only a condition for it. We take it as given for a reason of focus. Our question is how the prospect of a stable environment shapes the decision to adopt, not how that environment comes about, and separating the two keeps the comparative statics, which are the point of the analysis, clean and interpretable. We do not claim that stability is exogenous in practice, only that holding it fixed is the right starting point for the question we ask.

Allowing stability to respond to the adoption decision does not unsettle the result. Let stability take the value $\sigma_\lambda(0)$ without patronage and $\sigma_\lambda(1) \geq \sigma_\lambda(0)$ with it, reflecting the apparatus that accompanies the channel in the cases above. Adoption then compares the patronage payoff at the higher stability with the no-patronage payoff at the lower,

$$U^i_\alpha\Big(1, \sigma_\lambda(1)\Big) > U^i_\alpha\Big(0, \sigma_\lambda(0)\Big). \tag{68}$$

Against the benchmark that holds stability at $\sigma_\lambda(0)$ throughout, only the patronage payoff changes, and the adoption surplus gains

$$U^i_\alpha\Big(1, \sigma_\lambda(1)\Big) - U^i_\alpha\Big(1, \sigma_\lambda(0)\Big) = \int_{\sigma_\lambda(0)}^{\sigma_\lambda(1)} \frac{\partial U^i_\alpha(1, \sigma_\lambda)}{\partial \sigma_\lambda}\, d\sigma_\lambda. \tag{69}$$

For a strong incumbent this payoff rises with stability (see Lemma A.6 in Appendix A), so the added term is nonnegative and the adoption region only widens. Letting stability respond to adoption therefore reinforces rather than weakens the central result. A full treatment would let the incumbent produce stability at a cost, with the two decisions made together. This creates a loop: adoption raises stability, while higher stability, as in the current model, raises the value of adoption.

6 Conclusion

This paper asked when an incumbent chooses to adopt a durable patronage exchange with elites. The answer runs against a familiar intuition. Where patronage is read as a tool of the cornered, something an incumbent reaches for when competition threatens to end its time in office, we find that a durable channel is built by incumbents who are strong and expect to stay in power, not by those most at risk of losing it. The shift comes from taking durability seriously. Once the channel lasts, it is less a remedy for a hard election than a long-lived asset attached to office, and the choice to build it turns on how long the incumbent expects to draw on it.

The same durability that gives the channel its value also limits who will build it. Because the exchange outlives the politician who adopts it, the advantage it creates is not the adopter's to keep. It passes to whoever holds office next, and after turnover that is a rival. Strength therefore matters in two ways at once. A strong incumbent expects a long tenure in which to use the channel and runs little risk of handing it to a successor, so both effects of durability favor adoption. For a weaker incumbent the two pull apart, and this is where the conditional results come from. The effects of electoral stability and patience stop being uniform once the prospect of equipping a successor grows large enough to offset the gain from holding office today. The single decision at the center of the model ties the results together through one logic, the tension between using a durable asset now and losing it later.

This changes how durable patronage in hybrid regimes might be read. The arrangements we have in mind do not look like the hurried moves of incumbents under threat. They are built by incumbents who already command the state and expect to keep it, and who extend their reach by tying economic elites to office through a recurring flow of state resources. The profile the model associates with adoption, a strong incumbent in a stable environment looking far into the future, is the profile under which such channels take hold, and the cases we discuss show the same pattern. In each, the channel was built after the incumbent's position was secure rather than while it was contested. Read this way, the durable patronage exchange is a sign of political consolidation rather than of political weakness.

The model is deliberately simple, and the cases are not exact instances of it. They involve broader coalitions, more actors, and motives the model sets aside. What the model offers is a clear account of one mechanism, the choice to build a lasting channel whose worth depends on expected tenure, and the cases show that this mechanism is visible in how these incumbents acted. The fit lies in the pattern rather than in the details. Durable patronage appeared where incumbents were strong, where the electoral environment was stable, and where they expected to remain in office, and it was built once their position was secure. The contribution of the model is to explain why this, and not the opposite, is the pattern we should expect to find.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the author used *Claude* and *ChatGPT* to improve language and readability, help structure and refine arguments, support brainstorming, and obtain a preliminary impression of relevant literature. After using these tools, the author reviewed and edited the text as needed and takes full responsibility for it.

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Appendix A: Proofs and formalities

Lemma A.1. ***Comparative statics of U^i_α: The effects of office rent, transfer, and winning probabilities***
U^i_α in Equation (42) satisfies the following:

A.1.1 U^i_α is strictly increasing in R.

A.1.2 U^i_α is decreasing in x.

A.1.3 U^i_α is strictly increasing in p^i_α.

A.1.4 U^i_α is increasing in p^o_α.

Proof of Lemma A.1 By Assumption 4, $p^i_\alpha, p^o_\alpha \in [0,1]$, so $1-\delta(1-p^o_\alpha) \geq 1-\delta > 0$, and since $p^i_\alpha - p^o_\alpha \leq 1$, also $1-\delta(p^i_\alpha - p^o_\alpha) \geq 1-\delta > 0$, both using $\delta < 1$. Since $x \in [0,R)$ and $d \in \{0,1\}$, $R - dx \geq R - x > 0$. All denominators below are therefore strictly positive.

A.1.1: Differentiating Equation (42) with respect to R,

$$\frac{\partial U^i_\alpha}{\partial R} = \frac{1-\delta\left(1-p^o_\alpha\right)}{\left(1-\delta\right)\left(1-\delta\left(p^i_\alpha - p^o_\alpha\right)\right)} > 0,$$

since the numerator and both denominator factors are strictly positive. Hence U^i_α is strictly increasing in R.

A.1.2: Differentiating with respect to x,

$$\frac{\partial U^i_\alpha}{\partial x} = \frac{-d\left(1-\delta\left(1-p^o_\alpha\right)\right)}{\left(1-\delta\right)\left(1-\delta\left(p^i_\alpha - p^o_\alpha\right)\right)} \leq 0,$$

since $d \geq 0$ and the remaining factor is strictly positive, strict when $d = 1$. Hence U^i_α is decreasing in x.

A.1.3: Differentiating with respect to p^i_α,

$$\frac{\partial U^i_\alpha}{\partial p^i_\alpha} = \frac{\delta\left(R-dx\right)\left(1-\delta\left(1-p^o_\alpha\right)\right)}{\left(1-\delta\right)\left(1-\delta\left(p^i_\alpha - p^o_\alpha\right)\right)^2} > 0,$$

since $\delta > 0$, $R - dx > 0$, and the remaining factors are strictly positive. Hence U^i_α is strictly increasing in p^i_α.

A.1.4: Applying the quotient rule with respect to p^o_α,

$$\frac{\partial U^i_\alpha}{\partial p^o_\alpha} = \frac{R-dx}{1-\delta} \cdot \frac{\delta\left(1-\delta\left(p^i_\alpha - p^o_\alpha\right)\right) - \delta\left(1-\delta\left(1-p^o_\alpha\right)\right)}{\left(1-\delta\left(p^i_\alpha - p^o_\alpha\right)\right)^2}$$
$$= \frac{\delta^2\left(R-dx\right)\left(1-p^i_\alpha\right)}{\left(1-\delta\right)\left(1-\delta\left(p^i_\alpha - p^o_\alpha\right)\right)^2} \geq 0,$$

since $R - dx > 0$ and $1 - p^i_\alpha \geq 0$, strict when $p^i_\alpha < 1$. Hence U^i_α is increasing in p^o_α. □

Proof of Lemma 1

By Assumption 4, $2\sigma_\lambda c \leq 1$ and $\sigma_\lambda(c+\mu) \leq 1/2$. Combined with $\delta < 1$, this gives $1-\delta > 0$, $1-2\delta\sigma_\lambda c \geq 1-\delta > 0$, and $1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right) \geq 1-\delta > 0$, so all denominators below are strictly positive and every multiplication and division preserves the direction of the inequality. Substituting Equations (47) and (46) into $U_\alpha^i(1) > U_\alpha^i(0)$,

$$\frac{(R-x)\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)}{(1-\delta)(1-2\delta\sigma_\lambda c)} > \frac{R\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda\mu\right)\right)}{1-\delta}.$$

Multiplying both sides by $(1-\delta)(1-2\delta\sigma_\lambda c)$,

$$(R-x)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right) > R(1-2\delta\sigma_\lambda c)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda\mu\right)\right).$$

Expanding the left-hand side and moving the term in x to the right,

$$\begin{aligned} R\Big[&\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)-(1-2\delta\sigma_\lambda c)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda\mu\right)\right)\Big] \\ &> x\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right). \end{aligned}$$

The bracketed term simplifies as

$$\begin{aligned} &\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)-(1-2\delta\sigma_\lambda c)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda\mu\right)\right) \\ &\quad = -\delta\sigma_\lambda c + 2\delta\sigma_\lambda c\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda\mu\right)\right) \\ &\quad = \delta\sigma_\lambda c\,(1-\delta(1+2\sigma_\lambda\mu)), \end{aligned}$$

so the inequality becomes

$$R\,\delta\sigma_\lambda c\,(1-\delta(1+2\sigma_\lambda\mu)) > x\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right).$$

Dividing both sides by $1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)$,

$$\frac{R\,\delta\sigma_\lambda c\,(1-\delta(1+2\sigma_\lambda\mu))}{1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)} > x,$$

which is $x < R\tilde{x} = \bar{x}$, with $\tilde{x}$ defined in Equation (49). □

Lemma A.2. ***Bounds on the adoption threshold***
$\tilde{x}$ in Equation (49) satisfies the following:

A.2.1 If $\mu \leq 0$, then $\tilde{x} \in (0, 1)$.

A.2.2 If $\mu > 0$, then $\tilde{x} \in [0, 1)$ if and only if $\sigma_\lambda \leq \dfrac{1-\delta}{2\delta\mu}$.

Proof of Lemma A.2

By Assumption 4, $2\sigma_\lambda c \leq 1$ and $\sigma_\lambda(c+\mu) \leq 1/2$. Combined with $\delta < 1$, this gives $1 - 2\delta\sigma_\lambda c \geq 1 - \delta > 0$ and $1 - \delta\left(\frac{1}{2} + \sigma_\lambda(c+\mu)\right) \geq 1 - \delta > 0$, so the denominator of $\tilde{x}$ is strictly positive. We first show that $\tilde{x} < 1$ for either sign of μ. As the denominator is strictly positive, this is equivalent to the denominator strictly exceeding the numerator. A direct calculation gives

$$\begin{aligned}&\left[1 - \delta\left(\tfrac{1}{2} + \sigma_\lambda(c+\mu)\right)\right] - \delta\sigma_\lambda c\left[1 - \delta\left(1 + 2\sigma_\lambda\mu\right)\right]\\&\quad= \tfrac{1}{2}\left(1 - 2\delta\sigma_\lambda c\right)\left[(2-\delta) - 2\delta\sigma_\lambda\mu\right].\end{aligned}$$

The factor $1 - 2\delta\sigma_\lambda c$ is strictly positive. For the second factor, Assumption 4 gives $\sigma_\lambda\mu \leq \frac{1}{2} - \sigma_\lambda c < \frac{1}{2}$, where the strict inequality uses $\sigma_\lambda c > 0$ from Assumption 2, so with $\delta < 1$,

$$(2-\delta) - 2\delta\sigma_\lambda\mu > (2-\delta) - \delta = 2(1-\delta) > 0.$$

Both factors are strictly positive, so the denominator exceeds the numerator and $\tilde{x} < 1$. It remains to sign $\tilde{x}$. The denominator is strictly positive and $\delta\sigma_\lambda c > 0$ by Assumption 2, so the sign of $\tilde{x}$ coincides with the sign of $1 - \delta\left(1 + 2\sigma_\lambda\mu\right)$.

A.2.1: If $\mu \leq 0$, then $2\delta\sigma_\lambda\mu \leq 0 < 1 - \delta$, so $1 - \delta\left(1 + 2\sigma_\lambda\mu\right) > 0$ and $\tilde{x} > 0$. Hence $\tilde{x} \in (0, 1)$.

A.2.2: If $\mu > 0$, then $1 - \delta\left(1 + 2\sigma_\lambda\mu\right) \geq 0$ is equivalent to $2\delta\sigma_\lambda\mu \leq 1 - \delta$, so

$$\tilde{x} \geq 0 \iff \sigma_\lambda \leq \frac{1-\delta}{2\delta\mu}.$$

This condition is exactly Assumption 5, under which $\tilde{x} \geq 0$, so together with $\tilde{x} < 1$ it gives $\tilde{x} \in [0, 1)$. $\square$

Lemma A.3. ***Comparative statics of no-patronage***
$U^i_\alpha(0)$ in Equation (46) satisfies the following:

A.3.1 $U^i_\alpha(0)$ is strictly decreasing in μ.

A.3.2 $U^i_\alpha(0)$ is strictly decreasing in σ_λ if $\mu > 0$, strictly increasing in σ_λ if $\mu < 0$, and independent of σ_λ if $\mu = 0$.

A.3.3 $U^i_\alpha(0)$ is strictly increasing in δ.

Proof of Lemma A.3

A.3.1: Differentiating Equation (46) with respect to μ,

$$\frac{\partial U^i_\alpha(0)}{\partial \mu} = -\frac{R\delta\sigma_\lambda}{1-\delta} < 0,$$

since $R > 0$, $\delta \in (0,1)$, and $\sigma_\lambda > 0$. Hence $U^i_\alpha(0)$ is strictly decreasing in μ.

A.3.2: Differentiating with respect to σ_λ,

$$\frac{\partial U^i_\alpha(0)}{\partial \sigma_\lambda} = -\frac{R\delta\mu}{1-\delta}.$$

Since $R > 0$ and $\delta \in (0,1)$, the sign is that of $-\mu$, so the derivative is negative when $\mu > 0$, positive when $\mu < 0$, and zero when $\mu = 0$.

A.3.3: Differentiating with respect to δ,

$$\frac{\partial U^i_\alpha(0)}{\partial \delta} = \frac{R\left(\frac{1}{2} - \sigma_\lambda\mu\right)}{(1-\delta)^2}.$$

By Assumption 4 with $c > 0$, $\sigma_\lambda\mu < \sigma_\lambda(c+\mu) \leq 1/2$, so $\frac{1}{2} - \sigma_\lambda\mu > 0$. The denominator is strictly positive, hence

$$\frac{\partial U^i_\alpha(0)}{\partial \delta} > 0,$$

so $U^i_\alpha(0)$ is strictly increasing in δ. $\square$

Lemma A.4. ***Comparative statics of patronage: The effect of incumbent strength and discount factor***
$U_\alpha^i(1)$ in Equation (47) satisfies the following:

A.4.1 $U_\alpha^i(1)$ is strictly decreasing in μ.

A.4.2 $U_\alpha^i(1)$ is increasing in δ.

Proof of Lemma A.4

By Assumption 4, $2\sigma_\lambda c \leq 1$ and $\sigma_\lambda(c+\mu) \leq 1/2$. Combined with $\delta < 1$, this gives $1 - 2\delta\sigma_\lambda c \geq 1 - \delta > 0$, so all denominators below are strictly positive.

A.4.1: Differentiating Equation (47) with respect to μ,

$$\frac{\partial U_\alpha^i(1)}{\partial \mu} = -\frac{(R-x)\,\delta\sigma_\lambda}{(1-\delta)\,(1-2\delta\sigma_\lambda c)} < 0,$$

since $R - x > 0$, $\delta \in (0,1)$, and $\sigma_\lambda > 0$. Hence $U_\alpha^i(1)$ is strictly decreasing in μ.

A.4.2: Applying the quotient rule with respect to δ and simplifying,

$$\begin{aligned}\frac{\partial U_\alpha^i(1)}{\partial \delta} = & \frac{(R-x)}{(1-\delta)^2\,(1-2\delta\sigma_\lambda c)^2} \\ & \cdot \left[2\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\sigma_\lambda c\,\delta^2 - 4\sigma_\lambda c\,\delta + 2\sigma_\lambda c + 1 - \left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right].\end{aligned} \tag{A.1}$$

The denominator is strictly positive, so the sign is that of the bracketed polynomial in δ. Its second derivative in δ is $4(\frac{1}{2}+\sigma_\lambda(c+\mu))\sigma_\lambda c \geq 0$, so the polynomial is convex, with first derivative in δ vanishing at

$$\delta^* = \frac{1}{\frac{1}{2}+\sigma_\lambda(c+\mu)} \geq 1,$$

where the bound uses $\frac{1}{2}+\sigma_\lambda(c+\mu) \leq 1$ from Assumption 4. Convexity then makes the polynomial decreasing on $[0,1]$, so its minimum there is at $\delta = 1$, where it equals

$$\left(1-\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)(1-2\sigma_\lambda c) \geq 0,$$

both factors being non-negative by Assumption 4. The bracketed polynomial is therefore non-negative on $[0, 1]$, hence

$$\frac{\partial U_{\alpha}^{i}(1)}{\partial \delta} \geq 0,$$

so $U_{\alpha}^{i}(1)$ is increasing in δ. $\square$

Lemma A.5. ***Comparative statics of patronage: The effect of the elite contribution***

$U^i_\alpha(1)$ in Equation (47) is increasing in c. The increase is strict unless $\mu > 0$ and $\sigma_\lambda = \dfrac{1-\delta}{2\delta\mu}$.

Proof of Lemma A.5

Applying the quotient rule to Equation (47),

$$\begin{aligned}
\frac{\partial U^i_\alpha(1)}{\partial c} &= \frac{R-x}{1-\delta} \cdot \frac{(-\delta\sigma_\lambda)(1-2\delta\sigma_\lambda c) + 2\delta\sigma_\lambda\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)}{(1-2\delta\sigma_\lambda c)^2} \\
&= \frac{(R-x)\,\delta\sigma_\lambda\,(1-\delta\,(1+2\sigma_\lambda\mu))}{(1-\delta)\,(1-2\delta\sigma_\lambda c)^2}.
\end{aligned}$$

By Assumption 4, $2\sigma_\lambda c \le 1$, so with $\delta < 1$ the denominator is strictly positive. Since $R - x > 0$, $\delta \in (0,1)$, and $\sigma_\lambda > 0$, the sign of the derivative is that of $1-\delta(1+2\sigma_\lambda\mu)$. If $\mu \le 0$, then $1+2\sigma_\lambda\mu \le 1$, so $1-\delta(1+2\sigma_\lambda\mu) > 0$. If $\mu > 0$, Assumption 5 gives $\sigma_\lambda \le (1-\delta)/(2\delta\mu)$, equivalently $1-\delta(1+2\sigma_\lambda\mu) \ge 0$, which is strict for $\sigma_\lambda < (1-\delta)/(2\delta\mu)$ and vanishes only at $\sigma_\lambda = (1-\delta)/(2\delta\mu)$. Hence $U^i_\alpha(1)$ is increasing in c, strict unless $\mu > 0$ and $\sigma_\lambda = (1-\delta)/(2\delta\mu)$. □

Lemma A.6. ***Comparative statics of patronage: The effect of electoral stability***

$U^i_\alpha(1)$ in Equation (47) satisfies the following:

A.6.1 If $\mu \le 0$, then $U^i_\alpha(1)$ is strictly increasing in σ_λ.

A.6.2 If $\mu > 0$, define

$$c^{\sigma_\lambda} = \frac{\mu}{1-\delta}. \tag{A.2}$$

Then $U^i_\alpha(1)$ is strictly increasing in σ_λ if $c > c^{\sigma_\lambda}$ and strictly decreasing in σ_λ if $c < c^{\sigma_\lambda}$. The threshold c^{σ_λ} is strictly increasing in μ and in δ.

Proof of Lemma A.6

Differentiating Equation (47) in σ_λ by the quotient rule,

$$\begin{aligned}\frac{\partial U^i_\alpha(1)}{\partial \sigma_\lambda} &= \frac{R-x}{1-\delta} \cdot \frac{\left(-\delta(c+\mu)\right)\left(1-2\delta\sigma_\lambda c\right) + 2\delta c\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)}{\left(1-2\delta\sigma_\lambda c\right)^2} \\ &= \frac{(R-x)\,\delta\left(c\left(1-\delta\right)-\mu\right)}{\left(1-\delta\right)\left(1-2\delta\sigma_\lambda c\right)^2}.\end{aligned}$$

The denominator is again strictly positive, as $2\sigma_\lambda c \le 1$ and $\delta < 1$. With $R - x > 0$ and $\delta \in (0, 1)$, the sign of the derivative is that of $c(1-\delta) - \mu$.

A.6.1: Suppose $\mu \le 0$. Then $c(1-\delta) - \mu > 0$, since $c > 0$ by Assumption 2 and $\delta < 1$. Hence $U^i_\alpha(1)$ is strictly increasing in σ_λ.

A.6.2: Suppose $\mu > 0$. Then $c(1-\delta) - \mu > 0$ if and only if $c > \mu/(1-\delta)$. Defining $c^{\sigma_\lambda} = \mu/(1-\delta)$, it follows that $U^i_\alpha(1)$ is strictly increasing in σ_λ when $c > c^{\sigma_\lambda}$ and strictly decreasing when $c < c^{\sigma_\lambda}$. Differentiating the threshold,

$$\frac{\partial c^{\sigma_\lambda}}{\partial \mu} = \frac{1}{1-\delta} > 0, \qquad \frac{\partial c^{\sigma_\lambda}}{\partial \delta} = \frac{\mu}{(1-\delta)^2} > 0,$$

so c^{σ_λ} is strictly increasing in μ and in δ. □

Lemma A.7. ***Partial derivatives of the adoption threshold: Electoral stability and the discount factor***

Suppose $\tilde{x} \in (0,1)$. Then

$$\frac{\partial \tilde{x}}{\partial \sigma_\lambda} = \frac{2c\delta\, \Pi_{\sigma_\lambda}}{\left(2 - \delta\left(1 + 2\sigma_\lambda(c+\mu)\right)\right)^2}, \tag{A.3}$$

and

$$\frac{\partial \tilde{x}}{\partial \delta} = \frac{2c\sigma_\lambda\, \Pi_{\delta}}{\left(2 - \delta\left(1 + 2\sigma_\lambda(c+\mu)\right)\right)^2}, \tag{A.4}$$

where

$$\Pi_{\sigma_\lambda} = 4\delta^2\mu(\mu+c)\sigma_\lambda^2 + 4\delta\mu(\delta-2)\sigma_\lambda + (1-\delta)(2-\delta), \tag{A.5}$$

and

$$\Pi_\delta = (1+2\sigma_\lambda\mu)\left(1+2\sigma_\lambda(\mu+c)\right)\delta^2 - 4\left(1+2\sigma_\lambda\mu\right)\delta + 2. \tag{A.6}$$

Proof of Lemma A.7

We use the expression for $\tilde{x}$ in Equation (49). By Assumption 4, $\sigma_\lambda(c+\mu) \leq 1/2$, so its denominator $1 - \delta\left(\frac{1}{2} + \sigma_\lambda(c+\mu)\right) \geq 1 - \delta > 0$ is strictly positive on the maintained region.

Effect of σ_λ: Differentiating the numerator of Equation (49) with respect to σ_λ,

$$\begin{aligned}\frac{\partial}{\partial \sigma_\lambda}\left[\delta\sigma_\lambda c\left(1 - \delta\left(1 + 2\sigma_\lambda\mu\right)\right)\right] &= \delta c\left(1 - \delta\left(1+2\sigma_\lambda\mu\right)\right) - 2\delta^2\sigma_\lambda c\mu \\ &= \delta c\left(1 - \delta - 4\delta\sigma_\lambda\mu\right).\end{aligned}$$

The derivative of the denominator with respect to σ_λ is $-\delta(c+\mu)$. Applying the quotient rule,

$$\frac{\partial \tilde{x}}{\partial \sigma_\lambda} = \frac{1}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2}$$
$$\cdot \Bigg[\delta c\left(1-\delta-4\delta\sigma_\lambda\mu\right)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)$$
$$+\delta^2\sigma_\lambda c(c+\mu)\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\Bigg].$$

Factoring out $c\delta$,

$$\frac{\partial \tilde{x}}{\partial \sigma_\lambda} = \frac{c\delta}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2}$$
$$\cdot \Bigg[\left(1-\delta-4\delta\sigma_\lambda\mu\right)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)$$
$$+\delta\sigma_\lambda(c+\mu)\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\Bigg].$$

Since

$$1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right) = \frac{1}{2}\left(2-\delta\left(1+2\sigma_\lambda(c+\mu)\right)\right),$$

the squared denominator equals $\frac{1}{4}\left(2-\delta\left(1+2\sigma_\lambda(c+\mu)\right)\right)^2$. Substituting,

$$\frac{\partial \tilde{x}}{\partial \sigma_\lambda} = \frac{4c\delta}{\left(2-\delta\left(1+2\sigma_\lambda(c+\mu)\right)\right)^2}$$
$$\cdot \Bigg[\left(1-\delta-4\delta\sigma_\lambda\mu\right)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)$$
$$+\delta\sigma_\lambda(c+\mu)\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\Bigg].$$

Expanding the bracketed term and collecting powers of σ_λ,

$$
\begin{aligned}
&(1-\delta-4\delta\sigma_\lambda\mu)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)+\delta\sigma_\lambda(c+\mu)\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\\
&\quad=\frac{1}{2}\left[4\delta^2\mu(\mu+c)\sigma_\lambda^2+4\delta\mu(\delta-2)\sigma_\lambda+(1-\delta)(2-\delta)\right]=\frac{1}{2}\Pi_{\sigma_\lambda}.
\end{aligned}
$$

Substituting into the derivative yields Equation (A.3).

Effect of δ: Differentiating the numerator of Equation (49) with respect to δ,

$$
\begin{aligned}
\frac{\partial}{\partial\delta}\left[\delta\sigma_\lambda c\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\right]&=\sigma_\lambda c\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)-\delta\sigma_\lambda c\left(1+2\sigma_\lambda\mu\right)\\
&=\sigma_\lambda c\left(1-2\delta\left(1+2\sigma_\lambda\mu\right)\right).
\end{aligned}
$$

The derivative of the denominator with respect to δ is $-\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)$. Applying the quotient rule,

$$
\begin{aligned}
\frac{\partial\tilde{x}}{\partial\delta}=&\frac{1}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2}\\
&\cdot\left[\sigma_\lambda c\left(1-2\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)\right.\\
&\left.+\delta\sigma_\lambda c\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right].
\end{aligned}
$$

Factoring out $c\sigma_\lambda$,

$$
\begin{aligned}
\frac{\partial\tilde{x}}{\partial\delta}=&\frac{c\sigma_\lambda}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2}\\
&\cdot\left[\left(1-2\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)\right.\\
&\left.+\delta\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right].
\end{aligned}
$$

Using the same denominator identity,

$$
\begin{aligned}
\frac{\partial \tilde{x}}{\partial \delta} = &\frac{4c\sigma_\lambda}{\left(2-\delta\left(1+2\sigma_\lambda(c+\mu)\right)\right)^2} \\
&\cdot \Bigg[\left(1-2\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(1-\delta\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right)\right) \\
&+ \delta\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(\tfrac{1}{2}+\sigma_\lambda(c+\mu)\right) \Bigg].
\end{aligned}
$$

Expanding the bracketed term and collecting powers of δ,

$$
\begin{aligned}
&\left(1-2\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right) \\
&\quad + \delta\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right) \\
&\quad = \frac{1}{2}\Big[\left(1+2\sigma_\lambda\mu\right)\left(1+2\sigma_\lambda(c+\mu)\right)\delta^2 - 4\left(1+2\sigma_\lambda\mu\right)\delta + 2\Big] \\
&\quad = \frac{1}{2}\Pi_\delta.
\end{aligned}
$$

Substituting into the derivative yields Equation (A.4). □

Proof of Lemma 2

By Assumption 4, $\sigma_\lambda(c+\mu) \le 1/2$, so the denominator $1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right) \ge 1-\delta > 0$ of $\tilde{x}$ in Equation (49) is strictly positive on the maintained region.

2.1: Differentiating with respect to μ,

$$\begin{aligned}
\frac{\partial \tilde{x}}{\partial \mu} &= \frac{-2\delta^2\sigma_\lambda^2 c\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)+\delta^2\sigma_\lambda^2 c\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2} \\
&= \frac{\delta^2\sigma_\lambda^2 c\left(2\delta\sigma_\lambda c-1\right)}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2}.
\end{aligned}$$

By Assumption 4, $2\sigma_\lambda c \le 1$, and since $\delta < 1$ this gives $2\delta\sigma_\lambda c - 1 < 0$. The remaining factors are strictly positive, so $\partial\tilde{x}/\partial\mu < 0$ and $\tilde{x}$ is strictly decreasing in μ.

2.2: Differentiating with respect to c,

$$\begin{aligned}
\frac{\partial \tilde{x}}{\partial c} &= \frac{\delta\sigma_\lambda\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2} \\
&\quad + \frac{\delta^2\sigma_\lambda^2 c\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2} \\
&= \frac{\delta\sigma_\lambda\left(1-\delta\left(1+2\sigma_\lambda\mu\right)\right)\left(1-\frac{\delta}{2}-\delta\sigma_\lambda\mu\right)}{\left(1-\delta\left(\frac{1}{2}+\sigma_\lambda(c+\mu)\right)\right)^2}.
\end{aligned}$$

By Assumption 4 with $c > 0$, $\sigma_\lambda\mu < \sigma_\lambda(c+\mu) \le 1/2$, so $1-\frac{\delta}{2}-\delta\sigma_\lambda\mu > 1-\delta > 0$, and $\delta\sigma_\lambda$ with the denominator is positive. The sign of $\partial\tilde{x}/\partial c$ is thus that of $1-\delta(1+2\sigma_\lambda\mu)$, which Assumption 5 keeps non-negative. It is strictly positive whenever $\mu \le 0$, and for $\mu > 0$ vanishes only at $\sigma_\lambda = (1-\delta)/(2\delta\mu)$. Hence $\tilde{x}$ is increasing in c, strict unless $\mu > 0$ and $\sigma_\lambda = (1-\delta)/(2\delta\mu)$. □

Proof of Lemma 3 By Lemma A.7 and Equation (A.3),

$$\frac{\partial \tilde{x}}{\partial \sigma_\lambda} = \frac{2c\delta\,\Pi_{\sigma_\lambda}}{\left(2-\delta\left(1+2\sigma_\lambda(c+\mu)\right)\right)^2},$$
$$\Pi_{\sigma_\lambda} = 4\delta^2\mu(\mu+c)\,\sigma_\lambda^2 - 4\delta\mu(2-\delta)\,\sigma_\lambda + (1-\delta)(2-\delta).$$

By Assumption 4, $\sigma_\lambda(c+\mu) \leq 1/2$ on the admissible range, so the denominator is strictly positive, and $2c\delta > 0$. The sign of $\partial\tilde{x}/\partial\sigma_\lambda$ is therefore the sign of Π_{σ_λ}. At the left endpoint,

$$\Pi_{\sigma_\lambda}(0) = (1-\delta)(2-\delta) > 0.$$

We determine the sign of Π_{σ_λ} on $(0, \bar{\sigma}_\lambda]$, then translate its single sign change into the conditions of the lemma. Throughout we write $\tilde{\sigma}_\lambda = (1-\delta)/(2\delta\mu)$ for the second term of $\bar{\sigma}_\lambda$ in the case $\mu > 0$ of Equation (54), so that $\bar{\sigma}_\lambda \leq \tilde{\sigma}_\lambda$ when $\mu > 0$.

Strong incumbent, $\mu \leq 0$: If $\mu = 0$, then $\Pi_{\sigma_\lambda} = (1-\delta)(2-\delta) > 0$. Suppose $\mu < 0$. The linear coefficient $-4\delta\mu(2-\delta)$ is positive. If $\mu + c \leq 0$, then the leading coefficient $4\delta^2\mu(\mu+c)$ is non-negative, every coefficient of Π_{σ_λ} is non-negative, and the constant term is strictly positive, so $\Pi_{\sigma_\lambda} > 0$. If $\mu + c > 0$, equivalently $-\mu < c$, then Π_{σ_λ} is concave, so its minimum on $[0, \bar{\sigma}_\lambda]$ is attained at an endpoint. It is strictly positive at $\sigma_\lambda = 0$, and at the upper endpoint $\bar{\sigma}_\lambda = 1/(2(c-\mu))$, using $2\sigma_\lambda c \leq 1$ from Assumption 4 together with $-\mu < c$,

$$4\delta^2\mu(\mu+c)\,\sigma_\lambda^2 + 4\delta\mu(\delta-2)\,\sigma_\lambda \geq 2\delta(4-3\delta)(-\mu)\,\sigma_\lambda > 0.$$

With the positive constant term, $\Pi_{\sigma_\lambda}(\bar{\sigma}_\lambda) > 0$. In either subcase $\Pi_{\sigma_\lambda} > 0$, so $\partial\tilde{x}/\partial\sigma_\lambda > 0$ and $\tilde{x}$ is strictly increasing.

Weak incumbent, $\mu > 0$: The leading coefficient $4\delta^2\mu(\mu+c)$ is strictly positive, so Π_{σ_λ} is convex. Its discriminant is

$$\begin{aligned}
&\Big(4\delta\mu(2-\delta)\Big)^2 - 16\delta^2\mu(\mu+c)(1-\delta)(2-\delta) \\
&\quad = 16\delta^2\mu(2-\delta)\Big[\mu(2-\delta) - (\mu+c)(1-\delta)\Big] \\
&\quad = 16\delta^2\mu(2-\delta)\Big(\mu - c(1-\delta)\Big),
\end{aligned}$$

whose sign is the sign of $\mu - c(1-\delta)$. If $0 < \mu \leq c(1-\delta)$, the discriminant is non-positive, so Π_{σ_λ} has no real sign change. Since $\Pi_{\sigma_\lambda}(0) > 0$ and Π_{σ_λ} is convex, $\Pi_{\sigma_\lambda} \geq 0$ throughout, so $\partial\tilde{x}/\partial\sigma_\lambda \geq 0$ and $\tilde{x}$ is strictly increasing. If $\mu > c(1-\delta)$, the discriminant is strictly positive, so Π_{σ_λ} has two distinct real roots. The square root of the discriminant is $4\delta\sqrt{\mu(2-\delta)\big(\mu - c(1-\delta)\big)}$, and the roots are minus the linear coefficient plus or minus this square root, over twice the leading coefficient,

$$\begin{aligned}\sigma_\lambda^* &= \frac{4\delta\mu(2-\delta) - 4\delta\sqrt{\mu(2-\delta)\big(\mu - c(1-\delta)\big)}}{2\cdot 4\delta^2\mu(\mu+c)} \\ &= \frac{\mu(2-\delta) - \sqrt{\mu(2-\delta)\big(\mu - c(1-\delta)\big)}}{2\delta\mu(\mu+c)},\end{aligned}$$

$$\begin{aligned}\sigma_\lambda^{**} &= \frac{4\delta\mu(2-\delta) + 4\delta\sqrt{\mu(2-\delta)\big(\mu - c(1-\delta)\big)}}{2\cdot 4\delta^2\mu(\mu+c)} \\ &= \frac{\mu(2-\delta) + \sqrt{\mu(2-\delta)\big(\mu - c(1-\delta)\big)}}{2\delta\mu(\mu+c)},\end{aligned}$$

where the common factor 4δ cancels and σ_λ^* is the smaller root, equal to Equation (55). By Vieta's formulas, for a quadratic the product of the roots equals the constant term divided by the leading coefficient and the sum equals the linear coefficient divided by the leading coefficient with the sign reversed, so

$$\begin{aligned}\sigma_\lambda^*\,\sigma_\lambda^{**} &= \frac{(1-\delta)(2-\delta)}{4\delta^2\mu(\mu+c)}, \\ \sigma_\lambda^* + \sigma_\lambda^{**} &= \frac{4\delta\mu(2-\delta)}{4\delta^2\mu(\mu+c)} = \frac{2-\delta}{\delta(\mu+c)},\end{aligned}$$

both strictly positive, so both roots are positive. Evaluating Π_{σ_λ} at $\tilde{\sigma}_\lambda$, with $2\delta\mu\tilde{\sigma}_\lambda = 1-\delta$,

$$\Pi_{\sigma_\lambda}(\tilde{\sigma}_\lambda) = (1-\delta)\,\frac{c(1-\delta)-\mu}{\mu} < 0,$$

so $\tilde{\sigma}_\lambda$ lies strictly between the roots, $\sigma_\lambda^* < \tilde{\sigma}_\lambda < \sigma_\lambda^{**}$. Since $\bar{\sigma}_\lambda \leq \tilde{\sigma}_\lambda$, the larger root satisfies $\sigma_\lambda^{**} > \bar{\sigma}_\lambda$, so on $(0, \bar{\sigma}_\lambda]$ the only possible sign change of Π_{σ_λ} is at σ_λ^*, from positive to negative. Hence $\tilde{x}$ is single-peaked with a unique interior maximum at σ_λ^* when $\sigma_\lambda^* < \bar{\sigma}_\lambda$, and strictly increasing when $\sigma_\lambda^* \geq \bar{\sigma}_\lambda$. We express $\sigma_\lambda^* < \bar{\sigma}_\lambda$ in primitives. Since for $\mu > 0$ Equation (54) gives $\bar{\sigma}_\lambda$ as the minimum of $1/(2(c+\mu))$ and $\tilde{\sigma}_\lambda$, and we have shown $\sigma_\lambda^* < \tilde{\sigma}_\lambda$, the comparison with the minimum reduces to the first term,

$$\sigma_\lambda^* < \bar{\sigma}_\lambda \quad \Longleftrightarrow \quad \sigma_\lambda^* < \frac{1}{2(c+\mu)}.$$

Multiply by $2\delta\mu(\mu+c) > 0$. The right side becomes $2\delta\mu(\mu+c)/(2(c+\mu)) = \delta\mu$, so

$$\mu(2-\delta) - \sqrt{\mu(2-\delta)\Big(\mu - c(1-\delta)\Big)} < \delta\mu.$$

Moving the square root across and combining the remaining terms, with $\mu(2-\delta) - \delta\mu = 2\mu(1-\delta)$,

$$2\mu(1-\delta) < \sqrt{\mu(2-\delta)\Big(\mu - c(1-\delta)\Big)}.$$

Both sides are positive, so squaring preserves the inequality, and dividing by $\mu > 0$,

$$4\mu(1-\delta)^2 < (2-\delta)\Big(\mu - c(1-\delta)\Big).$$

Collecting the terms in μ and using $(2-\delta) - 4(1-\delta)^2 = 7\delta - 4\delta^2 - 2$,

$$\mu\Big(7\delta - 4\delta^2 - 2\Big) > c(1-\delta)(2-\delta).$$

The right side is strictly positive, so this requires $7\delta - 4\delta^2 - 2 > 0$. The roots of $4\delta^2 - 7\delta + 2$ are

$$\delta = \frac{7 \pm \sqrt{49-32}}{8} = \frac{7 \pm \sqrt{17}}{8},$$

that is $\hat{\delta} = (7-\sqrt{17})/8$ and $(7+\sqrt{17})/8 > 1$. Since $4\delta^2 - 7\delta + 2$ equals 2 at $\delta = 0$ and has only the root $\hat{\delta}$ in $(0,1)$, it is positive for $\delta < \hat{\delta}$ and negative for $\delta > \hat{\delta}$, so $7\delta - 4\delta^2 - 2 > 0$ if and only if $\delta > \hat{\delta}$. Given $\delta > \hat{\delta}$, dividing by $7\delta - 4\delta^2 - 2 > 0$ yields $\mu > \hat{\mu}$, and the steps reverse, so the inequality is equivalent to $\delta > \hat{\delta}$ and $\mu > \hat{\mu}$. Finally, since $(2-\delta) - \Big(7\delta - 4\delta^2 - 2\Big) = 4(1-\delta)^2 \geq 0$, the denominator in $\hat{\mu}$ does not exceed $2-\delta$, so $\hat{\mu} \geq c(1-\delta)$ and $\mu > \hat{\mu}$ implies $\mu > c(1-\delta)$, the regime in which the roots are real. Therefore, for $\mu > 0$,

$$\sigma_\lambda^* < \bar{\sigma}_\lambda \quad \Longleftrightarrow \quad \delta > \hat{\delta} \text{ and } \mu > \hat{\mu}.$$

$\tilde{x}$ is single-peaked on $(0, \bar{\sigma}_\lambda]$ with maximum at σ_λ^* when $\delta > \hat{\delta}$ and $\mu > \hat{\mu}$, and is strictly increasing in σ_λ otherwise, which proves Parts 3.1 and 3.2. Suppose $\delta > \hat{\delta}$ and $\mu > \hat{\mu}$, so $\mu > c(1-\delta)$ holds strictly and the discriminant is strictly positive. Define

$$F(\sigma_\lambda,\mu,\delta,c) = 4\delta^2\mu(\mu+c)\,\sigma_\lambda^2 - 4\delta\mu(2-\delta)\,\sigma_\lambda + (1-\delta)(2-\delta).$$

As a polynomial in its arguments, F is continuously differentiable on the open region where $\mu,\delta,c>0$, and $F(\sigma_\lambda^*,\mu,\delta,c)=0$. Its partial derivative in σ_λ is

$$\frac{\partial F}{\partial \sigma_\lambda} = 8\delta^2\mu(\mu+c)\,\sigma_\lambda - 4\delta\mu(2-\delta),$$

and at the smaller root it equals $-4\delta\sqrt{\mu(2-\delta)\big(\mu-c(1-\delta)\big)}$, the negative of the square root of the discriminant, hence strictly negative and in particular nonzero. The implicit function theorem therefore applies, so in a neighborhood of any such point σ_λ^* is a continuously differentiable function of (μ,δ,c), and for each parameter $k\in\{\mu,\delta,c\}$,

$$\frac{\partial \sigma_\lambda^*}{\partial k} = -\left.\frac{\partial F/\partial k}{\partial F/\partial \sigma_\lambda}\right|_{\sigma_\lambda=\sigma_\lambda^*}.$$

Because the denominator is negative, the sign of $\partial\sigma_\lambda^*/\partial k$ is the sign of $\partial F/\partial k$ at σ_λ^*. We use throughout that $\sigma_\lambda^* < \tilde\sigma_\lambda = (1-\delta)/(2\delta\mu)$.

Effect of μ:

$$\frac{\partial F}{\partial \mu} = 4\delta\sigma_\lambda\Big(\delta(2\mu+c)\sigma_\lambda - (2-\delta)\Big).$$

The bracket is strictly increasing in σ_λ, so at $\sigma_\lambda^* < \tilde\sigma_\lambda$ it is bounded above by its value at $\tilde\sigma_\lambda$,

$$\delta(2\mu+c)\,\tilde\sigma_\lambda - (2-\delta) = -1 + \frac{(1-\delta)c}{2\mu}.$$

Since $\mu > c(1-\delta)$ gives $(1-\delta)c<\mu<2\mu$, this is negative, so the bracket is negative at σ_λ^* and $\partial F/\partial\mu<0$. Hence $\partial\sigma_\lambda^*/\partial\mu<0$.

Effect of c:

$$\frac{\partial F}{\partial c} = 4\delta^2\mu\,\sigma_\lambda^2,$$

which is positive, so $\partial\sigma_\lambda^*/\partial c>0$.

Effect of δ:

$$\frac{\partial F}{\partial \delta} = 8\delta\mu(\mu+c)\,\sigma_\lambda^2 - 8\mu(1-\delta)\,\sigma_\lambda - (3-2\delta).$$

At σ_λ^*, the identity $F(\sigma_\lambda^*) = 0$ gives

$$4\delta^2\mu(\mu + c)\,\sigma_\lambda^{*2} = 4\delta\mu(2-\delta)\,\sigma_\lambda^* - (1-\delta)(2-\delta),$$

so that

$$8\delta\mu(\mu + c)\,\sigma_\lambda^{*2} = \frac{2}{\delta}\Big(4\delta\mu(2-\delta)\,\sigma_\lambda^* - (1-\delta)(2-\delta)\Big).$$

Substituting and simplifying,

$$\left.\frac{\partial F}{\partial \delta}\right|_{\sigma_\lambda^*} = 8\mu\,\sigma_\lambda^* + 3 - \frac{4}{\delta}.$$

This is strictly increasing in σ_λ^*, and at $\sigma_\lambda^* = \tilde{\sigma}_\lambda$ it equals

$$8\mu \cdot \frac{1-\delta}{2\delta\mu} + 3 - \frac{4}{\delta} = \frac{4(1-\delta)}{\delta} + 3 - \frac{4}{\delta} = -1.$$

Since $\sigma_\lambda^* < \tilde{\sigma}_\lambda$, the value at σ_λ^* is strictly below -1, hence negative, so $\partial\sigma_\lambda^*/\partial\delta < 0$. $\square$

Proof of Lemma 4

By Lemma A.7 and Equation (A.4),

$$\frac{\partial \tilde{x}}{\partial \delta} = \frac{2c\sigma_\lambda \, \Pi_\delta}{\left(2 - \delta\,(1 + 2\sigma_\lambda(c+\mu))\right)^2},$$

$$\Pi_\delta = (1 + 2\sigma_\lambda\mu)(1 + 2\sigma_\lambda(\mu + c))\,\delta^2 - 4(1 + 2\sigma_\lambda\mu)\,\delta + 2.$$

By Assumption 4, $\sigma_\lambda(c+\mu) \le 1/2$ on the admissible range, so the denominator is strictly positive, and $2c\sigma_\lambda > 0$. The sign of $\partial\tilde{x}/\partial\delta$ is therefore the sign of Π_δ. At the left endpoint,

$$\Pi_\delta(0) = 2 > 0.$$

We determine the sign of Π_δ on the admissible range, then translate its single sign change into the conditions of the lemma. The polynomial Π_δ is quadratic in δ with leading coefficient $(1 + 2\sigma_\lambda\mu)(1 + 2\sigma_\lambda(\mu + c))$, linear coefficient $-4(1 + 2\sigma_\lambda\mu)$, and constant term 2. By Assumption 4, $2\sigma_\lambda(c + |\mu|) \le 1$. Since $|\mu| \le c + |\mu|$, this gives $2\sigma_\lambda|\mu| \le 1$, so $1 + 2\sigma_\lambda\mu > 0$, strictly because $c > 0$. Since $c - \mu \le c + |\mu|$, it gives $2\sigma_\lambda(c - \mu) \le 1$, so $1 + 2\sigma_\lambda(\mu - c) \ge 0$. The remaining factor satisfies $1 + 2\sigma_\lambda(\mu + c) > 1 + 2\sigma_\lambda\mu > 0$. Hence the leading coefficient $(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c))$ and the magnitude of the linear coefficient $4(1 + 2\sigma_\lambda\mu)$ are strictly positive, Π_δ is convex, and the factor $1 + 2\sigma_\lambda(\mu - c)$ is non-negative. The discriminant of Π_δ is

$$\Big(4(1+2\sigma_\lambda\mu)\Big)^2 - 4(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c))\cdot 2 = 8(1+2\sigma_\lambda\mu)\Big(1+2\sigma_\lambda(\mu-c)\Big),$$

which is non-negative by the coefficient signs, so Π_δ has real roots. Its square root is $2\sqrt{2(1 + 2\sigma_\lambda\mu)(1 + 2\sigma_\lambda(\mu - c))}$, and the roots are minus the linear coefficient plus or minus this square root, over twice the leading coefficient,

$$\begin{aligned}\delta^* &= \frac{4(1+2\sigma_\lambda\mu) - 2\sqrt{2(1+2\sigma_\lambda\mu)\big(1+2\sigma_\lambda(\mu-c)\big)}}{2(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c))}\\ &= \frac{2(1+2\sigma_\lambda\mu) - \sqrt{2(1+2\sigma_\lambda\mu)\big(1+2\sigma_\lambda(\mu-c)\big)}}{(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c))},\end{aligned}$$

$$\begin{aligned}\delta^{**} &= \frac{4(1+2\sigma_\lambda\mu) + 2\sqrt{2(1+2\sigma_\lambda\mu)\big(1+2\sigma_\lambda(\mu-c)\big)}}{2(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c))}\\ &= \frac{2(1+2\sigma_\lambda\mu) + \sqrt{2(1+2\sigma_\lambda\mu)\big(1+2\sigma_\lambda(\mu-c)\big)}}{(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c))},\end{aligned}$$

where the common factor 2 cancels and δ^* is the smaller root, equal to Equation (59). By Vieta's formulas, for a quadratic the product of the roots equals the constant term divided by the leading coefficient and the sum equals the linear coefficient divided by the leading coefficient with the sign reversed, so

$$\begin{aligned}\delta^*\,\delta^{**} &= \frac{2}{(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c))},\\ \delta^*+\delta^{**} &= \frac{4(1+2\sigma_\lambda\mu)}{(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c))} = \frac{4}{1+2\sigma_\lambda(\mu+c)},\end{aligned}$$

both strictly positive, so both roots are positive and δ^* is the unique candidate interior maximum. Since $\Pi_\delta(0) > 0$ and Π_δ is convex with two positive roots, Π_δ is positive on $(0, \delta^*)$, negative on (δ^*, δ^{**}), and positive beyond δ^{**}.

Strong incumbent, $\mu \le 0$: The admissible range is $(0, 1)$. The larger root satisfies $\delta^{**} \ge 1$ if and only if $\sqrt{8(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu-c))} \ge 2(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c)) - 4(1+2\sigma_\lambda\mu)$. The right side equals

$$2(1+2\sigma_\lambda\mu)(1+2\sigma_\lambda(\mu+c)) - 4(1+2\sigma_\lambda\mu) = 2(1+2\sigma_\lambda\mu)\big(2\sigma_\lambda(\mu+c)-1\big) \le 0,$$

where the inequality uses $1+2\sigma_\lambda\mu > 0$ and $2\sigma_\lambda(\mu+c) \le 1$ from Assumption 4, so it is non-positive while the square root is non-negative, and the inequality holds. Hence $\delta^{**} \ge 1$, so it never lies in $(0, 1)$, and the only possible sign change on $(0, 1)$ is at δ^*. Evaluating Π_δ at the right endpoint,

$$\Pi_\delta(1) = 2 - (1+2\sigma_\lambda\mu)\big(3 - 2\sigma_\lambda(\mu+c)\big),$$

so $\delta^* < 1$ if and only if $\Pi_\delta(1) < 0$, that is if and only if $(1 + 2\sigma_\lambda\mu)(3 - 2\sigma_\lambda(\mu + c)) > 2$. If $2\sigma_\lambda c = 1$, which Assumption 4 permits only at $\mu = 0$ since $2\sigma_\lambda(c + |\mu|) \le 1$, then $1 + 2\sigma_\lambda(\mu - c) = 1 - 2\sigma_\lambda c = 0$, the discriminant vanishes, $\delta^* = \delta^{**} = 2/\Big(1 + 2\sigma_\lambda(\mu + c)\Big) = 2/(1 + 2\sigma_\lambda c) = 1$, and $\Pi_\delta = 2(\delta - 1)^2 > 0$ on $(0, 1)$, so $\tilde{x}$ is strictly increasing. Suppose then $2\sigma_\lambda c < 1$. Write $g(\mu) = (1 + 2\sigma_\lambda\mu)(3 - 2\sigma_\lambda(\mu + c))$ for the left side. Differentiating,

$$g'(\mu) = 4\sigma_\lambda\Big(1 - \sigma_\lambda(2\mu + c)\Big),$$

and for $\mu \le 0$ we have $\sigma_\lambda(2\mu + c) \le \sigma_\lambda c \le \frac{1}{2}$ by Assumption 4, so $g'(\mu) \ge 2\sigma_\lambda > 0$ and g is strictly increasing on $(-\infty, 0]$. At $\mu = 0$, $g(0) = 3 - 2\sigma_\lambda c > 2$ since $\sigma_\lambda c < \frac{1}{2}$, while $g(\mu) \to -\infty$ as $\mu \to -\infty$, so g crosses the value 2 exactly once on $(-\infty, 0]$. To locate the crossing, expand $g(\mu) = 2$ as a quadratic in μ,

$$-4\sigma_\lambda^2\,\mu^2 + 4\sigma_\lambda(1 - \sigma_\lambda c)\,\mu + (1 - 2\sigma_\lambda c) = 0,$$

or, multiplying by -1,

$$4\sigma_\lambda^2\,\mu^2 - 4\sigma_\lambda(1 - \sigma_\lambda c)\,\mu - (1 - 2\sigma_\lambda c) = 0.$$

Its discriminant is

$$16\sigma_\lambda^2(1 - \sigma_\lambda c)^2 + 16\sigma_\lambda^2(1 - 2\sigma_\lambda c) = 16\sigma_\lambda^2\Big(\sigma_\lambda^2 c^2 - 4\sigma_\lambda c + 2\Big),$$

with square root $4\sigma_\lambda\sqrt{\sigma_\lambda^2 c^2 - 4\sigma_\lambda c + 2}$, so the two roots are

$$\mu = \frac{4\sigma_\lambda(1 - \sigma_\lambda c) \pm 4\sigma_\lambda\sqrt{\sigma_\lambda^2 c^2 - 4\sigma_\lambda c + 2}}{8\sigma_\lambda^2} = \frac{(1 - \sigma_\lambda c) \pm \sqrt{\sigma_\lambda^2 c^2 - 4\sigma_\lambda c + 2}}{2\sigma_\lambda},$$

where the common factor $4\sigma_\lambda$ cancels. The single crossing on $(-\infty, 0]$ is the smaller root,

$$\mu_\delta = \frac{1 - \sigma_\lambda c - \sqrt{\sigma_\lambda^2 c^2 - 4\sigma_\lambda c + 2}}{2\sigma_\lambda},$$

which is real because $\sigma_\lambda c \le \frac{1}{2}$ keeps $\sigma_\lambda^2 c^2 - 4\sigma_\lambda c + 2$ positive, and strictly negative because $\sigma_\lambda^2 c^2 - 4\sigma_\lambda c + 2 - (1 - \sigma_\lambda c)^2 = 1 - 2\sigma_\lambda c > 0$ makes the square root exceed $1 - \sigma_\lambda c$. Since g is strictly increasing, $g(\mu) > 2$ for $\mu_\delta < \mu \le 0$ and $g(\mu) \le 2$ for $\mu \le \mu_\delta$. Hence $\tilde{x}$ is single-peaked with maximum at δ^* when $\mu_\delta < \mu \le 0$, and $\Pi_\delta \ge 0$ on $(0, 1)$ so $\tilde{x}$ is strictly increasing when $\mu \le \mu_\delta$.

The threshold therefore turns from single-peaked to strictly increasing as the incumbent grows stronger past μ_δ.

Weak incumbent, $\mu > 0$: The admissible range is $\left(0, \bar{\delta}\right]$ with $\bar{\delta} = 1/(1+2\sigma_\lambda \mu) < 1$. Evaluating Π_δ at $\bar{\delta}$,

$$\Pi_\delta\left(\frac{1}{1+2\sigma_\lambda\mu}\right) = -\frac{1+2\sigma_\lambda(\mu-c)}{1+2\sigma_\lambda\mu} < 0,$$

since $\mu > 0$ gives $2\sigma_\lambda(c-\mu) < 2\sigma_\lambda(c+\mu) \leq 1$ by Assumption 4, so $1+2\sigma_\lambda(\mu - c) > 0$, while $1+2\sigma_\lambda\mu > 0$ from the coefficient signs. So $\bar{\delta}$ lies strictly between the roots, $\delta^* < \bar{\delta} < \delta^{**}$, and the only sign change on $\left(0, \bar{\delta}\right]$ is at δ^*, from positive to negative. Hence $\tilde{x}$ is single-peaked with interior maximum at δ^*, and $\delta^* < \bar{\delta}$ holds throughout this regime. In both regimes the threshold is single-peaked with maximum at δ^* exactly when $\delta^* < \bar{\delta}$, where $\bar{\delta} = 1$ for $\mu \leq 0$, and is strictly increasing otherwise, which proves Parts 4.1 and 4.2. Suppose $\delta^* < \bar{\delta}$. The discriminant is then strictly positive: for $\mu > 0$ this follows from $1 + 2\sigma_\lambda(\mu - c) > 0$, shown above, while for $\mu \leq 0$ a zero discriminant would give $1 + 2\sigma_\lambda(\mu - c) = 0$, hence $1 + 2\sigma_\lambda(\mu + c) = 2\left(1 + 2\sigma_\lambda\mu\right)$ and $\delta^* = 1/\left(1 + 2\sigma_\lambda\mu\right) \geq 1 \geq \bar{\delta}$, contradicting $\delta^* < \bar{\delta}$. Define

$$G(\delta, \mu, \sigma_\lambda, c) = (1 + 2\sigma_\lambda\mu)(1 + 2\sigma_\lambda(\mu + c))\,\delta^2 - 4(1 + 2\sigma_\lambda\mu)\,\delta + 2.$$

As a polynomial in its arguments, G is continuously differentiable on the open region where $\sigma_\lambda, c > 0$, and $G(\delta^*, \mu, \sigma_\lambda, c) = 0$. Its partial derivative in δ is

$$\frac{\partial G}{\partial \delta} = 2(1 + 2\sigma_\lambda\mu)(1 + 2\sigma_\lambda(\mu + c))\,\delta - 4(1 + 2\sigma_\lambda\mu),$$

and at the smaller root it equals $-2\sqrt{2(1 + 2\sigma_\lambda\mu)\big(1 + 2\sigma_\lambda(\mu - c)\big)}$, the negative of the square root of the discriminant, hence strictly negative and in particular nonzero. The implicit function theorem therefore applies, so in a neighborhood of any such point δ^* is a continuously differentiable function of (μ, σ_λ, c), and for each parameter $k \in \{\mu, \sigma_\lambda, c\}$,

$$\frac{\partial \delta^*}{\partial k} = -\left.\frac{\partial G/\partial k}{\partial G/\partial \delta}\right|_{\delta = \delta^*}.$$

Because the denominator is negative, the sign of $\partial\delta^*/\partial k$ is the sign of $\partial G/\partial k$ at δ^*.

Effect of μ:

$$\frac{\partial G}{\partial \mu} = 4\sigma_\lambda\,\delta\Big(\delta(2\mu + c)\sigma_\lambda - (2 - \delta)\Big).$$

For $\mu > 0$, the bracket is increasing in δ, since its slope in δ is $(2\mu+c)\sigma_\lambda+1 > 0$, so on the admissible range it is bounded above by its value at the upper limit $\bar{\delta} = 1/(1+2\sigma_\lambda\mu)$, where it equals $(c\sigma_\lambda-2\mu\sigma_\lambda-1)/(1+2\sigma_\lambda\mu)$, and $c\sigma_\lambda \leq \frac{1}{2}-\sigma_\lambda\mu$ from Assumption 4 makes the numerator at most $-\frac{1}{2} - 3\sigma_\lambda\mu < 0$. For $\mu \leq 0$, directly, $\delta(2\mu + c)\sigma_\lambda \leq \delta\sigma_\lambda c \leq \sigma_\lambda c \leq \frac{1}{2}$ by Assumption 4, while $-(2-\delta) < -1$, so the bracket lies below $-\frac{1}{2}$. In both cases the bracket is negative at δ^*, so $\partial G/\partial\mu < 0$ and $\partial\delta^*/\partial\mu < 0$.

Effect of c:

$$\frac{\partial G}{\partial c} = 2\sigma_\lambda(1 + 2\sigma_\lambda\mu)\,\delta^2,$$

which is positive, so $\partial\delta^*/\partial c > 0$.

Effect of σ_λ:

$$\frac{\partial G}{\partial \sigma_\lambda} = 2\delta\bigg(\Big(c + 2\mu + 4\sigma_\lambda\mu(\mu + c)\Big)\delta - 4\mu\bigg).$$

Unlike the partials in c and μ, this expression has no fixed sign on the admissible range. The term -4μ is positive for a strong incumbent and negative for a weak one, while the coefficient $c + 2\mu + 4\sigma_\lambda\mu(\mu + c)$ multiplying δ can take either sign, so the bracket is positive in some cases and negative in others. Two admissible points with an interior peak make this concrete. At $\mu = -0.22$, $c = 1.54$, $\sigma_\lambda = 0.016$, the bracket is positive and $\partial G/\partial\sigma_\lambda > 0$, while at $\mu = 1.13$, $c = 1.92$, $\sigma_\lambda = 0.027$, it is negative and $\partial G/\partial\sigma_\lambda < 0$. Since $\partial\delta^*/\partial\sigma_\lambda$ has the sign of $\partial G/\partial\sigma_\lambda$, it is positive at the first point and negative at the second. The effect of σ_λ on δ^* is therefore ambiguous. $\square$