Hopf Algebras and Root Systems

István Heckenberger
Hans-Jürgen Schneider

Dedicated to Nicolás Andruskiewitsch

Contents

Preface

This book is an introduction to Hopf algebras in braided monoidal categories with applications to Hopf algebras in the usual sense, that is, in the category of vector spaces. By now there exists a wide variety of deep results in this area, and we don't aim to provide a complete overview. We will discuss some of these topics in Chapter 17.

Our main goal is to present from scratch and with complete proofs the theory of Nichols algebras (or quantum symmetric algebras) and the surprising relationship between Nichols algebras and (generalized) root systems. Hopefully our book makes the vast literature in the area more accessible, and it is useful for future research.

Since its beginnings some 70 years ago, the theory of Hopf algebras has developed rapidly into various directions. Its origins came from algebraic topology, algebraic and formal groups, and operator algebras. The influential book of Sweedler from 1969 [**Swe69**] laid the foundations of a general theory of abstract (non-commutative and non-cocommutative) Hopf algebras. After the work of Drinfeld and Jimbo on quantum groups, and Drinfeld's report "Quantum groups" [**Dri87**] at the International Congress of Mathematicians 1986, the interest in the topic drastically increased.

Quantum groups are prominent examples of pointed Hopf algebras (their irreducible comodules are one-dimensional). Several years after their discovery, general classification results for pointed Hopf algebras were obtained ([**AS02**]; [**AS04**], [**AA08**], [**AS10**] depending on [**Ros98**], [**Kha99**], [**Hec06**], [**Hec08**]). In these papers, the classical theory of quantum groups and of the small quantum groups as developed in [**Lus93**] is applied.

Although quantum groups are intrinsically related to Lie theoretical structures, it is not at all obvious to which extent this is true for general pointed Hopf algebras. The lifting method introduced in [**AS98**] showed that the classification of Nichols algebras is an essential step in the classification theory of pointed Hopf algebras. And here, in the theory of Nichols algebras, the combinatorics of root systems and Weyl groups, or better Weyl groupoids, plays an important role. Weyl groupoids were introduced in [**Hec06**] for diagonal braidings using Kharchenko's PBW basis [**Kha99**] based on the theory of Lyndon words, and in [**AHS10**] in general.

Nichols algebras as a special class of braided pointed Hopf algebras are studied in great detail in this book. They appeared first in [**Nic78**], independently as braided algebras in [**Wor89**]. It follows from the work of Lusztig [**Lus93**] that $U_q^+(\mathfrak{g})$, $\mathfrak{g}$ symmetrizable Kac-Moody Lie algebra, q transcendental, is a Nichols algebra; see [**Ros98**] (where a dual description of Nichols algebras as quantum shuffle algebras is used), [**Gre97**], and [**Sch96**].

We emphasize categorical constructions and one-sided coideal subalgebras. The introduction of Nichols systems, which are generalizations of Nichols algebras together with a grading by a free abelian group, allows us to develop the theory in a very general setting. We do not use the theory of Lyndon words, and we do not assume results from quantum groups. Our theory can be applied to quantum groups, and some of our results on right coideal subalgebras are new also in the special case of quantum groups.

Prerequisites. The reader is expected to be familiar with linear algebra and algebra on the graduate level including tensor products of modules, basic noncommutative algebra, and the language of categories, functors, and natural transformations. For a better understanding, a course in semisimple Lie algebras would be helpful but is not strictly necessary.

We now describe the contents of the book in more detail.

(1) Foundations. We begin in Chapter 1 with a quick introduction to Nichols algebras. Our goal is to give a complete exposition of the basics of Nichols algebras which are scattered over various papers.

The most important example of a braided monoidal category in this book is the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over some Hopf algebra H with bijective antipode. If $H = \Bbbk G$ is the group algebra of a group G over a field $\Bbbk$, then an object in ${}^H_H\mathcal{YD}$ is a G-graded vector space $V = \bigoplus_{g\in G} V_g$ with a G-action such that for all $g,h \in G$, $g \cdot V_h = V_{ghg^{-1}}$. The braiding $c_{V,W}$ between objects $V, W \in {}^H_H\mathcal{YD}$ is given by

$$c_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto g \cdot w \otimes v, \quad v \in V_g,\ w \in W.$$

The maps $c_{V,W}$ are G-graded and G-linear, where the monoidal structure is given by the usual grading and diagonal action on the tensor product $V \otimes W$. For any object $V \in {}^H_H\mathcal{YD}$, the Nichols algebra $\mathcal{B}(V)$ is defined as follows. We want an $\mathbb{N}_0$-graded Hopf algebra R in the braided category ${}^H_H\mathcal{YD}$ in which the elements of V are primitive and generators of the algebra. Moreover, R should be minimal in the sense that there are no other primitive elements than those in V. Of course, the tensor algebra $T(V)$ is an $\mathbb{N}_0$-graded Hopf algebra generated by V, where the elements of V are primitive. But in general there are more primitive elements in higher degrees. We define the **Nichols algebra** $\mathcal{B}(V)$ by

$$\mathcal{B}(V) = T(V)/I(V), \;\; I(V) \text{ the largest coideal in degree } \geq 2.$$

This is an $\mathbb{N}_0$-graded braided quotient Hopf algebra of the tensor algebra. Thus the Nichols algebra is defined by a universal property, which means that it is very often quite difficult to really compute $\mathcal{B}(V)$. In Corollary 1.9.7 we prove that the relations of the Nichols algebra can be described by the quantum symmetrizer maps defined by the action of the braid group. This is an important theoretical result. However, it does not immediately help, for example, to decide which Nichols algebras are finite-dimensional.

Let A be a Hopf algebra whose coradical $A_0 = H$ is a Hopf subalgebra, and let $\operatorname{gr} A$ be the associated $\mathbb{N}_0$-graded Hopf algebra with respect to the coradical filtration. Then the Nichols algebra over H appears naturally as a subalgebra of $\operatorname{gr} A$ (see Corollary 7.1.17). Hence Nichols algebras are essential for the classification problem of such Hopf algebras A.

Chapter 2 is a collection of fairly standard results in the theory of Hopf algebras which we will need later on or which motivate more general constructions later.

In Chapter 3 the theory of Hopf algebras in braided (strict) monoidal categories $\mathcal{C}$ is presented, partly with new proofs. To our knowledge, this theory didn't appear so far in a textbook. Sections 3.8 and 3.10 contain detailed proofs of the Radford-Majid-Bespalov theory of bosonization and Hopf algebras with a projection in braided categories. Theorem 3.10.6 on left and right coinvariant subobjects seems to be new; it is used to prove the existence of the Hopf algebra isomorphism T in Theorem 12.3.3, which in this book plays the role of the Lusztig automorphisms of quantum groups.

In Chapter 4 we specialize Chapter 3 to the braided category ${}^H_H\mathcal{YD}$. By Theorem 4.4.11, a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$ has bijective antipode and is a Frobenius algebra. This was shown in the pioneering paper [**LS69**] for usual Hopf algebras.

In Chapter 5 a fairly general theory of filtrations by abelian monoids is presented, which will be applied in particular to $\mathbb{N}_0^\theta$, $\theta \geq 2$, to obtain appropriate gradings of Nichols algebras. In addition we study the coradical filtration assuming standard results from the theory of the Jacobson radical of algebras.

Chapters 6 and 7 deal with general braided vector spaces and their Nichols algebras. They are rather independent of the remaining parts of the book. In Corollary 7.2.8 we establish the fundamental non-degenerate pairing between $B(V^*)$ and $B(V)$, where V is a finite-dimensional object in ${}^H_H\mathcal{YD}$.

In Chapter 8 we discuss quantized enveloping algebras and, more generally, linkings of Nichols algebras. We define Hopf algebras $U(\mathcal{D}, \lambda)$ which generalize the quantum groups $U_q(\mathfrak{g})$; they are given by the Serre relations in each connected component of the Dynkin diagram and linking relations such as the relations between the E_i and F_i for quantum groups (introduced in [**AS02**]).

(2) The main motivating problem. Lusztig in [**Lus93**] defines the positive part U_q^+ of the deformed universal enveloping algebra of a Kac-Moody Lie algebra by a universal property which is easily seen to be an alternative description of the Nichols algebra of the degree one part V of U_q^+. In this case V is a Yetter-Drinfeld module over the group algebra of a free abelian group G with basis $K_1, \dots, K_n$, and

$$V = \bigoplus_{i=1}^n \Bbbk E_i, \quad E_i \in V_{K_i}, \quad K_i \cdot E_j = q^{d_i a_{ij}} \text{ for all } i, j.$$

Here, q is not a root of unity, and $(d_i a_{ij})_{1 \leq i,j \leq n}$ is the symmetrized Cartan matrix. (In Lusztig's book, q is transcendental, and $\mathrm{char}(\Bbbk) = 0$.) The Nichols algebras of the summands $\Bbbk E_i$ are simply polynomial algebras in the variable E_i. Much later in his book, Lusztig shows that U_q^+ is explicitly given by the quantum Serre relations.

Assume more generally that

$$V = \bigoplus_{i=1}^\theta M_i \in {}^H_H\mathcal{YD}$$

is a finite direct sum of finite-dimensional irreducible objects $M_i \in {}^H_H\mathcal{YD}$, where H is a Hopf algebra with bijective antipode. If H is the group algebra of a finite group, and if the characteristic of the field does not divide the order of the group, then any finite-dimensional object V in ${}^H_H\mathcal{YD}$ is semisimple. The Nichols algebra $\mathcal{B}(V)$ has an additional important structure. It is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. We denote the standard basis of $\mathbb{Z}^\theta$ by $\alpha_1, \dots, \alpha_\theta$, and define the degree of M_i as α_i. Suppose we know the $\mathcal{B}(M_i)$. Which additional information is needed to understand $\mathcal{B}(V)$? For example, when is $\mathcal{B}(V)$ finite-dimensional? Is there an analog of Lusztig's PBW-basis depending on the longest element in the Weyl group of a semisimple Lie algebra?

Note that in our general situation no Cartan matrix is given a priori. The key to the missing information will be the root system and the Weyl groupoid of the tuple $M = (M_1, \dots, M_\theta)$. We define the Nichols algebra of the tuple by $\mathcal{B}(M) = \mathcal{B}(V)$.

(3) The combinatorics of Cartan graphs and their Weyl groupoids. This is a generalization of the notion of a Cartan matrix and its Weyl group to a family of Cartan matrices. Right now there are several approaches to this theory. Nevertheless we restrict ourselves in Part 2 of the book to a presentation based on families of Cartan matrices, since this approach appears to be most useful to explain the combinatorics in the theory of Nichols algebras. Part 2 is independent of the theory of Nichols algebras.

Let $\theta \geq 1$ be a natural number, $\mathbb{I} = \{1, \dots, \theta\}$, $\mathcal{X}$ a non-empty set, $(r_i)_{i \in \mathbb{I}}$ a family of maps $r_i : \mathcal{X} \to \mathcal{X}$, and $(A^X)_{X \in \mathcal{X}}$ a family of (generalized) Cartan matrices. The quadruple $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, (r_i), (A^X))$ is called a **semi-Cartan graph** if the following axioms hold.

(CG1) For all $i \in \mathbb{I}$, $r_i^2 = \mathrm{id}_{\mathcal{X}}$.

(CG2) For all $i \in \mathbb{I}$, $X \in \mathcal{X}$, A^X and $A^{r_i(X)}$ have the same i-th row.

For all $X \in \mathcal{X}$ and $i \in \mathbb{I}$ let $s_i^X \in \mathrm{Aut}(\mathbb{Z}^\theta)$ be the reflection map defined by $s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i$ for all $j \in I$. Let $\mathcal{W}(\mathcal{G})$ be the groupoid with objects $\mathcal{X}$ and morphisms generated by formal maps $s_i^X : X \to r_i(X)$. Composition of such morphism is given by multiplication in $\mathrm{Aut}(\mathbb{Z}^\theta)$. Note that $\mathcal{W}(\mathcal{G})$ is a groupoid (a category where every morphism is an isomorphism), since $s_i^{r_i(X)}$ is inverse to s_i^X. The **real roots** of X are the elements in $\mathbb{Z}^\theta$ which can be written as $w(\alpha_i)$ for some morphism $w : Y \to X$ and $i \in \mathbb{I}$ ($w(\alpha_i) = f(\alpha_i)$, where w is given by $f \in \mathrm{Aut}(\mathbb{Z}^\theta)$).

The axioms of a semi-Cartan graph are not yet strong enough to be useful. For example, we want that the real roots are positive or negative, that is, in $\mathbb{N}_0^\theta$ or in $-\mathbb{N}_0^\theta$. We define in Definition 9.1.14 a **Cartan graph** by two additional axioms (CG3) and (CG4). If $\mathcal{G}$ is a Cartan graph, we call $\mathcal{W}(\mathcal{G})$ the **Weyl groupoid** of $\mathcal{G}$. The importance of the axioms of a Cartan graph $\mathcal{G}$ comes from Theorem 9.4.8, where we show that the Weyl groupoid of a Cartan graph $\mathcal{G}$ is a **Coxeter groupoid** (in a different language this is a result of [**HY08**]), that is, the Weyl groupoid has defining relations of the same type as Coxeter groups have.

Most of the results in Part 2 have been already published in [**HY08**], [**CH09b**], [**CH09a**], and [**CH12**]. However, in Section 9.2 we present new axioms (CG3') and (CG4') of a Cartan graph in terms of reduced sequences. These axioms are those appearing most naturally for semi-Cartan graphs of Nichols systems.

(4) The Cartan graph of a Nichols algebra. Let $M = (M_1, \dots, M_\theta)$ as above. First we have to define reflection operators on tuples of Yetter-Drinfeld modules. For each $i \in \mathbb{I}$ let $R_i(M) = (M'_1, \ldots, M'_\theta)$, where

$$M'_j = \begin{cases} M_i^* & \text{if } j = i, \\ (\operatorname{ad} M_i)^{-a_{ij}^M}(M_j) & \text{if } j \neq i, \end{cases}$$

and where we assume that $a_{ij}^M = -\max\{m \in \mathbb{N}_0 \mid (\operatorname{ad} M_i)^m(M_j) \neq 0\}$ exists. The i-th component is the dual Yetter-Drinfeld module M_i^*, and ad is the braided adjoint action in the Nichols algebra $\mathcal{B}(M) = \mathcal{B}\big(\bigoplus_{i=1}^\theta M_i\big)$. By Lemma 13.4.4, $(a_{ij}^M)_{i,j\in\mathbb{I}}$ is a (generalized) Cartan matrix, when we set $a_{ii}^M = 2$. By Corollary 13.4.3, the components of $R_i(M)$ are again irreducible. Note the formal similarity with Lusztig's isomorphisms T_i of quantum groups, where

$$T_i(E_j) = \begin{cases} -F_iK_i & \text{if } j = i, \\ (\operatorname{ad} E_i)^{(-a_{ij})}(E_j) & \text{if } j \neq i. \end{cases}$$

The set of points $\mathcal{X}$ of $\mathcal{G}(M)$ is the set of isomorphism classes of all $R_{i_n} \cdots R_{i_1}(M)$, $n \geq 0$, which we assume to exist. We have attached to each $X = [M] \in \mathcal{X}$ a Cartan matrix $A^X = (a_{ij}^M)_{i,j\in\mathbb{I}}$, and we have defined maps $r_i : \mathcal{X} \to \mathcal{X}$, $[M] \mapsto [R_i(M)]$ ($[M]$ denotes the isomorphism class of M). By Theorem 13.6.2, $\mathcal{G}(M)$ is a semi-Cartan graph. This result was first obtained in [**AHS10**] with a different proof.

In order to implement the remaining axioms of a Cartan graph, sequences of graded right coideal subalgebras of Nichols algebras and their compatibility with reflections are studied in Chapter 14. Important results in this respect are Theorem 14.1.4, and in particular Theorem 14.1.9. The latter relates sequences of right coideal subalgebras of Nichols algebras to reduced sequences in the semi-Cartan graph. In Section 14.2 we introduce the notion of an exact factorization of bialgebras and Nichols systems. With this tool we prove in Theorem 14.2.12 that the semi-Cartan graph of a Nichols algebra admitting all reflections is indeed a Cartan graph. This is a new result; it was first shown in [**HS10b**] for finite semi-Cartan graphs $\mathcal{G}(M)$. It is more general than what was shown in the existing approaches, where the root system of the Nichols algebra, usually based on the theory of Lyndon words, was assumed.

(5) Categorical tools, and the role of the Lusztig isomorphisms. The proofs of these results on the Cartan graph $\mathcal{G}(M)$ depend on Chapters 12 and 13. For all $i \in \mathbb{I}$, let $K_i^{\mathcal{B}(M)}$ be the set of right coinvariant elements of the canonical projection $\mathcal{B}(M) \to \mathcal{B}(M_i)$. By the braided version of the Theorem of Radford on projections of Hopf algebras, $K_i^{\mathcal{B}(M)}$ is a Hopf algebra in the braided category ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, where $\mathcal{C} = {}^H_H\mathcal{YD}$, and $\mathcal{B}(M)$ is isomorphic to the smash product Hopf algebra $K_i^{\mathcal{B}(M)}\#\mathcal{B}(M_i)$. In Theorem 12.3.2 (which first appeared in [**HS13b**] in an equivalent version and with a very different proof) we show that there is a braided isomorphism

$$(\Omega, \omega) : {}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^{\mathcal{B}(M_i^*)}_{\mathcal{B}(M_i^*)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}.$$

Hence $\Omega(K_i^{\mathcal{B}(M)})$ is a Hopf algebra in ${}^{\mathcal{B}(M_i^*)}_{\mathcal{B}(M_i^*)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, and we may consider its bosonization $\Omega(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*)$. By Theorem 13.4.9, this bosonization is isomorphic to $\mathcal{B}(R_i(M))$. The deeper results on $\mathcal{B}(R_i(M))$ depend on this isomorphism.

Theorem 12.3.3 is another categorical result on the isomorphism (Ω, ω). It implies a very close relationship between $\mathcal{B}(M)$ and $\mathcal{B}(R_i(M))$. There is an isomorphism of braided Hopf algebras

$$T_i^{\mathcal{B}(M)} : L_i^{\mathcal{B}(R_i(M))} \to K_i^{\mathcal{B}(M)}$$

between the left coinvariants $L_i^{\mathcal{B}(R_i(M))}$ of the projection

$$\mathcal{B}(R_i(M)) \cong \Omega(K_i^{\mathcal{B}(M)}) \# \mathcal{B}(M_i^*) \to \mathcal{B}(M_i^*)$$

and the right coinvariants $(K_i^{\mathcal{B}(M)})^{\mathrm{cop}}$ of $\mathcal{B}(M)$. To make sense, this Hopf algebra isomorphism has to be understood in the formulation of Theorem 12.3.3 which did not appear in print before.

The isomorphisms $T_i^{\mathcal{B}(M)}$ play the role of the Lusztig automorphisms to construct a PBW basis of U_q^+. Since the maps $T_i^{\mathcal{B}(M)}$ can be seen as isomorphisms of Hopf algebras, they can be used in Theorem 14.1.9 to construct right coideal subalgebras in $\mathcal{B}(M)$ stepwise (Lusztig's isomorphisms are maps of algebras not of coalgebras).

If the Cartan graph $\mathcal{G}(M)$ is finite, that is, there are only finitely many real roots, then we obtain by this procedure in Corollary 14.5.3 a tensor decomposition

$$\mathcal{B}(M_{\beta_m}) \otimes \cdots \otimes \mathcal{B}(M_{\beta_1}) \cong \mathcal{B}(M), \tag{0.0.1}$$

depending on the longest element in $\mathrm{Hom}(\mathcal{W}(M), [M])$, where $M_{\beta_m}, \ldots, M_{\beta_1}$ are irreducible subobjects of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$ which correspond to the higher root vectors of quantum groups, and $\deg(M_{\beta_i}) = \beta_i \in \mathbb{N}_0^\theta$ for all i. For all $1 \le l \le m$, the image of $\mathcal{B}(M_{\beta_l}) \otimes \cdots \otimes \mathcal{B}(M_{\beta_1})$ in $\mathcal{B}(M)$ is a right coideal subalgebra.

Assume that the components M_i of M are one-dimensional. Then the M_{β_l} in (0.0.1) are one-dimensional, the algebras $\mathcal{B}(M_{\beta_l})$ are polynomial rings or truncated polynomial rings. Thus we have constructed a PBW basis of $\mathcal{B}(M)$. In particular, we obtain Lusztig's PBW basis of $U_q^+(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, without any case by case considerations; see also Remark 16.2.6. The Levendorskii-Soibelman commutation relations are also shown in the general context of Nichols algebras over any field; see Theorem 14.1.12 and Theorem 16.3.16.

In Corollary 14.5.3 we prove that $\mathcal{G}(M)$ must be finite if $\mathcal{B}(M)$ is finite-dimensional.

Assume that $\mathcal{G}(M)$ is finite. In Corollary 14.6.8 we prove that the construction of right coideal subalgebras mentioned above defines a bijection

$$\mathrm{Hom}(\mathcal{W}(M), [M]) \to \mathcal{K}(\mathcal{B}(M))$$

between morphisms in the Weyl groupoid ending in $[M]$ and the set of all graded right coideal subalgebras of $\mathcal{B}(M)$. Kharchenko [**Kha11**] conjectured that the number of such right coideal subalgebras in $U_q^+(\mathfrak{g})$ (for simple Lie algebras) is equal to the order of the Weyl group. Our work on right coideal subalgebras in [**HS13a**] was motivated by this conjecture, which is now proved as a special case of Corollary 14.6.8. As a novelty, in Theorem 14.6.6 we generalize the correspondence in Corollary 14.6.8 to tuples with not necessarily finite Cartan graph.

The categorical results in Chapter 12 are very general. They can be applied to any Hopf algebra K in ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, not just to $K_i^{\mathcal{B}(M)}$. This leads to a new and substantial extension of the theory of Nichols algebras in Section 13.5. There we introduce Nichols systems and define reflection operators for Nichols systems. The stepwise construction of right coideal subalgebras in Section 14.1 works for Nichols systems.

We use Nichols systems to establish criteria when a given pre-Nichols algebra is Nichols. By Theorem 14.5.4, any pre-Nichols system admitting all reflections and having a finite Cartan graph is in fact a Nichols algebra. Theorem 14.5.4 is fundamental for several proofs later on in the book. We would like to highlight Theorem 15.5.1 (finite-dimensional pre-Nichols algebras of diagonal type are Nichols), Theorem 16.2.5(2) (the positive part U_q^+ of a quantum group attached to a Cartan matrix of finite type, q not a root of 1, is a Nichols algebra), Theorem 16.4.23(2) (a pre-Nichols algebra with finite Gelfand-Kirillov dimension of a braided vector space of quasi-generic Cartan type is the Nichols algebra U_q^+), and Corollary 16.4.24 (a braided vector space of diagonal type with a Nichols algebra being a domain of finite Gelfand-Kirillov dimension is quasi-generic of finite Cartan type); see below for more details.

(6) Applications. After some basic observations on reflections of Yetter-Drinfeld modules of diagonal type in Section 15.1, we study root vector sequences in pre-Nichols systems. In the special case of usual quantum groups, the root vectors of Lusztig are shown later in Remark 16.2.6 to form root vector sequences. This has advantages for both approaches: Lusztig's root vectors satisfy integrality properties, and root vector sequences are defined by defining properties which can be used to develop new methods (such as braided commutators associated to Lyndon words) to construct them. Further important differences in the two approaches to quantum groups are that our root vectors are only unique up to scalar multiples, we don't use an analog of the braid relations for Lusztig's automorphisms, and we don't need to perform case by case analysis (except in Remark 16.2.6 to prove the correspondence). Note that root vector sequences, similarly to Lusztig's root vectors, are defined for any reduced decomposition of an element of the Weyl group(oid).

Using root vector sequences, Theorem 15.2.7 describes a basis of any right coideal subalgebra of a Nichols system attached to a reduced decomposition of an element of the Weyl groupoid.

Following [**HW15**], in Theorem 15.3.1 we classify two-dimensional braided vector spaces of diagonal type which have a finite Cartan graph, where the field $\Bbbk$ has characteristic 0. This classification uses explicitly the combinatorics of finite Cartan graphs of rank two from Section 10.3. The classification in [**Hec09**] of all finite-dimensional braided vector spaces of diagonal type and with finite Cartan graph is beyond the scope of this book.

Angiono in [**Ang15**] (using the results on right coideal subalgebras in Corollary 14.6.8) and [**Ang13**] found a celebrated presentation of the Nichols algebras appearing in [**Hec09**] in terms of generators and relations, where the ground field is algebraically closed of characteristic 0.

A conjecture in [**AS00a**] says that any finite-dimensional pointed Hopf algebra H over an algebraically closed field of characteristic 0 is generated as an algebra by group-like and skew-primitive elements. In Theorem 15.5.1 we prove that finite-dimensional pre-Nichols algebras of diagonal type over a field of characteristic 0 are Nichols algebras. This proves the conjecture when the group of group-like elements of H is abelian. This theorem was originally proved by I. Angiono in [**Ang13**] using his list of defining relations of the finite-dimensional Nichols algebras classified in [**Hec09**]. In contrast, our proof is based on the aforementioned Theorem 14.5.4 and some results in rank two and partially in rank three.

In Chapter 16, especially in Theorems 16.2.5 and 16.3.17, we recover the results of Angiono on generators and relations for Nichols algebras of finite Cartan type (which include the algebras studied by Lusztig when the Cartan matrix is of finite type) except for a few cases with parameters of small order. In the discussed cases the Nichols algebras are presented by the quantum Serre relations and by root vector relations. The proof of Theorem 16.2.5, where the braiding matrix is quasi-generic, is a more or less direct application of Theorem 14.5.4. A proof of Theorem 16.3.17 along the same line, where the entries of the braiding matrix are roots of unity, appears to be problematic since the root vector relations depend on the choice of a presentation of the longest element of the Weyl group. Instead, we provide first in Theorem 16.3.14 a basis of the Hopf algebra $U_{\boldsymbol{q}}^+$ defined by the quantum Serre relations by analyzing root vector sequences. This together with an easy dimension argument yields the claim.

It is known that for the excluded exceptional cases additional defining relations are needed.

In Section 16.4 we study Nichols algebras of diagonal type, which are domains of finite Gelfand-Kirillov dimension. By Corollary 16.4.24, these are the Nichols algebras of finite Cartan type, where the diagonal entries of the braiding are 1 (only in characteristic 0) or not roots of 1.

In Theorem 16.5.10 we show that the pointed Hopf algebras with abelian coradical, generic infinitesimal braiding, and finite Gelfand-Kirillov dimension are exactly the Hopf algebras $U(\mathcal{D},\lambda)$ defined in Section 8.3 generalizing the quantum groups $U_q(\mathfrak{g})$. This was shown in [**AS04**] for positive braidings using [**Ros98**], and extended in [**AA08**] to the general case using [**Hec06**].

In Chapter 17 Nichols algebras over non-abelian groups are studied. Among others we prove in Corollary 17.1.5 (partly following [**HS10b**]) that the Nichols algebra of a non-zero non-simple Yetter-Drinfeld module over a finite simple group is necessarily infinite-dimensional. A similar result for the symmetric groups $\mathbb{S}_n$ with $n \geq 3$ is shown in Corollary 17.1.8.

The theory of reflections does not give direct information about Nichols algebras of irreducible Yetter-Drinfeld modules over groups. However, it can be helpful to prove that a given Nichols algebra of an irreducible Yetter-Drinfeld module is infinite-dimensional by finding a braided subspace which can be realized over some other group with decomposable Yetter-Drinfeld module and which has infinite-dimensional Nichols algebra. This is demonstrated in Corollary 17.1.11 which led to the definition of racks of type D. The rack theoretical formulation of Corollary 17.1.11 (finite racks of type D collapse) was used for example in [**AF$^+$11a**] to show that any finite-dimensional pointed Hopf algebra H over $\mathbb{C}$ with group $G(H) \cong \mathbb{A}_n$, $n \geq 5$, is isomorphic to the group algebra $\mathbb{C}\mathbb{A}_n$ of the alternating group. (Racks of type D were not used for $\mathbb{A}_5$.)

We collect the known finite-dimensional examples of Nichols algebras of irreducible Yetter-Drinfeld modules over groups in characteristic 0 in Section 17.2 without proofs. Finally, in Section 17.3 the finite-dimensional Nichols algebras of direct sums of two simple Yetter-Drinfeld modules are listed without proof; this classification uses the finiteness of the corresponding Cartan graph by Corollary 14.5.3. For references, see Chapter 17.

In the notes in the end of each chapter we refer to the relevant literature. We do this to the best of our knowledge, and we apologize to all authors whose work we have unintentionally not mentioned appropriately.

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Part 1

Hopf algebras, Nichols algebras, braided monoidal categories, and quantized enveloping algebras

CHAPTER 1

A quick introduction to Nichols algebras

The structure theory of Nichols algebras is a central theme throughout the book. In this chapter we introduce the concepts which are needed to deal with Nichols algebras of group type and also in the general case later in Chapters 6 and 7.

In Section 1.3 we study $\mathbb{N}_0$-graded connected coalgebras which are strictly graded, that is, the only primitive elements are in degree 1. For any $\mathbb{N}_0$-graded connected coalgebra C, let $I_C(n)$ be the kernel of

$$C(n) \subseteq C \xrightarrow{\Delta^{n-1}} C^{\otimes n} \xrightarrow{\pi_1^{\otimes n}} C(1)^{\otimes n}.$$

Then $I_C = \bigoplus_{n\geq 2} I_C(n)$ is the largest coideal of C in degree ≥ 2, and $\mathcal{B}(C) = C/I_C$ is a universally defined strictly graded coalgebra quotient of C which coincides with C in degree 0 and 1.

The tensor algebra of a Yetter-Drinfeld module V (over a group algebra or in the general case in Chapter 7) is a braided Hopf algebra, where the elements in V are primitive. In Section 1.6 we define the Nichols algebra of V by

$$\mathcal{B}(V) = \mathcal{B}(T(V)) = T(V)/I_{T(V)}.$$

This is a braided Hopf algebra quotient of the tensor algebra. In Section 1.9 we describe the comultiplication of the tensor algebra $T(V)$ by braided shuffle maps, and the relations of the Nichols algebra as the kernels of the braided symmetrizer maps.

In the last section we will discuss several important examples and mention others with reference to a proof.

1.1. Algebras, coalgebras, modules and comodules

Convention. The ground field is denoted by $\Bbbk$. This is an arbitrary field. If we use additional assumptions on the field, we will say so explicitly.

We write $\Bbbk^\times$ for the subgroup of non-zero elements of $\Bbbk$. Vector spaces are vector spaces over $\Bbbk$, and linear maps between vector spaces are $\Bbbk$-linear maps. If V, W are vector spaces, then $\mathrm{Hom}(V, W)$ is the set of all linear maps from V to W, and $V \otimes W = V \otimes_\Bbbk W$ is the tensor product over $\Bbbk$. In this book we will use the following convention. If U, V, W are vector spaces, then we will identify

$$(U \otimes V) \otimes W = U \otimes (V \otimes W)$$

using the natural isomorphism

$$(U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W),\ (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w).$$

Hence we will omit the brackets in tensor products of several vector spaces. Occasionally we will also suppress the natural isomorphisms

$$\Bbbk \otimes V \xrightarrow{\cong} V,\ \alpha \otimes v \mapsto \alpha v,\ \ V \otimes \Bbbk \xrightarrow{\cong} V,\ v \otimes \alpha \mapsto \alpha v.$$

Thus we will write $V = \Bbbk \otimes V$ and $V = V \otimes \Bbbk$.

The dual of a vector space V is denoted by $V^* = \mathrm{Hom}(V, \Bbbk)$.

Let A be a vector space, and $\mu : A \times A \to A$ a map (called **multiplication**) whose images will be denoted by $\mu(a,b) = ab$ for all $a, b \in A$. Then A together with μ is an **algebra** (with unit element) if there exists an element $1_A = 1 \in A$ such that for all $a, b, c \in A$ and $\alpha \in \Bbbk$,

$$\begin{aligned}
a(bc) &= (ab)c,\\
a(b+c) &= ab + ac,\ (a+b)c = ac + bc,\\
\alpha(ab) &= (\alpha a)b = a(\alpha b),\\
1a &= a = a1.
\end{aligned}$$

The unit element 1_A of an algebra is uniquely determined. It defines a linear map $\eta : \Bbbk \to A,\ \alpha \mapsto \alpha 1_A$. The multiplication map μ is a $\Bbbk$-bilinear map. Hence it is given by a linear map

$$\mu : A \otimes A \to A,\ a \otimes b \mapsto ab.$$

Let V, W be vector spaces. The linear map

$$\tau_{V,W} : V \otimes W \to W \otimes V,\ v \otimes w \mapsto w \otimes v,$$

is called the **flip map** of V and W.

Let A, B be algebras. The tensor product of vector spaces $A \otimes B$ is an algebra with unit $1 \otimes 1$ and multiplication given by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' \tag{1.1.1}$$

for all $a, a' \in A$, $b, b' \in B$. Thus the multiplication map of $A \otimes B$ is the composition

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\mathrm{id}_A \otimes \tau_{B,A} \otimes \mathrm{id}_B} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\mu_A \otimes \mu_B} A \otimes B. \tag{1.1.2}$$

This algebra structure on $A \otimes B$ is called the **tensor product of the algebras** A and B. Note that for algebras A, B, C, the canonical isomorphism

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

is an isomorphism of algebras, and following our convention, we will identify these algebras.

The **opposite algebra** A^{op} is A as a vector space, where the elements are denoted by $a^{\mathrm{op}} = a \in A$, and where the multiplication is given by

$$a^{\mathrm{op}} b^{\mathrm{op}} = (ba)^{\mathrm{op}}$$

for all $a, b \in A$.

An **algebra homomorphism** (or **algebra map**) $\rho : A \to B$ is a linear map satisfying $\rho(1) = 1$ and $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in A$. An **algebra anti-homomorphism** $\rho : A \to B$ is an algebra homomorphism $\rho : A \to B^{\mathrm{op}}$. We write $\mathrm{Alg}(A, B)$ for the set of algebra homomorphisms from A to B.

An algebra can equivalently be defined as a triple (A, μ, η), where A is a vector space and $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ are linear maps such that the following diagrams commute.

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \tag{1.1.3}$$ (associativity)

$$\begin{tikzcd} A \otimes \Bbbk \arrow[r, "\mathrm{id}_A\otimes\eta"] \arrow[dr, "="'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd} \quad \begin{tikzcd} \Bbbk \otimes A \arrow[r, "\eta\otimes\mathrm{id}_A"] \arrow[dr, "="'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd} \tag{1.1.4}$$ (unit)

Let A and B be algebras. An algebra homomorphism $\rho : A \to B$ is a linear map such that the following diagrams commute.

$$\begin{tikzcd} A \otimes A \arrow[r, "\rho\otimes\rho"] \arrow[d, "\mu_A"] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "\rho"] & B \end{tikzcd} \tag{1.1.5}$$

$$\begin{tikzcd} A \arrow[rr, "\rho"] & & B \\ & \Bbbk \arrow[ul, "\eta_A"] \arrow[ur, "\eta_B"'] & \end{tikzcd} \tag{1.1.6}$$

We introduce coalgebras by formally inverting the arrows in the definiton of an algebra.

Definition 1.1.1. Let C be a vector space, and let $\Delta : C \to C \otimes C$, $\varepsilon : C \to \Bbbk$ be linear maps called **comultiplication** and **counit**. Then (C, Δ, ε) or simply C is a **coalgebra** if the following diagrams commute.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{id}_C\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}_C"] & C \otimes C \otimes C \end{tikzcd} \tag{1.1.7}$$ (coassociativity)

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "="'] & C \otimes C \arrow[d, "\mathrm{id}_C\otimes\varepsilon"] \\ & C \otimes \Bbbk \end{tikzcd} \quad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "="'] & C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}_C"] \\ & \Bbbk \otimes C \end{tikzcd} \tag{1.1.8}$$ (counit)

A subspace D of a coalgebra C is called a **subcoalgebra** if $\Delta(D) \subseteq D \otimes D$.

Let C, D be coalgebras. The vector space $C \otimes D$ is a coalgebra with counit $\varepsilon_C \otimes \varepsilon_D$ and comultiplication

$$C \otimes D \xrightarrow{\Delta_C\otimes\Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}_C\otimes\tau_{C,D}\otimes\mathrm{id}_D} (C \otimes D) \otimes (C \otimes D). \tag{1.1.9}$$

This coalgebra structure on $C\otimes D$ is called the **tensor product of the coalgebras** C and D.

A linear map $\varphi : C \to D$ is a **coalgebra homomorphism** or a **coalgebra map** if the following diagrams commute.

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\Delta_C"] & D \arrow[d, "\Delta_D"] \\ C\otimes C \arrow[r, "\varphi\otimes\varphi"] & D\otimes D \end{tikzcd} \tag{1.1.10}$$

$$\begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & \Bbbk & \end{tikzcd} \tag{1.1.11}$$

We denote by $\mathrm{Coalg}(C, D)$ the set of all coalgebra homomorphisms from C to D.

The coalgebra C is called **cocommutative** if the diagram

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & C\otimes C \arrow[d, "\tau_{C,C}"] \\ & C\otimes C \end{tikzcd} \qquad \text{(cocommutativity)} \tag{1.1.12}$$

commutes.

The **coopposite coalgebra** C^{cop} is C as a vector space with comultiplication $\tau_{C,C}\Delta$ and counit ε. A **coalgebra anti-homomorphism** $f : C \to D$ is a coalgebra homomorphism $f : C \to D^{\mathrm{cop}}$.

Example 1.1.2. Let Γ be a set and $\Bbbk\Gamma$ the vector space with basis Γ. Then $\Bbbk\Gamma$ is a coalgebra with $\Delta(g) = g\otimes g$, $\varepsilon(g) = 1$ for all elements $g \in \Gamma$.

Example 1.1.3. Let C be a 3-dimensional vector space with basis g, h, x. Define linear maps $\Delta : C \to C\otimes C$ and $\varepsilon : C \to \Bbbk$ on the basis of C by

$$\begin{aligned} &\Delta(g) = g\otimes g, && \Delta(h) = h\otimes h, && \Delta(x) = g\otimes x + x\otimes h,\\ &\varepsilon(g) = 1, && \varepsilon(h) = 1, && \varepsilon(x) = 0. \end{aligned}$$

It is easily checked by direct computation that C is a coalgebra.

Definition 1.1.4. Let C be a coalgebra.

(1) An element $g \in C$ is called **group-like** if $\Delta(g) = g\otimes g$ and $\varepsilon(g) = 1$. Let $G(C) = \{g \in C \mid g \text{ is group-like}\}$.
(2) Let $g, h \in G(C)$. Let $P_{g,h}(C) = \{x \in C \mid x \text{ is } (g, h)\text{-primitive}\}$, where $x \in C$ is called (g, h)**-primitive** if $\Delta(x) = g\otimes x + x\otimes h$.
(3) An element $x \in C$ is called **skew-primitive** if there are group-like elements $g, h \in G(C)$ with $x \in P_{g,h}(C)$.

Note that $g \in C$ is group-like if $\Delta(g) = g\otimes g$ and $g \neq 0$, since $g = \varepsilon(g)g$. The sets $P_{g,h}(C)$ with $g, h \in G(C)$ are subspaces of C. If $x \in P_{g,h}(C)$, then $\varepsilon(x) = 0$, since $x = \varepsilon(g)x + \varepsilon(x)h$ because of the counit axiom.

Example 1.1.5. Let $n \in \mathbb{N}$ and let $C = M_n(\Bbbk)^*$ denote the dual space of the vector space of n by n matrices. Let $(u_{ij})_{1\le i,j\le n}$ be the dual basis of the standard basis $(E_{ij})_{1\le i,j\le n}$ of $M_n(\Bbbk)$, where E_{ij} is a matrix having entry 1 in the

i-th row and j-th column, and zeros elsewhere. Then C together with the linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$,

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}$$

for all $i, j \in \{1, \dots, n\}$, is a coalgebra.

The next result is a version of Dedekind's Lemma in Galois theory on the linear independency of characters.

Proposition 1.1.6. *Let C be a coalgebra. Then $G(C)$ is a linearly independent subset of C.*

Proof. We show by induction on n that each subset of $G(C)$ of n elements is linearly independent. This is clear for $n = 1$. Assume that each subset of $G(C)$ of n elements is linearly independent. Let $g_1, \dots, g_{n+1} \in G(C)$ be pairwise distinct elements. Assume that there are non-zero scalars $\alpha_1, \dots, \alpha_{n+1} \in \Bbbk$ with $\sum_{i=1}^{n+1} \alpha_i g_i = 0$. Then $g_{n+1} = \sum_{i=1}^{n} \beta_i g_i$, where $\beta_i = -\frac{\alpha_i}{\alpha_{n+1}}$ for all $1 \le i \le n$. By applying Δ to this equation we get

$$\begin{aligned} \sum_{1\le i\le n} \beta_i g_i \otimes g_i =& \Delta\Big(\sum_{1\le i\le n} \beta_i g_i \Big) \\ =& \Delta(g_{n+1}) = g_{n+1} \otimes g_{n+1} = \sum_{1\le i,j\le n} \beta_i \beta_j g_i \otimes g_j. \end{aligned}$$

Hence $n = 1$ and $\beta_1 = 1$ by linear independency of $g_1, \dots, g_n$. This is a contradiction to $g_1 \neq g_2$. Hence $g_1, \dots, g_{n+1}$ are linearly independent. □

Lemma 1.1.7. *Let C, D be vector spaces and let $A \subseteq C$, $B \subseteq D$ be subspaces. Then*

$$\begin{aligned} A \otimes B = \{t \in C \otimes D \,|\, &(\mathrm{id}_C \otimes g)(t) \in A \textit{ for all } g \in D^*, \\ &(f \otimes \mathrm{id}_D)(t) \in B \textit{ for all } f \in C^*\}. \end{aligned}$$

Proof. The inclusion $\subseteq$ is clear. Conversely, any $t \in C \otimes D$ can be written as $t = \sum_{i=1}^{n} c_i \otimes d_i$ with $n \in \mathbb{N}_0$, $c_1, \dots, c_n \in C$, and $d_1, \dots, d_n \in D$. Take such a presentation of t for a minimal n. Then both $c_1, \dots, c_n$ and $d_1, \dots, d_n$ are linearly independent. If $(f \otimes \mathrm{id}_D)(t) \in B$ for all $f \in C^*$, then $d_i \in B$ for all $i \in \{1, \dots, n\}$. Similarly, if $(\mathrm{id}_C \otimes g)(t) \in A$ for all $g \in D^*$ then $c_i \in A$ for all i. This implies the inclusion $\supseteq$. □

Lemma 1.1.8. *A subspace D of a coalgebra C is a subcoalgebra if and only if $(\mathrm{id}_C \otimes f)\Delta(x) \in D$, $(f \otimes \mathrm{id}_C)\Delta(x) \in D$ for all $x \in D$, $f \in C^*$.*

Proof. The subspace D of C is a subcoalgebra if and only if $\Delta(x) \in D \otimes D$ for all $x \in D$. Thus the claim follows from Lemma 1.1.7. □

Proposition 1.1.9. *The intersection of subcoalgebras of a given coalgebra is a subcoalgebra.*

Proof. Apply Lemma 1.1.8 with D the intersection of subcoalgebras. □

If $X \subseteq C$ is a subspace of a coalgebra C, by Proposition 1.1.9 we can define the **subcoalgebra of C generated by X** as the intersections of all subcoalgebras of C containing X.

REMARK 1.1.10. For all elements c in a coalgebra C it is useful to symbolically write

$$\Delta(c) = c_{(1)} \otimes c_{(2)}. \qquad \text{(Sweedler notation)}$$

In this notation the axioms of a coalgebra are equivalent to the equations

$$\Delta(c_{(1)}) \otimes c_{(2)} = c_{(1)} \otimes \Delta(c_{(2)}), \qquad \text{(coassociativity)} \tag{1.1.13}$$

$$\varepsilon(c_{(1)})c_{(2)} = c = c_{(1)}\varepsilon(c_{(2)}) \qquad \text{(counit)} \tag{1.1.14}$$

for all $c \in C$. Let $c \in C$. Choose finitely many elements $c_{1i}, c_{2i} \in C$, $1 \leq i \leq n$, with $\Delta(c) = \sum_{i=1}^{n} c_{1i} \otimes c_{2i}$. Then the symbolic equations (1.1.13) and (1.1.14) say that

$$\sum_{i=1}^{n} \Delta(c_{1i}) \otimes c_{2i} = \sum_{i=1}^{n} c_{1i} \otimes \Delta(c_{2i}),$$

$$\sum_{i=1}^{n} \varepsilon(c_{1i})c_{2i} = c = \sum_{i=1}^{n} c_{1i}\varepsilon(c_{2i}).$$

Let C be a coalgebra. The iterations Δ^n, $n \geq 0$, of Δ are defined inductively by

$$\Delta^0 = \mathrm{id}_C : C \to C, \ \Delta^n = (\mathrm{id}_C \otimes \Delta^{n-1})\Delta : C \to C^{\otimes(n+1)} \tag{1.1.15}$$

for all $n \geq 1$. We extend the symbolic notation above to the iterations of Δ. For all $c \in C$ and $n \geq 1$, we write

$$\begin{aligned} \Delta(c) &= c_{(1)} \otimes c_{(2)}, \\ \Delta^2(c) &= c_{(1)} \otimes c_{(2)} \otimes c_{(3)}, \quad \dots \\ \Delta^n(c) &= c_{(1)} \otimes \cdots \otimes c_{(n+1)}. \end{aligned}$$

This notation is useful since implicitly it expresses the axiom of coassociativity. Thus for an element c in a coalgebra,

$$\Delta(c_{(1)}) \otimes c_{(2)} = c_{(1)} \otimes \Delta(c_{(2)}) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}.$$

Note that $c_{(1)}$ alone does not make sense. But if $F : \underbrace{C \times \cdots \times C}_{n} \to V$ is an n-fold multilinear function to a vector space V, where $n \geq 2$, then

$$F(c_{(1)}, \dots, c_{(n)}) = f(\Delta^{n-1}(c))$$

is a well-defined expression, where $f : C^{\otimes n} \to V$ is the linear map defined by F.

Let C, D be coalgebras. The formulas for the comultiplication and counit of the tensor product coalgebra $C \otimes D$ are

$$\Delta(c \otimes d) = (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}), \ \ \varepsilon(c \otimes d) = \varepsilon(c)\varepsilon(d) \tag{1.1.16}$$

for all $c \in C$, $d \in D$.

Quotients of algebras are described by ideals. We define coideals to describe coalgebra quotients.

We first note a lemma on the tensor product of linear maps.

LEMMA 1.1.11. *Let $f : V \to X$, $g : W \to Y$ be linear maps between vector spaces V, W, X, Y. Then* $\ker(f \otimes g) = V \otimes \ker(g) + \ker(f) \otimes W$.

PROOF. Choose subspaces $V' \subseteq V$, $W' \subseteq W$ such that $V = \ker(f) \oplus V'$ and $W = \ker(g) \oplus W'$. Then

$$V \otimes W = (V \otimes \ker(g)) \oplus (\ker(f) \otimes W') \oplus (V' \otimes W'),$$

and the restriction of $f \otimes g$ to $V' \otimes W'$ is injective. □

DEFINITION 1.1.12. Let C be a coalgebra. A vector subspace $I \subseteq C$ is a **coideal** if

$$\Delta(I) \subseteq I \otimes C + C \otimes I, \quad \varepsilon(I) = 0.$$

PROPOSITION 1.1.13. *Let C, D be coalgebras, $f : C \to D$ a coalgebra map.*

(1) *If $I \subseteq C$ is a coideal, then $f(I) \subseteq D$ is a coideal, and the quotient vector space C/I is a coalgebra with*

$$\Delta(\overline{x}) = \overline{x_{(1)}} \otimes \overline{x_{(2)}}, \quad \varepsilon(\overline{x}) = \varepsilon(x)$$

for all $x \in C$, where $\overline{x} = x + I$ is the residue class of x in C/I. The quotient map $C \to C/I$ is a coalgebra homomorphism.

(2) *Let $I = \ker(f)$, and let $\overline{f} : C/I \to D$ be the map induced by f. Then I is a coideal of C, and $\overline{f}$ is an injective coalgebra homomorphism.*

(3) *If $J \subseteq D$ is a coideal, then $f^{-1}(J) \subseteq C$ is a coideal.*

PROOF. (1) is clear from the definition, and (2) follows from Lemma 1.1.11, since $\Delta(\ker(f)) \subseteq \ker(f \otimes f)$. (3) follows from (2) applied to the composition $C \xrightarrow{f} D \to D/J$. □

The next lemma demonstrates another setting in which coideals appear naturally.

LEMMA 1.1.14. *Let C be a coalgebra and let $B \subseteq C$ be a subspace satisfying $\Delta(B) \subseteq B \otimes C$ or $\Delta(B) \subseteq C \otimes B$. Then $B^+ = \ker(\varepsilon : B \to \Bbbk)$ is a coideal of C, and $B \neq B^+$ if $B \neq 0$.*

PROOF. Assume that $B \neq 0$ and $\Delta(B) \subseteq B \otimes C$. By the counit axiom there exists $b \in B$ with $\varepsilon(b) = 1$. Hence $B = \Bbbk b \oplus B^+$. Let $x \in B^+$. Then

$$\Delta(x) \in b \otimes y + B^+ \otimes C$$

for some $y \in C$, and $y = x$ by applying $\varepsilon \otimes \mathrm{id}_C$ to the above formula. Thus $\Delta(B^+) \subseteq C \otimes B^+ + B^+ \otimes C$. If $\Delta(B) \subseteq C \otimes B$, then the claim is shown similarly. □

Let V be a vector space, (A, μ, η) an algebra, and $\lambda : A \otimes V \to V$ a linear map. The pair (V, λ) is a left A-module if the following diagrams commute.

$$(1.1.17) \qquad \begin{tikzcd} A \otimes A \otimes V \arrow[r, "\mu \otimes \mathrm{id}_V"] \arrow[d, "\mathrm{id}_A \otimes \lambda"'] & A \otimes V \arrow[d, "\lambda"] \\ A \otimes V \arrow[r, "\lambda"] & V \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \otimes V \arrow[r, "\eta \otimes \mathrm{id}_V"] \arrow[dr, "="'] & A \otimes V \arrow[d, "\lambda"] \\ & V \end{tikzcd}$$

Let V, W be left A-modules. An A-module homomorphism $f : V \to W$ is a linear map such that the following diagram commutes.

$$\begin{tikzcd} A \otimes V \arrow[r, "\mathrm{id}_A \otimes f"] \arrow[d, "\lambda_V"] & A \otimes W \arrow[d, "\lambda_W"] \\ V \arrow[r, "f"] & W \end{tikzcd} \tag{1.1.18}$$

We denote the category of left A-modules with A-linear maps as morphisms by ${}_A\mathcal{M}$. The category of right A-modules, defined analogously, is denoted by $\mathcal{M}_A$.

We introduce comodules over a coalgebra by formally inverting the diagrams defining a module over an algebra.

Definition 1.1.15. Let (C, Δ, ε) be a coalgebra, V a vector space, and let $\delta : V \to C \otimes V$ be a linear map. Then (V, δ) or simply V is a **left C-comodule** if the following diagrams commute.

$$\begin{tikzcd} V \arrow[r, "\delta"] \arrow[d, "\delta"] & C \otimes V \arrow[d, "\Delta \otimes \mathrm{id}_V"] \\ C \otimes V \arrow[r, "\mathrm{id}_C \otimes \delta"] & C \otimes C \otimes V \end{tikzcd} \qquad \text{(coassociativity)} \tag{1.1.19}$$

$$\begin{tikzcd} V \arrow[r, "\delta"] \arrow[dr, "="'] & C \otimes V \arrow[d, "\varepsilon \otimes \mathrm{id}_V"] \\ & \Bbbk \otimes V \end{tikzcd} \qquad \text{(counit)} \tag{1.1.20}$$

If (V, δ_V) and (W, δ_W) are left C-comodules, and $f : V \to W$ is a linear map, then f is a **left C-comodule homomorphism** or a **left C-colinear** map if the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\delta_V"] & W \arrow[d, "\delta_W"] \\ C \otimes V \arrow[r, "\mathrm{id}_C \otimes f"] & C \otimes W \end{tikzcd} \tag{1.1.21}$$

Let (V, δ) be a left C-comodule. A **subcomodule** of V is a subspace $U \subseteq V$ with $\delta(U) \subseteq C \otimes U$.

The category of left C-comodules with C-colinear maps as morphisms is denoted by ${}^C\mathcal{M}$. Right C-comodules and right C-colinear maps are defined similarly. Their category is denoted by $\mathcal{M}^C$.

We write $\mathrm{Hom}^C(V, W)$ for the set of all left (or right) C-colinear maps between two left (or right) C-comodules V, W.

Remark 1.1.16. Comodules over a coalgebra C form an abelian category like modules over an algebra. In particular, let $(V, \delta_V) \in {}^C\mathcal{M}$, and let $U \subseteq V$ be a subcomodule. Let V/U be the quotient vector space, and let $\pi : V \to V/U$ be the quotient map. Then $(V/U, \delta_{V/U})$ is a left C-comodule, where the comodule structure is uniquely defined by the commutative diagram

$$\begin{tikzcd} V \arrow[r, "\delta_V"] \arrow[d, "\pi"] & C \otimes V \arrow[d, "\mathrm{id}\otimes\pi"] \\ V/U \arrow[r, "\delta_{V/U}"] & C \otimes V/U \end{tikzcd}$$

If $V, W \in {}^C\mathcal{M}$, and $f : V \to W$ is left C-colinear, then $\ker(f) \subseteq V$ and $\mathrm{im}(f) \subseteq W$ are subcomodules, and $V/\ker(f) \xrightarrow{\cong} \mathrm{im}(f)$, $\overline{v} \mapsto f(v)$, is an isomorphism in ${}^C\mathcal{M}$.

Let Γ be a set. Comodules over $\Bbbk\Gamma$ are given by Γ-graded vector spaces. A **Γ-grading** of a vector space V is a family $\mathcal{V} = (V(g))_{g\in\Gamma}$ of subspaces of V such that

$$V = \bigoplus_{g\in\Gamma} V(g).$$

A **Γ-graded vector space** is a pair $(V, \mathcal{V})$, where V is a vector space with a grading (or a **gradation**) $\mathcal{V}$. For a graded vector space $V = (V, \mathcal{V})$ we denote by $\pi_g^V : V \to V(g)$, $g \in \Gamma$, the canonical projection. An element $v \in V$ is called **homogeneous of degree** $g \in \Gamma$ if $v \in V(g)$. We write $\deg(v) = g$, if $v \in V(g)$.

We also use the notation $V_g = V(g)$, in particular, when G is a monoid or a group.

Let $\Gamma\text{-}\mathrm{Gr}\,\mathcal{M}_\Bbbk$ be the category of Γ-graded vector spaces, where a morphism $f : (V, \mathcal{V}) \to (W, \mathcal{W})$ is a **graded map** or a **homogeneous map** (of degree 0), that is a $\Bbbk$-linear map with $f(V(g)) \subseteq W(g)$ for all $g \in \Gamma$.

PROPOSITION 1.1.17. *Let Γ be a set. The functor*

$$F : \Gamma\text{-}\mathrm{Gr}\,\mathcal{M}_\Bbbk \to {}^{\Bbbk\Gamma}\mathcal{M},\ (V, (V(g))_{g\in\Gamma}) \mapsto \Big(\bigoplus_{g\in\Gamma} V(g), \delta\Big),$$

where $\delta(v) = g \otimes v$ for all $v \in V(g)$, $g \in \Gamma$, and where morphisms f are mapped onto f, is an isomorphism of categories. The inverse functor maps a comodule (V, δ) onto V with grading $V(g) = V_g = \{v \in V \mid \delta(v) = g \otimes v\}$ for all $g \in \Gamma$.

PROOF. Let (V, δ) be a left $\Bbbk\Gamma$-comodule. We prove that $V = \bigoplus_{g\in\Gamma} V_g$, where

$$V_g = \{v \in V \mid \delta(v) = g \otimes v\} \text{ for all } g \in \Gamma.$$

For any $v \in V$ there are elements $v_g \in V$, $g \in \Gamma$, such that $v_g \neq 0$ only for finitely many $g \in \Gamma$ and such that $\delta(v) = \sum_{g\in\Gamma} g \otimes v_g$. By coassociativity,

$$\sum_{g\in\Gamma} g \otimes \delta(v_g) = \sum_{g\in\Gamma} g \otimes g \otimes v_g.$$

Hence $\delta(v_g) = g \otimes v_g$ for all $g \in \Gamma$. Moreover, $v = \sum_{g\in\Gamma} \varepsilon(g) v_g = \sum_{g\in\Gamma} v_g$. Hence $V = \sum_{g\in\Gamma} V_g$. Let now $(v_g)_{g\in\Gamma}$ be a family of elements of V, where $v_g \in V(g)$ for all $g \in \Gamma$ and $v_g \neq 0$ only for finitely many $g \in \Gamma$. Assume that $\sum_{g\in\Gamma} v_g = 0$. Applying δ gives $\sum_{g\in\Gamma} g \otimes v_g = 0$, hence $v_g = 0$ for all $g \in \Gamma$.

The isomorphism of categories now follows easily. □

REMARK 1.1.18. If (V, δ) is a right C-comodule, we define inductively

$$\delta^n : V \to V \otimes C^{\otimes n} \text{ for all } n \geq 0$$

by $\delta^0 = \mathrm{id}_V$, $\delta^1 = \delta_V$, and $\delta^n = (\delta \otimes \mathrm{id}_{C^{\otimes(n-1)}})\delta^{n-1}$ for all $n \geq 2$. Extending the Sweedler notation to comodules we write

$$\begin{aligned}
\delta(v) &= v_{(0)} \otimes v_{(1)}, \\
\delta^2(v) &= v_{(0)} \otimes \Delta(v_{(1)}) = v_{(0)} \otimes v_{(1)} \otimes v_{(2)}, \quad \dots \\
\delta^n(v) &= v_{(0)} \otimes v_{(1)} \otimes \cdots \otimes v_{(n)}
\end{aligned}$$

for all $v \in V$. For left C-comodules (V, δ) we use negative indices.

$$\begin{aligned}
\delta(v) &= v_{(-1)} \otimes v_{(0)}, \\
\delta^2(v) &= \Delta(v_{(-1)}) \otimes v_{(0)} = v_{(-2)} \otimes v_{(-1)} \otimes v_{(0)}, \quad \dots \\
\delta^n(v) &= v_{(-n)} \otimes \cdots \otimes v_{(-1)} \otimes v_{(0)}
\end{aligned}$$

for all $v \in V$.

1.2. Bialgebras and Hopf algebras

We continue with the introduction of bialgebras, Hopf algebras, quotients of them, and their graded versions.

Definition 1.2.1. Let H be a vector space, and let

$$\mu : H \otimes H \to H, \quad \eta : \Bbbk \to H, \quad \Delta : H \to H \otimes H, \quad \varepsilon : H \to \Bbbk$$

be linear maps. Then $(H, \mu, \eta, \Delta, \varepsilon)$ is a **bialgebra** if (H, μ, η) is an algebra, (H, Δ, ε) is a coalgebra, and Δ and ε are algebra maps.

Let H, H' be bialgebras. A **bialgebra homomorphism** $\varphi : H \to H'$ is an algebra and a coalgebra homomorphism. A **subbialgebra** of a bialgebra is a subalgebra and a subcoalgebra.

Proposition 1.2.2. *Let H be a vector space, and let*

$$\mu : H \otimes H \to H, \quad \eta : \Bbbk \to H, \quad \Delta : H \to H \otimes H, \quad \varepsilon : H \to \Bbbk$$

be linear maps. Assume that (H, μ, η) is an algebra and (H, Δ, ε) is a coalgebra. Then the following are equivalent.

(1) *Δ and ε are algebra maps.*
(2) *μ and η are coalgebra maps.*

Proof. By definition, (1) is equivalent to the commutativity of the diagrams (1.1.5) and (1.1.6) for Δ and ε, and (2) is equivalent to the commutativity of the diagrams (1.1.10) and (1.1.11) for μ and η.

Let $\tau = \tau_{H,H} : H \otimes H \to H \otimes H$ be the flip map. Then

$$\mu_{H\otimes H}(\Delta \otimes \Delta) = (\mu \otimes \mu)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta) = (\mu \otimes \mu)\Delta_{H\otimes H}.$$

Hence (1.1.5) for Δ and (1.1.10) for μ coincide. Obviously, the diagrams (1.1.6) for Δ and (1.1.10) for η, (1.1.5) for ε and (1.1.11) for μ, as well as (1.1.6) for ε and (1.1.11) for η coincide. $\square$

Example 1.2.3. Let G be a monoid, that is a set G together with an associative map $G \times G \to G$ and a unit element e. The **monoid algebra** $\Bbbk G$ (or **group algebra**, if G is a group) is the vector space with basis G. Its algebra structure $\mu : \Bbbk G \otimes \Bbbk G \to \Bbbk G$, $\eta : \Bbbk \to \Bbbk G$, is given by $\mu(g, h) = gh$ (the product of g and h in G) for all $g, h \in G$ and by $\eta(1) = e$. Then $\Bbbk G$ is a bialgebra where the elements of G are group-like. The bialgebra axioms are trivially verified on the basis.

Definition 1.2.4. Let H be a bialgebra.

(1) Let $V, W \in {}_H\mathcal{M}$. The map
$$H \otimes V \otimes W \to V \otimes W,\ h \otimes v \otimes w \mapsto h_{(1)}v \otimes h_{(2)}w,$$
is called the **diagonal action** of H on $V \otimes W$. The **trivial action** of H on $\Bbbk$ is defined by $H \otimes \Bbbk \to \Bbbk,\ h \otimes 1 \mapsto \varepsilon(h)$.

(2) Let $V, W \in {}^H\mathcal{M}$. The map
$$V \otimes W \to H \otimes V \otimes W,\ v \otimes w \mapsto v_{(-1)}w_{(-1)} \otimes v_{(0)} \otimes w_{(0)},$$
is called the **diagonal coaction** of H on $V \otimes W$. The **trivial coaction** of H on $\Bbbk$ is defined by $\Bbbk \to H \otimes \Bbbk,\ 1 \mapsto \eta(1) \otimes 1$.

For modules over $\Bbbk G$, G a monoid, the diagonal action is given by the familiar formulas from representation theory of groups:
$$g(v \otimes w) = gv \otimes gw,\ g\alpha = \alpha,$$
for all $v \in V$, $w \in W$, $\alpha \in \Bbbk$.

It is a fundamental consequence of the axioms of a bialgebra that modules and comodules over a bialgebra can be multiplied in the sense of the following proposition.

Proposition 1.2.5. *Let H be a bialgebra. The tensor product of two left H-(co)modules is a left H-(co)module with diagonal (co)action. Moreover, for all $U, V, W \in {}_H\mathcal{M}$ (for all $U, V, W \in {}^H\mathcal{M}$, respectively) the canonical isomorphisms*
$$(U \otimes V) \otimes W \to U \otimes (V \otimes W), \qquad \Bbbk \otimes V \to V, \qquad V \otimes \Bbbk \to V,$$
are left H-(co)linear.

Proof. This is easily checked using the Sweedler notation. □

Of course, the same result holds for right modules and right comodules where the diagonal action and coaction is defined in a similar way.

The next remark shows that in fact the last proposition gives a natural explanation of the axioms of a bialgebra.

Remark 1.2.6. Let H be an algebra together with algebra maps
$$\Delta : H \to H \otimes H,\ \varepsilon : H \to \Bbbk.$$
We will again write $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for all $h \in H$.

The trivial one-dimensional left H-module is the vector space $\Bbbk$ with H-action $h1_{\Bbbk} = \varepsilon(h)$ for all $h \in H$.

Let V, W be left H-modules. Then $V \otimes W$ is a left $H \otimes H$-module by
$$(x \otimes y)(v \otimes w) = xv \otimes yw$$
for all $x, y \in H$, $v \in V$, $w \in W$. Hence $V \otimes W$ is a left H-module induced by the algebra map Δ. Thus
$$h(v \otimes w) = h_{(1)}v \otimes h_{(2)}w$$
for all $h \in H$, $v \in V$, $w \in W$.

The coalgebra axioms in the definition of a bialgebra can now be explained in a very natural way.

(1) The map Δ satisfies (1.1.7) if and only if for all left H-modules U, V, W the canonical isomorphism

$$(U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

is left H-linear.

(2) The map ε satisfies (1.1.8) if and only if for all left H-modules V the canonical isomorphisms $V \otimes \Bbbk \to V$ and $\Bbbk \otimes V \to V$ are left H-linear.

DEFINITION 1.2.7. Let Γ be a monoid and let V, W be Γ-graded vector spaces. Then $V \otimes W$ is a Γ-graded vector space by

$$(V \otimes W)(g) = \bigoplus_{\substack{(a,b)\in\Gamma^2\\ ab=g}} V(a) \otimes W(b), \quad \text{for all } g \in \Gamma.$$

This grading on $V \otimes W$ is called the **diagonal Γ-grading**. The **trivial grading** on a vector space V is defined by $V(e) = \Bbbk$, e the unit element of Γ, that is, $V(g) = 0$ for all $e \neq g \in \Gamma$.

REMARK 1.2.8. Let Γ be a monoid.

(1) For all Γ-graded vector spaces U, V, W the canonical isomorphisms

$$(U \otimes V) \otimes W \to U \otimes (V \otimes W), \qquad \Bbbk \otimes V \to V, \qquad V \otimes \Bbbk \to V,$$

are Γ-graded. The flip maps $\tau_{V,W} : V \otimes W \to W \otimes V$ are only graded for all V, W if Γ is commutative.

(2) The category isomorphism $F : \Gamma\text{-}\mathrm{Gr}\,\mathcal{M}_\Bbbk \to {}^{\Bbbk\Gamma}\mathcal{M}$ of Proposition 1.1.17 preserves the trivial objects and the tensor product with diagonal structure, that is, $F(\Bbbk) = \Bbbk$, and for all Γ-graded vector spaces V, W,

$$F(V \otimes W) = F(V) \otimes F(W) \text{ in } {}^{\Bbbk\Gamma}\mathcal{M}.$$

The following algebra structure on $\mathrm{Hom}(C, A)$ for a coalgebra C and an algebra A will be an important tool to study the existence of the antipode of a bialgebra.

DEFINITION 1.2.9. Let C be a coalgebra, A an algebra, and $f, g \in \mathrm{Hom}(C, A)$ linear maps. The **convolution** $f * g \in \mathrm{Hom}(C, A)$ of f and g is defined by

$$(f * g)(c) = f(c_{(1)})g(c_{(2)})$$

for all $c \in C$, that is by the composition

$$f * g = (C \xrightarrow{\Delta} C \otimes C \xrightarrow{f\otimes g} A \otimes A \xrightarrow{\mu} A).$$

The coassociativity of the comultiplication Δ of C and the associativity of the multiplication map μ of A imply that the convolution product of $\mathrm{Hom}(C, A)$ is associative. Thus $\mathrm{Hom}(C, A)$ is an algebra with unit element $\eta\varepsilon$.

In the next proposition we will identify $\mathrm{Hom}(C, A)$ with an algebra of endomorphisms. This will give very useful information about the structure of the inverse of an element in $\mathrm{Hom}(C, A)$. We define

$$\mathrm{End}_A^C(A \otimes C) = \{f : A \otimes C \to A \otimes C \mid f \text{ left } A\text{-linear and right } C\text{-colinear}\},$$

where $A \otimes C$ is a left A-module by $\mu \otimes \mathrm{id}_C$, and a right C-comodule by $\mathrm{id}_A \otimes \Delta$. Then $\mathrm{End}_A^C(A \otimes C)$ becomes an algebra with composition of maps as multiplication.

LEMMA 1.2.10. *Let C be a coalgebra, and X a vector space. For any right C-comodule V, the map*

$$\mathrm{Hom}^C(V, X \otimes C) \xrightarrow{\cong} \mathrm{Hom}(V, X), \quad f \mapsto (\mathrm{id} \otimes \varepsilon)f,$$

is bijective with inverse given by $\varphi \mapsto (\varphi \otimes \mathrm{id})\delta_V$. Here, $X \otimes C$ is a right C-comodule with comodule structure $\mathrm{id}_X \otimes \Delta$.

PROOF. Let $f \in \mathrm{Hom}^C(V, X \otimes C)$. Then $f(v_{(0)}) \otimes v_{(1)} = (\mathrm{id}_X \otimes \Delta)f(v)$ for all $v \in V$, since f is C-colinear. By applying $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}$ to this equation we obtain $\varphi(v_{(0)}) \otimes v_{(1)} = f(v)$, where $\varphi = (\mathrm{id} \otimes \varepsilon)f$. Conversely, let $\varphi \in \mathrm{Hom}(V, X)$ and define $f = (\varphi \otimes \mathrm{id})\delta_V$. Then $f \in \mathrm{Hom}^C(V, X \otimes C)$ by coassociativity of δ_V. Moreover, $((\mathrm{id} \otimes \varepsilon)f)(v) = \varphi(v_{(0)})\varepsilon(v_{(1)}) = \varphi(v)$ for all $v \in V$. $\square$

Lemma 1.2.10 implies that the functor $\mathcal{M}_{\Bbbk} \to \mathcal{M}^C$, $X \mapsto X \otimes C$, is right adjoint to the forgetful functor $\mathcal{M}^C \to \mathcal{M}_{\Bbbk}$.

PROPOSITION 1.2.11. *Let C be a coalgebra and A an algebra.*

(1) *The map $\Phi : \mathrm{Hom}(C, A) \xrightarrow{\cong} \mathrm{End}_A^C(A \otimes C)$ given by*

$$f \mapsto (A \otimes C \xrightarrow{\mathrm{id} \otimes \Delta} A \otimes C \otimes C \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} A \otimes A \otimes C \xrightarrow{\mu \otimes \mathrm{id}} A \otimes C)$$

is an algebra anti-isomorphism, where $\mathrm{Hom}(C, A)$ is an algebra with convolution as multiplication.

(2) *Let $f \in \mathrm{Hom}(C, A)$. Then f is invertible if and only if $\Phi(f)$ is an isomorphism. If $\Phi(f)$ is an isomorphism with inverse map $\Phi(f)^{-1}$, then*

$$f^{-1} = (C = \Bbbk \otimes C \xrightarrow{\eta \otimes \mathrm{id}_C} A \otimes C \xrightarrow{\Phi(f)^{-1}} A \otimes C \xrightarrow{\mathrm{id} \otimes \varepsilon} A)$$

is the inverse of f in $\mathrm{Hom}(C, A)$.

PROOF. (1) Let $V = A \otimes C$ and $X = A$ in Lemma 1.2.10. Since the comodule structure $\delta_V = \mathrm{id} \otimes \Delta$ of V is left A-linear, the isomorphism in Lemma 1.2.10 restricts to an isomorphism $\Phi_1 : \mathrm{Hom}_A^C(A \otimes C, A \otimes C) \to \mathrm{Hom}_A(A \otimes C, A)$. Let

$$\Phi : \mathrm{Hom}(C, A) \xrightarrow{\cong} \mathrm{Hom}_A(A \otimes C, A) \xrightarrow{\Phi_1^{-1}} \mathrm{Hom}_A^C(A \otimes C, A \otimes C)$$

be the composition of Φ_1^{-1} with the isomorphism

$$\mathrm{Hom}(C, A) \xrightarrow{\cong} \mathrm{Hom}_A(A \otimes C, A), \; f \mapsto (a \otimes c \mapsto af(c)).$$

Then

$$\Phi(f)(a \otimes c) = af(c_{(1)}) \otimes c_{(2)} \quad \text{for all } f \in \mathrm{Hom}(C, A),\, a \in A,\, c \in C.$$

Hence for all $f, f' \in \mathrm{Hom}(C, A)$ and $a \in A$, $c \in C$,

$$\begin{aligned}(\Phi(f)\Phi(f'))\,(a \otimes c) &= \Phi(f)(af'(c_{(1)}) \otimes c_{(2)}) \\ &= af'(c_{(1)})f(c_{(2)}) \otimes c_{(3)} = \Phi(f' * f)(a \otimes c).\end{aligned}$$

The inverse of Φ is given by

$$\Phi^{-1} : \mathrm{End}_A^C(A \otimes C) \to \mathrm{Hom}(C, A), \; F \mapsto (\mathrm{id} \otimes \varepsilon)F(\eta_A \otimes \mathrm{id}_C).$$

(2) follows from (1). $\square$

Let C be a coalgebra. The algebra $C^* = \mathrm{Hom}(C, \Bbbk)$ in Definition 1.2.9 with $A = \Bbbk$ is called the **dual algebra** of C. It is easy to see that for any coalgebra map $\varphi : C \to D$ the map $\varphi^* : D^* \to C^*$, $f \mapsto f \circ \varphi$, is an algebra homomorphism.

EXAMPLE 1.2.12. Let $G = \{g_1, \dots, g_n\}$ be a finite set of n elements. The vector space $\Bbbk G$ with basis G is a coalgebra with $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ for all $g \in G$. Let $(e_i)_{1 \le i \le n}$ be the dual basis of $(g_i)_{1 \le i \le n}$. Then $e_i e_j = \delta_{ij} e_i$ for all i, j, and $\sum_{i=1}^n e_i = 1$. Hence $C^* \cong \Bbbk^n$ as algebras.

EXAMPLE 1.2.13. Let $C = M_n(\Bbbk)^*$ be the coalgebra in Example 1.1.5. For all $i, j \in \{1, \dots, n\}$ consider $E_{ij} \in M_n(\Bbbk)$ as an element in C^* via the natural isomorphism $M_n(\Bbbk)^{**} \cong M_n(\Bbbk)$, that is, $E_{ij}(u_{kl}) = \delta_{ik}\delta_{jl}$ for all $i, j, k, l \in \{1, \dots, n\}$. Then

$$(E_{ij} * E_{kl})(u_{rs}) = \sum_{m=1}^n E_{ij}(u_{rm}) E_{kl}(u_{ms}) = \delta_{ir}\delta_{jk}\delta_{ls} = \delta_{jk} E_{il}(u_{rs})$$

for all $i, j, k, l, r, s \in \{1, 2, \dots, n\}$. Hence the natural isomorphism

$$C^* \to M_n(\Bbbk)$$

is an algebra isomorphism, where the multiplication in C^* is the convolution product.

DEFINITION 1.2.14. A **Hopf algebra** H is a bialgebra such that id_H is invertible in the convolution algebra $\mathrm{Hom}(H, H)$. The inverse $\mathcal{S}$ (or $\mathcal{S}_H$) of id_H is called the **antipode of** H. A **Hopf algebra homomorphism** between two Hopf algebras is a bialgebra homomorphism. A **Hopf subalgebra** of a Hopf algebra H is a subbialgebra $H' \subseteq H$ such that $\mathcal{S}(H') \subseteq H'$.

REMARK 1.2.15. Let H be a bialgebra. Then H is a Hopf algebra (with antipode $\mathcal{S}$) if there is a linear map $\mathcal{S} : H \to H$ such that

$$h_{(1)}\mathcal{S}(h_{(2)}) = \varepsilon(h)1 = \mathcal{S}(h_{(1)})h_{(2)} \tag{1.2.1} \quad \text{(antipode)}$$

for all $h \in H$, or equivalently such that the following diagrams commute.

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\eta\varepsilon"] & H \otimes H \arrow[d, "\mathcal{S}\otimes\mathrm{id}_H"] \\ H & H \otimes H \arrow[l, "\mu"] \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\eta\varepsilon"] & H \otimes H \arrow[d, "\mathrm{id}_H\otimes\mathcal{S}"] \\ H & H \otimes H \arrow[l, "\mu"] \end{tikzcd} \tag{1.2.2} \quad \text{(antipode)}$$

By uniqueness of inverses, each bialgebra has at most one antipode.

EXAMPLE 1.2.16. Let G be a group. Then the bialgebra $\Bbbk G$ of the monoid G in Example 1.2.3 is a Hopf algebra with antipode defined by $\mathcal{S}(g) = g^{-1}$ for all $g \in G$.

PROPOSITION 1.2.17. *Let H be a Hopf algebra with antipode $\mathcal{S}$.*

(1) *The antipode $\mathcal{S}$ is an algebra anti-homomorphism and a coalgebra anti-homomorphism, that is, for all $x, y \in H$*
   (a) $\mathcal{S}(xy) = \mathcal{S}(y)\mathcal{S}(x)$, $\mathcal{S}(1) = 1$,
   (b) $\Delta(\mathcal{S}(x)) = \mathcal{S}(x_{(2)}) \otimes \mathcal{S}(x_{(1)})$, $\varepsilon(\mathcal{S}(x)) = \varepsilon(x)$.

(2) *Let H' be a Hopf algebra, and let $\varphi : H \to H'$ be a bialgebra map. Then $\mathcal{S}_{H'}\varphi = \varphi\mathcal{S}_H$.*

PROOF. (1) (a) Define $F, G \in \mathrm{Hom}(H \otimes H, H)$ by

$$F(x \otimes y) = \mathcal{S}(xy), \quad G(x \otimes y) = \mathcal{S}(y)\mathcal{S}(x)$$

for all $x, y \in H$. Then both F and G are convolution inverses of μ_H. Indeed, $\mu_H * F = \eta\varepsilon$ and $\mu_H * G = \eta\varepsilon$ since

$$x_{(1)}y_{(1)}\mathcal{S}(x_{(2)}y_{(2)}) = \varepsilon(x)\varepsilon(y),$$
$$x_{(1)}y_{(1)}\mathcal{S}(y_{(2)})\mathcal{S}(x_{(2)}) = \varepsilon(x)\varepsilon(y)$$

for all $x, y \in H$. Similarly, $F * \mu_H = G * \mu_H = \eta\varepsilon$. Hence $F = G$. Further, $\mathcal{S}(1) = 1$ since $1\mathcal{S}(1) = \varepsilon(1)1$.

(b) Define $F, G \in \mathrm{Hom}(H, H \otimes H)$ by

$$F(x) = \Delta(\mathcal{S}(x)),\ G(x) = \mathcal{S}(x_{(2)}) \otimes \mathcal{S}(x_{(1)})$$

for all $x \in H$. Then both F and G are convolution inverses of Δ_H. Indeed, $\Delta * F = (\eta \otimes \eta)\varepsilon$ and $\Delta * G = (\eta \otimes \eta)\varepsilon$ since

$$\Delta(x_{(1)})F(x_{(2)}) = \Delta(x_{(1)}\mathcal{S}(x_{(2)})) = \varepsilon(x)1 \otimes 1,$$
$$\Delta(x_{(1)})G(x_{(2)}) = x_{(1)}\mathcal{S}(x_{(4)}) \otimes x_{(2)}\mathcal{S}(x_{(3)}) = x_{(1)}\mathcal{S}(x_{(2)}) \otimes 1 = \varepsilon(x)1 \otimes 1$$

for all $x \in H$. Similarly, $F * \Delta = G * \Delta = (\eta \otimes \eta)\varepsilon$. Hence $F = G$. Further, $\varepsilon \circ \mathcal{S} = \varepsilon$, since both are convolution inverses of ε.

(2) Both $\mathcal{S}_{H'}\varphi$ and $\varphi\mathcal{S}_H$ are convolution inverses of $\varphi \in \mathrm{Hom}(H, H')$. □

REMARK 1.2.18. Let H be a bialgebra, and $\mathcal{S} : H \to H$ an algebra anti-homomorphism. For any left H-module V, the dual space $V^* = \mathrm{Hom}(V, \Bbbk)$ is a right H-module in the natural way by $(fh)(v) = f(hv)$ for all $h \in H$, $f \in V^*$, $v \in V$. Since $\mathcal{S}$ is an algebra anti-homomorphism, V^* becomes a left H-module by

$$(hf)(v) = f(\mathcal{S}(h)v)$$

for all $h \in H$, $f \in V^*$, $v \in V$. If V is a right H-module, then the dual vector space V^* is a right H-module by

$$(fh)(v) = f(v\mathcal{S}(h))$$

for all $h \in H$, $f \in V^*$, $v \in V$.

The map $\mathcal{S}$ satisfies (1.2.1) if and only if for all left H-modules V and all right H-modules W the evaluation maps

$$V^* \otimes V \to \Bbbk,\ p \otimes v \mapsto p(v), \quad W \otimes W^* \to \Bbbk,\ w \otimes q \mapsto q(w),$$

are left H-linear and right H-linear, respectively.

Bialgebras are generalizations of monoids and Hopf algebras are generalizations of groups. Proposition 1.2.17(1) says that $(gh)^{-1} = h^{-1}g^{-1}$ for all elements g, h of a group. By Proposition 1.2.17(2), a monoid homomorphism between groups preserves inverses.

However, the rule $(g^{-1})^{-1} = g$ for the elements g of a group does not generalize to Hopf algebras. In general, the antipode $\mathcal{S}$ of a Hopf algebra does not satisfy $\mathcal{S}^2 = \mathrm{id}$. There are (rather pathological) Hopf algebras whose antipode is not bijective. If the antipode is bijective, then its order as a vector space automorphism could be infinite.

A monoid M is a group if and only if the canonical map

$$M \times M \to M \times M,\ (x, y) \mapsto (xy, y),$$

is bijective. We note the corresponding characterization for Hopf algebras.

PROPOSITION 1.2.19. *Let H be a bialgebra. We denote the "Galois map" by*

$$\mathcal{G} = (H \otimes H \xrightarrow{\mathrm{id} \otimes \Delta} H \otimes H \otimes H \xrightarrow{\mu \otimes \mathrm{id}} H \otimes H).$$

(1) *The following are equivalent.*
  (a) *H is a Hopf algebra.*
  (b) *$\mathcal{G} : H \otimes H \to H \otimes H$ is an isomorphism.*

(2) *If $\mathcal{G}$ is an isomorphism with inverse $\mathcal{G}^{-1}$, then*

$$\mathcal{S} = (H \xrightarrow{\eta \otimes \mathrm{id}} H \otimes H \xrightarrow{\mathcal{G}^{-1}} H \otimes H \xrightarrow{\mathrm{id} \otimes \varepsilon} H)$$

is the antipode of H.

PROOF. Note that $\mathcal{G} \in \mathrm{End}_H^H(H \otimes H)$. The isomorphism

$$\Phi : \mathrm{Hom}(H, H) \xrightarrow{\cong} \mathrm{End}_H^H(H \otimes H)$$

of Proposition 1.2.11 maps the identity onto $\mathcal{G}$. Hence the claim follows from Proposition 1.2.11(2). □

By slight altering of the multiplication or comultiplication one can get new bialgebras and Hopf algebras. We will discuss this phenomenon in a more general setting in Proposition 3.2.15.

DEFINITION 1.2.20. Let H be a bialgebra. Then H^{op} with comultiplication Δ_H and counit ε_H is called the **opposite bialgebra**. Similarly, H^{cop} with multiplication μ_H and unit η_H is called the **coopposite bialgebra**.

It is easy to check that for any bialgebra H, H^{op} and H^{cop} are again bialgebras. Moreover, if H is a Hopf algebra then H^{op} and H^{cop} are Hopf algebras if and only if $\mathcal{S}$ is bijective. In this case, $\mathcal{S}^{-1}$ is the antipode of H^{op} and of H^{cop}.

To define quotients of bialgebras and Hopf algebras we introduce the subobjects which are the kernels of the corresponding quotient maps.

An **ideal** or **two-sided ideal** I in an algebra A is a linear subspace $I \subseteq A$ such that $ax \in I$ and $xa \in I$ for all $x \in I$ and $a \in A$.

DEFINITION 1.2.21. Let H be a bialgebra. A subspace $I \subseteq H$ is a **bi-ideal** of H if $I \subseteq H$ is an ideal and a coideal.

Let H be a Hopf algebra. A **Hopf ideal** of H is a bi-ideal I of H with $\mathcal{S}(I) \subseteq I$.

PROPOSITION 1.2.22. *Let H and H' be bialgebras, $I \subseteq H$ a bi-ideal, and let $\varphi : H \to H'$ a morphism of bialgebras.*

(1) *The quotient coalgebra and quotient algebra $\overline{H} = H/I$ is a bialgebra. If H is a Hopf algebra, and $I \subseteq H$ is a Hopf ideal, then $\overline{H}$ is a Hopf algebra with antipode $\mathcal{S}_{\overline{H}}(\overline{x}) = \overline{\mathcal{S}_H(x)}$ for all $x \in H$.*

(2) $\ker(\varphi) \subseteq H$ *is a bi-ideal, and the natural map $\overline{\varphi} : H/\ker(\varphi) \to H'$ is an injective bialgebra homomorphism. If H and H' are Hopf algebras, then* $\ker(\varphi)$ *is a Hopf ideal of H.*

PROOF. (1) follows directly from the definitions, and (2) follows from Proposition 1.1.13 and 1.2.17(2). □

It can be quite difficult or impossible to verify the axioms of a Hopf algebra on a vector space basis, since usually there is no easy formula for the comultiplication on all elements of a basis.

However, it is sufficient to check the axioms on algebra generators. We say that a subset M of an algebra A is a **set of algebra generators**, or that M generates A as an algebra, if any element of A is a $\Bbbk$-linear combination of products of elements of M. We write $A = \Bbbk[M]$ if M is a set of algebra generators.

PROPOSITION 1.2.23. *Let H be an algebra and $M \subseteq H$ a set of algebra generators. Let*

$$\varDelta : H \to H \otimes H, \quad \varepsilon : H \to \Bbbk, \quad \mathcal{S} : H \to H^{\mathrm{op}}$$

be algebra maps. Assume that the diagrams (1.1.7)*,* (1.1.8) *and* (1.2.2) *commute for all $h \in M$. Then $(H, \varDelta, \varepsilon, \mathcal{S})$ is a Hopf algebra.*

PROOF. In the diagrams in (1.1.7) and (1.1.8) for (H, Δ, ϵ) all maps are algebra maps. Hence the diagrams commute, since they commute when applied to elements of M.

But the maps in the diagrams in (1.2.2) are in general not algebra maps. Let H' be the subset of all elements of H on which the first diagram in (1.2.2) commutes. Thus $H' = \{h \in H \mid \mathcal{S}(h_{(1)})h_{(2)} = \varepsilon(h)1\}$ is a subspace of H containing the unit element 1 of H. Let $x, y \in H'$. Then $xy \in H'$, since

$$\begin{aligned}
\mathcal{S}((xy)_{(1)})(xy)_{(2)} &= \mathcal{S}(x_{(1)}y_{(1)})x_{(2)}y_{(2)} && (\varDelta \text{ is an algebra map})\\
&= \mathcal{S}(y_{(1)})\mathcal{S}(x_{(1)})x_{(2)}y_{(2)} \\
&= \mathcal{S}(y_{(1)})\varepsilon(x)y_{(2)} && (\text{since } x \in H')\\
&= \varepsilon(x)\varepsilon(y) && (\text{since } y \in H')\\
&= \varepsilon(xy), && (\varepsilon \text{ is an algebra map})
\end{aligned}$$

where the second equality holds since $\mathcal{S}$ is an algebra anti-homomorphism.

Hence H' is a subalgebra of H. This shows that $H' = H$, since $M \subseteq H'$. In the same way it follows that the second diagram in (1.2.2) commutes. $\square$

For the next example we need the notion of shuffle permutations. We will study them in more detail in Section 1.8.

Let n be a natural number, and $i \in \{0, 1, \ldots, n\}$. A permutation $w \in \mathbb{S}_n$ is called an $(i, n-i)$-**shuffle** or simply an i-shuffle if

$$w(1) < \cdots < w(i), \text{ and } w(i+1) < \cdots < w(n).$$

Note that any $(0, n)$- or $(n, 0)$-shuffle is the identity.

EXAMPLE 1.2.24. Let X be a set which we view as an alphabet. Let $\Bbbk\langle X\rangle$ be the **free algebra** in the alphabet X. If $X = \{a_1, \ldots, a_m\}$ is a finite set of m elements, we write $\Bbbk\langle X\rangle = \Bbbk\langle a_1, \ldots, a_m\rangle$.

The formal words

$$x_1 \cdots x_n, \text{ where } x_1, \ldots, x_n \in X,\, n \in \mathbb{N}_0,$$

form a basis of the vector space $\Bbbk\langle X\rangle$, and the multiplication is defined by concatenation of words. By definition, the length of the word $x_1 \cdots x_n$ is n, where $x_1, \ldots, x_n \in X$, $n \in \mathbb{N}_0$. The empty word is the unit element.

The free algebra has the following universal property: Let A be an algebra and $(a_x)_{x \in X}$ a family of elements $a_x \in A$. Then there is exactly one algebra map $\varphi : \Bbbk\langle X\rangle \to A$ such that $\varphi(x) = a_x$ for all $x \in X$.

Using the universal property, we define algebra maps

$$\Delta : \Bbbk\langle X\rangle \to \Bbbk\langle X\rangle \otimes \Bbbk\langle X\rangle, \quad \varepsilon : \Bbbk\langle X\rangle \to \Bbbk, \quad \mathcal{S} : \Bbbk\langle X\rangle \to \Bbbk\langle X\rangle^{\mathrm{op}}$$

with

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad \mathcal{S}(x) = -x$$

for all $x \in X$. It follows from Proposition 1.2.23 that $(\Bbbk\langle X\rangle, \Delta, \varepsilon, \mathcal{S})$ is a Hopf algebra. Explicitly, one obtains for all $x_1, \dots, x_n \in X$, $n \geq 1$,

$$\begin{aligned}\Delta(x_1 \cdots x_n) &= (1 \otimes x_1 + x_1 \otimes 1) \cdots (1 \otimes x_n + x_n \otimes 1)\\ &= \sum_{i=0}^{n} \sum_{w\ i\text{-shuffle}} x_{w(1)} \cdots x_{w(i)} \otimes x_{w(i+1)} \cdots x_{w(n)}.\end{aligned}$$

This formula follows easily since the elements $1 \otimes x_i$ and $x_j \otimes 1$ commute for all i, j.

Example 1.2.25. Let V be a vector space. For all natural numbers $n \geq 0$ let $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}$, where $V^{\otimes 0} = \Bbbk$. The **tensor algebra** of V is the vector space

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

with multiplication given by

$$V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes(m+n)}, \ x \otimes y \mapsto x \otimes y,$$

for all $m, n \geq 0$. We also write $T^n(V)$ for $V^{\otimes n}$ for all $n \geq 0$. Up to an isomorphism depending on the choice of a basis $(x_i)_{i \in I}$ of V, the tensor algebra is the free algebra in $X = \{x_i \mid i \in I\}$. The algebra map

$$\Bbbk\langle X\rangle \to T(V), \quad x_i \mapsto x_i, \quad i \in I,$$

is an isomorphism.

As in Example 1.2.24, $T(V)$ is a Hopf algebra with

$$\Delta(v) = 1 \otimes v + v \otimes 1, \quad \varepsilon(v) = 0, \quad \mathcal{S}(v) = -v$$

for all $v \in V$.

We end this section with some general definitions.

Definition 1.2.26. (1) An $\mathbb{N}_0$**-graded coalgebra** is a pair $(C, \mathcal{C})$, where C is a coalgebra, $(C, \mathcal{C})$ is an $\mathbb{N}_0$-graded vector space, and

$$\Delta(C(n)) \subseteq \bigoplus_{r+s=n} C(r) \otimes C(s) \text{ for all } n \geq 0, \tag{1.2.3}$$

$$\varepsilon(C(n)) = 0 \text{ for all } n > 0. \tag{1.2.4}$$

We write

$$\Delta_{m,n} : C(m+n) \subseteq C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi_m^C \otimes \pi_n^C} C(m) \otimes C(n), \ m, n \in \mathbb{N}_0,$$

for the **components of the comultiplication** Δ.

(2) An $\mathbb{N}_0$**-graded algebra** is a pair $(A, \mathcal{A})$, where A is an algebra, $(A, \mathcal{A})$ is an $\mathbb{N}_0$-graded vector space, and

$$A(m)A(n) \subseteq A(m+n) \text{ for all } m, n \geq 0, \tag{1.2.5}$$

$$1_A \in A(0). \tag{1.2.6}$$

The **components of the multiplication** are

$$\mu_{m,n} : A(m) \otimes A(n) \to A(m+n),\ x \otimes y \mapsto xy,\ m, n \geq 0.$$

(3) An $\mathbb{N}_0$**-graded bialgebra** H is a bialgebra and an $\mathbb{N}_0$-graded vector space $(H, \mathcal{H})$ such that H is an $\mathbb{N}_0$-graded algebra and an $\mathbb{N}_0$-graded coalgebra with respect to $\mathcal{H}$. An $\mathbb{N}_0$**-graded Hopf algebra** is an $\mathbb{N}_0$-graded bialgebra which is a Hopf algebra.

Corollary 1.2.27. *Let C be an $\mathbb{N}_0$-graded coalgebra, and A an $\mathbb{N}_0$-graded algebra. If $f \in \mathrm{Hom}(C, A)$ is an invertible graded map, then its inverse f^{-1} is graded.*

Proof. By Proposition 1.2.11, $\Phi(f)$ and f^{-1} are graded. □

By Corollary 1.2.27, the antipode of an $\mathbb{N}_0$-graded Hopf algebra is graded.

We note that in Example 1.2.25, $T(V)$ is an $\mathbb{N}_0$-graded Hopf algebra with grading $(T^n(V))_{n \geq 0}$.

1.3. Strictly graded coalgebras

Definition 1.3.1. An $\mathbb{N}_0$**-filtered coalgebra** is a pair $(C, \mathcal{F}(C))$, where C is a coalgebra and $\mathcal{F}(C) = (F_n(C))_{n \geq 0}$ is a family of subspaces $F_n(C) \subseteq C$, $n \geq 0$, such that

$$F_m(C) \subseteq F_n(C) \text{ for all } 0 \leq m \leq n, \tag{1.3.1}$$

$$C = \bigcup_{n \geq 0} F_n(C), \tag{1.3.2}$$

$$\Delta(F_n(C)) \subseteq \sum_{r+s \leq n} F_r(C) \otimes F_s(C) \text{ for all } n \geq 0. \tag{1.3.3}$$

Note that the subspaces $F_n(C) \subseteq C$, $n \geq 0$, of a filtered coalgebra are subcoalgebras. If $(C, (C(n))_{n \geq 0})$ is an $\mathbb{N}_0$-graded coalgebra, then $(C, \mathcal{F}(C))$ is an $\mathbb{N}_0$-filtered coalgebra with $F_n(C) = \bigoplus_{m=0}^{n} C(m)$ for all $n \geq 0$.

We want to prove two useful results about filtered coalgebras. We first look at their simple subcoalgebras. A coalgebra C is called **simple** if $C \neq 0$, and if 0 and C are the only subcoalgebras of C.

Proposition 1.3.2. *Let $(C, \mathcal{F}(C))$ be an $\mathbb{N}_0$-filtered coalgebra. Then any simple subcoalgebra of C is contained in $F_0(C)$.*

Proof. Let $D \subseteq C$ be a simple subcoalgebra. Since $F_0(C) \cap D$ is a subcoalgebra of C by Proposition 1.1.9, it is enough to prove that $F_0(C) \cap D$ is non-zero. Let $n \geq 0$ be minimal such that $F_n(C) \cap D \neq 0$, and let $x \in F_n(C) \cap D$ with $x \neq 0$. If $\Delta(x) \in F_0(C) \otimes D$, then $x = (\mathrm{id} \otimes \varepsilon)\Delta(x) \in F_0(C)$, and we are done. If $\Delta(x) \notin F_0(C) \otimes D$, then there exists $f \in C^* = \mathrm{Hom}(C, \Bbbk)$ such that $f(x_{(1)})x_{(2)} \neq 0$ and $f(F_0(C)) = 0$. Since $f(x_{(1)})x_{(2)} \in F_{n-1}(C) \cap D$, we obtain a contradiction to the minimality of n. □

We introduce at this point a basic coalgebra notion.

Definition 1.3.3. A coalgebra C is called **pointed** if every simple subcoalgebra of C is one-dimensional.

If C is a one-dimensional coalgebra, then there is a unique group-like element 1_C in C, and $C = \Bbbk 1_C$. In this section we study pointed coalgebras with a unique group-like element.

The main examples of coalgebras and Hopf algebras which appear in this book are pointed. We will say more on pointed coalgebras and Hopf algebras in Sections 2.4 and 5.4.

COROLLARY 1.3.4. *Let $(C, \mathcal{F}(C))$ be an $\mathbb{N}_0$-filtered coalgebra. If $F_0(C)$ is one-dimensional, then $F_0(C)$ is the unique simple subcoalgebra of C. The coalgebra C then has a unique group-like element which spans $F_0(C)$.*

PROOF. The subcoalgebra $F_0(C)$ is one-dimensional, hence simple. Thus the claim follows from Proposition 1.3.2. $\square$

We prove Takeuchi's criterion for invertibility in $\mathrm{Hom}(C, A)$.

PROPOSITION 1.3.5. *Let $(C, \mathcal{F})$ be a filtered coalgebra and assume that $F_0(C)$ is one-dimensional with unique group-like element 1_C. Let A be an algebra and $f : C \to A$ a linear map with $f(1_C) = 1$. Then f is invertible in* $\mathrm{Hom}(C, A)$ *with respect to convolution, and its inverse is*

$$f^{-1} = \sum_{n \geq 0} (\eta\varepsilon - f)^n.$$

PROOF. Let $g = \eta\varepsilon - f$. We first show that $\sum_{n\geq 0} g^n$ is well-defined. Let $m \geq 0$, and $x \in F_m(C)$. Then for all $n > m$,

$$g^n(x) \in \sum_{k_1+\cdots+k_n \leq m} g(F_{k_1}(C)) \cdots g(F_{k_n}(C)) = 0,$$

since $g(F_0(C)) = 0$. Hence $\sum_{n\geq 0} g^n(x) = \sum_{n=0}^{m} g^n(x)$. Then in the algebra $\mathrm{Hom}(C, A)$,

$$\begin{aligned}
\Big(f \sum_{n\geq 0} (\eta\varepsilon - f)^n\Big)(x) &= \Big((\eta\varepsilon - g) \sum_{n\geq 0} g^n\Big)(x) \\
&= (\varepsilon(x_{(1)}) - g(x_{(1)})) \sum_{n=0}^{m} g^n(x_{(2)}) \\
&= \sum_{n=0}^{m} g^n(x) - \sum_{n=0}^{m} g^{n+1}(x) \\
&= \eta\varepsilon(x).
\end{aligned}$$

The equation $(\sum_{n\geq 0} (\eta\varepsilon - f))f = \eta\varepsilon$ follows in the same way. $\square$

Let C be a coalgebra with exactly one group-like element, which we call $1_C = 1$. The space of **primitive elements** of C is defined by

$$P(C) = P_{1,1}(C) = \{x \in C \mid \Delta(x) = 1 \otimes x + x \otimes 1\}.$$

Note that $\varepsilon(x) = 0$ for each $x \in P(C)$ by the counit axiom.

The **primitive elements of a bialgebra** H are the elements in

$$P(H) = P_{1,1}(H) = \{x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1\}.$$

Let C be an $\mathbb{N}_0$-graded coalgebra. We call C **connected** if $C(0)$ is one-dimensional. Then $F_n(C) = \bigoplus_{i=0}^{n} C(i)$, $n \geq 0$, is a coalgebra filtration of C

with one-dimensional $F_0(C) = \Bbbk 1$, and 1 is the unique group-like element of C. If C is connected, then $P(C) \subseteq C$ is a graded subspace, since $P(C)$ is the kernel of the graded map $C \to C \otimes C$, $x \mapsto \Delta(x) - 1 \otimes x - x \otimes 1$.

LEMMA 1.3.6. (1) *Let $(C, \mathcal{F}(C))$ be an $\mathbb{N}_0$-filtered coalgebra. Assume that $F_0(C) = \Bbbk 1$ is one-dimensional. Let $n \geq 1$ and $x \in F_n(C)$. Then*

$$\Delta(x) \in 1 \otimes x + x \otimes 1 + F_{n-1}(C) \otimes F_{n-1}(C).$$

(2) *Let C be a connected $\mathbb{N}_0$-graded coalgebra. Then*

$$\Delta(x) \in 1 \otimes x + x \otimes 1 + \bigoplus_{i=1}^{n-1} C(i) \otimes C(n-i)$$

for all $n \geq 1$ and $x \in C(n)$. In particular, $C(1) \subseteq P(C)$.

(3) *Let C be an $\mathbb{N}_0$-graded coalgebra. Then the maps $\Delta_{0,n}$ and $\Delta_{n,0}$ are injective for all $n \geq 0$.*

PROOF. (1) Since $\mathcal{F}(C)$ is a coalgebra filtration with $F_0(C) = \Bbbk 1$, there exist $y, z \in F_n(C)$ such that $\Delta(x) - 1 \otimes y - z \otimes 1 \in F_{n-1}(C) \otimes F_{n-1}(C)$. Then

$$\Delta(x) - 1 \otimes x - x \otimes 1 - 1 \otimes (y - x) - (z - x) \otimes 1 \in F_{n-1}(C) \otimes F_{n-1}(C).$$

By the counit axioms, $x - y - \varepsilon(z)1 \in F_{n-1}(C)$ and $x - z - \varepsilon(y)1 \in F_{n-1}(C)$. Since $n \geq 1$, this implies (1).

(2) Let $n \geq 1$ and $x \in C(n)$. Since C is a connected graded coalgebra, there exist $y, z \in C(n), w \in \bigoplus_{i=1}^{n-1} C(i) \otimes C(n-i)$ such that $\Delta(x) = 1 \otimes y + z \otimes 1 + w$. By applying $\mathrm{id} \otimes \varepsilon$ and $\varepsilon \otimes \mathrm{id}$ to this equation we see that $x = y = z$. In particular, $C(1) \subseteq P(C)$.

(3) Let $n \geq 0$ and $x \in C(n)$. Then $\Delta(x) = \sum_{i=0}^{n} \Delta_{i,n-i}(x)$, hence

$$x = (\mathrm{id}_C \otimes \varepsilon)\Delta(x) = (\mathrm{id}_C \otimes \varepsilon)(\Delta_{n,0}(x)) = (\varepsilon \otimes \mathrm{id}_C)(\Delta_{0,n}(x))$$

since $\varepsilon(C(i)) = 0$ for all $i \geq 1$. This implies the claim. □

In general, a connected $\mathbb{N}_0$-graded coalgebra has non-zero primitive elements in degrees ≥ 2.

EXAMPLE 1.3.7. If H is a bialgebra, then for all $x, y \in P(H)$, the commutator $[x, y] = xy - yx$ is a primitive element in H. In particular, in the free algebra in Example 1.2.24 iterated commutators of the primitive generators are primitive.

EXAMPLE 1.3.8. Let $H = \Bbbk[x]$ be the polynomial algebra in one variable x. Then H is an $\mathbb{N}_0$-graded coalgebra (and bialgebra) with

$$H(n) = \Bbbk x^n,\ \Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i},\ \varepsilon(x^n) = \delta_{0n} \quad \text{for all } n \geq 0.$$

Note that H is the universal enveloping algebra of the one-dimensional abelian Lie algebra. Assume that the characteristic of $\Bbbk$ is 0. Then it is easy to see (and follows from the Theorem of Poincaré, Birkhoff, Witt) that $P(H) = H(1)$. But if the characteristic of $\Bbbk$ is $p > 0$, then for all $m \geq 1$ the binomial coefficients $\binom{p^m}{i}$ are zero for all $1 \leq i \leq p^m - 1$, hence x^{p^m} is primitive.

DEFINITION 1.3.9. ([**Swe69**, Section 11.2]) An $\mathbb{N}_0$-graded coalgebra is called **strictly graded** if it is connected with $P(C) = C(1)$.

The next proposition is a very special case of the following theorem of Heynemann and Radford: If $f : C \to D$ is a homomorphism of coalgebras such that the restriction of f to the first part C_1 of the coradical filtration is injective, then f is injective. See [**Mon93**, Theorem 5.3.1] for a proof of this result.

PROPOSITION 1.3.10. *Let $(C, \mathcal{F}(C))$ be an $\mathbb{N}_0$-filtered coalgebra and assume that $F_0(C) = \Bbbk 1$ is one-dimensional.*

(1) *Let $0 \neq I \subseteq C$ be a coideal. Then $I \cap P(C) \neq 0$.*
(2) *Let D be a coalgebra, and $f : C \to D$ a coalgebra homomorphism such that $f|P(C)$ is injective. Then f is injective.*

PROOF. The homomorphism theorem for coalgebras, Proposition 1.1.13, implies that (1) and (2) are equivalent. We prove (2). We show by induction on n that $f|F_n(C)$ is injective for all n. If $n = 0$, then $f|F_0(C)$ is injective, since $1 = \varepsilon(f(1))$. Let $n \geq 1$ and assume that $f|F_{n-1}(C)$ is injective. Let $x \in F_n(C)$ with $f(x) = 0$. By Lemma 1.3.6(1) there is an element $w \in F_{n-1}(C) \otimes F_{n-1}(C)$ such that $\Delta(x) = 1 \otimes x + x \otimes 1 + w$. Then

$$0 = \Delta(f(x)) = f(1) \otimes f(x) + f(x) \otimes f(1) + (f \otimes f)(w).$$

Thus $(f \otimes f)(w) = 0$, and hence $w = 0$ by Lemma 1.1.11 and by induction. Therefore $x \in P(C)$ and then $x = 0$ by the injectivity of $f|P(C)$. $\square$

COROLLARY 1.3.11. *Let C be a strictly graded coalgebra.*

(1) *Let $0 \neq I \subseteq C$ be a coideal. Then $I \cap C(1) \neq 0$.*
(2) *Let D be a coalgebra, and $f : C \to D$ a coalgebra homomorphism such that $f|C(1)$ is injective. Then f is injective.*
(3) *Let $0 \neq E \subseteq C$ be a subspace with $E \cap C(1) = 0$. Assume $\Delta(E) \subseteq E \otimes C$ or $\Delta(E) \subseteq C \otimes E$. Then $E = \Bbbk 1_C$.*

PROOF. (1) and (2) follow from Proposition 1.3.10 using the coalgebra filtration $\mathcal{F}(C)$ with $F_n(C) = \bigoplus_{i=0}^n C(n)$ for all $n \geq 0$, since $P(C) = C(1)$.

(3) By Lemma 1.1.14, $E \cap \ker(\varepsilon)$ is a coideal of C and $E \not\subseteq \ker(\varepsilon)$. Then $E \cap \ker(\varepsilon) = 0$ by (1), and hence E is one-dimensional. Since C is connected, we conclude that $E = \Bbbk 1_C$. $\square$

We will characterize strictly graded coalgebras in terms of the components of the graded map Δ and of its iterations.

DEFINITION 1.3.12. Let $C = \bigoplus_{n \in \mathbb{N}_0} C(n)$ be a graded coalgebra with projections $\pi_n = \pi_n^C$ for all $n \geq 0$. For all $n \geq 1$ we denote the $(1, \ldots, 1)$-th component of Δ^{n-1} by

$$\Delta_{1^n} : C(n) \subseteq C \xrightarrow{\Delta^{n-1}} C^{\otimes n} \xrightarrow{\pi_1^{\otimes n}} C(1)^{\otimes n}. \tag{1.3.4}$$

Let $I_C(n) = \ker(\Delta_{1^n})$ for all $n \geq 1$, and

$$I_C = \bigoplus_{n \geq 1} I_C(n) = \bigoplus_{n \geq 2} I_C(n).$$

Note that $I_C(1) = 0$ since $\Delta_1 = \mathrm{id}$.

LEMMA 1.3.13. *Let C be an $\mathbb{N}_0$-graded coalgebra.*

(1) (a) *Let* $n \geq 1$ *and* $m \geq 0$. *Then*

$$\pi_1^{\otimes n} \Delta^{n-1}|C(m) = \begin{cases} \Delta_{1^n} & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

(b) *Let* $1 \leq i \leq n-1$. *Then* $\Delta_{1^n} = (\Delta_{1^i} \otimes \Delta_{1^{n-i}})\Delta_{i,n-i}$.

(2) *Assume that* C *is connected. Then* $I_C \subseteq C$ *is a coideal of* C.

PROOF. (1)(a) Since Δ is graded,

$$\Delta^{n-1}(C(m)) \subseteq \bigoplus_{i_1+\cdots+i_n=m} C(i_1) \otimes \cdots \otimes C(i_n).$$

Thus $\pi_1^{\otimes n}\Delta^{n-1}|C(m) = 0$ if $m \neq n$.

To prove (1)(b) let $n \geq 2$ and $x \in C(n)$. Then $\Delta(x) = \sum_{j=0}^n \Delta_{j,n-j}(x)$ by definition of the components of Δ. Note that $\Delta^{n-1} = (\Delta^{i-1} \otimes \Delta^{n-i-1})\Delta$ for all $1 \leq i \leq n-1$ by coassociativity. Hence

$$\begin{aligned}
\Delta_{1^n}(x) &= \pi_1^{\otimes n}\Delta^{n-1}(x) \\
&= \pi_1^{\otimes n}(\Delta^{i-1} \otimes \Delta^{n-i-1})\Big(\sum_{j=0}^n \Delta_{j,n-j}(x)\Big) \\
&= \sum_{j=0}^n (\pi_1^{\otimes i}\Delta^{i-1} \otimes \pi_1^{\otimes(n-i)}\Delta^{n-i-1})(\Delta_{j,n-j}(x)) \\
&= (\Delta_{1^i} \otimes \Delta_{1^{n-i}})\Delta_{i,n-i}(x),
\end{aligned}$$

where the last equality holds by (1)(a).

(2) Let $n \geq 2$, $x \in I_C(n)$ and $i \in \{1, \ldots, n-1\}$. By (1)(b),

$$0 = \Delta_{1^n}(x) = (\Delta_{1^i} \otimes \Delta_{1^{n-i}})\Delta_{i,n-i}(x).$$

Hence $\Delta_{i,n-i}(x) \in \ker(\Delta_{1^i} \otimes \Delta_{1^{n-i}}) = C(i) \otimes I_C(n-i) + I_C(i) \otimes C(n-i)$ by Lemma 1.1.11. Therefore

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \sum_{i=1}^{n-1} \Delta_{i,n-i}(x) \in C \otimes I_C + I_C \otimes C$$

by Lemma 1.3.6(2). □

PROPOSITION 1.3.14. *Let* C *be an* $\mathbb{N}_0$*-graded coalgebra.*

(1) *The following are equivalent.*
  (a) *For all* $n \geq 2$, $\Delta_{1^n} : C(n) \to C(1)^{\otimes n}$ *is injective.*
  (b) *For all* $i, j \geq 0$, $\Delta_{i,j} : C(i+j) \to C(i) \otimes C(j)$ *is injective.*
  (c) *For all* $n \geq 2$, $\Delta_{n-1,1} : C(n) \to C(n-1) \otimes C(1)$ *is injective.*
  (d) *For all* $n \geq 2$, $\Delta_{1,n-1} : C(n) \to C(1) \otimes C(n-1)$ *is injective.*

(2) *Assume that* C *is connected. Then the following are equivalent.*
  (a) C *is strictly graded.*
  (b) *Conditions (a) – (d) in (1).*
  (c) $I_C = 0$.

PROOF. (1) (a) ⇒ (b): By Lemma 1.3.13(1b), $\Delta_{i,j}$ is injective for all $i, j \geq 1$. This proves (b) by Lemma 1.3.6(3).

(b) ⇒ (c) and (b) ⇒ (d) are trivial.

(d) $\Rightarrow$ (a) follows by induction on n, since by Lemma 1.3.13(1b),

$$\Delta_{1^n} = (\mathrm{id}_{C(1)} \otimes \Delta_{1^{n-1}})\Delta_{1,n-1}$$

for all $n \geq 2$. The implication (c) $\Rightarrow$ (a) is shown similarly.

(2) By definition of I_C, (1a) holds if and only if $I_C = 0$. Assume that C is strictly graded. By Lemma 1.3.13(2), I_C is a coideal of C. Hence $I_C = 0$ by Corollary 1.3.11(1). Conversely, assume that $I_C = 0$. Then for all $n \geq 2$ and $x \in C(n) \cap P(C)$, $\Delta_{n-1,1}(x) = 0$, and $x = 0$ by (1c). Thus C is strictly graded. $\square$

DEFINITION 1.3.15. Let C be a connected $\mathbb{N}_0$-graded coalgebra. The coalgebra $\mathcal{B}(C) = C/I_C$ is called the **associated strictly graded coalgebra** to C. Let $\pi_C : C \to \mathcal{B}(C)$ denote the canonical graded coalgebra map.

The next theorem gives a characterization of the coalgebra $\mathcal{B}(C)$.

THEOREM 1.3.16. *Let C be a connected $\mathbb{N}_0$-graded coalgebra.*

(1) *The coideal I_C is the only graded coideal I of C such that*
   (a) *C/I is strictly graded, and*
   (b) *$\pi(1) : C(1) \to (C/I)(1)$ is bijective, where $\pi : C \to C/I$ is the canonical map.*
(2) *The coideal I_C is the largest coideal of C contained in $\bigoplus_{n\geq 2} C(n)$.*
(3) *The coideal I_C is the only coideal I of C contained in $\bigoplus_{n\geq 2} C(n)$ such that $P(C/I) = C(1)$.*
(4) *Let D be an $\mathbb{N}_0$-graded coalgebra and $\pi : C \to D$ a surjective graded coalgebra map such that $\pi(1) : C(1) \to D(1)$ is bijective. Then there is exactly one graded coalgebra map $\widetilde{\pi} : D \to \mathcal{B}(C)$ with $\pi_C = \widetilde{\pi}\pi$.*

PROOF. We first show that I_C satisfies (1)(a) and (1)(b). By Lemma 1.3.13(2), $I_C \subseteq C$ is a graded coideal of C. By definition, the grading of $\mathcal{B}(C)$ is given by $\mathcal{B}(C) = \Bbbk 1 \oplus C(1) \oplus \bigoplus_{n\geq 2} C(n)/I_C(n)$. Thus (1)(b) holds. To prove that $\mathcal{B}(C)$ is strictly graded we use Proposition 1.3.14(2). We show that $\Delta^{\mathcal{B}(C)}_{1^n}$ is injective for all $n \geq 2$. Let $n \geq 2$. Since $\pi_C : C \to \mathcal{B}(C) = C/I_C$ is a graded coalgebra map and $C(1) = \mathcal{B}(C)(1)$,

$$\Delta^C_{1^n} = \left(C(n) \xrightarrow{\pi_C(n)} C(n)/I_C(n) \xrightarrow{\Delta^{\mathcal{B}(C)}_{1^n}} C(1)^{\otimes n} \right).$$

Hence $\Delta^{\mathcal{B}(C)}_{1^n}$ is injective, since by definition, $I_C(n) = \ker(\Delta^C_{1^n})$.

(2) Let $J \subseteq C$ be the sum of all coideals of C contained in $\bigoplus_{n\geq 2} C(n)$. Then J is the largest coideal of C contained in $\bigoplus_{n\geq 2} C(n)$. Hence $I_C \subseteq J$, and the induced map $f : C/I_C \to C/J$ is a coalgebra map which is injective when restricted to $C/I_C(1) = C(1)$. Since C/I_C is strictly graded, f is injective by Corollary 1.3.11(2). Thus $I_C = J$.

(3) By the first paragraph of the proof, $P(C/I_C) = C(1)$. Let I be a coideal of C contained in $\bigoplus_{n\geq 2} C(n)$ with $P(C/I) = C(1)$. Then $I \subseteq I_C$ by (2). The induced coalgebra homomorphism $C/I \to C/I_C$ is injective by Proposition 1.3.10(2), since it is injective on $P(C/I)$. Note that the image of the natural filtration of C is a coalgebra filtration of C/I with one-dimensional $F_0(C/I)$.

(4) Let $I = \ker(\pi)$. Then $I \subseteq C$ is a graded coideal. By assumption, $I(1) = 0$. Further, $I(0) = 0$ since C is connected and $\varepsilon(1_C) = 1$. Hence $I \subseteq I_C$ by (2). This proves existence and the uniqueness of $\widetilde{\pi}$, since π is surjective.

To finish the proof of (1), we have to show that each coideal I of C satisfying (a) and (b) coincides with I_C. Let $I \subseteq C$ be such a coideal. Then $I \subseteq I_C$ by (2), and the induced map $C/I \to C/I_C$ is bijective by Corollary 1.3.11(2). Hence $I = I_C$. $\square$

We finally note a useful property of the tensor product of strictly graded coalgebras.

PROPOSITION 1.3.17. *Let C, D be strictly $\mathbb{N}_0$-graded coalgebras. Assume that the tensor product $C \otimes D$ of the vector spaces C, D has a coalgebra structure with comultiplication $\Delta_{C\otimes D}$ and counit $\varepsilon_{C\otimes D} = \varepsilon_C \otimes \varepsilon_D$ such that*

(1) *$(C \otimes D, \Delta_{C\otimes D}, \varepsilon_{C\otimes D})$ is an $\mathbb{N}_0$-graded coalgebra with grading*
$$(C \otimes D)(n) = \bigoplus_{i+j=n} C(i) \otimes D(j) \text{ for all } n \geq 0,$$
(2) *$(\mathrm{id}_C \otimes \varepsilon_D \otimes \varepsilon_C \otimes \mathrm{id}_D)\Delta_{C\otimes D} = \mathrm{id}_{C\otimes D}$,*
(3) *$\mathrm{id}_C \otimes \varepsilon_D : C \otimes D \to C \otimes \Bbbk \cong C$ and $\varepsilon_C \otimes \mathrm{id}_D : C \otimes D \to \Bbbk \otimes D \cong D$ are coalgebra maps.*

Then $C \otimes D$ is a strictly graded coalgebra.

PROOF. Let $n \geq 2$ and $x \in (C \otimes D)(n)$ a primitive element. We write
$$x = 1_C \otimes d + y + c \otimes 1_D,\ c \in C(n),\ d \in D(n),\ y \in \bigoplus_{i=1}^{n-1} C(i) \otimes D(n-i).$$
By assumption,
$$\Delta(x) = x \otimes 1_C \otimes 1_D + 1_C \otimes 1_D \otimes x \in C \otimes D \otimes C \otimes D.$$
We apply $f = \mathrm{id}_C \otimes \varepsilon_D \otimes \varepsilon_C \otimes \mathrm{id}_D$ to both sides of this equation. Then by (2), $f\Delta(x) = x$. Hence $x = 1_C \otimes d + c \otimes 1_D$. Moreover, $c = (\mathrm{id}_C \otimes \varepsilon_D)(x) \in P(C)$ and $d = (\varepsilon_C \otimes \mathrm{id}_D)(x) \in P(C)$ by (3). Hence $c = 0$, $d = 0$ and $x = 0$, since C and D are strictly graded. $\square$

Proposition 1.3.17 can be applied to the usual tensor product of coalgebras, but also to more general "braided tensor products".

1.4. Yetter-Drinfeld modules over a group algebra

In this section, let G be a group. We write $g \triangleright h = ghg^{-1}$, $g, h \in G$, for the adjoint action of G on itself. The center of G is denoted by $Z(G)$.

If V is a left $\Bbbk G$-module, and $\chi \in \widehat{G} = \mathrm{Gr}\,(G, \Bbbk^\times)$ is a character of G, we define $V^\chi = \{v \in V \mid gv = \chi(g)v \text{ for all } g \in G\}$.

DEFINITION 1.4.1. A **Yetter-Drinfeld module over the group algebra** $\Bbbk G$ is a G-graded vector space $V = \bigoplus_{g\in G} V_g$, and a left $\Bbbk G$-module with module structure $\Bbbk G \otimes V \to V$, $g \otimes v \mapsto g \cdot v$, where $g \in G$, such that

$$g \cdot V_h \subseteq V_{g\triangleright h} \text{ for all } g, h \in G. \tag{1.4.1}$$

We denote the category of Yetter-Drinfeld modules over the group algebra $\Bbbk G$ by ${}^G_G\mathcal{YD}$. Objects of ${}^G_G\mathcal{YD}$ are the Yetter-Drinfeld modules over $\Bbbk G$, morphisms are the G-graded and G-linear maps. Let ${}^G_G\mathcal{YD}^{\mathrm{fd}}$ be the full subcategory of ${}^G_G\mathcal{YD}$ of finite-dimensional objects.

If V is a Yetter-Drinfeld module over $\Bbbk G$, then $g \cdot V_h = V_{g \triangleright h}$ for all $g, h \in G$, since $g \cdot V_h \subseteq V_{g \triangleright h}$ and $g^{-1} \cdot V_{g \triangleright h} \subseteq V_h$. If G is abelian, then Yetter-Drinfeld modules over $\Bbbk G$ are G-graded vector spaces and G-modules such that each homogeneous component is stable under the action of G.

EXAMPLE 1.4.2. Assume that G is abelian. Let $h \in G$. Then any $\Bbbk G$-module U is a Yetter-Drinfeld module over $\Bbbk G$ with $U = U_h$. On the other hand, let V be a non-zero Yetter-Drinfeld module over $\Bbbk G$. Then there is an $h \in G$ such that $V_h \neq 0$. Moreover, for any $h \in G$ the subspace V_h is a Yetter-Drinfeld submodule of V and any subspace of V_h is a $\Bbbk G$-submodule of V_h if and only if it is a Yetter-Drinfeld submodule. In particular, the set of isomorphism classes of irreducible Yetter-Drinfeld modules over $\Bbbk G$ is in bijection to $G \times \operatorname{Irrep} G$, where $\operatorname{Irrep} G$ is the set of isomorphism classes of simple $\Bbbk G$-modules.

EXAMPLE 1.4.3. Let us determine one-dimensional Yetter-Drinfeld modules $V = \Bbbk x \in {}^G_G\mathcal{YD}$. The action on V and the degree of x are given by a character $\chi \in \widehat{G} = \operatorname{Gr}(G, \Bbbk^\times)$ and an element $g \in G$ with

$$h \cdot x = \chi(h)x, \quad x \in V_g,$$

for all $h \in G$. The Yetter-Drinfeld condition (1.4.1) holds if and only if for all $h \in G$, $hgh^{-1} = \deg(h \cdot x) = \deg(\chi(h)x) = g$, that is, if and only if $g \in Z(G)$. Thus there is a bijection between the set of isomorphism classes of one-dimensional Yetter-Drinfeld modules in ${}^G_G\mathcal{YD}$ and $Z(G) \times \widehat{G}$.

EXAMPLE 1.4.4. Assume that G is abelian, and $\Bbbk$ is algebraically closed. Let V be a finite-dimensional irreducible $\Bbbk G$-module, and let $\rho : \Bbbk G \to \operatorname{End}(V)$ be the representation of V. Then there is a common eigenvector for the set $\rho(\Bbbk G)$ of pairwise commuting endomorphisms. Hence V is one-dimensional.

It follows from the two previous examples that the finite-dimensional irreducible objects in ${}^G_G\mathcal{YD}$ are one-dimensional and given by elements in $G \times \widehat{G}$.

LEMMA 1.4.5. *Let G be an abelian group and $V \in {}^G_G\mathcal{YD}$. Then the following are equivalent:*

(1) *V is a direct sum of one-dimensional Yetter-Drinfeld modules in ${}^G_G\mathcal{YD}$.*
(2) *V is a direct sum of one-dimensional G-modules.*

PROOF. Clearly, (1) implies (2). Assume now (2). Since G is abelian, the comodule decomposition $V = \bigoplus_{g \in G} V_g$ is a decomposition of G-modules. By (2), all direct summands V_g, $g \in G$, are direct sums of one-dimensional Yetter-Drinfeld modules. □

PROPOSITION 1.4.6. *Let G be a finite abelian group and $V \in {}^G_G\mathcal{YD}^{\mathrm{fd}}$. Assume that $\Bbbk$ is algebraically closed and that* $\operatorname{char}(\Bbbk)$ *does not divide the order of G.*

(1) *Any finite-dimensional $\Bbbk G$-module is a direct sum of one-dimensional $\Bbbk G$-modules.*
(2) *Any $V \in {}^G_G\mathcal{YD}^{\mathrm{fd}}$ is the direct sum of one-dimensional Yetter-Drinfeld modules.*

PROOF. (1) is well-known (and follows from the Theorem of Maschke and Example 1.4.4), and (2) follows from (1) and Lemma 1.4.5. □

EXAMPLE 1.4.7. We denote the symmetric group of n elements $\{1, \dots, n\}$ by $\mathbb{S}_n$. Let $\mathcal{O}_2 = \{(i\,j) \mid 1 \le i < j \le n\}$ be the set of all transpositions in $\mathbb{S}_n$, $n \ge 3$. Let V_n be the Yetter-Drinfeld module in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$ with basis x_t, $t \in \mathcal{O}_2$, and

$$\deg(x_t) = t,\ s \cdot x_t = \operatorname{sign}(s) x_{s \triangleright t} \text{ for all } t \in \mathcal{O}_2,\ s \in \mathbb{S}_n.$$

Note that V_n is irreducible in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$, since any non-zero subobject contains x_t for some t, and the elements $g \cdot x_t$ with $g \in \mathbb{S}_n$ span V_n, since $\mathcal{O}_2$ is a conjugacy class of $\mathbb{S}_n$.

REMARK 1.4.8. Yetter-Drinfeld modules V in ${}^G_G\mathcal{YD}$ can equivalently be defined as left $\Bbbk G$-modules with a left $\Bbbk G$-comodule structure

$$\delta : V \to \Bbbk G \otimes V,\ v \mapsto v_{(-1)} \otimes v_{(0)}, \text{ such that}$$
$$\delta(g \cdot v) = g v_{(-1)} g^{-1} \otimes g \cdot v_{(0)}$$

for all $v \in V$, $g \in G$. This follows from the category isomorphism between G-graded vector spaces and $\Bbbk G$-comodules in Proposition 1.1.17.

Let $V, W \in {}^G_G\mathcal{YD}$. Note that $V \otimes W$ is an object in ${}^G_G\mathcal{YD}$ with diagonal action and diagonal coaction of G. The trivial object $\Bbbk$ with grading $\Bbbk = \Bbbk_e$ and G-action $g \cdot 1 = 1$ for all $g \in G$ is an object in ${}^G_G\mathcal{YD}$.

PROPOSITION 1.4.9. (1) *Let $V, W, V', W' \in {}^G_G\mathcal{YD}$. Then for all morphisms $f : V \to V'$ and $g : W \to W'$ in ${}^G_G\mathcal{YD}$, the tensor product $f \otimes g : V \otimes W \to V' \otimes W'$ is a morphism in ${}^G_G\mathcal{YD}$.*

(2) *For all $U, V, W \in {}^G_G\mathcal{YD}$ the canonical isomorphisms*

$$(U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W),\ \ \Bbbk \otimes V \xrightarrow{\cong} V,\ \ V \otimes \Bbbk \xrightarrow{\cong} V$$

are morphisms in ${}^G_G\mathcal{YD}$.

PROOF. (1) is clear from the definition, and (2) is a special case of Proposition 1.2.5. $\square$

Let H be a bialgebra. Suppose that the canonical isomorphism of vector spaces

$$\tau_{V,W} : V \otimes W \xrightarrow{\cong} W \otimes V,\ v \otimes w \mapsto w \otimes v,$$

is H-linear for all left H-modules V, W and the diagonal action. Then H is cocommutative. Similarly, H is commutative, if $\tau_{V,W}$ is H-colinear for all left H-comodules V, W with the diagonal coaction.

Hence it is quite remarkable that a commutativity rule for objects in ${}^G_G\mathcal{YD}$ does exist. It is not the flip map $\tau_{V,W}$, but it is a natural isomorphism in ${}^G_G\mathcal{YD}$ which behaves like a commutativity law.

DEFINITION 1.4.10. For all $V, W \in {}^G_G\mathcal{YD}$ the linear map

$$c_{V,W} : V \otimes W \to W \otimes V \tag{1.4.2}$$

defined by $c_{V,W}(v \otimes w) = g \cdot w \otimes v$ for all $g \in G$, $v \in V_g$, and $w \in W$, is called the **braiding of** V, W.

PROPOSITION 1.4.11. (1) *For all $V, W \in {}^G_G\mathcal{YD}$, $c_{V,W} : V \otimes W \to W \otimes V$ is an isomorphism in ${}^G_G\mathcal{YD}$.*

(2) *For all objects U, V, W, V', W' in ${}^G_G\mathcal{YD}$ and all morphisms $f : V \to V'$, $g : W \to W'$ in ${}^G_G\mathcal{YD}$, the following diagrams commute.*

$$\begin{tikzcd} V \otimes W \arrow[r, "c_{V,W}"] \arrow[d, "f\otimes g"] & W \otimes V \arrow[d, "g\otimes f"] \\ V' \otimes W' \arrow[r, "c_{V',W'}"] & W' \otimes V' \end{tikzcd} \tag{1.4.3}$$

$$\begin{tikzcd} U \otimes V \otimes W \arrow[rr, "c_{U,V\otimes W}"] \arrow[dr, "c_{U,V}\otimes \mathrm{id}"'] & & V \otimes W \otimes U \\ & V \otimes U \otimes W \arrow[ur, "\mathrm{id}\otimes c_{U,W}"'] & \end{tikzcd} \tag{1.4.4}$$

$$\begin{tikzcd} U \otimes V \otimes W \arrow[rr, "c_{U\otimes V,W}"] \arrow[dr, "\mathrm{id}\otimes c_{V,W}"'] & & W \otimes U \otimes V \\ & U \otimes W \otimes V \arrow[ur, "c_{U,W}\otimes \mathrm{id}"'] & \end{tikzcd} \tag{1.4.5}$$

$$\begin{tikzcd} \Bbbk \otimes V \arrow[r, "c_{\Bbbk,V}"] \arrow[d, "\cong"] & V \otimes \Bbbk \arrow[d, "\cong"] \\ V \arrow[r, "="] & V \end{tikzcd} \qquad \begin{tikzcd} V \otimes \Bbbk \arrow[r, "c_{V,\Bbbk}"] \arrow[d, "\cong"] & \Bbbk \otimes V \arrow[d, "\cong"] \\ V \arrow[r, "="] & V \end{tikzcd} \tag{1.4.6}$$

(Note that Proposition 1.4.9 *is used in the formulation of* (2)*.)*

We will meet the diagrams of Proposition 1.4.11 later in Section 3.2 in the axioms of a braided monoidal category.

Proof. (1) To see that $c_{V,W}$ is G-linear and G-graded, let $g, h \in G$, and let $v \in V_g$, $w \in W_h$ be homogeneous elements. Then for all $a \in G$,

$$\begin{aligned} c_{V,W}(a \cdot (v \otimes w)) &= c_{V,W}(a \cdot v \otimes a \cdot w) \\ &= aga^{-1}a \cdot w \otimes a \cdot v = a \cdot c_{V,W}(v \otimes w), \\ \deg(c_{V,W}(v \otimes w)) &= \deg(g \cdot w \otimes v) = ghg^{-1}g = \deg(v \otimes w). \end{aligned}$$

The map $c_{V,W}$ is an isomorphism with inverse

$$c^{-1}_{V,W} : W \otimes V \to V \otimes W, \quad w \otimes v \mapsto v \otimes g^{-1} \cdot w,$$

for all $v \in V_g$, $g \in G$, and $w \in W$.

(2) The commutativity of the diagrams is easily checked on homogeneous elements. $\square$

Definition 1.4.12. Let G be an abelian group, and $\chi : G \times G \to \Bbbk^\times$ a **bicharacter of** G, that is, a mapping χ such that for all $f, g, h \in G$

$$\chi(f + g, h) = \chi(f, h)\chi(g, h), \quad \chi(f, g + h) = \chi(f, g)\chi(f, h).$$

Let ${}^G_\chi\mathcal{YD}$ be the full subcategory of ${}^G_G\mathcal{YD}$ whose objects are G-graded vector spaces $V = \bigoplus_{g\in G} V_g$ with G-action defined by $g \cdot v = \chi(g, h)v$ for all $v \in V_h$, $g, h \in G$.

Note that a bicharacter χ satisfies $\chi(g,0) = 1 = \chi(0,g)$ for all $g \in G$.

Let G be a free abelian group with basis $(\alpha_i)_{i\in I}$, and let $(q_{ij})_{i,j\in I}$ be a family of non-zero scalars in $\Bbbk$. Then

$$\chi : G \times G \to \Bbbk^\times,\ (\alpha_i, \alpha_j) \mapsto q_{ij} \text{ for all } i, j \in I,$$

defines a bicharacter of G.

PROPOSITION 1.4.13. *Let G be an abelian group and χ a bicharacter of G. Let $V, W \in {}^G_\chi\mathcal{YD}$.*

(1) *$V \otimes W \in {}^G_\chi\mathcal{YD}$ with diagonal G-grading and G-action. The trivial object $\Bbbk$ of ${}^G_G\mathcal{YD}$ is an object of ${}^G_\chi\mathcal{YD}$.*

(2) *The braiding $c = c_{V,W} : V \otimes W \to W \otimes V$ in ${}^G_G\mathcal{YD}$ is given by*

$$c(v \otimes w) = \chi(g, h)w \otimes v$$

for all $v \in V_g$, $w \in W_h$, $g, h \in G$.

PROOF. Let $f, g, h \in G$, and $v \in V_g$, $w \in W_h$. Then

$$f \cdot (v \otimes w) = f \cdot v \otimes f \cdot w = \chi(f, g)v \otimes \chi(f, h)w = \chi(f, g + h)v \otimes w.$$

This proves that $V \otimes W \in {}^G_\chi\mathcal{YD}$, and the remaining claims are obvious. □

If χ is a bicharacter of an abelian group, then Proposition 1.4.13 says that the subcategory ${}^G_\chi\mathcal{YD} \subseteq {}^G_G\mathcal{YD}$ is closed under tensor products.

EXAMPLE 1.4.14. Let $G = \mathbb{Z}/(2)$ and $\chi : \mathbb{Z}/(2) \times \mathbb{Z}/(2) \to \Bbbk^\times$ the non-trivial bicharacter with $\chi(\bar{i}, \bar{j}) = (-1)^{ij}$, $i, j \in \{0, 1\}$. Assume that $\operatorname{char}(\Bbbk) \neq 2$. Then $\mathcal{S} = {}^G_\chi\mathcal{YD}$ is called the category of **super vector spaces**. Objects of $\mathcal{S}$ are $\mathbb{Z}/(2)$-graded vector spaces $V = V_0 \oplus V_1$, where $V_i = V_{\bar{i}}$, $i \in \{0, 1\}$. For a homogeneous element $v \in V_i$ we write $|v| = i$. If $V, W \in \mathcal{S}$, then the grading of $V \otimes W$ is given by

$$(V \otimes W)_0 = V_0 \otimes W_0 \oplus V_1 \otimes W_1,\ \ (V \otimes W)_1 = V_0 \otimes W_1 \oplus V_1 \otimes W_0,$$

and the braiding $c_{V,W} : V \otimes W \to W \otimes V$ by

$$c(v \otimes w) = (-1)^{|v||w|}w \otimes v$$

for homogeneous elements $v \in V$, $w \in W$.

In the remainder of this section, we want to construct the objects in ${}^G_G\mathcal{YD}$ explicitly for arbitrary groups.

For an element $g \in G$ we denote the centralizer of g by

$$G^g = \{h \in G \mid hg = gh\},$$

and the conjugacy class of g by

$$\mathcal{O}_g = \{h \triangleright g \mid h \in G\}.$$

Let $\{\mathcal{O}_l \mid l \in L\}$ be the set of all conjugacy classes of G, and assume that $\mathcal{O}_k \neq \mathcal{O}_l$ for all $k \neq l$ in L.

Any Yetter-Drinfeld module $M \in {}^G_G\mathcal{YD}$ has a decomposition

$$M = \bigoplus_{l\in L} \bigoplus_{s\in\mathcal{O}_l} M_s \tag{1.4.7}$$

into a direct sum of Yetter-Drinfeld modules $\bigoplus_{s\in\mathcal{O}_l} M_s$, $l \in L$.

We first consider one conjugacy class $\mathcal{O} \subseteq G$. We denote by ${}^G_G\mathcal{YD}(\mathcal{O})$ the full subcategory ${}^G_G\mathcal{YD}(\mathcal{O})$ of ${}^G_G\mathcal{YD}$ consisting of all $M \in {}^G_G\mathcal{YD}$ with $M = \bigoplus_{s\in\mathcal{O}} M_s$. Choose an element $g \in G$. Thus $\mathcal{O} = \mathcal{O}_g$, and the map

$$G/G^g \to \mathcal{O}_g,\ \overline{h} = hG^g \mapsto h \triangleright g,$$

is bijective. Recall that $M_{h\triangleright g} = h \cdot M_g$ for all $M \in {}^H_H\mathcal{YD}(\mathcal{O}_g)$ and $h \in G$. We will see that M is completely determined by the G^g-module M_g.

DEFINITION 1.4.15. Let $g \in G$, and let V be a left $\Bbbk G^g$-module. Define

$$M(g, V) = \Bbbk G \otimes_{\Bbbk G^g} V$$

as an object in ${}^G_G\mathcal{YD}(\mathcal{O}_g)$, where $M(g, V)$ is the induced $\Bbbk G$-module, and the G-grading is given by

$$\deg(h \otimes v) = h \triangleright g \text{ for all } h \in G, v \in V.$$

Note that the grading is well-defined and $M(g, V)$ is a Yetter-Drinfeld module over G, since for all $v \in V$, $h \in G$ and $a \in G^g$,

$$\deg(ha \otimes v) = (ha) \triangleright g = h \triangleright g = \deg(h \otimes a \cdot v),$$

and since for all $v \in V$ and $h, h' \in G$,

$$\deg(h' \cdot (h \otimes v)) = \deg(h'h \otimes v) = (h'h) \triangleright g = h' \triangleright \deg(h \otimes v).$$

Let V, W be left $\Bbbk G^g$-modules, and $f : V \to W$ a left $\Bbbk G^g$-linear map. Then $\mathrm{id} \otimes f : M(g, V) \to M(g, W)$ is a morphism in ${}^G_G\mathcal{YD}$.

Thus we have defined a functor

$$F_g : {}_{\Bbbk G^g}\mathcal{M} \to {}^G_G\mathcal{YD}(\mathcal{O}_g)$$

with $F_g(V) = M(g, V)$ and $F_g(f) = \mathrm{id} \otimes f$ for all left $\Bbbk G^g$-modules V, W and all left $\Bbbk G^g$-linear maps $f : V \to W$.

LEMMA 1.4.16. *Let $g \in G$, $V \in {}_{\Bbbk G^g}\mathcal{M}$, and $M \in {}^G_G\mathcal{YD}(\mathcal{O}_g)$.*

(1) *The decomposition of $M(g, V)$ into G-homogeneous components is given by*

$$M(g, V) = \bigoplus_{s\in\mathcal{O}_g} M(g, V)_s, \quad M(g, V)_{h\triangleright g} = h \otimes V \text{ for all } h \in G.$$

(2) *$V \xrightarrow{\cong} M(g, V)_g$, $v \mapsto 1 \otimes v$, is a left $\Bbbk G^g$-linear isomorphism.*

(3) *$M(g, M_g) \xrightarrow{\cong} M$, $h \otimes m \mapsto h \cdot m$, is an isomorphism of Yetter-Drinfeld modules in ${}^G_G\mathcal{YD}$.*

PROOF. Let $(h_x)_{x\in X}$ be a complete set of representatives of the cosets in G/G^g, where X is a set of the same cardinality as $\mathcal{O}_g$. We can assume that $h_{x_0} = 1$ for some $x_0 \in X$. Since $\Bbbk G$ is a free right $\Bbbk G^g$-module with basis $(h_x)_{x\in X}$,

$$M(g, V) = \Bbbk G \otimes_{\Bbbk G^g} V = \bigoplus_{x\in X} h_x \otimes V. \tag{1.4.8}$$

By (1.4.8), $M(g, V)_{h_x\triangleright g} = h_x \otimes V$, since $h_x \otimes V \subseteq M(g, V)_{h_x\triangleright g}$ for all $x \in X$. In particular, $M(g, V)_g = 1 \otimes V$, and $V \xrightarrow{\cong} 1 \otimes V$, $v \mapsto 1 \otimes v$, is a $\Bbbk G^g$-linear isomorphism. This proves (1) and (2).

(3) The map $f : M(g, M_g) = \Bbbk G \otimes_{\Bbbk G^g} M_g \to M$, $h \otimes m \mapsto h \cdot m$, is a morphism in ${}^H_H\mathcal{YD}(\mathcal{O}_g)$. By (2), f induces an isomorphism

$$f_g : M(g, M_g)_g \to M_g$$

of left G^g-modules. Hence for all $h \in G$, f induces a bijection

$$f_{h \triangleright g} : M(g, M_g)_{h \triangleright g} = hM(g, M_g)_g \to M_{h \triangleright g} = h \cdot M_g,$$

since $f(h \cdot m) = h \cdot f(m)$ for all $m \in M(g, M_g)_g$. Thus f is bijective. □

PROPOSITION 1.4.17. *Let $g \in G$. Then $F_g : {}_{\Bbbk G^g}\mathcal{M} \to {}^G_G\mathcal{YD}(\mathcal{O}_g)$ is an equivalence of categories with quasi-inverse functor given by $M \mapsto M_g$.*

PROOF. Let $F'_g : {}^G_G\mathcal{YD}(\mathcal{O}_g) \to {}_{\Bbbk G^g}\mathcal{M}$ be the functor given by $F'_g(M) = M_g$ for all $M \in {}^G_G\mathcal{YD}(\mathcal{O}_g)$. Since the isomorphisms in Lemma 1.4.16(2) and (3) are natural transformations in $V \in {}_{\Bbbk G^g}\mathcal{M}$ and in $M \in {}^G_G\mathcal{YD}(\mathcal{O}_g)$, $F'_g F_g \cong \mathrm{id}$ and $F_g F'_g \cong \mathrm{id}$. □

We choose for any conjugacy class $\mathcal{O}_l$, $l \in L$, an element $g_l \in \mathcal{O}_l$. It follows from Proposition 1.4.17 and (1.4.7) that there is a category equivalence

$$\prod_{l \in L} {}_{\Bbbk G^{g_l}}\mathcal{M} \xrightarrow{\cong} {}^G_G\mathcal{YD}. \tag{1.4.9}$$

COROLLARY 1.4.18. *There is a bijection between the disjoint union of the isomorphism classes of the simple left $\Bbbk G^{g_l}$-modules, $l \in L$, and the set of isomorphism classes of the simple Yetter-Drinfeld modules in ${}^G_G\mathcal{YD}$.*

PROOF. This follows from Proposition 1.4.17 and (1.4.7), where for all $l \in L$ and all simple left $\Bbbk G^{g_l}$-module V_l, the isomorphism class of V_l is mapped onto the isomorphism class of $M(g_l, V_l)$. □

EXAMPLE 1.4.19. Let $G = \mathbb{Z}$ and let g be a generator of G. For any $\lambda \in \Bbbk^\times$ and any $k \geq 2$, there is a $\Bbbk G$-module $V = V(\lambda, k)$ with $\dim V = k$ such that $(g - \lambda)^k V = 0$, $(g - \lambda)^{k-1} V \neq 0$, and any two such modules are isomorphic. Note that V is cyclic, indecomposable, and not irreducible as a $\Bbbk G$-module, since any non-zero submodule of V contains the one-dimensional eigenspace to the eigenvalue λ of the action of g. Since G is abelian, $F_g(V) = V$ as a G-module and the G-grading of $F_g(V)$ is given by $F_g(V) = F_g(V)_g$. By Proposition 1.4.17, $F_g(V(\lambda, k)) \in {}^{\mathbb{Z}}_{\mathbb{Z}}\mathcal{YD}$ is an indecomposable but not irreducible Yetter-Drinfeld module.

PROPOSITION 1.4.20. *Let G be a finite group, and assume that the characteristic of $\Bbbk$ does not divide the order of G. Then ${}^G_G\mathcal{YD}$ is a semisimple category. For any $M \in {}^G_G\mathcal{YD}$,*

$$M \cong \bigoplus_{\lambda \in \Lambda} M(g_\lambda, V_\lambda) \quad \text{in } {}^G_G\mathcal{YD},$$

where Λ is an index set, $g_\lambda \in G$, and V_λ is a simple left $\Bbbk G^{g_\lambda}$-module for all $\lambda \in \Lambda$.

PROOF. Let $M \in {}^G_G\mathcal{YD}$. It follows from Proposition 1.4.17 and (1.4.7) that M is a direct sum of Yetter-Drinfeld modules of the form $M(g, V)$, where $g \in G$ and $V \in {}_{\Bbbk G^g}\mathcal{M}$. By our assumption and the Theorem of Maschke, the group algebra $\Bbbk G^g$ is semisimple. Hence V is a direct sum of simple left $\Bbbk G^g$-modules. The functor F_g commutes with direct sums by the additivity of the tensor product. Hence M is a direct sum of Yetter-Drinfeld modules of the form $M(g, V)$, where $g \in G$ and V is a simple left $\Bbbk G^g$-module. This proves the claim by Corollary 1.4.18. □

We end the section with an invariant of irreducible Yetter-Drinfeld modules.

PROPOSITION 1.4.21. *Assume that* $\Bbbk$ *is an algebraically closed field. Let* V *be a finite-dimensional irreducible object in* ${}^{G}_{G}\mathcal{YD}$. *Then there exists* $q_V \in \Bbbk^\times$ *such that* $g \cdot v = q_V v$ *for all* $g \in G$ *and* $v \in V_g$.

PROOF. We may assume that $V \neq 0$. Let $h \in G$ with $V_h \neq 0$. Since V is irreducible, $V \in {}^{G}_{G}\mathcal{YD}(\mathcal{O}_h)$. Since V_h is finite-dimensional and $\Bbbk$ is algebraically closed, there exists $q_V \in \Bbbk^\times$ and $v \in V_h$ with $v \neq 0$, $h \cdot v = q_V v$. Let

$$W = \{w \in V_h \mid h \cdot w = q_V w\}.$$

Then $W \in {}_{\Bbbk G^h}\mathcal{M}$. Proposition 1.4.17 implies that $\Bbbk G \cdot W$ is a Yetter-Drinfeld submodule of V. Thus $W = V_h$ since V is irreducible and $(\Bbbk G \cdot W)_h = W$. Finally, for all $g \in G$ and $v \in V_h$,

$$ghg^{-1} \cdot (g \cdot v) = gh \cdot v = q_V g \cdot v$$

which implies the claim. $\square$

1.5. Braided vector spaces of group type

Let V be a vector space and $c : V \otimes V \to V \otimes V$ a linear endomorphism. For any natural number $n \geq 2$ and $1 \leq i \leq n-1$ we define $c_i \in \mathrm{End}(V^{\otimes n})$ by applying c at the i-th position, that is

$$c_i = \begin{cases} c \otimes \mathrm{id}_{V^{\otimes (n-2)}}, & \text{if } i = 1, \\ \mathrm{id}_{V^{\otimes (i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes (n-i-1)}}, & \text{if } 2 \leq i \leq n-2, \\ \mathrm{id}_{V^{\otimes (n-2)}} \otimes c, & \text{if } i = n-1. \end{cases} \tag{1.5.1}$$

Note that c_i depends on n. It will be clear from the context which n is meant.

DEFINITION 1.5.1. A **braided vector space** (V, c) is a pair consisting of a vector space V and a linear automorphism $c : V \otimes V \to V \otimes V$ satisfying

$$c_1 c_2 c_1 = c_2 c_1 c_2 \quad \text{in } \mathrm{End}(V^{\otimes 3}).$$

If (V, c) is a braided vector space, the automorphism c is called a **braiding** (or a **Yang-Baxter operator**). If (V, c) and (W, d) are braided vector spaces, a **braided linear map** (or a morphism of braided vector spaces) $f : (V, c) \to (W, d)$ is a linear map $f : V \to W$ with $(f \otimes f)c = d(f \otimes f)$.

Clearly, the inverse of a bijective braided linear map is braided linear.

COROLLARY 1.5.2. *Let* $V \in {}^{G}_{G}\mathcal{YD}$. *Then* $(V, c_{V,V})$ *is a braided vector space.*

PROOF. By (1.4.5), $c_1 c_2 = c_{V \otimes V, V}$. Hence we have to show that

$$c_{V \otimes V, V} c_1 = c_2 c_{V \otimes V, V}.$$

Since $c_1 = c \otimes \mathrm{id}_V$ and $c_2 = \mathrm{id}_V \otimes c$, this follows since by (1.4.3), $c_{V \otimes V, V}$ is a natural transformation with respect to endomorphisms of $V \otimes V$. $\square$

EXAMPLE 1.5.3. Assume that G is abelian. If $V \in {}^{G}_{G}\mathcal{YD}$, and $g \in G$, $\chi \in \widehat{G}$, we define

$$V_g^\chi = \{v \in V_g \mid h \cdot v = \chi(h) v\}. \tag{1.5.2}$$

Then $V_g^\chi \subseteq V$ is a subobject in ${}^{G}_{G}\mathcal{YD}$.

An important class of Yetter-Drinfeld modules over G is constructed as follows. Let I be an index set, and V a vector space with basis x_i, $i \in I$. For all $i \in I$, let $g_i \in G$, $\chi_i \in \widehat{G}$. Then

$$V = \bigoplus_{i \in I} \Bbbk x_i \in {}_G^G\mathcal{YD}, \quad \text{where } \Bbbk x_i \in V_{g_i}^{\chi_i} \text{ for all } i \in I. \tag{1.5.3}$$

By Definition 1.4.10, the braiding $c_{V,V}$ is given by

$$c_{V,V}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \;\; q_{ij} = \chi_j(g_i) \quad \text{for all } i, j \in I. \tag{1.5.4}$$

REMARK 1.5.4. Let I be an index set, and let $(q_{ij})_{i,j \in I}$ be a family of non-zero scalars in $\Bbbk$. Let V be a vector space with basis x_i, $i \in I$. We define a linear map $c : V \otimes V \to V \otimes V$ by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \quad \text{for all } i, j \in I. \tag{1.5.5}$$

Then c is a linear automorphism of $V \otimes V$, and for all $i, j, k \in I$,

$$c_1c_2c_1(x_i \otimes x_j \otimes x_k) = q_{ij} c_1c_2(x_j \otimes x_i \otimes x_k) = q_{ij}q_{ik}q_{jk} x_k \otimes x_j \otimes x_i,$$
$$c_2c_1c_2(x_i \otimes x_j \otimes x_k) = q_{jk} c_2c_1(x_i \otimes x_k \otimes x_j) = q_{jk}q_{ik}q_{ij} x_k \otimes x_j \otimes x_i.$$

Thus (V, c) always is a braided vector space. One says that (V, c) is a braided vector space of **diagonal type**, and that c is a **diagonal braiding**. The matrix $(q_{ij})_{i,j \in I}$ is called the **braiding matrix of** (V, c) with respect to the basis x_i, $i \in I$.

The braiding of a braided vector space (V, c) of diagonal type can be realized as the braiding of a Yetter-Drinfeld module over an abelian group. For example, let G be a free abelian group with basis g_i, $i \in I$. Define characters $\chi_i \in \widehat{G}$ by $\chi_j(g_i) = q_{ij}$ for all $i, j \in I$. Then $V \in {}_G^G\mathcal{YD}$ by (1.5.3) and $c_{V,V} = c$ by (1.5.4).

The following class of braided vector spaces was introduced by Takeuchi to characterize braidings of Yetter-Drinfeld modules over groups.

DEFINITION 1.5.5. Let (V, c) be a braided vector space. We call (V, c) of **group type** if there are a basis $(x_i)_{i \in I}$ of V and elements $g_i(x_j) \in V$ for all $i, j \in I$ such that

$$c(x_i \otimes x_j) = g_i(x_j) \otimes x_i \quad \text{for all } i, j \in I. \tag{1.5.6}$$

Note that it follows from the bijectivity of c, that the family of elements $g_i(x_j)$, $i, j \in I$, defines linear automorphisms $g_i \in \mathrm{Aut}(V)$ for all $i \in I$.

PROPOSITION 1.5.6. *Let (V, c) be a braided vector space. Then the following are equivalent:*

1. *(V, c) is of group type.*
2. *There are a group G and a $\Bbbk G$-module and a $\Bbbk G$-comodule structure on V such that $V \in {}_G^G\mathcal{YD}$ and $c = c_{V,V}$.*

PROOF. We prove first that (1) implies (2). Let $(x_i)_{i \in I}$ be a basis of V and let $(g_i)_{i \in I}$ be a family of linear automorphisms of V satisfying (1.5.6). For all $i, j, k \in I$ we compute

$$c_1c_2c_1(x_i \otimes x_j \otimes x_k) = c(g_i(x_j) \otimes g_i(x_k)) \otimes x_i,$$
$$c_2c_1c_2(x_i \otimes x_j \otimes x_k) = g_ig_j(x_k) \otimes g_i(x_j) \otimes x_i.$$

Since (V, c) is a braided vector space, we obtain that

$$c(g_i(x_j) \otimes g_i(x_k)) = g_ig_j(x_k) \otimes g_i(x_j) \quad \text{for all } i, j, k \in I. \tag{1.5.7}$$

Let $G \subseteq \mathrm{Aut}(V)$ be the subgroup generated by the automorphisms g_i, $i \in I$. Hence V is a G-module. We define a G-grading on V by

$$\deg(x_i) = g_i \quad \text{for all } i \in I.$$

Then V is a Yetter-Drinfeld module over G if

$$g_i(x_j) \in V_{g_i g_j g_i^{-1}} \quad \text{for all } i, j \in I. \tag{1.5.8}$$

Let $i, j \in I$, and write $g_i(x_j) = \sum_{l \in I'} \alpha_{ij}^l x_l$, where $I' \subseteq I$ is a non-empty finite subset, and $0 \neq \alpha_{ij}^l \in \Bbbk$ for all $l \in I'$. Then for all $k \in I$,

$$c(g_i(x_j) \otimes g_i(x_k)) = c\Big(\sum_{l \in I'} \alpha_{ij}^l x_l \otimes g_i(x_k)\Big) = \sum_{l \in I'} \alpha_{ij}^l g_l g_i(x_k) \otimes x_l.$$

Hence by (1.5.7), $g_l g_i(x_k) = g_i g_j(x_k)$ for all $k \in I$, $l \in I'$. Thus for all $l \in I'$, $g_l = g_i g_j g_i^{-1}$, and $g_i(x_j) \in V_{g_i g_j g_i^{-1}}$.

The equality $c = c_{V,V}$ is clear from the definition of $V \in {}^G_G\mathcal{YD}$.

Now we prove that (2) implies (1). Let G be a group and let $V \in {}^G_G\mathcal{YD}$ be such that $c = c_{V,V}$. Choose a basis $(x_i)_{i \in I}$ of V of G-homogeneous elements, that is, with $x_i \in V_{g_i}$ for all $i \in I$, where $g_i \in G$ for all $i \in I$. Then

$$c(x_i \otimes x_j) = g_i \cdot x_j \otimes x_i$$

for all $i, j \in I$ by Definition 1.4.10. This proves (1). □

In order to describe braided vector spaces of group type without referring to the group, the notions of racks and two-cocycles are very useful.

Definition 1.5.7. Let X be a non-empty set and $\triangleright : X \times X \to X$ a map denoted by $(x, y) \mapsto x \triangleright y$ for all $x, y \in X$. The pair $(X, \triangleright)$ is called a **rack** if

(1) For all $x \in X$, the map $\varphi_x : X \to X$, $y \mapsto x \triangleright y$, is bijective.
(2) The map $\triangleright$ is left self-distributive, that is, for all $x, y, z \in X$,

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

A rack $(X, \triangleright)$ is called a **quandle** if $x \triangleright x = x$ for all $x \in X$. Two racks (or quandles) $(X, \triangleright)$ and $(Y, \triangleright')$ are called **isomorphic** if there is a bijection $f : X \to Y$ such that $f(x \triangleright z) = f(x) \triangleright' f(z)$ for all $x, z \in X$.

Example 1.5.8. Let G be a group. The union X of any non-empty set of conjugacy classes of G is a quandle, where $x \triangleright y = xyx^{-1}$ for all $x, y \in X$ is the adjoint action of the group. The pair $(G, \triangleright')$ with $g \triangleright' h = gh^{-1}g$ for all $g, h \in G$ is a quandle.

Example 1.5.9. Let A be an abelian group. Let σ be an automorphism of A and let $\triangleright : A \times A \to A$, $x \triangleright y = x + \sigma(y - x)$. Then $(A, \triangleright)$ is a quandle and is called an **affine rack** or **affine quandle**. Indeed, for any $x \in A$ the inverse of φ_x is given by

$$\varphi_x^{-1}(y) = x + \sigma^{-1}(y - x).$$

Moreover,

$$\varphi_x \varphi_y(z) = \varphi_x(y + \sigma(z - y)) = x + \sigma(y - x) + \sigma^2(z - y) = \varphi_{x \triangleright y} \varphi_x(z)$$

for all $x, y, z \in A$.

EXAMPLE 1.5.10. Let G be a group, $g \in G$ and V a left $\Bbbk G^g$-module. As in the proof of Lemma 1.4.16, let $(h_x)_{x\in X}$ be a complete set of representatives of G/G^g. For all $x, y \in X$, define $x \triangleright y \in X$ and $u(x, y) \in G^g$ by the equation

$$(h_x \triangleright g)h_y = h_{x\triangleright y}u(x, y).$$

Then $(X, \triangleright)$ is a rack.

Condition (1) of Definition 1.5.7 clearly holds, since G/G^g is a left G-space, and left multiplication with $h_x \triangleright g$ is bijective. To check (2), let $x, y, z \in X$. By definition,

$$(h_x \triangleright g)h_z = h_{x\triangleright z}u(x, z), \qquad (h_x \triangleright g)h_{y\triangleright z} = h_{x\triangleright(y\triangleright z)}u(x, y \triangleright z),$$
$$(h_y \triangleright g)h_z = h_{y\triangleright z}u(y, z), \qquad (h_{x\triangleright y} \triangleright g)h_{x\triangleright z} = h_{(x\triangleright y)\triangleright(x\triangleright z)}u(x \triangleright y, x \triangleright z).$$

Hence

$$\begin{aligned} h_{x\triangleright(y\triangleright z)}u(x, y \triangleright z)u(y, z) &= (h_x \triangleright g)(h_y \triangleright g)h_z,\\ h_{(x\triangleright y)\triangleright(x\triangleright z)}u(x \triangleright y, x \triangleright z)u(x, z) &= (h_{x\triangleright y} \triangleright g)(h_x \triangleright g)h_z\\ &= (((h_x \triangleright g)h_y u(x, y)^{-1}) \triangleright g)(h_x \triangleright g)h_z\\ &= (h_x \triangleright g)(h_y \triangleright g)h_z, \end{aligned}$$

where the last equality holds since $u(x, y) \in G^g$. This proves (2). Moreover,

$$u(x \triangleright y, x \triangleright z)u(x, z) = u(x, y \triangleright z)u(y, z) \tag{1.5.9}$$

for all $x, y, z \in X$.

The braiding of $M(g, V) = \Bbbk G \otimes_{\Bbbk G^g} V$ can hence be written as

$$\begin{aligned} c(h_x \otimes v, h_y \otimes w) &= (h_x \triangleright g)h_y \otimes w \otimes h_x \otimes v\\ &= h_{x\triangleright y} \otimes u(x, y) \cdot w \otimes h_x \otimes v\\ &= h_{x\triangleright y} \otimes \boldsymbol{q}_{x,y}(w) \otimes h_x \otimes v \end{aligned}$$

for all $x, y \in X$, $v, w \in V$, where $\boldsymbol{q}_{x,y} \in \mathrm{Aut}(V)$, $\boldsymbol{q}_{x,y}(w) = u(x, y)\cdot w$ for all $w \in V$.

The braiding in Example 1.5.10 can easily be formulated for any rack.

DEFINITION 1.5.11. Let $(X, \triangleright)$ be a rack, and let $\boldsymbol{q} : X \times X \to H$ for some group H be a map which we write as $\boldsymbol{q}(x, y) = \boldsymbol{q}_{x,y}$ for all $x, y \in X$. Then $\boldsymbol{q}$ is called a **two-cocycle** if

$$\boldsymbol{q}_{x\triangleright y,x\triangleright z}\boldsymbol{q}_{x,z} = \boldsymbol{q}_{x,y\triangleright z}\boldsymbol{q}_{y,z} \tag{1.5.10}$$

for all $x, y, z \in X$. We say that $\boldsymbol{q}$ is **constant** if $H = \mathrm{Aut}(V)$ for some vector space V and there exists $\lambda \in \Bbbk$ such that $\boldsymbol{q}_{x,y} = \lambda\mathrm{id}_V$ for all $x, y \in X$.

A constant map $\boldsymbol{q} : X \times X \to \mathrm{Aut}(V)$ is always a two-cocycle. The map u in Example 1.5.10 is a two-cocycle with values in G^g by (1.5.9).

PROPOSITION 1.5.12. *Let X be a non-empty set, V be a vector space, and*

$$\triangleright : X \times X \to X, \quad \boldsymbol{q} : X \times X \to \mathrm{Aut}(V)$$

be maps. Let $M = \Bbbk X \otimes V$ and let $c^{\boldsymbol{q}} : M \otimes M \to M \otimes M$ be the linear map with

$$c^{\boldsymbol{q}}((x \otimes v) \otimes (y \otimes w)) = ((x \triangleright y) \otimes \boldsymbol{q}_{x,y}(w)) \otimes (x \otimes v) \tag{1.5.11}$$

for all $x, y \in X$, $v, w \in V$. Then $(M, c^{\boldsymbol{q}})$ is a braided vector space if and only if $(X, \triangleright)$ is a rack and $\boldsymbol{q}$ is a two-cocycle. In this case, $(M, c^{\boldsymbol{q}})$ is of group type.

PROOF. In the proof we write xv instead of $x \otimes v$ for all $x \in X$, $v \in V$. Let $x, y, z \in X$ and $v, w, u \in V$. Then

$$c_1c_2c_1(xv \otimes yw \otimes zu) = (x \triangleright y) \triangleright (x \triangleright z)\boldsymbol{q}_{x\triangleright y, x\triangleright z}(\boldsymbol{q}_{x,z}(u)) \otimes (x \triangleright y)\boldsymbol{q}_{x,y}(w) \otimes xv,$$
$$c_2c_1c_2(xv \otimes yw \otimes zu) = (x \triangleright (y \triangleright z))\boldsymbol{q}_{x, y\triangleright z}(\boldsymbol{q}_{y,z}(u)) \otimes (x \triangleright y)\boldsymbol{q}_{x,y}(w) \otimes xv.$$

This implies the first part of the claim. The rest is clear. □

EXAMPLE 1.5.13. Let $X = \{1, 2, 3, 4\}$ and let φ_i, $i \in X$, be the permutations

$$\varphi_1 = (2\,3\,4), \quad \varphi_2 = (1\,4\,3), \quad \varphi_3 = (1\,2\,4), \quad \varphi_4 = (1\,3\,2).$$

Then $(X, \triangleright)$ is a quandle, where $x \triangleright y = \varphi_x(y)$ for all $x, y \in X$. More precisely, consider the affine quandle structure on the field $\mathbb{F}_4$ with 4 elements and the automorphism determined by left multiplication with an element of multiplicative order 3 in $\mathbb{F}_4$. This quandle and $(X, \triangleright)$ are isomorphic.

Let V be a one-dimensional vector space, $(X, \triangleright)$ a rack, $M = \Bbbk X \otimes V \cong \Bbbk X$, and let $c^{\boldsymbol{q}}$ be as in Proposition 1.5.12, where $\lambda \in \Bbbk^\times$ and $\boldsymbol{q}$ is the constant two-cocycle with $\boldsymbol{q}_{x,y} = \lambda$ for all $x, y \in X$. Then

$$c^{\boldsymbol{q}}(x \otimes y) = \lambda(x \triangleright y) \otimes x$$

for all $x, y \in X$.

EXAMPLE 1.5.14. Let $m \geq 2$ be a positive integer and let $1 \leq i < m$ with $\gcd(m, i) = 1$. Multiplication with i in $\mathbb{Z}/(m)$ is an automorphism. Hence

$$\mathrm{Aff}(m, i) = (\mathbb{Z}/(m), \triangleright), \quad x \triangleright y = x + i(y - x),$$

is an affine quandle. For $i = 1$, $x \triangleright y = y$ for all $x, y \in \mathbb{Z}/(m)$.

1.6. Braided Hopf algebras and Nichols algebras over groups

Let again G be a group. To simplify the **notation**, we write $\mathcal{C} = {}^G_G\mathcal{YD}$.

The tensor product of two objects in $\mathcal{C}$ is an object in $\mathcal{C}$, the tensor product of two morphisms in $\mathcal{C}$ is a morphism in $\mathcal{C}$, and the canonical isomorphisms in Proposition 1.2.5 for $U, V, W \in \mathcal{C}$ are morphisms in $\mathcal{C}$ by Proposition 1.4.9.

Let $A \in \mathcal{C}$, and let $\mu : A \otimes A \to A$, $\eta : \Bbbk \to A$ be morphisms in $\mathcal{C}$. Then (A, μ, η) is an **algebra in** $\mathcal{C}$ if the diagrams (1.1.3) and (1.1.4) commute. If A, B are algebras in $\mathcal{C}$, and $\rho : A \to B$ is a morphism in $\mathcal{C}$, then ρ is a **morphism of algebras in** $\mathcal{C}$, if the diagrams (1.1.5) and (1.1.6) commute.

Let $C \in \mathcal{C}$, and let $\Delta : C \to C \otimes C$, $\varepsilon : C \to \Bbbk$ be morphisms in $\mathcal{C}$. The triple (C, Δ, ε) is a **coalgebra in** $\mathcal{C}$ if the diagrams (1.1.7) and (1.1.8) commute. If C, D are coalgebras in $\mathcal{C}$, and $\varphi : C \to D$ is a morphism in $\mathcal{C}$, then φ is a **morphism of coalgebras in** $\mathcal{C}$, if the diagrams (1.1.10) and (1.1.11) commute.

Thus algebras and coalgebras in $\mathcal{C}$ are algebras and coalgebras in the sense of Section 1.1 whose structure maps are morphisms in $\mathcal{C}$. In the same way **modules in** $\mathcal{C}$ and **comodules in** $\mathcal{C}$ are modules and comodules, respectively, whose structure maps are morphisms in $\mathcal{C}$.

COROLLARY 1.6.1. *Let C be a coalgebra in $\mathcal{C}$, A an algebra in $\mathcal{C}$, and f an invertible map in* $\mathrm{Hom}(C, A)$. *If f is a morphism in $\mathcal{C}$, then so is f^{-1}.*

PROOF. This is another application of Proposition 1.2.11. □

PROPOSITION 1.6.2. *Let $V \in \mathcal{C}$, and $T(V) = \bigoplus_{n \geq 0} T^n(V)$ the tensor algebra of the vector space V.*

(1) *$T(V)$ is an algebra in $\mathcal{C}$, where $T^n(V) = V^{\otimes n}$, $n \geq 0$, is the n-fold tensor product in $\mathcal{C}$.*
(2) *For any algebra A in $\mathcal{C}$ and any morphism $f : V \to A$ in $\mathcal{C}$, there is exactly one algebra morphism $\varphi : T(V) \to A$ in $\mathcal{C}$ extending f.*

PROOF. This is clear from the universal property of the tensor algebra (or the free algebra), since for all $n \geq 2$, $V^{\otimes n} \xrightarrow{f^{\otimes n}} A^{\otimes n} \xrightarrow{\mu^{n-1}} A$ is a morphism in $\mathcal{C}$, where μ^{n-1} is the $(n-1)$-fold iteration of the multiplication map μ. □

DEFINITION 1.6.3. (1) Let (A, μ_A, η_A) and (B, μ_B, η_B) be algebras in $\mathcal{C}$. Define $\mu_{A\otimes B}$ and $\eta_{A\otimes B}$ by

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\mathrm{id}\otimes c_{B,A}\otimes\mathrm{id}} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\mu_A\otimes\mu_B} A \otimes B,$$
$$\Bbbk \cong \Bbbk \otimes \Bbbk \xrightarrow{\eta_A\otimes\eta_B} A \otimes B.$$

Then $(A \otimes B, \mu_{A\otimes B}, \eta_{A\otimes B})$ is called the **tensor product of algebras in** $\mathcal{C}$.

(2) Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras in $\mathcal{C}$. Define $\Delta_{C\otimes D}$ and $\varepsilon_{C\otimes D}$ by

$$C \otimes D \xrightarrow{\Delta_C\otimes\Delta_D} (C \otimes C) \otimes (D \otimes D) \xrightarrow{\mathrm{id}\otimes c_{C,D}\otimes\mathrm{id}} (C \otimes D) \otimes (C \otimes D),$$
$$C \otimes D \xrightarrow{\varepsilon_C\otimes\varepsilon_D} \Bbbk \otimes \Bbbk \cong \Bbbk.$$

Then $(C \otimes D, \Delta_{C\otimes D}, \varepsilon_{C\otimes D})$ is called the **tensor product of coalgebras in** $\mathcal{C}$.

By Definition 1.4.10, the product $\mu_{A\otimes B}$ is defined for elements $a, x \in A$ and $b \in B_g$, $y \in B$, $g \in G$, by

$$(a \otimes b)(x \otimes y) = a(g \cdot x) \otimes by. \tag{1.6.1}$$

The unit element of $A \otimes B$ is $1_A \otimes 1_B$.

PROPOSITION 1.6.4. *Let A, B, C, D be algebras in $\mathcal{C}$.*

(1) *$(A \otimes B, \mu_{A\otimes B}, \eta_{A\otimes B})$ is an algebra in $\mathcal{C}$.*
(2) *The canonical isomorphism $(A\otimes B)\otimes C \cong A\otimes(B\otimes C)$ is an isomorphism of algebras in $\mathcal{C}$.*
(3) *Let $\varphi : A \to C$ and $\psi : B \to D$ be morphisms of algebras in $\mathcal{C}$. Then $\varphi \otimes \psi : A \otimes B \to C \otimes D$ is a morphism of algebras in $\mathcal{C}$.*

PROOF. (1) It is clear from the definition that $\mu_{A\otimes B}$ and $\eta_{A\otimes B}$ are morphisms in $\mathcal{C}$. To check associativity, consider elements $a, u, x \in A$ and $b \in B_g$, $v \in B_h$, $y \in B$, where $g, h \in G$. Then $\deg(bv) = gh$, since the multiplication map $B\otimes B \to B$ is G-graded. Hence

$$((a \otimes b)(u \otimes v))(x \otimes y) = (a(g \cdot u) \otimes bv)(x \otimes y) = a(g \cdot u)((gh) \cdot x) \otimes bvy,$$
$$(a \otimes b)((u \otimes v)(x \otimes y)) = (a \otimes b)(u(h \cdot x) \otimes vy) = a(g \cdot (u(h \cdot x))) \otimes bvy.$$

This proves associativity, since the multiplication map $A \otimes A \to A$ is left G-linear, hence $(g \cdot u)((gh) \cdot x) = g \cdot (u(h \cdot x))$.

(2) Let $a, x \in A$, $b \in B_g$, $y \in B$, $c \in C_h$, $z \in C$, where $g, h \in G$. We compute in $A \otimes (B \otimes C)$ and then in $(A \otimes B) \otimes C$,

$$\begin{aligned}(a \otimes (b \otimes c))(x \otimes (y \otimes z)) &= a((gh) \cdot x) \otimes (b \otimes c)(y \otimes z) \\ &= a((gh) \cdot x) \otimes b(h \cdot y) \otimes cz, \\ ((a \otimes b) \otimes c)((x \otimes y) \otimes z) &= (a \otimes b)(h \cdot x \otimes h \cdot y) \otimes cz \\ &= a((gh) \cdot x) \otimes b(h \cdot y) \otimes cz.\end{aligned}$$

(3) Let $a, u \in A$, $b, v \in B$, and assume that $b \in B_g$, $g \in G$. Then

$$\begin{aligned}(\varphi \otimes \psi)((a \otimes b)(u \otimes v)) &= \varphi(a(g \cdot u)) \otimes \psi(bv) \\ &= \varphi(a)(g \cdot \varphi(u)) \otimes \psi(b)\psi(v) \\ &= (\varphi(a) \otimes \psi(b))(\varphi(u) \otimes \psi(v)).\end{aligned}$$

This implies the claim. □

PROPOSITION 1.6.5. *Let C, D, E, F be coalgebras in $\mathcal{C}$.*

(1) $(C \otimes D, \Delta_{C \otimes D}, \varepsilon_{C \otimes D})$ *is a coalgebra in* $\mathcal{C}$.
(2) *The canonical isomorphism* $(C \otimes D) \otimes E \cong C \otimes (D \otimes E)$ *is an isomorphism of coalgebras in* $\mathcal{C}$.
(3) *Let* $\varphi : C \to E$ *and* $\psi : D \to F$ *be morphisms of coalgebras in* $\mathcal{C}$. *Then* $\varphi \otimes \psi : C \otimes D \to E \otimes F$ *is a morphism of coalgebras in* $\mathcal{C}$.

PROOF. This can be shown as in the proof of Proposition 1.6.4 by direct computation using the comodule description of Yetter-Drinfeld modules in Remark 1.4.8. □

We will see in Section 3.2 that Propositions 1.6.4 and 1.6.5 formally follow from the properties of the braiding in Proposition 1.4.11. Proposition 1.6.4 holds in braided monoidal categories, and Proposition 1.6.5 is Proposition 1.6.4 in the dual category.

DEFINITION 1.6.6. (1) Let R be an object in $\mathcal{C}$, and let

$$\mu : R \otimes R \to R, \quad \eta : \Bbbk \to R, \quad \Delta : R \to R \otimes R, \quad \varepsilon : R \to \Bbbk$$

be morphisms in $\mathcal{C}$. Then $(R, \mu, \eta, \Delta, \varepsilon)$ is a **bialgebra in** $\mathcal{C}$ if (R, μ, η) is an algebra in $\mathcal{C}$, (R, Δ, ε) is a coalgebra in $\mathcal{C}$, and Δ and ε are algebra maps in $\mathcal{C}$.

(2) Let R be a bialgebra in $\mathcal{C}$, and $\mathcal{S} : R \to R$ a morphism in $\mathcal{C}$. Then $(R, \mathcal{S})$ is a **Hopf algebra in** $\mathcal{C}$ with **antipode** $\mathcal{S}$, if the diagrams (1.2.2) commute.

(3) Let R, R' be bialgebras in $\mathcal{C}$, and $\varphi : R \to R'$ a morphism in $\mathcal{C}$. Then φ is a **bialgebra morphism in** $\mathcal{C}$, if φ is a morphism of algebras and coalgebras in $\mathcal{C}$. A **Hopf algebra morphism in** $\mathcal{C}$ between Hopf algebras in $\mathcal{C}$ is a bialgebra morphism in $\mathcal{C}$.

PROPOSITION 1.6.7. *Let R be an object in $\mathcal{C}$, and let*

$$\mu : R \otimes R \to R, \quad \eta : \Bbbk \to R, \quad \Delta : R \to R \otimes R, \quad \varepsilon : R \to \Bbbk$$

be morphisms in $\mathcal{C}$. Assume that (R, μ, η) is an algebra and (R, Δ, ε) is a coalgebra in $\mathcal{C}$. Then the following are equivalent.

(1) Δ *and* ε *are morphisms of algebras in* $\mathcal{C}$.
(2) μ *and* η *are morphisms of coalgebras in* $\mathcal{C}$.

PROOF. Replace in the proof of Proposition 1.2.2 the flip map $\tau_{R,R}$ by the braiding $c_{R,R}$. $\square$

REMARK 1.6.8. (1) Let $(R,\mathcal{S})$ be a Hopf algebra in $\mathcal{C}$. Then $\mathcal{S}$ is uniquely determined as the inverse of id in $\mathrm{Hom}(R,R)$.

(2) If R is a bialgebra in $\mathcal{C}$, and the inverse $\mathcal{S}$ of id in $\mathrm{Hom}(R,R)$ exists, then $\mathcal{S}$ is a morphism in $\mathcal{C}$ by Corollary 1.6.1, hence $(R,\mathcal{S})$ is a Hopf algebra in $\mathcal{C}$.

(3) Let R, R' be Hopf algebras in $\mathcal{C}$ an $\varphi : R \to R'$ a bialgebra morphism in $\mathcal{C}$. Then $\varphi\mathcal{S}_R = \mathcal{S}_{R'}\varphi$ by the proof of Proposition 1.2.17(2).

LEMMA 1.6.9. *Let R be a bialgebra in $\mathcal{C}$. Then $P(R) \subseteq R$ is a subobject in $\mathcal{C}$.*

PROOF. By definition, $P(R)$ is the kernel of the morphism

$$R \to R\otimes R,\ \ x \mapsto \Delta(x) - (x\otimes 1 + 1\otimes x)$$

in $\mathcal{C}$. This implies the claim. $\square$

An $\mathbb{N}_0$**-graded object in** $\mathcal{C}$ is an object $V \in \mathcal{C}$ with a family of subobjects $V(n) \subseteq V$, $n \geq 0$, in $\mathcal{C}$ such that $V = \bigoplus_{n\geq 0} V(n)$ in $\mathcal{C}$. The category of $\mathbb{N}_0$-graded objects in $\mathcal{C}$ with graded morphisms in $\mathcal{C}$ as morphisms is denoted by $\mathbb{N}_0$-$\mathrm{Gr}(\mathcal{C})$.

An $\mathbb{N}_0$**-graded algebra, coalgebra, bialgebra and Hopf algebra in** $\mathcal{C}$ is an algebra, coalgebra, bialgebra and Hopf algebra, respectively, in $\mathcal{C}$ with an $\mathbb{N}_0$-grading of subobjects in $\mathcal{C}$ such that the structure maps are graded.

For $V \in \mathcal{C}$, the tensor algebra $T(V)$ is an algebra in $\mathcal{C}$ by Proposition 1.6.2. The usual $\mathbb{N}_0$-grading with $T(V)(n) = T^n(V) = V^{\otimes n}$ for all $n \geq 0$ turns $T(V)$ into an $\mathbb{N}_0$-graded algebra in $\mathcal{C}$ by construction.

COROLLARY 1.6.10. *Let R be an $\mathbb{N}_0$-graded connected bialgebra in $\mathcal{C}$. Then R is an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$.*

PROOF. Since R is an algebra and a coalgebra, $\mathrm{Hom}(R,R)$ is an algebra with convolution product. The identity map in $\mathrm{Hom}(R,R)$ is invertible by Proposition 1.3.5. Hence the claim follows from Remark 1.6.8. $\square$

DEFINITION 1.6.11. Let $V \in \mathcal{C}$, and $T(V)$ the tensor algebra of V in $\mathcal{C}$. By Proposition 1.6.2, there are uniquely determined algebra morphisms in $\mathcal{C}$

$$\Delta : T(V) \to T(V)\otimes T(V),\ \ \varepsilon : T(V) \to \Bbbk$$

such that

$$\Delta(v) = v\otimes 1 + 1\otimes v,\ \ \varepsilon(v) = 0$$

for all $v \in V$, where $T(V)\otimes T(V)$ is the tensor product of algebras in $\mathcal{C}$.

EXAMPLE 1.6.12. Let $V = \bigoplus_{i\in I} \Bbbk x_i \in {}^G_G\mathcal{YD}$, where $x_i \in V_{g_i}^{\chi_i}$, $\chi_j(g_i) = q_{ij}$ for all $i,j \in I$. Then in $T(V)$ for all $i,j \in I$,

$$\begin{aligned}\Delta(x_ix_j) &= (x_i\otimes 1 + 1\otimes x_i)(x_j\otimes 1 + 1\otimes x_j)\\ &= x_ix_j\otimes 1 + x_i\otimes x_j + q_{ij}x_j\otimes x_i + 1\otimes x_ix_j.\end{aligned}$$

PROPOSITION 1.6.13. *Let $V \in \mathcal{C}$.*

(1) *The tensor algebra $T(V)$ is an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$ with comultiplication Δ and counit ε of Definition* 1.6.11.
(2) *Let R be a bialgebra in $\mathcal{C}$, and $f : V \to P(R)$ a morphism in $\mathcal{C}$. Then there is exactly one bialgebra map $\varphi : T(V) \to R$ in $\mathcal{C}$ extending f.*

(3) *Let R be an $\mathbb{N}_0$-graded connected bialgebra in $\mathcal{C}$, and $f : V \to R(1)$ a morphism in $\mathcal{C}$. Then there is exactly one bialgebra map $\varphi : T(V) \to R$ in $\mathcal{C}$ extending f, and φ is $\mathbb{N}_0$-graded.*

Proof. (1) Since Δ and ε are homogeneous on V, they are $\mathbb{N}_0$-graded algebra morphisms in $\mathcal{C}$. Then $(T(V), \Delta, \varepsilon)$ is an $\mathbb{N}_0$-graded coalgebra in $\mathcal{C}$, since by Proposition 1.6.4(2), the diagrams (1.1.7) and (1.1.8) are diagrams of algebra morphisms which commute on the generators $v \in V$. Thus $T(V)$ is an $\mathbb{N}_0$-graded bialgebra in $\mathcal{C}$. Then $T(V)$ is a Hopf algebra in $\mathcal{C}$ by Corollary 1.6.10.

(2) By Proposition 1.6.2, there is a unique algebra map $\varphi : T(V) \to R$ in $\mathcal{C}$ extending $f : V \to R$. It remains to show that φ is a coalgebra map, that is, the diagrams

$$\begin{tikzcd} T(V) \arrow[r, "\varphi"] \arrow[d, "\Delta"] & R \arrow[d, "\Delta"] \\ T(V) \otimes T(V) \arrow[r, "\varphi\otimes\varphi"] & R \otimes R \end{tikzcd} \qquad \begin{tikzcd} T(V) \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon"'] & & R \arrow[dl, "\varepsilon"] \\ & \Bbbk & \end{tikzcd}$$

commute. All maps in the diagrams are algebra maps, and it is enough to prove commutativity on the generators in V. It is clear from the assumption on f that both diagrams commute on elements of V.

(3) This follows from (2), since $R(1) \subseteq P(R)$ by Lemma 1.3.6(2). $\square$

Ideals, coideals, bi-ideals and Hopf ideals in $\mathcal{C}$ are subobjects in $\mathcal{C}$ which are ideals, coideals, bi-ideals and Hopf ideals, respectively. They describe quotients of algebras, coalgebras, bialgebras and Hopf algebras in $\mathcal{C}$ as in Propositions 1.1.13 and 1.2.22.

Lemma 1.6.14. *Let A be a bialgebra in $\mathcal{C}$, and $I \subseteq A$ a coideal in $\mathcal{C}$. Then AI and IA are coideals of A in $\mathcal{C}$.*

Proof. Since the multiplication map $A \otimes A \to A$ is a morphism in $\mathcal{C}$, AI is a subobject of A in $\mathcal{C}$. Since ε is an algebra map, $\varepsilon(AI) \subseteq \varepsilon(A)\varepsilon(I) = 0$. Since Δ is an algebra map,

$$\begin{aligned} \Delta(AI) &\subseteq \Delta(A)\Delta(I) \subseteq (A \otimes A)(I \otimes A + A \otimes I) \\ &= Ac(A \otimes I)A + Ac(A \otimes A)I = AI \otimes A + A \otimes AI. \end{aligned}$$

Hence AI is a coideal of A in $\mathcal{C}$. Similarly, $IA \subseteq A$ is a coideal of A in $\mathcal{C}$. $\square$

Corollary 1.6.15. *Let $R = \bigoplus_{n\geq 0} R(n)$ be an $\mathbb{N}_0$-graded connected Hopf algebra in $\mathcal{C}$, and let $I_R \subseteq R$ be the largest coideal contained in $\bigoplus_{n\geq 2} R(n)$. Then R/I_R is an $\mathbb{N}_0$-graded connected quotient Hopf algebra in $\mathcal{C}$ with*

$$P(R/I_R) = (R/I_R)(1) \cong R(1).$$

Proof. By Theorem 1.3.16, $I_R = \bigoplus_{n\geq 2} \ker(\Delta_{1^n})$, and R/I_R is strictly graded, that is, $P(R/I_R) = (R/I_R)(1) \cong R(1)$. For all $n \geq 2$, the maps $\Delta^R_{1^n}$ are $\mathbb{N}_0$-graded morphisms in $\mathcal{C}$. Hence $I_R \subseteq R$ is an $\mathbb{N}_0$-graded subobject in $\mathcal{C}$, and R/I_R is an $\mathbb{N}_0$-graded coalgebra quotient of R in $\mathcal{C}$. By the maximality of I_R and by Lemma 1.6.14, I_R is a bi-ideal of R. Then R/I_R is an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$ by Corollary 1.6.10. $\square$

DEFINITION 1.6.16. Let $V \in \mathcal{C}$. An $\mathbb{N}_0$-graded connected Hopf algebra R in $\mathcal{C}$ is a **pre-Nichols algebra of** V, if

(N1) $R(1) \cong V$ in $\mathcal{C}$,
(N2) R is generated as an algebra by $R(1)$.

A pre-Nichols algebra of V is a **Nichols algebra of** V, if

(N3) R is strictly graded, that is, $P(R) = R(1)$.

It is a remarkable fact that by Theorem 1.6.18 below the structure of a Nichols algebra of $V \in \mathcal{C}$ is completely determined by V. This is somewhat similar to the situation of irreducible cocommutative Hopf algebras U over a field of characteristic 0. The structure of U is completely determined by the Lie algebra of its primitive elements. In this analogy, the Nichols algebra corresponds to the universal enveloping algebra of a Lie algebra.

The Nichols algebra can be constructed as the smallest $\mathbb{N}_0$-graded Hopf algebra quotient of $T(V)$ which is isomorphic to V in degree one. Recall from Proposition 1.6.13 that $T(V)$ is an $\mathbb{N}_0$-graded connected coalgebra.

DEFINITION 1.6.17. Let $V \in \mathcal{C}$. Let $I(V)$ be the largest coideal of $T(V)$ contained in $\bigoplus_{n\geq 2} T^n(V)$. The **Nichols algebra** of V is defined by

$$\mathcal{B}(V) = T(V)/I(V).$$

Note that $I(V) = \bigoplus_{n\geq 2} \ker(\Delta_{1^n}^{T(V)}) = I_{T(V)}$ by Theorem 1.3.16.

THEOREM 1.6.18. *Let $V \in \mathcal{C}$.*

(1) *$\mathcal{B}(V)$ is a Nichols algebra of V.*
(2) *Let R be a pre-Nichols algebra of V, $f : R(1) \xrightarrow{\cong} V$ an isomorphism in $\mathcal{C}$.*
   (a) *There is exactly one morphism $\pi : R \to \mathcal{B}(V)$ of $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$ such that f is the restriction of π to $R(1)$, and π is surjective.*
   (b) *π is bijective if and only if R is a Nichols algebra of V.*

PROOF. (1) follows from Corollary 1.6.15.

(2) (a) Let $\varphi : T(V) \to R$ be the surjective $\mathbb{N}_0$-graded braided bialgebra map extending f^{-1} by Proposition 1.6.13(3). Then $\ker(\varphi) \subseteq I(V)$, since φ is bijective in degree 0 and 1. The induced map

$$\pi : R \cong T(V)/\ker(\varphi) \to T(V)/I(V) = \mathcal{B}(V)$$

is a surjective map of $\mathbb{N}_0$-graded braided Hopf algebras with $\pi(1) = f$.

(b) If $P(R) = R(1)$, then π in (1) is bijective by Proposition 1.3.10(2). Conversely, if $R \cong \mathcal{B}(V)$, then $P(R) = R(1)$ by (1). $\square$

REMARK 1.6.19. Let $U, V \in \mathcal{C}$, and $f : U \to V$ a morphism in $\mathcal{C}$. Then f induces a morphism $T(f) : T(U) \to T(V)$ of $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$. Since $T(f)$ is a coalgebra morphism, $T(f)(I(U)) \subseteq I(V)$. Hence the construction of the Nichols algebra is a functor from $\mathcal{C}$ to the category of $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$. Clearly, f is surjective if and only if $\mathcal{B}(f)$ is surjective.

Suppose that f is injective. Then $T(f)^{-1}(I(V)) \subseteq \bigoplus_{n\geq 2} T^n(U)$. Hence $T(f)^{-1}(I(V)) = I(U)$, and $\mathcal{B}(f)$ is injective.

REMARK 1.6.20. Direct sum decompositions of Yetter-Drinfeld modules give rise to important gradings of the Nichols algebra, see Corollary 7.1.15.

Let $\theta \geq 1$ be an integer. Then $\mathbb{N}_0^\theta$ is a monoid with componentwise addition of natural numbers. The standard basis of $\mathbb{Z}^\theta$ is denoted by $\alpha_1, \dots, \alpha_\theta$. Thus for $\alpha = (a_1, \dots, a_\theta) \in \mathbb{N}_0^\theta$, $\alpha = \sum_{i=1}^\theta a_i \alpha_i$.

Let $V \in \mathcal{C}$ with subobjects $V_i \subseteq V$ in $\mathcal{C}$ such that $V = \bigoplus_{1 \leq i \leq \theta} V_i$. Then $\mathcal{B}(V)$ is an $\mathbb{N}_0^\theta$-graded Hopf algebra in $\mathcal{C}$, where for all $1 \leq i \leq \theta$, $\deg(V_i) = \alpha_i$.

1.7. Braid group and braided vector spaces

We begin by recalling some general facts about the symmetric group. Let W be a group and $S \subseteq W$ a subset of elements of order 2. In particular, S does not contain the identity element 1 of W. For all $s, s' \in S$ let $m(s,s')$ be the order of ss'. The pair (W, S) is called a **Coxeter system**, and W is called a **Coxeter group** [**Bou68**, Ch. IV, §1, 1.3], if

$$\langle S \mid (ss')^{m(s,s')} = 1 \quad \text{for all } s, s' \in S \text{ with } m(s,s') < \infty \rangle \xrightarrow{\cong} W, \quad s \mapsto s,$$

is a group isomorphism, that is, if W is generated by the set S with (only) relations $(ss')^{m(s,s')} = 1$ for all $s, s' \in S$ with $m(s,s') < \infty$.

Let $n \geq 2$. We denote the elementary transpositions of the symmetric group $\mathbb{S}_n$ by $s_i = (i\, i+1)$ for all $1 \leq i \leq n-1$. Note that

$$\operatorname{ord}(s_i s_j) = \begin{cases} 1, & \text{if } i = j, \\ 3, & \text{if } |i-j| = 1, \\ 2, & \text{if } |i-j| > 1. \end{cases}$$

THEOREM 1.7.1. *For all $n \geq 2$, $(\mathbb{S}_n, \{s_1, \dots, s_{n-1}\})$ is a Coxeter system, that is, $\mathbb{S}_n$ is generated by $s_1, \dots, s_{n-1}$ with defining relations*

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for all } 1 \leq i \leq n-2, \tag{1.7.1}$$

$$s_i s_j = s_j s_i \quad \text{for all } 1 \leq i, j \leq n-1, |i-j| > 1, \tag{1.7.2}$$

$$s_i^2 = 1 \quad \text{for all } i. \tag{1.7.3}$$

PROOF. For $n = 2$ the claim is trivial. Assume that $n \geq 3$. Let W_n denote the Coxeter group given by generators $s_1, \dots, s_{n-1}$ and relations (1.7.1)–(1.7.3). The elementary transpositions of $\mathbb{S}_n$ satisfy Equations (1.7.1)–(1.7.3), hence there is a surjective map $W_n \to \mathbb{S}_n$. On the other hand,

$$W_n = \{w, ws_{n-1}, ws_{n-1}s_{n-2}, \dots, ws_{n-1}s_{n-2}\cdots s_1 \mid w \in \langle s_1, \dots, s_{n-2} \rangle\}.$$

Indeed, let $i, j \in \{1, \dots, n-1\}$. Then

$$(s_{n-1}s_{n-2}\cdots s_i)s_j = \begin{cases} s_{j-1}(s_{n-1}s_{n-2}\cdots s_i) & \text{if } j > i, \\ s_{n-1}s_{n-2}\cdots s_{i+1} & \text{if } j = i, \\ s_{n-1}s_{n-2}\cdots s_{i-1} & \text{if } j = i-1, \\ s_j(s_{n-1}s_{n-2}\cdots s_i) & \text{if } j < i-1. \end{cases}$$

Hence $\{w, ws_{n-1}, ws_{n-1}s_{n-2}, \dots, ws_{n-1}s_{n-2}\cdots s_1 \mid w \in \langle s_1, \dots, s_{n-2} \rangle\}$ is a subgroup of W_n containing all generators of W_n and hence coincides with W_n. We conclude that $|W_n| \leq n|W_{n-1}|$ and hence $|W_n| \leq n!$ by induction on n. Therefore $W_n \cong \mathbb{S}_n$ since $|\mathbb{S}_n| = n!$. □

Let

$$\begin{aligned}\Delta &= \{(a,b) \in \mathbb{N}^2 \mid 1 \le a,b \le n,\ a \ne b\},\\ \Delta_+ &= \{(a,b) \in \Delta \mid a < b\},\\ \Delta_- &= \{(a,b) \in \Delta \mid a > b\},\end{aligned}$$

and define

$$\alpha_1 = (1,2),\ \alpha_2 = (2,3),\ \ldots,\ \alpha_{n-1} = (n-1,n) \in \Delta_+.$$

The symmetric group $\mathbb{S}_n$ acts on Δ by

$$\mathbb{S}_n \times \Delta \to \Delta,\ (w,(a,b)) \mapsto (w(a), w(b)).$$

For $w \in \mathbb{S}_n$ let

$$\Delta_w = \{\alpha \in \Delta_+ \mid w(\alpha) \in \Delta_-\}.$$

The elements of Δ_w are called **inversions** of w.

The **length** $\ell(w)$ of a permutation $w \in \mathbb{S}_n$ is defined as the smallest natural number $l \in \mathbb{N}_0$ such that there exist $1 \le i_1, \ldots, i_l \le n-1$ with $w = s_{i_1} \cdots s_{i_l}$. A sequence $(i_1, \ldots, i_l)$ with $1 \le i_1, \ldots, i_l \le n-1$ is called a **reduced decomposition of** w if $w = s_{i_1} \cdots s_{i_l}$, and if $l = \ell(w)$.

In practice, the length of a permutation is computed by counting the number of its inversions.

THEOREM 1.7.2. *Let $w \in \mathbb{S}_n$ and let $i \in \mathbb{N}$ with $i \le n-1$.*

(1) *$\ell(ws_i) = \ell(w) + 1$ if and only if $w(i) < w(i+1)$.*
(2) *$\ell(ws_i) = \ell(w) - 1$ if and only if $w(i) > w(i+1)$.*
(3) *For any reduced decomposition $(i_1, \ldots, i_l)$ of w,*

$$\Delta_w = \{s_{i_l} \cdots s_{i_2}(\alpha_{i_1}), s_{i_l} \cdots s_{i_3}(\alpha_{i_2}), \ldots, s_{i_l}(\alpha_{i_{l-1}}), \alpha_{i_l}\}$$

and $l = \ell(w) = |\Delta_w|$.

PROOF. (a) Let $v \in W$, $1 \le m < n$, and $1 \le j < k \le n-1$. If $j = m$ and $k = m+1$, then (j,k) is an inversion of v if and only if it is not an inversion of vs_m. Otherwise, (j,k) is an inversion of v if and only if $(s_m(j), s_m(k))$ is an inversion of vs_m. Therefore

$$\alpha_m \in \Delta_v \Rightarrow \Delta_{vs_m} = s_m(\Delta_v \setminus \{\alpha_m\}),\ |\Delta_{vs_m}| = |\Delta_v| - 1, \tag{1.7.4}$$

$$\alpha_m \notin \Delta_v \Rightarrow \Delta_{vs_m} = s_m(\Delta_v) \cup \{\alpha_m\},\ |\Delta_{vs_m}| = |\Delta_v| + 1. \tag{1.7.5}$$

(b) Clearly, $w = \mathrm{id}_{\mathbb{S}_n}$ if and only if $\Delta_w = \emptyset$. By induction on $\ell(w)$, it follows from (1.7.4) and (1.7.5) that $|\Delta_w| \le \ell(w)$. On the other hand, if $\Delta_w \ne \emptyset$ then there exists $1 \le m < n$ such that $w(m) > w(m+1)$. Then $|\Delta_{ws_m}| = |\Delta_w| - 1$ by (1.7.4). By induction on $|\Delta_w|$ it follows that there exist $j_1, \ldots, j_l$ with $l = |\Delta_w|$ such that $\Delta_{ws_{j_1} \cdots s_{j_l}} = \emptyset$, and hence $w = s_{j_l} \cdots s_{j_1}$. Thus $\ell(w) \le |\Delta_w|$. Therefore $\ell(w) = |\Delta_w|$.

(c) Since $\ell(w) = |\Delta_w|$ by (b), (1) and (2) follow from (1.7.4) and (1.7.5) with $v = w$, $m = i$. Finally, (3) follows by induction on $\ell(w)$ from (1) and (1.7.5). □

DEFINITION 1.7.3. Let $n \ge 1$ be a natural number. The Artin **braid group** $\mathbb{B}_n$ is the group generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } 1 \le i \le n-2, \tag{1.7.6}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for all } 1 \le i,j \le n-1,\ |i-j| > 1. \tag{1.7.7}$$

Thus $\mathbb{B}_1$ is the trivial group with one element, and $\mathbb{B}_2 \cong \mathbb{Z}$.

It follows from the description of $\mathbb{S}_n$ in Theorem 1.7.1 that

$$\mathbb{B}_n \to \mathbb{S}_n, \ \ \sigma_i \mapsto s_i, \ 1 \le i \le n-1,$$

defines a surjective group homomorphism.

The following Theorem, attributed to Matsumoto, is a special case of an important tool in the theory of Coxeter groups. Here it will be used to describe the components of the comultiplication of the tensor algebra of a braided vector space, see e. g. Theorem 1.9.1.

THEOREM 1.7.4. *Let $n \ge 2$. Then*

$$\sigma : \mathbb{S}_n \to \mathbb{B}_n, \ \ w = s_{i_1} \cdots s_{i_l} \mapsto \sigma_{i_1} \cdots \sigma_{i_l},$$

where $(i_1, \dots, i_l)$ is a reduced decomposition of w, is a well-defined map.

PROOF. Let $w \in \mathbb{S}_n$, $l = \ell(w)$, and let $(i_1, \dots, i_l)$, $(j_1, \dots, j_l)$ be two reduced decompositions of w. We have to show that

$$\sigma_{i_1} \cdots \sigma_{i_l} = \sigma_{j_1} \cdots \sigma_{j_l}. \tag{1.7.8}$$

We proceed by induction on l. If $l \le 1$ then (1.7.8) clearly holds. Assume that $l \ge 2$. If $i_l = j_l$ then $(i_1, \dots, i_{l-1})$ and $(j_1, \dots, j_{l-1})$ are reduced decompositions of ws_{i_l} and hence (1.7.8) holds by induction hypothesis.

Assume that $i_l < j_l - 1$. Then (i_l, i_{l+1}) and (j_l, j_{l+1}) are inversions of w. Theorem 1.7.2(2) implies that $w = us_{j_l}s_{i_l} = us_{i_l}s_{j_l}$ for some $u \in \mathbb{S}_n$ $\ell(u) = l-2$. Therefore

$$\sigma_{i_1} \cdots \sigma_{i_l} = \sigma(u)\sigma_{j_l}\sigma_{i_l} = \sigma(u)\sigma_{i_l}\sigma_{j_l} = \sigma_{j_1} \cdots \sigma_{j_l}$$

by induction hypothesis and by (1.7.7).

Assume that $j_l = i_l + 1$. Then (i_l, i_{l+1}) and (i_{l+1}, i_{l+2}) are inversions of w. Hence $(i_l, i_{l+2}) \in \Delta_w$. Theorem 1.7.2(2) implies that $w = us_{i_l}s_{j_l}s_{i_l}$ for some $u \in \mathbb{S}_n$ such that $\ell(u) = l-3$. Then $w = us_{j_l}s_{i_l}s_{j_l}$ and

$$\sigma_{i_1} \cdots \sigma_{i_l} = \sigma(u)\sigma_{i_l}\sigma_{j_l}\sigma_{i_l} = \sigma(u)\sigma_{j_l}\sigma_{i_l}\sigma_{j_l} = \sigma_{j_1} \cdots \sigma_{j_l}.$$

by induction hypothesis and by (1.7.6). □

The map σ in Theorem 1.7.4 is a section of the canonical map $\pi : \mathbb{B}_n \to \mathbb{S}_n$, that is, $\pi\sigma = \mathrm{id}_{\mathbb{S}_n}$. It is called the **Matsumoto section**.

Recall the notation $c_i : V^{\otimes n} \to V^{\otimes n}$, $n \ge 2$, $1 \le i \le n-1$, in (1.5.1) for a vector space V with endomorphism $c : V \otimes V \to V \otimes V$. By abuse of notation we thus identify c_i with $c_i \otimes \mathrm{id}_{V^{\otimes m}}$ for all $m \ge 0$.

LEMMA 1.7.5. *Let (V, c) be a braided vector space, and $n \ge 2$. Then*

$$\mathbb{B}_n \to \mathrm{Aut}(V^{\otimes n}), \quad \sigma_i \mapsto c_i, \ 1 \le i \le n-1,$$

defines a group homomorphism.

PROOF. This follows from the definition of the braid group, since the automorphisms c_i satisfy the relations of the generators σ_i of $\mathbb{B}_n$. □

The action of $\mathbb{B}_n$ on $V^{\otimes n}$ defined in Lemma 1.7.5 will be denoted by

$$\Bbbk\mathbb{B}_n \otimes V^{\otimes n} \to V^{\otimes n}, \ \sigma \otimes x \mapsto \sigma x, \tag{1.7.9}$$

for all $\sigma \in \mathbb{B}_n$, $x \in V^{\otimes n}$.

DEFINITION 1.7.6. Let (V, c) be a braided vector space, and $n \geq 2$. For all $w \in \mathbb{S}_n$ we denote the image of w under the composition

$$\mathbb{S}_n \xrightarrow{\sigma} \mathbb{B}_n \to \mathrm{Aut}(V^{\otimes n})$$

by $c_w = c_{i_1} \cdots c_{i_l}$, if $(i_1, \dots, i_l)$ is a reduced decomposition of w.

COROLLARY 1.7.7. *Let (V, c) be a braided vector space, and $n \geq 2$. Then $c_{\mathrm{id}} = \mathrm{id}_{V^{\otimes n}}$, $c_{s_i} = c_i$ for all $1 \leq i \leq n-1$, and $c_{w_1 w_2} = c_{w_1} c_{w_2}$ for any $w_1, w_2 \in \mathbb{S}_n$ with $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.*

PROOF. This follows from Lemma 1.7.5 and Theorem 1.7.4. $\square$

If c is the flip map, Definition 1.7.6 describes the natural left action of the symmetric group $\mathbb{S}_n$ on $V^{\otimes n}$ with

$$c_w(x_1 \otimes \cdots \otimes x_n) = x_{w^{-1}(1)} \otimes \cdots \otimes x_{w^{-1}(n)}$$

for all $n \geq 2$ and $x_i \in V$ for all $1 \leq i \leq n$. More generally, there is an explicit formula for c_w in the case of diagonal braidings.

PROPOSITION 1.7.8. *Let V be a vector space with basis $(x_i)_{i \in I}$ and braiding c given by*

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \; i, j \in I,$$

where the q_{ij}, $i, j \in I$, are non-zero scalars in $\Bbbk$. Then for all $n \geq 1$, $w \in \mathbb{S}_n$ and all functions $k : \{a \in \mathbb{N} \mid 1 \leq a \leq n\} \to I$,

$$c_w(x_{k(1)} \otimes \cdots \otimes x_{k(n)}) = \prod_{\substack{a<b,\\ w(a)>w(b)}} q_{k(a),k(b)} x_{k(w^{-1}(1))} \otimes \cdots \otimes x_{k(w^{-1}(n))}.$$

PROOF. For $w = s_i$, $1 \leq i \leq n-1$, the claim holds by definition of $c_{s_i} = c_i$, and since $\Delta_{s_i} = \{(i, i+1)\}$. If the length of w is $l \geq 2$, let $(i_1, \dots, i_l)$ be a reduced decomposition of w. Write $w = s_{i_1} u$, $u = s_{i_2} \cdots s_{i_l}$. By induction on the length of w we may assume that the formula holds for u. Let

$$x_k = x_{k(1)} \otimes \cdots \otimes x_{k(n)}, \; k : \{a \in \mathbb{N} \mid 1 \leq a \leq n\} \to I.$$

We know from Theorem 1.7.2(3) that $\Delta_w = \Delta_u \cup \{(u^{-1}(i_1), u^{-1}(i_1 + 1))\}$ and $|\Delta_w| = |\Delta_u| + 1$. Therefore

$$\begin{aligned} c_w(x_k) = c_{i_1} c_u(x_k) = c_{i_1} \Big(\prod_{\substack{a<b,\\ u(a)>u(b)}} q_{k(a),k(b)} x_{ku^{-1}} \Big) \\ = \prod_{\substack{a<b,\\ u(a)>u(b)}} q_{k(a),k(b)} q_{ku^{-1}(i_1), ku^{-1}(i_1+1)} x_{ku^{-1} s_{i_1}} \\ = \prod_{\substack{a<b,\\ w(a)>w(b)}} q_{k(a),k(b)} x_{kw^{-1}}. \end{aligned}$$

This proves the claim. $\square$

We introduce the following useful notation. For all natural numbers $2 \leq m \leq n$ and $0 \leq i \leq n-m$ there are embeddings of groups

$$\mathrm{sh}^i_{m,n} : \mathbb{S}_m \to \mathbb{S}_n, \; s_j \mapsto s_{j+i}, \; 1 \leq j \leq m-1. \tag{1.7.10}$$

We will write

$$\mathrm{sh}^i_{m,n}(w) = w^{\uparrow i},\ w \in \mathbb{S}_m. \tag{1.7.11}$$

Thus we identify $\mathbb{S}_m$ with $\{w \in \mathbb{S}_n \mid w(j) = j \text{ for all } m+1 \le j \le n\}$. The **shift operators** $\uparrow i$ can also be defined for the braid group. There are group homomorphisms $\mathrm{sh}^i_{m,n} : \mathbb{B}_m \to \mathbb{B}_n$, $\sigma_j \mapsto \sigma_{j+i}$, $1 \le j \le m-1$. These maps are embeddings, but we will not use this fact. However, we will write

$$\mathrm{sh}^i_{m,n}(\sigma) = \sigma^{\uparrow i},\ \sigma \in \mathbb{B}_m. \tag{1.7.12}$$

Another type of shift operators are defined for automorphisms:

$$\mathrm{Aut}(V^{\otimes m}) \to \mathrm{Aut}(V^{\otimes n}),\ f \mapsto f^{\uparrow i} = \mathrm{id}_{V^{\otimes i}} \otimes f \otimes \mathrm{id}_{V^{\otimes n-m-i}}. \tag{1.7.13}$$

Then ${c_j}^{\uparrow i} = c_{j+i}$ and ${c_w}^{\uparrow i} = c_{w^{\uparrow i}}$ for all $1 \le j \le m-1$ and $w \in \mathbb{S}_m$.

DEFINITION 1.7.9. Let (V, c) be a braided vector space. For $m, n \ge 1$ let

$$s_{m,n} = \left(\begin{smallmatrix} 1 & 2 & \dots & m & m+1 & m+2 & \dots & m+n \\ n+1 & n+2 & \dots & n+m & 1 & 2 & \dots & n \end{smallmatrix}\right) \in \mathbb{S}_{m+n},$$
$$c_{m,n} = c_{s_{m,n}} \in \mathrm{Aut}(V^{\otimes m+n}).$$

We write $\Bbbk = V^{\otimes 0}$, and denote for all $n \ge 0$ by $c_{n,0} : V^{\otimes n} \otimes \Bbbk \to \Bbbk \otimes V^{\otimes n}$ and $c_{0,n} : \Bbbk \otimes V^{\otimes n} \to V^{\otimes n} \otimes \Bbbk$ the canonical isomorphisms. By abuse of notation we again identify $c_{m,n}$ with $c_{m,n} \otimes \mathrm{id}_{V^{\otimes p}}$ for all $p \ge 0$.

COROLLARY 1.7.10. *Let (V, c) be a braided vector space, and $l, m, n \ge 1$. Then*

(1) $c_{m,n} = (c_n c_{n-1} \cdots c_1)(c_n c_{n-1} \cdots c_1)^{\uparrow 1} \cdots (c_n c_{n-1} \cdots c_1)^{\uparrow m-1}$,
(2) $c_{m,n} = (c_1 c_2 \cdots c_m)^{\uparrow n-1}(c_1 c_2 \cdots c_m)^{\uparrow n-2} \cdots (c_1 c_2 \cdots c_m)$,
(3) $(c_{m,n})^{-1} = (c^{-1})_{n,m}$,
(4) $c_{l+m,n} = c_{l,n}{c_{m,n}}^{\uparrow l}$,
(5) $c_{l,m+n} = {c_{l,n}}^{\uparrow m} c_{l,m}$.

In particular, for all $n \ge 1$ we obtain that

$$c_{1,n} = c_n c_{n-1} \cdots c_1, \qquad c_{n,1} = c_1 c_2 \cdots c_n. \tag{1.7.14}$$

PROOF. By counting the inversions of $s_{m,n}$ we see that $\ell(s_{m,n}) = mn$. Hence

$$(n, n-1, \dots, 1, n+1, n, \dots, 2, \dots, n+m-1, n+m-2, \dots, m),$$
$$(n, n+1, \dots, n+m-1, \dots, 2, 3, \dots, m+1, 1, 2, \dots, m)$$

are reduced decompositions of $s_{m,n}$. Thus (1) and (2) follow from Theorem 1.7.4. The equality in (3) follows by computing the left-hand side with (1) and the right-hand side with (2). The equations in (4) and (5) follow from the formulas in (1) and (2). □

For any group G and any $V \in {}^G_G\mathcal{YD}$ (or V in a braided strict monoidal category, see Section 3.2), the braid group acts on tensor powers of V as in Lemma 1.7.5. The maps $c_{m,n}$ arise naturally in this context.

LEMMA 1.7.11. *Let G be a group, and $V \in {}^G_G\mathcal{YD}$ with braiding $c = c_{V,V}$. Then for all $m, n \ge 1$,*

$$c_{V^{\otimes m}, V^{\otimes n}} = c_{m,n}.$$

Proof. By Corollary 1.7.10(2) it suffices to show that for all $m, n \geq 1$,

$$c_{V^{\otimes m}, V^{\otimes n}} = (c_n c_{n+1} \cdots c_{n+m-1}) \cdots (c_2 c_3 \cdots c_{m+1})(c_1 c_2 \cdots c_m).$$

(1) By induction on m we first prove that $c_{V^{\otimes m}, V} = c_1 c_2 \cdots c_m$ for all $m \geq 1$. This is clear for $m = 1$. Let $m \geq 1$. Then

$$c_{V^{\otimes m+1}, V} = c_{V^{\otimes m} \otimes V, V} = (c_{V^{\otimes m}, V} \otimes \mathrm{id}_V) c_{m+1}$$

by (1.4.5), and the claim follows by induction.

(2) Now we show for fixed m by induction on n that $c_{V^{\otimes m}, V^{\otimes n}} = c_{m,n}$ for all $n \geq 1$. For $n = 1$ this holds by (1). Let $n \geq 1$. Then by (1) and (1.4.4),

$$\begin{aligned} c_{V^{\otimes m}, V^{\otimes (n+1)}} &= (\mathrm{id}_{V^{\otimes n}} \otimes c_{V^{\otimes m}, V})(c_{V^{\otimes m}, V^{\otimes n}} \otimes \mathrm{id}_V) \\ &= (c_{n+1} c_{n+2} \cdots c_{n+m})(c_{V^{\otimes m}, V^{\otimes n}} \otimes \mathrm{id}_V), \end{aligned}$$

and the claim follows by induction. $\square$

1.8. Shuffle permutations and braided shuffle elements

Recall the notion of a shuffle permutation from Section 1.2.

Definition 1.8.1. Let n be a natural number, and $0 \leq i \leq n$. A permutation $w \in \mathbb{S}_n$ is called an $(i, n-i)$-**shuffle** or simply an i-shuffle if

$$w(1) < \cdots < w(i), \text{ and } w(i+1) < \cdots < w(n).$$

Let $\mathbb{S}_{i,n-i}$ denote the set of all i-shuffles in $\mathbb{S}_n$.

Note that $\mathbb{S}_{0,n} = \{\mathrm{id}\} = \mathbb{S}_{n,0}$. The cardinality of $\mathbb{S}_{i,n-i}$ is $\binom{n}{i}$. To obtain all $(n-1,1)$- and $(1,n-1)$-shuffles, one looks at the image of n and 1, respectively.

Let $1 \leq i \leq n$. Then

$$s_i s_{i+1} \cdots s_{n-1} = (i\, i+1 \dots n) = \left(\begin{smallmatrix} 1 & 2 & \dots & i-1 & i & i+1 & \dots & n-1 & n \\ 1 & 2 & \dots & i-1 & i+1 & i+2 & \dots & n & i \end{smallmatrix}\right)$$

is an $(n-1,1)$-shuffle of length $n-i$, and

$$s_{i-1} s_{i-2} \cdots s_1 = (i\, i-1 \dots 1) = \left(\begin{smallmatrix} 1 & 2 & 3 & \dots & i-1 & i & i+1 & \dots & n \\ i & 1 & 2 & \dots & i-2 & i-1 & i+1 & \dots & n \end{smallmatrix}\right)$$

is a $(1, n-1)$-shuffle of length $i-1$. Thus

$$\mathbb{S}_{n-1,1} = \{\mathrm{id}\} \cup \{s_i s_{i+1} \cdots s_{n-1} \mid 1 \leq i \leq n-1\}, \tag{1.8.1}$$

$$\mathbb{S}_{1,n-1} = \{\mathrm{id}\} \cup \{s_i s_{i-1} \cdots s_1 \mid 1 \leq i \leq n-1\}. \tag{1.8.2}$$

Shuffle permutations can be described inductively.

Proposition 1.8.2. *Let $n \geq 2$ and $1 \leq i \leq n-1$.*

(1) *$\mathbb{S}_{i,n-i} = \mathbb{S}_{i,n-1-i} \cup \mathbb{S}_{i-1,n-i} s_{n-1} s_{n-2} \cdots s_i$ (disjoint union).*
(2) *Let $w \in \mathbb{S}_{n-1}$. Then $\ell(w s_{n-1} s_{n-2} \cdots s_i) = \ell(w) + n - i$.*

Proof. Let $u \in \mathbb{S}_{i,n-i}$. If $u(n) = n$, then $u \in \mathbb{S}_{i,n-1-i}$. If $u(n) \neq n$, then $u(i) = n$, since u is an i-shuffle. Note that $s_{n-1} s_{n-2} \cdots s_i = (n\, n-1 \cdots i)$. Define $u_1 = u(i\, i+1 \dots n)$. Then $u_1(n) = n$,

$$u_1(1) < u_1(2) < \cdots < u_1(i-1)$$

and

$$u_1(i) = u(i+1) < u_1(i+1) = u(i+2) < \cdots < u_1(n-1) = u(n).$$

Hence $u = u_1 s_{n-1} s_{n-2} \cdots s_i$, and $u_1 \in \mathbb{S}_{i-1,n-i}$. This proves the inclusion $\subseteq$ in (1), and the other inclusion follows similarly.

We prove (2) by induction on $n-i$. Let $u_1 = ws_{n-1}\cdots s_{i+1}$ and $u = u_1 s_i$. Then $\ell(u_1) = \ell(w)+n-i-1$ by induction hypothesis. As $u_1(i) = w(i) < n = u_1(i+1)$, we conclude that $\ell(u) = \ell(u_1)+1$ by Theorem 1.7.2(1). This implies the claim. □

In the next Proposition we show that the i-shuffles are a complete set of representatives of $\mathbb{S}_n$ modulo the subgroup

$$\langle s_{i+1},\dots,s_{n-1}\rangle\langle s_1,\dots,s_{i-1}\rangle \cong \mathbb{S}_{n-i}\times\mathbb{S}_i.$$

PROPOSITION 1.8.3. *Let $n\geq 2$ and $1\leq i\leq n-1$.*

(1) *The map*

$$\mathbb{S}_{i,n-i}\times\mathbb{S}_{n-i}\times\mathbb{S}_i\to\mathbb{S}_n,\quad (u,s,t)\mapsto us^{\uparrow i}t,$$

is bijective.

(2) *Let $u\in\mathbb{S}_{i,n-i}$, $s\in\mathbb{S}_{n-i}$, $t\in\mathbb{S}_i$. Then $\ell(us^{\uparrow i}t) = \ell(u)+\ell(s)+\ell(t)$.*

PROOF. (1) Let $w\in\mathbb{S}_n$. Total orderings of the sets $\{w(l)\mid 1\leq l\leq i\}$ and $\{w(l)\mid i+1\leq l\leq n\}$ define permutations v_1 of $\{1,\dots,i\}$ and v_2 of $\{i+1,\dots,n\}$ with

$$wv_1(1)<\cdots<wv_1(i) \text{ and } wv_2(i+1)<\cdots<wv_2(n).$$

Thus $v_1\in\langle s_1,\dots,s_{i-1}\rangle$, $v_2\in\langle s_{i+1},\dots,s_{n-1}\rangle$, and $wv_1v_2\in\mathbb{S}_{i,n-i}$. Set $u = wv_1v_2$, $t = v_1^{-1}$ and $s\in\mathbb{S}_{n-i}$ such that $s^{\uparrow i} = v_2^{-1}$. Then $w = us^{\uparrow i}t$. Hence the map $\mathbb{S}_{i,n-i}\times\mathbb{S}_{n-i}\times\mathbb{S}_i\to\mathbb{S}_n$ in (1) is surjective. It is bijective since

$$|\mathbb{S}_{i,n-i}\times\mathbb{S}_{n-i}\times\mathbb{S}_i| = n! = |\mathbb{S}_n|.$$

To prove (2), we count the inversions of $w = us^{\uparrow i}t$. Let $1\leq k<l\leq n$. We distinguish three cases. If $l\leq i$, then (k,l) is an inversion of w if and only if (k,l) is an inversion of t. If $i+1\leq k$, then (k,l) is an inversion of w if and only if (k,l) is an inversion of $s^{\uparrow i}$. If $k\leq i<l$, then (k,l) is an inversion of w if and only if $(t(k),s^{\uparrow i}(l))$ is an inversion of u. This implies (2) by Theorem 1.7.2(3). □

COROLLARY 1.8.4. *Let $n\geq 2$.*

(1) *The multiplication map $\mathbb{S}_{n-1,1}\times\mathbb{S}_{n-2,1}\times\cdots\times\mathbb{S}_{1,1}\to\mathbb{S}_n$ is bijective.*

(2) *Let $w_i\in\mathbb{S}_{i,1}$ for all $1\leq i\leq n-1$. Then*

$$\ell(w_{n-1}w_{n-2}\cdots w_1) = \ell(w_{n-1})+\ell(w_{n-2})+\cdots+\ell(w_1).$$

PROOF. By Proposition 1.8.3, the multiplication map $\mathbb{S}_{n-1,1}\times\mathbb{S}_{n-1}\to\mathbb{S}_n$ is bijective, and $\ell(ut) = \ell(u)+\ell(t)$ for all $u\in\mathbb{S}_{n-1,1}$ and $t\in\mathbb{S}_{n-1}$. Hence the claim follows by induction on n. □

COROLLARY 1.8.5. *Let $n\geq 2$. Then $\mathbb{S}_{i,n-i}\mathbb{S}_{n-i}{}^{\uparrow i} = \mathbb{S}_{n-1,1}\mathbb{S}_{n-2,1}\cdots\mathbb{S}_{i,1}$ for any $1\leq i<n$.*

PROOF. Both subsets $\mathbb{S}_{i,n-i}\mathbb{S}_{n-i}{}^{\uparrow i}$ and $\mathbb{S}_{n-1,1}\mathbb{S}_{n-2,1}\cdots\mathbb{S}_{i,1}$ of $\mathbb{S}_n$ have cardinality $n(n-1)\cdots(i+1)$ by Proposition 1.8.3(1) and Corollary 1.8.4(1). Moreover, both sets consist of representatives of minimal length of the left $\mathbb{S}_i$ cosets of $\mathbb{S}_n$ by Proposition 1.8.3(2) and Corollary 1.8.4. □

REMARK 1.8.6. Using Corollary 1.8.4 together with (1.8.1) one obtains reduced decompositions for any element of $\mathbb{S}_n$. In particular,

$$(1,2,\dots,n-1,1,2,\dots,n-2,\dots,1,2,1)$$

is a reduced decomposition of the unique longest element

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$$

in $\mathbb{S}_n$, and w_0 has length $\frac{n(n-1)}{2}$ and order two. Conjugation with w_0 in $\mathbb{S}_n$ is the inner automorphism

$$\alpha_n : \mathbb{S}_n \to \mathbb{S}_n, \;\; s_i \mapsto s_{n-i}, \; 1 \le i \le n-1. \tag{1.8.3}$$

Since the map α_n permutes the elementary reflections, it preserves the length of elements in $\mathbb{S}_n$.

Theorem 1.7.2(1) implies that any reduced decomposition of an element $w \in \mathbb{S}_n$ can be extended to a reduced decomposition of w_0. Hence

$$\ell(w_0) = \ell(w) + \ell(w^{-1}w_0) \tag{1.8.4}$$

for all $w \in \mathbb{S}_n$.

We introduce the following important elements in the group algebra $\mathbb{Z}\mathbb{B}_n$ of the braid group with integer coefficients. Recall the Matsumoto section $\sigma : \mathbb{S}_n \to \mathbb{B}_n$ of Theorem 1.7.4.

Definition 1.8.7. Let $n \ge 2$ and $0 \le i \le n$. We define the **braided symmetrizer** and the **braided shuffle** elements in $\mathbb{Z}\mathbb{B}_n$ by

$$S_n = \sum_{w \in \mathbb{S}_n} \sigma(w), \;\; S_{i,n-i} = \sum_{w \in \mathbb{S}_{i,n-i}} \sigma(w^{-1}).$$

Note that $S_{0,n} = 1 = S_{n,0}$, and by (1.8.1) and (1.8.2),

$$S_{1,n-1} = 1 + \sigma_1 + \sigma_1\sigma_2 + \cdots + \sigma_1\sigma_2\cdots\sigma_{n-1}, \tag{1.8.5}$$

$$S_{n-1,1} = 1 + \sigma_{n-1} + \sigma_{n-1}\sigma_{n-2} + \cdots + \sigma_{n-1}\cdots\sigma_2\sigma_1. \tag{1.8.6}$$

We define an algebra automorphism of $\mathbb{Z}\mathbb{B}_n$ by

$$\alpha_n : \mathbb{Z}\mathbb{B}_n \to \mathbb{Z}\mathbb{B}_n, \;\; \sigma_i \mapsto \sigma_{n-i}, \; 1 \le i \le n-1, \tag{1.8.7}$$

and an algebra antiautomorphism by

$$\beta_n : \mathbb{Z}\mathbb{B}_n \to \mathbb{Z}\mathbb{B}_n, \;\; \sigma_i \mapsto \sigma_i, \; 1 \le i \le n-1. \tag{1.8.8}$$

Applying α_n, β_n or $\beta_n\alpha_n$ gives new representations of elements in $\mathbb{Z}\mathbb{B}_n$. In particular, by (1.8.5) and (1.8.6), $\alpha_n(S_{1,n-1}) = S_{n-1,1}$.

For all natural numbers $2 \le m \le n$, and $0 \le i \le n-m$ the shift operation of the braid groups extends to an algebra map

$$\mathbb{Z}\mathbb{B}_m \to \mathbb{Z}\mathbb{B}_n, \; \sigma_j \mapsto \sigma_{i+j}, \; 1 \le j \le m-1.$$

Let $x^{\uparrow i}$ denote the image of $x \in \mathbb{Z}\mathbb{B}_m$ under this map. For $i = 0$ we write x instead of $x^{\uparrow 0}$. With this convention, expressions like $S_i S_{n-i}{}^{\uparrow i} S_{i,n-i}$ for $1 \le i \le n-1$ make sense in $\mathbb{Z}\mathbb{B}_k$ for all $k \ge n$, see Corollary 1.8.8 below.

By Theorem 1.7.4, the reduced decompositions of permutations we have obtained above translate directly into equalities in the group algebra $\mathbb{Z}\mathbb{B}_n$.

Corollary 1.8.8. *Let $n \ge 2$ and $1 \le i < n$. Then*

(1) $S_{i,n-i} = S_{i,n-1-i} + \sigma_i\sigma_{i+1}\cdots\sigma_{n-1}S_{i-1,n-i}$,
(2) $S_n = S_i S_{n-i}{}^{\uparrow i} S_{i,n-i}$,
(3) $S_n = S_{1,1}S_{2,1}\cdots S_{n-1,1}$,
(4) $S_{n-i}{}^{\uparrow i} S_{i,n-i} = S_{i,1}S_{i+1,1}\cdots S_{n-1,1}$.

PROOF. (1), (2), and (3) follow from Proposition 1.8.2, Proposition 1.8.3, and Corollary 1.8.4, respectively. (4) follows from Corollary 1.8.5, Proposition 1.8.3(2), and Corollary 1.8.4(2). $\square$

REMARK 1.8.9. By applying α_n, β_n and $\beta_n\alpha_n$ to the product decomposition of S_n in Corollary 1.8.8(3) we obtain three more formulas. In particular,

$$\begin{aligned}
S_1 &= 1,\\
S_2 &= 1+\sigma_1,\\
S_3 &= (1+\sigma_1)(1+\sigma_2+\sigma_2\sigma_1),\\
&= (1+\sigma_2)(1+\sigma_1+\sigma_1\sigma_2),\\
&= (1+\sigma_2+\sigma_1\sigma_2)(1+\sigma_1),\\
&= (1+\sigma_1+\sigma_2\sigma_1)(1+\sigma_2),\\
S_4 &= (1+\sigma_1)(1+\sigma_2+\sigma_2\sigma_1)(1+\sigma_3+\sigma_3\sigma_2+\sigma_3\sigma_2\sigma_1),\\
&= (1+\sigma_3)(1+\sigma_2+\sigma_2\sigma_3)(1+\sigma_1+\sigma_1\sigma_2+\sigma_1\sigma_2\sigma_3),\\
&= (1+\sigma_3+\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_3)(1+\sigma_2+\sigma_1\sigma_2)(1+\sigma_1),\\
&= (1+\sigma_1+\sigma_2\sigma_1+\sigma_3\sigma_2\sigma_1)(1+\sigma_2+\sigma_3\sigma_2)(1+\sigma_3).
\end{aligned}$$

The braided symmetrizer and the braided shuffle elements in $\mathbb{Z}\mathbb{B}_n$ define endomorphism on n-fold tensor products of braided vector spaces (V,c). Recall the $\mathbb{Z}\mathbb{B}_n$-module structure

$$\mathbb{Z}\mathbb{B}_n \otimes V^{\otimes n} \to V^{\otimes n}, \quad \sigma_i \mapsto c_i,\ 1 \le i \le n,$$

of $V^{\otimes n}$ in Lemma 1.7.5.

DEFINITION 1.8.10. Let (V,c) be a braided vector space. Let $n \ge 2$, and $1 \le i \le n-1$. The **braided shuffle** map $S_{i,n-i}^{(V,c)} = S_{i,n-i} : V^{\otimes n} \to V^{\otimes n}$ and the **braided symmetrizer** map $S_n^{(V,c)} = S_n : V^{\otimes n} \to V^{\otimes n}$ are defined by

$$S_{i,n-i} = \sum_{w \in \mathbb{S}_{i,n-i}} c_{w^{-1}}, \quad S_n = \sum_{w \in \mathbb{S}_n} c_w.$$

The inductive description of the braided shuffle map and the braided symmetrizer map in the next corollary is an immediate consequence of Corollary 1.8.8(1) and (2).

COROLLARY 1.8.11. *Let (V,c) be a braided vector space. Let $1 \le i < n$. Then the following equations hold in* $\mathrm{End}(V^{\otimes n})$*:*

$$S_{i,n-i} = S_{i,n-1-i} \otimes \mathrm{id}_V + c_i c_{i+1} \cdots c_{n-1}(S_{i-1,n-i} \otimes \mathrm{id}_V), \tag{1.8.9}$$

$$S_n = (S_i \otimes S_{n-i})S_{i,n-i}. \tag{1.8.10}$$

The braided shuffle elements $S_{n-1,1}$ in $\mathbb{Z}\mathbb{B}_n$ have an interesting description as rational functions. For the proof we need an easy commutation rule in the braid group.

LEMMA 1.8.12. *Let $n \ge 2$, and $p_{n-1} = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1 \in \mathbb{B}_n$. Then*

$$\sigma_{i-1}p_{n-1} = p_{n-1}\sigma_i$$

for all $2 \le i \le n-1$.

PROOF. Using the relations of the braid group we compute

$$\begin{aligned} p_{n-1}\sigma_i &= \sigma_{n-1}\cdots\sigma_{i+1}\sigma_i\sigma_{i-1}\sigma_{i-2}\cdots\sigma_1\sigma_i \\ &= \sigma_{n-1}\cdots\sigma_{i+1}\sigma_i\sigma_{i-1}\sigma_i\sigma_{i-2}\cdots\sigma_1 && \text{(by (1.7.7))} \\ &= \sigma_{n-1}\cdots\sigma_{i+1}\sigma_{i-1}\sigma_i\sigma_{i-1}\sigma_{i-2}\cdots\sigma_1 && \text{(by (1.7.6))} \\ &= \sigma_{i-1}p_{n-1}. && \text{(by (1.7.7))} \end{aligned}$$

This proves the Lemma. □

PROPOSITION 1.8.13. *Let $n \geq 2$. Then*

(1) $S_{n-1,1}(1-\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1) = (1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)S_{n-2,1}{}^{\uparrow 1}$.
(2) $S_{n-1,1}(1-\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(1-\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(1-\sigma_{n-1})$
$= (1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)(1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2)\cdots(1-\sigma_{n-1}^2)$.

PROOF. (1) Let $p_{n-1} = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$. It follows from (1.8.6) that

$$\sigma_{n-1}S_{n-2,1} = S_{n-1,1} - 1, \tag{1.8.11}$$

$$S_{n-2,1}{}^{\uparrow 1} + p_{n-1} = S_{n-1,1}. \tag{1.8.12}$$

It follows from Lemma 1.8.12 that

$$p_{n-1}S_{n-2,1}{}^{\uparrow 1} = S_{n-2,1}p_{n-1}. \tag{1.8.13}$$

Then

$$\begin{aligned} (1-\sigma_{n-1}p_{n-1})S_{n-2,1}{}^{\uparrow 1} &= S_{n-2,1}{}^{\uparrow 1} - \sigma_{n-1}S_{n-2,1}p_{n-1} && \text{(by (1.8.13))} \\ &= S_{n-2,1}{}^{\uparrow 1} - (S_{n-1,1}-1)p_{n-1} && \text{(by (1.8.11))} \\ &= S_{n-2,1}{}^{\uparrow 1} - S_{n-1,1}p_{n-1} + p_{n-1} \\ &= S_{n-1,1}(1-p_{n-1}) && \text{(by (1.8.12))}. \end{aligned}$$

(2) follows from (1). □

COROLLARY 1.8.14. *For all $n \geq 1$ let $p_n = \sigma_n\sigma_{n-1}\cdots\sigma_1$ and*

$$T_n = (1-\sigma_n^2\sigma_{n-1}\cdots\sigma_1)\cdots(1-\sigma_n^2\sigma_{n-1})(1-\sigma_n^2) \in \mathbb{ZB}_{n+1}, \tag{1.8.14}$$

$$\varphi_n = \beta_{n+1}(S_{1,n-1}) - \beta_{n+1}(S_{n-1,1})\sigma_n p_n \in \mathbb{ZB}_{n+1}. \tag{1.8.15}$$

Let $\varphi_0 = 0$. Then the following hold for all $n \geq 1$.

(1) $T_n = S_{n,1}(1-\sigma_n\sigma_{n-1}\cdots\sigma_1)\cdots(1-\sigma_n\sigma_{n-1})(1-\sigma_n)$.
(2) $S_nT_n = S_{n+1}(1-\sigma_n\sigma_{n-1}\cdots\sigma_1)\cdots(1-\sigma_n\sigma_{n-1})(1-\sigma_n)$.
(3) $\varphi_n = 1 - \beta_{n+1}(p_n)p_n + \varphi_{n-1}{}^{\uparrow 1}\sigma_1$.
(4) $S_{n+1}T_{n+1} = \varphi_{n+1}S_n{}^{\uparrow 1}T_n{}^{\uparrow 1} = \varphi_{n+1}\varphi_n{}^{\uparrow 1}\cdots\varphi_1{}^{\uparrow n}$.

REMARK 1.8.15. For $1 \leq n \leq 3$ the definition of φ_n says that

$$\begin{aligned} \varphi_1 &= 1-\sigma_1^2, \\ \varphi_2 &= 1+\sigma_1-\sigma_2^2\sigma_1-\sigma_1\sigma_2^2\sigma_1, \\ \varphi_3 &= 1+\sigma_1+\sigma_2\sigma_1-\sigma_3^2\sigma_2\sigma_1-\sigma_2\sigma_3^2\sigma_2\sigma_1-\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1. \end{aligned}$$

PROOF OF COROLLARY 1.8.14. (1) holds by Proposition 1.8.13(2), and (2) follows from (1), since $S_{n+1} = S_nS_{n,1}$ by Corollary 1.8.8(2).

(3) holds for $n = 1$ by definition, since $\varphi_1 = 1 - \sigma_1^2$. For $n \geq 2$ the claim is obtained from (1.8.11) and (1.8.12) using the maps α_n and β_n. Indeed,

$$\begin{aligned}
\varphi_n &= \beta_{n+1}\alpha_n(S_{n-1,1}) - \beta_{n+1}(S_{n-1,1})\sigma_n p_n \\
&= \beta_{n+1}\alpha_n(1 + \sigma_{n-1}S_{n-2,1}) - \beta_{n+1}(S_{n-2,1}{}^{\uparrow 1} + p_{n-1})\sigma_n p_n \\
&= 1 + \beta_{n+1}(S_{1,n-2}{}^{\uparrow 1})\sigma_1 \\
&\quad - \beta_{n+1}(S_{n-2,1}{}^{\uparrow 1})(\sigma_{n-1}p_{n-1})^{\uparrow 1}\sigma_1 - \beta_{n+1}(\sigma_n p_{n-1})p_n \\
&= 1 - \beta_{n+1}(p_n)p_n + \varphi_{n-1}{}^{\uparrow 1}\sigma_1.
\end{aligned}$$

(4) To prove the first equation, by definition of T_{n+1} it suffices to show that $S_{n+1}(1 - \sigma_{n+1}p_{n+1}) = \varphi_{n+1}S_n{}^{\uparrow 1}$. We obtain that

$$\begin{aligned}
S_{n+1}(1 - \sigma_{n+1}p_{n+1}) &= \beta_{n+1}(S_{1,n})S_n{}^{\uparrow 1} - \beta_{n+1}(S_{n,1})S_n\sigma_{n+1}p_{n+1} \\
&= \beta_{n+1}(S_{1,n})S_n{}^{\uparrow 1} - \beta_{n+1}(S_{n,1})\sigma_{n+1}p_{n+1}S_n{}^{\uparrow 1} = \varphi_{n+1}S_n{}^{\uparrow 1},
\end{aligned}$$

where the first equation follows from Corollary 1.8.8(2), the second equation from Lemma 1.8.12, and the third from (1.8.15).

The second equation in (4) follows by induction from the first one and from $S_1T_1 = 1 - \sigma_1^2 = \varphi_1$. □

1.9. Braided symmetrizer and Nichols algebras

In this section we fix a braided vector space (V, c), where $V \in {}^G_G\mathcal{YD}$, G a group, and $c = c_{V,V}$. In Section 6.4 we will see that the results in this section hold for any braided vector space (V, c) with exactly the same proofs.

Recall that by Proposition 1.6.13 the tensor algebra $T(V)$ is an $\mathbb{N}_0$-graded Hopf algebra in ${}^G_G\mathcal{YD}$ with braiding given for all $m, n \geq 0$ by

$$c_{m,n} : V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}.$$

In the next theorem we prove an explicit formula for the components of the comultiplication in terms of the braiding of V. This formula is similar to the one for the usual comultiplication of $T(V)$ in Example 1.2.24. However, the case of a non-trivial braiding is more involved.

THEOREM 1.9.1. *For all $n \geq 2$ and $1 \leq i \leq n-1$,*

$$\Delta_{i,n-i} = S_{i,n-i} : T^n(V) = V^{\otimes n} \to T^i(V) \otimes T^{n-i}(V) = V^{\otimes n},$$

where Δ is the comultiplication of $T(V)$.

PROOF. Let $n \geq 1$ and $v_1, \dots, v_n \in V$. For clarity we will write $v_1 v_2 \cdots v_n$ for the element $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$. We show by induction on n that

$$\Delta(v_1 \cdots v_n) = 1 \otimes v_1 \cdots v_n + \sum_{i=1}^{n-1} S_{i,n-i}(v_1 \cdots v_n) + v_1 \cdots v_n \otimes 1. \tag{1.9.1}$$

For $n = 1$ the formula clearly holds.

Let $n \geq 2$ and $v_1, \dots, v_n \in V$. By induction hypothesis,

$$\begin{aligned}\Delta(v_1 \cdots v_n) &= \Delta(v_1 \cdots v_{n-1})\Delta(v_n) \\ &= \Big(1 \otimes v_1 \cdots v_{n-1} + \sum_{i=1}^{n-2} S_{i,n-1-i}(v_1 \cdots v_{n-1}) + v_1 \cdots v_{n-1} \otimes 1\Big) \\ &\quad \times (1 \otimes v_n + v_n \otimes 1).\end{aligned}$$

Multiplication of the first factor with $1 \otimes v_n$ gives the sum

$$1 \otimes v_1 \cdots v_n + \sum_{i=1}^{n-2} S_{i,n-1-i}(v_1 \cdots v_{n-1})v_n + v_1 \cdots v_{n-1} \otimes v_n.$$

For the multiplication with $v_n \otimes 1$ we need the braiding. First,

$$(v_1 \cdots v_{n-1} \otimes 1)(v_n \otimes 1) = v_1 \cdots v_n \otimes 1,$$

and by Lemma 1.7.11 and (1.7.14),

$$(1 \otimes v_1 \cdots v_{n-1})(v_n \otimes 1) = c_1 \cdots c_{n-1}(v_1 \otimes \cdots \otimes v_n).$$

To compute the middle terms

$$S_{i,n-1-i}(v_1 \cdots v_{n-1})(v_n \otimes 1) \in (T^i(V) \otimes T^{n-1-i}(V))(T^1(V) \otimes 1)$$

for $1 \leq i \leq n-2$, we note that by Lemma 1.7.11 and (1.7.14) in $T(V) \otimes T(V)$ for all $x \in T^i(V)$, $y \in T^{n-1-i}(V)$,

$$(x \otimes y)(v_n \otimes 1) = c_{n-1-i,1}{}^{\uparrow i}(x \otimes y \otimes v_n) = c_{i+1}c_{i+2} \cdots c_{n-1}(x \otimes y \otimes v_n).$$

Hence

$$\begin{aligned}&\sum_{i=1}^{n-2} S_{i,n-1-i}(v_1 \cdots v_{n-1})(v_n \otimes 1) \\ &\qquad = \sum_{i=1}^{n-2} c_{i+1}c_{i+2} \cdots c_{n-1}(S_{i,n-1-i} \otimes \mathrm{id}_V)(v_1 \otimes \cdots \otimes v_n) \\ &\qquad = \sum_{i=2}^{n-1} c_i c_{i+1} \cdots c_{n-1}(S_{i-1,n-i} \otimes \mathrm{id}_V)(v_1 \otimes \cdots \otimes v_n).\end{aligned}$$

By adding up and reordering the summands we obtain

$$\Delta(v_1 \cdots v_n) = 1 \otimes v_1 \cdots v_n + v_1 \cdots v_n \otimes 1 + A + B + C,$$

where

$$\begin{aligned}A &= \sum_{i=2}^{n-2} \big(S_{i,n-1-i} \otimes \mathrm{id}_V + c_i \cdots c_{n-1}(S_{i-1,n-i} \otimes \mathrm{id}_V)\big)(v_1 \cdots v_n), \\ B &= (S_{1,n-2} \otimes \mathrm{id}_V)(v_1 \cdots v_n) + c_1 \cdots c_{n-1}(v_1 \cdots v_n), \\ C &= v_1 \cdots v_{n-1} \otimes v_n + c_{n-1}(S_{n-2,1} \otimes \mathrm{id}_V)(v_1 \cdots v_n).\end{aligned}$$

By (1.8.9),

$$A = \sum_{i=2}^{n-2} S_{i,n-i}(v_1 \cdots v_n), \quad B = S_{1,n-1}(v_1 \cdots v_n), \quad C = S_{n-1,1}(v_1 \cdots v_n)$$

which implies (1.9.1). □

We note that Theorem 1.9.1 is related to the q-binomial formula.

DEFINITION 1.9.2. Let $\mathbb{Q}(v)$ be the field of rational functions in the indeterminate v over the rational numbers. For all natural numbers $n \geq 0$ and $0 \leq i \leq n$ define elements in $\mathbb{Q}(v)$ by

$$\begin{aligned}(n)_v &= 1 + v + v^2 + \cdots + v^{n-1} = \frac{v^n - 1}{v - 1},\\ (n)^!_v &= (1)_v(2)_v \cdots (n)_v, \quad (0)^!_v = 1,\\ \binom{n}{i}_v &= \frac{(n)^!_v}{(i)^!_v(n-i)^!_v}.\end{aligned}$$

For all $i < 0$ and all $i > n$ let $\binom{n}{i}_v = 0$.

LEMMA 1.9.3. *Let $n \geq 0$.*

(1) *For all $0 \leq i \leq n$, $\binom{n}{i}_v = \binom{n}{n-i}_v = \binom{n-1}{i-1}_v + v^i\binom{n-1}{i}_v$.*
(2) *For all $0 \leq i \leq n$, $\binom{n}{i}_v = v^{n-i}\binom{n-1}{i-1}_v + \binom{n-1}{i}_v$.*
(3) *For all $0 \leq i \leq n$, $\binom{n}{i}_v \in \mathbb{Z}[v]$.*

PROOF. The first equation in (1) holds by definition, and the second is clear for $i = 0$ and for $i = n$. For $0 < i < n$, (1) follows by direct computation:

$$\begin{aligned}\binom{n-1}{i-1}_v + v^i\binom{n-1}{i}_v &= \frac{(n-1)^!_v(i)_v}{(i)^!_v(n-i)^!_v} + v^i\frac{(n-1)^!_v(n-i)_v}{(i)^!_v(n-i)^!_v}\\ &= \frac{(n-1)^!_v}{(i)^!_v(n-i)^!_v}\big((i)_v + v^i(n-i)_v\big)\end{aligned}$$

which clearly equals $\binom{n}{i}_v$. (2) follows from (1) with i replaced by $n - i$, and (3) follows from (1) by induction on n. □

Let q be any element in $\Bbbk$, and let $n, i \in \mathbb{N}_0$ with $i \leq n$. Lemma 1.9.3 allows us to define the q-numbers and q-binomial numbers $(n)_q$ and $\binom{n}{i}_q$ in $\Bbbk$ as the images of $(n)_v$ and $\binom{n}{i}_v$ under the ring homomorphism $\mathbb{Z}[v] \to \Bbbk$ mapping v onto q.

LEMMA 1.9.4. *Let $n \geq 2$ and let $q \in \Bbbk$ be a primitive n-th root of unity. Then $\binom{n}{i}_q = 0$ for all $0 < i < n$.*

PROOF. By assumption, $q \neq 1$. Hence $(m)_q = (q^m - 1)/(q - 1)$ for any $m \in \mathbb{N}_0$. Let $0 < i < n$. Then $(i)^!_q, (n-i)^!_q \neq 0$ in $\Bbbk$ by assumption. Hence

$$\binom{n}{i}_q = \frac{(n)^!_q}{(i)^!_q(n-i)^!_q} = 0$$

in $\Bbbk$, since $(n)_q = 0$. □

For any ring A, let $Z(A) = \{a \in A \mid ax = xa \text{ for all } x \in A\}$ denote its **center**.

PROPOSITION 1.9.5. *Let A be an algebra, $q \in Z(A)$, and $x, y \in A$. Assume that $yx = qxy$. For all $0 \leq i \leq n$, let $\binom{n}{i}_q \in A$ be the image of $\binom{n}{i}_v$ under the ring homomorphism $\mathbb{Z}[v] \to A$ mapping v onto q. Then for all $n \geq 0$,*

$$(x + y)^n = \textstyle\sum_{i=0}^n \binom{n}{i}_q x^i y^{n-i} = \sum_{i=0}^n \binom{n}{i}_q x^{n-i} y^i.$$

PROOF. This follows by induction on n as in the proof of the usual binomial formula using $yx^i = q^i x^i y$ for all $i \geq 0$, and Lemma 1.9.3(1). □

EXAMPLE 1.9.6. Let us consider the special case of Theorem 1.9.1 when $V = \Bbbk x$ is one-dimensional. Then there is a non-zero scalar $q \in \Bbbk$ such that the braiding is given by $c(x \otimes x) = qx \otimes x$. Let $n \geq 2$ and $w \in \mathbb{S}_n$. The linear map $c_w : V^{\otimes n} \to V^{\otimes n}$ is multiplication with the scalar $q^{\ell(w)}$, and by (1.8.9) we see that

$$S_{i,n-i} = S_{i,n-i-1} + q^{n-i} S_{i-1,n-i}$$

in $\operatorname{End}(V^{\otimes n})$ for all $1 \leq i \leq n-1$. These formulas are the recursion formulas for the q-binomial coefficients, see Lemma 1.9.3(2). Hence

$$S_{i,n-i} = \binom{n}{i}_q \operatorname{id} \text{ for all } 0 \leq i \leq n, \tag{1.9.2}$$

$$S_n = (n)^!_q \operatorname{id}, \tag{1.9.3}$$

where the second formula follows from (1.8.10).

By Theorem 1.9.1,

$$\Delta_{i,n-i}(x^n) = S_{i,n-i}(x^n) = \binom{n}{i}_q x^i \otimes x^{n-i}.$$

The same result follows from the q-binomial formula in Proposition 1.9.5. Indeed, $(1 \otimes x)(x \otimes 1) = q(x \otimes 1)(1 \otimes x)$ and hence

$$\Delta(x^n) = (x \otimes 1 + 1 \otimes x)^n = \sum_{i=0}^{n} \binom{n}{i}_q x^i \otimes x^{n-i}. \tag{1.9.4}$$

We will now see that explicit relations of the Nichols algebra are given by the braided symmetrizer maps.

COROLLARY 1.9.7. *Let $n \geq 2$, and let $S_n = S_n^{(V,c)} : V^{\otimes n} \to V^{\otimes n}$ be the braided symmetrizer map.*

(1) *$\Delta_{1^n} = S_n$ in $\operatorname{End}(V^{\otimes n})$, where Δ is the comultiplication of the tensor algebra $T(V)$.*
(2) *$\mathcal{B}(V) = \Bbbk \oplus V \oplus \bigoplus_{n \geq 2} V^{\otimes n} / \ker(S_n)$.*

PROOF. (1) We proceed by induction on n. The case when $n = 1$ is trivial. Let $n \geq 2$, and assume that $\Delta_{1^{n-1}} = S_{n-1}$. Then

$$\Delta_{1^n} = (\Delta_1 \otimes \Delta_{1^{n-1}})\Delta_{1,n-1} = (S_1 \otimes S_{n-1})S_{1,n-1} = S_n,$$

where the first equation holds by Lemma 1.3.13(1b), the second by induction and Theorem 1.9.1, and the third was shown in (1.8.10).

(2) follows from (1) and Definition 1.6.17. □

COROLLARY 1.9.8. *Let $n \geq 2$, $1 \leq i \leq n-1$, and for all $1 \leq j \leq n$ let $\pi_j : V^{\otimes j} \to V^{\otimes j} / \ker(S_j)$ be the canonical map. Then*

$$\ker(\Delta_{1^n}^{T(V)}) = \ker\big((\pi_i \otimes \pi_{n-i})S_{i,n-i}\big).$$

PROOF. The claim follows directly from Corollary 1.9.7 and (1.8.10), since $\ker(S_i \otimes S_{n-i}) = \ker(\pi_i \otimes \pi_{n-i})$. □

It is important to note that the Nichols algebra $\mathcal{B}(V) = T(V)/I(V)$ as an algebra and a coalgebra only depends on the braided vector space (V, c). Let G' be another group, and $V' \in {}^{G'}_{G'}\mathcal{YD}$ such that there is a linear isomorphism $f : V \to V'$ with

$$(f \otimes f)c_{V,V} = (f \otimes f)c_{V',V'}.$$

Then f induces an isomorphism $\mathcal{B}(V) \to \mathcal{B}(V')$ of algebras and coalgebras.

1.10. Examples of Nichols algebras

We are going to describe several examples of Nichols algebras.

Throughout we will use the following notation for algebras given by **generators and relations**. Let X be a set, and let $f_i, g_i \in \Bbbk\langle X\rangle$, $i \in I$, be elements in the free algebra, where I is some index set. Let $(f_i - g_i \mid i \in I)$ be the ideal of $\Bbbk\langle X\rangle$ generated by the elements $f_i - g_i$, $i \in I$. Then

$$\Bbbk\langle X \mid f_i = g_i \text{ for all } i \in I\rangle = \Bbbk\langle X\rangle/(f_i - g_i \mid i \in I)$$

is the algebra generated by X with relations $f_i = g_i$, $i \in I$. By abuse of notation we denote the residue class of $x \in X$ in $\Bbbk\langle X \mid f_i = g_i \text{ for all } i \in I\rangle$ by the same symbol x.

In the whole section let G be a group.

The Nichols algebra of a one-dimensional object $V \in {}^G_G\mathcal{YD}$ is easy to compute.

EXAMPLE 1.10.1. Let $V \in {}^G_G\mathcal{YD}$ be one-dimensional with basis $x \in V$, and $c = c_{V,V}$. Then there is a non-zero scalar q such that $c(x \otimes x) = qx \otimes x$. Let

$$N(q) = \begin{cases} \operatorname{ord}(q) & \text{if } q \neq 1 \text{ and } \operatorname{ord}(q) \text{ is finite,} \\ p & \text{if } q = 1 \text{ and } \operatorname{char}(\Bbbk) = p > 0, \\ \infty & \text{otherwise.} \end{cases} \tag{1.10.1}$$

Thus if $(m)_q = 0$ for some natural number $m \geq 2$, then $N(q)$ is the smallest such m; otherwise $N(q) = \infty$. We have seen in (1.9.3) that $S_n = (n)^!_q\mathrm{id}$. Hence $I(V) = \bigoplus_{n\geq N} \Bbbk x^n$ in $T(V) = \Bbbk[x]$, and

$$\mathcal{B}(V) \cong \begin{cases} \Bbbk[x]/(x^{N(q)}) & \text{if } N(q) \neq \infty, \\ \Bbbk[x] & \text{otherwise.} \end{cases}$$

EXAMPLE 1.10.2. Let $V \in {}^G_G\mathcal{YD}$ be finite-dimensional with basis $x_1, \dots, x_\theta$ and $x_i \in V^{\chi_i}_{g_i}$, $g_i \in G$, $\chi_i \in \widehat{G}$ for all $1 \leq i \leq \theta$. Assume that $\operatorname{char}(\Bbbk) = 0$, and that $\mathcal{B}(V)$ is finite-dimensional. Then for all $1 \leq i \leq \theta$, $\chi_i(g_i) \neq 1$. This follows from Example 1.10.1 and Remark 1.6.19.

In the next two examples we discuss Nichols algebras of irreducible but not one-dimensional Yetter-Drinfeld modules over non-abelian groups.

EXAMPLE 1.10.3. Let V_n, $n \geq 3$, be the irreducible Yetter-Drinfeld module in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$ in Example 1.4.7 with basis x_t, $t \in \mathcal{O}_2$. Then the quadratic relations of $\mathcal{B}(V_n)$ in $\ker(\mathrm{id}_{V^{\otimes 2}} + c)$ are

$$\begin{aligned} x_t^2 &= 0 \text{ for all } t \in \mathcal{O}_2, \\ x_s x_t + x_t x_s &= 0 \text{ for all } s, t \in \mathcal{O}_2 \text{ with } st = ts,\ s \neq t, \\ x_s x_t + x_t x_{t\triangleright s} + x_{t\triangleright s} x_s &= 0 \text{ for all } s, t \in \mathcal{O}_2 \text{ with } st \neq ts. \end{aligned}$$

Let $\widetilde{\mathcal{B}}(V_n) = T(V_n)/(x \in V_n^{\otimes 2} \mid c(x) = -x)$ be the algebra generated by x_t, $t \in \mathcal{O}_2$, with the above quadratic relations of the Nichols algebra only. It is known that

$$\dim \widetilde{\mathcal{B}}(V_3) = 12, \quad \dim \widetilde{\mathcal{B}}(V_4) = 576, \quad \dim \widetilde{\mathcal{B}}(V_5) = 8{,}294{,}400,$$

and that $\mathcal{B}(V_n) = \widetilde{\mathcal{B}}(V_n)$ for $n = 3, 4, 5$. For $n = 3, 4$ this was shown in [**MS00**], and for $n = 5$ by M. Graña (with help by J.-E. Roos). But for $n \geq 6$, the Nichols algebra of V_n is a mystery. It is not even known whether $\widetilde{\mathcal{B}}(V_n)$ is finite-dimensional for one $n \geq 6$.

EXAMPLE 1.10.4. Let $(X, \triangleright)$ and $\boldsymbol{q}$ be the rack and constant two-cocycle in Example 1.5.13 with $X = \{1, 2, 3, 4\}$ and $\lambda = -1$. We write x_i for the basis vector of $\Bbbk X$ corresponding to $i \in X$. Then $(\Bbbk X, c^{\boldsymbol{q}})$ is a braided vector space of group type by Proposition 1.5.12. By Proposition 1.5.6, $\Bbbk X \in {}^{G}_{G}\mathcal{YD}$ for some group G. The Nichols algebra of $\Bbbk X$ appeared first in [**Gn00b**]. We follow the presentation in [**HV18**]. The Nichols algebra $\mathcal{B}(\Bbbk X)$ can be presented as an algebra by generators x_i, $i \in X$, and relations

$$\begin{aligned} x_1^2 = x_2^2 = x_3^2 = x_4^2 &= 0, \\ x_1x_2 + x_2x_3 + x_3x_1 &= 0, \\ x_1x_3 + x_3x_4 + x_4x_1 &= 0, \\ x_1x_4 + x_4x_2 + x_2x_1 &= 0, \\ x_2x_4 + x_4x_3 + x_3x_2 &= 0, \\ (x_1 + x_2 + x_3)^6 &= 0. \end{aligned}$$

Let $y = x_1x_3 + x_3x_2 + x_2x_1 \in \mathcal{B}(\Bbbk X)$. The elements

$$x_1^{n_1}(x_1 + x_2)^{n_2} x_3^{n_3} y^{n_0} x_4^{n_4}, \text{ where } n_1, n_3, n_4 \in \{0, 1\}, n_2, n_0 \in \{0, 1, 2\},$$

form a basis of $\mathcal{B}(\Bbbk X)$. In particular, $\dim \mathcal{B}(\Bbbk X) = 72$. Note that the quadratic relations of $\mathcal{B}(\Bbbk X)$ can easily be obtained using Corollary 1.9.8.

For the next example we need the logarithm of an automorphism.

LEMMA 1.10.5. *Assume that* $\operatorname{char}(\Bbbk) = 0$. *Let V be a vector space and let $\mu : V \times V \to V$, $\mu(u, v) = uv$, be a bilinear map. Let σ be an automorphism of (V, μ). Assume that $\sigma - \mathrm{id}$ is locally nilpotent, that is, for all $v \in V$ there is $m \geq 0$ with $(\sigma - \mathrm{id})^m(v) = 0$. Then*

$$\log(\sigma) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (\sigma - \mathrm{id})^m \in \operatorname{End}(V)$$

is a derivation of (V, μ), that is, $\log(\sigma)(uv) = \log(\sigma)(u)v + u\log(\sigma)(v)$ for all $u, v \in V$.

PROOF. Let $x = \sigma - \mathrm{id}$. First note that for any $k \geq 1$,

$$\sigma^k \sum_{n=0}^{\infty} (-1)^n \binom{n+k-1}{k-1} x^n = \mathrm{id}_V \tag{1.10.2}$$

in $\operatorname{End}(V)$. Indeed the claim is true for $k = 1$, and for $k > 1$ it follows by induction on k by substituting $\sigma^k = \sigma^{k-1}(x + \mathrm{id}_V)$.

For any $v \in V$, $\log(\sigma)(v)$ is well-defined since x is locally nilpotent. Moreover, $x(uv) = x(u)\sigma(v) + ux(v)$ for all $u, v \in V$. Since x and σ are commuting endomorphisms, it follows for any $u, v \in V$ that

$$\begin{aligned}
\log(\sigma)(uv) &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m(uv) \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{k=0}^{m} \binom{m}{k} x^k(u)\sigma^k x^{m-k}(v) \\
&= u\log(\sigma)(v) + \sum_{k=1}^{\infty} x^k(u) \sum_{m=k}^{\infty} \frac{(-1)^{m+1}}{m} \binom{m}{k} \sigma^k x^{m-k}(v) \\
&= u\log(\sigma)(v) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k(u) \sum_{n=0}^{\infty} (-1)^n \binom{n+k-1}{k-1} \sigma^k x^n(v) \\
&= u\log(\sigma)(v) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k(u)v \\
&= u\log(\sigma)(v) + \log(\sigma)(u)v,
\end{aligned}$$

where (1.10.2) is used in the fifth equation. This proves the claim. $\square$

EXAMPLE 1.10.6. Assume that $\mathrm{char}(\Bbbk) = 0$. Let $J^+ = F_g(V(1,2)) \in {}^{\mathbb{Z}}_{\mathbb{Z}}\mathcal{YD}$ be the Yetter-Drinfeld module in Example 1.4.19. Thus $J^+ = J_g^+$, where g is a generator of $\mathbb{Z}$, and there is a basis v_1, v_2 of J^+ such that $g \cdot v_1 = v_1$, $g \cdot v_2 = v_2 + v_1$. We prove that

$$\mathcal{B}(J^+) = \Bbbk\langle v_1, v_2\rangle / \Big(v_2v_1 - v_1v_2 + \frac{1}{2}v_1^2\Big)$$

and that the monomials

$$v_1^k v_2^l, \quad k, l \geq 0, \tag{1.10.3}$$

form a basis of $\mathcal{B}(J^+)$.

Let $x = v_2v_1 - v_1v_2 + \frac{1}{2}v_1^2 \in T(J^+)$. Then

$$\begin{aligned}
\Delta_{T(J^+)}(x) =& x \otimes 1 + 1 \otimes x + v_2 \otimes v_1 + v_1 \otimes v_2 \\
& - v_1 \otimes v_2 - (v_1 + v_2) \otimes v_1 + v_1 \otimes v_1 \\
=& x \otimes 1 + 1 \otimes x.
\end{aligned}$$

Hence $x = 0$ in $\mathcal{B}(J^+)$. Hence $\mathcal{B}(J^+)$ is spanned by the monomials $v_1^k v_2^l$, $k, l \geq 0$. Let σ be the automorphism of the algebra $\mathcal{B}(J^+)$ with $\sigma(v) = g \cdot v$ for all $v \in \mathcal{B}(J^+)$. Then $\sigma - \mathrm{id}$ is locally nilpotent, and hence $\partial = \log(\sigma)$ is a derivation of $\mathcal{B}(J^+)$ by Lemma 1.10.5. By definition, $\partial(v_1) = 0$, $\partial(v_2) = v_1$. Let $i_1, \dots, i_m \in \{1, 2\}, m \geq 1$. Then by induction on n it follows that

$$\partial^n(v_{i_1} \cdots v_{i_m}) = n! v_1^m$$

where $n = i_1 + \cdots + i_m - m$. Let $(a_l)_{0 \leq l \leq m} \in \Bbbk^{m+1}$ with $\sum_{l=0}^{m} a_l v_1^{m-l} v_2^l = 0$, and let $0 \leq l_0 \leq m$ such that $a_l = 0$ for all $l > l_0$. Then

$$\partial^{l_0}\left(\sum_{l=0}^{m} a_l v_1^{m-l} v_2^l\right) = a_{l_0} l_0! v_1^m.$$

Since $v_1^m \neq 0$ by Example 1.10.1, it follows that $a_{l_0} = 0$, and hence $a_l = 0$ for all $0 \leq l \leq m$ by induction on $m - l$. Hence the monomials in (1.10.3) are linearly independent in $\mathcal{B}(J^+)$.

EXAMPLE 1.10.7. Assume that $\mathrm{char}(\Bbbk) = 0$. Let $J^- = F_g(V(-1,2)) \in {}^{\mathbb{Z}}_{\mathbb{Z}}\mathcal{YD}$ be the Yetter-Drinfeld module in Example 1.4.19. Thus $J^- = J_g^-$, where g is a generator of $\mathbb{Z}$, and there is a basis v_1, v_2 of J^- with $g \cdot v_1 = -v_1$, $g \cdot v_2 = -v_2 + v_1$. We prove that

$$\mathcal{B}(J^-) = \Bbbk\langle v_1, v_2\rangle/(v_1^2, v_2^2 v_1 - v_1 v_2^2 - v_1 v_2 v_1)$$

and that the monomials

$$v_1^{a_1}(v_2 v_1)^{a_2} v_2^{a_3}, \quad a_1 \in \{0,1\},\, a_2, a_3 \geq 0, \tag{1.10.4}$$

form a basis of $\mathcal{B}(J^-)$.

For all $k \geq 2$ let $\pi_k : (J^-)^{\otimes k} \to (J^-)^{\otimes k}/\ker(S_k)$ be the canonical map. Since $gv_1 = -v_1$, it follows that $c(v_1 \otimes v_1) = -v_1 \otimes v_1$ and hence $v_1^2 = 0$ in $\mathcal{B}(J^-)$. Let $x = v_2^2 v_1 - v_1 v_2^2 - v_1 v_2 v_1 \in T(J^-)$. By Corollary 1.9.8, $x \in \ker(\Delta_{1^3})$ if and only if $(\mathrm{id}_{J^-} \otimes \pi_2) S_{1,2}(x) = 0$. Since $S_{1,2} = \mathrm{id} + c_1 + c_1 c_2$ by (1.8.9), it follows that

$$\begin{aligned}(\mathrm{id}_{J^-} \otimes \pi_2) S_{1,2}(x) &= v_2 \otimes v_2 v_1 + g v_2 \otimes v_2 v_1 + g^2 v_1 \otimes v_2^2 - v_1 \otimes v_2^2 \\ &\quad - g v_2 \otimes v_1 v_2 - g^2 v_2 \otimes v_1 v_2 - v_1 \otimes v_2 v_1 - g v_2 \otimes v_1^2 - g^2 v_1 \otimes v_1 v_2 \\ &= 0\end{aligned}$$

because of $v_1^2 \in \ker(\pi_2)$. Hence $x = 0$ in $\mathcal{B}(J^-)$, and therefore $\mathcal{B}(J^-)$ is spanned by the monomials in (1.10.4). Below we will further need that

$$v_2(v_2 v_1)^k = (v_1 v_2)^k v_2 + k(v_1 v_2)^k v_1 \tag{1.10.5}$$

for all $k \geq 1$ which follows from $x = 0$ by induction on k.

Assume that there is a non-trivial linear combination of the monomials in (1.10.4) which is zero in $\mathcal{B}(J^-)$. By multiplying this with v_1 from the left or v_2 from the right if necessary, it follows that there is $m \geq 2$, m even, and a non-trivial linear combination of the monomials $v_1(v_2 v_1)^a v_2^{m-1-2a}$, $0 \leq a \leq (m-1)/2$, which is zero in $\mathcal{B}(J^-)$.

Let σ be the automorphism of the algebra $\mathcal{B}(J^-)$, where $\sigma(v) = (-1)^n g v$ for all $v \in \mathcal{B}(J^-)(n)$, $n \geq 0$. Then $\sigma(v_1) = v_1$, $\sigma(v_2) = v_2 - v_1$, and the map $\sigma - \mathrm{id}$ is locally nilpotent. Hence $\partial = -\log(\sigma)$ is a derivation of $\mathcal{B}(J^-)$ by Lemma 1.10.5. By definition, $\partial(v_1) = 0$, $\partial(v_2) = v_1$. For any $n \geq 1$ let

$$M_n = \{(i_1, i_2, \ldots, i_{2n}) \in \{1,2\}^{2n} \mid i_1 = 1, \forall 1 \leq k \leq n : i_{2k} = 2\}.$$

Since $v_1^2, x \in I(J^-)$, it follows by induction on n that

$$\forall n \geq 1, (1, i_2, \ldots, i_{2n}) \in \{1,2\}^{2n} \setminus M_n : v_1 v_{i_2} \cdots v_{i_{2n}} = 0 \tag{1.10.6}$$

in $\mathcal{B}(J^-)$. Then by induction on k it follows from (1.10.6) that

$$\partial^k(v_{i_1} \cdots v_{i_{2n}}) = k!(v_1 v_2)^n, \quad \partial^{k+1}(v_{i_1} \cdots v_{i_{2n}}) = 0 \tag{1.10.7}$$

in $\mathcal{B}(J^-)$ for any $(i_1, \ldots, i_{2n}) \in M_n$, where $k = \sum_{j=1}^n i_{2j-1} - n$ is the number of 2's at the odd positions. Let $(a_l)_{0 \leq l < m/2} \in \Bbbk^{m/2}$ be such that

$$\sum_{l=0}^{m/2-1} a_l (v_1 v_2)^{m/2-l} v_2^{2l} = 0$$

in $\mathcal{B}(J^-)$, and let $0 \le l_0 < m/2$ with $a_l = 0$ for all $l > l_0$. Then

$$\partial^{l_0}\left(\sum_{l=0}^{m/2-1} a_l (v_1v_2)^{m/2-l} v_2^{2l}\right) = a_{l_0} l_0! (v_1v_2)^{m/2}$$

by (1.10.7). We prove that for any $n \ge 1$, $(v_1v_2)^n$ and $(v_2v_1)^n$ are linearly independent in $\mathcal{B}(J^-)$. Then it follows that $a_{l_0} = 0$, and hence $a_l = 0$ for all $0 \le l < m/2$ by induction on $m/2 - l$. Hence the monomials in (1.10.4) are linearly independent in $\mathcal{B}(J^-)$.

Since

$$S_{1,1}(v_1v_2) = v_1 \otimes v_2 + (-v_2 + v_1) \otimes v_1, \tag{1.10.8}$$

$$S_{1,1}(v_2v_1) = v_2 \otimes v_1 - v_1 \otimes v_2, \tag{1.10.9}$$

the monomials v_1v_2 and v_2v_1 are linearly independent in $\mathcal{B}(J^-)$. Let now $n \ge 1$ and assume that $(v_1v_2)^n$ and $(v_2v_1)^n$ are linearly independent. Then

$$(\mathrm{id} \otimes \pi_{2n}) S_{1,2n}((v_1v_2)^n v_1) = v_1 \otimes (v_2v_1)^n + g^{2n} v_1 \otimes (v_1v_2)^n$$

by (1.10.6), and hence $(v_1v_2)^n v_1 \ne 0$ by (1.8.9).

Let now $\lambda_1, \lambda_2 \in \Bbbk$. Then

$$\begin{aligned}
&(\mathrm{id} \otimes \pi_{2n+1}) S_{1,2n+1}(\lambda_1 (v_1v_2)^{n+1} + \lambda_2 (v_2v_1)^{n+1})\\
&= \lambda_1\big(v_1 \otimes (v_2v_1)^n v_2 + g^{2n} v_1 \otimes (v_1v_2)^n v_2 + g^{2n+1} v_2 \otimes (v_1v_2)^n v_1\big)\\
&\quad + \lambda_2\big(v_2 \otimes (v_1v_2)^n v_1 + g v_1 \otimes v_2 (v_2v_1)^n + g^{2n+1} v_1 \otimes (v_2v_1)^n v_2\big)\\
&= (\lambda_2 - \lambda_1) v_2 \otimes (v_1v_2)^n v_1 + (\lambda_1 - \lambda_2) v_1 \otimes (v_2v_1)^n v_2\\
&\quad + (\lambda_1 - \lambda_2) v_1 \otimes (v_1v_2)^n v_2 + (\lambda_1(2n+1) - \lambda_2 n) v_1 \otimes (v_1v_2)^n v_1,
\end{aligned}$$

where the first equation follows from (1.10.6), and the second from (1.10.5). Since $(v_1v_2)^n v_1 \ne 0$, we conclude from (1.8.9) that $(v_1v_2)^{n+1}$ and $(v_2v_1)^{n+1}$ are linearly independent. This finishes the proof.

As Example 1.10.1 shows, it can happen that the tensor algebra of an object $V \in {}^G_G\mathcal{YD}$ is strictly graded, or equivalently that $I(V) = 0$. In the next proposition we find general necessary conditions for $I(V) \ne 0$.

Lemma 1.10.8. *Let (V, c) be a braided vector space, $n \ge 2$, and assume that $S^{(V,c)}_{n-1,1} \ne 0$ is not an isomorphism. Then*

$$\ker(\mathrm{id}_{V^{\otimes m}} - c^2_{m-1} c_{m-2} \cdots c_1) \ne 0$$

for some $2 \le m \le n$.

Proof. The identity of Proposition 1.8.13(2) in the group algebra of the braid group implies the following equation in $\mathrm{Aut}(V^{\otimes n})$, $n \ge 2$.

$$\begin{aligned}
&S_{n-1,1}(\mathrm{id}_{V^{\otimes n}} - c_{n-1}c_{n-2}\cdots c_1)(\mathrm{id}_{V^{\otimes n}} - c_{n-1}c_{n-2}\cdots c_2)\cdots(\mathrm{id}_{V^{\otimes n}} - c_{n-1})\\
&\quad = (\mathrm{id}_{V^{\otimes n}} - c^2_{n-1}c_{n-2}\cdots c_1)(\mathrm{id}_{V^{\otimes n}} - c^2_{n-1}c_{n-2}\cdots c_2)\cdots(\mathrm{id}_{V^{\otimes n}} - c^2_{n-1}).
\end{aligned}$$

Thus $\ker(\mathrm{id}_{V^{\otimes n}} - c^2_{n-1}c_{n-2}\cdots c_i) \ne 0$ for some $1 \le i \le n-1$, since $S_{n-1,1}$ is not an isomorphism. The lemma follows with $m = n - i + 1$. □

Proposition 1.10.9. *Let $V \in {}^G_G\mathcal{YD}$ be finite-dimensional with $\dim V = d$, $c = c_{V,V}$, and assume that $I(V) \ne 0$.*

(1) *There exists* $n \geq 2$*, such that* $\ker(\mathrm{id}_{V^{\otimes n}} - c_{n-1}^2 c_{n-2} \cdots c_1) \neq 0$.
(2) *If the braiding is diagonal with matrix* $(q_{a,b})_{1 \leq a,b \leq d}$*, then there is an integer* $n \geq 2$ *and a sequence* $(k_1, \dots, k_n) \in \{1, \dots, d\}^n$ *such that*

$$\prod_{1 \leq i < j \leq n} q_{k_i,k_j} q_{k_j,k_i} = 1.$$

PROOF. (1) The tensor algebra $T(V)$ is not strictly graded, since $I(V) \neq 0$. Hence by Proposition 1.3.14 and Theorem 1.9.1,

$$\Delta_{n-1,1} = S_{n-1,1} : V^{\otimes n} \to V^{\otimes n}$$

is not injective for some $n \geq 2$. Thus (1) follows from Lemma 1.10.8.

(2) By (1) there is an integer $n \geq 2$ and a non-zero element $x \in V^{\otimes n}$ such that $c_{n-1}^2 c_{n-2} \cdots c_1(x) = x$. Let $x_1, \dots, x_d$ be a basis of V such that

$$c(x_a \otimes x_b) = q_{a,b} x_b \otimes x_a \text{ for all } a, b \in \{1, \dots, d\}.$$

Then there is a unique presentation of x, $x = \sum_{k=(k_1,\dots,k_n) \in \{1,\dots,d\}^n} \alpha_k x_k$ with $\alpha_k \in \Bbbk$ for all $k \in \{1, \dots, d\}^n$, where $x_k = x_{k_1} \otimes \cdots \otimes x_{k_n}$. Now

$$\begin{aligned} &c_{n-1}^2 c_{n-2} \cdots c_1(x_{k_1} \otimes \cdots \otimes x_{k_n}) = \\ &q_{k_1,k_2} q_{k_1,k_3} \cdots q_{k_1,k_{n-1}} q_{k_1,k_n} q_{k_n,k_1} x_{k_2} \otimes x_{k_3} \otimes \cdots \otimes x_{k_{n-1}} \otimes x_{k_1} \otimes x_{k_n}, \end{aligned}$$

$$(c_{n-1}^2 c_{n-2} \cdots c_1)^{n-1}(x_{k_1} \otimes \cdots \otimes x_{k_n}) = \prod_{1 \leq i < j \leq n} q_{k_i,k_j} q_{k_j,k_i} x_{k_1} \otimes \cdots \otimes x_{k_n}.$$

Since $c_{n-1}^2 c_{n-2} \cdots c_1(x) = x$, it follows that $(c_{n-1}^2 c_{n-2} \cdots c_1)^{n-1}(x) = x$, which implies (2) by the above equations. $\square$

EXAMPLE 1.10.10. Let $0 \neq V \in {}^G_G\mathcal{YD}$ be finite-dimensional, $c = c_{V,V}$, such that $c(x \otimes y) = qy \otimes x$ for all $x, y \in V$, where $0 \neq q \in \Bbbk$. Then by Example 1.10.1 and Proposition 1.10.9(2), the following are equivalent.

(1) $\mathcal{B}(V) = T(V)$.
(2) (a) q is not a root of 1, or
    (b) $q = 1$, $\dim V = 1$, and $\mathrm{char}(\Bbbk) = 0$.

One of the main problems we want to discuss in this book is the structure of the Nichols algebra of a direct sum of objects in ${}^G_G\mathcal{YD}$.

We now study the easy case of a direct sum $V_1 \oplus V_2$ of subobjects V_1, V_2 of V in ${}^G_G\mathcal{YD}$ such that

$$\mathrm{id}_{V_i \otimes V_j} = (V_i \otimes V_j \xrightarrow{c_{V_i,V_j}} V_j \otimes V_i \xrightarrow{c_{V_j,V_i}} V_i \otimes V_j)$$

for all $i \neq j$.

For a Hopf algebra H in ${}^G_G\mathcal{YD}$ let

$$\begin{aligned} \mathrm{ad} = (H \otimes H &\xrightarrow{\Delta_H \otimes id_H} H \otimes H \otimes H \xrightarrow{\mathrm{id}_H \otimes c_{H,H}} H \otimes H \otimes H \\ &\xrightarrow{\mathrm{id}_H \otimes \mathrm{id}_H \otimes \mathcal{S}_H} H \otimes H \otimes H \xrightarrow{\mu_r(\mathrm{id}_H \otimes \mu_H)} H) \end{aligned}$$

be the **braided adjoint action**.

For elements $x, y \in H$, we write

$$\mathrm{ad}\,(x \otimes y) = \mathrm{ad}\, x(y) = \mathrm{ad}_c x(y), \text{ where } c = c_{H,H}.$$

If $x \in P(H)$, $y \in H$, then $\operatorname{ad} x(y) = xy - (x_{(-1)} \cdot y)x_{(0)}$ is the **braided commutator** of x and y. If $x \in P(H)$ is homogeneous of degree $g \in G$, then

$$\operatorname{ad} x(y) = xy - (g \cdot y)x.$$

LEMMA 1.10.11. *Let H be a Hopf algebra in ${}^G_G\mathcal{YD}$ with braiding $c = c_{H,H}$, and $x, y \in P(H)$. Then*

$$\Delta(\operatorname{ad} x(y)) = \operatorname{ad} x(y) \otimes 1 + 1 \otimes \operatorname{ad} x(y) + (\mathrm{id}_{H\otimes H} - c^2)(x \otimes y)$$

PROOF. Let x be homogeneous of degree $g \in G$. Then

$$\begin{aligned}
\Delta(\operatorname{ad} x(y)) &= \Delta(x)\Delta(y) - (g \cdot \Delta(y))\Delta(x)\\
&= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y)\\
&\qquad - (g \cdot y \otimes 1 + 1 \otimes g \cdot y)(x \otimes 1 + 1 \otimes x)\\
&= xy \otimes 1 + x \otimes y + g \cdot y \otimes x + 1 \otimes xy\\
&\qquad - (g \cdot y)x \otimes 1 - g \cdot y \otimes x - c(g \cdot y \otimes x) - 1 \otimes (g \cdot y)x\\
&= \operatorname{ad} x(y) \otimes 1 + 1 \otimes \operatorname{ad} x(y) + (\mathrm{id}_{H\otimes H} - c^2)(x \otimes y).
\end{aligned}$$

□

PROPOSITION 1.10.12. *Let $V_1, V_2 \in {}^G_G\mathcal{YD}$, $V = V_1 \oplus V_2$, and $c = c_{V,V}$. For all $1 \le i \le 2$, identify $\mathcal{B}(V_i)$ with the image of the injective map $\mathcal{B}(V_i) \to \mathcal{B}(V)$ induced by the inclusion $V_i \subseteq V$ (see Remark 1.6.19).*

(1) *The multiplication map $\mathcal{B}(V_1) \otimes \mathcal{B}(V_2) \xrightarrow{\mu_{12}} \mathcal{B}(V)$ is an injective map of $\mathbb{N}_0$-graded coalgebras in ${}^G_G\mathcal{YD}$, where $\mathcal{B}(V_1) \otimes \mathcal{B}(V_2)$ is the tensor product of coalgebras in ${}^G_G\mathcal{YD}$.*
(2) *The following are equivalent.*
   (a) *μ_{12} is bijective.*
   (b) *$c^2|V_2 \otimes V_1 = \mathrm{id}_{V_2\otimes V_1}$.*
   (c) *$\mathcal{B}(V_1) \otimes \mathcal{B}(V_2)$ is a Hopf algebra in ${}^G_G\mathcal{YD}$, where the coalgebra and algebra structure is the tensor product of coalgebras and of algebras in ${}^G_G\mathcal{YD}$.*

   If (c) holds, then μ_{12} is an isomorphism of Hopf algebras in ${}^G_G\mathcal{YD}$.

Proposition 1.10.12 and its proof below generalize directly to pairs of Yetter-Drinfeld modules over Hopf algebras with bijective antipode using the definitions in Section 7.1.

PROOF. (1) By Remark 1.6.19, the inclusion maps $V_i \subseteq V$, $1 \le i \le 2$, define injective mophisms of $\mathbb{N}_0$-graded Hopf algebras $\mathcal{B}(V_i) \to \mathcal{B}(V)$ in ${}^G_G\mathcal{YD}$ which we view as inclusions. The map

$$\mu_{12} = \big(\mathcal{B}(V_1) \otimes \mathcal{B}(V_2) \subseteq \mathcal{B}(V) \otimes \mathcal{B}(V) \xrightarrow{\mu} \mathcal{B}(V)\big)$$

is a coalgebra homomorphism by Proposition 1.6.7. Hence the tensor product coalgebra $\mathcal{B}(V_1) \otimes \mathcal{B}(V_2)$ in ${}^G_G\mathcal{YD}$ is an $\mathbb{N}_0$-graded subcoalgebra of $\mathcal{B}(V) \otimes \mathcal{B}(V)$. By Proposition 1.3.17, the coalgebra $\mathcal{B}(V_1) \otimes \mathcal{B}(V_2)$ is strictly graded with

$$P(\mathcal{B}(V_1) \otimes \mathcal{B}(V_2)) = V_1 \otimes 1 + 1 \otimes V_2.$$

Since μ_{12} defines an isomorphism $V_1 \otimes 1 + 1 \otimes V_2 \to V_1 \oplus V_2$, we conclude with Corollary 1.3.11 that μ_{12} is injective.

(2) (a) $\Rightarrow$ (b). By (a), $\mathcal{B}(V_1)\mathcal{B}(V_2) = \mathcal{B}(V)$, and $V_1V_2 + V_1^2 + V_2^2 = \mathcal{B}^2(V)$. By Definition 1.6.17 and Corollary 1.9.7, the symmetrizer maps $S_n : T^n(V) \to T^n(V)$ factorize over the Nichols algebra $\mathcal{B}(V)$. Thus there are linear maps π, φ such that

$$\begin{tikzcd} T^2(V) \arrow[rr, "S_2"] \arrow[dr, "\pi"'] & & T^2(V) \\ & \mathcal{B}^2(V) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes, where π is the second component of the quotient map $T(V) \to \mathcal{B}(V)$. Then $S_2(T^2(V)) = S_2(V_1 \otimes V_2 + V_1 \otimes V_1 + V_2 \otimes V_2)$.

Recall that $S_2 = \mathrm{id} + c$. Let $a \in V_2 \otimes V_1$. Then

$$(\mathrm{id} - c^2)(a) = S_2(\mathrm{id} - c)(a) = S_2(b + u_1 + u_2)$$

for some $b \in V_1 \otimes V_2$, $u_1 \in V_1 \otimes V_1$, $u_2 \in V_2 \otimes V_2$. Since $c^2(a) \in V_2 \otimes V_1$ and $c(b) \in V_2 \otimes V_1$, it follows that $b = 0$ and $(\mathrm{id} - c^2)(a) = 0$.

(b) $\Rightarrow$ (c). Assume that $c^2|V_2 \otimes V_1 = \mathrm{id}_{V_2\otimes V_1}$. Let $x \in V_1$, $y \in V_2$. By Lemma 1.10.11, $\mathrm{ad}\, y(x)$ is primitive, hence

$$0 = \mathrm{ad}\, y(x) = yx - \mu_{\mathcal{B}(V)}c(y \otimes x)$$

in $\mathcal{B}(V)$, and $\mu_{12} : \mathcal{B}(V_1) \otimes \mathcal{B}(V_2) \to \mathcal{B}(V)$ is an algebra map, where $\mathcal{B}(V_1) \otimes \mathcal{B}(V_2)$ is the tensor product algebra. Then μ_{12} is an isomorphism, since the algebra $\mathcal{B}(V)$ is generated by V_1 and V_2. This proves (c).

(c) $\Rightarrow$ (a). By (c), $R = \mathcal{B}(V_1)\otimes\mathcal{B}(V_2)$ is a pre-Nichols algebra of $V_1 \otimes 1 \oplus 1 \otimes V_2$. By Theorem 1.6.18, there is a surjective map $\pi : R \to \mathcal{B}(V)$ of Hopf algebras in ${}^G_G\mathcal{YD}$, where $\pi(1)$ is the isomorphism $V_1 \otimes 1 \oplus 1 \otimes V_2 \cong V$. Then $\pi = \mu_{12}$ is surjective. $\square$

We combine Example 1.10.1 with Proposition 1.10.12.

Example 1.10.13. Let $(q_{ij})_{1\le i,j\le n}$, $n \ge 2$, be a family of non-zero scalars in $\Bbbk$ with $q_{ij}q_{ji} = 1$ for all $i \ne j$. For all $1 \le i \le n$, we define $N_i = N(q_{ii})$. Let $V \in {}^G_G\mathcal{YD}$ be a vector space with basis $x_1, \dots, x_n$ and diagonal braiding given by $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all $1 \le i, j \le n$. Assume that the elements $x_1, \dots, x_n$ span one-dimensional subobjects of V in ${}^G_G\mathcal{YD}$. The braided vector space (V, c) is called a quantum linear space. Note that $c^2(x_i \otimes x_j) = x_i \otimes x_j$ for all $i \ne j$, and $\mathrm{ad}\, x_i(x_j) = x_ix_j - q_{ij}x_jx_i$ for all i, j. Hence for all $i \ne j$, $x_ix_j = q_{ij}x_jx_i$ in $\mathcal{B}(V)$ by Lemma 1.10.11. Let

$$A = \Bbbk\langle x_1, \dots, x_n \mid x_ix_j = q_{ij}x_jx_i, \quad x_k^{N_k} = 0 \text{ for all } i, j, k, i < j, N_k < \infty\rangle.$$

By Example 1.10.1, there is a well-defined algebra map

$$\varphi : A \to \mathcal{B}(V), \quad x_i \mapsto x_i \text{ for all } 1 \le i \le n.$$

It is clear from the relations that the elements $x_1^{t_1} \cdots x_n^{t_n}$, $0 \le t_i < N_i, 1 \le i \le n$, span the vector space A. (Here, $t < \infty$ for all $t \in \mathbb{N}_0$.) Their images under φ are a basis of $\mathcal{B}(V)$, since the multiplication map $\mathcal{B}(\Bbbk x_1) \otimes \cdots \otimes \mathcal{B}(\Bbbk x_n) \to \mathcal{B}(V)$ is bijective by Proposition 1.10.12. Hence φ is an isomorphism.

EXAMPLE 1.10.14. Assume in Example 1.10.13 that $q_{ij} = 1$ for all i, j, that is, $c(x \otimes y) = y \otimes x$ for all $x, y \in V$. Then by Example 1.10.13,

$$\mathcal{B}(V) \cong \begin{cases} S(V), \text{the symmetric algebra of } V, & \text{if } \operatorname{char}(\Bbbk) = 0; \\ S(V)/(v^p \mid v \in V), & \text{if } \operatorname{char}(\Bbbk) = p > 0. \end{cases}$$

EXAMPLE 1.10.15. (a) Assume in Example 1.10.13 that $q_{ij} = -1$ for all i, j, that is, $c(x \otimes y) = -y \otimes x$ for all $x, y \in V$. By Example 1.10.13,

$$\mathcal{B}(V) \cong \Bbbk\langle x_1, \dots, x_n \mid x_i^2 = 0, x_i x_j + x_j x_i = 0 \text{ for all } i \neq j\rangle \cong \Lambda(V)$$

is the exterior algebra of V of dimension 2^n. By Example 1.4.14, the braiding can be realized by a Yetter-Drinfeld module V over the group G of order 2.

(b) Let $\operatorname{char}(\Bbbk) = 0$, $G = \{1, g\}$ the group with two elements, and $\widehat{G} = \{\varepsilon, \chi\}$, where $\chi(g) = -1$. Let $V \in {}^G_G\mathcal{YD}$. Then $V = V^\varepsilon \oplus V^\chi$ as $\Bbbk G$-module. Assume that $\mathcal{B}(V)$ is finite-dimensional. Then, by Example 1.10.2, $V = V_g^\chi$ as Yetter-Drinfeld module. Hence $\mathcal{B}(V) \cong \Lambda(V)$ by (a).

EXAMPLE 1.10.16. Let $\operatorname{char}(\Bbbk) = 0$, and let $V = V_0 \oplus V_1$ be a finite-dimensional super vector space. By Example 1.4.14, $V \in {}^G_G\mathcal{YD}$, where $G = \mathbb{Z}/(2)$, and the braiding is given by

$$\begin{aligned} c_{V_0,V} &= \tau : V_0 \otimes V \to V \otimes V_0, \ c_{V,V_0} = \tau : V \otimes V_0 \to V_0 \otimes V, \\ c_{V_1,V_1} &= -\tau : V_1 \otimes V_1 \to V_1 \otimes V_1, \end{aligned}$$

where τ is the flip map. Then by Examples 1.10.13, 1.10.14 and 1.10.15,

$$\mathcal{B}(V) \cong S(V_0) \otimes \Lambda(V_1)$$

is the graded-symmetric algebra of $V_0 \oplus V_1$.

If the assumption on the braiding in Proposition 1.10.12(2) is not satisfied, then the description of $\mathcal{B}(V_1 \oplus V_2)$ is much more difficult.

Without proof we mention the fundamental example of a Nichols algebra $\mathcal{B}(V)$ coming from the theory of quantum groups. Here, the braiding of V is given by a Yetter-Drinfeld structure over a free abelian group of finite rank, and V is a direct sum of finitely many one-dimensional Yetter-Drinfeld modules V_i. The Nichols algebras of each summand V_i are simply polynomial algebras in one variable, but $\mathcal{B}(V)$ is given by the complicated quantum Serre relations.

DEFINITION 1.10.17. Let I be a non-empty finite set. Recall from [**Kac90**, §1.1] that a (generalized) **Cartan matrix** $A = (a_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(1) $a_{ii} = 2$ and $a_{jk} \leq 0$ for all $i, j, k \in I$ with $j \neq k$,
(2) if $i, j \in I$ and $a_{ij} = 0$, then $a_{ji} = 0$.

A Cartan matrix $A = (a_{ij})_{i,j\in I}$ is called **symmetrizable**, if there are integers $d_i \geq 1$ for all $i \in I$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$. A Cartan matrix $(a_{ij})_{i,j\in I}$ is called of **finite type** if it is symmetrizable and if the symmetric bilinear form $(\cdot,\cdot) : \mathbb{R}^I \times \mathbb{R}^I \to \mathbb{R}$, $(x, y) \mapsto \sum_{i,j\in I} x_i d_i a_{ij} y_j$, is positive definite.

The classification of Cartan matrices of finite type is well-known and is easily obtained from the definition by induction on the cardinality of I. We follow the convention in [**Kac90**, §4.8].

THEOREM 1.10.18. *Let $l \geq 1$. Then up to a bijection of the index set, the indecomposable Cartan matrices of finite type in $\mathbb{Z}^{l\times l}$, see Definition* 10.1.15, *are the following.*

(1) *Type A_l, $l \geq 1$:* $a_{ij} = \begin{cases} -1, & \text{if } |i-j| = 1, \\ 0, & \text{if } |i-j| \geq 2. \end{cases}$
Then $d_i = 1$ for all $1 \leq i \leq l$.

(2) *Type B_l, $l \geq 2$:* $a_{ij} = \begin{cases} -1, & \text{if } |i-j| = 1, i \neq l, \\ -2, & \text{if } i = l, j = l-1, \\ 0, & \text{if } |i-j| \geq 2. \end{cases}$
Then $d_i = 2$ for all $1 \leq i \leq l-1$ and $d_l = 1$.

(3) *Type C_l, $l \geq 3$:* $a_{ij} = \begin{cases} -1, & \text{if } |i-j| = 1, j \neq l, \\ -2, & \text{if } i = l-1, j = l, \\ 0, & \text{if } |i-j| \geq 2. \end{cases}$
Then $d_i = 1$ for all $1 \leq i \leq l-1$ and $d_l = 2$.

(4) *Type D_l, $l \geq 4$: $a_{ij} = -1$ if $|i-j| = 1$, $i,j < l$; $a_{l-2\,l} = a_{l,l-2} = -1$; $a_{ij} = 0$ otherwise, whenever $i \neq j$. Then $d_i = 1$ for all $1 \leq i \leq l$.*

(5) *Type E_l, $6 \leq l \leq 8$: $a_{ij} = -1$ if $|i-j| = 1$, $i,j < l$; $a_{l-3\,l} = a_{l\,l-3} = -1$; $a_{ij} = 0$ otherwise, whenever $i \neq j$. Then $d_i = 1$ for all $1 \leq i \leq l$.*

(6) *Type F_4, $l = 4$:* $a_{ij} = \begin{cases} -1, & \text{if } |i-j| = 1, (i,j) \neq (3,2), \\ 0, & \text{if } |i-j| \geq 2, \end{cases}$
$a_{32} = -2$. Then $d_1 = d_2 = 2$, $d_3 = d_4 = 1$.

(7) *Type G_2, $l = 2$: $a_{12} = -1$, $a_{21} = -3$. Then $d_1 = 3$, $d_2 = 1$.*

In particular, for any such Cartan matrix A there exist unique integers d_i, $1 \leq i \leq r$, such that $d_i a_{ij} = d_j a_{ji}$ for all $1 \leq i,j \leq r$, and $\{d_i \mid 1 \leq i \leq r\}$ is one of the sets $\{1\}$, $\{1,2\}$, $\{1,3\}$.

The following example is an immediate consequence of Theorem 1.10.18.

EXAMPLE 1.10.19. A Cartan matrix $A \in \mathbb{Z}^{2\times 2}$ is of finite type if and only if $a_{12}a_{21} \in \{0,1,2,3\}$. An indecomposable Cartan matrix $A \in \mathbb{Z}^{3\times 3}$ is of finite type if there exist $i,j,k \in \{1,2,3\}$ such that $a_{ik} = a_{ki} = 0$, $a_{ij} = a_{ji} = -1$, and $a_{jk}a_{kj} \in \{1,2\}$.

EXAMPLE 1.10.20. Let $q \in \Bbbk$ be non-zero and not a root of one, $G = \mathbb{Z}^n$ a free abelian group of rank $n \geq 1$ with basis $K_1, \dots, K_n$, and $(a_{ij})_{1\leq i,j\leq n}$ a Cartan matrix of finite type, where $(d_i a_{ij})$ is symmetric and $d_i \in \{1,2,3\}$ for all i. We define a Yetter-Drinfeld module $V \in {}^G_G\mathcal{YD}$ with basis $x_i \in V^{\chi_i}_{K_i}$, $1 \leq i \leq n$, where $\chi_1, \dots, \chi_n$ are characters of $\mathbb{Z}^n$ with

$$\chi_i(K_j) = q^{d_i a_{ij}} \text{ for all } 1 \leq i,j \leq n,$$

that is

$$\deg(x_i) = K_i,\ g \cdot x_i = \chi_i(g)x_i \text{ for all } g \in G, 1 \leq i \leq n.$$

Then $V = \Bbbk x_1 \oplus \cdots \oplus \Bbbk x_n$ is the direct sum of one-dimensional Yetter-Drinfeld modules kx_i. We prove in Theorem 16.2.5 that

$$\mathcal{B}(V) \cong \Bbbk\langle x_1, \dots, x_n \mid (\operatorname{ad} x_i)^{1-a_{ij}}(x_j) = 0 \text{ for all } i \neq j\rangle$$

is given by the quantum Serre relations. Thus $\mathcal{B}(V) = U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is the semisimple Lie algebra defined by the matrix $(a_{ij})_{1\leq i,j\leq n}$.

We note that the elements $(\operatorname{ad} x_i)^{1-a_{ij}}(x_j) \in T(V)$, $i \neq j$, are primitive by Proposition 4.3.12, hence $\Bbbk\langle x_1, \dots, x_n \mid (\operatorname{ad} x_i)^{1-a_{ij}}(x_j) = 0 \text{ for all } i \neq j\rangle$ is a Hopf algebra in ${}^G_G\mathcal{YD}$.

REMARK 1.10.21. Nichols algebras of Yetter-Drinfeld modules play an important role in the classification theory of Hopf algebras. They appear naturally as subalgebras of graded Hopf algebras associated to the coradical filtration, see Corollary 7.1.17.

1.11. Notes

1.1. The first comultiplication appeared in a paper by Heinz Hopf [**Hop41**] written in German and published in the Ann. of Math. in 1941.

1.4. Yetter-Drinfeld modules over a Hopf algebra, in particular over a group algebra, together with their braiding were introduced 1990 by Yetter in [**Yet90**].

The explicit description of Yetter-Drinfeld modules over groups was given in the equivalent category of Hopf bimodules in several early papers, beginning with [**Nic78**] over finite abelian groups in the semisimple case, [**DPR90**] over finite groups over the complex numbers as modules over the Drinfeld double of the group algebra, and in the general case in [**CR97**].

1.5. The fruitful idea to describe braided vector spaces of group type by racks was introduced in [**AGn03**].

1.6. Nichols defined in [**Nic78**] a bialgebra of type one as the image of a canonical map from the tensor algebra to the cotensor algebra of a Hopf bimodule. Bialgebras of type one contain Nichols algebras as subalgebras. Hopf bimodules are equivalent to Yetter-Drinfeld modules, see Notes to Section 3.7. It was shown independently in several papers ([**Sch96**], [**Ros95**], [**Róż96**], [**BD97**]) that the Nichols algebra can be seen as the image of a canonical map from the tensor algebra to the shuffle algebra of the braided vector space. See the notes to Section 6.4 for the definition of the shuffle algebra which is dual to the braided tensor algebra.

1.7. We have found the notation $\uparrow i$ for the shift operator in [**IO09**].

1.8. The equations in Proposition 1.8.13 appeared in [**DK$^+$97**, Lemma 6.12].

1.9. Theorem 1.9.1 about the comultiplication of the tensor algebra already was shown in [**HH92**, Proposition 4.8].

The braided (anti)symmetrizer map was introduced by Woronowicz in [**Wor89**], where he defined the braiding for Hopf bimodules (which he called bicovariant bimodules). Corollary 1.9.7 describing the relations of the Nichols algebra as a Hopf algebra by the braided symmetrizer map was shown in the papers mentioned in the notes to Section 1.6, since the canonical map from the tensor algebra to the shuffle algebra is given by the quantum symmetrizer.

1.10. Proposition 1.10.9(2) is shown in [**Fd$^+$01**, Corollary (5.2.b)], by a different method. Example 1.10.10 is a very special case of the main result of [**HZ18**], where the finite-dimensional braided vector spaces V of diagonal type satisfying $\mathcal{B}(V) = T(V)$ are determined.

Proposition 1.10.12 also holds for the general braidings in Chapter 7. The equivalence of (a) and (b) was first shown in [**Gn00a**] for finite-dimensional Nichols algebras.

CHAPTER 2

Basic Hopf algebra theory

In the book we will need many basic properties of coalgebras and Hopf algebras, which are collected mainly in this chapter. In particular, module and comodule algebras will appear frequently. Two-cocycle deformations of bialgebras are a standard tool in the theory which we will use later in the discussion of quantized enveloping algebras and of linkings of Nichols algebras.

2.1. Finiteness properties of coalgebras and comodules

We start with a characterization of right comodules.

LEMMA 2.1.1. *Let C be a coalgebra, V a vector space, and $\delta_V : V \to V \otimes C$ a linear map. Let $(v_i)_{i\in I}$ be a basis of V, and $\delta_V(v_j) = \sum_{i\in I} v_i \otimes c_{ij}$ for all j, where $(c_{ij})_{i,j\in I}$ is a family of elements of C such that for all $j \in I$, $c_{ij} \neq 0$ only for finitely many indices $i \in I$. Then the following are equivalent.*

(1) *(V, δ_V) is a right C-comodule.*

(2) *For all $i, j \in I$, $\Delta(c_{ij}) = \sum_{k\in I} c_{ik} \otimes c_{kj}$, $\varepsilon(c_{ij}) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$*

The subspace $C(V) \subseteq C$ spanned by the elements c_{ij}, $i, j \in I$, is the smallest subspace $C' \subseteq C$ such that $\delta_V(V) \subseteq V \otimes C'$, and it is a subcoalgebra of C.

PROOF. By definition, (V, δ_V) is a right C-comodule, if for all $j \in I$,

$$\sum_{i\in I} \delta_V(v_i) \otimes c_{ij} = \sum_{i\in I} v_i \otimes \Delta(c_{ij}),$$

$$\sum_{i\in I} v_i \varepsilon(c_{ij}) = v_j.$$

Since $\delta_V(v_i) = \sum_{l\in I} v_l \otimes c_{li}$ for all $i \in I$, the equivalence of (1) and (2) follows by comparing coefficients. The claim about $C(V)$ is obvious. □

Note the special case of Lemma 2.1.1 when V is finite-dimensional with basis $v_1, \dots, v_n$. By Lemma 2.1.1 there is a bijection

$$\{\delta_V \mid (V, \delta_V) \text{ is a right } C\text{-comodule } \} \cong \mathrm{Coalg}(M_n(\Bbbk)^*, C),$$

where $M_n(\Bbbk)^*$ is the coalgebra in Example 1.1.5.

LEMMA 2.1.2. *Let C be a coalgebra and (V, δ_V) a right C-comodule. Let $v \in V$, and let $(c_i)_{i\in I}$ be a basis of C. Write*

$$\Delta(c_i) = \sum_{j\in I} c_{ij} \otimes c_j \text{ for all } i \in I, \qquad \delta_V(v) = \sum_{i\in I} v_i \otimes c_i,$$

where for all $i, j \in I$, $c_{ij} \in C$, $v_i \in V$, such that for all $i \in I$, $c_{ij} \neq 0$ only for finitely many $j \in I$, and where $v_i \neq 0$ only for finitely many $i \in I$. Then

$$\delta_V(v_j) = \sum_{i\in I} v_i \otimes c_{ij} \text{ for all } j \in I.$$

Proof. By coassociativity of δ_V,

$$\sum_{i\in I} \delta_V(v_i) \otimes c_i = (\delta_V \otimes \mathrm{id})\delta_V(v) = \sum_{i\in I} v_i \otimes \Delta(c_i) = \sum_{i,j\in I} v_i \otimes c_{ij} \otimes c_j.$$

The claim follows by comparing coefficients. □

Coalgebras and comodules over coalgebras are easier objects than algebras and modules over algebras, since they satisfy the following finiteness property.

Theorem 2.1.3 (Finiteness Theorem). *Let C be a coalgebra and V a right C-comodule. Then V is the union of its finite-dimensional subcomodules, and C is the union of its finite-dimensional subcoalgebras.*

Proof. We have to show the following.

(1) Any element of V is contained in a finite-dimensional subcomodule.
(2) Any element of C is contained in a finite-dimensional subcoalgebra.

(1) Let $v \in V$. By Lemma 2.1.2, the vector space V' spanned by the elements v_i, $i \in I$, is a finite-dimensional subcomodule of V. Moreover, $v = \sum_{i\in I} v_i\varepsilon(c_i) \in V'$.

(2) Let $c \in C$. By (1) applied to C as a right C-comodule via Δ, there is a finite-dimensional subspace $V \subseteq C$ with $c \in V$, $\Delta(V) \subseteq V \otimes C$. By Lemma 2.1.1, $\Delta(V) \subseteq V \otimes C(V)$, and $C(V)$ is a finite-dimensional subcoalgebra of C. Moreover, $c \in C(V)$ since $c = (\varepsilon \otimes \mathrm{id}_C)(\Delta(c))$. □

The unions in Theorem 2.1.3 are ascending unions, that is, finitely many finite-dimensional subcomodules, respectively subcoalgebras, are always contained in a finite-dimensional subcomodule, respectively subcoalgebra, namely in their sum. Thus comodules and coalgebras are **direct limits of finite-dimensional subobjects**.

An algebra A is called **residually finite-dimensional** if there exists a family of ideals of A of finite codimension whose intersection is zero.

Corollary 2.1.4. *Let C be a coalgebra.*

(1) *The dual algebra C^* is residually finite-dimensional.*
(2) *Let $f \in C^*$, and assume that for all finite-dimensional subcoalgebras D of C, the image of f under the restriction map $C^* \to D^*$ is invertible in D^*. Then f is invertible in C^*.*

Proof. (1) For all finite-dimensional subcoalgebras $D \subseteq C$ the kernel of the restriction map $\pi_D : C^* \to D^*$ is an ideal of C^* of finite codimension, and by Theorem 2.1.3,

$$\bigcap\{\ker(\pi_D) \mid D \subseteq C \text{ a finite-dimensional subcoalgebra}\} = 0.$$

(2) For all finite-dimensional subcoalgebras $D \subseteq C$ let $g_D \in D^*$ be the inverse of $f|D$. Let $g : C \to \Bbbk$, $x \mapsto g_F(x)$, where $F \subseteq C$ is a finite-dimensional subcoalgebra containing x which exists by Theorem 2.1.3. Then g is well-defined since for all finite-dimensional subcoalgebras $E \subseteq F$, $g_F|E = g_E$ by uniqueness of the inverse. Hence f is invertible in C^* with inverse g. □

It is clear from Corollary 2.1.4 that not any algebra is of the form C^* for some coalgebra C. In particular, infinite-dimensional algebras which are simple, that is, have no proper non-zero ideals, are not residually finite-dimensional. Examples of infinite-dimensional simple algebras are infinite field extensions or the Weyl algebra over a field of characteristic zero (see Example 2.6.16).

2.2. Duality

If A is an algebra and V is a right A-module, then the dual vector space $V^* = \mathrm{Hom}(V, \Bbbk)$ is a left A-module in a natural way. This also works for comodules which are finite-dimensional.

LEMMA 2.2.1. *Let X, Y be vector spaces. The linear map*

$$\varphi_{X,Y} : X^* \otimes Y \to \mathrm{Hom}(X, Y),\ f \otimes y \mapsto (x \mapsto f(x)y),$$

is injective, and it is bijective if X is finite-dimensional.

PROOF. We leave the elementary proof to the reader. □

For a coalgebra C, we denote the category of finite-dimensional right or left C-comodules with C-colinear maps as morphisms by $\mathcal{M}^{\mathrm{fd},C}$ and ${}^C\mathcal{M}^{\mathrm{fd}}$, respectively. A **duality** between categories is a contravariant equivalence.

PROPOSITION 2.2.2. *Let C be a coalgebra.*

(1) *Let $V \in \mathcal{M}^{\mathrm{fd},C}$. Then $V^* = \mathrm{Hom}(V, \Bbbk)$ is a left C-comodule, where the comodule structure $\delta_{V^*} : V^* \to C \otimes V^*$, $f \mapsto f_{(-1)} \otimes f_{(0)}$, is defined by the equations*

$$f_{(-1)}f_{(0)}(v) = f(v_{(0)})v_{(1)} \text{ for all } v \in V.$$

(2) *The functor*

$$\mathcal{M}^{\mathrm{fd},C} \to {}^C\mathcal{M}^{\mathrm{fd}},\ (V, \delta_V) \mapsto (V^*, \delta_{V^*}),$$

where comodule maps f are mapped onto f^*, is a duality.

PROOF. (1) By Lemma 2.2.1, the map

$$C \otimes V^* \to \mathrm{Hom}(V, C), \quad c \otimes f \mapsto (v \mapsto f(v)c),$$

is bijective. For any $f \in V^*$ let $\delta_{V^*}(f) = f_{(-1)} \otimes f_{(0)} \in C \otimes V^*$ with

$$f_{(-1)}f_{(0)}(v) = f(v_{(0)})v_{(1)}$$

for all $v \in V$. This defines a linear map $\delta_{V^*} : V^* \to C \otimes V^*$. To prove that (V^*, δ_{V^*}) is a left C-comodule, we have to show for all $f \in V^*$,

$$f_{(-1)} \otimes \delta_{V^*}(f_{(0)}) = \Delta(f_{(-1)}) \otimes f_{(0)} \in C \otimes C \otimes V^*, \tag{2.2.1}$$

$$\varepsilon(f_{(-1)})f_{(0)} = f. \tag{2.2.2}$$

Using Lemma 2.2.1, we check the equality (2.2.1) by evaluating on elements of V. By evaluation of the left-hand side of (2.2.1) on $v \in V$ we get

$$f_{(-1)} \otimes f_{(0)}(v_{(0)})v_{(1)} = f(v_{(0)})v_{(1)} \otimes v_{(2)}.$$

On the other hand

$$\Delta(f_{(-1)})f_{(0)}(v) = \Delta(f_{(-1)}f_{(0)}(v)) = \Delta(f(v_{(0)})v_{(1)}) = f(v_{(0)})v_{(1)} \otimes v_{(2)}.$$

Finally,

$$\varepsilon(f_{(-1)})f_{(0)}(v) = \varepsilon(f_{(-1)}f_{(0)}(v)) = \varepsilon(f(v_{(0)})v_{(1)}) = f(v_{(0)})\varepsilon(v_{(1)}) = f(v)$$

for all $v \in V$, which proves (2.2.2).

(2) follows easily from (1), since for all $V \in \mathcal{M}^{\mathrm{fd},C}$ the natural isomorphism

$$V \to V^{**},\ v \mapsto (f \mapsto f(v)),$$

is an isomorphism of right C-comodules. □

LEMMA 2.2.3. *Let X, Y be vector spaces. Then the map*

$$\varphi_{X,Y} : X^* \otimes Y^* \to (X \otimes Y)^*,\ f \otimes g \mapsto (x \otimes y \mapsto f(x)g(y)),$$

is injective, and a natural transformation in both variables X and Y. If X or Y are finite-dimensional, then $\varphi_{X,Y}$ is an isomorphism.

PROOF. The proof of this Lemma is rather elementary as well, and is left to the reader. □

We note that in the cases when the maps $\varphi_{X,Y}$ of Lemma 2.2.1 and Lemma 2.2.3 are isomorphisms, there is no natural way (that is, without using bases) to write down a formula for their inverses.

In the next proposition we write $\mathcal{C}^{\mathrm{fd}}$ for the category of finite-dimensional coalgebras with coalgebra homomorphisms as morphisms, and $\mathcal{A}^{\mathrm{fd}}$ for the category of finite-dimensional algebras with algebra maps as morphisms.

PROPOSITION 2.2.4.
(1) *For any finite-dimensional algebra A, the dual vector space A^* is a coalgebra with $\varepsilon(f) = f(1)$, $\Delta(f) = f_{(1)} \otimes f_{(2)}$, $f_{(1)}(x)f_{(2)}(y) = f(xy)$ for all $f \in A^*$, $x, y \in A$. It is called the* **dual coalgebra** *of A.*
(2) *For any algebra map $\rho : A \to B$ between finite-dimensional algebras A, B, the map $\rho^* : B^* \to A^*$, $f \mapsto f \circ \rho$, is a coalgebra map.*
(3) *The functor $\mathcal{C}^{\mathrm{fd}} \to \mathcal{A}^{\mathrm{fd}}$ mapping a coalgebra C to its dual algebra C^*, and a coalgebra homomorphism f to f^*, is a duality. The inverse functor $\mathcal{A}^{\mathrm{fd}} \to \mathcal{C}^{\mathrm{fd}}$ sends an algebra A to its dual coalgebra A^*, and an algebra map ρ to the coalgebra map ρ^*.*

PROOF. (1) By definition, the comultiplication of A^* is defined by

$$A^* \xrightarrow{\mu_A^*} (A \otimes A)^* \xrightarrow{\varphi_{A,A}^{-1}} A^* \otimes A^*,$$

where $\varphi_{A,A}$ is the isomorphism in Lemma 2.2.3. To check coassociativity of Δ, we use the isomorphism

$$A^* \otimes A^* \otimes A^* \to (A \otimes A \otimes A)^*,\ f \otimes g \otimes h \mapsto (x \otimes y \otimes z \mapsto f(x)g(y)h(z)),$$

which is a consequence of Lemma 2.2.3. Let $f \in A^*$ and $x, y, z \in A$. Then

$$f_{(1)(1)}(x)f_{(1)(2)}(y)f_{(2)}(z) = f_{(1)}(xy)f_{(2)}(z) = f(xyz),$$
$$f_{(1)}(x)f_{(2)(1)}(y)f_{(2)(2)}(z) = f_{(1)}(x)f_{(2)}(yz) = f(xyz).$$

The counit axioms are checked similarly.

(2) For any $f \in B^*$, $x, y \in A$, one has

$$\rho^*(f)(xy) = f(\rho(xy)) = f(\rho(x)\rho(y)) = \rho^*(f_{(1)})\rho^*(f_{(2)})$$

and $\rho^*(f)(1) = f(\rho(1)) = f(1)$.

(3) The functorial isomorphism $X \to X^{**}$, $x \mapsto (f \mapsto f(x))$, for finite-dimensional vector spaces X defines a coalgebra isomorphism $C \to C^{**}$ for any finite-dimensional coalgebra C, and an algebra isomorphism $A \to A^{**}$ for any finite-dimensional algebra A. □

Let A be an algebra. We denote by $\mathcal{M}_A^{\mathrm{fd}}$ the category of finite-dimensional right A-modules with A-linear maps as morphisms. Proposition 2.2.4 follows essentially from Lemma 2.2.3. The next proposition is shown in the same way.

PROPOSITION 2.2.5. *Let C be a coalgebra and $A = C^*$ its dual algebra.*

(1) *Let (V, δ_V) be a right C-comodule. Then V^* is a right A-module with module structure*

$$\lambda_{V^*} = \big(V^* \otimes C^* \xrightarrow{\varphi_{V,C}} (V \otimes C)^* \xrightarrow{\delta_V^*} V^*\big).$$

(2) *Assume that C is finite-dimensional. The functor*

$$\mathcal{M}^{C,\mathrm{fd}} \to \mathcal{M}_A^{\mathrm{fd}},\ (V, \delta_V) \mapsto (V^*, \lambda_{V^*}),$$

where a comodule map f is mapped onto f^*, is a duality.

The duality functor in Proposition 2.2.4 induces a bijective correspondence between subcoalgebras of a finite-dimensional coalgebra and ideals or quotient algebras of the dual algebra.

REMARK 2.2.6. For any vector space V there is a correspondence between subspaces of V and of the dual space V^*. If $U \subseteq V$ and $X \subseteq V^*$ are subspaces, we define subspaces $U^\perp \subseteq V^*$ and $X^\perp \subseteq V$ with respect to the pairing $V^* \otimes V \to \Bbbk$, $f \otimes v \mapsto f(v)$, by

$$U^\perp = \{f \in V^* \mid f(u) = 0 \text{ for all } u \in U\},$$
$$X^\perp = \{v \in V \mid f(v) = 0 \text{ for all } f \in X\}.$$

By definition, $U^\perp$ is the kernel of the restriction map $V^* \to U^*$, and $X^\perp$ is the kernel of the map $\rho_X : V \to X^*$, $v \mapsto (f \mapsto f(v))$. If V is finite-dimensional, then $X^\perp$ is canonically isomorphic to $(V^*/X)^*$. The following rules are easy to check.

(1) If $U \subseteq V$ is a subspace, then $U^{\perp\perp} = U$.
(2) Assume that V is finite-dimensional. Then

$$\{U \mid U \subseteq V \text{a subspace}\} \to \{X \mid X \subseteq V^* \text{ a subspace}\},\ U \mapsto U^\perp,$$

is bijective and inclusion reversing with inverse given by $X \mapsto X^\perp$.

A non-zero coalgebra C is **simple** if 0 and C are the only subcoalgebras of C. By Theorem 2.1.3, simple coalgebras are finite-dimensional, and by Proposition 2.2.4 a coalgebra C is simple if and only if C^* is a finite-dimensional simple algebra, that is if it has no non-trivial quotient algebras.

EXAMPLE 2.2.7. The coalgebra $M_n(\Bbbk)^*$ in Example 1.1.5 is simple since by Example 1.2.13 its dual is isomorphic to the matrix algebra $M_n(\Bbbk)$ which is a simple algebra.

We denote the set of all (two-sided) maximal ideals of an algebra A by $\mathrm{Max}(A)$.

COROLLARY 2.2.8. *Let C be a finite-dimensional coalgebra. The maps*

$$\{D \mid D \text{ subcoalgebra of } C\} \to \{I \mid I \text{ ideal of } C^*\}$$
$$\{D \mid D \text{ simple subcoalgebra of } C\} \to \operatorname{Max}(C^*),$$

defined by $D \mapsto D^\perp$ are bijective.

PROOF. The bijectivity of the first map in the claim follows by duality from Proposition 2.2.4. Since the map $D \mapsto D^\perp$ is inclusion reversing, simple subcoalgebras correspond to maximal ideals. □

Our next goal is to prove a dual version of Nakayama's lemma using the duality principle in Proposition 2.2.5.

DEFINITION 2.2.9. Let C be a coalgebra, $V \in \mathcal{M}^C$, and $W \in {}^C\mathcal{M}$ with structure maps $\delta_V : V \to V \otimes C$, $\delta_W : W \to C \otimes W$.

(1) The **cotensor product** $V \square_C W$ is defined as the kernel of

$$\delta_V \otimes \mathrm{id}_W - \mathrm{id}_V \otimes \delta_W : V \otimes W \to V \otimes C \otimes W.$$

(2) Let $D \subseteq C$ be a subcoalgebra. We define $V(D) = \delta_V^{-1}(V \otimes D)$.

REMARK 2.2.10. (1) Let $f : V \to V'$ and $g : W \to W'$ be a map of right and left C-comodules, respectively. Then $f \otimes g : V \otimes W \to V' \otimes W'$ induces a linear map $f \square_C g : V \square_C W \to V' \square_C W'$. Thus the cotensor product is a functor in two variables $\square_C : \mathcal{M}^C \times {}^C\mathcal{M} \to \mathcal{M}_\Bbbk$.

(2) The cotensor product commutes with arbitrary direct sums in both variables, that is, if $V \in \mathcal{M}^C$ and $(W_i)_{i\in I}$ is a family of left C-comodules, then the map $\bigoplus_{i\in I}(V \square_C W_i) \to V \square_C (\bigoplus_{i\in I} W_i)$, defined for all $i \in I$ on the summand $V \square_C W_i$ by $\mathrm{id} \square_C \iota_i$, is an isomorphism, where $\iota_i : W_i \to \bigoplus_{i\in I} W_i$ is the inclusion map; in the same way $\bigoplus_{i\in I}(V_i \square_C W) \cong (\bigoplus_{i\in I} V_i) \square_C W$, where $(V_i)_{i\in I}$ is a family of right C-comodules, and $W \in {}^C\mathcal{M}$.

(3) It follows from the coassociativity of δ_V (or from Lemma 2.1.2) that $V(D)$ in Definition 2.2.9(2) is a right D-comodule by restriction of δ_V. It is easy to see that δ_V induces an isomorphism $\delta_{V(D)} : V(D) \to V \square_C D$, where D is a left C-comodule via Δ. The inverse map is induced from $V \otimes D \to V$, $v \otimes d \mapsto v\varepsilon(d)$.

(4) Let A be an algebra and $V \in \mathcal{M}_A$, $W \in {}_A\mathcal{M}$ A-modules with structure maps $\mu_V : V \otimes A \to V$, $\mu_W : A \otimes W \to W$. Then the tensor product $V \otimes_A W$ can be defined as the cokernel of the map

$$\mu_V \otimes \mathrm{id} - \mathrm{id} \otimes \mu_W : V \otimes A \otimes W \to V \otimes W.$$

Thus the cotensor product for comodules over a coalgebra is dual to the tensor product of modules over an algebra.

LEMMA 2.2.11. *Let C be a coalgebra, $D \subseteq C$ a subcoalgebra, and V a finite-dimensional right C-comodule. Let I be the kernel of the restriction map $C^* \to D^*$. Then I is an ideal in C^*, and $V(D)^* \cong V^*/V^*I$ as right modules over $C^*/I \cong D^*$.*

PROOF. By definition, $V(D)$ is the kernel of the map

$$V \xrightarrow{\delta_V} V \otimes C \xrightarrow{\mathrm{id}_V \otimes \mathrm{can}} V \otimes C/D.$$

Since $I \cong (C/D)^*$, the claim follows by duality using Lemma 2.2.3. □

The following remark is a standard result in algebra. The reader may use it as a motivation or (together with Lemma 2.2.11) for an alternative proof of Proposition 2.2.14 below by duality.

REMARK 2.2.12. Let A be a finite-dimensional algebra and M a right A-module. By Wedderburn-Artin, there are finitely many maximal ideals $P_1, \dots, P_n$ of A, and $\bigcap_{i=1}^n P_i = \mathrm{Rad}(A)$ is the Jacobson radical of A. By the Chinese remainder theorem, there is a right A-linear isomorphism

$$M/M\mathrm{Rad}(A) \cong \prod_{i=1}^{n} M/MP_i$$

given by the diagonal map.

Let M, N be finite-dimensional right A-modules, and $f : M \to N$ a right A-linear map. By Nakayama's Lemma, f is surjective if and only if the induced map $M/M\mathrm{Rad}(A) \to N/N\mathrm{Rad}(A)$ is surjective. Thus by the Chinese remainder theorem, f is surjective if and only if for all maximal ideals $P \subseteq A$, the induced map $M/MP \to N/NP$ is surjective.

PROPOSITION 2.2.13. *Let C be a coalgebra, and $V \in \mathcal{M}^C$.*

(1) *If V is simple, then $C(V)$ is a simple subcoalgebra of C.*
(2) *If $V \neq 0$, then there is a simple subcoalgebra $D \subseteq C$ such that $V(D) \neq 0$.*

PROOF. (1) By Theorem 2.1.3, V is finite-dimensional. Let $v_1, \dots, v_n$ be a basis of V and $c_{ij} \in C(V)$ with $1 \le i, j \le n$ as in Lemma 2.1.1. Then Lemma 2.1.1 implies that for any $1 \le k \le n$ the linear map $f_k : V \to C(V)$, $v_i \mapsto c_{ki}$, is a comodule map, where $C(V)$ is a right $C(V)$-comodule via Δ. Hence $C(V)$ is a sum of simple C-comodules, each of them isomorphic to V. Thus $C(U) = C(V)$ for all simple subcomodules U of $C(V)$. It follows that $C(V)$ has no proper subcoalgebra.

(2) Theorem 2.1.3 implies that V has a simple subcomodule W. By (1) applied to W, $D = C(W)$ is a simple subcoalgebra of C and $W \subseteq V(D)$. □

PROPOSITION 2.2.14. *Let C be a coalgebra, $V, W \in \mathcal{M}^C$, and $f : V \to W$ a C-colinear map. Then the following are equivalent.*

(1) *The map $f : V \to W$ is injective.*
(2) *For all simple subcoalgebras $D \subseteq C$, the map $V(D) \to W(D)$ induced by f is injective.*

PROOF. Clearly, (1) implies (2). Assume now that $\ker(f) \neq 0$. Let U be a simple subcomodule of $\ker(f)$. Then $D = C(U)$ is a simple subcoalgebra of C by Proposition 2.2.13, and $U \subseteq V(D)$. Hence $f|V(D)$ is not injective. □

DEFINITION 2.2.15. Let C be a coalgebra, and V a right C-comodule with comodule structure $\delta_V : V \to V \otimes C$. Then $\mu_V : C^* \otimes V \to V$ defined by

$$fv = \mu_V(f \otimes v) = f(v_{(1)})v_{(0)}$$

for all $f \in C^*$, $v \in V$, is called the **adjoint C^*-module structure** to δ_V.

It is easy to see that V is indeed a left C^*-module with the adjoint module structure.

LEMMA 2.2.16. *Let C be a finite-dimensional coalgebra, and V a vector space. There is a bijection*

$$\{\delta_V \mid (V, \delta_V) \text{ is a right } C\text{-comodule}\} \to \{\mu_V \mid (V, \mu_V) \text{ is a left } C^*\text{-module}\}$$

where a right C-comodule structure is mapped onto its adjoint left C^*-module structure.

PROOF. Let $c_1, \dots, c_n$ be a basis of C, and $f_1, \dots, f_n$ its dual basis in C^*. The linear map

$$\mathrm{Hom}(V, V \otimes C) \to \mathrm{Hom}(C^* \otimes V, V),\ \delta_V \mapsto \mu_V,$$

where $\mu_V(f_i \otimes v) = v_i$ for all $1 \leq i \leq n$ and $v \in V$ with $\delta_V(v) = \sum_{j=1}^n v_j \otimes c_j$, is bijective. Note that if δ_V is mapped onto μ_V, and if we write $\delta_V(v) = v_{(0)} \otimes v_{(1)}$, then $\mu_V(f \otimes v) = f(v_{(1)})v_{(0)}$ for all $v \in V$, $f \in C^*$.

Then one checks that under this bijection comodule structures correspond to module structures. □

Let A be an algebra. A left A-module V is called **locally finite** if any element of V is contained in some finite-dimensional A-submodule. The full subcategory of ${}_A\mathcal{M}$ consisting of locally finite A-modules is denoted by ${}_A\mathcal{M}^{\mathrm{lf}}$.

PROPOSITION 2.2.17. *Let C be a coalgebra.*

(1) *The functor $\mathcal{M}^C \to {}_{C^*}\mathcal{M}^{\mathrm{lf}}$, which maps a comodule V to V with the adjoint module structure, and a comodule homomorphism f to f, is fully faithful.*
(2) *If C is finite-dimensional, then the functor $\mathcal{M}^C \to {}_{C^*}\mathcal{M}$ in (1) is an isomorphism of categories.*

PROOF. (1) It follows from Theorem 2.1.3 that for any right C-comodule V, the left C^*-module V with the adjoint module structure is locally finite.

Let $V, W \in \mathcal{M}^C$ and let $F : V \to W$ be a linear map. We have to show that F is right C-colinear if and only if F is left C^*-linear. Colinearity of F means that $F(v)_{(0)} \otimes F(v)_{(1)} = F(v_{(0)}) \otimes v_{(1)}$ for all $v \in V$, or equivalently

$$f(F(v)_{(1)})F(v)_{(0)} = f(v_{(1)})F(v_{(0)})$$

for all $v \in V$ and $f \in C^*$. The claim follows, since

$$\begin{aligned} f(F(v)_{(1)})F(v)_{(0)} &= fF(v), \\ f(v_{(1)})F(v_{(0)}) &= F(f(v_{(1)})v_{(0)}) = F(fv). \end{aligned}$$

(2) follows from (1) and Lemma 2.2.16. □

COROLLARY 2.2.18. *Let C be a coalgebra, $V \in \mathcal{M}^C$, and $X \subseteq V$ a subspace. Then $X \subseteq V$ is a right C-subcomodule if and only if it is a left C^*-submodule with respect to the adjoint C^*-module structure of V. In particular, C^*X is the* **C-subcomodule of V generated by** *X, that is, the smallest subcomodule containing X.*

PROOF. This follows from Proposition 2.2.17. Alternatively we give a direct proof. Let $\delta : V \to V \otimes C,\ v \mapsto v_{(0)} \otimes v_{(1)}$, be the comodule structure of V. If X is a subcomodule of V, then it is obviously a submodule. Conversely, assume that $X \subseteq V$ is a C^*-submodule. Then X is a subcomodule of V, that is, $\delta(X) \subseteq X \otimes C$, since $x_{(0)}f(x_{(1)}) = fx \in X$ for all $x \in X$, $f \in C^*$. □

Finally we extend the duality between finite-dimensional algebras and coalgebras to Hopf algebras.

PROPOSITION 2.2.19. *Let H be a finite-dimensional bialgebra. Then H^* is a bialgebra with structure maps defined for all $f, g \in H^*$ and all $x, y \in H$ by*

$$(fg)(x) = f(x_{(1)})g(x_{(2)}),\ \varepsilon(f) = f(1),\ f_{(1)}(x)f_{(2)}(y) = f(xy),\ 1_{H^*} = \varepsilon_H.$$

If H is a Hopf algebra, then H^* is a Hopf algebra with antipode

$$\mathcal{S}(f)(x) = f(\mathcal{S}(x))$$

for all $f \in H^*$ and $x \in H$.

PROOF. We know from the previous section that H^* is an algebra and a coalgebra. The bialgebra axiom holds, since for all $f, g \in H^*$ and $x, y \in H$,

$$\begin{aligned}(f_{(1)}g_{(1)})(x)(f_{(2)}g_{(2)})(y) &= f_{(1)}(x_{(1)})g_{(1)}(x_{(2)})f_{(2)}(y_{(1)})g_{(2)}(y_{(2)}) \\ &= f(x_{(1)}y_{(1)})g(x_{(2)}y_{(2)}) \\ &= (fg)(xy).\end{aligned}$$

Moreover, $\varepsilon(fg) = (fg)(1) = f(1)g(1) = \varepsilon(f)\varepsilon(g)$.

If H has an antipode, then for all $f \in H^*$ and $x \in H$,

$$f_{(1)}(x_{(1)})f_{(2)}(\mathcal{S}(x_{(2)})) = f(x_{(1)}\mathcal{S}(x_{(2)})) = f(\varepsilon(x)1_H) = \varepsilon(f)1_{H^*}(x).$$

Hence $f_{(1)}\mathcal{S}(f_{(2)}) = \varepsilon(f)1_{H^*}$. Similarly, $\mathcal{S}(f_{(1)})f_{(2)} = \varepsilon(f)1_{H^*}$. $\square$

EXAMPLE 2.2.20. Let G be a finite group and $\Bbbk G$ the group algebra as a Hopf algebra defined in Example 1.2.16. The dual Hopf algebra $(\Bbbk G)^*$ can be identified with the function algebra $\Bbbk^G$. Let e_g, $g \in G$, be the dual basis in $(\Bbbk G)^*$ of the basis G. Then for all $g \in G$,

$$e_g e_h = \delta_{gh} e_g,\ \ \Delta(e_g) = \sum_{\substack{a,b \in G \\ ab = g}} e_a \otimes e_b,\ \ \varepsilon(e_g) = \delta_{g1},\ \ \mathcal{S}(e_g) = e_{g^{-1}},$$

and $1_{(\Bbbk G)^*} = \sum_{g \in G} e_g$.

2.3. The restricted dual

In many situations it is helpful to consider dual objects of infinite dimensional (Hopf) algebras. In this section we discuss elements of the corresponding theory.

LEMMA 2.3.1. *Let X, Y be vector spaces such that X is finite-dimensional. Then*

$$\varphi_{X,Y} : X \otimes Y^* \to \mathrm{Hom}(X, Y)^*,\ x \otimes f \mapsto \big(F \mapsto f(F(x))\big),$$

is an isomorphism.

PROOF. The map $\varphi_{X,Y}$ is the composition of the isomorphisms

$$X \otimes Y^* \cong X^{**} \otimes Y^* \cong (X^* \otimes Y)^* \cong \mathrm{Hom}(X, Y)^*,$$

where the first isomorphism is induced from the canonical map $X \to X^{**}$, the second is the isomorphism of Lemma 2.2.3, and the third is the dual of the isomorphisms of Lemma 2.2.1. $\square$

DEFINITION 2.3.2. Let A be an algebra and V a finite-dimensional left A-module. For all $v \in V$ and $f \in V^*$ let $c^V_{f,v} = c_{f,v} \in A^*$ be defined by

$$c_{f,v}(x) = f(xv)$$

for all $x \in A$. The linear function $c_{f,v}$ is called a **matrix coefficient** of V. Let C^V be the $\Bbbk$-linear span of all matrix coefficients $c_{f,v}$, $f \in V^*$, $v \in V$.

LEMMA 2.3.3. *Let A be an algebra, and V a finite-dimensional left A-module with representation $\rho : A \to \mathrm{End}(V)$, $x \mapsto (v \mapsto xv)$, and annihilator $I = \ker(\rho)$.*

(1) $C^V = \mathrm{im}(\rho^*) = \{f \in A^* \mid f(I) = 0\} \cong (A/I)^*$.
(2) *C^V is a coalgebra which is isomorphic to the dual coalgebra of the finite-dimensional algebra A/I by (1). Let $F, F_{1i}, F_{2i} \in C^V$ for all $i \in \{1, \dots, n\}$, $n \geq 1$. Then the following are equivalent.*
  (a) $\Delta_{C^V}(F) = \sum_{i=1}^n F_{1i} \otimes F_{2i}$.
  (b) $F(xy) = \sum_{i=1}^n F_{1i}(x)F_{2i}(y)$ *for all* $x, y \in A$.

PROOF. (1) Note that for all $v \in V$, $f \in V^*$, the matrix coefficient $c_{f,v}$ is the image of $v \otimes f$ under the composition $V \otimes V^* \xrightarrow{\varphi_{V,V}} \mathrm{End}(V)^* \xrightarrow{\rho^*} A^*$, where $\varphi_{V,V}$ is the isomorphism of Lemma 2.3.1.

(2) By Proposition 2.2.4(1) and by (1), $\mathrm{End}(V)^*$ and C^V are coalgebras. The rest follows from the definition of Δ_{C^V}. □

LEMMA 2.3.4. *Let H be an algebra, and V, W finite-dimensional left H-modules.*

(1) *If $V \cong W$, then $C^V = C^W$. If $V \subseteq W$ is a left A-submodule, then $C^V \subseteq C^W$ is a subcoalgebra.*
(2) $C^{V \oplus W} = C^V + C^W$.
(3) *Let H be a bialgebra. Then $C^{V \otimes W} = C^V C^W$, where the product in H^* is the convolution product.*

PROOF. (1) is clear by Lemma 2.3.3.

(2) Let $f \in (V \oplus W)^* \cong V^* \oplus W^*$. Then $c_{f,v+w} = c_{f|V,v} + c_{f|W,w}$ for all $v \in V$ and $w \in W$.

(3) Let $f \in (V \otimes W)^*$, $f_1, \dots, f_n \in V^*$ and $g_1, \dots, g_n \in W^*, n \geq 1$, with $f(v \otimes w) = \sum_{i=1}^n f_i(v)g_i(w)$ for all $v \in V$, $w \in W$. Then for all $v \in V$, $w \in W$, $c_{f,v\otimes w} = \sum_{i=1}^n c_{f_i,v}c_{g_i,w}$. Hence the claim follows from Lemma 2.2.3. □

REMARK 2.3.5. Let A be an algebra and $V \in {}_A\mathcal{M}^{\mathrm{fd}}$. Let $v_1, \dots, v_n$, $n \geq 1$, be a basis of V and $f_1, \dots, f_n$ the dual basis of V^*. Then for all $x \in A$, $v \in V$, $f \in V^*$, and $j \in \{1, \dots, n\}$,

$$xv_j = \sum_{i=1}^n c_{f_i,v_j}(x)v_i, \quad \Delta_{C^V}(c_{f,v}) = \sum_{i=1}^n c_{f,v_i} \otimes c_{f_i,v}.$$

DEFINITION 2.3.6. Let H be an algebra, and $\mathcal{C} \subseteq {}_H\mathcal{M}^{\mathrm{fd}}$ a class of finite-dimensional left H-modules. Let $H^0_{\mathcal{C}} = \sum_{V \in \mathcal{C}} C^V \subseteq H^*$.

We define the following conditions for $\mathcal{C}$, where H is assumed to be a bialgebra for ($\mathcal{C}2$) and ($\mathcal{C}3$), and a Hopf algebra for ($\mathcal{C}4$).

($\mathcal{C}1$) If $V, W \in \mathcal{C}$, then $V \oplus W \in \mathcal{C}$.
($\mathcal{C}2$) If $V, W \in \mathcal{C}$, then $V \otimes W \in \mathcal{C}$.

($\mathcal{C}3$) ${}_\varepsilon\Bbbk \in \mathcal{C}$, where ${}_\varepsilon\Bbbk = \Bbbk$ is the trivial H-module with $x1 = \varepsilon(x)1$ for all $x \in H$.

($\mathcal{C}4$) If $V \in \mathcal{C}$, then $V^* \in \mathcal{C}$.

PROPOSITION 2.3.7. *Let H be an algebra and $\mathcal{C}$ a class of finite-dimensional left H-modules.*

(1) *Assume that $\mathcal{C}$ satisfies ($\mathcal{C}1$). Then $H^0_{\mathcal{C}}$ is a coalgebra with comultiplication and counit given by $\Delta_{H^0_{\mathcal{C}}}(F) = \Delta_{C^V}(F)$, $\varepsilon_{H^0_{\mathcal{C}}}(F) = F(1)$ for all $F \in C^V$, $V \in \mathcal{C}$.*

(2) *Let H be a bialgebra. Assume that $\mathcal{C}$ satisfies ($\mathcal{C}1$), ($\mathcal{C}2$) and ($\mathcal{C}3$). Then $H^0_{\mathcal{C}}$ is a bialgebra, where $H^0_{\mathcal{C}} \subseteq H^*$ is a subalgebra of the dual algebra of the coalgebra H, and where the coalgebra structure of $H^0_{\mathcal{C}}$ is defined in* (1).

(3) *Let H be a Hopf algebra. Assume that $\mathcal{C}$ satisfies ($\mathcal{C}1$) – ($\mathcal{C}4$). Then $H^0_{\mathcal{C}}$ is a Hopf algebra with antipode defined by $\mathcal{S}_{H^0_{\mathcal{C}}}(F) = F \circ \mathcal{S}_H$ for all $F \in H^0_{\mathcal{C}}$.*

PROOF. (1) Since the subspaces C^V are coalgebras by Lemma 2.3.3, we have to show that the definition of $\Delta_{H^0_{\mathcal{C}}}(F)$ does not depend on the choice of V. Let $V, V' \in \mathcal{C}$ with $F \in C^V$ and $F \in C^{V'}$. By ($\mathcal{C}1$) and Lemma 2.3.4(1) and (2), $W := V \oplus V' \in \mathcal{C}$, and C^V and $C^{V'}$ are subcoalgebras of C^W. Hence it follows that $\Delta_{C^V}(F) = \Delta_{C^W}(F) = \Delta_{C^{V'}}(F)$.

(2) Let $F \in C^V$ and $G \in C^W$, where $V, W \in \mathcal{C}$. Choose $F_{1i}, F_{2i} \in C^V$ and $G_{1i}, G_{2i} \in C^W$, $i \in \{1, \dots, n\}$, $n \geq 1$ with $\Delta_{C^V}(F) = \sum_{i=1}^n F_{1i} \otimes F_{2i}$ and $\Delta_{C^W}(G) = \sum_{i=1}^n G_{1i} \otimes G_{2i}$. Then $FG, F_{1i}G_{1j}, F_{2i}G_{2j} \in C^{V \otimes W}$ for all elements i, j in $\{1, \dots, n\}$ by Lemma 2.3.4(3). The computation in the proof of Proposition 2.2.19 shows that

$$\Delta_{C^{V\otimes W}}(FG) = \sum_{1 \leq i,j \leq n} F_{1i}G_{1j} \otimes F_{2i}G_{2j} = \Delta_{C^V}(F)\Delta_{C^W}(G).$$

Hence $\Delta_{H^0_{\mathcal{C}}}(FG) = \Delta_{H^0_{\mathcal{C}}}(F)\Delta_{H^0_{\mathcal{C}}}(G)$. By ($\mathcal{C}3$), $c_{\mathrm{id}_\Bbbk,1} = \varepsilon_H \in H^0_{\mathcal{C}}$ is the identity element of the algebra $H^0_{\mathcal{C}}$. Since ε_H is an algebra map, $\Delta_{H^0_{\mathcal{C}}}$ is unitary.

(3) Let $F = c^V_{f,v}$, where $V \in \mathcal{C}$, $v \in V$, and $f \in V^*$. By ($\mathcal{C}4$), $V^* \in \mathcal{C}$, where for all $f \in V^*$, $x \in H$ and $v \in V$, $(xf)(v) = f(\mathcal{S}_H(x)v)$. Let $V \to V^{**}$, $v \mapsto \varphi_v$, be the canonical isomorphism with $\varphi_v(f) = f(v)$ for all $f \in V^*$. Then $\mathcal{S}_{H^0_{\mathcal{C}}}(c^V_{f,v}) = c^{V^*}_{\varphi_v,f}$, and hence $\mathcal{S}_{H^0_{\mathcal{C}}}(F) \in H^0_{\mathcal{C}}$. As in the proof of Proposition 2.2.19, it follows that $\mathcal{S}_{H^0_{\mathcal{C}}}$ is the antipode of $H^0_{\mathcal{C}}$. □

DEFINITION 2.3.8. Let H be an algebra, and $H^0 = H^0_{\mathcal{C}}$, where $\mathcal{C} = {}_H\mathcal{M}^{\mathrm{fd}}$. The coalgebra H^0 of Proposition 2.3.7(1) is called the **dual coalgebra** of H. If H is a bialgebra or a Hopf algebra, H^0 of Proposition 2.3.7(3) is called the **dual bialgebra** or the **dual Hopf algebra** of H.

To characterize the elements of H^0, we note

LEMMA 2.3.9. *Let A be an algebra. Then any left or right ideal of A of finite codimension contains an ideal of A of finite codimension.*

PROOF. Let $I \subseteq A$ be a left ideal, and assume that I is of finite codimension, that is, $\dim A/I < \infty$. Let $\rho : A \to \mathrm{End}(A/I)$, $a \mapsto (\overline{x} \mapsto \overline{ax})$, be the natural representation of A over A/I. Then the kernel of ρ is an ideal of A of finite codimension which is contained in I. The proof for right ideals is similar. □

COROLLARY 2.3.10. *Let H be an algebra, and H^0 the dual coalgebra.*

(1) *For any $F \in H^*$, $F \in H^0$ if and only if $F(I) = 0$ for some ideal I of H of finite codimension.*
(2) *Let $F, F_{1i}, F_{2i} \in H^0$, $i \in \{1, \dots, n\}$, $n \geq 1$. Then the following are equivalent.*
   (a) $\Delta_{H^0}(F) = \sum_{i=1}^n F_{1i} \otimes F_{2i}$.
   (b) $F(xy) = \sum_{i=1}^n F_{1i}(x) F_{2i}(y)$ *for all* $x, y \in H$.
(3) *Let $F, F_{1i}, F_{2i} \in H^*$, $i \in \{1, \dots, n\}$, $n \geq 1$, such that* (2)(b) *holds. Then $F \in H^0$.*

PROOF. (1) and (2) are clear from Proposition 2.3.7 and Lemma 2.3.3.

(3) Let $I = \bigcap_{i=1}^n \ker(F_{2i})$. Then I has finite codimension in H, since finite intersections of subspaces of finite codimension have finite codimension. By (2)(b), $F(HI) = 0$. Hence $F \in H^0$ by (1) and Lemma 2.3.9. $\square$

For algebras A, B, a triple (M, λ, ρ) is an (A, B)-**bimodule** if $(M, \lambda) \in {}_A\mathcal{M}$, $(M, \rho) \in \mathcal{M}_B$, and if $\rho(\lambda \otimes \mathrm{id}) = \lambda(\mathrm{id} \otimes \rho)$ as maps $A \otimes M \otimes B \to M$, that is, $(am)b = a(mb)$ for all $a \in A$, $b \in B$, $m \in M$.

Let A be an algebra and let M be an (A, A)-bimodule. A linear map $d : A \to M$ is called a **derivation** if for all $x, y \in A$, $d(xy) = xd(y) + d(x)y$.

Let A, B be algebras, and $\sigma, \tau \in \mathrm{Alg}(A, B)$. Let ${}_\sigma B_\tau$ be the vector space B with (A, A)-bimodule structure given by $A \otimes B \to B$, $(a, b) \mapsto \sigma(a)b$, and $B \otimes A \to B$, $(b, a) \mapsto b\tau(a)$. A (σ, τ)-**derivation** (or a **skew derivation**) $d : A \to B$ is a derivation from A to the (A, A)-bimodule ${}_\sigma B_\tau$, that is, a linear map $d : A \to B$ such that

$$d(xy) = \sigma(x)d(y) + d(x)\tau(y) \text{ for all } x, y \in A.$$

Let (σ, τ)-$\mathrm{Der}(A, B)$ be the set of all (σ, τ)-derivations $d : A \to B$. The next obvious lemma is useful to construct skew derivations.

LEMMA 2.3.11. *Let A and B be algebras, $\sigma, \tau : A \to B$ algebra homomorphisms, and $d : A \to B$ a linear map. Then the following are equivalent.*

(1) *d is a (σ, τ)-derivation.*
(2) *The map*

$$A \to M_2(B),\ x \mapsto \begin{pmatrix} \sigma(x) & d(x) \\ 0 & \tau(x) \end{pmatrix},$$

is an algebra homomorphism.

Skew derivations are related to skew-primitive elements of a coalgebra.

COROLLARY 2.3.12. *Let H be an algebra, and H^0 the dual coalgebra.*

(1) $G(H^0) = \mathrm{Alg}(H, \Bbbk)$.
(2) *Let $\sigma, \tau \in \mathrm{Alg}(H, \Bbbk)$. Then $P_{\sigma,\tau}(H^0) = (\sigma, \tau)$-$\mathrm{Der}(A, \Bbbk)$.*

PROOF. This follows from Corollary 2.3.10. $\square$

2.4. Basic Hopf algebra examples

Group-like and skew-primitive elements play a fundamental role in many Hopf algebras. We discuss some examples and some theory from this perspective.

PROPOSITION 2.4.1. *Let H be a Hopf algebra.*

(1) *The set* $G(H)$ *is a subgroup of the group of invertible elements of* H. *The subalgebra of* H *generated by* $G(H)$ *is isomorphic to the group algebra of* $G(H)$. *Moreover,* $\mathcal{S}(g) = g^{-1}$ *for each* $g \in G(H)$.
(2) *Let* $g, h \in G(H)$ *and* $x \in P_{g,h}(H)$. *Then* $\mathcal{S}(x) = -g^{-1}xh^{-1}$.

PROOF. (1) Clearly, $G(H)$ is a submonoid of H. Let $g \in G(H)$. By definition of the antipode, $1 = g\mathcal{S}(g) = \mathcal{S}(g)g$. Hence $g^{-1} = \mathcal{S}(g) \in H$, and $g^{-1} \in G(H)$. The remaining claim follows from Proposition 1.1.6.

(2) Since $\Delta(x) = g \otimes x + x \otimes h$, and $\varepsilon(x) = 0$, we obtain that

$$0 = \mathcal{S}(x_{(1)})x_{(2)} = g^{-1}x + \mathcal{S}(x)h.$$

Hence $\mathcal{S}(x) = -g^{-1}xh^{-1}$. $\square$

PROPOSITION 2.4.2. *Let* $0 \neq q \in \Bbbk$. *Let* H *be a bialgebra,* $g, h \in G(H)$, *and* $x \in P_{g,1}(H)$, $y \in P_{h,1}(H)$. *Then*

(1) $g - h \in P_{g,h}(H)$.
(2) *If* $gh = hg$, *then* $hx, xh \in P_{gh,h}(H)$.
(3) *If* $gy = qyg$, $hx = q^{-1}xh$, *and* $gh = hg$, *then* $xy - qyx \in P_{gh,1}(H)$.
(4) *If* $x, y \in P(H)$, *then* $xy - yx \in P(H)$. *If the characteristic of* $\Bbbk$ *is* $p > 0$, *then* $x^p \in P(H)$.
(5) *Let* $n \geq 2$. *If* $(n-1)^!_q \neq 0$ *and* $gx = qxg$, *then* $x^n \in P_{g^n,1}(H)$ *if and only if* $(n)_q = 0$.

PROOF. (1) follows from the computation

$$\Delta(g - h) = g \otimes g - h \otimes h = g \otimes (g - h) + (g - h) \otimes h.$$

Regarding (2), note that $hx \in P_{hg,h}(H)$ and $xh \in P_{gh,h}(H)$.

For (3) we compute

$$\begin{aligned}
\Delta(xy - qyx) &= (g \otimes x + x \otimes 1)(h \otimes y + y \otimes 1) \\
&\quad - q(h \otimes y + y \otimes 1)(g \otimes x + x \otimes 1) \\
&= gh \otimes xy + gy \otimes x + xh \otimes y + xy \otimes 1 \\
&\quad - qhg \otimes yx - qhx \otimes y - qyg \otimes x - qyx \otimes 1 \\
&= gh \otimes (xy - qyx) + (xy - qyx) \otimes 1 \\
&\quad + (xh - qhx) \otimes y + (gy - qyg) \otimes x,
\end{aligned}$$

where we have used $gh = hg$ in the last equality. This implies (3).

The first part of (4) is a special case of (3). The second part of (4) follows from the binomial formula, since in characteristic p

$$\Delta(x^p) = (1 \otimes x + x \otimes 1)^p = 1 \otimes x^p + x^p \otimes 1.$$

(5) holds by Proposition 1.9.5(2) since $(g \otimes x)(x \otimes 1) = q(x \otimes 1)(g \otimes x)$. $\square$

The next claim is an important generalization of Proposition 2.4.2(3). We will apply it in Proposition 4.3.12 where H is the bosonization of a braided Hopf algebra and $x^m \triangleright y$ is an iterated adjoint action. The skew-primitive elements of the form $x^m \triangleright y$ will also be used in the construction of quantum groups, see Proposition 8.1.3.

PROPOSITION 2.4.3. *Let H be a bialgebra, $q, r, s \in \Bbbk$, $g, h \in G(H)$ with $gh = hg$, and $x \in P_{g,1}(H)$, $y \in P_{h,1}(H)$. Assume that $gx = qxg$, $gy = ryg$, and $hx = sxh$. For all $m \in \mathbb{N}_0$ let*

$$x^m \triangleright y = \sum_{k=0}^{m} (-r)^k q^{k(k-1)/2} \binom{m}{k}_q x^{m-k} y x^k.$$

(1) *For any $m \in \mathbb{N}_0$,*

$$\Delta(x^m \triangleright y) = x^m \triangleright y \otimes 1 + \sum_{k=0}^{m} \binom{m}{k}_q \Big(\prod_{l=k}^{m-1} (1 - q^l rs) \Big) x^{m-k} g^k h \otimes x^k \triangleright y.$$

(2) *Let $m \in \mathbb{N}_0$. If $q^m rs = 1$, then $x^{m+1} \triangleright y \in P_{g^{m+1}h,1}$.*

PROOF. (1) We proceed by induction on m. Clearly, $x^0 \triangleright y = y$. Therefore the claim holds for $m = 0$ since $y \in P_{h,1}$. Let $m \in \mathbb{N}_0$. Lemma 1.9.3(2) implies that

$$x^{m+1} \triangleright y = x(x^m \triangleright y) - q^m r (x^m \triangleright y) x.$$

For $0 \le k \le m$ let $a_k = \binom{m}{k}_q \Big(\prod_{l=k}^{m-1} (1 - q^l rs) \Big)$. Then induction hypothesis implies that

$$\begin{aligned}
&(x \otimes 1)\Delta(x^m \triangleright y) - q^m r \Delta(x^m \triangleright y)(x \otimes 1) \\
&= x^{m+1} \triangleright y \otimes 1 + \sum_{k=0}^{m} a_k (1 - q^{m+k} rs) x^{m+1-k} g^k h \otimes x^k \triangleright y
\end{aligned}$$

and that

$$\begin{aligned}
&(g \otimes x)\Delta(x^m \triangleright y) - q^m r \Delta(x^m \triangleright y)(g \otimes x) \\
&= \sum_{k=0}^{m} a_k x^{m-k} g^{k+1} h \otimes (q^{m-k} x (x^k \triangleright y) - q^m r (x^k \triangleright y) x) \\
&= \sum_{k=0}^{m} a_k q^{m-k} x^{m-k} g^{k+1} h \otimes x^{k+1} \triangleright y \\
&= \sum_{k=1}^{m+1} a_{k-1} q^{m+1-k} x^{m+1-k} g^k h \otimes x^k \triangleright y.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta(x^{m+1} \triangleright y) &= x^{m+1} \triangleright y \otimes 1 \\
&+ \sum_{k=0}^{m+1} \big(a_k (1 - q^{m+k} rs) + a_{k-1} q^{m+1-k} \big) x^{m+1-k} g^k h \otimes x^k \triangleright y,
\end{aligned}$$

where $a_{-1} = 0$ and $a_{m+1} = 0$. From Lemma 1.9.3(1),(2) we obtain that

$$\begin{aligned}
&a_k(1 - q^{m+k}rs) + a_{k-1}q^{m+1-k} \\
&= \binom{m}{k}_q (1 - q^{m+k}rs) \prod_{l=k}^{m-1}(1 - q^l rs) + \binom{m}{k-1}_q q^{m+1-k} \prod_{l=k-1}^{m-1}(1 - q^l rs) \\
&= \left(\binom{m}{k}_q (1 - q^{m+k}rs) + \binom{m}{k-1}_q q^{m+1-k}(1 - q^{k-1}rs)\right) \prod_{l=k}^{m-1}(1 - q^l rs) \\
&= \binom{m+1}{k}_q (1 - q^m rs) \prod_{l=k}^{m-1}(1 - q^l rs)
\end{aligned}$$

for $0 \le k \le m+1$. This implies the formula for $\Delta(x^{m+1} \triangleright y)$.

(2) is a direct consequence of (1). □

In the next proposition we describe a standard method to construct bi-ideals and Hopf ideals.

PROPOSITION 2.4.4. *Let H be a bialgebra, and $X \subseteq H$ a subset of skew-primitive elements. Let (X) denote the ideal of H generated by X. Then (X) is a bi-ideal of H. If H is a Hopf algebra, then (X) is a Hopf ideal of H and $H/(X)$ is a Hopf algebra.*

PROOF. Any element of (X) is a sum of elements of the form axb with $a, b \in H$ and $x \in X$. To see that (X) is a bi-ideal it is enough to show that for any $x \in X$ and $a, b \in H$, $\Delta(axb)$ is contained in $(X) \otimes H + H \otimes (X)$. For any $x \in X$, there are $g, h \in G(H)$ such that $\Delta(x) = g \otimes x + x \otimes h$. Then

$$\Delta(axb) = a_{(1)}gb_{(1)} \otimes a_{(2)}xb_{(2)} + a_{(1)}xb_{(1)} \otimes a_{(2)}hb_{(2)} \in H \otimes (X) + (X) \otimes H.$$

If H is a Hopf algebra, then Propositions 1.2.17(1) and 2.4.1(2) imply that

$$\begin{aligned}
\mathcal{S}(axb) &= \mathcal{S}(b)\mathcal{S}(x)\mathcal{S}(a) \\
&= -\mathcal{S}(b)g^{-1}xh^{-1}\mathcal{S}(a) \in (X).
\end{aligned}$$

Hence (X) is a Hopf ideal. Finally, $H/(X)$ is a Hopf algebra by Proposition 1.2.22. □

EXAMPLE 2.4.5. Recall that a **Lie algebra** is a vector space $\mathfrak{g}$ together with a $\Bbbk$-bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x, y) \mapsto [x, y],$$

called the **Lie bracket**, such that

$$\begin{aligned}
[x, x] &= 0, \\
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0
\end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

The **universal enveloping algebra** of $\mathfrak{g}$ is the quotient algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/I,$$

where I is the ideal of $T(\mathfrak{g})$ generated by the elements $x \otimes y - y \otimes x - [x, y]$ with $x, y \in \mathfrak{g}$. We view $T(\mathfrak{g})$ as a Hopf algebra by Example 1.2.25. Then $U(\mathfrak{g})$ is a quotient Hopf algebra of the tensor algebra by Proposition 2.4.4, since I is generated by primitive elements by Proposition 2.4.2(4).

If $\mathfrak{g}$ is a finite-dimensional Lie algebra with basis $x_1, \dots, x_n$ and multiplication table

$$[x_i, x_j] = \sum_{k=1}^{n} \alpha_{ij}^k x_k,$$

where $\alpha_{ij}^k \in \Bbbk$ for all i, j, k, then by definition

$$U(\mathfrak{g}) \cong \Bbbk\langle x_1, \dots, x_n \mid x_i x_j - x_j x_i = \sum_{k=1}^{n} \alpha_{ij}^k x_k \text{ for all } 1 \leq i, j \leq n\rangle,$$

and the elements $x_1, \dots, x_n$ are primitive.

EXAMPLE 2.4.6. Let $\mathfrak{sl}_2$ be the Lie algebra of 2×2-matrices with trace 0, and with Lie bracket $[x, y] = xy - yx$ for all $x, y \in \mathfrak{sl}_2$. Then $\mathfrak{sl}_2$ is 3-dimensional with basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence $U(\mathfrak{sl}_2) \cong \Bbbk\langle e, f, h \mid ef - fe = h,\ he - eh = 2e,\ hf - fh = -2f\rangle$.

EXAMPLE 2.4.7. Let $A = \Bbbk[x_{ij}]_{1 \leq i,j \leq n}$ be the commutative polynomial algebra in n^2 variables x_{ij}, $1 \leq i, j \leq n$, where $n \geq 1$. Using the universal property of A one shows quickly that A is a bialgebra, where Δ and ε are given by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}$$

for all $1 \leq i, j \leq n$. Let X, X_1, X_2 be the $n \times n$-matrices with entries x_{ij}, $x_{ij} \otimes 1$ and $1 \otimes x_{ij}$ in the i-th row and j-th column, respectively. Then $\Delta(X) = X_1 X_2$, where $\Delta(X) = (\Delta(x_{ij}))_{1 \leq i,j \leq n}$. The determinant $d = \det(X) \neq 0$ of X is group-like. Indeed, $\varphi(\det(X)) = \det(\varphi(X))$ for any commutative algebra B and any algebra map $\varphi : \Bbbk[x_{ij}]_{1 \leq i,j \leq n} \to B$, where $\varphi(X) = (\varphi(x_{ij}))_{1 \leq i,j \leq n}$. Hence

$$\begin{aligned} \Delta(\det(X)) &= \det(\Delta(X)) \\ &= \det(X_1 X_2) = \det(X_1)\det(X_2) = \det(X) \otimes \det(X). \end{aligned}$$

Thus $A/(d-1)$ is a bialgebra by Propositions 2.4.4 and 2.4.2(1).

For any commutative algebra R, the bijective map

$$\mathrm{Alg}(A/(d-1), R) \to \mathrm{SL}_n(R),\ \varphi \mapsto (\varphi(\overline{x_{ij}}))_{1 \leq i,j \leq n},$$

is a homomorphism of monoids, where $\mathrm{Alg}(A/(d-1), R)$ is a monoid under convolution, and the multiplication in $\mathrm{SL}_n(R)$ is matrix multiplication. Hence the monoid $\mathrm{Alg}(A/(d-1), R)$ is a group, since $\mathrm{SL}_n(R)$ is. Thus $\mathrm{id}_{A/(d-1)}$ is convolution invertible, and $A/(d-1)$ is a Hopf algebra.

EXAMPLE 2.4.8. Let $0 \neq q \in \Bbbk$, and $n \geq 1$ a natural number. The free algebra $\Bbbk\langle g, x\rangle$ is a bialgebra with

$$\begin{aligned} \Delta(g) &= g \otimes g, & \varepsilon(g) &= 1, \\ \Delta(x) &= g \otimes x + x \otimes 1, & \varepsilon(x) &= 0. \end{aligned}$$

This is easily checked on the generators. Hence the algebras

$$\Bbbk\langle g, x \mid gx = qxg\rangle, \quad \Bbbk\langle g, x \mid g^n = 1, gx = qxg\rangle$$

are quotient bialgebras of the free algebra by Proposition 2.4.2(1),(2). The bialgebra $\Bbbk\langle g, x \mid gx = qxg, g^n = 1\rangle$ is a Hopf algebra, since the antipode can be defined as the algebra anti-homomorphism $\mathcal{S}$ with

$$\mathcal{S}(g) = g^{n-1}, \quad \mathcal{S}(x) = -g^{n-1}x.$$

To see that $\mathcal{S}$ is well-defined, one has to check that

$$(\mathcal{S}(g))^n = 1, \quad \mathcal{S}(x)\mathcal{S}(g) = q\mathcal{S}(g)\mathcal{S}(x).$$

EXAMPLE 2.4.9. Let $0 \neq q \in \Bbbk$. The free algebra $\Bbbk\langle g, g^{-1}, x\rangle$ is a bialgebra with

$$\begin{aligned} &\Delta(g) = g \otimes g, && \Delta(g^{-1}) = g^{-1} \otimes g^{-1}, && \Delta(x) = x \otimes 1 + g \otimes x, \\ &\varepsilon(g) = 1, && \varepsilon(g^{-1}) = 1, && \varepsilon(x) = 0. \end{aligned}$$

It admits an antipode, and hence a Hopf algebra structure, such that

$$\mathcal{S}(g) = g^{-1}, \quad \mathcal{S}(g^{-1}) = g, \quad \mathcal{S}(x) = -g^{-1}x.$$

The elements $gg^{-1} - 1$, $g^{-1}g - 1$, and $gx - qxg$ are skew-primitive by Proposition 2.4.2(1),(2). Therefore

$$H_q = \Bbbk\langle g, g^{-1}, x \,|\, gg^{-1} = 1, g^{-1}g = 1, gx = qxg\rangle$$

becomes a Hopf algebra by Proposition 2.4.4.

EXAMPLE 2.4.10. Let $n \geq 2$ be an integer, and $q \in \Bbbk$ a primitive n-th root of unity. Then

$$T_{q,n} = \Bbbk\langle g, x \mid g^n = 1, gx = qxg, x^n = 0\rangle$$

is a Hopf algebra with

$$\begin{aligned} &\Delta(g) = g \otimes g, && \varepsilon(g) = 1, && \mathcal{S}(g) = g^{n-1}, \\ &\Delta(x) = g \otimes x + x \otimes 1, && \varepsilon(x) = 0, && \mathcal{S}(x) = -g^{n-1}x \end{aligned}$$

and is known as the **Taft Hopf algebra**. By Proposition 2.4.2(5), $T_{q,n}$ is a quotient Hopf algebra of the Hopf algebra $\Bbbk\langle g, x \mid gx = qxg, g^n = 1\rangle$ in Example 2.4.8.

EXAMPLE 2.4.11. Let $0 \neq q \in \Bbbk$ with $q^2 \neq 1$. Then

$$\begin{aligned} U_q(\mathfrak{sl}_2) = \Bbbk\Big\langle E, F, K, K^{-1} \;\mid\; & KK^{-1} = 1 = K^{-1}K, \\ & KE = q^2EK, KF = q^{-2}FK, EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\Big\rangle \end{aligned}$$

is a Hopf algebra with

$$\begin{aligned} \Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, & \varepsilon(K^{\pm 1}) &= 1, & \mathcal{S}(K^{\pm 1}) &= K^{\mp 1}, \\ \Delta(E) &= K \otimes E + E \otimes 1, & \varepsilon(E) &= 0, & \mathcal{S}(E) &= -K^{-1}E, \\ \Delta(F) &= 1 \otimes F + F \otimes K^{-1}, & \varepsilon(F) &= 0, & \mathcal{S}(F) &= -FK. \end{aligned}$$

As in Example 2.4.8, it follows from Proposition 2.4.2 that

$$\begin{aligned} \widetilde{U_q}(\mathfrak{sl}_2) = \Bbbk\langle E, F, K, K^{-1} \,|\, & KK^{-1} = 1 = K^{-1}K, \\ & KE = q^2EK, KF = q^{-2}FK\rangle \end{aligned}$$

is a Hopf algebra, where Δ, ε and $\mathcal{S}$ are defined on the generators by the same formulas as for $U_q(\mathfrak{sl}_2)$. Let $\widetilde{F} = FK$ in $U_q(\mathfrak{sl}_2)$. Then $\widetilde{F}$ is $(K,1)$-primitive and $E\widetilde{F} - q^{-2}\widetilde{F}E$ is $(K^2,1)$-primitive by Proposition 2.4.2. Moreover,

$$E\widetilde{F} - q^{-2}\widetilde{F}E = EFK - q^{-2}FKE = EFK - FEK = \frac{K^2-1}{q-q^{-1}},$$

and $K^2 - 1$ is $(K^2,1)$-primitive by Proposition 2.4.2(1). Therefore

$$U_q(\mathfrak{sl}_2) \cong \widetilde{U_q}(\mathfrak{sl}_2) \Big/ \left(E\widetilde{F} - q^{-2}\widetilde{F}E - \frac{K^2-1}{q-q^{-1}} \right)$$

is a Hopf algebra.

The Hopf algebras in Examples 2.4.8 and 2.4.9 are special cases of the general class of Hopf algebras A_χ in the next example.

EXAMPLE 2.4.12. Let X be a set, G a group and $(g_x)_{x\in X}$ a family of elements in G. Assume that $X \cap G = \emptyset$. Let $\tilde{A} = \Bbbk\langle X \cup G\rangle$. By the universal property of the tensor algebra, $\tilde{A}$ has a unique bialgebra structure such that

$$\begin{aligned} \Delta(x) &= g_x \otimes x + x \otimes 1, & \varepsilon(x) &= 0 \quad \text{for all } x \in X,\\ \Delta(g) &= g \otimes g, & \varepsilon(g) &= 1 \quad \text{for all } g \in G. \end{aligned}$$

Since products of group-like elements are group-like, the elements

$$1_{\tilde{A}} - 1_G,\ \mu_{\tilde{A}}(g \otimes h) - \mu_G(g,h)$$

with $g, h \in G$ are skew-primitive by Proposition 2.4.2(1). Thus the ideal $\tilde{I}$ generated by them is a bi-ideal and $A = \tilde{A}/\tilde{I}$ is a bialgebra by Proposition 2.4.4. We denote by $\mathcal{S} : \tilde{A} \to \tilde{A}^{\mathrm{op}}$ the algebra map with

$$\mathcal{S}(x) = -g_x^{-1}x, \quad \mathcal{S}(g) = g^{-1}$$

for all $x \in X$, $g \in G$. Since $\mathcal{S}(1_{\tilde{A}} - 1_G) = 1_{\tilde{A}} - 1_G$ and

$$\mathcal{S}(\mu_{\tilde{A}}(g \otimes h) - \mu_G(g,h)) = \mu_{\tilde{A}}(h^{-1} \otimes g^{-1}) - \mu_G(h^{-1}, g^{-1}) \in \tilde{I}$$

for all $g, h \in \tilde{I}$, the map $\mathcal{S}$ induces an algebra map $\mathcal{S} : A \to A^{\mathrm{op}}$ which fulfills the equations

$$\mathcal{S}(x_{(1)})x_{(2)} = \mathcal{S}(g_x)x + \mathcal{S}(x)1 = g_x^{-1}x - g_x^{-1}x = 0$$

for all $x \in X$. Similarly, $x_{(1)}\mathcal{S}(x_{(2)}) = \varepsilon(x)$ for all $x \in X$,

$$g_{(1)}\mathcal{S}(g_{(2)}) = \mu_A(g \otimes g^{-1}) = \mu_G(g, g^{-1}) = 1$$

and $\mathcal{S}(g_{(1)})g_{(2)} = \varepsilon(g)1$ for all $g \in G$. Hence A is a Hopf algebra by Proposition 1.2.23. The group algebra of G is contained in A, since there is a well-defined surjective algebra map $A \to \Bbbk G$ mapping the residue classes of $g \in G$ and $x \in X$ onto g and 0, respectively. Thus the images of the elements $g \in G$ are linearly independent in A.

Assume that G is abelian. Let $\chi : X \to \widehat{G}$, $x \mapsto \chi_x$, be a map and let A_χ be the quotient algebra

$$A_\chi = A/(gx - \chi_x(g)xg \mid g \in G, x \in X).$$

By Proposition 2.4.2(2), for any $g \in G$, $x \in X$ the element $gx - \chi_x(g)xg \in A$ is (gg_x, g)-primitive and hence A_χ is a Hopf algebra by Proposition 2.4.4. Note that

$$A_\chi = \Bbbk\langle g, x \mid g \in G, x \in X,\ 1 = 1_G,\ gh = \mu_G(g,h) \text{ for all } g,h \in G,\\ gx = \chi_x(g)xg \text{ for all } g \in G, x \in X\rangle$$

with $\Delta(g) = g \otimes g$, $\Delta(x) = g_x \otimes x + x \otimes 1$ for all $g \in G$, $x \in X$.

REMARK 2.4.13. Let $n \in \mathbb{N}$ and let $A = (a_{ij})_{i,j\in\{1,\dots,n\}}$ be a Cartan matrix. Let $\mathfrak{h}$ be a complex vector space of dimension $2n - \operatorname{rank} A$ and let $\alpha_i^\vee \in \mathfrak{h}$, $\alpha_i \in \mathfrak{h}^*$ for $1 \le i \le n$ be elements with $\alpha_i(\alpha_j^\vee) = a_{ji}$. Assume that $\alpha_1, \dots, \alpha_n$ and $\alpha_1^\vee, \dots, \alpha_n^\vee$ are linearly independent in $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively.

Let $h_1, \dots, h_{\dim \mathfrak{h}}$ be a basis of $\mathfrak{h}$. Let $\tilde{\mathfrak{g}}(A)$ be the complex Lie algebra given by generators h_j, e_i, f_i, where $1 \le j \le \dim \mathfrak{h}$, $1 \le i \le n$, and relations

(1) $[h_j, h_k] = 0$,
(2) $[h_j, e_i] = \alpha_i(h_j)e_i$, $[h_j, f_i] = -\alpha_i(h_j)f_i$,
(3) $[e_i, f_m] = \delta_{im}\alpha_i^\vee$

for all $i, m \in \{1, \dots, n\}$, $j, k \in \{1, \dots, \dim \mathfrak{h}\}$. There is a unique maximal ideal $\mathfrak{r}$ of $\tilde{\mathfrak{g}}(A)$ having trivial intersection with $\mathfrak{h} = \sum_{j=1}^{\dim \mathfrak{h}} \mathbb{C}h_j$. The quotient Lie algebra $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{r}$ is called a **Kac-Moody algebra**.

The Lie algebra $\tilde{\mathfrak{g}}(A)$ has a triangular decomposition

$$\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-,$$

where $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ are the Lie subalgebras of $\tilde{\mathfrak{g}}(A)$ generated by $e_1, \dots, e_n$ and $f_1, \dots, f_n$, respectively. Then $\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-)$ and

$$\mathfrak{r} \subseteq [\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+] \oplus [\tilde{\mathfrak{n}}_-, \tilde{\mathfrak{n}}_-].$$

It is reasonable and fruitful to view the Hopf algebra A_χ in Example 2.4.12 as the analog of $\tilde{\mathfrak{n}}_+ \oplus \mathfrak{h}$ with the abelian Lie algebra $\mathfrak{h}$ replaced by an abelian group G. The analog of $\mathfrak{h} \oplus \tilde{\mathfrak{n}}_+/(\mathfrak{r} \cap \tilde{\mathfrak{n}}_+)$ then will be the quotient Hopf algebra of A_χ by the maximal Hopf ideal contained in (X^2).

2.5. Coinvariant elements

The main topics in this section are Hopf modules, one-sided coideal subalgebras and coinvariant elements.

DEFINITION 2.5.1. Let C be a coalgebra with a distinguished group-like element $1_C \in G(C)$. Let V be a left C-comodule with comodule structure $\delta_V : V \to C \otimes V$, and let W be a right C-comodule with comodule structure $\delta_W : W \to W \otimes C$. The **$C$-coinvariant elements** of V and W with respect to 1_C are defined by

$${}^{\mathrm{co}\,C}V = \{v \in V \mid \delta_V(v) = 1_C \otimes v\},$$
$$W^{\mathrm{co}\,C} = \{w \in W \mid \delta_W(w) = w \otimes 1_C\}.$$

LEMMA 2.5.2. *Let C be a coalgebra with a distinguished group-like element 1_C, and let X be a vector space. Then the linear maps*

$$X \to {}^{\mathrm{co}\,C}(C \otimes X),\ x \mapsto 1_C \otimes x,$$
$$X \to (X \otimes C)^{\mathrm{co}\,C},\ x \mapsto x \otimes 1_C,$$

are bijective, where the C-comodule structures of $C \otimes X$ and $X \otimes C$ are $\Delta \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes \Delta$, respectively.

PROOF. We only consider $C \otimes X$. For all $x \in X$, $1_C \otimes x$ is coinvariant since 1_C is group-like. Conversely, let $\sum_{i=1}^n c_i \otimes x_i \in {}^{\mathrm{co}\, C}(C \otimes X)$. Then

$$\sum_{i=1}^n c_{i(1)} \otimes c_{i(2)} \otimes x_i = \sum_{i=1}^n 1_C \otimes c_i \otimes x_i.$$

Applying $\mathrm{id}_C \otimes \varepsilon \otimes \mathrm{id}_X$ to this equation gives $\sum_{i=1}^n c_i \otimes x_i = 1_C \otimes \sum_{i=1}^n \varepsilon(c_i)x_i$, hence $\sum_{i=1}^n c_i \otimes x_i \in 1_C \otimes X$. □

If H is a bialgebra, we define H-coinvariant elements of H-comodules with respect to the unit element $1 \in H$.

DEFINITION 2.5.3. Let H be a Hopf algebra, V a vector space, $(V, \rho) \in \mathcal{M}_H$, and $(V, \delta) \in \mathcal{M}^H$. Then (V, ρ, δ) is a **right Hopf module over** H if $\delta : V \to V \otimes H$, $v \mapsto v_{(0)} \otimes v_{(1)}$, is right H-linear, that is,

$$\delta(v \cdot h) = v_{(0)} \cdot h_{(1)} \otimes v_{(1)} h_{(2)}$$

for all $h \in H$ and $v \in V$, where $\rho(v \otimes h) = v \cdot h$ for all $v \in V$, $h \in H$. The category $\mathcal{M}_H^H$ of right Hopf modules over H has right Hopf modules over H as objects and right H-linear and right H-colinear maps as morphisms.

Let M be a vector space. Then $(M \otimes H, \mathrm{id}_M \otimes \mu, \mathrm{id}_M \otimes \Delta)$ is a right Hopf module over H.

The following result is also known as the fundamental theorem of Hopf modules.

THEOREM 2.5.4 (Larson-Sweedler). *Let H be a Hopf algebra, and (V, ρ, δ) a right Hopf module over H.*

(1) *The map $\vartheta : V \to V^{\mathrm{co}\, H}$, $v \mapsto v_{(0)}\mathcal{S}(v_{(1)})$, is well-defined.*
(2) *Let $h \in H$ and $v \in V$. Then $\vartheta(vh) = \vartheta(v)\varepsilon(h)$.*
(3) *The multiplication map $V^{\mathrm{co}\, H} \otimes H \to V$, $v \otimes h \mapsto vh$, is an isomorphism of right Hopf modules over H with inverse given by $v \mapsto \vartheta(v_{(0)}) \otimes v_{(1)}$.*

PROOF. (1) Let $v \in V$. Then $\vartheta(v) \in V^{\mathrm{co}\, H}$, since

$$\delta(v_{(0)}\mathcal{S}(v_{(1)})) = v_{(0)}\mathcal{S}(v_{(3)}) \otimes v_{(1)}\mathcal{S}(v_{(2)}) = v_{(0)}\mathcal{S}(v_{(1)}) \otimes 1.$$

(2) For all $v \in V$, $h \in H$,

$$\vartheta(vh) = v_{(0)}h_{(1)}\mathcal{S}(v_{(1)}h_{(2)}) = v_{(0)}h_{(1)}\mathcal{S}(h_{(2)})\mathcal{S}(v_{(1)}) = \vartheta(v)\varepsilon(h).$$

(3) follows easily from (1) and (2). □

We note that by Theorem 2.5.4 and by Lemma 2.5.2, the functor

$$\mathcal{M}_\Bbbk \to \mathcal{M}_H^H,\ M \mapsto M \otimes H,$$

mapping a linear function f onto $f \otimes \mathrm{id}_H$, is an equivalence of categories.

DEFINITION 2.5.5. Let C be a coalgebra, and $B \subseteq C$ a subspace. Then B is called a **right coideal** of C if $\Delta(B) \subseteq B \otimes C$ (that is, B is stable under the right coaction of C). It is called a **left coideal of** C if $\Delta(B) \subseteq C \otimes B$.

LEMMA 2.5.6. *Let C be a coalgebra. Let $I \subseteq C$ be a coideal with canonical coalgebra map $\pi : C \to C/I$, $c \mapsto \overline{c}$, and let $u \in C$ be a fixed element. Let $\varepsilon_u : C \to C/I$, $c \mapsto \varepsilon(c)\overline{u}$.*

(1) *Define*

$$C^{\mathrm{co}\,C/I} = \{c \in C \mid c_{(1)} \otimes \overline{c_{(2)}} = c \otimes \overline{u}\}.$$

Then $C^{\mathrm{co}\,C/I}$ *is a left coideal of* C, *and* $\pi|C^{\mathrm{co}\,C/I} = \varepsilon_u|C^{\mathrm{co}\,C/I}$. *Moreover, any left coideal* D *of* C *such that* $\pi|D = \varepsilon_u|D$ *is contained in* $C^{\mathrm{co}\,C/I}$.

(2) *Define*

$${}^{\mathrm{co}\,C/I}C = \{c \in C \mid \overline{c_{(1)}} \otimes c_{(2)} = \overline{u} \otimes c\}.$$

Then ${}^{\mathrm{co}\,C/I}C$ *is a right coideal of* C, *and* $\pi|{}^{\mathrm{co}\,C/I}C = \varepsilon_u|{}^{\mathrm{co}\,C/I}C$. *Moreover, any right coideal* D *of* C *such that* $\pi|D = \varepsilon_u|D$ *is contained in* ${}^{\mathrm{co}\,C/I}C$.

Proof. (1) Define linear maps $\overline{\Delta}$, $i_1 : C \to C \otimes C/I$ by

$$\overline{\Delta}(c) = c_{(1)} \otimes \overline{c_{(2)}}, \quad i_1(c) = c \otimes \overline{u}$$

for all $c \in C$. Thus $C^{\mathrm{co}\,C/I} = \ker(\overline{\Delta} - i_1)$. Since $\overline{\Delta}$ and i_1 are left C-colinear, $C^{\mathrm{co}\,C/I}$ is a left coideal of C. Note that

$$\pi(c) = \pi(\varepsilon(c_{(1)})c_{(2)}) = \varepsilon(c_{(1)})\pi(c_{(2)}) = \varepsilon(c)\overline{u} = \varepsilon_u(c)$$

for any $c \in C^{\mathrm{co}\,C/I}$.

Let now $D \subseteq C$ be a left coideal with $\pi|D = \varepsilon_u|D$. Then

$$d_{(1)} \otimes \overline{d_{(2)}} = d_{(1)} \otimes \pi(d_{(2)}) = d_{(1)} \otimes \varepsilon(d_{(2)})\overline{u} = d \otimes \overline{u}$$

for any $d \in D$, and hence $D \subseteq C^{\mathrm{co}\,C/I}$.

The proof of (2) is analogous to the one of (1). □

If G is a group and $G' \subseteq G$ is a subgroup, then the quotient set G/G' of left cosets is in general not a group but just a set on which G acts from the left. We now define homogeneous spaces such as G/G' for Hopf algebras or bialgebras. Thus we have to define general quotient objects and dually general subobjects of a bialgebra.

Definition 2.5.7. Let A be a bialgebra and $B \subseteq A$ a subspace. Then B is a **right (left) coideal subalgebra** of A if B is a subalgebra and a right (left) coideal of A.

There is a correspondence between right or left coideal subalgebras and quotient coalgebras and left or right modules of a bialgebra. These are the quotients and subobjects of a Hopf algebra which generalize homogeneous spaces for groups.

Proposition 2.5.8. *Let* A *be a bialgebra.*

(1) *Let* B *be a right or left coideal subalgebra of* A. *Let* $B^+ = \ker(\varepsilon|B)$. *Then* A/AB^+ *is a quotient coalgebra and a quotient left* A*-module of* A, *and* A/B^+A *is a quotient coalgebra and a quotient right* A*-module of* A.

(2) *Let* I *be a coideal and a left or right ideal of* A. *Then*

$$A^{\mathrm{co}\,A/I} = \{a \in A \mid a_{(1)} \otimes \overline{a_{(2)}} = a \otimes \overline{1}\}$$

is a left coideal subalgebra of A, *and*

$${}^{\mathrm{co}\,A/I}A = \{a \in A \mid \overline{a_{(1)}} \otimes a_{(2)} = \overline{1} \otimes a\}$$

is a right coideal subalgebra of A.

Proof. (1) By Lemma 1.1.14, B^+ is a coideal of A, hence AB^+ is a coideal and a left A-submodule of A. Then A/AB^+ is a quotient coalgebra and a quotient left A-module of A. Similarly, A/B^+A is a quotient coalgebra and a quotient right A-module of A.

(2) Let I be a coideal and a left ideal of A. By Lemma 2.5.6(1), $A^{\mathrm{co}\,A/I}$ is a left coideal of A. It is also a subalgebra of A. Indeed,

$$(aa')_{(1)} \otimes \overline{(aa')_{(2)}} = a_{(1)}a'_{(1)} \otimes \overline{a_{(2)}a'_{(2)}} = a_{(1)}a'_{(1)} \otimes a_{(2)}\overline{a'_{(2)}}$$

for all $a, a' \in A$, since A/I is a left A-module. If $a, a' \in A^{\mathrm{co}\,A/I}$, then

$$(aa')_{(1)} \otimes \overline{(aa')_{(2)}} = a_{(1)}a' \otimes a_{(2)}\overline{1} = a_{(1)}a' \otimes \overline{a_{(2)}} = aa' \otimes \overline{1}.$$

Similarly it is shown that $A^{\mathrm{co}\,A/I}$ is a left coideal subalgebra of A if I is a coideal and a right ideal, and that ${}^{\mathrm{co}\,A/I}A$ is a right coideal subalgebra of A if I is a coideal and a left or right ideal of A. $\square$

Example 2.5.9. Let G be a group, $G' \subseteq G$ a subgroup and G/G' the set of left residue classes $\overline{g} = gG'$, $g \in G$. Then the vector space $\Bbbk G/G'$ with basis $\overline{g}$, $g \in G$, is a left $\Bbbk G$-module and a coalgebra by

$$x\overline{g} = \overline{xg},\ \Delta_{\Bbbk G/G'}(\overline{g}) = \overline{g} \otimes \overline{g}$$

for all $x, g \in G$.

Since $(\Bbbk G')^+ = \ker(\varepsilon : \Bbbk G' \to \Bbbk)$ is the subspace of $\Bbbk G'$ spanned by the elements $g' - 1$, $g' \in G'$, we see that $\Bbbk G(\Bbbk G')^+ = (\Bbbk G')^+$. Hence

$$\Bbbk G/\Bbbk G(\Bbbk G')^+ \xrightarrow{\cong} \Bbbk G/G',\ \overline{g} \mapsto \overline{g} \text{ for all } g \in G,$$

is an isomorphism of left $\Bbbk G$-modules and of coalgebras.

Thus the group algebra $\Bbbk G'$ is not only the vector space kernel of the quotient map $\Bbbk G \to \Bbbk G/G'$, but if $A = \Bbbk G$ and $B = \Bbbk G'$, then

$$B = A^{\mathrm{co}\,A/AB^+} = {}^{\mathrm{co}\,A/AB^+}A.$$

We will see in Theorem 6.3.2 that pointed Hopf algebras have a rich quotient theory. There is a one-to-one correspondence between all quotient objects of H and a large class of subobjects.

In the following example we will use the notion of the coequalizer of two morphisms.

Definition 2.5.10. Let $\mathcal{C}$ be any category and $f, g : X \to Y$ be morphisms. An **equalizer of** f and g is a morphism $e : E \to X$ such that $fe = ge$, and for each morphism $e' : E' \to X$ with $fe' = ge'$ there is a unique morphism $h : E' \to E$ with $eh = e'$. The diagram

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

is called the equalizer diagram. Dually, a **coequalizer of** f and g is a morphism $c : Y \to C$ such that $cf = cg$, and for each morphism $c' : Y \to C'$ satisfying $c'f = c'g$ there is a unique morphism $h : C \to C'$ with $hc = c'$. The corresponding diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "c"] & C \end{tikzcd}$$

is called the coequalizer diagram.

If they exist, both equalizers and coequalizers are known to be unique up to unique isomorphisms. Moreover, the morphism e in the equalizer diagram is a monomorphism, that is, if $d_1, d_2 : D \to E$ with $ed_1 = ed_2$, then $d_1 = d_2$. Similarly, the morphism c in the coequalizer diagram is an epimorphism.

If $\mathcal{C}$ is the category of abelian groups, the kernel of $f - g$ with the inclusion map e is an equalizer of f and g, and the cokernel of $f - g$ with its quotient map c is a coequalizer of f and g.

EXAMPLE 2.5.11. Let A be a bialgebra and $B \subseteq A$ a left coideal subalgebra. Let $p_1 \in \mathrm{Hom}(A \otimes B, A)$, $a \otimes b \mapsto a\varepsilon(b)$, and let $\mu : A \otimes B \to A$ denote the multiplication map. Then the canonical map $\pi : A \to A/AB^+$ is the coequalizer of p_1 and μ. If A is finite-dimensional, then by duality, B^* is a coalgebra and left A^*-module quotient of A^*, and $\pi^* : (A/AB^+)^* \to A^*$ is the equalizer of the maps $p_1^*, \mu^* : A^* \to (A \otimes B)^* \cong A^* \otimes B^*$. Thus $(A/AB^+)^*$ is the left coideal subalgebra of right B^*-coinvariant elements of A^*. In particular, if G is a finite group and $G' \subseteq G$ is a subgroup, then $\Bbbk^{G/G'} \cong (\Bbbk G/G')^*$ is naturally embedded into $\Bbbk^G \cong (\Bbbk G)^*$ as the left coideal subalgebra of right $\Bbbk^{G'}$-coinvariant elements of $\Bbbk^G$.

EXAMPLE 2.5.12. Let $n \geq 2$ be an integer and $q \in \Bbbk$ a primitive n-th root of unity. The following subalgebras of the Taft Hopf algebra $T_{q,n}$ are left coideal subalgebras.

(1) $R = \Bbbk[x]$,
(2) $\Bbbk[g^m]$, $1 \leq m \leq n$, $m|n$,
(3) $\Bbbk[g^m, x]$, $1 \leq m < n$, $m|n$,
(4) $R_\alpha = \Bbbk[x + \alpha g]$, $0 \neq \alpha \in \Bbbk$.

The only proper Hopf subalgebras in this list are in (2). Moreover, $R_\alpha \neq R_\beta$ in (4) for all $0 \neq \alpha, \beta \in \Bbbk$, $\alpha \neq \beta$. One can show that this list contains all left coideal subalgebras of $T_{q,n}$.

2.6. Actions and coactions

Abstract groups are studied via their actions on sets, that is, as transformation groups. Hopf algebras form the natural framework to describe actions on algebras.

DEFINITION 2.6.1. Let H be a bialgebra, and A an algebra. Assume that A is a left H-module with module structure $\lambda : H \otimes A \to A$, $h \otimes a \mapsto h \cdot a$. Then (A, λ) is called a **left H-module algebra** if for all $h \in H$ and $a, b \in A$,

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \tag{2.6.1}$$

$$h \cdot 1 = \varepsilon(h)1. \tag{2.6.2}$$

If no confusion is possible, we suppress λ in the notation.

Equations (2.6.1) and (2.6.2) should be read as a very general Leibniz rule. Indeed, according to them, primitive elements act by derivations of A.

REMARK 2.6.2. Let H be a bialgebra and (A, λ) a left H-module algebra.

(1) Let $g \in G(H)$. Then $A \to A$, $a \mapsto g \cdot a$, is an algebra homomorphism. If g is invertible, then the same map is an algebra automorphism of A.
(2) Let $g, h \in G(H)$ and define $\sigma, \tau \in \mathrm{Alg}(A, A)$ by $\sigma(a) = g \cdot a$, $\tau(a) = h \cdot a$ for all $a \in A$. If $x \in P_{g,h}(H)$ then $A \to A$, $a \mapsto x \cdot a$, is a (σ, τ)-derivation.

This follows from the explicit formulas of the comultiplication.

Let $H = \Bbbk G$ be the group algebra of a group G. Then by (1), there is a bijection between all left $\Bbbk G$-module algebra structures $\Bbbk G \otimes A \to A$ and all group homomorphisms $G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of algebra automorphisms of A.

Let $H = U(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then by (2) (with $g = h = 1$), there is a bijection between all left $U(\mathfrak{g})$-module algebra structures on A and all Lie algebra homomorphisms $\mathfrak{g} \to \mathrm{Der}(A)$, where $\mathrm{Der}(A)$ is the Lie algebra of derivations of A with commutator of derivations as Lie bracket.

EXAMPLE 2.6.3. Let H be a Hopf algebra. Then H acts on itself via the **left adjoint action**

$$H \otimes H \to H, \quad h \otimes x \mapsto \mathrm{ad}\, h(x) = h_{(1)} x \mathcal{S}(h_{(2)}).$$

With this action, H becomes a left H-module algebra, since

$$\mathrm{ad}\, h(xy) = h_{(1)} xy \mathcal{S}(h_{(2)}) = h_{(1)} x \mathcal{S}(h_{(2)}) h_{(3)} y \mathcal{S}(h_{(4)}) = \mathrm{ad}\, h_{(1)}(x)\, \mathrm{ad}\, h_{(2)}(y)$$

for all $h \in H$ and $x, y \in A$.

EXAMPLE 2.6.4. Let A be an algebra, H a Hopf algebra, and $\gamma : H \to A$ an algebra morphism. Define

$$\mathrm{ad}_\gamma : H \otimes A \to A, \quad h \otimes a \mapsto \gamma(h_{(1)}) a \gamma(\mathcal{S}(h_{(2)})).$$

Then A is a left H-module algebra with action ad_γ.

PROPOSITION 2.6.5. *Let H be a bialgebra and A an algebra which has a left H-module structure $H \otimes A \to A$, $h \otimes a \mapsto h \cdot a$. Assume that the algebra H is generated by a subset $M \subseteq H$ such that for all $h \in M$ and $a, b \in A$*

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \quad h \cdot 1 = \varepsilon(h) 1.$$

Then A is a left H-module algebra.

PROOF. As in the proof of Proposition 1.2.23, let

$$H' = \{h \in H \mid h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b) \text{ for all } a, b \in A,\ h \cdot 1 = \varepsilon(h) 1\}.$$

Then $M \subseteq H'$. We show that H' is a subalgebra of H. Clearly, $1 \in H'$ and H' is a subspace of H. If $g, h \in H'$, then $gh \in H'$ since

$$\begin{aligned}(gh) \cdot (ab) &= g \cdot (h \cdot (ab)) \\ &= g \cdot ((h_{(1)} \cdot a)(h_{(2)} \cdot b)) \\ &= ((g_{(1)} h_{(1)}) \cdot a)((g_{(2)} h_{(2)}) \cdot b) \\ &= ((gh)_{(1)} \cdot a)((gh)_{(2)} \cdot b)\end{aligned}$$

for all $a, b \in A$, and $(gh) \cdot 1 = g \cdot (h \cdot 1) = \varepsilon(g)\varepsilon(h) 1 = \varepsilon(gh) 1$. $\square$

LEMMA 2.6.6. *Let H be a bialgebra, A a left H-module algebra, and $V \subseteq A$ a subspace of A. Then the subalgebra of A generated by $H \cdot V$ is an H-module subalgebra of A.*

PROOF. Let $h_1, \dots, h_n \in H, v_1, \dots, v_n \in V, n \geq 1$, and $h \in H$. Then

$$h \cdot ((h_1 \cdot v_1) \cdots (h_n \cdot v_n)) = ((h_{(1)} h_1) \cdot v_1) \cdots ((h_{(n)} h_n) \cdot v_n).$$

Thus the subalgebra of A generated by $H \cdot V$ is an H-submodule of A. $\square$

LEMMA 2.6.7. *Let $H = \Bbbk\langle g, x\rangle$ be the free algebra as a bialgebra in Example 2.4.8 with $g \in G(H)$ and $x \in P_{g,1}(H)$. Let A be an algebra, $\sigma : A \to A$ an algebra endomorphism, and $\delta : A \to A$ a (σ, id_A)-derivation. Then A is a left H-module algebra with $g \cdot a = \sigma(a)$, $x \cdot a = \delta(a)$ for all $a \in A$.*

PROOF. Since H is the free algebra in g, x, a left A-module structure on A, that is, an algebra homomorphism $H \to \mathrm{Hom}(A, A)$, is given by any action of g and x. By Proposition 2.6.5, A is an H-module algebra since the axioms (2.6.1) and (2.6.2) are satisfied for $g, x \in H$ by Remark 2.6.2. $\square$

DEFINITION 2.6.8. Let H be a bialgebra and A a left H-module algebra. The **smash product** algebra $A\#H$ is $A \otimes H$ with the algebra structure

$$(a\#x)(b\#y) = a(x_{(1)} \cdot b)\#x_{(2)}y, \quad \eta_{A\#H}(1) = 1\#1 \tag{2.6.3}$$

for $a, b \in A$, $x, y \in H$, where we write $a\#h = a\otimes h$ to indicate the algebra structure.

PROPOSITION 2.6.9. *Let H be a bialgebra and (A, λ) a left H-module algebra. Then $A\#H$ is an algebra. The embeddings*

$$A \to A\#H,\ a \mapsto a\#1, \quad H \to A\#H,\ h \mapsto 1\#h,$$

are injective algebra homomorphisms, and the multiplication map $A \otimes H \to A\#H$, $a \otimes h \mapsto (a\#1)(1\#h)$, is bijective.

PROOF. The multiplication map $A\#H \otimes A\#H \to A\#H$ is well-defined since it can be written as a composition of linear maps

$$\begin{aligned}
A \otimes H \otimes A \otimes H &\xrightarrow{\mathrm{id}_A\otimes\Delta\otimes\mathrm{id}_A\otimes\mathrm{id}_H} A \otimes H \otimes H \otimes A \otimes H\\
&\xrightarrow{\mathrm{id}_A\otimes\mathrm{id}_H\otimes\tau_{H,A}\otimes\mathrm{id}_H} A \otimes H \otimes A \otimes H \otimes H\\
&\xrightarrow{\mathrm{id}_A\otimes\lambda\otimes\mathrm{id}_H\otimes\mathrm{id}_H} A \otimes A \otimes H \otimes H \xrightarrow{\mu_A\otimes\mu_H} A \otimes H.
\end{aligned}$$

To check associativity, let $a, b, c \in A$ and $x, y, z \in H$. Then

$$\begin{aligned}
(a\#x)((b\#y)(c\#z)) &= (a\#x)(b(y_{(1)} \cdot c)\#y_{(2)}z)\\
&= a(x_{(1)} \cdot (b(y_{(1)} \cdot c)))\#x_{(2)}y_{(2)}z\\
&= a(x_{(1)} \cdot b)(x_{(2)}y_{(1)} \cdot c)\#x_{(3)}y_{(2)}z,\\
((a\#x)(b\#y))(c\#z) &= (a(x_{(1)} \cdot b)\#x_{(2)}y)(c\#z)\\
&= a(x_{(1)} \cdot b)(x_{(2)}y_{(1)} \cdot c)\#x_{(3)}y_{(2)}z.
\end{aligned}$$

The remaining claims are obvious. $\square$

REMARK 2.6.10. There is a natural left action of the smash product algebra $A\#H$ in Proposition 2.6.9 on A defined by

$$A\#H \otimes A \to A, \quad a\#h \otimes x \mapsto a(h \cdot x).$$

It corresponds to the natural left action of $A\#H$ on $(A\#H)/(A\otimes H^+)$. Thus there is a natural algebra homomorphism

$$A\#H \to \mathrm{End}(A),$$

of $A\#H$ into the algebra of linear endomorphisms of A.

We will follow the **convention** to write ah instead of $a\#h$ in $A\#H$ for all $a \in A$, $h \in H$. Thus we identify A and H with subalgebras of $A\#H$. The multiplication in $A\#H$ is then determined by the rule

$$ha = (h_{(1)} \cdot a)h_{(2)} \tag{2.6.4}$$

for all $a \in A$, $h \in H$.

Smash products generalize several familiar constructions in algebra.

EXAMPLE 2.6.11. Let G be a group, A an algebra, and $G \to \mathrm{Aut}(A)$ a group homomorphism. Thus A is a left $\Bbbk G$-module algebra. The smash product algebra $A * G = A\#\Bbbk G$ is called the **skew group algebra**.

EXAMPLE 2.6.12. Let $m, n \geq 2$ be natural numbers, and $0 \neq q \in \Bbbk$ with $q^n = 1$. Let $G = \langle g \rangle$ be a cyclic group of order n with generator g, and $\Bbbk[x]$ the polynomial algebra in the indeterminate x. Then the quotient algebra $\Bbbk[x]/(x^m)$ is a left $\Bbbk G$-module algebra with G-action given by the group homomorphism

$$G \to \mathrm{Aut}(\Bbbk[x]/(x^m)), \quad g \mapsto (\overline{x} \mapsto q\overline{x}).$$

The algebra map

$$\Bbbk\langle g, x \mid g^n = 1, x^m = 0, gx = qxg\rangle \to \Bbbk[x]/(x^m)\#\Bbbk[g],\ g \mapsto 1\#g,\ x \mapsto \overline{x}\#1,$$

is bijective, since the elements $x^i g^j$, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ span the vector space on the left-hand side, and their images are a vector space basis in $\Bbbk[x]/(x^m)\#\Bbbk G$.

In particular, we have found a vector space basis of n^2 elements of the Taft Hopf algebra $T_{q,n}$ in Example 2.4.10. As an application, we can now prove that the order of the linear automorphism $\mathcal{S}^2$ of the Taft Hopf algebra is n. Indeed, $\mathcal{S}(x) = -g^{-1}x$ and $\mathcal{S}^2(x) = g^{-1}xg = q^{-1}x$.

EXAMPLE 2.6.13. The argument in Example 2.6.12 easily extends to the general case of the Hopf algebras A_χ in Example 2.4.12. The free algebra $\Bbbk\langle X\rangle$ is a left $\Bbbk G$-module algebra by the group homomorphism

$$G \to \mathrm{Aut}(\Bbbk\langle X\rangle), \quad g \mapsto (x \mapsto \chi_x(g)x \text{ for all } x \in X),$$

and the algebra map

$$A_\chi \to \Bbbk\langle X\rangle\#\Bbbk G, \quad g \mapsto 1\#g,\ x \mapsto x\#1 \text{ for all } g \in G,\ x \in X,$$

is bijective.

Smash products allow us to define Ore extensions and to prove their associativity in a natural way.

REMARK 2.6.14. Let A be an algebra, $\sigma : A \to A$ an algebra endomorphism, and $\delta : A \to A$ a (σ, id_A)-derivation. Let $H = \Bbbk\langle g, x\rangle$ be the free algebra, and A the left H-module algebra defined in Lemma 2.6.7. Then the subalgebra of H generated by x is the polynomial algebra $\Bbbk[x]$. Since x is $(g, 1)$-primitive,

$$A\#\Bbbk[x] \subseteq A\#\Bbbk\langle g, x\rangle$$

is a subalgebra. We define the **Ore-extension** $A[\theta; \sigma, \delta]$ of A as the subalgebra $A\#\Bbbk[x]$ of the smash product, where we write θ instead of x. By Proposition 2.6.9,

$A\#\Bbbk[x]$ is a free left A-module with basis x^i, $i \geq 0$. With other words, the elements of $A[\theta;\sigma,\delta]$ can be written in a unique way as left polynomials

$$\sum_{i=0}^{n} a_i\theta^i,\ a_i \in A,\ 0 \leq i \leq n.$$

Multiplication is determined by the rule

$$\theta a = \sigma(a)\theta + \delta(a) \tag{2.6.5}$$

for all $a \in A$.

In case σ is the identity of A, we write $A[\theta;\delta] = A[\theta;\mathrm{id},\delta]$. This algebra is a **formal differential operator algebra**.

In case $\delta = 0$, we write $A[\theta;\sigma] = A[\theta;\sigma,0]$.

Ore extensions, in particular iterations of them, are often suitable to construct vector space bases of algebras.

Definition 2.6.15. Let A be an algebra, and B a subalgebra of A. Let $n \geq 1$, and $x_1,\dots,x_n \in A$. Let I be a subset of $\{1,\dots,n\}$ and let $N : I \to \mathbb{N}$ be a map with $N(i) \geq 2$ for each $i \in I$. If A is a free left B-module with basis

$$x_1^{a_1}\cdots x_n^{a_n},\quad a_1,\dots,a_n \geq 0,\ a_i < N(i) \text{ for all } i \in I,$$

then this basis is called a **restricted PBW basis of A over B**. If I is the empty set, then the basis is said to be a PBW basis of A over B. If $B = \Bbbk 1$ then one talks about a (restricted) **PBW basis of A**.

Example 2.6.16. Let $\Bbbk[t]$ be the polynomial algebra in the indeterminate t, and let $\delta : \Bbbk[t] \to \Bbbk[t]$ be the derivation $\delta(f) = \frac{df}{dt}$ for all $f \in \Bbbk[t]$. Then $A_1 = \Bbbk[t][\theta;\delta]$ is the **Weyl algebra**. Note that

$$\Bbbk\langle x, y \mid xy - yx = 1\rangle \to A_1,\quad x \mapsto \theta,\ y \mapsto t,$$

is an algebra isomorphism, since the elements x^iy^j, $i,j \geq 0$, span the vector space $\Bbbk\langle x,y \mid xy-yx=1\rangle$, and their images form a PBW basis of A_1. Under the action defined in Remark 2.6.10, x acts on $\Bbbk[t]$ as the derivative $\frac{d}{dt}$, and y as multiplication with t.

Example 2.6.17. We describe the quantum group $U_q(\mathfrak{sl}_2)$ of Example 2.4.11 as an iterated Ore extension. First let

$$A = \Bbbk\langle F, K, K^{-1} \mid KK^{-1} = 1 = K^{-1}K, KFK^{-1} = q^{-2}F\rangle.$$

The algebra $\Bbbk[K,K^{-1}] = \Bbbk\langle K, K^{-1} \mid KK^{-1} = 1 = K^{-1}K\rangle$ is the group algebra of the infinite cyclic group generated by K. Let

$$\sigma_1 : \Bbbk[K,K^{-1}] \to \Bbbk[K,K^{-1}]$$

be the algebra automorphism given by $\sigma_1(K) = q^2K$. Then the algebra homomorphism

$$A \to \Bbbk[K,K^{-1}][\theta;\sigma_1],\quad F \mapsto \theta,\ K \mapsto K,\ K^{-1} \mapsto K^{-1},$$

is bijective, since the elements K^iF^j, $i,j \in \mathbb{Z}$, $j \geq 0$, span A as a vector space, and their images in the Ore extension are a basis.

The map

$$\sigma : A \to A,\quad F \mapsto F,\ K^{\pm 1} \mapsto q^{\mp 2}K^{\pm 1},$$

is a well-defined algebra automorphism. By Lemma 2.3.11, it is easy to check that there is a (σ, id)-derivation

$$\delta : A \to A, \quad \delta(K) = 0,\ \delta(F) = \frac{K - K^{-1}}{q - q^{-1}}.$$

Then the algebra homomorphism

$$U_q(\mathfrak{sl}_2) \to A[\theta; \sigma, \delta], \quad K^{\pm 1} \mapsto K^{\pm 1},\ F \mapsto F,\ E \mapsto \theta,$$

is bijective. Again this follows since the elements

$$K^i F^j E^k,\ i, j, k \in \mathbb{Z},\ j, k \geq 0,$$

span $U_q(\mathfrak{sl}_2)$ and their images form a vector space basis of the Ore extension.

In particular, we have found a PBW basis of $U_q(\mathfrak{sl}_2)$ over $\Bbbk[K, K^{-1}]$.

For the complete picture, in addition to actions on algebras we have to consider coactions of bialgebras on algebras.

Definition 2.6.18. Let H be a bialgebra and A an algebra which is a right H-comodule with structure map $\delta : A \to A \otimes H$, $a \mapsto a_{(0)} \otimes a_{(1)}$. Then (A, δ) (or simply A) is called a **right H-comodule algebra** if the structure map δ is an algebra homomorphism, where $A \otimes H$ is the usual tensor product of algebras. In terms of elements this means that for all $a, b \in A$,

$$\delta(ab) = a_{(0)} b_{(0)} \otimes a_{(1)} b_{(1)}, \tag{2.6.6}$$

$$\delta(1) = 1 \otimes 1. \tag{2.6.7}$$

Left H-comodule algebras are defined similarly.

Remark 2.6.19. For any bialgebra H and right H-comodule algebra A,

$$A^{\operatorname{co} H} = \{a \in A \mid a_{(0)} \otimes a_{(1)} = a \otimes 1\}$$

is the set of right H-coinvariant elements. It is a subalgebra of A. If A is a left H-comodule algebra,

$${}^{\operatorname{co} H}A = \{a \in A \mid a_{(-1)} \otimes a_{(0)} = 1 \otimes a\}$$

is the subalgebra of left H-coinvariant elements of A.

Example 2.6.20. Let A, H be bialgebras, and $\pi : A \to H$ a bialgebra homomorphism. Then A is a right H-comodule algebra with structure map

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id}_A \otimes \pi} A \otimes H.$$

Example 2.6.21. Let $H = \Bbbk[x_{ij}]_{1 \leq i,j \leq n}$ be the bialgebra in Example 2.4.7. The commutative polynomial algebra $\Bbbk[x_1, \dots, x_n]$ is a right H-comodule algebra with structure map

$$\Bbbk[x_1, \dots, x_n] \xrightarrow{\delta} \Bbbk[x_1, \dots, x_n] \otimes H, \quad x_j \mapsto \sum_{i=1}^{n} x_i \otimes x_{ij},\ 1 \leq j \leq n.$$

The map δ represents multiplication of $n \times n$-matrices on the n-dimensional affine space, since

$$(\delta(x_1), \dots, \delta(x_n)) = (x_1, \dots, x_n) \otimes (x_{ij})_{1 \leq i,j \leq n}.$$

In general, actions of affine group schemes on affine schemes are given by commutative comodule algebras of commutative Hopf algebras.

Smash product algebras $A\#H$ have an essential additional structure. They are right H-comodule algebras.

LEMMA 2.6.22. *Let H be a bialgebra and A a left H-module algebra. Then $A\#H$ is a right H-comodule algebra with comodule structure map*

$$\delta = \mathrm{id}_A \otimes \Delta : A\#H \to A\#H \otimes H,$$

and $(A\#H)^{\mathrm{co}\,H} = A \otimes \Bbbk 1 \cong A$.

PROOF. To see that δ is an algebra map, let $a, b \in A$ and $x, y \in H$. Then

$$\begin{aligned}\delta((a\#x)(b\#y)) &= \delta(a(x_{(1)} \cdot b)\#x_{(2)}y)\\ &= a(x_{(1)} \cdot b)\#x_{(2)}y_{(1)} \otimes x_{(3)}y_{(2)},\\ \delta(a\#x)\delta(b\#y) &= (a\#x_{(1)} \otimes x_{(2)})(b\#y_{(1)} \otimes y_{(2)})\\ &= a(x_{(1)} \cdot b)\#x_{(2)}y_{(1)} \otimes x_{(3)}y_{(2)}.\end{aligned}$$

The equality $(A\#H)^{\mathrm{co}\,H} = A \otimes \Bbbk 1 \cong A$ follows from Lemma 2.5.2. $\square$

It is easy to see that H-module algebras and H-comodule algebras can be defined alternatively as algebras A whose structure maps $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ are H-linear and H-colinear, respectively.

In the next theorem we formulate a necessary and sufficient condition for a comodule algebra to be a smash product.

THEOREM 2.6.23. *Let H be a Hopf algebra and (A, δ) a right H-comodule algebra with $\delta : A \to A \otimes H$, $a \mapsto a_{(0)} \otimes a_{(1)}$.*

(1) *Assume that there is an algebra map $\gamma : H \to A$ which is right H-colinear, where H is a right H-comodule via Δ. Then*

$$R = A^{\mathrm{co}\,H} = \{a \in A \mid a_{(0)} \otimes a_{(1)} = a \otimes 1\}$$

is a left H-module algebra with H-action

$$\mathrm{ad}_R : H \otimes R \to R, \quad h \otimes r \mapsto \gamma(h_{(1)})r\gamma(\mathcal{S}(h_{(2)})).$$

The map

$$\vartheta : A \to R, \quad a \mapsto a_{(0)}\gamma(\mathcal{S}(a_{(1)})),$$

is a well-defined left R-linear map with $\vartheta|R = \mathrm{id}_R$. The maps

$$\Phi : R\#H \to A,\ r\#h \mapsto r\gamma(h), \quad \Psi : A \to R\#H,\ a \mapsto \vartheta(a_{(0)})\#a_{(1)},$$

are mutually inverse right H-colinear algebra isomorphisms.

(2) *Conversely, assume that there is a left H-module algebra R and a right H-colinear algebra isomorphism $\Phi : R\#H \to A$. Then*

$$\gamma : H \to A,\ h \mapsto \Phi(1\#h),$$

is a right H-colinear algebra map.

PROOF. (1) We first show that R is a left H-module algebra. By Example 2.6.4, A is a left H-module algebra under the action ad_γ. For all $h \in H$,

$$\delta(\gamma(h)) = \gamma(h_{(1)}) \otimes h_{(2)},$$

since γ is right H-colinear. Hence for all $h \in H$, $r \in R$,

$$\begin{aligned}\delta(\gamma(h_{(1)})r\gamma(\mathcal{S}(h_{(2)}))) &= \delta(\gamma(h_{(1)}))\delta(r)\delta(\mathcal{S}(h_{(2)}))\\ &= \gamma(h_{(1)})r\gamma(\mathcal{S}(h_{(4)})) \otimes h_{(2)}1\mathcal{S}(h_{(3)})\\ &= \gamma(h_{(1)})r\gamma(\mathcal{S}(h_{(2)})) \otimes 1.\end{aligned}$$

Thus the map $\mathrm{ad}_\gamma : H \otimes A \to A$ restricts to $\mathrm{ad}_R : H \otimes R \to R$.

The vector space A is a Hopf module in $\mathcal{M}_H^H$ with H-comodule structure δ and H-module structure $A \otimes H \to A$, $a \otimes h \mapsto a\gamma(h)$. By Theorem 2.5.4, $\vartheta : A \to R$ is a well-defined map, and Φ, Ψ are inverse isomorphisms.

The map Φ is clearly right H-colinear, and it is an algebra map, since for all $g, h \in H$ and $r, s \in R$,

$$\begin{aligned}\Phi(r\#g)\Phi(s\#h) &= r\gamma(g)s\gamma(h)\\ &= r\gamma(g_{(1)})s\gamma(\mathcal{S}(g_{(2)}))\gamma(g_{(3)})\gamma(h)\\ &= \Phi(r(g_{(1)} \cdot s)\#g_{(2)}h)\\ &= \Phi((r\#g)(s\#h)).\end{aligned}$$

(2) is obvious. □

Remark 2.6.24. In the situation of Theorem 2.6.23, we note the following rules for ϑ which are easily checked. For all $a \in A$, $h \in H$,

(1) $\vartheta(a\gamma(h)) = \vartheta(a)\varepsilon(h)$,
(2) $\vartheta(\gamma(h)a) = h \cdot \vartheta(a)$.

Here is a useful tool to compute $R = A^{\mathrm{co}\,H}$.

Lemma 2.6.25. *Under the assumptions of Theorem* 2.6.23*, let $W \subseteq R$ be a vector subspace such that A as an algebra is generated by W and $\gamma(H)$. Then the algebra R is generated by* $(\mathrm{ad}_R\gamma(H))(W)$.

Proof. Let R' be the subalgebra of R generated by $(\mathrm{ad}_R\gamma(H))(W)$. By Lemma 2.6.6, R' is an H-module subalgebra under the adjoint action. Hence $R'\#H$ is a subalgebra of $R\#H$. The restriction of the isomorphism Φ in Theorem 2.6.23(1) to $R'\#H$ is surjective, since W and $\gamma(H)$ generate A. Thus $R' = R$. □

2.7. Cleft objects and two-cocycles

We have seen in Theorem 2.6.23 that smash products have an elegant description as right H-comodule algebras which admit a right H-colinear algebra map $\gamma : H \to A$. In this section we study a more general situation.

Definition 2.7.1. Let H be a Hopf algebra and (A, δ) with $\delta : A \to A \otimes H$ a right H-comodule algebra. Then A is **H-cleft** if there is a right H-colinear map $\gamma : H \to A$ which is invertible with respect to convolution. Then γ is called a **section** if $\gamma(1) = 1$. An **H-cleft object** is an H-cleft right H-comodule algebra with $A^{\mathrm{co}\,H} = \Bbbk 1$.

We note that in the definition, γ can always be assumed to be a section by replacing γ by $\gamma\gamma(1)^{-1}$. Let R be a left H-module algebra, and $A = R\#H$ the smash product. The map $H \to A$, $h \mapsto 1 \otimes h$, is a right H-colinear algebra map. Hence A is H-cleft, since any algebra map $\gamma : H \to A$ is invertible with inverse $\gamma\mathcal{S}$.

The explicit description of H-cleft H-comodule algebras as an algebra structure on $R \otimes H$ is much more complicated than for smash products. It involves some kind of general two-cocycle. In this section we will only consider H-cleft objects. They are completely described by two-cocycles defined as follows.

DEFINITION 2.7.2. Let H be a bialgebra over a field $\Bbbk$. A map $\sigma : H \otimes H \to \Bbbk$ is called a **two-cocycle for** H, if it is convolution invertible and satisfies

$$\sigma(x_{(1)} \otimes y_{(1)})\sigma(x_{(2)}y_{(2)} \otimes z) = \sigma(y_{(1)} \otimes z_{(1)})\sigma(x \otimes y_{(2)}z_{(2)}) \tag{2.7.1}$$

for all $x, y, z \in H$. We say that σ is **normalized** if $\sigma(1 \otimes 1) = 1$.

REMARK 2.7.3. (1) By Definition 1.2.9, a linear map $\sigma : H \otimes H \to \Bbbk$ is convolution invertible if and only if there is a linear map $\sigma^{-1} : H \otimes H \to \Bbbk$ such that

$$\sigma(x_{(1)} \otimes y_{(1)})\sigma^{-1}(x_{(2)} \otimes y_{(2)}) = \sigma^{-1}(x_{(1)} \otimes y_{(1)})\sigma(x_{(2)} \otimes y_{(2)}) = \varepsilon(x)\varepsilon(y)$$

for all $x, y \in H$.

(2) For any two-cocycle σ for a Hopf algebra H and for any $\lambda \in \Bbbk$ with $\lambda \neq 0$, the map $\lambda\sigma$ is a two-cocycle for H with convolution inverse $\lambda^{-1}\sigma^{-1}$. The invertibility of σ implies that $\sigma(1 \otimes 1) \neq 0$. Therefore, any two-cocycle for H is a multiple of a normalized two-cocycle.

(3) Let H be a bialgebra and let σ be a two-cocycle for H. Then the map $\sigma^{\mathrm{op}} : H \otimes H \to \Bbbk$, $x \otimes y \mapsto \sigma(y \otimes x)$, is a two-cocycle for H^{op}. The convolution inverse of σ^{op} is $(\sigma^{-1})^{\mathrm{op}}$.

(4) The inverse of a two-cocycle σ for a bialgebra H is a two-cocycle for H^{cop}. Indeed, (2.7.1) is equivalent to

$$(\sigma \otimes \varepsilon) * \sigma(\mu \otimes \mathrm{id}) = (\varepsilon \otimes \sigma) * \sigma(\mathrm{id} \otimes \mu)$$

in $\mathrm{Hom}(H \otimes H \otimes H, \Bbbk)$. Convolution multiplication of the latter from the left with $\varepsilon \otimes \sigma^{-1}$ and from the right with $\sigma^{-1}(\mu \otimes \mathrm{id})$ results in

$$\sigma^{-1}(y_{(1)} \otimes z)\sigma(x \otimes y_{(2)}) = \sigma(x_{(1)} \otimes y_{(1)}z_{(1)})\sigma^{-1}(x_{(2)}y_{(2)} \otimes z_{(2)}) \tag{2.7.2}$$

for all $x, y, z \in H$. Then additional convolution multiplication from the left with $\sigma^{-1}(\mathrm{id} \otimes \mu)$ and from the right with $\sigma^{-1} \otimes \varepsilon$ yields

$$\sigma^{-1}(x \otimes y_{(1)}z_{(1)})\sigma^{-1}(y_{(2)} \otimes z_{(2)}) = \sigma^{-1}(x_{(1)}y_{(1)} \otimes z)\sigma^{-1}(x_{(2)} \otimes y_{(2)})$$

for all $x, y, z \in H$.

(5) Let σ be a two-cocycle for a bialgebra H. Then

$$\sigma(x \otimes 1) = \sigma(1 \otimes x) = \varepsilon(x)\sigma(1 \otimes 1), \tag{2.7.3}$$

$$\sigma^{-1}(x \otimes 1) = \sigma^{-1}(1 \otimes x) = \varepsilon(x)\sigma^{-1}(1 \otimes 1) \tag{2.7.4}$$

for any $x \in H$. Indeed, $\sigma(x \otimes 1) = \varepsilon(x)\sigma(1 \otimes 1)$ by (2.7.2) with $y = z = 1$ and by the definition of σ^{-1}. Then $\sigma(1 \otimes x) = \varepsilon(x)\sigma(1 \otimes 1)$ for any $x \in H$ using the latter equation for the bialgebra H^{op} with the two-cocycle σ^{op}. The equations in (2.7.4) follow from (2.7.3) applied to H^{cop} and σ^{-1}.

REMARK 2.7.4. Let G be a group with neutral element e and $\Bbbk G$ the group algebra. A function $\sigma : G \times G \to \Bbbk^{\times}$ is a normalized two-cocycle of the group G (with respect to the trivial action), if for all $x, y, z \in G$,

$$\sigma(x, y)\sigma(xy, z) = \sigma(y, z)\sigma(x, yz),$$
$$\sigma(z, e) = \sigma(e, z) = 1.$$

A linear map $\sigma : \Bbbk G \otimes \Bbbk G \to \Bbbk$ is a normalized two-cocycle for the Hopf algebra $\Bbbk G$ if and only if the restriction of σ defines a normalized two-cocycle of the group G. Note that for any two-cocycle σ for $\Bbbk G$, $\sigma(g \otimes h) \neq 0$ for all $g, h \in G$, since σ is convolution invertible.

Let G be abelian. Then any bilinear form $\sigma : G \times G \to \Bbbk^\times$ is a normalized two-cocycle.

Let G be a free abelian group with basis $g_1, \dots, g_\theta$. Then any family $(\sigma_{ij})_{1 \le i,j \le \theta}$ of non-zero elements in $\Bbbk$ defines a normalized two-cocycle $\sigma : \Bbbk G \otimes \Bbbk G \to \Bbbk$ which is determined by the bilinear form $\sigma : G \times G \to \Bbbk^\times$ given by $\sigma(g_i, g_j) = \sigma_{ij}$ for all $i, j \in \{1, \dots, \theta\}$.

LEMMA 2.7.5. *Let H be a Hopf algebra and σ a two-cocycle for H. Then for all $x \in H$,*

$$\sigma(x_{(1)} \otimes S(x_{(2)}))\sigma^{-1}(S(x_{(3)}) \otimes x_{(4)}) = \varepsilon(x). \tag{2.7.5}$$

PROOF. Equation (2.7.2) with $x \otimes y \otimes z = x_{(1)} \otimes \mathcal{S}(x_{(2)}) \otimes x_{(3)}$ yields

$$\begin{aligned} &\sigma^{-1}(\mathcal{S}(x_{(3)}) \otimes x_{(4)})\sigma(x_{(1)} \otimes \mathcal{S}(x_{(2)})) \\ &\quad = \sigma(x_{(1)} \otimes \mathcal{S}(x_{(4)})x_{(5)})\sigma^{-1}(x_{(2)}\mathcal{S}(x_{(3)}) \otimes x_{(6)}) \end{aligned} \tag{2.7.6}$$

for all $x \in H$. The left hand side of (2.7.6) is just the left hand side of (2.7.5). The right hand side of (2.7.6) equals $\varepsilon(x)$ because of the antipode and counit axioms and Remark 2.7.3(5). $\square$

LEMMA 2.7.6. *Let H be a Hopf algebra, and let (A, δ) be an H-cleft object with section $\gamma : H \to A$. Then*

(1) $\delta(\gamma(x)) = \gamma(x_{(1)}) \otimes x_{(2)}$ *for all $x \in H$,*
(2) $\delta(\gamma^{-1}(x)) = \gamma^{-1}(x_{(2)}) \otimes \mathcal{S}(x_{(1)})$ *for all $x \in H$.*

PROOF. (1) just says that γ is right H-colinear, and (2) follows since δ induces an algebra map $\mathrm{Hom}(H, A) \to \mathrm{Hom}(H, A \otimes H)$ with respect to convolution, and the formula in (2) is an expression for $\delta\gamma^{-1}(x)$. $\square$

REMARK 2.7.7. The axiom of a two-cocycle is explained by the following equivalence which is easily checked.

Let H be a bialgebra and let $\sigma : H \otimes H \to \Bbbk$ be a linear map. Define a new product on the vector space H by

$$\mu_{(\sigma)} : H \otimes H \to H, \quad x \otimes y \mapsto \sigma(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}.$$

Then H with $\mu_{(\sigma)}$ is an associative algebra with the old unit 1 if and only if σ satisfies (2.7.1), (2.7.3), and if $\sigma(1 \otimes 1) = 1$.

DEFINITION 2.7.8. Let H be a bialgebra and σ a normalized two-cocycle for H. We denote by $H_{(\sigma)}$ the vector space H with algebra structure given by

$$H \otimes H \to H, \quad x \otimes y \mapsto \sigma(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}. \tag{2.7.7}$$

The next theorem shows that H-cleft objects are given by two-cocycles, and that two-cocycles can be constructed by finding a section of an H-cleft object.

THEOREM 2.7.9. *Let H be a bialgebra.*

(1) *Let σ be a normalized two-cocycle for H. Then $H_{(\sigma)}$ is an H-cleft object with H-comodule algebra structure $\Delta : H_{(\sigma)} \to H_{(\sigma)} \otimes H$ and section $\gamma = \mathrm{id} : H \to H_{(\sigma)}$.*

(2) *Let A be an H-cleft object with section γ and comodule algebra structure $\delta : A \to A \otimes H$, $a \mapsto a_{(0)} \otimes a_{(1)}$. Let*

$$\sigma(x \otimes y) = \gamma(x_{(1)})\gamma(y_{(1)})\gamma^{-1}(x_{(2)}y_{(2)})$$

for all $x, y \in H$. Then σ is a normalized two-cocycle for H, and the map $\gamma : H_{(\sigma)} \to A$ is a right H-colinear algebra isomorphism.

Proof. (1) By Remark 2.7.7 and Definition 2.7.8, $H_{(\sigma)}$ is a right H-comodule algebra. Lemma 2.7.5 implies that $\gamma = \mathrm{id}$ is invertible with inverse

$$\gamma^{-1}(x) = \sigma^{-1}(\mathcal{S}(x_{(2)}) \otimes x_{(3)})\mathcal{S}(x_{(1)})$$

for all $x \in H$.

(2) Using Lemma 2.7.6 it follows easily that for all $x, y \in H$ and $a \in A$, the elements $\sigma(x \otimes y) = \gamma(x_{(1)})\gamma(y_{(1)})\gamma^{-1}(x_{(2)}y_{(2)})$ and $a_{(0)}\gamma^{-1}(a_{(1)})$ are in $A^{\mathrm{co}\,H} = \Bbbk 1$. Hence σ defines a multiplication μ' in $H_{(\sigma)}$, and

$$\lambda : A \to H_{(\sigma)}, \quad a \mapsto a_{(0)}\gamma^{-1}(a_{(1)})a_{(2)},$$

is a well-defined linear map. Now it is easy to check that $\gamma\lambda = \mathrm{id}_A$, $\lambda\gamma = \mathrm{id}_H$, that $\sigma : H \otimes H \to \Bbbk$ is invertible with inverse given by

$$\sigma^{-1}(x \otimes y) = \gamma(x_{(1)}y_{(1)})\gamma^{-1}(y_{(2)})\gamma^{-1}(x_{(2)})$$

for all $x, y \in H$. Moreover, for all $x, y \in H_{(\sigma)}$,

$$\begin{aligned}\gamma(\mu'(x \otimes y)) &= \gamma(\sigma(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}) \\ &= \gamma(x_{(1)})\gamma(y_{(1)})\gamma^{-1}(x_{(2)}y_{(2)})\gamma(x_{(3)}y_{(3)}) = \gamma(x)\gamma(y)\end{aligned}$$

by definition of σ. Thus, $\gamma : H_{(\sigma)} \to A$ commutes with the multiplication. Hence $H_{(\sigma)}$ is an associative algebra, and σ is a two-cocycle by Remark 2.7.7. This proves (2). □

2.8. Two-cocycle deformations and Drinfeld double

Two-cocycles play an important role for the construction of new bialgebras.

Definition 2.8.1. Let H be a bialgebra and σ a two-cocycle for H. Let $H_\sigma = H$ as a coalgebra with multiplication

$$\mu_\sigma : H_\sigma \otimes H_\sigma \to H_\sigma,\ x \otimes y \mapsto \sigma(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)} \otimes y_{(3)}).$$

Theorem 2.8.2. *Let H be a bialgebra and let σ be a two-cocycle for H. Then H_σ is a bialgebra. If H is a Hopf algebra, then H_σ is a Hopf algebra with antipode $\mathcal{S}_\sigma$, where*

$$\mathcal{S}_\sigma(x) = \sigma(x_{(1)} \otimes \mathcal{S}(x_{(2)}))\,\mathcal{S}(x_{(3)})\,\sigma^{-1}(\mathcal{S}(x_{(4)}) \otimes x_{(5)})$$

for all $x \in H_\sigma$.

Proof. (1) We first show that H_σ is a bialgebra. For any $x, y, z \in H$ we obtain that

$$\begin{aligned}
\mu_\sigma(x \otimes \mu_\sigma(y \otimes z)) &= \sigma(y_{(1)} \otimes z_{(1)})\mu_\sigma(x \otimes y_{(2)}z_{(2)})\sigma^{-1}(y_{(3)} \otimes z_{(3)}) \\
&= \sigma(y_{(1)} \otimes z_{(1)})\sigma(x_{(1)} \otimes y_{(2)}z_{(2)})x_{(2)}y_{(3)}z_{(3)} \\
&\quad \sigma^{-1}(x_{(3)} \otimes y_{(4)}z_{(4)})\sigma^{-1}(y_{(5)} \otimes z_{(5)}) \\
&= \sigma(x_{(1)} \otimes y_{(1)})\sigma(x_{(2)}y_{(2)} \otimes z_{(1)})x_{(3)}y_{(3)}z_{(2)} \\
&\quad \sigma^{-1}(x_{(4)}y_{(4)} \otimes z_{(3)})\sigma^{-1}(x_{(5)} \otimes y_{(5)}) \\
&= \mu_\sigma(\sigma(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)} \otimes y_{(3)}) \otimes z) \\
&= \mu_\sigma(\mu_\sigma \otimes \mathrm{id})(x \otimes y \otimes z)
\end{aligned}$$

by Remark 2.7.3(4). Therefore μ_σ is associative.

The unit $1 \in H$ is a unit for H_σ. Indeed,

$$\begin{aligned}
\mu_\sigma(x \otimes 1) &= \sigma(x_{(1)} \otimes 1)x_{(2)}\sigma^{-1}(x_{(3)} \otimes 1) \\
&= \varepsilon(x_{(1)})\sigma(1 \otimes 1)x_{(2)}\varepsilon(x_{(3)})\sigma^{-1}(1 \otimes 1) \\
&= x
\end{aligned}$$

for all $x \in H$ by Remark 2.7.3(5). Similarly, $\mu_\sigma(1 \otimes x) = x$ for all $x \in H$.

Clearly, the counit of H_σ is an algebra map. Finally, the comultiplication of H_σ is an algebra map. Indeed, for any $x, y \in H_\sigma$ we obtain that

$$\begin{aligned}
\Delta(\mu_\sigma(x \otimes y)) &= \sigma(x_{(1)} \otimes y_{(1)})\Delta(x_{(2)}y_{(2)})\sigma^{-1}(x_{(3)} \otimes y_{(3)}) \\
&= \sigma(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)} \otimes x_{(3)}y_{(3)}\sigma^{-1}(x_{(4)} \otimes y_{(4)}) \\
&= \sigma(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)} \otimes y_{(3)}) \\
&\quad \otimes \sigma(x_{(4)} \otimes y_{(4)})x_{(5)}y_{(5)}\sigma^{-1}(x_{(6)} \otimes y_{(6)}) \\
&= \mu_\sigma(x_{(1)} \otimes y_{(1)}) \otimes \mu_\sigma(x_{(2)} \otimes y_{(2)}).
\end{aligned}$$

(2) Now let H be a Hopf algebra. Let $x \in H$. Then

$$\begin{aligned}
&\mu_\sigma(x_{(1)} \otimes \mathcal{S}_\sigma(x_{(2)})) \\
&\quad = \sigma(x_{(2)} \otimes \mathcal{S}(x_{(3)}))\mu_\sigma(x_{(1)} \otimes \mathcal{S}(x_{(4)}))\sigma^{-1}(\mathcal{S}(x_{(5)}) \otimes x_{(6)}) \\
&\quad = \underline{\sigma(x_{(4)} \otimes \mathcal{S}(x_{(5)}))}\sigma(x_{(1)} \otimes \mathcal{S}(x_{(8)})) \\
&\qquad\quad x_{(2)}\mathcal{S}(x_{(7)})\underline{\sigma^{-1}(x_{(3)} \otimes \mathcal{S}(x_{(6)}))}\sigma^{-1}(\mathcal{S}(x_{(9)}) \otimes x_{(10)}).
\end{aligned}$$

The underlined factors can be simplified to $\varepsilon(x_{(3)})\varepsilon(x_{(4)})\varepsilon(x_{(5)})\varepsilon(x_{(6)})1$ by the definition of σ^{-1}. Therefore the expression simplifies further to

$$\begin{aligned}
&\sigma(x_{(1)} \otimes \mathcal{S}(x_{(4)}))x_{(2)}\mathcal{S}(x_{(3)})\sigma^{-1}(\mathcal{S}(x_{(5)}) \otimes x_{(6)}) \\
&\quad = \sigma(x_{(1)} \otimes \mathcal{S}(x_{(2)}))\sigma^{-1}(\mathcal{S}(x_{(3)}) \otimes x_{(4)}) = \varepsilon(x),
\end{aligned}$$

where the last equation holds by (2.7.5). The equation

$$\mu_\sigma(\mathcal{S}_\sigma(x_{(1)}) \otimes x_{(2)}) = \varepsilon(x)$$

is proven analogously. □

The bialgebra H_σ in Theorem 2.8.2 is called a **two-cocycle deformation** of H.

REMARK 2.8.3. Let H be a bialgebra, and σ_0, σ_1 two-cocycles for H. Using Remark 2.7.3(4) it is easy to see that the convolution product $\rho = \sigma_1 * \sigma_0^{-1}$ is a two-cocycle for H_{σ_0}. Then, by Definition 2.8.1,

$$H_{\sigma_1} = (H_{\sigma_0})_\rho.$$

If H is the tensor product of two bialgebras, then two-cocycles can be constructed via skew pairings.

DEFINITION 2.8.4. Let A, U be bialgebras over a field $\Bbbk$. A **skew pairing** of A and U is a linear map $\tau : A \otimes U \to \Bbbk$ satisfying the equations

$$\tau(a \otimes 1) = \varepsilon(a), \quad \tau(1 \otimes x) = \varepsilon(x), \tag{2.8.1}$$

$$\tau(ab \otimes x) = \tau(a \otimes x_{(1)})\tau(b \otimes x_{(2)}), \tag{2.8.2}$$

$$\tau(a \otimes xy) = \tau(a_{(1)} \otimes y)\tau(a_{(2)} \otimes x) \tag{2.8.3}$$

for any $a, b \in A$ and $x, y \in U$.

REMARK 2.8.5. Let A, U be bialgebras. A skew pairing τ of A and U is nothing but a bialgebra homomorphism φ from A^{cop} to the dual bialgebra U^0 of U. The correspondence is given by the equation

$$\langle \varphi(a), x \rangle = \tau(a \otimes x)$$

for any $a \in A$ and $x \in U$, where $\langle\ ,\ \rangle$ denotes evaluation. Therefore, very often skew pairings can be constructed explicitly, if the algebra A is given by generators and relations. We will show in Proposition 2.8.7 below that any invertible skew pairing defines a two-cocycle. This is a very elegant way to actually find two-cocycles.

LEMMA 2.8.6. *Let A, U be bialgebras, and $\tau : A \otimes U \to \Bbbk$ a skew pairing. If A is a Hopf algebra, or U is a Hopf algebra with bijective antipode, then τ is invertible, and for all $a \in A$, $x \in U$,*

$$\tau^{-1}(a \otimes x) = \tau(\mathcal{S}(a) \otimes x), \quad \tau^{-1}(a \otimes x) = \tau(a \otimes \mathcal{S}^{-1}(x)),$$

respectively.

PROOF. Assume that A is a Hopf algebra. Then

$$\begin{aligned}
&\tau^{-1}(a_{(1)} \otimes x_{(1)})\tau(a_{(2)} \otimes x_{(2)}) \\
&\quad = \tau(\mathcal{S}(a_{(1)}) \otimes x_{(1)})\tau(a_{(2)} \otimes x_{(2)}) = \tau(\mathcal{S}(a_{(1)})a_{(2)}1 \otimes x) = \varepsilon(a)\varepsilon(x)
\end{aligned}$$

for all $a, b \in A$, $u \in U$, where the second and third equations follow from Definition 2.8.4. The equation $\tau\tau^{-1} = \varepsilon \otimes \varepsilon$ is proven analogously.

If U is a Hopf algebra with bijective antipode, the proof is similar. □

PROPOSITION 2.8.7. *Let A, U be bialgebras and let $H = A \otimes U$. For any invertible skew pairing τ of A and U, the map*

$$\sigma : H \otimes H \to \Bbbk, \quad \sigma((a \otimes x) \otimes (b \otimes y)) = \varepsilon(a)\tau(b \otimes x)\varepsilon(y),$$

is a two-cocycle for H. The inverse of σ is given by

$$\sigma^{-1}((a \otimes x) \otimes (b \otimes y)) = \varepsilon(a)\tau^{-1}(b \otimes x)\varepsilon(y)$$

for all $a, b \in A$ and $x, y \in U$.

PROOF. Let $\sigma^{-1} : H \otimes H \to \Bbbk$ be as in the proposition. We check first that σ^{-1} is the inverse of σ. For any $a, b \in A$ and $x, y \in U$ we obtain that

$$\begin{aligned}
&\sigma((a_{(1)} \otimes x_{(1)}) \otimes (b_{(1)} \otimes y_{(1)}))\sigma^{-1}((a_{(2)} \otimes x_{(2)}) \otimes (b_{(2)} \otimes y_{(2)}))\\
&\quad = \varepsilon(a_{(1)})\tau(b_{(1)} \otimes x_{(1)})\varepsilon(y_{(1)})\varepsilon(a_{(2)})\tau^{-1}(b_{(2)} \otimes x_{(2)})\varepsilon(y_{(2)})\\
&\quad = \varepsilon(a)\varepsilon(b)\varepsilon(x)\varepsilon(y)
\end{aligned}$$

and hence $\sigma\sigma^{-1} = \varepsilon$. Similarly, $\sigma^{-1}\sigma = \varepsilon$.

Now we verify (2.7.1). Let $a, b, c \in A$ and $x, y, z \in U$. Then

$$\begin{aligned}
&\sigma((a_{(1)} \otimes x_{(1)}) \otimes (b_{(1)} \otimes y_{(1)}))\sigma((a_{(2)}b_{(2)} \otimes x_{(2)}y_{(2)}) \otimes (c \otimes z))\\
&\quad = \varepsilon(a_{(1)})\tau(b_{(1)} \otimes x_{(1)})\varepsilon(y_{(1)})\varepsilon(a_{(2)}b_{(2)})\tau(c \otimes x_{(2)}y_{(2)})\varepsilon(z)\\
&\quad = \tau(b \otimes x_{(1)})\tau(c \otimes x_{(2)}y)\varepsilon(a)\varepsilon(z)\\
&\quad = \tau(b \otimes x_{(1)})\tau(c_{(1)} \otimes y)\tau(c_{(2)} \otimes x_{(2)})\varepsilon(a)\varepsilon(z).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sigma((b_{(1)} \otimes y_{(1)}) \otimes (c_{(1)} \otimes z_{(1)}))\sigma((a \otimes x) \otimes (b_{(2)}c_{(2)} \otimes y_{(2)}z_{(2)}))\\
&\quad = \varepsilon(b_{(1)})\tau(c_{(1)} \otimes y_{(1)})\varepsilon(z_{(1)})\varepsilon(a)\tau(b_{(2)}c_{(2)} \otimes x)\varepsilon(y_{(2)}z_{(2)})\\
&\quad = \tau(c_{(1)} \otimes y)\tau(bc_{(2)} \otimes x)\varepsilon(a)\varepsilon(z)\\
&\quad = \tau(c_{(1)} \otimes y)\tau(b \otimes x_{(1)})\tau(c_{(2)} \otimes x_{(2)})\varepsilon(a)\varepsilon(z).
\end{aligned}$$

This proves the claim. $\square$

COROLLARY 2.8.8. *Let A, U be bialgebras, $\tau : A \otimes U \to \Bbbk$ an invertible skew pairing, and let σ be the two-cocycle for the bialgebra $A \otimes U$ defined by τ in Proposition 2.8.7.*

(1) *$(A \otimes U)_\sigma$ is a bialgebra with the comultiplication of $A \otimes U$. The maps*

$$A \to (A \otimes U)_\sigma,\ a \mapsto a \otimes 1, \qquad U \to (A \otimes U)_\sigma,\ x \mapsto 1 \otimes x,$$

are injective bialgebra maps. For all $a \in A$, $x \in U$, in $(A \otimes U)_\sigma$,

$$\begin{aligned}
(a \otimes 1)(1 \otimes x) &= a \otimes x,\\
(1 \otimes x)(a \otimes 1) &= \tau(a_{(1)} \otimes x_{(1)})a_{(2)} \otimes x_{(2)}\tau^{-1}(a_{(3)} \otimes x_{(3)}).
\end{aligned}$$

(2) *If A and U are Hopf algebras, then $(A \otimes U)_\sigma$ is a Hopf algebra with antipode $\mathcal{S}_\sigma$, and for all $a \in A$, $x \in U$,*

$$\mathcal{S}_\sigma(a \otimes x) = \tau(\mathcal{S}(a_{(1)}) \otimes x_{(1)})(\mathcal{S}(a_{(2)}) \otimes \mathcal{S}(x_{(2)}))\tau^{-1}(a_{(3)} \otimes \mathcal{S}(x_{(3)})).$$

PROOF. (1) By Theorem 2.8.2 and Proposition 2.8.7, $(A \otimes U)_\sigma$ is a bialgebra. For all $a, b \in A$, the product of $a \otimes 1$ and $b \otimes 1$ in $(A \otimes U)_\sigma$ is given by

$$\begin{aligned}
&\sigma((a_{(1)} \otimes 1) \otimes (b_{(1)} \otimes 1))(a_{(2)} \otimes 1)(b_{(2)} \otimes 1)\sigma^{-1}((a_{(3)} \otimes 1) \otimes (b_{(3)} \otimes 1))\\
&\quad = \tau(b_{(1)} \otimes 1)ab_{(2)} \otimes 1\tau^{-1}(b_{(3)} \otimes 1)\\
&\quad = ab \otimes 1.
\end{aligned}$$

Similarly, $U \to (A \otimes U)_\sigma,\ x \mapsto 1 \otimes x$, is an algebra map. Moreover, for any $a \in A$ and $x \in U$, $(a \otimes 1)(1 \otimes x) = a \otimes x$, and

$$\begin{aligned}
&(1 \otimes x)(a \otimes 1)\\
&\quad = \sigma((1 \otimes x_{(1)}) \otimes (a_{(1)} \otimes 1))a_{(2)} \otimes x_{(2)}\sigma^{-1}((1 \otimes x_{(3)}) \otimes (a_{(3)} \otimes 1))\\
&\quad = \tau(a_{(1)} \otimes x_{(1)})a_{(2)} \otimes x_{(2)}\tau^{-1}(a_{(3)} \otimes x_{(3)}).
\end{aligned}$$

(2) follows from Theorem 2.8.2 and Proposition 2.8.7. □

The bialgebra $(A \otimes U)_\sigma$ in Corollary 2.8.8 is known as **Drinfeld's quantum double of A and U.**

REMARK 2.8.9. Let U be a finite-dimensional Hopf algebra. Then the evaluation map $\tau : U^* \otimes U \to \Bbbk$ is an invertible skew pairing of $(U^*)^{\mathrm{cop}}$ and U by Lemma 2.8.6. Let σ be the two-cocycle given by τ as in Proposition 2.8.7. Then $((U^*)^{\mathrm{cop}} \otimes U)_\sigma$ is called the **Drinfeld double of U.** It is a Hopf algebra by the results of this section.

We now discuss two ways to define an algebra map on $(A \otimes U)_\sigma$.

LEMMA 2.8.10. *Let C be an algebra and let A, U be subalgebras of C such that the multiplication map $A \otimes U \to C$ is bijective. Assume that A and U are given by generators $(a_i)_{i \in I_A}$ and $(b_k)_{k \in I_U}$ and relations $r_j((a_i)_{i \in I_A})$, $j \in J_A$, and $s_j((b_k)_{k \in I_U})$, $j \in J_U$, respectively. Let $V_A = \mathrm{span}_\Bbbk\{1, a_i \mid i \in I_A\}$, and $V_U = \mathrm{span}_\Bbbk\{1, b_k \mid k \in I_U\}$. Assume that $V_U V_A \subseteq V_A V_U$. Then C is canonically isomorphic to $\langle a_i, b_k \mid i \in I_A, k \in I_U\rangle/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by $r_j((a_i)_{i \in I_A})$, $j \in J_A$, $s_j((b_k)_{k \in I_U})$, $j \in J_U$, and the quadratic relations of C in $V_U V_A + V_A V_U$.*

PROOF. The algebra C is generated by the set $\{a_i, b_k \mid i \in I_A, k \in I_U\}$. Let $\overline{C} = \langle a_i, b_k \mid i \in I_A, k \in I_U\rangle/\mathcal{I}$. Then, by construction, there is a surjective algebra map $f : \overline{C} \to C$ with $f(a_i) = a_i$, $f(b_k) = b_k$ for all $i \in I_A$, $k \in I_U$. Let $\overline{A}$ and $\overline{U}$ be the subalgebras of $\overline{C}$ generated by $(a_i)_{i \in I_A}$ and $(b_k)_{k \in I_U}$, respectively. Then $f|\overline{A} : \overline{A} \to A$ and $f|\overline{U} : \overline{U} \to U$ are bijective by construction. Moreover, $b_k V_A^n \subseteq V_A^n V_U$ for all $k \in I_U$ and $n \in \mathbb{N}$, and hence $\overline{C} = \overline{AU}$. Thus the diagram

$$\begin{tikzcd}
\overline{A} \otimes \overline{U} \arrow[r, "\mathrm{mult}"] \arrow[d, "f|\overline{A} \otimes f|\overline{U}"'] & \overline{C} \arrow[d, "f"] \\
A \otimes U \arrow[r, "\mathrm{mult}"'] & C
\end{tikzcd}$$

of surjective maps commutes, where mult denotes the multiplication map. Hence $f : \overline{C} \to C$ is bijective. □

PROPOSITION 2.8.11. *Let A, U be bialgebras, T an algebra, $\varphi_A : A \to T$ and $\varphi_U : U \to T$ algebra maps, and $\tau : A \otimes U \to \Bbbk$ an invertible skew pairing with corresponding two-cocycle σ for $A \otimes U$. Let $\mathcal{P} \subseteq A \times U$ be the subset of all pairs $(a, x) \in A \times U$ satisfying*

$$\varphi_U(x_{(1)})\varphi_A(a_{(1)})\tau(a_{(2)} \otimes x_{(2)}) = \tau(a_{(1)} \otimes x_{(1)})\varphi_A(a_{(2)})\varphi_U(x_{(2)}).$$

Let $(a_k)_{k\in K}$ and $(x_l)_{l\in L}$ be generators of the algebras A and U, respectively, and let $C = \operatorname{span}_{\Bbbk}\{a_k \mid k \in K\}$. Assume

(1) *$C \subseteq A$ is a subcoalgebra,*
(2) *$(a_k, x_l) \in \mathcal{P}$ for all $k \in K$, $l \in L$.*

Then the map

$$\varphi : (A \otimes U)_\sigma \to T, \quad a \otimes x \mapsto \varphi_A(a)\varphi_U(x),$$

is an algebra map. If T is a bialgebra, and φ_A, φ_U are bialgebra maps, then φ is a bialgebra map.

Proof. Let $D = \operatorname{span}_{\Bbbk}\{x_l \mid l \in L\}$. Note that by (2), $(a, x) \in \mathcal{P}$ for all $a \in C$, $x \in D$.

Let $x, y \in U$, and assume that for all $a \in C$, $(a, x) \in \mathcal{P}$ and $(a, y) \in \mathcal{P}$. Then for all $a \in C$, $(a, xy) \in \mathcal{P}$, since

$$\begin{aligned}
&\varphi_U(x_{(1)}y_{(1)})\varphi_A(a_{(1)})\tau(a_{(2)} \otimes x_{(2)}y_{(2)})\\
&= \varphi_U(x_{(1)})\varphi_U(y_{(1)})\varphi_A(a_{(1)})\tau(a_{(2)} \otimes y_{(2)})\tau(a_{(3)} \otimes x_{(2)})\\
&= \varphi_U(x_{(1)})\tau(a_{(1)} \otimes y_{(1)})\varphi_A(a_{(2)})\varphi_U(y_{(2)})\tau(a_{(3)} \otimes x_{(2)})\\
&= \tau(a_{(2)} \otimes x_{(1)})\varphi_A(a_{(3)})\varphi_U(x_{(2)})\tau(a_{(1)} \otimes y_{(1)})\varphi_U(y_{(2)})\\
&= \tau(a_{(1)} \otimes x_{(1)}y_{(1)})\varphi_A(a_{(2)})\varphi_U(x_{(2)}y_{(2)}),
\end{aligned}$$

where the first equality follows from (2.8.3), the second and the third, since the pairs $(a_{(1)}, y), (a_{(2)}, x)$ are elements in $\mathcal{P}$, and the last again from (2.8.3).

It follows that $C \times U \subseteq \mathcal{P}$, since the elements $(x_l)_{l\in L}$ generate U. Since C generates the algebra A, a similar computation using (2.8.2) proves that $A \times U = \mathcal{P}$.

Hence the formula for the multiplication in $(A\otimes U)_\sigma$ in Corollary 2.8.8(1) shows that φ is an algebra map. If T is a bialgebra and φ_A, φ_U are bialgebra maps, then φ is a bialgebra map, since A and U are subbialgebras of $(A \otimes U)_\sigma$, and $(A \otimes U)_\sigma$ is generated by $A \cup U$. $\square$

2.9. Notes

For general Hopf algebra theory, we refer to the books [**Swe69**], [**Mon93**], [**Rad12**].

2.4. The Hopf algebras $T_{q,n}$ in Example 2.4.10 have been introduced by Taft in [**Taf71**]. The algebra $U_q(\mathfrak{sl}_2)$ in Example 2.4.11 was introduced by Kulish and Reshetikhin in [**KR81**], its Hopf algebra structure in 1985 by Sklyanin. The small quantum group $u_q(\mathfrak{sl}_2)$, q a root of unity of order 3, was already defined by Nichols in [**Nic78**].

2.5. Hopf modules have been introduced for abstract Hopf algebras by Larson and Sweedler in [**LS69**].

2.7. For general cleft extensions, see [**Mon93**, Section 7.2] and the references therein.

2.8. For the Drinfeld double of U see [**Dri87**], or [**DT94**, Remark 2.3]. We follow the exposition by Doi and Takeuchi in [**DT94**].

CHAPTER 3

Braided monoidal categories

Throughout the book, braidings of different type appear and have a strong impact on many structures. Most of the braidings arise naturally in categories of Yetter-Drinfeld modules of vector spaces or in other braided categories. In this chapter we present the general theory of braided (strict) monoidal categories. We extend basic notions and results from Chapter 1 and Chapter 2 to braided monoidal categories. This is usually possible, but the proofs can be much more involved. In Section 3.8 we discuss bosonization in this general context, and in Section 3.10 we prove the important theorem of Radford, Majid and Bespalov which can be viewed as an extension of the theory of semidirect products of groups.

3.1. Monoidal categories

Let $\mathcal{C}$ be a category. We write $X \in \mathcal{C}$, if X is an object of $\mathcal{C}$. The class of morphisms $f : X \to Y$ between objects X, Y is denoted by $\mathcal{C}(X,Y)$ or by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. Let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a functor. As for the tensor product of vector spaces, we denote the image under $\otimes$ of a pair (X,Y) of objects of $\mathcal{C}$ by $X \otimes Y$, and the image of a pair of morphisms $(f : X \to X', g : Y \to Y')$ by $f \otimes g$. Let

$$a = (a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z))_{X,Y,Z \in \mathcal{C}}$$

be a natural isomorphism, called an **associativity constraint**. One says that a satifies the **pentagon axiom**, if for all $W, X, Y, Z \in \mathcal{C}$ the diagram

$$\begin{tikzcd}[column sep=small]
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W\otimes X,Y,Z}"'] \arrow[dr, "a_{W,X,Y}\otimes \mathrm{id}_Z"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] \\
W \otimes (X \otimes (Y \otimes Z)) & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ll, "\mathrm{id}_W \otimes a_{X,Y,Z}"]
\end{tikzcd} \tag{3.1.1}$$

commutes.

Let $I \in \mathcal{C}$ be an object, called the **unit object**, and let

$$l = (l_X : I \otimes X \to X)_{X \in \mathcal{C}}, \quad r = (r_X : X \otimes I \to X)_{X \in \mathcal{C}}$$

be natural isomorphisms, called **unit constraints**. They satisfy the **triangle axiom** with respect to I, if for all $X, Y \in \mathcal{C}$ the diagram

(3.1.2)
$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a_{X,I,Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

commutes.

Definition 3.1.1. A collection $(\mathcal{C}, \otimes, I, a, l, r)$ consisting of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object I, an associativity constraint a, and unit constraints l, r is called a **monoidal category**, if the pentagon axiom and the triangle axiom hold. Occasionally, such a collection is abbreviated by $\mathcal{C}$.

The pentagon and triangle axioms in a monoidal category imply the commutativity of any diagram constructed from a, l, r and identity maps by tensoring and composition. This follows from Mac Lane's coherence theorem, see [**Kas95**, Theorem XI.5.3].

Example 3.1.2. The category ${}_{\Bbbk}\mathcal{M}$ of vector spaces over the field $\Bbbk$ is monoidal, where $\otimes$ is the tensor product of vector spaces, $I = \Bbbk$, and a, l, r are the standard associativity and unit constraints.

Example 3.1.3. Let H be a bialgebra. The category ${}_H\mathcal{M}$ of left H-modules is monoidal, where the tensor product of $V, W \in {}_H\mathcal{M}$ is the tensor product $V \otimes W$ of the underlying vector spaces as a left H-module with the diagonal action defined in Definition 1.2.4. The unit object is $I = \Bbbk$ with trivial action defined by $hv = \varepsilon(h)v$ for all $h \in H$, $v \in V$. The associativity and unit constraints are the same as for vector spaces. In the same way, the category $\mathcal{M}_H$ of right H-modules is monoidal.

Example 3.1.4. This example is dual to Example 3.1.3. The category $\mathcal{M}^H$ of right H-comodules (and similarly the category of left H-comodules) over a bialgebra H is monoidal, where the tensor product of right H-comodules is the underlying vector space of the tensor product of the vector spaces with diagonal coaction defined in Definition 1.2.4. The unit object is $I = \Bbbk$ together with the H-coaction $\Bbbk \to \Bbbk \otimes H$, $1 \mapsto 1 \otimes 1$. The associativity and unit constraints are the same as for vector spaces.

A monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ is called **strict** if the maps $a_{X,Y,Z}, l_X$ and r_X are the identity maps for all $X, Y, Z \in \mathcal{C}$. In this book the monoidal categories of interest are all categories of vector spaces with an additional algebraic structure and with associativity and unit constraints as for vector spaces. We follow the convention to suppress the associativity and unit constraints for these examples, that is, we view the category of vector spaces and related monoidal categories as strict monoidal categories.

In many cases it is justified to prove a result for general monoidal categories by assuming that the categories are strict, see [**Kas95**, Section XI.5].

Let $(\mathcal{C}, \otimes, I)$ be a strict monoidal category.

The **dual category** $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ with reversed arrows. Thus for all objects X, Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. We write $f^{\mathrm{op}} : X \to Y$ for the morphism $f : Y \to X$. Composition of morphisms is defined by

$$g^{\mathrm{op}} f^{\mathrm{op}} = (fg)^{\mathrm{op}},$$

where $f : X \to Y$ and $g : Z \to X$ are morphisms in $\mathcal{C}$. The dual category $\mathcal{C}^{\mathrm{op}}$ is strict monoidal with the same tensor product on objects as $\mathcal{C}$ and with $f^{\mathrm{op}} \otimes g^{\mathrm{op}} = (f \otimes g)^{\mathrm{op}}$ for morphisms f, g. We call $(\mathcal{C}^{\mathrm{op}}, \otimes, I)$ the **dual monoidal category** of $(\mathcal{C}, \otimes, I)$.

The reversed tensor product $\otimes^{\mathrm{rev}}$ is defined by

$$X \otimes^{\mathrm{rev}} Y = Y \otimes X, \qquad f \otimes^{\mathrm{rev}} g = g \otimes f$$

for objects X, Y and morphisms f, g in $\mathcal{C}$. The monoidal category $\mathcal{C}^{\mathrm{rev}} = (\mathcal{C}, \otimes^{\mathrm{rev}}, I)$ is called the **reversed category** of $\mathcal{C}$.

Algebras, modules, coalgebras and comodules and their morphisms in a strict monoidal category $\mathcal{C}$ are defined as in Chapter 1 in the category of vector spaces.

An **algebra in** $\mathcal{C}$ is a triple (A, μ, η), where A is an object in $\mathcal{C}$ with morphisms $\mu : A \otimes A \to A$, $\eta : I \to A$ such that the following diagrams commute.

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \tag{3.1.3}$$

$$\begin{tikzcd} I \otimes A \arrow[rr, "\eta\otimes\mathrm{id}"] \arrow[dr, "="'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A \otimes I \arrow[rr, "\mathrm{id}\otimes\eta"] \arrow[dr, "="'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \tag{3.1.4}$$

Let A, B be algebras in $\mathcal{C}$ and $\rho : A \to B$ a morphism in $\mathcal{C}$. Then ρ is an **algebra morphism** if the diagrams

$$\begin{tikzcd} A \otimes A \arrow[r, "\rho\otimes\rho"] \arrow[d, "\mu_A"] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "\rho"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\rho"] & & B \\ & I \arrow[ul, "\eta_A"] \arrow[ur, "\eta_B"'] & \end{tikzcd} \tag{3.1.5}$$

commute.

Let A be an algebra in $\mathcal{C}$, V an object in $\mathcal{C}$, and $\lambda : A \otimes V \to V$ a morphism. Then (V, λ) is a left A**-module** if the diagrams

$$\begin{tikzcd} A \otimes A \otimes V \arrow[r, "\mathrm{id}\otimes\lambda"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes V \arrow[d, "\lambda"] \\ A \otimes V \arrow[r, "\lambda"] & V \end{tikzcd} \qquad \begin{tikzcd} I \otimes V \arrow[rr, "\eta\otimes\mathrm{id}"] \arrow[dr, "="'] & & A \otimes V \arrow[dl, "\lambda"] \\ & V & \end{tikzcd}$$

commute. Let (V, λ_V) and (W, λ_W) be left A-modules, and $f : V \to W$ a morphism in $\mathcal{C}$. Then f is a morphism of left A-modules if

$$\begin{tikzcd} A \otimes V \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "\lambda_V"] & A \otimes W \arrow[d, "\lambda_W"] \\ V \arrow[r, "f"] & W \end{tikzcd} \tag{3.1.6}$$

commutes.

Right A-modules and their morphisms are defined similarly.

A **coalgebra** (C, Δ, ε) in $\mathcal{C}$, where $C \in \mathcal{C}$ and $\Delta : C \to C \otimes C$, $\varepsilon : C \to I$ are morphisms, is an algebra in $\mathcal{C}^{\mathrm{op}}$. If C is a coalgebra, $V \in \mathcal{C}$, and $\delta : V \to C \otimes V$ is a morphism in $\mathcal{C}$, then (V, δ) is a left C**-comodule** in $\mathcal{C}$ if (V, δ) is a left C-module in $\mathcal{C}^{\mathrm{op}}$. Right comodules are defined similarly, and morphisms of coalgebras and comodules are defined dually to morphisms of algebras and modules.

If C is a coalgebra in $\mathcal{C}$, and A is an algebra in $\mathcal{C}$, we denote by ${}^C\mathcal{C}$ and $\mathcal{C}^C$ the categories of left and of right C-comodules, and by ${}_A\mathcal{C}$ and $\mathcal{C}_A$ the categories of left and of right A-modules in $\mathcal{C}$, respectively.

Lemma 3.1.5. *Let $(A, \mu, \eta), (A, \mu, \eta')$ be algebras and $(C, \Delta, \varepsilon), (C, \Delta, \varepsilon')$ coalgebras in $\mathcal{C}$. Then $\eta = \eta'$ and $\varepsilon = \varepsilon'$.*

Proof. By (3.1.4), $\eta = \mu(\mathrm{id} \otimes \eta')(\eta \otimes \mathrm{id}) = \mu(\eta \otimes \mathrm{id})(\mathrm{id} \otimes \eta') = \eta'$. The equality $\varepsilon = \varepsilon'$ follows by duality. □

Definition 3.1.6. Let C be a coalgebra and A an algebra in $\mathcal{C}$. The **convolution product** of morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ is defined by

$$f * g = (C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu_A} A).$$

It follows easily from the algebra and coalgebra axioms that $\mathrm{Hom}_{\mathcal{C}}(C, A)$ is a monoid with product $*$ and unit $C \xrightarrow{\varepsilon} I \xrightarrow{\eta} A$.

Definition 3.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be strict monoidal categories. A **monoidal functor** from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, φ_0, φ) consisting of a functor $F : \mathcal{C} \to \mathcal{D}$, an isomorphism $\varphi_0 : I_{\mathcal{D}} \to F(I_{\mathcal{C}})$, and a natural isomorphism

$$\varphi = (\varphi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y))_{X,Y \in \mathcal{C}}$$

such that for all objects $X, Y, Z \in \mathcal{C}$, the diagrams

$$\begin{tikzcd} F(X) \otimes F(Y) \otimes F(Z) \arrow[r, "\mathrm{id} \otimes \varphi_{Y,Z}"] \arrow[d, "\varphi_{X,Y} \otimes \mathrm{id}"'] & F(X) \otimes F(Y \otimes Z) \arrow[d, "\varphi_{X, Y \otimes Z}"] \\ F(X \otimes Y) \otimes F(Z) \arrow[r, "\varphi_{X \otimes Y, Z}"] & F(X \otimes Y \otimes Z) \end{tikzcd} \tag{3.1.7}$$

$$\begin{tikzcd} F(X) \otimes I \arrow[r, "="] \arrow[d, "\mathrm{id} \otimes \varphi_0"'] & F(X) \arrow[d, "="] & I \otimes F(X) \arrow[r, "="] \arrow[d, "\varphi_0 \otimes \mathrm{id}"'] & F(X) \arrow[d, "="] \\ F(X) \otimes F(I) \arrow[r, "\varphi_{X,I}"] & F(X \otimes I) & F(I) \otimes F(X) \arrow[r, "\varphi_{I,X}"] & F(I \otimes X) \end{tikzcd} \tag{3.1.8}$$

commute. The pair (φ_0, φ) is called a **monoidal structure** of F if (F, φ_0, φ) is a monoidal functor.

A **monoidal equivalence** (respectively **isomorphism**) is a monoidal functor (F, φ_0, φ) where F is an equivalence (respectively an isomorphism) of categories. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is called an equivalence (respectively an isomorphism) if there is a functor $G : \mathcal{D} \to \mathcal{C}$ with $FG \cong \mathrm{id}_{\mathcal{D}}$, $GF \cong \mathrm{id}_{\mathcal{C}}$ (respectively $FG = \mathrm{id}_{\mathcal{D}}$, $GF = \mathrm{id}_{\mathcal{C}}$).

In many cases φ_0 is the identity. Then the axioms in (3.1.8) say that

$$\varphi_{I,X} = \mathrm{id}_{F(X)} = \varphi_{X,I}. \tag{3.1.9}$$

We denote the monoidal functor $(F, \mathrm{id}, \varphi)$ by (F, φ) and call φ the monoidal structure of F.

A monoidal functor (F, φ) is called **strict** if $\varphi = \mathrm{id}$.

If $(F, \varphi) : \mathcal{C} \to \mathcal{D}$ and $(G, \psi) : \mathcal{D} \to \mathcal{E}$ are monoidal functors, then the composition

$$(GF, \lambda) : \mathcal{C} \to \mathcal{E}, \quad \lambda_{X,Y} = G(\varphi_{X,Y})\psi_{F(X),F(Y)} \text{ for all } X, Y \in \mathcal{C}, \tag{3.1.10}$$

is a monoidal functor.

Let $(F, \varphi) : \mathcal{C} \to \mathcal{D}$ be a monoidal isomorphism of categories with inverse functor $G : \mathcal{D} \to \mathcal{C}$. Then (G, ψ) is a monoidal functor with

$$\psi_{U,V} = G(\varphi_{G(U),G(V)})^{-1} : G(U) \otimes G(V) \to G(U \otimes V) \tag{3.1.11}$$

for all $U, V \in \mathcal{D}$.

REMARK 3.1.8. A monoidal functor (F, φ_0, φ) from $\mathcal{C}$ to $\mathcal{D}$ preserves algebraic structures defined in terms of the tensor product, in particular algebras, coalgebras, their modules and comodules, and the convolution product.

(1) Let (A, μ, η) be an algebra in $\mathcal{C}$ and (V, λ) a left A-module. Then $F(A)$ is an algebra in $\mathcal{D}$ with multiplication and unit

$$F(A) \otimes F(A) \xrightarrow{\varphi_{A,A}} F(A \otimes A) \xrightarrow{F(\mu)} F(A), \quad I \xrightarrow{\varphi_0} F(I) \xrightarrow{F(\eta)} F(A),$$

denoted by $(F, \varphi_0, \varphi)(A)$, and $F(V)$ is a left $F(A)$-module with module structure

$$F(A) \otimes F(V) \xrightarrow{\varphi_{A,V}} F(A \otimes V) \xrightarrow{F(\lambda)} F(V),$$

denoted by $(F, \varphi_0, \varphi)(V)$. For a coalgebra (C, Δ, ε) and a left C-comodule (V, δ), $F(C)$ is a coalgebra with comultiplication and counit

$$F(C) \xrightarrow{F(\Delta)} F(C \otimes C) \xrightarrow{\varphi_{C,C}^{-1}} F(C) \otimes F(C), \quad F(C) \xrightarrow{F(\varepsilon)} F(I) \xrightarrow{\varphi_0^{-1}} I,$$

denoted by $(F, \varphi_0, \varphi)(C)$, and $F(V)$ is a left $F(C)$-comodule with comodule structure

$$F(V) \xrightarrow{F(\delta)} F(C \otimes V) \xrightarrow{\varphi_{C,V}^{-1}} F(C) \otimes F(V),$$

denoted by $(F, \varphi_0, \varphi)(V)$.

(2) Let A be an algebra and C a coalgebra in $\mathcal{C}$. Then

$$\mathrm{Hom}_{\mathcal{C}}(C, A) \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(A)), \quad f \mapsto F(f),$$

is a monoid homomorphism with respect to convolution.

EXAMPLE 3.1.9. The duality functor $\mathcal{M}_{\Bbbk}^{\mathrm{fd}} \to \mathcal{M}_{\Bbbk}^{\mathrm{fd}}$, $V \mapsto V^*$, is a monoidal equivalence with monoidal structure $\varphi_{X,Y} : X^* \otimes Y^* \to (X \otimes Y)^*$ in Lemma 2.2.3, and $\varphi_0 : \Bbbk \to \Bbbk^*$, $1 \mapsto \mathrm{id}_{\Bbbk}$. This explains the duality between finite-dimensional algebras and coalgebras.

Here is an example of a monoidal isomorphism which is far from being strict. Let H be a bialgebra, and $\sigma : H \otimes H \to \Bbbk$ a convolution invertible linear map. Recall from Definition 2.7.2 and Remark 2.7.3(5) that σ is a normalized two-cocycle for H if and only if for all $x, y, z \in H$,

$$\sigma(x_{(1)} \otimes y_{(1)})\sigma(x_{(2)}y_{(2)} \otimes z) = \sigma(y_{(1)} \otimes z_{(1)})\sigma(x \otimes y_{(2)}z_{(2)}), \tag{3.1.12}$$

$$\sigma(z \otimes 1) = \sigma(1 \otimes z) = \varepsilon(z). \tag{3.1.13}$$

Proposition 3.1.10. *Let H be a bialgebra and $\sigma : H \otimes H \to \Bbbk$ a normalized two-cocycle for H. Let $F : {}^H\mathcal{M} \to {}^{H_\sigma}\mathcal{M}$ be the identity functor. For all X, Y in ${}^H\mathcal{M}$ let*

$$\varphi_{\sigma X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y), \quad x \otimes y \mapsto \sigma(x_{(-1)} \otimes y_{(-1)}) x_{(0)} \otimes y_{(0)}.$$

Then $(F, \varphi_\sigma) : {}^H\mathcal{M} \to {}^{H_\sigma}\mathcal{M}$ is a monoidal isomorphism.

Proof. Let $X, Y \in {}^H\mathcal{M}$. For simplicity, let $\varphi_{X,Y} = \varphi_{\sigma X,Y}$. The H_σ-comodule structures of $F(X) \otimes F(Y)$ and of $F(X \otimes Y)$ are denoted by $\delta_{F(X)\otimes F(Y)}$ and $\delta_{F(X\otimes Y)}$. To prove that $\varphi_{X,Y}$ is a morphism in ${}^{H_\sigma}\mathcal{M}$, let $x \in X$ and $y \in Y$. Then

$$\begin{aligned}
\delta_{F(X)\otimes F(Y)}(x \otimes y) &= \mu_\sigma(x_{(-1)} \otimes y_{(-1)}) \otimes x_{(0)} \otimes y_{(0)} \\
&= \sigma(x_{(-3)} \otimes y_{(-3)}) x_{(-2)} y_{(-2)} \sigma^{-1}(x_{(-1)} \otimes y_{(-1)}) \otimes x_{(0)} \otimes y_{(0)}, \\
\delta_{F(X\otimes Y)}(x \otimes y) &= x_{(-1)} y_{(-1)} \otimes x_{(0)} \otimes y_{(0)}.
\end{aligned}$$

Hence

$$\begin{aligned}
(\mathrm{id}_{H_\sigma} \otimes \varphi_{X,Y}) \delta_{F(X)\otimes F(Y)}(x \otimes y) &= \sigma(x_{(-2)} \otimes y_{(-2)}) x_{(-1)} y_{(-1)} \otimes x_{(0)} \otimes y_{(0)} \\
&= \delta_{F(X\otimes Y)} \varphi_{X,Y}(x \otimes y).
\end{aligned}$$

The linear map $\varphi_{X,Y}$ is bijective with inverse

$$X \otimes Y \to X \otimes Y, \quad x \otimes y \mapsto \sigma^{-1}(x_{(-1)} \otimes y_{(-1)}) x_{(0)} \otimes y_{(0)}.$$

The axioms of the monoidal structure of φ_σ are equivalent to the axioms of a normalized two-cocycle, since the commutativity of the diagrams (3.1.7) and the identities (3.1.9) are equivalent to (3.1.12) and (3.1.13). $\square$

3.2. Braided monoidal categories and graphical calculus

Many important monoidal categories, in particular categories of Yetter-Drinfeld modules, are braided. We fix here the terminology and introduce the graphical calculus, which typically improves the clarity of proofs.

Definition 3.2.1. Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category, and

$$c = (c_{X,Y} : X \otimes Y \to Y \otimes X)_{X,Y \in \mathcal{C}}$$

be a family of natural isomorphisms, that is, for all objects X, Y, X', Y' and morphisms $f : X \to X'$, $g : Y \to Y'$ in $\mathcal{C}$, $c_{X,Y} : X \otimes Y \to Y \otimes X$ is an isomorphism in $\mathcal{C}$ and the diagram

(3.2.1)
$$\begin{tikzcd}
X \otimes Y \arrow[r, "c_{X,Y}"] \arrow[d, "f\otimes g"] & Y \otimes X \arrow[d, "g\otimes f"] \\
X' \otimes Y' \arrow[r, "c_{X',Y'}"] & Y' \otimes X'
\end{tikzcd}$$

commutes. Then c is called a **braiding of** $(\mathcal{C}, \otimes, I, a, l, r)$ if for all objects X, Y, Z in $\mathcal{C}$ the following diagrams commute.

$$
\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] \arrow[dl, "a^{-1}_{X,Y,Z}"'] & (Y \otimes Z) \otimes X & \\
(X \otimes Y) \otimes Z \arrow[dr, "c_{X,Y} \otimes \mathrm{id}_Z"'] & & & Y \otimes (Z \otimes X) \arrow[ul, "a^{-1}_{Y,Z,X}"'] \\
& (Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"] & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes c_{X,Z}"'] &
\end{tikzcd}
\tag{3.2.2}
$$

$$
\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] \arrow[dl, "a_{X,Y,Z}"'] & Z \otimes (X \otimes Y) & \\
X \otimes (Y \otimes Z) \arrow[dr, "\mathrm{id}_X \otimes c_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \arrow[ul, "a_{Z,X,Y}"'] \\
& X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}
\tag{3.2.3}
$$

Let c be a braiding of $(\mathcal{C}, \otimes, I, a, l, r)$. Then $(\mathcal{C}, \otimes, I, a, l, r, c)$ is called a **braided monoidal category**.

We note that in a braided monoidal category $\mathcal{C}$, for all $X \in \mathcal{C}$, the following diagrams commute.

$$
\begin{tikzcd}
X \otimes I \arrow[r, "c_{X,I}"] \arrow[d, "r_X"] & I \otimes X \arrow[d, "l_X"] \\
X \arrow[r, "\mathrm{id}_X"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes X \arrow[r, "c_{I,X}"] \arrow[d, "l_X"] & X \otimes I \arrow[d, "r_X"] \\
X \arrow[r, "\mathrm{id}_X"] & X
\end{tikzcd}
\tag{3.2.4}
$$

For a proof, see [**Kas95**, Proposition XIII.1.2].

A **braided strict monoidal category** is a quadruple $(\mathcal{C}, \otimes, I, c)$ such that $(\mathcal{C}, \otimes, I)$ is strict monoidal and c is a braiding of $\mathcal{C}$. Then the axioms (3.2.2) and (3.2.3) say that for all $X, Y, Z \in \mathcal{C}$ the diagrams

$$
\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "c_{X,Y\otimes Z}"] \arrow[dr, "c_{X,Y} \otimes \mathrm{id}_Z"'] & & Y \otimes Z \otimes X \\
& Y \otimes X \otimes Z \arrow[ur, "\mathrm{id}_Y \otimes c_{X,Z}"'] &
\end{tikzcd}
\tag{3.2.5}
$$

$$
\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "c_{X\otimes Y,Z}"] \arrow[dr, "\mathrm{id}_X \otimes c_{Y,Z}"'] & & Z \otimes X \otimes Y \\
& X \otimes Z \otimes Y \arrow[ur, "c_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}
\tag{3.2.6}
$$

commute. The commutativity of the diagrams (3.2.4) reduces to the equations

$$c_{I,X} = \mathrm{id}_X = c_{X,I}. \tag{3.2.7}$$

Note that (3.2.7) follows immediately from (3.2.5) with $(X, Y, Z) = (X, I, I)$ and (3.2.6) with $(X, Y, Z) = (I, I, X)$.

In the concrete examples of braided monoidal categories in this book which are not strict monoidal, the objects are vector spaces with additional structure and the monoidal structure is the same as for the underlying vector spaces. In these examples, the diagrams in (3.2.4) are clearly commutative. We will therefore view them as braided strict monoidal categories, where the equalities in (3.2.7) are to be interpreted as the commutative diagrams in (3.2.4).

Let $(\mathcal{C}, \otimes, I, c)$ be a braided strict monoidal category. The **dual category** of $(\mathcal{C}, \otimes, I, c)$ is the braided strict monoidal category $(\mathcal{C}^{\mathrm{op}}, \otimes, I, c^{\mathrm{op}})$ with braiding given by $c^{\mathrm{op}}_{X,Y} = (c^{\mathcal{C}}_{Y,X})^{\mathrm{op}}$ for objects X, Y.

The **mirror category** of $(\mathcal{C}, \otimes, I, c)$ is the braided strict monoidal category $\overline{\mathcal{C}} = (\mathcal{C}, \otimes, I, \overline{c})$, where $\overline{c}_{X,Y} = (c_{Y,X})^{-1}$ for all $X, Y \in \mathcal{C}$.

The **reversed category** of $(\mathcal{C}, \otimes, I, c)$ is the braided strict monoidal category $\mathcal{C}^{\mathrm{rev}} = (\mathcal{C}, \otimes^{\mathrm{rev}}, I, c^{\mathrm{rev}})$ with braiding $c^{\mathrm{rev}}_{X,Y} = c_{Y,X}$ for all $X, Y \in \mathcal{C}$. We note that by the left-right symmetry of the axioms, algebras, coalgebras, bialgebras and Hopf algebras in $\mathcal{C}$ are algebras, coalgebras, bialgebras and Hopf algebras in $\mathcal{C}^{\mathrm{rev}}$.

DEFINITION 3.2.2. If $\mathcal{C}$ and $\mathcal{D}$ are braided strict monoidal categories, then a monoidal functor (F, φ_0, φ) is **braided** if for all $X, Y \in \mathcal{C}$ the diagram

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "c_{F(X),F(Y)}"'] & F(X \otimes Y) \arrow[d, "F(c_{X,Y})"'] \\ F(Y) \otimes F(X) \arrow[r, "\varphi_{Y,X}"] & F(Y \otimes X) \end{tikzcd} \tag{3.2.8}$$

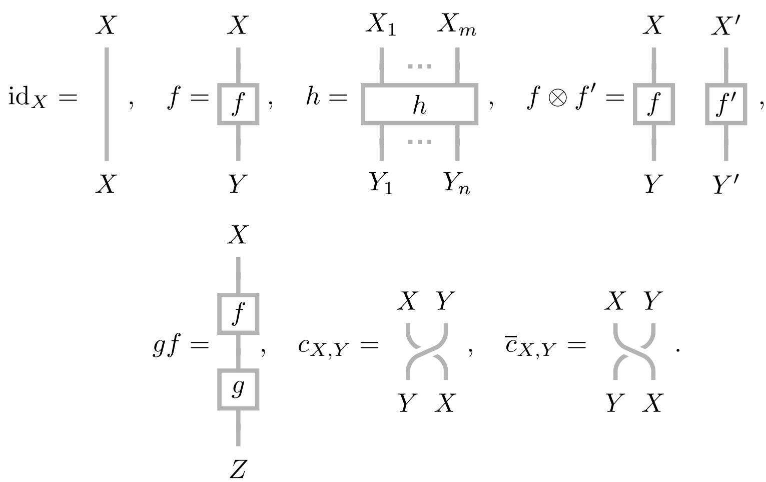
commutes. A **braided monoidal equivalence** (**isomorphism**, respectively) is a monoidal equivalence (isomorphism, respectively) (F, φ_0, φ) such that (F, φ_0, φ) is a braided monoidal functor.

REMARK 3.2.3. Sometimes it is useful to consider a more general situation. A **prebraiding** of $\mathcal{C}$ is a family $c = (c_{V,W} : V \otimes W \to W \otimes V)_{V,W \in \mathcal{C}}$ of natural morphisms (not assumed to be isomorphisms) satisfying (3.2.5), (3.2.6) and (3.2.7). Prebraided strict monoidal categories and prebraided monoidal functors, equivalences and isomorphisms are defined in the obvious way.

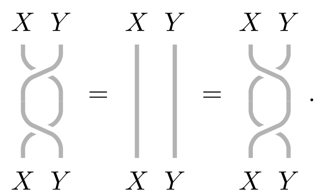
Let $\mathcal{C} = (\mathcal{C}, \otimes, I, c)$ be a braided strict monoidal category. We use the following convention for the graphical calculus. Diagrams are read from top to bottom. Let $f : X \to Y$, $g : Y \to Z$, $f' : X' \to Y'$ and

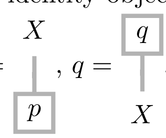
$$h : X_1 \otimes \cdots \otimes X_m \to Y_1 \otimes \cdots \otimes Y_n,$$

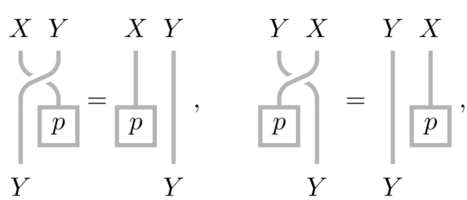
$m, n \geq 1$, be morphisms in $\mathcal{C}$. We denote the identity morphism $\mathrm{id}_X : X \to X$, the morphisms f, h, the tensor product $f \otimes f' : X \otimes X' \to Y \otimes Y'$, the composition $gf : X \to Z$, the braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ and the inverse braiding $\overline{c}_{X,Y}$ by

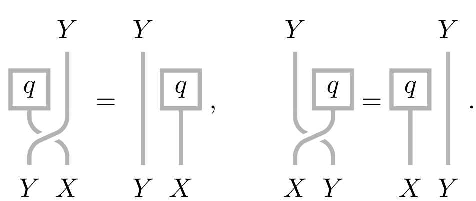


By definition of the inverse braiding,

(3.2.9) 

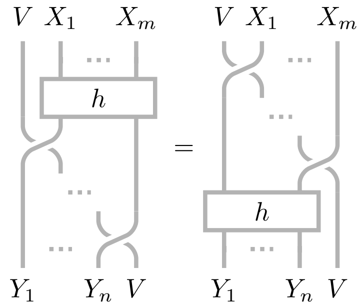
By (3.2.7), the braiding acts trivially on the identity object I. Hence for any morphisms $p : X \to I$, $q : I \to X$, denoted by $p =$ ,

(3.2.10) 

(3.2.11) 

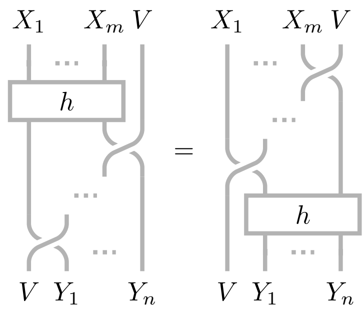
Let $V \in \mathcal{C}$. Axioms (3.2.5) and (3.2.6) and the naturality of the braiding (3.2.1) imply the following important rules.

(3.2.12)



,

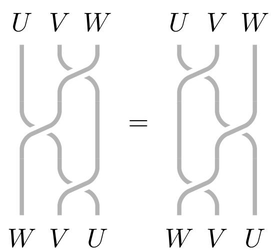
(3.2.13)



.

Let $U, V, W \in \mathcal{C}$. We note the special case of (3.2.12) with $h = c_{V,W}$:

(3.2.14)

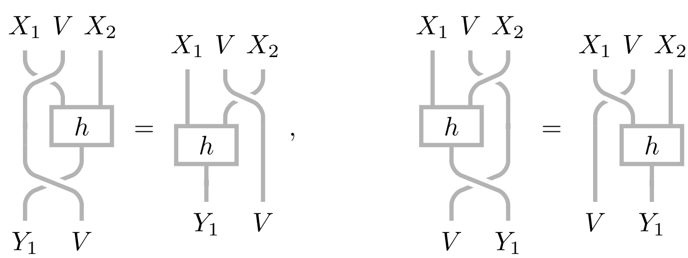


.

Let $U = V = W$. Then (3.2.14) is the braid equation $c_1c_2c_1 = c_2c_1c_2$, $c_1 = c_{V,V} \otimes \mathrm{id}$, $c_2 = \mathrm{id}_V \otimes c_{V,V}$. In knot theory, (3.2.9) and (3.2.14) are known as the second and the third Reidemeister move.

Here is an application of the rules above.

(3.2.15)



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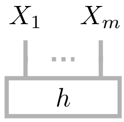
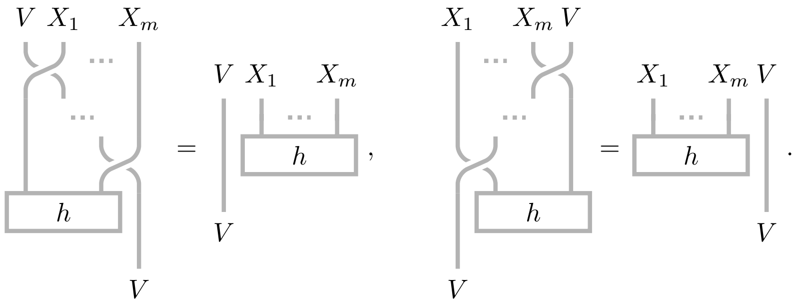
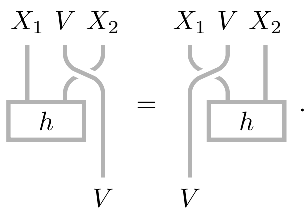
To prove the first equality in (3.2.15), apply (3.2.12) with the inverse braiding $\overline{c}$ to the lower part of the left-hand side, and then use (3.2.9); the second equality follows in the same way from (3.2.13).

Finally we want to mention the case of morphisms $h : X_1 \otimes \cdots \otimes X_m \to I$ which we denote by $h =$ 

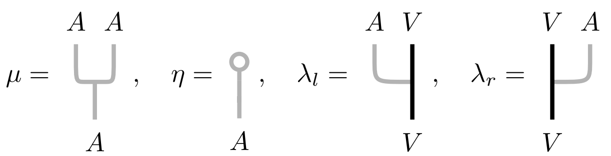
. By (3.2.12), (3.2.13) and (3.2.10),

(3.2.16)


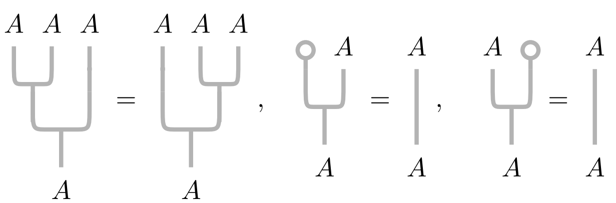
Moreover, by (3.2.15) and (3.2.10).

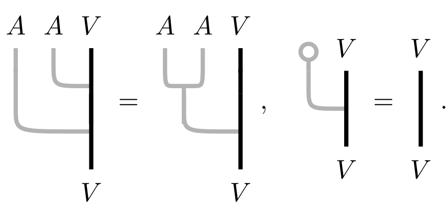
(3.2.17)


We denote the structure maps of an algebra (A, μ, η), a left A-module (V, λ_l), and a right A-module (V, λ_r) by



respectively. Then the axioms of an algebra and a left module are

(3.2.18)


(3.2.19)


PROPOSITION 3.2.4. *Let A, B, C, D be algebras and $\varphi : A \to C$, $\psi : B \to D$ algebra morphisms in $\mathcal{C}$, V a left A-module and W a left B-module in $\mathcal{C}$.*

(1) $(A \otimes B, \mu_{A\otimes B}, \eta_{A\otimes B})$ *is an algebra in* $\mathcal{C}$ *with unit* $\eta_A \otimes \eta_B$ *and multiplication*

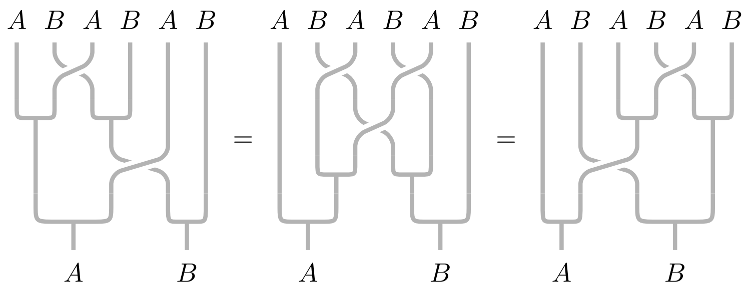
$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B.$$

(2) $\varphi \otimes \psi : A \otimes B \to C \otimes D$ *is an algebra morphism in* $\mathcal{C}$.
(3) $V \otimes W$ *is a left* $A \otimes B$*-module with module structure*

$$A \otimes B \otimes V \otimes W \xrightarrow{\mathrm{id}_A \otimes c_{B,V} \otimes \mathrm{id}_W} A \otimes V \otimes B \otimes W \xrightarrow{\lambda_V \otimes \lambda_W} V \otimes W.$$

(4) *The algebra structures on* $(A \otimes B) \otimes C$ *and* $A \otimes (B \otimes C)$ *defined by* (1) *coincide.*

Proof. (1) It is easy to see that $\eta_{A\otimes B}$ is a unit. To prove associativity we write $\mu = \mu_{A\otimes B}$. The equality $\mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu)$ is shown by



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where the first equality follows from associativity of A and from (3.2.13) with $h = \mu_B$, and the second from associativity of B and (3.2.12) with $h = \mu_A$.

(2) follows easily from (3.2.13) with $h = \psi$.

(3) follows from the proof in (1) by replacing the third pair (A, B) by (V, W), and the multiplications (μ_A, μ_B) by the module structures (λ_V, λ_W).

(4) The equality of the algebra structures is equivalent to the equality of the morphisms

$$B \otimes C \otimes A \otimes B \xrightarrow{\mathrm{id} \otimes c_{C,A\otimes B}} B \otimes A \otimes B \otimes C \xrightarrow{c_{B,A} \otimes \mathrm{id} \otimes \mathrm{id}} A \otimes B \otimes B \otimes C,$$

$$B \otimes C \otimes A \otimes B \xrightarrow{c_{B\otimes C,A} \otimes \mathrm{id}} A \otimes B \otimes C \otimes B \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes c_{C,B}} A \otimes B \otimes B \otimes C,$$

which follows easily from the axioms of a braiding. □

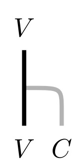
By Proposition 3.2.4, the category of algebras in $\mathcal{C}$ with algebra morphisms as morphisms is strict monoidal with $\otimes$ defined in Proposition 3.2.4(1) and (2). The unit object is the algebra $(I, \mathrm{id}, \mathrm{id})$.

We now dualize. The structure maps of a coalgebra (C, Δ, ε), a left C-comodule (V, δ_l) and a right C-comodule (V, δ_r) are denoted by

$\Delta =$ 

, $\varepsilon =$ 

, $\delta_l =$ 

, $\delta_r =$ 

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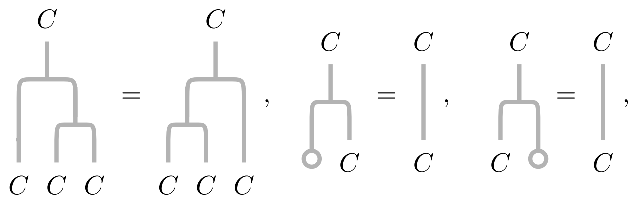
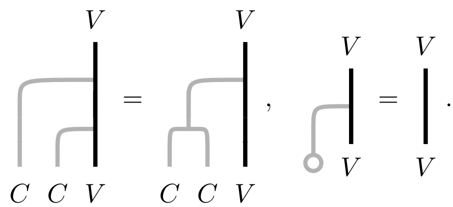
The axioms of a coalgebra and a left comodule are

(3.2.20)



(3.2.21)



We next show that the category of coalgebras in $\mathcal{C}$ with coalgebra morphisms as morphisms is strict monoidal. The unit object is $(I, \mathrm{id}, \mathrm{id})$.

PROPOSITION 3.2.5. *Let C, D, E, F be coalgebras and $\varphi : C \to E$, $\psi : D \to F$ coalgebra morphisms in $\mathcal{C}$, V a left C-comodule and W a left D-comodule in $\mathcal{C}$.*

(1) *$(C \otimes D, \Delta_{C\otimes D}, \varepsilon_{C\otimes D})$ is a coalgebra in $\mathcal{C}$ with counit $\varepsilon_C \otimes \varepsilon_D$ and comultiplication*

$$C \otimes D \xrightarrow{\Delta_C\otimes\Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}_C\otimes c_{C,D}\otimes\mathrm{id}_D} C \otimes D \otimes C \otimes D.$$

(2) *$\varphi \otimes \psi : C \otimes D \to E \otimes F$ is a coalgebra morphism in $\mathcal{C}$.*
(3) *$V \otimes W$ is a left $C \otimes D$-comodule with comodule structure*

$$V \otimes W \xrightarrow{\delta_V\otimes\delta_W} C \otimes V \otimes D \otimes W \xrightarrow{\mathrm{id}_C\otimes c_{V,D}\otimes\mathrm{id}_W} C \otimes D \otimes V \otimes W.$$

(4) *The coalgebra structures on $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ defined by* (1) *coincide.*

PROOF. Apply Proposition 3.2.4 to the dual braided category. $\square$

The tensor product of algebras and of coalgebras will always be equipped with the algebra and coalgebra structure of Propositions 3.2.4 and 3.2.5.

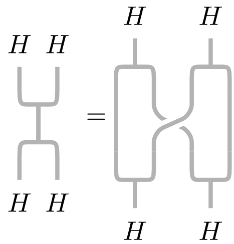
DEFINITION 3.2.6. Let $H \in \mathcal{C}$. Assume that (H, μ, η) is an algebra and (H, Δ, ε) is a coalgebra in $\mathcal{C}$. Then $H = (H, \mu, \eta, \Delta, \varepsilon)$ is a **bialgebra** in $\mathcal{C}$ if the following equivalent conditions hold.

(1) Δ and ε are algebra morphisms in $\mathcal{C}$.
(2) μ and η are coalgebra morphisms in $\mathcal{C}$.

Let H, H' be bialgebras in $\mathcal{C}$. A morphism $\varphi : H \to H'$ in $\mathcal{C}$ is a **morphism of bialgebras** if it is a morphism of algebras and of coalgebras in $\mathcal{C}$.

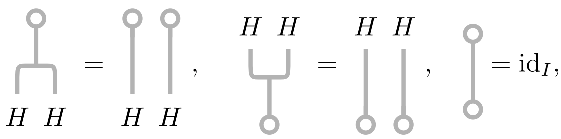
It is clear that (1) and (2) in Definition 3.2.6 are both equivalent to

(3.2.22)



and

(3.2.23)



where (3.2.23) are the pictures of the equations

$$\Delta_H \eta_H = \eta_{H\otimes H}, \quad \varepsilon_H \mu_H = \varepsilon_{H\otimes H}, \quad \varepsilon_H \eta_H = \mathrm{id}_I. \tag{3.2.24}$$

If $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra in $\mathcal{C}$, then $(H, \Delta^{\mathrm{op}}, \varepsilon^{\mathrm{op}}, \mu^{\mathrm{op}}, \eta^{\mathrm{op}})$ is a bialgebra in $\mathcal{C}^{\mathrm{op}}$. Indeed, reading the axioms of a bialgebra in the graphical calculus from bottom to top gives the same axioms up to a permutation.

The next proposition says that the category of left H-modules over a bialgebra H is strict monoidal.

Proposition 3.2.7. *Let H be a bialgebra in $\mathcal{C}$. The category ${}_H\mathcal{C}$ of left H-modules in $\mathcal{C}$ is strict monoidal, where*

(1) *for all $V, W \in {}_H\mathcal{C}$, the tensor product of V, W in ${}_H\mathcal{C}$ is the object $V \otimes W$ in $\mathcal{C}$ with module structure*

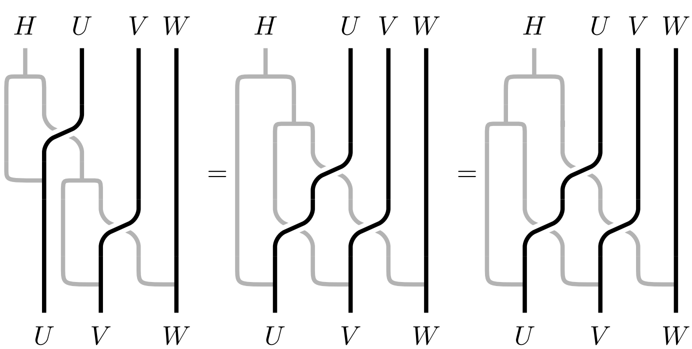
$$\lambda_{V\otimes W} = \big(H \otimes V \otimes W \xrightarrow{\Delta\otimes\mathrm{id}} H \otimes H \otimes V \otimes W \\ \xrightarrow{\mathrm{id}\otimes c_{H,V}\otimes\mathrm{id}} H \otimes V \otimes H \otimes W \xrightarrow{\lambda_V\otimes\lambda_W} V \otimes W\big),$$

(2) *the identity object is $(I, \varepsilon \otimes \mathrm{id})$, and*

(3) *for all morphisms f, g in ${}_H\mathcal{C}$, the tensor product $f \otimes g$ in $\mathcal{C}$ is the tensor product of f and g in ${}_H\mathcal{C}$.*

Proof. (a) Since $\Delta : H \to H \otimes H$ is an algebra morphisms, it follows from Proposition 3.2.4(3) that $(V \otimes W, \lambda_{V\otimes W})$ is a left H-module.

(b) Let $U, V, W \in {}_H\mathcal{C}$. Then $U \otimes (V \otimes W) = (U \otimes V) \otimes W$ as left H-modules, since



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where the first equality follows from (3.2.13) with $h = \Delta_H$, and the second from coassociativity of H.

(c) Let $f : V \to X$ and $g : W \to Y$ be morphisms in ${}_H\mathcal{C}$. Then the morphism $f \otimes g : V \otimes W \to X \otimes Y$ is left H-linear, since $(f \otimes \mathrm{id}_H)c_{H,V} = c_{H,X}(\mathrm{id}_H \otimes f)$, and since f, g are H-linear.

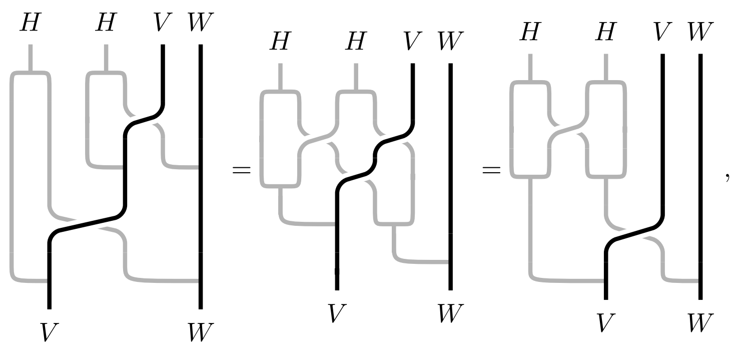
(d) It is easy to check that for all $V \in {}_H\mathcal{C}$, $I \otimes V = V = V \otimes I$, where I is the trivial left H-module with module structure $\varepsilon \otimes \mathrm{id}$. □

Conversely, the diagonal action in Proposition 3.2.7 can be used to check the bialgebra axiom.

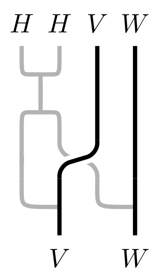
PROPOSITION 3.2.8. *Let H be an object of $\mathcal{C}$, and (H, μ, η) an algebra and (H, Δ, ε) a coalgebra in $\mathcal{C}$. Assume that $\varepsilon : H \to I$ is an algebra morphism. Then the following are equivalent.*

(1) *H is a bialgebra.*
(2) *Let $(V, \lambda_V), (W, \lambda_W) \in {}_H\mathcal{C}$. Then $(V \otimes W, \lambda_{V\otimes W}) \in {}_H\mathcal{C}$, where $\lambda_{V\otimes W}$ is the diagonal action defined in Proposition* 3.2.7.
(3) (2) *holds for $V = W = H$ with left module structure μ.*

PROOF. By Proposition 3.2.7, it suffices to prove that (3) implies (1). Let $V, W \in {}_H\mathcal{C}$. Then $\lambda_{V\otimes W}(\mathrm{id}_H \otimes \lambda_{V\otimes W})$ is equal to



where the first equality follows from naturality of the braiding (3.2.12) with $h = \lambda_V$, and from the module axioms for V and W, and the second from (3.2.13) with $h = \mu$. On the other hand,

$$\lambda_{V\otimes W}(\mu \otimes \mathrm{id}_V \otimes \mathrm{id}_W) =$$  .

Assume (3). Then $\lambda_{V\otimes W}(\mathrm{id}_H \otimes \lambda_{V\otimes W}) = \lambda_{V\otimes W}(\mu \otimes \mathrm{id}_V \otimes \mathrm{id}_W)$ for $V = W = H$. Hence

$$\lambda_{V\otimes W}(\mathrm{id}_H \otimes \lambda_{V\otimes W})(\mathrm{id}_H \otimes \mathrm{id}_H \otimes \eta \otimes \eta) =$$
$$\lambda_{V\otimes W}(\mu \otimes \mathrm{id}_V \otimes \mathrm{id}_W)(\mathrm{id}_H \otimes \mathrm{id}_H \otimes \eta \otimes \eta),$$

which is the bialgebra axiom (3.2.22). The first bialgebra axiom in (3.2.23), that is, Δ is unitary, follows since the H-module $H \otimes H$ is unitary. $\square$

PROPOSITION 3.2.9. *Let H be a bialgebra in $\mathcal{C}$. The category ${}^H\mathcal{C}$ of left H-comodules in $\mathcal{C}$ is strict monoidal, where*

(1) *for all $V, W \in {}^H\mathcal{C}$, the tensor product of V, W in ${}^H\mathcal{C}$ is the object $V \otimes W$ in $\mathcal{C}$ with comodule structure*

$$\delta_{V\otimes W} = \big(V \otimes W \xrightarrow{\delta_V \otimes \delta_W} H \otimes V \otimes H \otimes W$$
$$\xrightarrow{\mathrm{id}\otimes c_{V,H}\otimes \mathrm{id}} H \otimes H \otimes V \otimes W \xrightarrow{\mu\otimes\mathrm{id}} H \otimes V \otimes W\big),$$

(2) *the identity object is $(I, \eta \otimes \mathrm{id})$, and*

(3) *for all morphisms f, g in ${}^H\mathcal{C}$, the tensor product $f \otimes g$ in $\mathcal{C}$ is the tensor product of f and g in ${}^H\mathcal{C}$.*

PROOF. Apply Proposition 3.2.7 to the dual category. $\square$

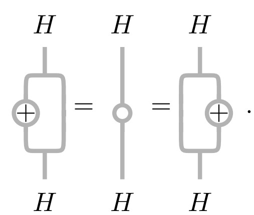
We note that Propositions 3.2.7 and 3.2.9 have obvious versions for right modules and for right comodules.

DEFINITION 3.2.10. Let H be a bialgebra in $\mathcal{C}$, and $\mathcal{S} : H \to H$ a morphism in $\mathcal{C}$. Then $H = (H, \mathcal{S})$ is a **Hopf algebra** with **antipode** $\mathcal{S}$, if $\mathcal{S}$ is the convolution inverse of id_H in the monoid $\mathrm{Hom}_{\mathcal{C}}(H, H)$.

The antipode $\mathcal{S} : H \to H$ of a Hopf algebra H in $\mathcal{C}$, and its inverse $\mathcal{S}^{-1}$ if $\mathcal{S}$ is an isomorphism in $\mathcal{C}$, are denoted by

$\mathcal{S} =$  , $\mathcal{S}^{-1} =$  .

Thus the axioms of the antipode are

(3.2.25) 

Let $(H, \mu, \eta, \Delta, \varepsilon, \mathcal{S})$ be a Hopf algebra in $\mathcal{C}$. Then $(H, \Delta^{\mathrm{op}}, \varepsilon^{\mathrm{op}}, \mu^{\mathrm{op}}, \eta^{\mathrm{op}}, \mathcal{S}^{\mathrm{op}})$ is a Hopf algebra in $\mathcal{C}^{\mathrm{op}}$.

LEMMA 3.2.11. *Let H, H' be Hopf algebras, and $\varphi : H \to H'$ a morphism of bialgebras in $\mathcal{C}$. Then $\mathcal{S}_{H'}\varphi = \varphi\mathcal{S}_H$.*

PROOF. It is easy to see that in the convolution algebra $\mathrm{Hom}_{\mathcal{C}}(H, H')$,

$$\mathcal{S}_{H'}\varphi * \varphi = \varphi * \mathcal{S}_{H'}\varphi = \eta\varepsilon,$$

since φ is a morphism of coalgebras. By duality,

$$\varphi\mathcal{S}_H * \varphi = \varphi * \varphi\mathcal{S}_H = \eta\varepsilon,$$

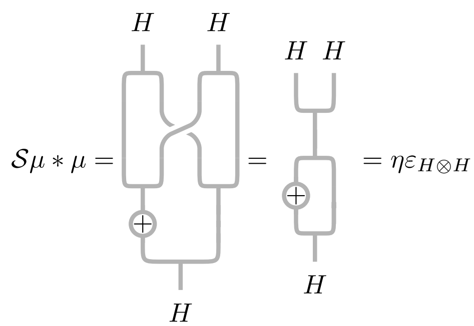
since φ is a morphism of algebras. Hence φ is invertible in the convolution algebra with inverse $\mathcal{S}_{H'}\varphi = \varphi\mathcal{S}_H$. $\square$

Proposition 3.2.12. *Let* $(H, \mu, \eta, \Delta, \varepsilon, \mathcal{S})$ *be a Hopf algebra in* $\mathcal{C}$. *Then*

(1) $c_{H\otimes H}(\mathcal{S}\otimes\mathcal{S}) = (\mathcal{S}\otimes\mathcal{S})c_{H,H}$,
(2) $\mathcal{S}\mu = \mu c_{H,H}(\mathcal{S}\otimes\mathcal{S})$,
(3) $\Delta\mathcal{S} = (\mathcal{S}\otimes\mathcal{S})c_{H,H}\Delta$,
(4) $\mathcal{S}\eta = \eta$,
(5) $\varepsilon\mathcal{S} = \varepsilon$.

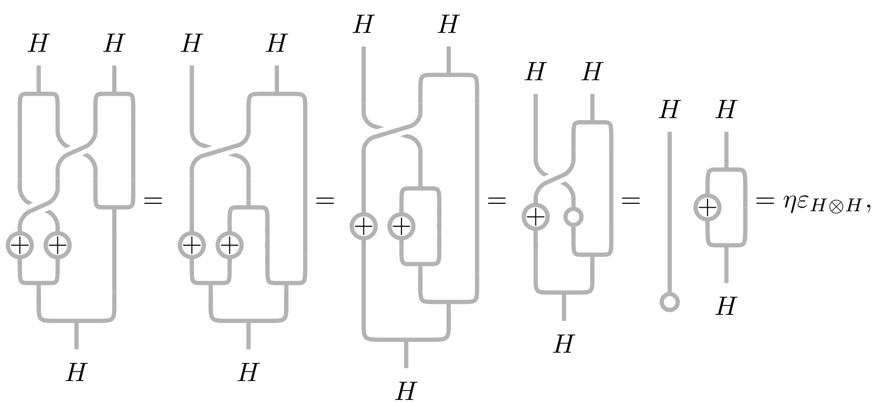
Proof. (1) follows since the braiding is a natural transformation.

(2) We prove (2) by showing that both sides of (2) are convolution inverse to μ in $\mathrm{Hom}_{\mathcal{C}}(H\otimes H, H)$. This is easy for $\mathcal{S}\mu$:

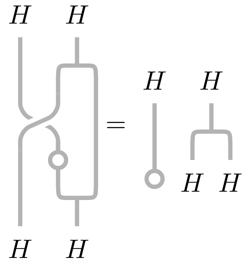


by (3.2.22), (3.2.25), and (3.2.23). The equality $\mu * \mathcal{S}\mu = \eta\varepsilon_{H\otimes H}$ follows in the same way.

We compute $\mu(\mathcal{S}\otimes\mathcal{S})c_{H,H} * \mu$.



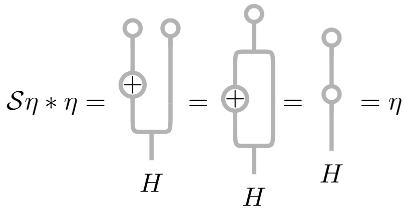
where the first equality follows from (3.2.13) with $h = \Delta$, the second from associativity, and the third and the last from the axiom of the antipode. To prove the fourth equality, note that



by the algebra axiom for the unit and (3.2.10).

The equality $\mu * \mu(\mathcal{S} \otimes \mathcal{S})c_{H,H} = \eta\varepsilon_{H\otimes H}$ follows similarly.

(4) In the convolution algebra $\mathrm{Hom}_{\mathcal{C}}(I, H)$ with product $*$,

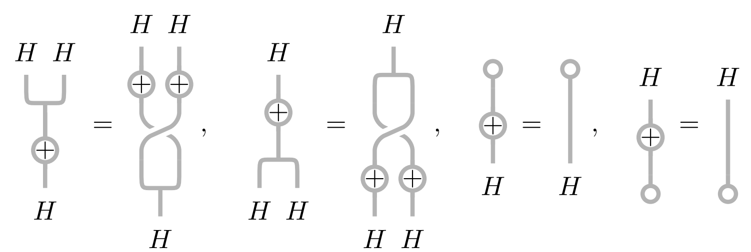


by (3.2.23) and the axiom of the antipode. Hence $\mathcal{S}\eta = \eta$, since the unit element in the algebra $\mathrm{Hom}_{\mathcal{C}}(I, H)$ is $\eta\varepsilon_I = \eta$.

(3) and (5) follow by duality from (2) and (4). □

The pictures for the rules of the antipode in Proposition 3.2.12 are

(3.2.26)



REMARK 3.2.13. Braided monoidal functors preserve bialgebras and Hopf algebras. They are an important machinery for constructing new Hopf algebras.

Let $\mathcal{D}$ be a braided strict monoidal category, and $(F, \varphi) : \mathcal{C} \to \mathcal{D}$ a braided monoidal functor.

If A, B are algebras in $\mathcal{C}$, then

$$\varphi_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$$

is an algebra morphisms in $\mathcal{D}$, where $F(A), F(B)$ and $F(A \otimes B)$ are the algebras $(F, \varphi)(A)$, $(F, \varphi)(B)$, and $(F, \varphi)(A \otimes B)$, respectively. In the same way, for coalgebras C, D in $\mathcal{C}$,

$$\varphi_{C,D} : F(C) \otimes F(D) \to F(C \otimes D)$$

is a morphisms of coalgebras in $\mathcal{D}$.

If $H = (H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra in $\mathcal{C}$, then

$$(F, \varphi)(H) = (F(H), F(\mu)\varphi_{H,H}, F(\eta), \varphi^{-1}_{H,H}F(\Delta), F(\varepsilon))$$

is a bialgebra in $\mathcal{D}$. If H has an antipode $\mathcal{S}$, then $(F,\varphi)(H)$ is a Hopf algebra with antipode $F(\mathcal{S})$.

We next extend the notions of the opposite algebra and coopposite coalgebra to braided monoidal categories. This can be done in different ways. We fix one of the possible definitions.

DEFINITION 3.2.14. For a bialgebra $H = (H,\mu,\eta,\Delta,\varepsilon)$ in $\mathcal{C}$ let

$$H^{\mathrm{op}} = (H, \mu\overline{c}_{H,H}, \eta, \Delta, \varepsilon),$$
$$H^{\mathrm{cop}} = (H, \mu, \eta, \overline{c}_{H,H}\Delta, \varepsilon).$$

It turns out that for a bialgebra H, H^{op} and H^{cop} are not bialgebras in $\mathcal{C}$ but in $\overline{\mathcal{C}}$.

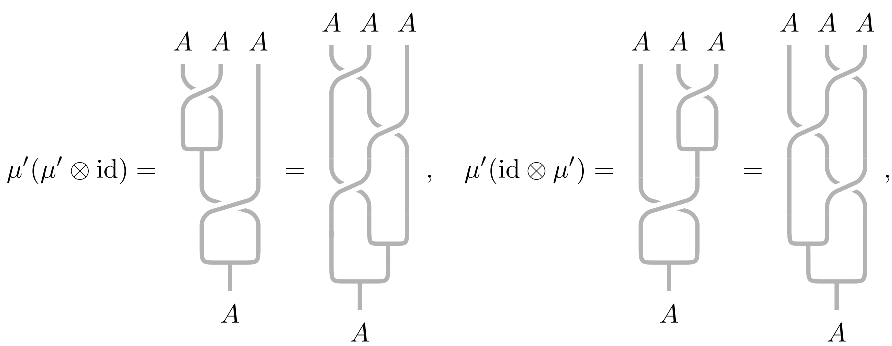
PROPOSITION 3.2.15. (1) *Let (A,μ,η) and (C,Δ,ε) be an algebra and a coalgebra in $\mathcal{C}$. Then $(A,\mu c_{A,A},\eta)$ is an algebra and $(C, c_{C,C}\Delta,\varepsilon)$ is a coalgebra in $\mathcal{C}$.*

(2) *Let H be a bialgebra in $\mathcal{C}$. Then H^{op} and H^{cop} are bialgebras in $\overline{\mathcal{C}}$.*

(3) *Let H be a Hopf algebra in $\mathcal{C}$. Then the following are equivalent.*
   (a) *The antipode $\mathcal{S}$ of H is an isomorphism in $\mathcal{C}$.*
   (b) *H^{op} is a Hopf algebra in $\overline{\mathcal{C}}$.*
   (c) *H^{cop} is a Hopf algebra in $\overline{\mathcal{C}}$.*

In this case, $\mathcal{S}^{-1}$ is the antipode of H^{op} and of H^{cop}.

PROOF. (1) We prove associativity of $\mu' = \mu c_{A,A}$.



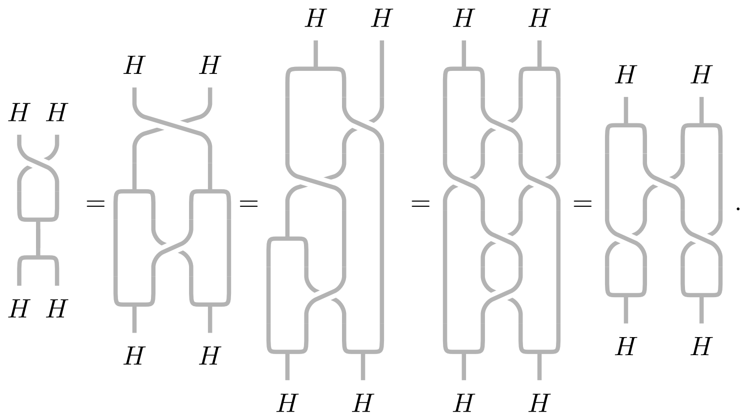
by (3.2.13) and (3.2.12) with $h = \mu$. Hence associativity of $\mu c_{A,A}$ follows from associativity of A and (3.2.14).

By (3.2.11), η is a unit for $\mu c_{A,A}$, since η is a unit for μ.

The coalgebra axioms for $(C, c_{C,C}\Delta,\varepsilon)$ follow by duality.

(2) By assumption, H is an algebra and a coalgebra in $\mathcal{C}$, and hence in $\overline{\mathcal{C}}$. By (1), H^{op} is an algebra and a coalgebra in $\overline{\mathcal{C}}$. We prove the bialgebra axiom (3.2.22)

for H^{op} in $\overline{\mathcal{C}}$.



The first equality in this proof follows from the bialgebra axiom (3.2.22) for H, the second and the third from (3.2.13) and (3.2.12) with $h = \mu$, and the last from (3.2.9).

The bialgebra axiom (3.2.23) for H^{op} is easy to check, and the claim for H^{cop} follows by duality.

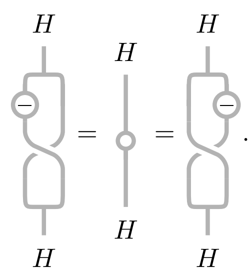
(3) Assume (a). We show that $\mu \overline{c}_{H,H}(\mathcal{S}^{-1} \otimes \mathrm{id})\Delta = \eta\varepsilon$, which is half of the antipode axiom for H^{op}. By Proposition 3.2.12(3), $c_{H,H}(\mathcal{S} \otimes \mathcal{S})\Delta = \Delta\mathcal{S}$. Hence $\mu \overline{c}_{H,H}(\mathcal{S}^{-1} \otimes \mathrm{id})\Delta = \mu(\mathrm{id} \otimes \mathcal{S})\Delta\mathcal{S}^{-1} = \eta\varepsilon$ by the properties of the antipode of H. The other half of the axiom of the antipode follows similarly. Thus (a) implies (b), and $\mathcal{S}^{-1}$ is the antipode of H^{op}.

Assume (b) and let T be the antipode of H^{op}. Similar computations as in the previous paragraph show that $T\mathcal{S}$ and $\mathcal{S}T$ are convolution inverse to $\mathcal{S}$. Hence $T = \mathcal{S}^{-1}$. Thus (b) implies (a).

The equivalence of (a) and (c) follows by duality. □

Let H be a Hopf algebra in $\mathcal{C}$ with antipode $\mathcal{S}$, and assume that $\mathcal{S}$ is an isomorphism. By Proposition 3.2.15, $\mathcal{S}^{-1}$ is the antipode of H^{op}. Hence

(3.2.27)



Corollary 3.2.16. *Let H be a Hopf algebra in $\mathcal{C}$, and assume that the antipode $\mathcal{S}$ of H is an isomorphism in $\mathcal{C}$.*

(1) *$\mathcal{S} : H^{\mathrm{op}} \to H^{\mathrm{cop}}$ is an isomorphism of Hopf algebras in $\overline{\mathcal{C}}$.*
(2) *$(H^{\mathrm{op}})^{\mathrm{cop}}$ and $(H^{\mathrm{cop}})^{\mathrm{op}}$ are Hopf algebras in $\mathcal{C}$ with antipode $\mathcal{S}$, and*

$$\mathcal{S} : H \to (H^{\mathrm{cop}})^{\mathrm{op}}, \quad \mathcal{S} : (H^{\mathrm{op}})^{\mathrm{cop}} \to H$$

are isomorphisms of Hopf algebras in $\mathcal{C}$.

Proof. This follows from Propositions 3.2.15 and 3.2.12. □

REMARK 3.2.17. Let A, B be bialgebras in $\mathcal{C}$. Then in general $A \otimes B$ (with the algebra and coalgebra structure of the tensor product) is not a bialgebra in $\mathcal{C}$, see Proposition 1.10.12.

3.3. Modules and comodules over braided Hopf algebras

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, c)$ be a braided strict monoidal category.

The braiding can be used to change the sides of modules and comodules.

PROPOSITION 3.3.1. *Let H be a Hopf algebra in $\mathcal{C}$, and assume that the antipode $\mathcal{S}$ of H is an isomorphism in $\mathcal{C}$. Then the functors*

$$F_{lr} : {}_H\mathcal{C} \to \overline{\mathcal{C}}_{H^{\mathrm{op}}},\ (V, \lambda) \mapsto (V, \lambda \overline{c}_{V,H}),$$

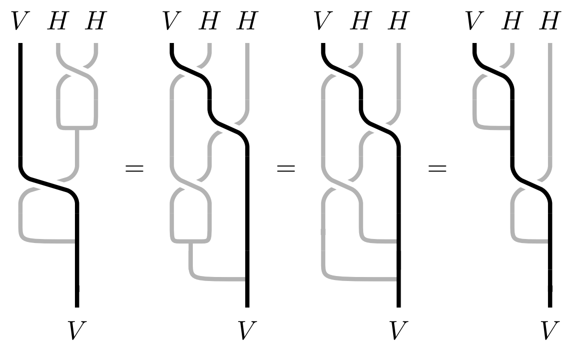
$$F_{rl} : \mathcal{C}_H \to {}_{H^{\mathrm{op}}}\overline{\mathcal{C}},\ (V, \lambda) \mapsto (V, \lambda \overline{c}_{H,V}),$$

where morphisms f are mapped onto f, are strict monoidal isomorphisms.

PROOF. Let $F = F_{lr}$. We first show that F is a strict monoidal functor.

Let (V, λ) be a left H-module, and $\lambda_r = \lambda \overline{c}_{V,H}$. Then (V, λ_r) is a right H^{op}-module in $\overline{\mathcal{C}}$ (and in $\mathcal{C}$), since

$\lambda_r(\mathrm{id} \otimes \mu) =$

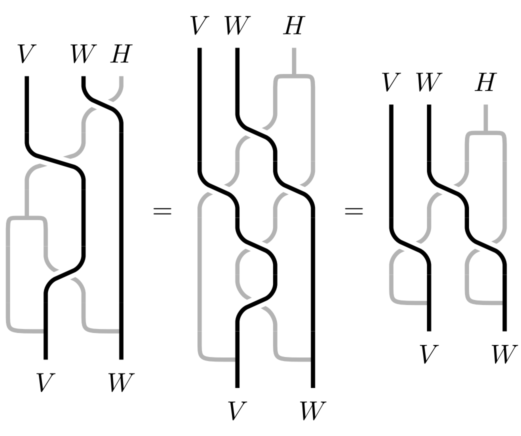


$= \lambda_r(\lambda_r \otimes \mathrm{id}),$

where the second equality follows from (3.2.12) with $h = \mu \overline{c}_{H,H}$, the third from the module axiom for (V, λ), and the fourth from (3.2.13), where h is the upper module action λ. Note that (V, λ_r) is unitary by (3.2.11).

To show that F is strict monoidal, let V, W be left H-modules. Then

$\lambda_{F(V \otimes W)} =$



$= \lambda_{F(V) \otimes F(W)},$

where the second equality follows from (3.2.12) with $h = \Delta$, and the third from (3.2.9). By (3.2.7), $F(I) = I$.

In the same way it follows that F_{rl} is a strict monoidal functor. Both functors are isomorphisms, since F_{lr} for H and F_{rl} for H^{op} are inverse functors. $\square$

The next proposition follows by duality from Proposition 3.3.1.

PROPOSITION 3.3.2. *Let H be a Hopf algebra in $\mathcal{C}$, and assume that the antipode $\mathcal{S}$ of H is an isomorphism in $\mathcal{C}$. Then the functors*

$$F^{lr} : {}^H\mathcal{C} \to \overline{\mathcal{C}}^{H^{\mathrm{cop}}},\ (V, \delta) \mapsto (V, \overline{c}_{H,V}\delta),$$
$$F^{rl} : \mathcal{C}^H \to {}^{H^{\mathrm{cop}}}\overline{\mathcal{C}},\ (V, \delta) \mapsto (V, \overline{c}_{V,H}\delta),$$

where morphisms f are mapped onto f, are strict monoidal isomorphisms.

Let A, B be algebras in $\mathcal{C}$, and $\varphi : A \to B$ an algebra morphism. We define the obvious **restriction functors**

$$\varphi_\downarrow : {}_B\mathcal{C} \to {}_A\mathcal{C}, \quad (V, \lambda) \mapsto (V, \lambda(\varphi \otimes \mathrm{id})), \tag{3.3.1}$$
$$\varphi_\downarrow : \mathcal{C}_B \to \mathcal{C}_A, \quad (V, \lambda) \mapsto (V, \lambda(\mathrm{id} \otimes \varphi)). \tag{3.3.2}$$

For coalgebras C, D and coalgebra morphisms $\varphi : C \to D$ in $\mathcal{C}$ we let

$$\varphi^\uparrow : {}^C\mathcal{C} \to {}^D\mathcal{C}, \quad (V, \delta) \mapsto (V, (\varphi \otimes \mathrm{id})\delta), \tag{3.3.3}$$
$$\varphi^\uparrow : \mathcal{C}^C \to \mathcal{C}^D, \quad (V, \delta) \mapsto (V, (\mathrm{id} \otimes \varphi)\delta). \tag{3.3.4}$$

In each case, morphisms f are mapped onto f. It is clear that $\varphi_\downarrow$ and $\varphi^\uparrow$ are well-defined functors. If A, B are bialgebras, and $\varphi : A \to B$ is a bialgebra morphism, then the functors $\varphi_\downarrow$ and $\varphi^\uparrow$ are strict monoidal.

We use the notation $c^{-1} = \overline{c}$ for the braiding of $\overline{\mathcal{C}}$.

DEFINITION 3.3.3. Let H be a Hopf algebra in $\mathcal{C}$, and assume that the antipode $\mathcal{S}$ of H is an isomorphism.

(1) For $(V, \lambda) \in {}_H\mathcal{C}$, let

$$\lambda_\pm = \big(V \otimes H \xrightarrow{\mathrm{id}\otimes\mathcal{S}^{\pm 1}} V \otimes H \xrightarrow{(c^{\pm 1})_{V,H}} H \otimes V \xrightarrow{\lambda} V\big).$$

(2) For $(V, \lambda) \in \mathcal{C}_H$, let

$$\lambda_\pm = \big(H \otimes V \xrightarrow{\mathcal{S}^{\pm 1}\otimes\mathrm{id}} H \otimes V \xrightarrow{(c^{\pm 1})_{H,V}} V \otimes H \xrightarrow{\lambda} V\big).$$

COROLLARY 3.3.4. *Let H be a Hopf algebra in $\mathcal{C}$ such that the antipode $\mathcal{S}$ of H is an isomorphism in $\mathcal{C}$. Then the functors changing sides of modules in $\mathcal{C}$,*

$${}_H\mathcal{C} \xrightarrow{F_{lr}^-} \overline{\mathcal{C}}_{H^{\mathrm{cop}}}, \quad {}_{H^{\mathrm{cop}}}\overline{\mathcal{C}} \xrightarrow{F_{lr}^+} \mathcal{C}_H,$$
$$\mathcal{C}_H \xrightarrow{F_{rl}^-} {}_{H^{\mathrm{cop}}}\overline{\mathcal{C}}, \quad \overline{\mathcal{C}}_{H^{\mathrm{cop}}} \xrightarrow{F_{rl}^+} {}_H\mathcal{C},$$

with $F_{lr}^\pm(V, \lambda) = (V, \lambda_\pm)$ for all modules $(V, \lambda) \in {}_H\mathcal{C}$, and $F_{rl}^\pm(V, \lambda) = (V, \lambda_\pm)$ for all modules $(V, \lambda) \in \mathcal{C}_H$, and where morphisms f are mapped onto f, are strict monoidal isomorphisms.

PROOF. By Corollary 3.2.16, $\mathcal{S}^{-1} : H^{\mathrm{cop}} \to H^{\mathrm{op}}$ is an isomorphism of Hopf algebras in $\overline{\mathcal{C}}$. Since

$$F_{lr}^- = \big({}_H\mathcal{C} \xrightarrow{F_{lr}} \overline{\mathcal{C}}_{H^{\mathrm{op}}} \xrightarrow{(\mathcal{S}^{-1})_\downarrow} \overline{\mathcal{C}}_{H^{\mathrm{cop}}}\big),$$

it follows from Proposition 3.3.1 that F_{lr}^{-} is a strict monoidal isomorphism. The same argument works for F_{rl}^{-}. It follows that F_{lr}^{+} and F_{rl}^{+} are strict monoidal isomorphisms, since F_{lr}^{+} and F_{rl}^{+} are the functors F_{lr}^{-} and F_{rl}^{-} with H replaced by H^{cop}. $\square$

DEFINITION 3.3.5. Let H be a Hopf algebra in $\mathcal{C}$, and assume that the antipode $\mathcal{S}$ of H is an isomorphism.

(1) For $(V,\delta) \in {}^H\mathcal{C}$, let

$$\delta_{\pm} = \big(V \xrightarrow{\delta} H \otimes V \xrightarrow{(c^{\pm 1})_{H,V}} V \otimes H \xrightarrow{\mathrm{id}\otimes\mathcal{S}^{\pm 1}} V \otimes H\big).$$

(2) For $(V,\lambda) \in \mathcal{C}^H$, let

$$\delta_{\pm} = \big(V \xrightarrow{\delta} V \otimes H \xrightarrow{(c^{\pm 1})_{V,H}} H \otimes V \xrightarrow{\mathcal{S}^{\pm 1}\otimes\mathrm{id}} H \otimes V\big).$$

The next result follows by duality from Corollary 3.3.4.

COROLLARY 3.3.6. *Let H be a Hopf algebra in $\mathcal{C}$ such that the antipode $\mathcal{S}$ of H is an isomorphism in $\mathcal{C}$. Then the functors changing sides of comodules in $\mathcal{C}$,*

$$ {}^H\mathcal{C} \xrightarrow{F_{-}^{lr}} \overline{\mathcal{C}}^{H^{\mathrm{op}}}, \quad {}^{H^{\mathrm{op}}}\overline{\mathcal{C}} \xrightarrow{F_{+}^{lr}} \mathcal{C}^H,$$

$$ \mathcal{C}^H \xrightarrow{F_{-}^{rl}} {}^{H^{\mathrm{op}}}\overline{\mathcal{C}}, \quad \overline{\mathcal{C}}^{H^{\mathrm{op}}} \xrightarrow{F_{+}^{rl}} {}^H\mathcal{C},$$

with $F_{\pm}^{lr}(V,\delta) = (V,\delta_{\pm})$ for all comodules $(V,\delta) \in {}^H\mathcal{C}$, and $F_{\pm}^{rl}(V,\delta) = (V,\delta_{\pm})$ for all comodules $(V,\delta) \in \mathcal{C}^H$, and where morphisms f are mapped onto f, are strict monoidal isomorphisms.

A fundamental construction in Hopf algebra theory is the module structure over the dual algebra C^* of a comodule over a coalgebra C in Definition 2.2.15. This construction is based on the evaluation pairing $C^* \otimes C \to \Bbbk$. To generalize it to braided categories we formulate the natural axioms for an abstract pairing.

DEFINITION 3.3.7. Let A and B be bialgebras in $\mathcal{C}$. A morphism

$$p : A \otimes B \to I$$

in $\mathcal{C}$ is called a **Hopf pairing**, if the following diagrams commute.

$$\begin{tikzcd}
A \otimes B \otimes B \arrow[r, "\mathrm{id}\otimes\mu_B"] \arrow[d, "\Delta_A\otimes\mathrm{id}\otimes\mathrm{id}"'] & A \otimes B \arrow[dd, "p"] \\
A \otimes A \otimes B \otimes B \arrow[d, "\mathrm{id}\otimes p\otimes\mathrm{id}"'] & \\
A \otimes B \arrow[r, "p"] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes B \arrow[r, "\mu_A\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\Delta_B"'] & A \otimes B \arrow[dd, "p"] \\
A \otimes A \otimes B \otimes B \arrow[d, "\mathrm{id}\otimes p\otimes\mathrm{id}"'] & \\
A \otimes B \arrow[r, "p"] & I
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes B \arrow[rr, "\eta_A\otimes\mathrm{id}"] \arrow[dr, "\varepsilon_B"'] & & A \otimes B \arrow[dl, "p"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes I \arrow[rr, "\mathrm{id}\otimes\eta_B"] \arrow[dr, "\varepsilon_A"'] & & A \otimes B \arrow[dl, "p"] \\
& I &
\end{tikzcd}$$

Let A, B be bialgebras in $\mathcal{M}_{\Bbbk}$, and

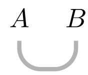
$$p : A \otimes B \to \Bbbk,\ a \otimes b \mapsto p(a, b) = p(a \otimes b)$$

a linear map. In Sweedler notation the axioms of a Hopf pairing are

$$p(a, bb') = p(a_{(1)}, b')p(a_{(2)}, b), \qquad p(1, b) = \varepsilon(b),$$
$$p(aa', b) = p(a, b_{(2)})p(a', b_{(1)}), \qquad p(a, 1) = \varepsilon(a)$$

for all $a, a' \in A$ and $b, b' \in B$. Thus for a finite-dimensional bialgebra H, the evaluation map $H^{*\mathrm{op}\,\mathrm{cop}} \otimes H \to \Bbbk$ is a Hopf pairing.

In Section 7.2 we will define an important Hopf pairing between the Nichols algebra of the dual of a Yetter-Drinfeld module V and the Nichols algebra of V.

A Hopf pairing $p : A \otimes B \to I$ is denoted by $p =$ 

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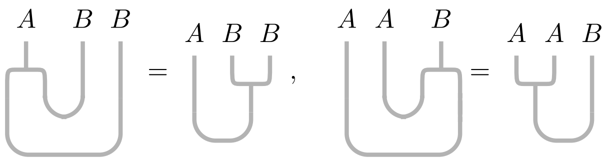
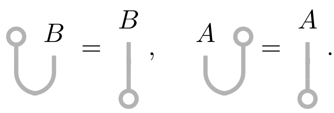
By definition of a Hopf pairing,

(3.3.5) 

(3.3.6) 

In addition we note the rules (3.2.16) and (3.2.17) when $h = p$ is a Hopf pairing.

PROPOSITION 3.3.8. *Let A and B be Hopf algebras in $\mathcal{C}$, and $p : A \otimes B \to I$ a Hopf pairing.*

(1) $p(\mathcal{S}_A \otimes \mathrm{id}) = p(\mathrm{id} \otimes \mathcal{S}_B) : A \otimes B \to I.$

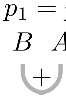
(2) $p^+ = \big(B \otimes A \xrightarrow{\mathcal{S}_B \otimes \mathcal{S}_A} B \otimes A \xrightarrow{c_{B,A}} A \otimes B \xrightarrow{p} I\big)$ *is a Hopf pairing.*

(3) *Assume that the antipodes of A and B are isomorphisms. Then*

$$p^{\mathrm{cop}} = p(\mathcal{S}_A^{-1} \otimes \mathrm{id}_B) : A^{\mathrm{cop}} \otimes B^{\mathrm{cop}} \to I$$

is a Hopf pairing of $A^{\mathrm{cop}}, B^{\mathrm{cop}}$ in $\overline{\mathcal{C}}$, and $p^{\mathrm{cop}} = p(\mathrm{id}_A \otimes \mathcal{S}_B^{-1})$.

PROOF. (1) For all $f, g \in \mathrm{Hom}_{\mathcal{C}}(A \otimes B, I)$ let $f \cdot g = g(\mathrm{id} \otimes f \otimes \mathrm{id})(\Delta_A \otimes \Delta_B)$. Since $\mathcal{C}$ is a monoidal category and A, B are coalgebras in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A \otimes B, I)$ is a monoid with product $\cdot$ and unit $\epsilon = \varepsilon_A \otimes \varepsilon_B$. Let $p_1 = p(\mathcal{S}_A \otimes \mathrm{id})$ and $p_2 = p(\mathrm{id} \otimes \mathcal{S}_B)$. Then $p_1 \cdot p = \varepsilon_A \otimes \varepsilon_B = p \cdot p_2$, hence $p_1 = p_2$.

(2) See Figure 3.3.1 with $p^+ =$ 

. The first equality follows from the definition of p^+ and (3.2.26), the second from (3.2.12) with $h = \mu_A c_{A,A}$, the third from axiom (3.3.5) of a Hopf pairing, the fourth from (3.2.13) with $h = \Delta_B$ and (3.2.26), the fifth from (3.2.16) with $h = p$, and finally the sixth from (3.2.16) with $h = pc_{B,A}$.

The second equation in (3.3.5) is shown in the same way, and (3.3.6) is easy to check.

(3) The first part of the claim follows from the rules of the antipode in Proposition 3.2.12, and the second from (1). □

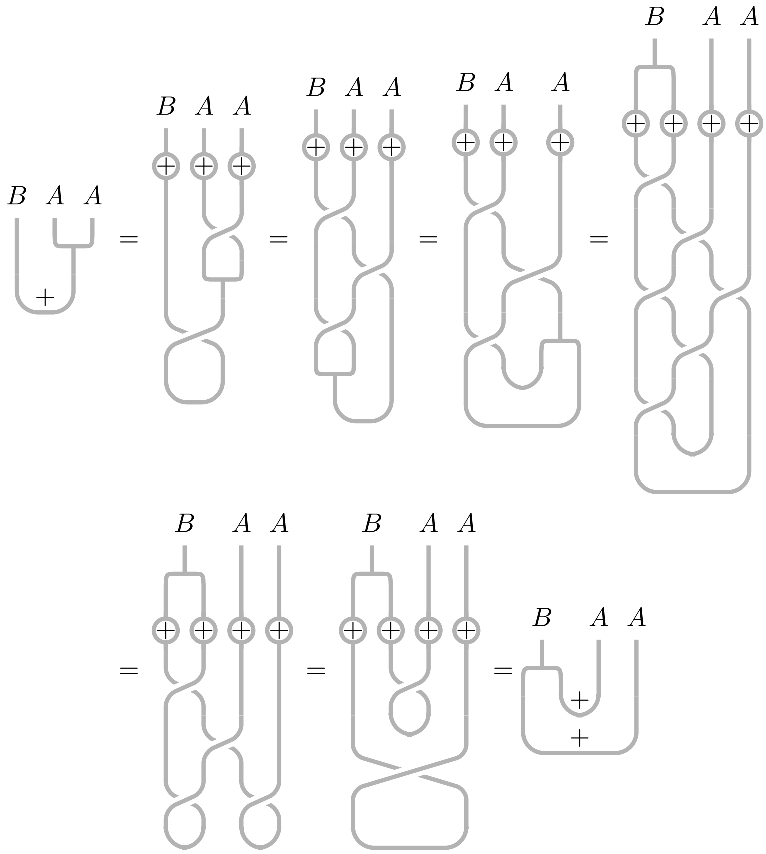


FIGURE 3.3.1. Proof that p^+ is a Hopf pairing

PROPOSITION 3.3.9. *Let A and B be bialgebras in $\mathcal{C}$, and $p : A \otimes B \to I$ a Hopf pairing. The functors*

$$D^l : {}^{B^{\mathrm{op}}}\overline{\mathcal{C}} \to {}_A\mathcal{C},\ (V,\delta) \mapsto (V,\lambda), \qquad \overline{D}^l : {}^B\mathcal{C} \to {}_{A^{\mathrm{cop}}}\overline{\mathcal{C}},\ (V,\delta) \mapsto (V,\lambda),$$

$$\textit{with } \lambda = \big(A \otimes V \xrightarrow{\mathrm{id}\otimes\delta} A \otimes B \otimes V \xrightarrow{p\otimes\mathrm{id}} V\big),$$

$$D^r : \overline{\mathcal{C}}^{A^{\mathrm{op}}} \to \mathcal{C}_B,\ (V,\delta) \mapsto (V,\lambda), \qquad \overline{D}^r : \mathcal{C}^A \to \overline{\mathcal{C}}_{B^{\mathrm{cop}}},\ (V,\delta) \mapsto (V,\lambda),$$

$$\textit{with } \lambda = \big(V \otimes B \xrightarrow{\delta\otimes\mathrm{id}} V \otimes A \otimes B \xrightarrow{\mathrm{id}\otimes p} V\big),$$

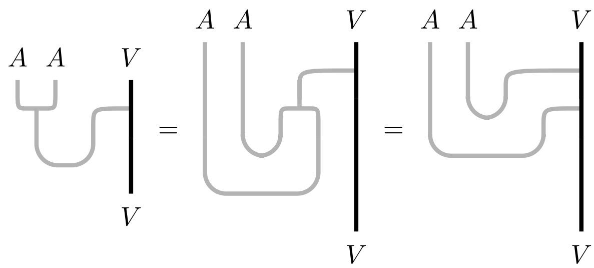
where in all cases morphisms f are mapped onto f, are strict monoidal.

PROOF. Forgetting the monoidal structure, we first show that the functor

$$D^l = \overline{D}^l : {}^B\mathcal{C} \to {}_A\mathcal{C}$$

is well-defined.

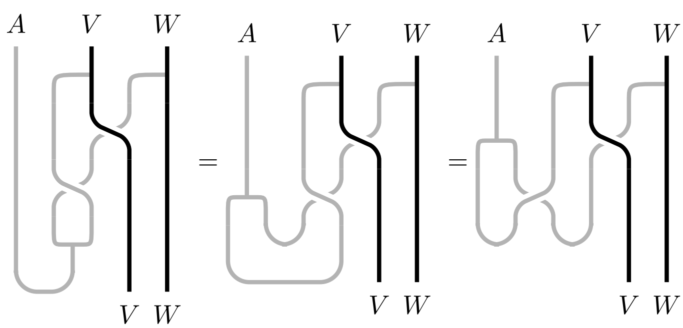
For any object $(V, \delta) \in {}^{B}\mathcal{C}$, $D^l(V, \delta) = (V, (p \otimes \mathrm{id}_V)(\mathrm{id}_A \otimes \delta))$ is a left A-module, since

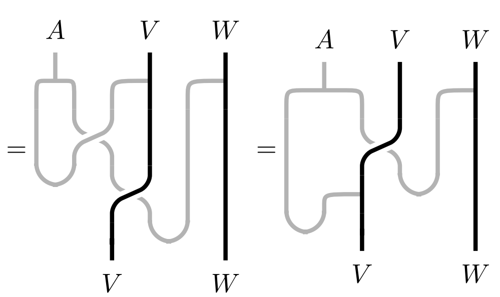


by (3.3.5) and coassociativity of δ. Thus the A-action of $D^l(V, \delta)$ is associative. By (3.3.6), $D^l(V, \delta)$ is unitary.

Let $V, W \in^{B} \mathcal{C}$, and let $f : V \to W$ be a morphism in ${}^{B}\mathcal{C}$. It is easy to see that $f : D^l(V) \to D^l(W)$ is a morphism in ${}_{A}\mathcal{C}$.

To prove that the functor $D^l : {}^{B^{op}}\overline{(\mathcal{C})} \to {}_{A}\mathcal{C}$ is strict monoidal, let (V, δ_V) and (W, δ_W) be objects in ${}^{B}\mathcal{C}$. Then

$\lambda_{D^l(V \otimes W)} =$ 

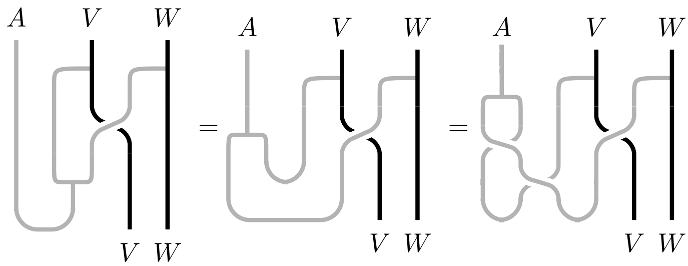


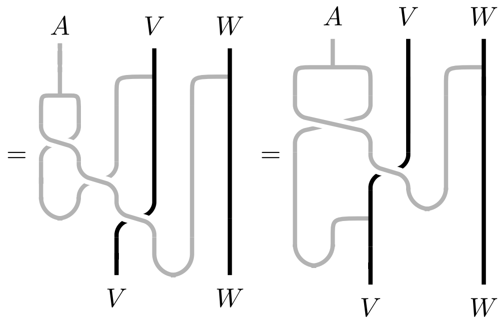
$= \lambda_{D^l(V) \otimes D^l(W)}$,

where the second equality follows from (3.3.5), the third and the fourth from (3.2.17), and the fifth from (3.2.12) with $h = \delta_V$.

By somewhat different arguments,

$\lambda_{\overline{D}^l(V\otimes W)} =$





$= \lambda_{\overline{D}^l(V)\otimes\overline{D}^l(W)}$,

where the second equality follows from (3.3.5), the third from (3.2.16), the fourth from (3.2.17), and the fifth from (3.2.12) with $h = \delta_V$.

Note that $D^l(I) = \overline{D}^l(I) = I$ by (3.3.6). We have shown that D^l and $\overline{D}^l$ are strict monoidal. The claims for D^r and $\overline{D}^r$ follow in the same way. □

Corollary 3.3.10. *Let A and B be Hopf algebras in $\mathcal{C}$, and $p : A \otimes B \to I$ a Hopf pairing. Then the functors*

$$D^{rl} = \left(\mathcal{C}^B \xrightarrow{F^{rl}_-} {}^{B^{\mathrm{op}}}\overline{\mathcal{C}} \xrightarrow{D^l} {}_A\mathcal{C}\right),$$
$$D^{lr} = \left({}^A\mathcal{C} \xrightarrow{F^{lr}_-} \overline{\mathcal{C}}^{A^{\mathrm{op}}} \xrightarrow{D^r} \mathcal{C}_B\right)$$

are strict monoidal.

Proof. The claim follows from Proposition 3.3.9 and Corollary 3.3.6. □

3.4. Yetter-Drinfeld modules

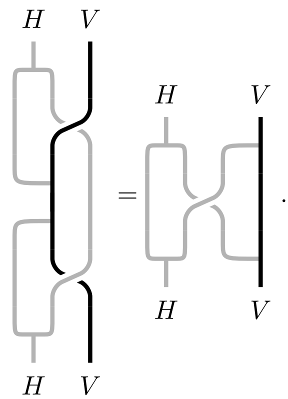
Let $\mathcal{C} = (\mathcal{C}, \otimes, I, c)$ be a braided strict monoidal category.

Let $H = (H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra in $\mathcal{C}$. Yetter-Drinfeld modules over H are left or right H-modules and left or right H-comodules satisfying a compatibility condition. Hence there are four different types of Yetter-Drinfeld modules. We will need two of them.

Definition 3.4.1. Let V be an object in $\mathcal{C}$ and let $\lambda : H \otimes V \to V$ and $\delta : V \to H \otimes V$ be morphisms. The triple (V, λ, δ) is a **left Yetter-Drinfeld module** over H if $(V, \lambda) \in {}_H\mathcal{C}$, $(V, \delta) \in {}^H\mathcal{C}$, and in $\mathrm{Hom}_{\mathcal{C}}(H \otimes V, H \otimes V)$,

$$(\mu \otimes \mathrm{id})(\mathrm{id} \otimes c_{V,H})(\delta\lambda \otimes \mathrm{id})(\mathrm{id} \otimes c_{H,V})(\Delta \otimes \mathrm{id}) =$$
$$(\mu \otimes \lambda)(\mathrm{id} \otimes c_{H,H} \otimes \mathrm{id})(\Delta \otimes \delta), \text{ that is,}$$

(3.4.1)



Note that (3.4.1) is upside-down symmetric.

If (V, λ, δ) is a left Yetter-Drinfeld module over H, then $(V, \delta^{\mathrm{op}}, \lambda^{\mathrm{op}})$ is a left Yetter-Drinfeld module over $(H, \Delta^{\mathrm{op}}, \varepsilon^{\mathrm{op}}, \mu^{\mathrm{op}}, \eta^{\mathrm{op}})$ in $\mathcal{C}^{\mathrm{op}}$.

REMARK 3.4.2. We look at the special case of bialgebras in $\mathcal{C} = \mathcal{M}_{\Bbbk}$. In Sweedler notation, (3.4.1) is equivalent to the following condition. For all $h \in H$, $v \in V$,

$$(h_{(1)} \cdot v)_{(-1)} h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)} = h_{(1)} v_{(-1)} \otimes h_{(2)} \cdot v_{(0)}. \tag{3.4.2}$$

If H is a Hopf algebra, then (3.4.2) is equivalent to

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)} \tag{3.4.3}$$

for all $h \in H$, $v \in V$. Thus Yetter-Drinfeld modules over the group algebra in the sense of Definition 1.4.1 and Remark 1.4.8 are left Yetter-Drinfeld modules.

EXAMPLE 3.4.3. We determine one-dimensional Yetter-Drinfeld modules in the category $\mathcal{C} = \mathcal{M}_{\Bbbk}$. If H is a group algebra, Yetter-Drinfeld modules over H have been determined in Example 1.4.3. Let H be a bialgebra, V a one-dimensional vector space, and let $\lambda : H \otimes V \to V$ and $\delta : V \to H \otimes V$ be maps. Let $x \in V$, $g \in H$, $\chi : H \to \Bbbk$ be such that

$$x \neq 0, \quad \delta(x) = g \otimes x, \quad \lambda(h \otimes x) = \chi(h)x \text{ for all } h \in H.$$

Then $(V, \lambda) \in {}_H\mathcal{C}$ if and only if $\lambda \in \mathrm{Alg}(H, \Bbbk)$. Moreover, $(V, \delta) \in {}^H\mathcal{C}$ if and only if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. Finally, (V, λ, δ) is a Yetter-Drinfeld module over H if and only if additionally

$$\chi(h_{(1)}) g h_{(2)} = h_{(1)} g \chi(h_{(2)})$$

for all $h \in H$.

Assume that (V, λ, δ) as above is a Yetter-Drinfeld module over H. Then $\chi(h)gh = hg\chi(h)$ for each group-like element $h \in H$. If h is an invertible group-like element, then

$$1 = \chi(1) = \chi(hh^{-1}) = \chi(h)\chi(h^{-1}),$$

and hence $\chi(h) \neq 0$ and $gh = hg$. Let $\xi \in \mathrm{Alg}(H, \Bbbk)$. Then

$$\chi(h_{(1)})\xi(g)\xi(h_{(2)}) = \xi(h_{(1)})\xi(g)\chi(h_{(2)})$$

and hence $(\chi\xi)(h)\xi(g) = (\xi\chi)(h)\xi(g)$ for all $h \in H$. In particular, if g is invertible then $\xi(g) \neq 0$ and $\chi\xi = \xi\chi$.

Definition 3.4.4. For all $(X, \delta) \in {}^H\mathcal{C}$ and $(Y, \lambda) \in {}_H\mathcal{C}$ let

$$c^{\mathcal{YD}(\mathcal{C})}_{X,Y} = \big(X \otimes Y \xrightarrow{\delta\otimes\mathrm{id}} H \otimes X \otimes Y \xrightarrow{\mathrm{id}\otimes c_{X,Y}} H \otimes Y \otimes X \xrightarrow{\lambda\otimes\mathrm{id}} Y \otimes X\big),$$

(3.4.4) $$c^{\mathcal{YD}(\mathcal{C})}_{X,Y} = c^{\mathcal{YD}}_{X,Y} =$$  .

The definition of $c^{\mathcal{YD}}_{X,Y}$ is upside-down symmetric. Hence

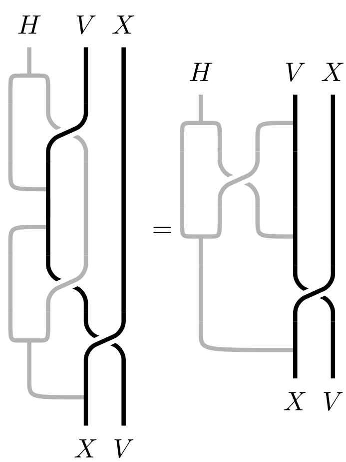
$$c^{\mathcal{YD}(\mathcal{C}^{\mathrm{op}})}_{Y,X} = (c^{\mathcal{YD}(\mathcal{C})}_{X,Y})^{\mathrm{op}}.$$

In the next proposition we characterize the Yetter-Drinfeld condition (3.4.1) by properties of the morphisms $c^{\mathcal{YD}}_{X,Y}$.

Proposition 3.4.5. *Let V be an object in $\mathcal{C}$, $(V, \lambda) \in {}_H\mathcal{C}$, and $(V, \delta) \in {}^H\mathcal{C}$. Then the following are equivalent.*

(1) *(V, λ, δ) is a left Yetter-Drinfeld module over H.*
(2) *For all $X \in {}_H\mathcal{C}$, $c^{\mathcal{YD}}_{V,X}$ is a morphism in ${}_H\mathcal{C}$.*
(3) *$c^{\mathcal{YD}}_{V,H}$ is a morphism in ${}_H\mathcal{C}$, where H is a left H-module by the multiplication in H.*
(4) *For all $X \in {}^H\mathcal{C}$, $c^{\mathcal{YD}}_{X,V}$ is a morphism in ${}^H\mathcal{C}$.*

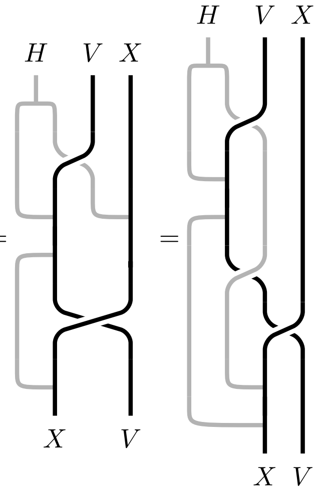
Proof. (1) $\Rightarrow$ (2). Let $(X, \lambda_X) \in {}_H\mathcal{C}$. Tensoring (3.4.1) with X from the right and braiding of V and X and action with H gives the equation

(3.4.5)  .

We will prove (2) by showing that $c^{\mathcal{YD}}_{V,X}\lambda_{V\otimes X}$ is equal to the left-hand side of (3.4.5), and $\lambda_{X\otimes V}(\mathrm{id} \otimes c^{\mathcal{YD}}_{V,X})$ to the right-hand side of (3.4.5).

By definition and (3.2.12) with $h = \lambda_X$,

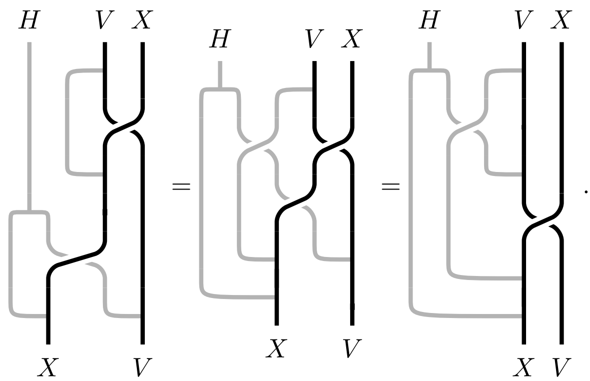
$$c^{\mathcal{YD}}_{V,X}\lambda_{V\otimes X} =$$



which is the left-hand side of (3.4.5) since $X \in {}_H\mathcal{C}$.

By definition and (3.2.12) with $h = \lambda_X$, and then by (3.2.13) with $h = \lambda_V$,

$$\lambda_{X\otimes V}(\mathrm{id} \otimes c^{\mathcal{YD}}_{V,X}) =$$



Since $X \in {}_H\mathcal{C}$, the last picture is the right-hand side of (3.4.5).

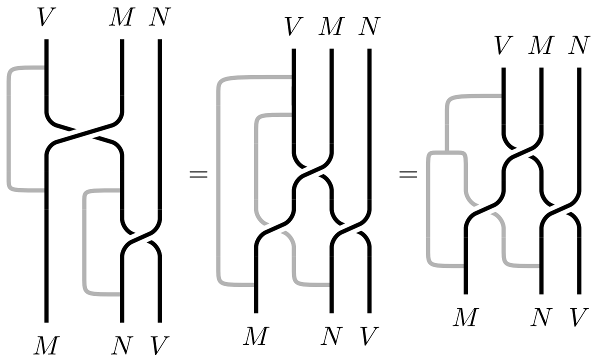
(3) $\Rightarrow$ (1). We have seen in the proof of (1) $\Rightarrow$ (2) that (2) is equivalent to (3.4.5) for all $X \in {}_H\mathcal{C}$. Let $X = H$ as a left H-module by multiplication. Then (3.4.5) for $X = H$ composed with $\mathrm{id}_H \otimes \mathrm{id}_V \otimes \eta$ implies (1).

(2) $\Rightarrow$ (3) is trivial, and (1) $\Leftrightarrow$ (4) follows by duality from (1) $\Leftrightarrow$ (2). $\square$

PROPOSITION 3.4.6. *Let $V, W \in {}^H\mathcal{C}$ and $M, N \in {}_H\mathcal{C}$.*

(1) $c^{\mathcal{YD}}_{V,M\otimes N} = (\mathrm{id}_M \otimes c^{\mathcal{YD}}_{V,N})(c^{\mathcal{YD}}_{V,M} \otimes \mathrm{id}_N)$.
(2) $c^{\mathcal{YD}}_{V\otimes W,M} = (c^{\mathcal{YD}}_{V,M} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes c^{\mathcal{YD}}_{W,M})$.
(3) $c^{\mathcal{YD}}_{V,I} = \mathrm{id}_V$, $c^{\mathcal{YD}}_{I,M} = \mathrm{id}_M$, *where the module structure of I is ε, and the comodule structure is η, respectively.*
(4) *Let $f : V \to W$ and $g : M \to N$ be morphisms of left H-comodules and of left H-modules. Then $(g \otimes f)c^{\mathcal{YD}}_{V,M} = c^{\mathcal{YD}}_{W,N}(f \otimes g)$.*

PROOF. (1) The composition $(\mathrm{id}_M \otimes c^{\mathcal{YD}}_{V,N})(c^{\mathcal{YD}}_{V,M} \otimes \mathrm{id}_N)$ equals



$= c^{\mathcal{YD}}_{V,M\otimes N}$,

where the first equality follows from (3.2.12) with $h = \delta_V$, and the second from coassociativity of V.

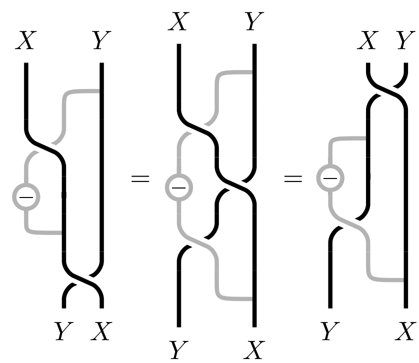
(2) is shown in the same way as (1), and (3) and (4) are easy to see. □

DEFINITION 3.4.7. Let H be a Hopf algebra in $\mathcal{C}$ with antipode $\mathcal{S}$, and assume that $\mathcal{S}$ is an isomorphism in $\mathcal{C}$. For all $X \in {}_H\mathcal{C}$ and $Y \in {}^H\mathcal{C}$ let

$$\overline{c}^{\mathcal{YD}}_{X,Y} = \big(X \otimes Y \xrightarrow{\mathrm{id}\otimes\delta_Y} X \otimes H \otimes Y \xrightarrow{\overline{c}_{X,H}\otimes\mathrm{id}} H \otimes X \otimes Y =$$
$$H \otimes X \otimes Y \xrightarrow{\mathcal{S}^{-1}\otimes\mathrm{id}\otimes\mathrm{id}} H \otimes X \otimes Y \xrightarrow{\lambda_X\otimes\mathrm{id}} X \otimes Y \xrightarrow{\overline{c}_{X,Y}} Y \otimes X\big),$$

The definition of $\overline{c}^{\mathcal{YD}}_{X,Y}$ does not look upside-down symmetric, but it is, and $\overline{c}^{\mathcal{YD}(\mathcal{C}^{\mathrm{op}})}_{Y,X} = (\overline{c}^{\mathcal{YD}(\mathcal{C})}_{X,Y})^{\mathrm{op}}$, since

$\overline{c}^{\mathcal{YD}}_{X,Y} =$



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by first (3.2.13) with $h = \lambda_X$, and then (3.2.12) with $h = \delta_Y$.

PROPOSITION 3.4.8. *Let H be a Hopf algebra in $\mathcal{C}$ with antipode $\mathcal{S}$, and assume that $\mathcal{S}$ is an isomorphism. Let $X \in {}^H\mathcal{C}$, $Y \in {}_H\mathcal{C}$. Then $c^{\mathcal{YD}}_{X,Y}$ is an isomorphism in $\mathcal{C}$ with inverse $\overline{c}^{\mathcal{YD}}_{Y,X}$.*

PROOF. We transform $\overline{c}^{\mathcal{YD}}_{Y,X} c^{\mathcal{YD}}_{X,Y}$ according to Figure 3.4.1, where the second equality follows from (3.2.13) with $h = \lambda_X$, the third from (3.2.15) with $h = \lambda_Y$, and the last from coassociativity and associativity of X and Y. The last picture is the identity of $X \otimes Y$ by (3.2.27), counitarity and unitarity of X and Y, and (3.2.9). The equation $c^{\mathcal{YD}}_{X,Y}\overline{c}^{\mathcal{YD}}_{Y,X} = \mathrm{id}_{Y\otimes X}$ follows by symmetry. □

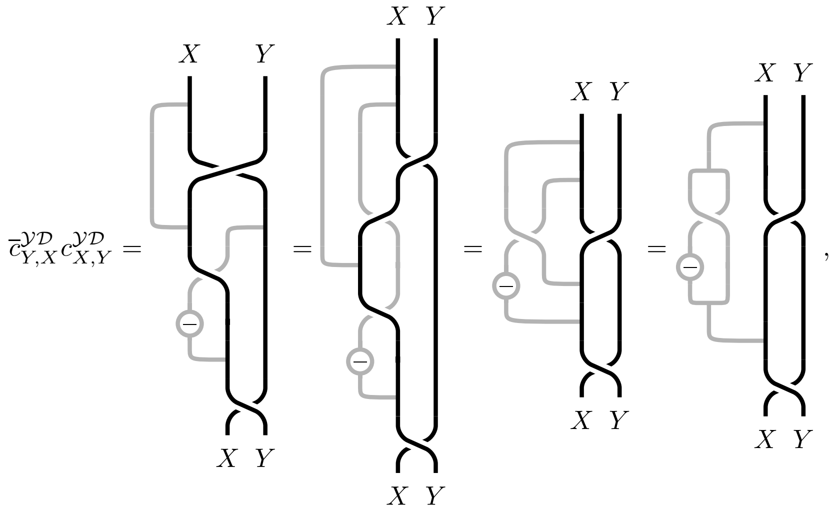


FIGURE 3.4.1. Part of proof of Proposition 3.4.8

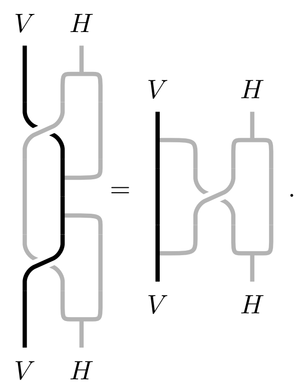
We now discuss the right version of left Yetter-Drinfeld modules.

DEFINITION 3.4.9. Let V be an object in $\mathcal{C}$ and let $\lambda : V \otimes H \to V$ and $\delta : V \to V \otimes H$ be morphisms. The triple (V, λ, δ) is a **right Yetter-Drinfeld module** over H if $(V, \lambda) \in \mathcal{C}_H$, $(V, \delta) \in \mathcal{C}^H$, and in $\mathrm{Hom}_{\mathcal{C}}(V \otimes H, V \otimes H)$,

$$(\mathrm{id} \otimes \mu)(c_{H,V} \otimes \mathrm{id})(\mathrm{id} \otimes \delta\lambda)(c_{V,H} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) = (\lambda \otimes \mu)(\mathrm{id} \otimes c_{H,H} \otimes \mathrm{id})(\delta \otimes \Delta),$$

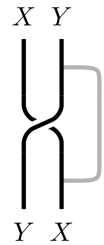
that is,

(3.4.6)



DEFINITION 3.4.10. For all $(X, \lambda) \in \mathcal{C}_H$ and $(Y, \delta) \in \mathcal{C}^H$ let

$$c^{\mathcal{YD}(\mathcal{C})}_{X,Y} = \big(X \otimes Y \xrightarrow{\mathrm{id} \otimes \delta} X \otimes Y \otimes H \xrightarrow{c_{X,Y} \otimes \mathrm{id}} Y \otimes X \otimes H \xrightarrow{\mathrm{id} \otimes \lambda} Y \otimes X\big),$$

(3.4.7) $$c^{\mathcal{YD}(\mathcal{C})}_{X,Y} = c^{\mathcal{YD}}_{X,Y} =$$ 

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PROPOSITION 3.4.11. *Let V be an object in $\mathcal{C}$, $(V,\lambda) \in \mathcal{C}_H$, and $(V,\delta) \in \mathcal{C}^H$. Then the following are equivalent.*

(1) *(V,λ,δ) is a right Yetter-Drinfeld module over H.*
(2) *For all $X \in \mathcal{C}_H$, $c^{\mathcal{YD}}_{X,V}$ is a morphism in $\mathcal{C}_H$.*
(3) *$c^{\mathcal{YD}}_{H,V}$ is a morphism in $\mathcal{C}_H$, where H is a right H-module by the multiplication in H.*
(4) *For all $X \in \mathcal{C}^H$, $c^{\mathcal{YD}}_{V,X}$ is a morphism in $\mathcal{C}^H$.*

PROOF. This follows from Proposition 3.4.5 by left-right symmetry. □

DEFINITION 3.4.12. Let H be a bialgebra in the braided strict monoidal category $\mathcal{C}$. The category of left Yetter-Drinfeld modules (right Yetter-Drinfeld modules, respectively) is denoted by ${}^H_H\mathcal{YD}(\mathcal{C})$ ($\mathcal{YD}(\mathcal{C})^H_H$, respectively). Morphisms in ${}^H_H\mathcal{YD}(\mathcal{C})$ and in $\mathcal{YD}(\mathcal{C})^H_H$ are morphisms of H-modules and H-comodules.

THEOREM 3.4.13. *Let H be a bialgebra in $\mathcal{C}$. Then ${}^H_H\mathcal{YD}(\mathcal{C})$ and $\mathcal{YD}(\mathcal{C})^H_H$ are prebraided strict monoidal categories, where the monoidal structure is the monoidal structure of modules and comodules defined in Section* 3.2, *and for all X, Y in ${}^H_H\mathcal{YD}(\mathcal{C})$ (X, Y in $\mathcal{YD}(\mathcal{C})^H_H$, respectively), the braiding is $c^{\mathcal{YD}}_{X,Y}$ defined in* (3.4.4) *(in* (3.4.7)*, respectively).*

If H is a Hopf algebra with antipode $\mathcal{S}$, and if $\mathcal{S}$ is an isomorphism, then the categories ${}^H_H\mathcal{YD}(\mathcal{C})$ and $\mathcal{YD}(\mathcal{C})^H_H$ are braided strict monoidal.

PROOF. Let $V, W \in {}^H_H\mathcal{YD}(\mathcal{C})$. Then $V \otimes W \in {}_H\mathcal{C}$, and $V \otimes W \in {}^H\mathcal{C}$ with diagonal action and coaction of Section 3.2. For all $X \in {}_H\mathcal{C}$, $c^{\mathcal{YD}}_{W,X}$ and $c^{\mathcal{YD}}_{V,X}$ are left H-module morphisms by Proposition 3.4.5. Hence by Proposition 3.4.6(2), $c^{\mathcal{YD}}_{V\otimes W,X}$ is a morphism of left H-modules, and $V \otimes W \in {}^H_H\mathcal{YD}(\mathcal{C})$ by Proposition 3.4.5.

By Proposition 3.4.5 and Proposition 3.4.6, the family $(c^{\mathcal{YD}}_{V,W})_{V,W\in {}^H_H\mathcal{YD}(\mathcal{C})}$ is a prebraiding. If H is a Hopf algebra, and the antipode of H is an isomorphism, then the prebraiding of ${}^H_H\mathcal{YD}(\mathcal{C})$ is a braiding by Proposition 3.4.8.

The claim for right Yetter-Drinfeld modules follows by left-right symmetry. □

To prove the next theorem we need the following easy identifications.

REMARK 3.4.14. (1) Let $\mathcal{C}, \mathcal{C}'$ be braided strict monoidal categories, and let $(F,\varphi) : \mathcal{C} \to \mathcal{C}'$ be a braided monoidal functor. Let H be a Hopf algebra in $\mathcal{C}$ and H' the Hopf algebra $(F,\varphi)(H)$ in $\mathcal{C}'$. Then (F,φ) induces a braided monoidal functor $\mathcal{YD}(F,\varphi)$ with functor

$$(3.4.8)\qquad \begin{aligned} &F : \mathcal{YD}(\mathcal{C})^H_H \to \mathcal{YD}(\mathcal{C}')^{H'}_{H'},\\ &(V,\lambda,\delta) \mapsto (F(V),\lambda',\delta') \text{ with } \lambda' = F(\lambda)\varphi_{V,H}, \delta' = \varphi^{-1}_{V,H}F(\delta), \end{aligned}$$

where morphisms f are mapped onto $F(f)$, and with monoidal structure φ.

(2) Let H, H' be Hopf algebras in $\mathcal{C}$ whose antipodes are isomorphisms, and let $\rho : H \to H'$ be an isomorphism of Hopf algebras. Then the functor

$$\begin{aligned} &\mathcal{YD}(\rho) : \mathcal{YD}(\mathcal{C})_H^H \to \mathcal{YD}(\mathcal{C})_{H'}^{H'}, \\ &\qquad (V, \lambda, \delta) \mapsto (V, \lambda', \delta') \text{ with } \lambda' = \lambda(\mathrm{id}_V \otimes \rho^{-1}), \delta' = (\mathrm{id}_V \otimes \rho)\delta, \end{aligned} \tag{3.4.9}$$

where morphisms f are mapped onto f, is a braided strict monoidal isomorphism. In the same way $\mathcal{YD}(\varphi)$ is defined for left Yetter-Drinfeld modules.

(3) Let H be a Hopf algebra whose antipode is an isomorphism. Then H is a Hopf algebra in $\mathcal{C}^{\mathrm{rev}}$. It is easy to see that the functors

$$\begin{aligned} (\mathcal{C}^{\mathrm{rev}})_H \to ({}_H\mathcal{C})^{\mathrm{rev}}, \quad (V, \lambda) \mapsto (V, \lambda), \\ (\mathcal{C}^{\mathrm{rev}})^H \to ({}^H\mathcal{C})^{\mathrm{rev}}, \quad (V, \delta) \mapsto (V, \delta), \end{aligned}$$

are strict monoidal isomorphisms, and that

$$\mathcal{YD}(\mathcal{C}^{\mathrm{rev}})_H^H \to ({}_H^H\mathcal{YD}(\mathcal{C}))^{\mathrm{rev}},\ (V, \lambda, \delta) \mapsto (V, \lambda, \delta), \tag{3.4.10}$$

is a braided strict monoidal isomorphism, where in each case morphisms f are mapped onto f.

(4) Let

$$\begin{aligned} F_{\mathcal{C}}^{\mathrm{rev}} = (\mathrm{id}, c) : \mathcal{C}^{\mathrm{rev}} \to \mathcal{C},\ \overline{F}_{\mathcal{C}}^{\mathrm{rev}} = (\mathrm{id}, \varphi) : \mathcal{C} \to \mathcal{C}^{\mathrm{rev}},\ \text{where} \\ \varphi_{X,Y} = \overline{c}_{Y,X} : X \otimes^{\mathrm{rev}} Y \to X \otimes Y, \end{aligned} \tag{3.4.11}$$

for all $X, Y \in \mathcal{C}$. It follows from the axioms of a braiding and (3.2.14) that $F_{\mathcal{C}}^{\mathrm{rev}}$ and $\overline{F}_{\mathcal{C}}^{\mathrm{rev}}$ are inverse braided monoidal isomorphisms. Replacing c by $\overline{c}$ defines another pair of inverse braided monoidal isomorphisms

$$\begin{aligned} F_{\mathcal{C},\overline{c}}^{\mathrm{rev}} = (\mathrm{id}, \overline{c}) : \mathcal{C}^{\mathrm{rev}} \to \mathcal{C},\ \overline{F}_{\mathcal{C},\overline{c}}^{\mathrm{rev}} = (\mathrm{id}, \varphi) : \mathcal{C} \to \mathcal{C}^{\mathrm{rev}},\ \text{where} \\ \varphi_{X,Y} = c_{Y,X} : X \otimes^{\mathrm{rev}} Y \to X \otimes Y, \end{aligned} \tag{3.4.12}$$

for all $X, Y \in \mathcal{C}$. Hence for a bialgebra (Hopf algebra, respectively) H in $\mathcal{C}$,

$$\overline{F}_{\mathcal{C}}^{\mathrm{rev}}(H) = (H^{\mathrm{op}})^{\mathrm{cop}}, \quad \overline{F}_{\mathcal{C},\overline{c}}^{\mathrm{rev}}(H) = (H^{\mathrm{cop}})^{\mathrm{op}},$$

are bialgebras (Hopf algebras, respectively) in $\mathcal{C}$. Note that Proposition 3.2.15 is not used in this argument.

Recall the notations λ_-, δ_+ in Definitions 3.3.3 and 3.3.5 for $(V, \lambda) \in \mathcal{C}_H$ and $(V, \delta) \in \mathcal{C}^H$.

Theorem 3.4.15. *Let H be a Hopf algebra in $\mathcal{C}$, and assume that the antipode of H is an isomorphism. Then the functors*

$$\begin{aligned} F_{rl}^{\mathcal{YD}} : \mathcal{YD}(\mathcal{C})_H^H \to {}_H^H\mathcal{YD}(\mathcal{C}),\ (V, \lambda, \delta) \mapsto (V, \lambda_-, \delta_+), \\ F_{lr}^{\mathcal{YD}} : {}_H^H\mathcal{YD}(\mathcal{C}) \to \mathcal{YD}(\mathcal{C})_H^H,\ (V, \lambda, \delta) \mapsto (V, \lambda_+, \delta_-), \end{aligned}$$

and where morphisms f are mapped onto f, are inverse isomorphisms, and

$$\begin{aligned} (F_{rl}^{\mathcal{YD}}, \rho) : \mathcal{YD}(\mathcal{C})_H^H \to {}_H^H\mathcal{YD}(\mathcal{C}), \quad \textit{where } \rho_{X,Y} = c_{Y,X}^{\mathcal{YD}(\mathcal{C})_H^H} \overline{c}_{X,Y}, \\ (F_{lr}^{\mathcal{YD}}, \psi) : {}_H^H\mathcal{YD}(\mathcal{C}) \to \mathcal{YD}(\mathcal{C})_H^H, \quad \textit{where } \psi_{U,V} = \overline{c}_{V,U}^{{}_H^H\mathcal{YD}(\mathcal{C})} c_{U,V}, \end{aligned}$$

for all for all $X, Y \in \mathcal{YD}(\mathcal{C})_H^H$ and all $U, V \in {}_H^H\mathcal{YD}(\mathcal{C})$, are inverse braided monoidal isomorphisms.

PROOF. (1) We first prove the claim for $(F_{rl}^{\mathcal{YD}}, \varphi)$. Let $(F, \overline{\varphi})$ be the composition of the following braided monoidal isomorphisms

$$\mathcal{YD}(\mathcal{C})_H^H \xrightarrow{\mathcal{YD}(\overline{F}_{\mathcal{C}}^{\mathrm{rev}})} \mathcal{YD}(\mathcal{C}^{\mathrm{rev}})_{(H^{\mathrm{op}})^{\mathrm{cop}}}^{(H^{\mathrm{op}})^{\mathrm{cop}}} \xrightarrow{\mathcal{YD}(\mathcal{S})} \mathcal{YD}(\mathcal{C}^{\mathrm{rev}})_H^H \xrightarrow{(3.4.10)} ({}_H^H\mathcal{YD}(\mathcal{C}))^{\mathrm{rev}}.$$

Recall that $\overline{F}_{\mathcal{C}}^{\mathrm{rev}}(H) = (H^{\mathrm{op}})^{\mathrm{cop}}$. The braided strict monoidal isomorphism $\mathcal{YD}(\mathcal{S})$ is induced from the isomorphism $\mathcal{S} : (H^{\mathrm{op}})^{\mathrm{cop}} \to H$ of Hopf algebras in $\mathcal{C}$ in Corollary 3.2.16.

Then for all $(X, \lambda, \delta) \in \mathcal{YD}(\mathcal{C})_H^H$, $F(X, \lambda, \delta) = (X, \lambda_-, \delta_+)$, and

$$\overline{\varphi}_{X,Y} = \overline{c}_{Y,X} : F(X) \otimes^{\mathrm{rev}} F(Y) \to F(X \otimes Y)$$

for all $X, Y \in \mathcal{YD}(\mathcal{C})_H^H$.

The theorem follows by composing $(F, \overline{\varphi})$ and

$$F_{{}_H^H\mathcal{YD}(\mathcal{C})}^{\mathrm{rev}} = (\mathrm{id}, c^{{}_H^H\mathcal{YD}(\mathcal{C})}) : ({}_H^H\mathcal{YD}(\mathcal{C}))^{\mathrm{rev}} \to {}_H^H\mathcal{YD}(\mathcal{C}).$$

Note that the monoidal structure of the composition is given by

$$\varphi_{X,Y} = \overline{c}_{Y,X} c_{F(X),F(Y)}^{{}_H^H\mathcal{YD}(\mathcal{C})} = c_{Y,X}^{\mathcal{YD}(\mathcal{C})_H^H} \overline{c}_{X,Y}$$

for all $X, Y \in \mathcal{YD}(\mathcal{C})_H^H$, since $(F, \overline{\varphi})$ is braided.

(2) It is clear that $F_{rl}^{\mathcal{YD}}$ and $F_{lr}^{\mathcal{YD}}$ are inverse functors. By (3.1.11), the inverse of $(F_{rl}^{\mathcal{YD}}, \varphi)$ is the monoidal functor $(G, \psi), G = F_{lr}^{\mathcal{YD}}$, with

$$\psi_{U,V} = G(\varphi_{G(U),G(V)})^{-1} = c_{V,U} \overline{c}_{G(U),G(V)}^{\mathcal{YD}(\mathcal{C})_H^H} \tag{3.4.13}$$

for all $U, V \in {}_H^H\mathcal{YD}(\mathcal{C})$, where we used the definition of φ in (1).

Since (G, ψ) is braided, for all $U, V \in {}_H^H\mathcal{YD}(\mathcal{C})$,

$$c_{U,V}^{{}_H^H\mathcal{YD}(\mathcal{C})} \psi_{U,V} = \psi_{V,U} c_{G(U),G(V)}^{\mathcal{YD}(\mathcal{C})_H^H},$$

hence by(3.4.13),

$$\begin{aligned}
c_{U,V}^{{}_H^H\mathcal{YD}(\mathcal{C})} c_{V,U} \overline{c}_{G(U),G(V)}^{\mathcal{YD}(\mathcal{C})_H^H} &= c_{U,V} \overline{c}_{G(V),G(U)}^{\mathcal{YD}(\mathcal{C})_H^H} c_{G(V),G(U)}^{\mathcal{YD}(\mathcal{C})_H^H} = c_{U,V}, \text{ or} \\
\psi_{U,V} = c_{V,U} \overline{c}_{G(U),G(V)}^{\mathcal{YD}(\mathcal{C})_H^H} &= \overline{c}_{V,U}^{{}_H^H\mathcal{YD}(\mathcal{C})} c_{U,V}.
\end{aligned}$$

This implies the claim. □

The monoidal isomorphism in Theorem 3.4.15 is not strict. However, by the next theorem there is a strict monoidal isomorphism between right Yetter-Drinfeld modules over H and left Yetter-Drinfeld modules over H^{cop}.

THEOREM 3.4.16. *Let H be a Hopf algebra in $\mathcal{C}$, and assume that the antipode of H is an isomorphism. Then the functors*

$$\begin{aligned}
\overline{F}_{rl}^{\mathcal{YD}} &: \overline{\mathcal{YD}(\mathcal{C})_H^H} \to {}_{H^{\mathrm{cop}}}^{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}}),\ (V, \lambda, \delta) \mapsto (V, \lambda_-, \overline{c}_{V,H}\delta), \\
\overline{F}_{lr}^{\mathcal{YD}} &: {}_{H^{\mathrm{cop}}}^{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}}) \to \overline{\mathcal{YD}(\mathcal{C})_H^H},\ (V, \lambda, \delta) \mapsto (V, \lambda_+, c_{H,V}\delta),
\end{aligned}$$

and where morphisms f are mapped onto f, are inverse, braided strict monoidal isomorphisms.

PROOF. (1) By Proposition 3.3.2 and Corollary 3.3.4, the functors

$$F_1 = F^-_{rl} : \mathcal{C}_H \to {}_{H^{\mathrm{cop}}}\overline{\mathcal{C}},\ (V,\lambda) \mapsto (V, \lambda_-),$$
$$F_2 = F^{rl} : \mathcal{C}^H \to {}^{H^{\mathrm{cop}}}\overline{\mathcal{C}},\ (V,\delta) \mapsto (V, \overline{c}_{V,H}\delta),$$

are strict monoidal isomorphisms.

Let $(X, \lambda_X) \in \mathcal{C}_H$, $(V, \delta) \in \mathcal{C}^H$, and define

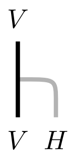
$$(X, \lambda'_X) = F_1(X, \lambda_X), \quad (V, \delta') = F_2(V, \delta).$$

We first prove the equality

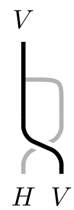
$$\overline{c}^{\mathcal{YD}}_{(X,\lambda'_X),(V,\delta')} = c^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)}, \tag{3.4.14}$$

where $\lambda'_X = (\lambda_X)_- = \lambda \overline{c}_{H,V}(\mathcal{S}_H^{-1} \otimes \mathrm{id}_V)$, $\delta' = \overline{c}_{V,H}\delta$, and hence

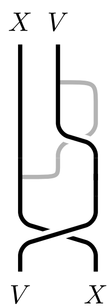
$$\overline{c}^{\mathcal{YD}}_{(X,\lambda'_X),(V,\delta')} = \big(X \otimes V \xrightarrow{\mathrm{id}\otimes\delta'} X \otimes H \otimes V \xrightarrow{\lambda_X\otimes\mathrm{id}} X \otimes V \xrightarrow{c_{X,V}} V \otimes X\big).$$

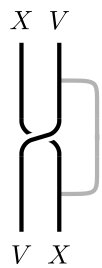
Let $\delta =$ 

, and $\lambda_X =$ 

. Then by definition, $\delta' =$ 

, and

$\overline{c}^{\mathcal{YD}}_{(X,\lambda'_X),(V,\delta')} =$ 

$=$ 

$= c^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)},$

where the second equality follows from (3.2.15).

(2) Let $V \in \mathcal{C}$, and define

$$\mathcal{P}^l(V) = \{(\lambda, \delta) \mid (V, \lambda) \in \mathcal{C}_H, (V, \delta) \in \mathcal{C}^H\},$$
$$\mathcal{P}^r(V) = \{(\lambda', \delta') \mid (V, \lambda') \in {}_{H^{\mathrm{cop}}}\overline{\mathcal{C}}, (V, \delta') \in {}^{H^{\mathrm{cop}}}\overline{\mathcal{C}}\}.$$

Then $\Phi : \mathcal{P}^l(V) \to \mathcal{P}^r(V)$, $(\lambda, \delta) \mapsto (\lambda', \delta')$, where

$$(V, \lambda') = F_1(V, \lambda), \quad (V, \delta') = F_2(V, \delta),$$

is bijective.

Let $(\lambda, \delta) \in \mathcal{P}^l(V)$, and $(\lambda', \delta') = \Phi(\lambda, \delta)$.

We claim that the following are equivalent.

(a) $(V, \lambda, \delta) \in \mathcal{YD}(\mathcal{C})^H_H$.
(b) $(V, \lambda', \delta') \in {}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$.
(c) For all $(X, \lambda_X) \in \mathcal{C}_H$, the morphism

$$c^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)} : (X, \lambda_X) \otimes (V, \lambda) \to (V, \lambda) \otimes (X, \lambda_X) \ \text{ is in } \mathcal{C}_H.$$

(d) For all $(X, \lambda'_X) \in {}_{H^{\mathrm{cop}}}\overline{\mathcal{C}}$, the morphism

$$\overline{c}^{\mathcal{YD}}_{(X,\lambda'_X),(V,\delta')} : (X, \lambda'_X) \otimes (V, \lambda') \to (V, \lambda') \otimes (X, \lambda'_X) \text{ is in } {}_{H^{\mathrm{cop}}}\overline{\mathcal{C}}.$$

By Proposition 3.4.11, (a) is equivalent to (c), and by Proposition 3.4.8 and Proposition 3.4.5, (b) is equivalent to (d). The equivalence of (c) and (d) follows from (3.4.14), since F_1 is a strict monoidal isomorphism.

(3) Since F_1 and F_2 are strict monoidal isomorphisms, it follows from (1) and (2) that $\overline{F}^{\mathcal{YD}}_{rl}$ is a strict monoidal isomorphism with inverse $\overline{F}^{\mathcal{YD}}_{lr}$.

We finally show that the functor $F = \overline{F}^{\mathcal{YD}}_{rl}$ is braided. Let $X = (X, \lambda_X, \delta_X)$ and $V = (V, \lambda, \delta)$ be Yetter-Drinfeld modules in $\mathcal{YD}(\mathcal{C})^H_H$, and $F(X) = (X', \lambda'_X, \delta'_X)$ and $F(V) = (V', \lambda', \delta')$ their images under F.

We write

$$F : \overline{\mathcal{A}} = \overline{\mathcal{YD}(\mathcal{C})^H_H} \to \mathcal{B} = {}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}}).$$

In the notation of (1), $c^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)} = c^{\mathcal{A}}_{X,V}$, and $\overline{c}^{\mathcal{YD}}_{(X',\lambda'_X),(V',\delta')} = \overline{c}^{\mathcal{B}}_{F(X),F(V)}$. By (3.4.14), $c^{\mathcal{A}}_{X,V} = \overline{c}^{\mathcal{B}}_{F(X),F(V)}$, hence

$$F(\overline{c}^{\mathcal{A}}_{V,X}) = \overline{c}^{\mathcal{A}}_{V,X} = c^{\mathcal{B}}_{F(V),F(X)}.$$

$\square$

Corollary 3.4.17. *Let H be a Hopf algebra in $\mathcal{C}$, and assume that the antipode of H is an isomorphism. Then the functor*

$$F : \overline{{}^H_H\mathcal{YD}(\mathcal{C})} \to {}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}}),\ (V, \lambda, \delta) \mapsto (V, \lambda, (\mathcal{S}_H^{-1} \otimes \mathrm{id})\overline{c}^2_{H,V}\delta),$$

and where morphisms f are mapped onto f, is an isomorphism, and

$$(F, \varphi) : \overline{{}^H_H\mathcal{YD}(\mathcal{C})} \to {}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}}),\ \text{where } \varphi_{X,Y} = \overline{c}^{{}^H_H\mathcal{YD}(\mathcal{C})}_{Y,X} c_{X,Y},$$

for all $X, Y \in \overline{{}^H_H\mathcal{YD}(\mathcal{C})}$, is a braided monoidal isomorphism.

Proof. This follows by composing the isomorphisms in Theorems 3.4.15 and 3.4.16, that is, we define $(F, \varphi) = \overline{F}^{\mathcal{YD}}_{rl}(F^{\mathcal{YD}}_{lr}, \psi)$. $\square$

3.5. Duality and Hopf modules

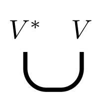
Let $\mathcal{C}$ be a strict monoidal category.

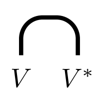
Definition 3.5.1. Let $V \in \mathcal{C}$. A **left dual** of V is a triple $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$, where V^* is an object in $\mathcal{C}$, and $\mathrm{ev}_V : V^* \otimes V \to I$ and $\mathrm{coev}_V : I \to V \otimes V^*$ are morphisms in $\mathcal{C}$ with

$$(V^* \otimes I \xrightarrow{\mathrm{id}\otimes\mathrm{coev}_V} V^* \otimes V \otimes V^* \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} I \otimes V^*) = \mathrm{id}_{V^*},$$

$$(I \otimes V \xrightarrow{\mathrm{coev}_V\otimes\mathrm{id}} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} V \otimes I) = \mathrm{id}_V.$$

We use the notations

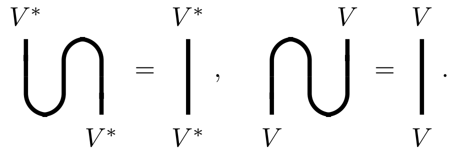
$\mathrm{ev}_V =$ 

, $\quad \mathrm{coev}_V =$ 

.

Hence by definition of a left dual,

(3.5.1)



REMARK 3.5.2. Let $V \in \mathcal{C}$ and $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$ a left dual of V.

(1) For all $X, Y \in \mathcal{C}$,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X \otimes V, Y) &\to \mathrm{Hom}_{\mathcal{C}}(X, Y \otimes V^*), \\ F &\mapsto (F \otimes \mathrm{id}_{V^*})(\mathrm{id}_X \otimes \mathrm{coev}_V),\end{aligned} \tag{3.5.2}$$

is bijective with inverse given by $G \mapsto (\mathrm{id}_Y \otimes \mathrm{ev}_V)(G \otimes \mathrm{id}_V)$, and

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(V^* \otimes X, Y) &\to \mathrm{Hom}_{\mathcal{C}}(X, V \otimes Y), \\ F &\mapsto (\mathrm{id}_V \otimes F)(\mathrm{coev}_V \otimes \mathrm{id}_X),\end{aligned} \tag{3.5.3}$$

is bijective with inverse given by $G \mapsto (\mathrm{ev}_V \otimes \mathrm{id}_Y)(\mathrm{id}_{V^*} \otimes G)$.

By (3.5.2), the pair (V, ev_V) satisfies the following universal property.

For all $X, Y \in \mathcal{C}$ and morphisms $F : X \otimes V \to Y$ there is exactly one morphism $G : X \to Y \otimes V^*$ such that the diagram

$$\begin{tikzcd} X \otimes V \arrow[rr, "F"] \arrow[dr, "G \otimes \mathrm{id}_V"'] & & Y \\ & Y \otimes V^* \otimes V \arrow[ur, "\mathrm{id}_Y \otimes \mathrm{ev}_V"'] & \end{tikzcd} \tag{3.5.4}$$

commutes. Explicitly, G is given by $G = (F \otimes \mathrm{id}_{V^*})(\mathrm{id}_X \otimes \mathrm{coev}_V)$.

(2) We note another universal property of the pair (V^*, ev_V) by setting $Y = I$ in (1).

For all $X \in \mathcal{C}$ and morphisms $F : X \otimes V \to I$ there is exactly one morphism $G : X \to V^*$ such that the diagram

$$\begin{tikzcd} X \otimes V \arrow[rr, "F"] \arrow[dr, "G \otimes \mathrm{id}_V"'] & & I \\ & V^* \otimes V \arrow[ur, "\mathrm{ev}_V"'] & \end{tikzcd} \tag{3.5.5}$$

commutes. Explicitly, G is given by $G = (F \otimes \mathrm{id}_{V^*})(\mathrm{id}_X \otimes \mathrm{coev}_V)$.

(3) If $f : V \to W$ is a morphism, and if W has a left dual $(W, \mathrm{ev}_W, \mathrm{coev}_W)$ we define a morphism $f^* : W^* \to V^*$ by (3.5.5) with $X = W^*$ and $F = \mathrm{ev}_W(\mathrm{id} \otimes f)$, $G = f^*$, that is by the commutative diagram

$$\begin{tikzcd} W^* \otimes V \arrow[r, "\mathrm{id}_{W^*} \otimes f"] \arrow[d, "f^* \otimes \mathrm{id}_V"'] & W^* \otimes W \arrow[d, "\mathrm{ev}_W"] \\ V^* \otimes V \arrow[r, "\mathrm{ev}_V"] & I \end{tikzcd} . \tag{3.5.6}$$

By the universal property, $\mathrm{id}_V^* = \mathrm{id}_{V^*}$, and $(fg)^* = g^* f^*$, if $g : U \to V$ is a morphism such that a left dual $(U, \mathrm{ev}_U, \mathrm{coev}_U)$ exists.

EXAMPLE 3.5.3. Let $\mathcal{C} = \mathcal{M}_{\Bbbk}$ be the monoidal category of vector spaces.

Suppose a vector space V has a left dual $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$. Choose elements $v_i \in V$, $f_i \in V^*$, $1 \le i \le n$, with $\mathrm{coev}_V(1) = \sum_{i=1}^n v_i \otimes f_i$. Then V is finite-dimensional, since for all $v \in V$, $v = \sum_{i=1}^n v_i \mathrm{ev}_V(f_i \otimes v)$.

A finite-dimensional vector space V has the left dual $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$, where $V^* = \mathrm{Hom}(V, \Bbbk)$ is the dual space, $\mathrm{ev}_V : V^* \otimes V \to \Bbbk$, $f \otimes v \mapsto f(v)$, is evaluation, and coev_V is defined by $\mathrm{coev}_V(1) = \sum_{i=1}^n v_i \otimes f_i$, where $(v_i)_{1 \le i \le n}$ and $(f_i)_{1 \le i \le n}$ are dual bases. If $f : V \to W$ is a linear map of finite-dimensional vector spaces, then f^* defined by (3.5.6) is $\mathrm{Hom}(f, \mathrm{id})$.

The left dual is uniquely determined (if it exists) in the sense of the next lemma.

LEMMA 3.5.4. *Let $\mathcal{C}$ be a strict monoidal category and let $f : V \to W$ be a morphism in $\mathcal{C}$. Assume that $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$ and $(V', \mathrm{ev}'_V, \mathrm{coev}'_V)$ are left duals of V, and that $(W^*, \mathrm{ev}_W, \mathrm{coev}_W)$ and $(W', \mathrm{ev}'_W, \mathrm{coev}'_W)$ are left duals of W.*

(1) *There is exactly one morphism $\varphi : V' \to V^*$ such that the diagram*

$$\begin{tikzcd} V' \otimes V \arrow[rr, "\mathrm{ev}'_V"] \arrow[dr, "\varphi \otimes \mathrm{id}"'] & & I \\ & V^* \otimes V \arrow[ur, "\mathrm{ev}_V"'] & \end{tikzcd}$$

commutes. Explicitly, $\varphi = (\mathrm{ev}'_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V'} \otimes \mathrm{coev}_V)$, and φ is an isomorphism.

(2) *$(\mathrm{id}_V \otimes \varphi)\mathrm{coev}'_V = \mathrm{coev}_V$.*

(3) *Let $\psi : W' \to W^*$ be the isomorphism φ in (1) for the duals of W. Let $f^* : W^* \to V^*$ and $f' : W' \to V'$ be the morphisms defined by the diagram (3.5.6) for the duals W^*, V^* and the duals W', V'. Then $f^*\psi = \varphi f'$.*

PROOF. (1) follows from the universal property (3.5.5).

(2) By definition of the dual and by (1),

$$\begin{aligned} \mathrm{id}_V &= (\mathrm{id}_V \otimes \mathrm{ev}'_V)(\mathrm{coev}'_V \otimes \mathrm{id}_V) \\ &= (\mathrm{id}_V \otimes \mathrm{ev}_V)(\mathrm{id}_V \otimes \varphi \otimes \mathrm{id}_V)(\mathrm{coev}'_V \otimes \mathrm{id}_V). \end{aligned}$$

Hence $(\mathrm{id}_V \otimes \varphi)\mathrm{coev}'_V = \mathrm{coev}_V$ by the uniqueness of G in (3.5.4) with $X = I$, $Y = V$, and $F = \mathrm{id}_V$.

(3) Define $\overline{f} : W' \to V'$ by the equation $f^*\psi = \varphi\overline{f}$. Then $\overline{f}$ satisfies the defining commutative diagram for f'. Hence $\overline{f} = f'$. □

LEMMA 3.5.5. *Let $\mathcal{C}$ be a braided strict monoidal category, and $V, W \in \mathcal{C}$. Assume that $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$ and $(W^*, \mathrm{ev}_W, \mathrm{coev}_W)$ are left duals of V and W, respectively.*

(1) *Let $\widetilde{\mathrm{ev}}_V = \mathrm{ev}_V c_{V,V^*}$, $\widetilde{\mathrm{coev}}_V = \overline{c}_{V,V^*}\mathrm{coev}_V$. Then $(V, \widetilde{\mathrm{ev}}_V, \widetilde{\mathrm{coev}}_V)$ is a left dual of V^*.*

(2) *Define $\widetilde{\mathrm{ev}}_{V,W}$ and $\widetilde{\mathrm{coev}}_{V,W}$ by the compositions*

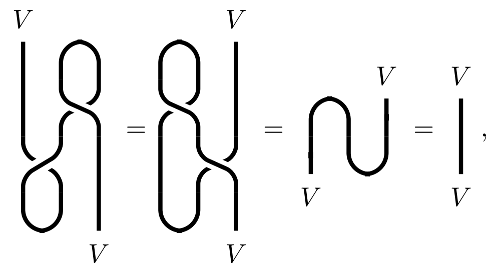
$$V^* \otimes W^* \otimes V \otimes W \xrightarrow{\mathrm{id} \otimes \overline{c}_{W^*,V} \otimes \mathrm{id}} V^* \otimes V \otimes W^* \otimes W \xrightarrow{\mathrm{ev}_V \otimes \mathrm{ev}_W} I \text{ and}$$

$$I \xrightarrow{\mathrm{coev}_V \otimes \mathrm{coev}_W} V \otimes V^* \otimes W \otimes W^* \xrightarrow{\mathrm{id} \otimes c_{V^*,W} \otimes \mathrm{id}} V \otimes W \otimes V^* \otimes W^*.$$

Then $(V^* \otimes W^*, \widetilde{\mathrm{ev}}_{V,W}, \widetilde{\mathrm{coev}}_{V,W})$ is a left dual of $V \otimes W$.

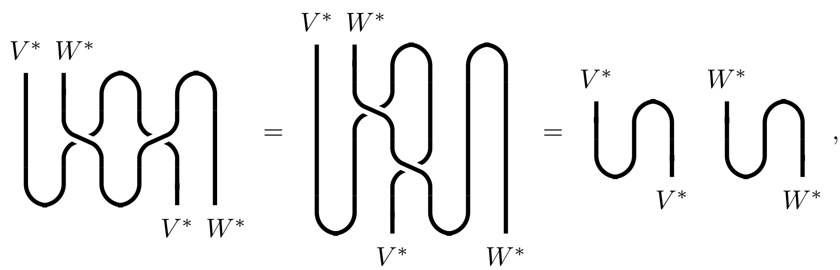
PROOF. In both cases we prove the first equation in (3.5.1), the second follows by symmetry.

(1)



where the first equality follows from (the upside-down version of) (3.2.17) and the second from (3.2.11).

(2)



where the first equality follows from (3.2.17) and the second from (3.2.11). □

DEFINITION 3.5.6. A braided strict monoidal category $\mathcal{C}$ is called **rigid**, if each object has a left dual.

Let $\mathcal{C}$ be a rigid braided strict monoidal category. For any $V \in \mathcal{C}$ we fix a left dual $(V, \mathrm{ev}_V, \mathrm{coev}_V)$. (For the left dual of I we take $(I, \mathrm{id}, \mathrm{id})$.) The contravariant functor

$$(\ )^* : \mathcal{C} \to \mathcal{C},\ V \mapsto V^*,$$

where morphisms f are mapped onto f^* is called the **left duality functor**.

REMARK 3.5.7. Let $\mathcal{C}$ be a strict monoidal category, and $V \in \mathcal{C}$. A **right dual** of V is a triple $({}^*V, \mathrm{ev}'_V, \mathrm{coev}'_V)$, where ${}^*V$ is an object in $\mathcal{C}$, and $\mathrm{ev}'_V : V \otimes {}^*V \to I$ and $\mathrm{coev}'_V : I \to {}^*V \otimes V$ are morphisms in $\mathcal{C}$ with

$$(I \otimes {}^*V \xrightarrow{\mathrm{coev}'_V \otimes \mathrm{id}} {}^*V \otimes V \otimes {}^*V \xrightarrow{\mathrm{id} \otimes \mathrm{ev}'_V} {}^*V \otimes I) = \mathrm{id}_{V^*},$$

$$(V \otimes I \xrightarrow{\mathrm{id} \otimes \mathrm{coev}'_V} V \otimes {}^*V \otimes V \xrightarrow{\mathrm{ev}'_V \otimes \mathrm{id}} I \otimes V) = \mathrm{id}_V.$$

The monoidal category $\mathcal{C}$ is called rigid, if each object has a left dual and a right dual. In this sense, a rigid braided strict monoidal category is rigid, since for each $V \in \mathcal{C}$ with left dual $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$ the triple $(V^*, \mathrm{ev}'_V, \mathrm{coev}'_V)$ is a right dual, where

$$\mathrm{ev}'_V = (V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \xrightarrow{\mathrm{ev}_V} I),$$

$$\mathrm{coev}'_V = (I \xrightarrow{\mathrm{coev}_V} V \otimes V^* \xrightarrow{\overline{c}_{V,V^*}} V^* \otimes V).$$

THEOREM 3.5.8. *Let $\mathcal{C}$ be a rigid braided strict monoidal category. For all $V, W \in \mathcal{C}$ let*

$$\varphi_{V,W} = (\widetilde{\mathrm{ev}}_{V,W} \otimes \mathrm{id}_{(V\otimes W)^*})(\mathrm{id}_{V^*\otimes W^*} \otimes \mathrm{coev}_{V\otimes W}), \tag{3.5.7}$$

$$\psi_V = (\widetilde{\mathrm{ev}}_V \otimes \mathrm{id}_{V^{**}})(\mathrm{id}_V \otimes \mathrm{coev}_{V^*}). \tag{3.5.8}$$

Then the families

$$\varphi = (\varphi_{V,W} : V^* \otimes W^* \to (V \otimes W)^*)_{V,W\in\mathcal{C}}, \quad \psi = (\psi_V : V \to V^{**})_{V\in\mathcal{C}}$$

are natural isomorphisms, and

$$((\ )^*, \varphi) : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$$

is a braided monoidal equivalence.

PROOF. Let $V, W \in \mathcal{C}$. By Lemma 3.5.5 and Lemma 3.5.4, ψ_V and $\varphi_{V,W}$ are the isomorphisms in Lemma 3.5.4(1) making the following diagrams commutative.

$$\begin{tikzcd} V \otimes V^* \arrow[rr, "\widetilde{\mathrm{ev}}_V"] \arrow[dr, "\psi_V\otimes\mathrm{id}"'] & & I \\ & V^{**} \otimes V^* \arrow[ur, "\mathrm{ev}_{V^*}"'] & \end{tikzcd} \tag{3.5.9}$$

$$\begin{tikzcd} V^* \otimes W^* \otimes V \otimes W \arrow[rr, "\widetilde{\mathrm{ev}}_{V,W}"] \arrow[dr, "\varphi_{V,W}\otimes\mathrm{id}"'] & & I\ . \\ & (V \otimes W)^* \otimes V \otimes W \arrow[ur, "\mathrm{ev}_{V\otimes W}"'] & \end{tikzcd} \tag{3.5.10}$$

It follows from Lemma 3.5.4 that ψ and φ are natural isomorphisms.

We next show that φ is a monoidal structure of the duality functor. Let $U, V, W \in \mathcal{C}$. The equalities $\varphi_{V,I} = \mathrm{id}_{V^*} = \varphi_{I,V}$ are obvious. To prove that the diagram

$$\begin{tikzcd} U^* \otimes V^* \otimes W^* \arrow[r, "\mathrm{id}\otimes\varphi_{V,W}"] \arrow[d, "\varphi_{U,V}\otimes\mathrm{id}"'] & U^* \otimes (V \otimes W)^* \arrow[d, "\varphi_{U,V\otimes W}"'] \\ (U \otimes V)^* \otimes W^* \arrow[r, "\varphi_{U\otimes V,W}"] & (U \otimes V \otimes W)^* \end{tikzcd}$$

commutes, by (3.5.4) we have to show that

$$\begin{aligned} &\mathrm{ev}_{U\otimes V\otimes W}(\varphi_{U\otimes V,W} \otimes \mathrm{id}_{U\otimes V\otimes W})(\varphi_{U,V} \otimes \mathrm{id}_{W^*} \otimes \mathrm{id}_{U\otimes V\otimes W}) \\ =&\mathrm{ev}_{U\otimes V\otimes W}(\varphi_{U,V\otimes W} \otimes \mathrm{id}_{U\otimes V\otimes W})(\mathrm{id}_{U^*} \otimes \varphi_{V,W} \otimes \mathrm{id}_{U\otimes V\otimes W}). \end{aligned}$$

This is easily checked using the defining diagrams of the φ-maps, the definition of the $\widetilde{\mathrm{ev}}$-maps, and the axioms of the braiding.

Finally, the diagram

$$\begin{tikzcd} V^* \otimes W^* \arrow[r, "\varphi_{V,W}"] \arrow[d, "c_{V^*,W^*}"'] & (V \otimes W)^* \arrow[d, "(c_{W,V})^*"'] \\ W^* \otimes V^* \arrow[r, "\varphi_{W,V}"] & (W \otimes V)^* \end{tikzcd}$$

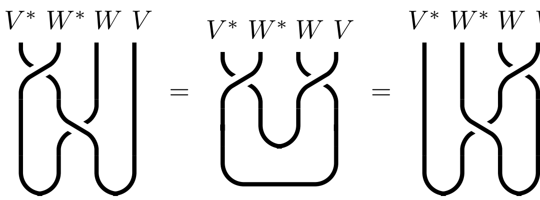
commutes. As before we have to prove that

$$\begin{aligned}&\mathrm{ev}_{W\otimes V}(\varphi_{W,V}\otimes \mathrm{id}_{W\otimes V})(c_{V^*,W^*}\otimes \mathrm{id}_{W\otimes V})\\ =&\mathrm{ev}_{W\otimes V}((c_{W,V})^*\otimes \mathrm{id}_{W\otimes V})(\varphi_{V,W}\otimes \mathrm{id}_{W\otimes V}).\end{aligned}$$

By the defining diagrams of $\varphi_{W,V}$, of $(c_{W,V})^*$ and of $\varphi_{V,W}$, the last equation is equivalent to

$$\widetilde{\mathrm{ev}}_{W,V}(c_{V^*,W^*}\otimes \mathrm{id}_{W\otimes V})=\widetilde{\mathrm{ev}}_{V,W}(\mathrm{id}_{V^*\otimes W^*}\otimes c_{W,V}),$$

and $\widetilde{\mathrm{ev}}_{W,V}(c_{V^*,W^*}\otimes \mathrm{id}_{W\otimes V})$ is equal to



$= \widetilde{\mathrm{ev}}_{W,V}(\mathrm{id}_{V^*\otimes W^*}\otimes c_{W,V})$,

where we moved ev_V twice to the left using (3.2.17). □

Remark 3.5.9. Let $\mathcal{C}$ be a braided strict monoidal and rigid category, and let $H=(H,\mu,\eta,\Delta,\varepsilon,\mathcal{S})$ be a Hopf algebra in $\mathcal{C}$. Then by Theorem 3.5.8, the dual Hopf algebra of H is the Hopf algebra $(H^*,\Delta^*\varphi_{H,H},\varepsilon^*,\varphi_{H,H}^{-1}\mu^*,\eta^*,\mathcal{S}^*)$.

Lemma 3.5.10. *Let $V\in\mathcal{C}$ with left dual $(V^*,\mathrm{ev}_V,\mathrm{coev}_V)$, C a coalgebra and A an algebra in $\mathcal{C}$.*

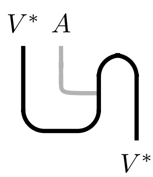
(1) *If $(V,\lambda)\in {}_A\mathcal{C}$, then $(V^*,\lambda_r)\in\mathcal{C}_A$, where λ_r is defined by*

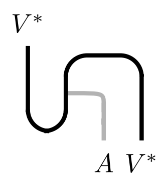
$$V^*\otimes A\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{coev}_V} V^*\otimes A\otimes V\otimes V^*\xrightarrow{\mathrm{id}\otimes\lambda_V\otimes\mathrm{id}} V^*\otimes V\otimes V^*\xrightarrow{\mathrm{ev}_V\otimes\mathrm{id}} V^*.$$

(2) *If $(V,\delta)\in\mathcal{C}^C$, then $(V^*,\delta_l)\in {}^C\mathcal{C}$, where δ_l is defined by*

$$V^*\xrightarrow{\mathrm{id}\otimes\mathrm{coev}_V} V^*\otimes V\otimes V^*\xrightarrow{\mathrm{id}\otimes\delta_V\otimes\mathrm{id}} V^*\otimes V\otimes C\otimes V^*\xrightarrow{\mathrm{ev}_V\otimes\mathrm{id}\otimes\mathrm{id}} C\otimes V^*.$$

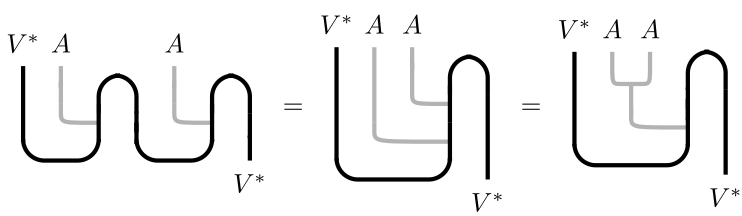
In graphical notation,

(3.5.11) $\lambda_r =$ 

, $\delta_l =$ 

.

Proof. (1) It is obvious that (V^*,λ_r) is unitary. By (3.5.1) and associativity,



that is, $\lambda_r(\lambda_r\otimes\mathrm{id}_A)=\lambda_r(\mathrm{id}\otimes\Delta)$.

(2) follows in the same way. □

In the remainder of this section, let $\mathcal{C}$ be a braided strict monoidal category.

DEFINITION 3.5.11. Let H be a bialgebra in $\mathcal{C}$.

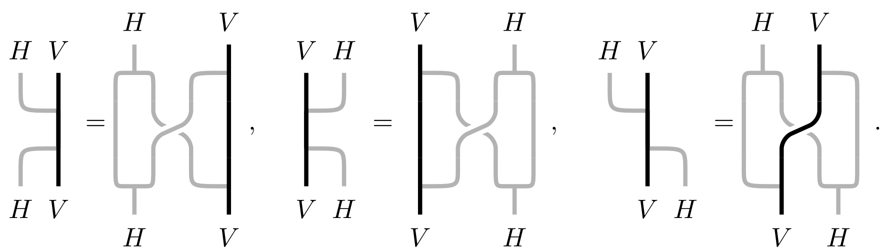
A **left (right)** H**-Hopf module** is a triple (V, λ, δ), where (V, λ) is a left (right) H-module, and (V, δ) is a left (right) H-comodule such that $\delta : V \to H \otimes V$ ($\delta : V \to V \otimes H$) is a morphism of left (right) H-modules.

A **left-right** H**-Hopf module** is a triple (V, λ, δ), where (V, λ) is a left H-module, and (V, δ) is a right H-comodule such that $\delta : V \to V \otimes H$ is a morphism of left H-modules. (Here, H is a left and right H-module by multiplication, and $H \otimes V$, $V \otimes H$ are H-modules with diagonal action.)

We denote the categories of left, right and of left-right H-Hopf modules by ${}^H_H\mathcal{C}$, $\mathcal{C}^H_H$, and ${}_H\mathcal{C}^H$, respectively. Their morphisms are H-module and H-comodule morphisms.

The pictures for left, right and left-right Hopf modules are

(3.5.12)



Note that the notion of a Hopf module is self-dual. The Hopf module axiom is equivalent to saying that the module structure is a morphism of H-comodules.

THEOREM 3.5.12. *Let H be a Hopf algebra in $\mathcal{C}$, and (V, λ, δ) a Hopf module in ${}_H\mathcal{C}^H$. Assume that V has a left dual $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$. Then $(V^*, \lambda_{r+}, \delta_l)$ is a Hopf module in ${}^H_H\mathcal{C}$, where λ_r and δ_l are defined in* (3.5.11)*, and*

$$\lambda_{r+} = \lambda_r c_{H,V^*}(\mathcal{S}_H \otimes \mathrm{id}).$$

PROOF. See Figure 3.5.1, where the first equality follows from the definition of λ_{r+}, the second from (3.2.12) with $h = \delta_l$, the third from the definition of δ_l, the fourth from duality (3.5.1), the fifth from the defining equation (3.5.12) of the left-right Hopf module V, the sixth from (3.2.13) with $h = \Delta_H$ and from associativity of H, the seventh from (3.2.17), the eighth from (3.2.26) and coassociativity, and the last equality from the definitions of δ_l and λ_{r+} and duality (3.5.1). $\square$

The next result is the fundamental theorem for one-sided Hopf modules of Larson and Sweedler extended to the braided situation. We will state it in Theorem 3.5.14 for left Hopf modules.

In the rest of the section let H be a Hopf algebra in $\mathcal{C}$.

DEFINITION 3.5.13. Let $(V, \delta) \in {}^H\mathcal{C}$. We say that $({}^{\mathrm{co}\,H}V, \iota)$ exists if

$$ {}^{\mathrm{co}\,H}V \xrightarrow{\ \iota\ } V \underset{\eta \otimes \mathrm{id}_V}{\overset{\delta}{\rightrightarrows}} H \otimes V$$

is an equalizer diagram in $\mathcal{C}$.

A left H-module (V, λ) is called **trivial**, if $\lambda = \varepsilon \otimes \mathrm{id} : H \otimes V \to V$. Any object $V \in \mathcal{C}$ is a trivial H-module via the action of ε.

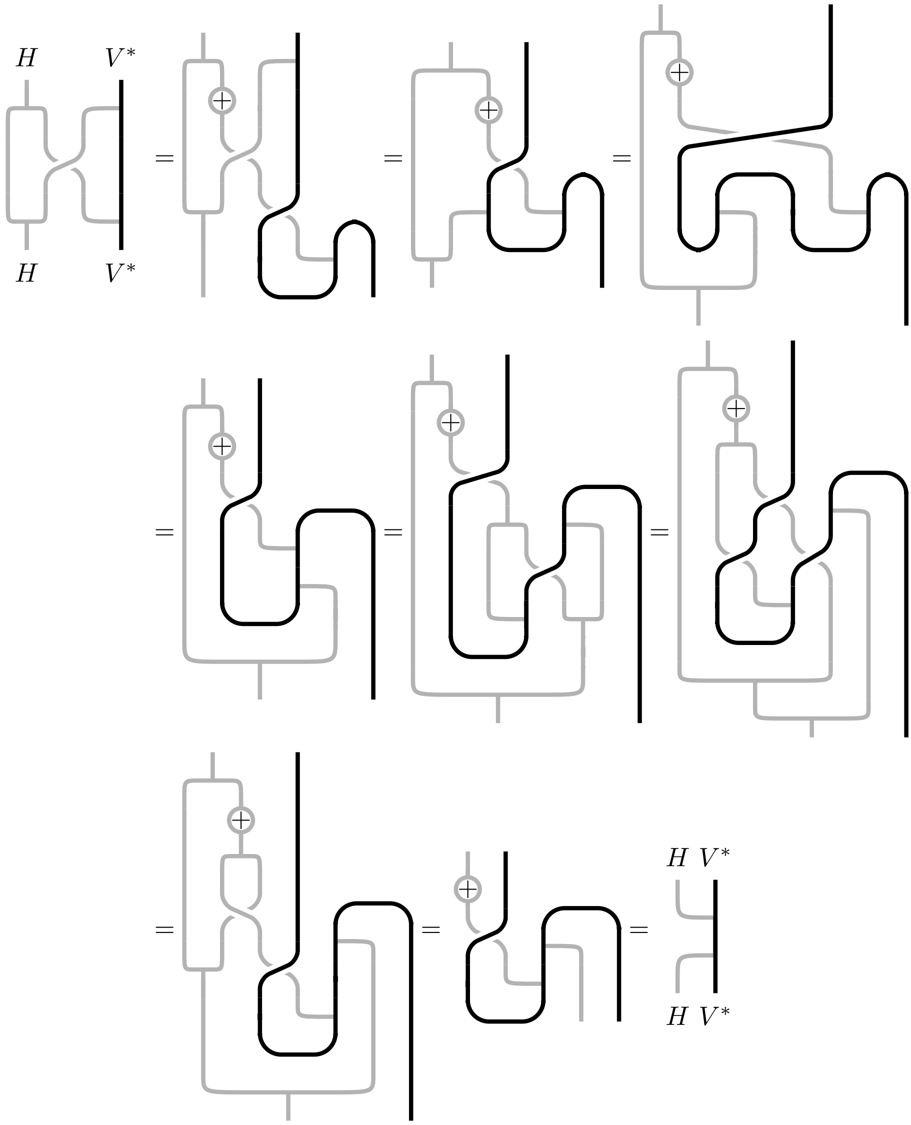


FIGURE 3.5.1. Proof of the Hopf module axiom for the dual

THEOREM 3.5.14. *Let* $(V, \lambda, \delta) \in {}^H_H\mathcal{C}$, *and assume that* $({}^{\mathrm{co}\,H}V, \iota)$ *exists.*

(1) (a) *There is a uniquely determined morphism* $\vartheta : V \to {}^{\mathrm{co}\,H}V$ *with*

$$(V \xrightarrow{\vartheta} {}^{\mathrm{co}\,H}V \xrightarrow{\iota} V) = (V \xrightarrow{\delta} H \otimes V \xrightarrow{\mathcal{S}_H \otimes \mathrm{id}_V} H \otimes V \xrightarrow{\lambda} V).$$

(b) $({}^{\mathrm{co}\,H}V \xrightarrow{\iota} V \xrightarrow{\vartheta} {}^{\mathrm{co}\,H}V) = \mathrm{id}_{{}^{\mathrm{co}\,H}V}$.

(c) $\vartheta : V \to {}^{\mathrm{co}\,H}V$ *is left* H*-linear, where* ${}^{\mathrm{co}\,H}V$ *is a trivial left* H*-module.*

(d) *The following is a coequalizer diagram.*

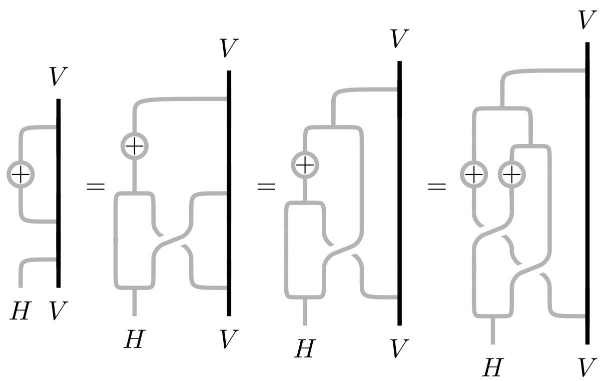
$$H \otimes V \underset{\varepsilon\otimes \mathrm{id}_V}{\overset{\lambda}{\rightrightarrows}} V \xrightarrow{\vartheta} {}^{\mathrm{co}\,H}V$$

(2) *The morphisms*

$$H \otimes {}^{\mathrm{co}\,H}V \xrightarrow{\mathrm{id}\otimes\iota} H\otimes V \xrightarrow{\lambda} V, \quad V \xrightarrow{\delta} H \otimes V \xrightarrow{\mathrm{id}\otimes\vartheta} H \otimes {}^{\mathrm{co}\,H}V$$

are inverse isomorphisms in ${}^H_H\mathcal{C}$, *where* $H \otimes {}^{\mathrm{co}\,H}V$ *is a Hopf module with module structure* $\mu_H \otimes \mathrm{id}_{{}^{\mathrm{co}\,H}V}$ *and comodule structure* $\Delta_H \otimes \mathrm{id}_{{}^{\mathrm{co}\,H}V}$.

Proof. Let $\Theta = (V \xrightarrow{\delta} H \otimes V \xrightarrow{\mathcal{S}_H\otimes\mathrm{id}_V} H\otimes V \xrightarrow{\lambda} V)$.
(1)(a) To prove that $\iota\vartheta = \Theta$, it suffices to show that $\delta\Theta = (\eta\otimes\mathrm{id})\Theta$.

$\delta\Theta =$  $= (\eta \otimes \mathrm{id})\Theta,$

where the second equality follows from the Hopf module axiom, the third from coassociativity of the comodule V, the fourth from (3.2.26) and coassociativity of H, and the last from the axiom of the antipode.

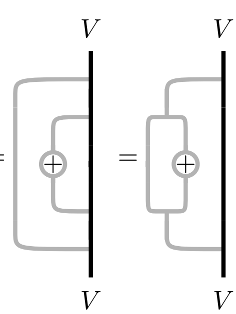
(1)(b) The equation $\delta\iota = (\eta\otimes\mathrm{id})\iota$ implies $\Theta\iota = \iota$. Hence $\iota\vartheta\iota = \Theta\iota = \iota$ by (a), and $\vartheta\iota = \mathrm{id}$, since ι is a monomorphism.

(1)(c),(d) The equation $\Theta\lambda = \Theta(\varepsilon\otimes\mathrm{id})$ follows by duality from (1)(a). Since ι is a monomorphism and $\Theta = \iota\vartheta$, $\vartheta\lambda = \vartheta(\varepsilon\otimes\mathrm{id}) = \varepsilon\otimes\vartheta$. Thus ϑ is left H-linear.

Let $Z \in \mathcal{C}$ and $h : V \to Z$ a morphism with $h\lambda = h(\varepsilon\otimes\mathrm{id})$. If there is a morphism $h' : {}^{\mathrm{co}\,H}V \to Z$ with $h = h'\vartheta$, then $h\iota = h'\vartheta\iota = h'$. It remains to show that $h = h\iota\vartheta = h\Theta$. By definition of Θ, and since $h\lambda = h(\varepsilon\otimes\mathrm{id})$,

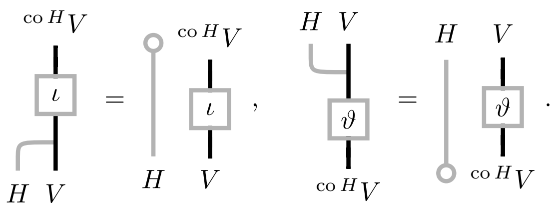
$$h\Theta = h\lambda(\mathcal{S}\otimes\mathrm{id}_V)\delta = h(\varepsilon\otimes\mathrm{id})(\mathcal{S}\otimes\mathrm{id}_V)\delta = h\mathrm{id}_V = h.$$

(2) By (1), associativity and coassociativity of V, and by the axiom of the antipode,

$\lambda(\mathrm{id}\otimes\iota)(\mathrm{id}\otimes\vartheta)\delta = \lambda(\mathrm{id}\otimes\Theta)\delta =$  $= \mathrm{id}_V.$

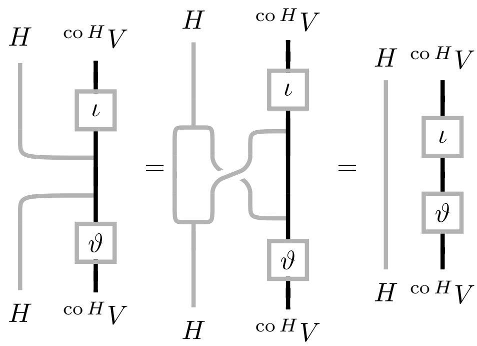
Note that by definition of ι and by (1)(c),

(3.5.13)



Then

$$(\mathrm{id}\otimes\vartheta)\delta\lambda(\mathrm{id}\otimes\iota) =$$



$$= \mathrm{id}_{H\otimes {}^{\mathrm{co}\,H}V},$$

where the second equality follows from the Hopf module axiom, the third from (3.5.13), and the last from (1)(b).

We have shown that the morphisms in (2) are inverse isomorphisms. They are morphisms of Hopf modules in ${}^H_H\mathcal{C}$, since $\lambda(\mathrm{id}\otimes\iota)$ is left H-linear, and $(\mathrm{id}\otimes\vartheta)\delta$ is left H-colinear. □

3.6. Smash products and smash coproducts

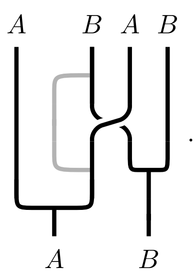
Let $\mathcal{C}$ be a braided strict monoidal category, and H a bialgebra in $\mathcal{C}$.

The Yetter-Drinfeld map in Definition 3.4.4 defines a **generalized smash product algebra**.

DEFINITION 3.6.1. Let A be an algebra in ${}_H\mathcal{C}$ and B an algebra in ${}^H\mathcal{C}$. Let $A\#B = (A\otimes B, \mu_{A\#B}, \eta_{A\#B})$, where $\eta_{A\#B} = \eta_A\otimes\eta_B$, and

$$\mu_{A\#B} = \big(A\otimes B\otimes A\otimes B \xrightarrow{\mathrm{id}_A\otimes c^{\mathcal{YD}}_{B,A}\otimes\mathrm{id}_B} A\otimes A\otimes B\otimes B \xrightarrow{\mu_A\otimes\mu_B} A\otimes B\big)$$

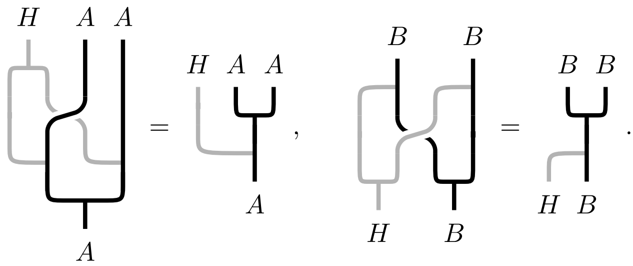
$$=$$



PROPOSITION 3.6.2. *Let A be an algebra in ${}_H\mathcal{C}$ and B an algebra in ${}^H\mathcal{C}$. Then $A\#B = (A\otimes B, \mu_{A\#B}, \eta_{A\#B})$ is an algebra in $\mathcal{C}$.*

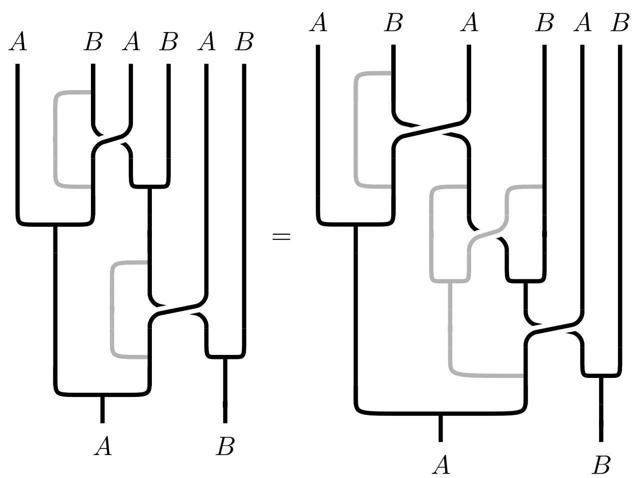
PROOF. Since μ_A is H-linear, and μ_B is H-colinear,

(3.6.1)



It is easy to see that $\eta_{A\#B}$ is a unit. We prove associativity of $\mu_{A\#B}$. Let λ_A be the module strucure of A and δ_B the comodule structure of B.

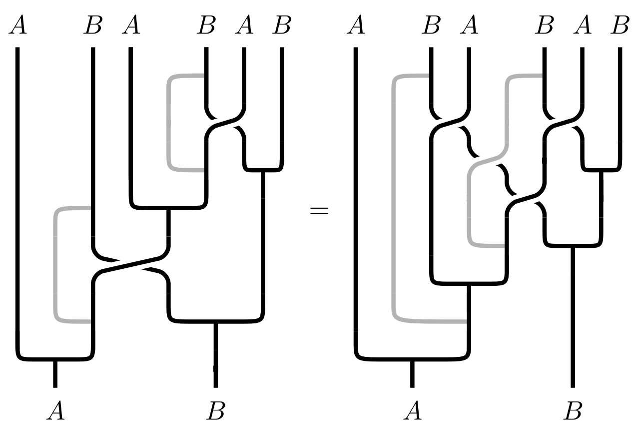
$\mu_{A\#B}(\mu_{A\#B} \otimes \mathrm{id}) =$



,

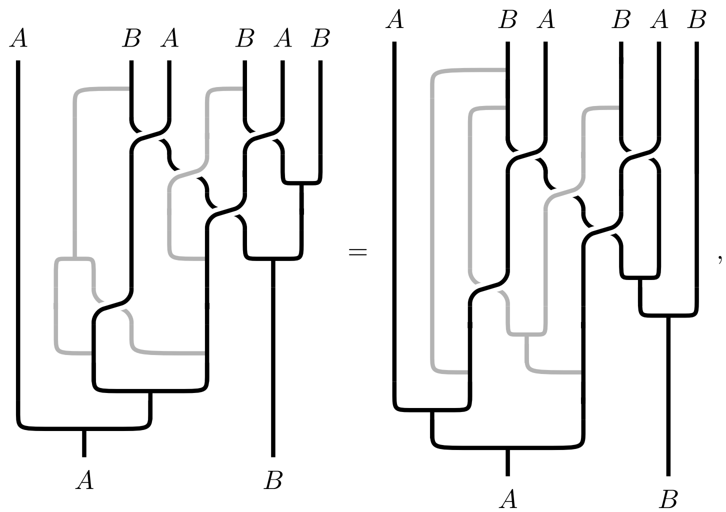
where the second equality follows from the second formula in (3.6.1).

$\mu_{A\#B}(\mathrm{id} \otimes \mu_{A\#B}) =$



,

where the second equation follows from (3.2.12) with $h = \mu_A(\mathrm{id}_A \otimes \lambda_A)$. Then the first formula in (3.6.1) gives the picture



where the last equality follows from associativity of A and B, and from the comodule axiom for δ_B and the module axiom for λ_A. Finally, associativity of $\mu_{A\#B}$ follows from (3.2.13) by moving the lower $\delta_B = h$ on the left-hand side to the right and the upper $\mu_B = h$ to the left in the last picture. □

We note that in Proposition 3.6.2,

$$\iota_1 = \mathrm{id}_A \otimes \eta_B : A \to A\#B, \qquad \iota_2 : \eta_A \otimes \mathrm{id}_B : B \to A\#B$$

are algebra morphisms, and the multiplication map

$$A \otimes B \xrightarrow{\iota_1 \otimes \iota_2} A\#B \otimes A\#B \xrightarrow{\mu_{A\#B}} A\#B$$

is the identity morphism.

A **left (right) H-module algebra** in $\mathcal{C}$ is an algebra in the monoidal category ${}_H\mathcal{C}$ (in $\mathcal{C}_H$, respectively). A **left (right) H-comodule algebra** in $\mathcal{C}$ is an algebra in ${}^H\mathcal{C}$ (in $\mathcal{C}^H$, respectively).

For any monoidal category $\mathcal{D}$ we denote by $\mathrm{ALG}(\mathcal{D})$ the **category of algebras in $\mathcal{D}$**. Objects in $\mathrm{ALG}(\mathcal{D})$ are the algebras in $\mathcal{D}$, and morphisms the algebra morphisms.

REMARK 3.6.3. Let A be an algebra in $\mathcal{C}$, and (A, δ_A) a left (right) H-comodule. Then A is a left (right) H-comodule algebra if and only if δ_A is a morphism of algebras in $\mathcal{C}$.

DEFINITION 3.6.4. Let A be a left H-module algebra in $\mathcal{C}$ with H-module structure λ_A. The **smash product algebra** $A\#H$ is the object $A \otimes H$ with multiplication and unit morphism

$$\mu_{A\#H} = (\mu_A \otimes \mathrm{id}_H)(\mathrm{id}_A \otimes \lambda_{A\otimes H}), \qquad \eta_{A\#H} = \eta_A \otimes \eta_H.$$

Here, $\lambda_{A\otimes H}$ is the left H-module structure on $A\otimes H$ given by the monoidal structure of ${}_H\mathcal{C}$, where A and H are H-modules by λ_A and μ, respectively.

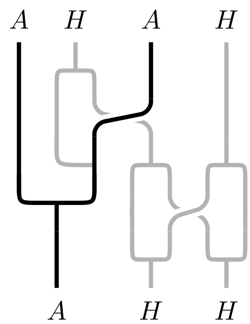
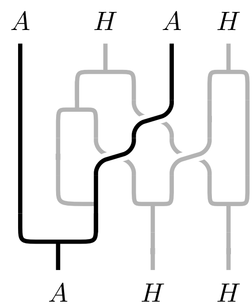
Thus $A\#H$ is the special case of Definition 3.6.1 with left H-comodule algebra $B = H$ via multiplication μ_H and H-comodule structure Δ_H, since

$$\mu_{A\#H} = (\mu_A \otimes \mu_H)(\mathrm{id}_A \otimes c^{\mathcal{YD}}_{H,A} \otimes \mathrm{id}_H). \tag{3.6.2}$$

PROPOSITION 3.6.5. (1) *Let A be a left H-module algebra in $\mathcal{C}$ with H-module structure λ_A. Then $(A \otimes H, \mu_{A\#H}, \eta_{A\#H})$ is a right H-comodule algebra in $\mathcal{C}$ with H-comodule structure $\delta_{A\#H} = \mathrm{id}_A \otimes \Delta_H$.*

(2) $\mathrm{ALG}({}_H\mathcal{C}) \to \mathrm{ALG}(\mathcal{C}^H)$, $(A, \lambda_A) \mapsto (A\#H, \delta_{A\#H})$, *and where morphisms φ are mapped onto $\varphi \otimes \mathrm{id}_H$, is a well-defined functor.*

PROOF. (1) By Proposition 3.6.2, $A\#H$ is an algebra. We prove that $\mu_{A\#H}$ is right H-colinear.

$\delta_{A\#H}\mu_{A\#H} =$  $=$  ,

where the first equality follows from the bialgebra axiom, and the second from (3.2.13) with $h = \Delta_H$, and from coassociativity. Hence $\mu_{A\#H}$ is H-colinear, since the second picture is $(\mu_{A\#H} \otimes \mathrm{id}_H)\delta_{(A\#H)\otimes(A\#H)}$.

(2) is easy to check. $\square$

PROPOSITION 3.6.6. *Let A be a left H-module algebra in $\mathcal{C}$. Then the functor*

$${}_A({}_H\mathcal{C}) \to_{A\#H} \mathcal{C},\ ((V, \lambda_H), \lambda_A) \mapsto (V, \lambda_A(\mathrm{id}_A \otimes \lambda_H)),$$

where morphisms f are mapped to f, is an isomorphism. The inverse functor is given by $(V, \lambda_{A\#H}) \mapsto ((V, \lambda_H), \lambda_A)$, where

$$\lambda_H = \lambda_{A\#H}(\gamma \otimes \mathrm{id}_V), \quad \lambda_A = \lambda_{A\#H}(\iota_1 \otimes \mathrm{id}_V).$$

PROOF. This follows directly from the definitions. $\square$

We now dualize. A **left (right) H-comodule coalgebra** is a coalgebra in ${}^H\mathcal{C}$ (in $\mathcal{C}^H$, respectively). A **left (right) H-module coalgebra** is a coalgebra in ${}_H\mathcal{C}$ (in $\mathcal{C}_H$, respectively).

For any monoidal category $\mathcal{D}$ we denote by $\mathrm{COALG}(\mathcal{D})$ the **category of coalgebras in $\mathcal{D}$**. Objects in $\mathrm{COALG}(\mathcal{D})$ are the coalgebras in $\mathcal{D}$, and morphisms the coalgebra morphisms.

REMARK 3.6.7. Let C be a coalgebra in $\mathcal{C}$, and (C, λ_C) a left (right) H-module. Then C is a left (right) H-module coalgebra if and only if λ_C is a morphism of coalgebras in $\mathcal{C}$.

DEFINITION 3.6.8. Let C be a left H-comodule coalgebra with H-comodule structure δ_C. The **smash coproduct coalgebra** $C\#H$ is the object $C \otimes H$ with comultiplication and counit morphism

$$\Delta_{C\#H} = \big(C \otimes H \xrightarrow{\Delta_H \otimes \mathrm{id}} C \otimes C \otimes H \xrightarrow{\mathrm{id} \otimes \delta_{C\otimes H}} C \otimes H \otimes C \otimes H\big),$$
$$\varepsilon_{C\#H} = \varepsilon_C \otimes \varepsilon_H.$$

Here, $\delta_{C\otimes H}$ is the left H-comodule structure on $C \otimes H$ given by the monoidal structure of ${}^H\mathcal{C}$, where C and H are H-comodules by δ_C and Δ_H, respectively.

Dually to (3.6.2), the smash coproduct of $C\#H$ can also be written as

(3.6.3) $$\Delta_{C\#H} = (\mathrm{id}_C \otimes c^{\mathcal{YD}}_{C,H} \otimes \mathrm{id}_H)(\Delta_C \otimes \Delta_H),$$

where $H \in {}_H\mathcal{C}$ via μ_H.

PROPOSITION 3.6.9. (1) *Let C be a left H-comodule coalgebra in $\mathcal{C}$ with H-comodule structure δ_C. Then the triple $(C\#H, \Delta_{C\#H}, \varepsilon_{C\#H})$ is a right H-module coalgebra in $\mathcal{C}$ with H-module structure $\lambda_{C\#H} = \mathrm{id}_C \otimes \mu_H$.*

(2) $\mathrm{COALG}({}^H\mathcal{C}) \to \mathrm{COALG}(\mathcal{C}_H)$, $(C, \delta_C) \mapsto (C\#H, \lambda_{C\#H})$, *and where morphisms φ are mapped onto $\varphi \otimes \mathrm{id}_C$, is a well-defined functor.*

PROOF. Dual to Proposition 3.6.5. □

PROPOSITION 3.6.10. *Let C be a left H-comodule coalgebra in $\mathcal{C}$. Then the functor*

$${}^C({}^H\mathcal{C}) \to {}^{C\#H}\mathcal{C}, \quad ((V, \delta_H), \delta_C) \mapsto (V, (\mathrm{id}_C \otimes \delta_H)\delta_C),$$

where morphisms f are mapped to f, is an isomorphism. The inverse functor is given by $(V, \delta_{C\#H}) \mapsto ((V, \delta_H), \delta_C)$, where

$$\delta_H = (\pi \otimes \mathrm{id}_V)\delta_{C\#H}, \quad \delta_C = (\vartheta \otimes \mathrm{id}_V)\delta_{C\#H}.$$

PROOF. Dual to Proposition 3.6.6. □

3.7. Adjoint action and adjoint coaction

Let $\mathcal{C}$ be a braided strict monoidal category, and H a Hopf algebra in $\mathcal{C}$. We discuss here the concept of the adjoint action in a general setting.

Let A be an algebra in $\mathcal{C}$, $V \in \mathcal{C}$, λ_l a left A-module structure and λ_r a right A-module structure on V. Then $(V, \lambda_l, \lambda_r)$ is an A-**bimodule** if the following diagram commutes:

$$\begin{tikzcd} A \otimes V \otimes A \arrow[r, "\lambda_l \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A \otimes \lambda_r"'] & V \otimes A \arrow[d, "\lambda_r"] \\ A \otimes V \arrow[r, "\lambda_l"] & V \end{tikzcd}$$

The category of A-bimodules in $\mathcal{C}$ is denoted by ${}_A\mathcal{C}_A$.

PROPOSITION 3.7.1. *The functor*

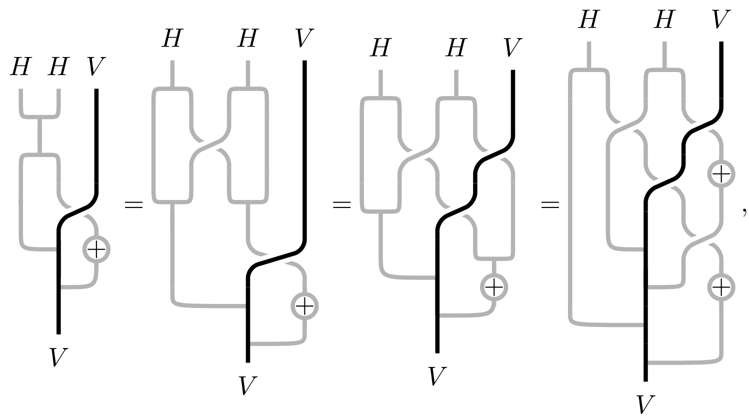
$$\mathrm{ad} : {}_H\mathcal{C}_H \to {}_H\mathcal{C}, \ (V, \lambda_l, \lambda_r) \mapsto (V, \mathrm{ad}),$$

where $\mathrm{ad} = H \otimes V \xrightarrow{\Delta_H \otimes \mathrm{id}_V} H \otimes H \otimes V \xrightarrow{\mathrm{id}_H \otimes c_{H,V}} H \otimes V \otimes H$
$$\xrightarrow{\lambda_l \otimes \mathcal{S}_H} V \otimes H \xrightarrow{\lambda_r} V,$$

with $\mathrm{ad}(f) = f$ for morphisms f of H-bimodules, is well-defined.

Note that in general the functor ad is not strict monoidal.

PROOF. (1) Let $(V, \lambda_l, \lambda_r)$ be an H-bimodule. We show that (V, ad) is a left H-module. Clearly, ad is a morphism in $\mathcal{C}$. The unit axiom for ad is easily checked. We have to prove the equality $\mathrm{ad}(\mu_H \otimes \mathrm{id}) = \mathrm{ad}(\mathrm{id}_H \otimes \mathrm{ad})$; $\mathrm{ad}(\mu_H \otimes \mathrm{id})$ equals



where the first equation follows from the bialgebra axiom for H, the second from functoriality of the braiding (3.2.13), and the third from the rules for the antipode (3.2.26) and the axioms of a module and a bimodule. By functoriality of the braiding (3.2.12) with $h = \lambda_r(\lambda_l \otimes \mathrm{id})$, the last picture is equal to $\mathrm{ad}(\mathrm{id}_H \otimes \mathrm{ad})$.

(2) Let $f : V \to W$ be a morphism of H-bimodules in $\mathcal{C}$. We have to show that f is a morphism in ${}_H\mathcal{C}$, that is, $f\mathrm{ad} = \mathrm{ad}(\mathrm{id} \otimes f)$. The latter is clear since f is a morphism of H-bimodules in $\mathcal{C}$. □

PROPOSITION 3.7.2. (1) *Let $(V, \lambda_l, \lambda_r)$ be an H-bimodule.*
- (a) $\lambda_l = \lambda_r(\mathrm{ad} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes c_{H,V})(\Delta_H \otimes \mathrm{id}_V)$, *and*
- (b) $\lambda_l(\mathcal{S}_H \otimes \mathrm{ad})(\Delta_H \otimes \mathrm{id}_V) = \lambda_r(\mathrm{id}_V \otimes \mathcal{S}_H)c_{H,V}$.

(2) *Let A be an algebra, and $\gamma : H \to A$ an algebra morphism in $\mathcal{C}$. Then $(A, \lambda_l, \lambda_r)$ is an H-bimodule with $\lambda_l = \mu(\gamma \otimes \mathrm{id}_A)$ and $\lambda_r = \mu(\mathrm{id}_A \otimes \gamma)$, and (A, ad) is a left H-module algebra.*

PROOF. (1)(a) and (b) follow from associativity of λ_r and λ_l, respectively, and from coassociativity of Δ_H and the axiom of the antipode.

(2) It is easy to check that (A, λ_l) is a left H-module, (A, λ_r) is a right H-module, the bimodule axiom holds, and that η_A is H-linear. We prove that the multiplication map $\mu_A : A \otimes A \to A$ is left H-linear with respect to ad, where the action on $A \otimes A$ is the diagonal action.

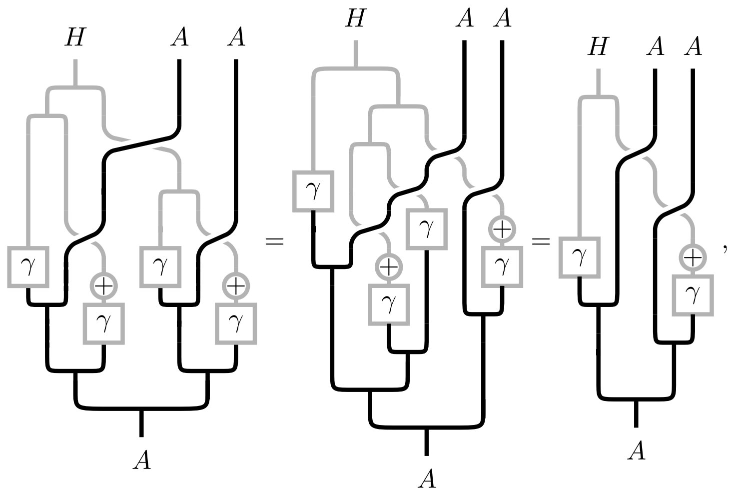
Let $A_l = (A, \lambda_l, \mathrm{id} \otimes \varepsilon)$ and $A_r = (A, \varepsilon \otimes \mathrm{id}, \lambda_r)$ as H-bimodules. Then

$$\mu_A : A_l \otimes A_r \to A$$

is a morphism of H-bimodules by associativity of μ_A. Thus

$$\mu_A \mathrm{ad} = \mathrm{ad}(\mathrm{id}_H \otimes \mu_A) : H \otimes A_l \otimes A_r \to A$$

for the functorial action ad by Proposition 3.7.1. It remains to prove the equation $\mu_A \mathrm{ad} = \mu_A \mathrm{ad}_{\mathrm{diag}}$, where $\mathrm{ad}_{\mathrm{diag}}$ is the diagonal action on $A \otimes A$. The latter holds since

$\mu_A \mathrm{ad}_{\mathrm{diag}} =$ 

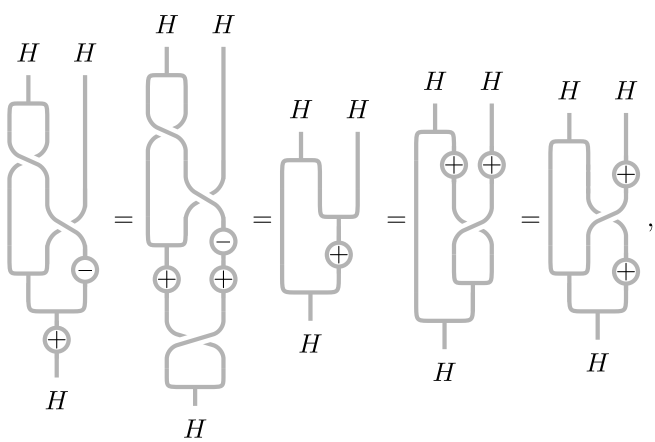
where the second equation follows from coassociativity of H, associativity of A, and (3.2.13), and the third from the axiom for the antipode of H and since γ is an algebra map. The last picture is $\mu_A\mathrm{ad}$. $\square$

DEFINITION 3.7.3. If A is an algebra and $\gamma : H \to A$ is a morphism of algebras in $\mathcal{C}$, then ad in Proposition 3.7.2(2) is called the **left adjoint action** of H on A with respect to γ, and we denote it by ad_γ. The left adjoint action of H on H with respect to id_H is denoted by $\mathrm{ad}_H : H \otimes H \to H$.

LEMMA 3.7.4. *If the antipode $\mathcal{S}_H$ of H is an isomorphism in $\mathcal{C}$, then*

$$\mathcal{S}_H \mathrm{ad}_{H^{\mathrm{cop}}} = \mathrm{ad}_H(\mathrm{id} \otimes \mathcal{S}_H) : H \otimes H \to H.$$

PROOF.



where the first equation follows from the rules of the antipode (3.2.26), the second from functoriality of the braiding (3.2.12) and (3.2.9), the third again from (3.2.26), and the fourth from associativity. $\square$

The monoidal structure of ${}_H\mathcal{C}$ and $\mathcal{C}_H$ defines a monoidal structure for the category ${}_H\mathcal{C}_H$ of H-bimodules in $\mathcal{C}$. It follows from an easy argument (using the functoriality of the braiding) that the tensor product of two H-bimodules is in fact an H-bimodule. The multiplication μ of H defines an H-bimodule structure on H. Then (H, Δ, ε) is a coalgebra in ${}_H\mathcal{C}_H$.

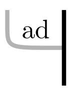
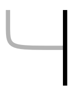
PROPOSITION 3.7.5. *The functor*

$$^H({}_H\mathcal{C}_H) \to {}^H_H\mathcal{YD}(\mathcal{C}),\ ((V, \lambda_l, \lambda_r), \delta) \mapsto (V, \mathrm{ad}, \delta),$$

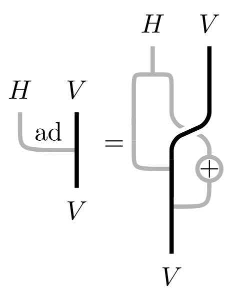
where $\mathrm{ad} : H \otimes V \to V$ *is the adjoint H-module structure of Proposition* 3.7.1, *and where morphisms f are mapped onto f, is well-defined.*

PROOF. We prove that (V, ad, δ) is an object in ${}^H_H\mathcal{YD}(\mathcal{C})$.

The module structures and the comodule structure of V are denoted by

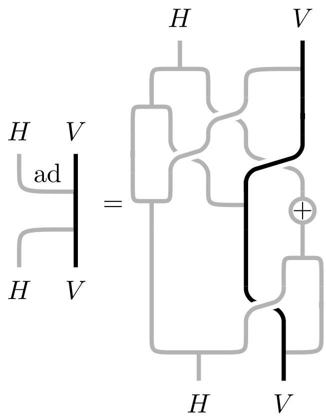
ad =  , $\lambda_l =$  , $\lambda_r =$  , $\delta =$  .

Let ${}_{\mathrm{ad}}V$ and ${}_{\mathrm{ad}}(H \otimes V)$ be the left H-modules of Proposition 3.7.1(1) for the bimodules V and $H \otimes V$, respectively. By definition,



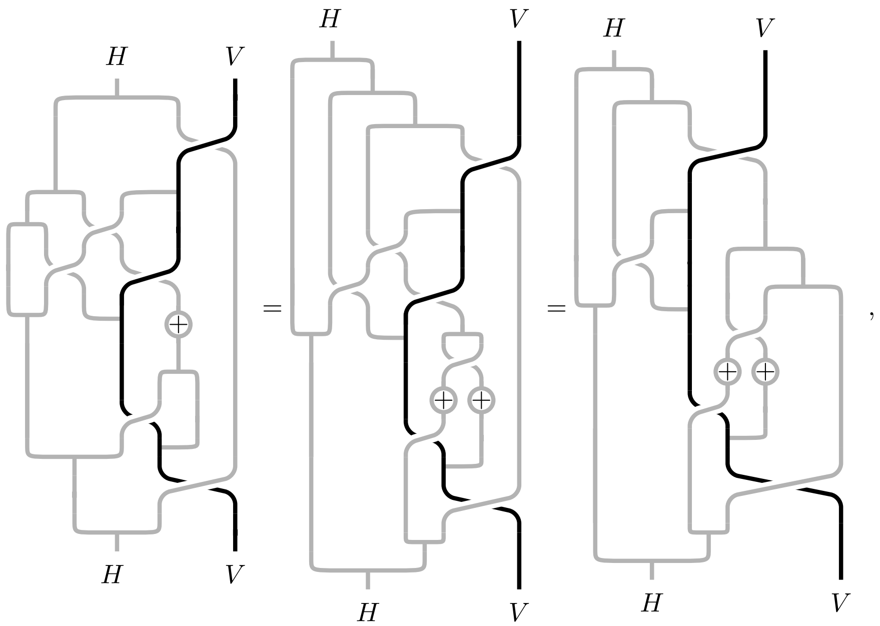
, and

(3.7.1)

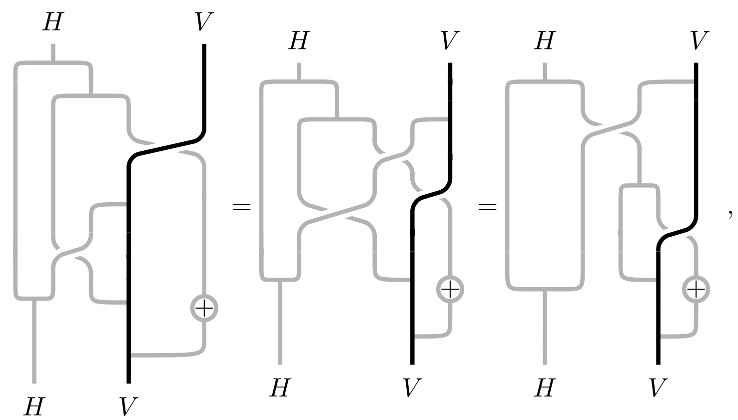


,

since $\delta : {}_{\mathrm{ad}}V \to {}_{\mathrm{ad}}(H \otimes V)$ is left H-linear. We prove the defining equality (3.4.1) of left Yetter-Drinfeld modules for (V, ad, δ). By (3.7.1), the left-hand side of (3.4.1) is equal to



where the first equality follows from associativity and coassociativity and the rule for the antipode in Proposition 3.2.12(3). To prove the second equation we move the third comultiplication across the braiding by (3.2.13) and then use coassociativity. By definition of the antipode, the picture simplifies to



where the first equality follows by moving δ across the braiding by (3.2.12), and the second by moving the second comultiplication across the braiding by (3.2.13). The last picture is the right-hand side of (3.4.1) for (V, ad, δ). $\square$

We dualize the previous notions. Let C be a coalgebra in $\mathcal{C}$, $V \in \mathcal{C}$, δ_l a left C-comodule structure and δ_r a right C-comodule structure on V. Then (V, δ_l, δ_r) is a C-**bicomodule** if $(\mathrm{id}_C \otimes \delta_r)\delta_l = (\delta_l \otimes \mathrm{id}_C)\delta_r : V \to C \otimes V \otimes C$. The category of C-bicomodules is denoted by ${}^C\mathcal{C}^C$.

PROPOSITION 3.7.6. *The functor*

$$\mathrm{coad} : {}^H\mathcal{C}^H \to {}^H\mathcal{C}\,, \quad (V,\delta_l,\delta_r) \mapsto (V,\mathrm{coad}\,),$$

$$\textit{where}\ \mathrm{coad} = \big(V \xrightarrow{\delta_r} V\otimes H \xrightarrow{\delta_l\otimes\mathcal{S}_H} H\otimes V\otimes H$$
$$\xrightarrow{\mathrm{id}_H\otimes c_{V,H}} H\otimes H\otimes V \xrightarrow{\mu_H\otimes\mathrm{id}_V} H\otimes V\big),$$

with $\mathrm{coad}(f) = f$ *for each morphism* f *of* H*-bicomodules, is well-defined.*

PROPOSITION 3.7.7. *Let* C *be a coalgebra, and* $\pi : C \to H$ *a coalgebra morphism in* $\mathcal{C}$*. Then* (C,δ_l,δ_r) *is an* H*-bicomodule with* $\delta_l = (\pi\otimes\mathrm{id}_C)\Delta$ *and* $\delta_r = (\mathrm{id}_C\otimes\pi)\Delta$*, and* (C,coad) *is a left* H*-comodule coalgebra, where* coad *is the left* H*-comodule structure defined in Proposition* 3.7.6 *based on the* H*-bicomodule* (C,δ_l,δ_r)*.*

DEFINITION 3.7.8. If C is a coalgebra and $\pi : C \to H$ is a morphism of coalgebras in $\mathcal{C}$, then coad in Proposition 3.7.7 is called the **left adjoint coaction** of H on C with respect to π, and we denote it by coad_π. If $C = H$ and $\pi = \mathrm{id}_H$, then we write coad_H for coad_π.

We note the dual of Proposition 3.7.5, where (H,μ,η) is an algebra in the category ${}^H\mathcal{C}^H$ of H-bicomodules.

PROPOSITION 3.7.9. *The functor*

$${}_H({}^H\mathcal{C}^H) \to {}^H_H\mathcal{YD}(\mathcal{C}),\ ((V,\delta_l,\delta_r),\lambda) \mapsto (V,\lambda,\mathrm{coad}\,),$$

where coad $: V \to H\otimes V$ *is the coadjoint* H*-comodule structure of Proposition* 3.7.6*, and where morphisms* f *are mapped onto* f*, is well-defined.*

REMARK 3.7.10. For any monoidal category $\mathcal{C}$, a coalgebra C and an algebra A in $\mathcal{C}$, there are functors

$$\mathcal{C} \to {}^C\mathcal{C},\ V \mapsto (C\otimes V, \Delta_C\otimes\mathrm{id}_V),$$
$$\mathcal{C} \to {}_A\mathcal{C},\ V \mapsto (A\otimes V, \mu_A\otimes\mathrm{id}_V),$$

where in both cases morphisms f are mapped onto $\mathrm{id}\otimes f$.

In particular, there are functors ${}_H\mathcal{C}_H \to {}^H({}_H\mathcal{C}_H)$ and ${}^H\mathcal{C}^H \to {}_H({}^H\mathcal{C}^H)$. By composition with the functors in Propositions 3.7.5 and 3.7.9, we obtain functors

$${}_H\mathcal{C}_H \to {}^H_H\mathcal{YD}(\mathcal{C}),\ V \mapsto (H\otimes V, \mathrm{ad}_{H\otimes V}, \Delta_H\otimes\mathrm{id}_V),$$
$${}^H\mathcal{C}^H \to {}^H_H\mathcal{YD}(\mathcal{C}),\ V \mapsto (H\otimes V, \mu_H\otimes\mathrm{id}_V, \mathrm{coad}_{\,H\otimes V}).$$

If $\mathcal{C} = \mathcal{M}_\Bbbk$, then adjoint action and coaction on H are given by

$$\mathrm{ad}_H : H\otimes H \to H,\ h\otimes x \mapsto h_{(1)}x\mathcal{S}(h_{(2)}),$$
$$\mathrm{coad}_H : H \to H\otimes H,\ h \mapsto h_{(1)}\mathcal{S}(h_{(3)})\otimes h_{(2)}.$$

Let V be an H-bimodule. Then $H\otimes V \in {}^H_H\mathcal{YD}$ with H-coaction $\Delta_H\otimes\mathrm{id}_V$ and H-action

$$\mathrm{ad}_{H\otimes V} : H\otimes H\otimes V \to H\otimes V,\ g\otimes h\otimes v \mapsto g_{(1)}h\mathcal{S}(g_{(4)})\otimes g_{(2)}v\mathcal{S}(g_{(3)}),$$

Let V be an H-bicomodule. Then $H\otimes V \in {}^H_H\mathcal{YD}$ with H-action $\mu_H\otimes\mathrm{id}_V$ and H-coaction

$$\mathrm{coad}_{\,H\otimes V} : H\otimes V \to H\otimes H\otimes V,\ h\otimes v \mapsto h_{(1)}v_{(-1)}\mathcal{S}(h_{(3)}v_{(1)})\otimes h_{(2)}\otimes v_{(0)}.$$

3.8. Bosonization

Let $\mathcal{C}$ be a braided strict monoidal category, and H a Hopf algebra in $\mathcal{C}$. We introduce the important process of bosonization which transforms a bialgebra (or Hopf algebra) in ${}^H_H\mathcal{YD}(\mathcal{C})$ into a bialgebra (or Hopf algebra) in $\mathcal{C}$.

Definition 3.8.1. Let R be a bialgebra in ${}^H_H\mathcal{YD}(\mathcal{C})$. In particular, R is an algebra in ${}_H\mathcal{C}$ and a coalgebra in ${}^H\mathcal{C}$. We denote the H-action and H-coaction of R by

$$\lambda_R : H \otimes R \to R, \quad \delta_R : R \to H \otimes R.$$

Let $(R\#H, \mu_{R\#H}, \eta_{R\#H})$ be the corresponding smash product algebra of Definition 3.6.4, and $(R\#H, \Delta_{R\#H}, \varepsilon_{R\#H})$ the corresponding smash coproduct coalgebra in $\mathcal{C}$ of Definition 3.6.8. We call

$$R\#H = (R \otimes H, \mu_{R\#H}, \eta_{R\#H}, \Delta_{R\#H}, \varepsilon_{R\#H})$$

the **bosonization** (or the **Radford biproduct**) of R. Let

$$\begin{aligned} \pi &= \varepsilon_R \otimes \mathrm{id}_H : R\#H \to H, & \gamma &= \eta_R \otimes \mathrm{id}_H : H \to R\#H, \\ \iota &= \mathrm{id}_R \otimes \eta_H : R \to R\#H, & \vartheta &= \mathrm{id}_R \otimes \varepsilon_H : R\#H \to R. \end{aligned}$$

We will see in Proposition 3.8.4 that $R\#H$ is in fact a bialgebra in $\mathcal{C}$. The next lemma is easily verified.

Lemma 3.8.2. *Let R be a bialgebra in ${}^H_H\mathcal{YD}(\mathcal{C})$ and $R\#H$ the bosonization. Then $R\#H$ is an algebra and a coalgebra in $\mathcal{C}$, and*

(1) *π and γ are algebra and coalgebra morphisms with $\pi\gamma = \mathrm{id}_H$.*
(2) *ι is an algebra and ϑ a coalgebra morphism with $\vartheta\iota = \mathrm{id}_R$.*
(3) *ϑ is right H-linear, where $R\#H$ is a right H-module induced by the algebra morphism γ, that is, by $\mathrm{id}_R \otimes \mu_H$, and R is the trivial H-module defined via ε_H.*
(4) *ϑ is left H-linear, where $R\#H$ is a left H-module induced by the algebra morphism γ, that is, with H-action $\mu_{R\#H}(\gamma \otimes \mathrm{id}_{R\#H})$, and R is a left H-module by the given H-action on R.*
(5) *$(\mathrm{id}_{R\#H} \otimes \pi)\Delta_{R\#H} = \mathrm{id}_R \otimes \Delta_H : R\#H \to R\#H \otimes H$.*
(6) *$(\pi \otimes \mathrm{id}_{R\#H})\Delta_{R\#H} : R\#H \to H \otimes R\#H$ is the diagonal H-coaction on $R \otimes H$.*

Moreover, the maps ι and $\mu_{R\#H}$ satisfy the claims dual to Lemma 3.8.2(3)-(6). The following diagram describes the situation of Lemma 3.8.2.

$$\begin{tikzcd} & & H \arrow[dl, "\gamma"'] \arrow[d, "="] \\ R \arrow[r, "\iota"] \arrow[d, "="'] & R\#H \arrow[r, "\pi"] \arrow[dl, "\vartheta"] & H \\ R & & \end{tikzcd}$$

DEFINITION 3.8.3. Let R be a bialgebra in ${}^H_H\mathcal{YD}(\mathcal{C})$ and $R\#H$ the bosonization of R. We define functors

$$F_1 : {}_R({}^H_H\mathcal{YD}(\mathcal{C})) \to {}_{R\#H}\mathcal{C}, \quad ((V,\lambda^H,\delta^H),\lambda^R) \mapsto (V,\lambda^{R\#H}),$$
$$F_2 : {}^R({}^H_H\mathcal{YD}(\mathcal{C})) \to {}^{R\#H}\mathcal{C}, \quad ((V,\lambda^H,\delta^H),\delta^R) \mapsto (V,\delta^{R\#H}),$$
$$\text{where } \lambda^{R\#H} = \lambda^R(\mathrm{id}_R \otimes \lambda^H) \text{ and } \delta^{R\#H} = (\mathrm{id}_R \otimes \delta^H)\delta^R,$$

and where morphisms f are mapped to f.

Note that F_1 is the composition

$${}_R({}^H_H\mathcal{YD}(\mathcal{C})) \to {}_R({}_H\mathcal{C}) \cong {}_{R\#H}\mathcal{C},$$

of the forgetful functor and the isomorphism of Proposition 3.6.6. Similarly, F_2 is the composition

$${}^R({}^H_H\mathcal{YD}(\mathcal{C})) \to {}^R({}^H\mathcal{C}) \cong {}^{R\#H}\mathcal{C},$$

of the forgetful functor and the isomorphism of Proposition 3.6.10.

PROPOSITION 3.8.4. *Let R be a bialgebra in ${}^H_H\mathcal{YD}(\mathcal{C})$ with bosonization $R\#H$. Let $R\otimes H$ be the tensor product in ${}^H_H\mathcal{YD}(\mathcal{C})$ of R and H, where $H \in {}^H_H\mathcal{YD}(\mathcal{C})$ via μ_H and* coad_H.

(1) $(R\otimes H, \mu_R \otimes \mathrm{id}_H) \in {}_R({}^H_H\mathcal{YD}(\mathcal{C}))$, *and $F_1(R\otimes H)$ is the regular representation of $R\#H$, that is, $R\#H$ as a left $R\#H$-module via $\mu_{R\#H}$.*
(2) *$R\#H$ is a bialgebra in $\mathcal{C}$.*
(3) *The functors F_1 and F_2 are strict monoidal.*

PROOF. (1) By Proposition 3.7.9, $(H,\mu_H,\mathrm{coad}_H) \in {}^H_H\mathcal{YD}(\mathcal{C})$, hence $R\otimes H$ is an object in ${}_R({}^H_H\mathcal{YD}(\mathcal{C}))$ with R-module structure $\mu_R \otimes \mathrm{id}_H$. It is obvious that $F_1(R\otimes H)$ is the regular representation of $R\#H$.

(2) and (3). (a) Let $V,W \in {}_R({}^H_H\mathcal{YD}(\mathcal{C}))$. Then the diagonal action of $R\#H$ on $F_1(V)\otimes F_1(W)$ is the action of $R\#H$ on $F_1(V\otimes W)$. This follows directly from the definitions. In particular, $F_1(V)\otimes F_1(W)$ with the diagonal $R\#H$-action $\lambda_{F_1(V)\otimes F_1(W)}$ is a left $R\#H$-module.

(b) It is easy to see that $\varepsilon_{R\#H}$ is an algebra morphism, since ε_R is. By (1), $F_1(R\otimes H)$ is the regular representation of $R\#H$. By (a), the diagonal action of $R\#H$ on $R\#H\otimes R\#H$ defines a left $R\#H$-module. Thus $R\#H$ is a bialgebra by Proposition 3.2.8. We have shown (2). Then (a) says that F_1 is strict monoidal, and the claim for F_2 follows dually. □

Let R be a bialgebra in $\mathcal{C}$, and $V \in {}^R\mathcal{C}$, $X \in {}_R\mathcal{C}$. Recall the Yetter-Drinfeld morphism $c^{\mathcal{YD}(\mathcal{C})}_{V,X} : V\otimes X \to X\otimes V$ in Definition 3.4.4. For clarity we will also write $c^{\mathcal{YD}(\mathcal{C})}_{V,X} = c^{\mathcal{YD}(\mathcal{C},R)}_{V,X}$.

LEMMA 3.8.5. *Let R be a bialgebra in ${}^H_H\mathcal{YD}(\mathcal{C})$, $(V,\delta) \in {}^R({}^H_H\mathcal{YD}(\mathcal{C}))$, and $(V,\lambda),(X,\lambda_X) \in {}_R({}^H_H\mathcal{YD}(\mathcal{C}))$. Define $\delta^{R\#H},\lambda^{R\#H},\lambda_X^{R\#H}$ by*

$$(V,\delta^{R\#H}) = F_2(V,\delta), \quad (V,\lambda^{R\#H}) = F_1(V,\lambda), \quad (X,\lambda_X^{R\#H}) = F_1(X,\lambda_X).$$

(1) $c^{\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C}),R)}_{(V,\delta),(X,\lambda_X)} = c^{\mathcal{YD}(\mathcal{C},R\#H)}_{(V,\delta^{R\#H}),(X,\lambda_X^{R\#H})} : V\otimes X \to X\otimes V$, *as morphisms in $\mathcal{C}$.*
(2) *Let $f : V\otimes X \to X\otimes V$ be the morphism in (1). The following are equivalent.*

(a) *f is left R-linear.*
(b) *f is left $R\#H$-linear, where V and X are left $R\#H$-modules by $\lambda^{R\#H}$ and $\lambda_X^{R\#H}$.*

PROOF. (1) follows directly from the definitions.

(2) (a) $\Rightarrow$ (b) follows by applying the strict monoidal functor F_1.

(b) $\Rightarrow$ (a) is clear, since for all $X, Y \in {}_R({}^H_H\mathcal{YD}(\mathcal{C}))$ the restriction via the morphism $\iota : R \to R\#H$ of the diagonal $R\#H$-action on $F_1(X) \otimes F_1(Y)$ is the diagonal R-action on $X \otimes Y$. $\square$

LEMMA 3.8.6. *Let R be a bialgebra in ${}^H_H\mathcal{YD}(\mathcal{C})$ with bosonization $R\#H$. Let $(V, \lambda, \delta) \in {}^{R\#H}_{R\#H}\mathcal{YD}(\mathcal{C})$, and define*

$$\lambda^H = \lambda(\gamma \otimes \mathrm{id}_V) : H \otimes V \to V, \qquad \delta^H = (\pi \otimes \mathrm{id}_V)\delta : V \to H \otimes V,$$
$$\lambda^R = \lambda(\iota \otimes \mathrm{id}_V) : R \otimes V \to V, \qquad \delta^R = (\vartheta \otimes \mathrm{id}_V)\delta : V \to R \otimes V.$$

Then $(V, \lambda^H, \delta^H) \in {}^H_H\mathcal{YD}(\mathcal{C})$, and

$$((V, \lambda^H, \delta^H), \lambda^R) \in {}_R({}^H_H\mathcal{YD}(\mathcal{C})), \quad ((V, \lambda^H, \delta^H), \delta^R) \in {}^R({}^H_H\mathcal{YD}(\mathcal{C})).$$

PROOF. It is clear (see Propositions 3.6.6 and 3.6.10) that (V, λ^H) and (V, λ^R) are modules, (V, δ^H) and (V, δ^R) are comodules, and

$$((V, \lambda^H), \lambda^R) \in {}_R({}_H\mathcal{C}), \quad ((V, \delta^H), \delta^R) \in {}^R({}^H\mathcal{C}).$$

We have to prove

(1) $(V, \lambda^H, \delta^H) \in {}^H_H\mathcal{YD}(\mathcal{C})$,
(2) $((V, \lambda^H), \lambda^R) \in {}_R({}^H\mathcal{C})$, that is, λ^R is H-colinear,
(3) $((V, \delta^H), \delta^R) \in {}^R({}_H\mathcal{C})$, that is, δ^R is left H-linear.

We denote the left-hand side of the defining equation (3.4.1) of the YD-module $(V, \lambda, \delta) \in {}^{R\#H}_{R\#H}\mathcal{YD}(\mathcal{C})$ by φ_l, and the right-hand side by φ_r.

(1) follows from $(\pi \otimes \mathrm{id}_V)\varphi_l(\gamma \otimes \mathrm{id}_V) = (\pi \otimes \mathrm{id}_V)\varphi_r(\gamma \otimes \mathrm{id}_V)$, since π, γ are bialgebra morphisms with $\pi\gamma = \mathrm{id}_H$.

(2) Note that

$$(\mathrm{id}_{R\#H} \otimes \pi)\Delta_{R\#H}\iota = \iota \otimes \eta_H, \tag{3.8.1}$$
$$(\pi \otimes \mathrm{id}_{R\#H})\Delta_{R\#H}\iota = \delta_R \otimes \eta_H, \tag{3.8.2}$$

by Lemma 3.8.2(5) and (6).

Let $\delta^H_{R\otimes V} : R \otimes V \to H \otimes R \otimes V$ be the diagonal H-coaction. (2) follows from $(\pi \otimes \mathrm{id}_V)\varphi_l(\iota \otimes \mathrm{id}_V) = (\pi \otimes \mathrm{id}_V)\varphi_r(\iota \otimes \mathrm{id}_V)$, since

$$(\pi \otimes \mathrm{id}_V)\varphi_l(\iota \otimes \mathrm{id}_V) = \delta^H\lambda^R,$$
$$(\pi \otimes \mathrm{id}_V)\varphi_r(\iota \otimes \mathrm{id}_V) = (\mathrm{id}_H \otimes \lambda^R)\delta^H_{R\otimes V},$$

by (3.8.1) and (3.8.2), and since π is a bialgebra morphism.

(3) Let $\lambda^H_{R\otimes V} : H \otimes R \otimes V \to R \otimes V$ be the diagonal H-action on $R \otimes H$. (3) follows from $(\vartheta \otimes \mathrm{id}_V)\varphi_l(\gamma \otimes \mathrm{id}_V) = (\vartheta \otimes \mathrm{id}_V)\varphi_r(\gamma \otimes \mathrm{id}_V)$, since

$$(\vartheta \otimes \mathrm{id}_V)\varphi_l(\gamma \otimes \mathrm{id}_V) = \delta^R\lambda^H,$$
$$(\vartheta \otimes \mathrm{id}_V)\varphi_r(\gamma \otimes \mathrm{id}_V) = \lambda^H_{R\otimes V}(\mathrm{id}_H \otimes \delta^R),$$

by Lemma 3.8.2(3) and (4), and since γ is a bialgebra morphism. $\square$

THEOREM 3.8.7. *Let R be a bialgebra in ${}^H_H\mathcal{YD}(\mathcal{C})$ with bosonization $R\#H$. The functor*

$$F : {}^R_R\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C})) \to {}^{R\#H}_{R\#H}\mathcal{YD}(\mathcal{C}),$$
$$((V,\lambda^H,\delta^H),\lambda^R,\delta^R) \mapsto (V,\lambda^{R\#H},\delta^{R\#H}),$$

where $\lambda^{R\#H} = \lambda^R(\mathrm{id}_R \otimes \lambda^H)$, and $\delta^{R\#H} = (\mathrm{id}_V \otimes \delta^H)\delta^R$, and where morphisms f are mapped to f, is a prebraided strict monoidal isomorphism.

PROOF. (1) We first show that the functor F is well-defined.

Let $V \in {}^R_R\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C}))$. Then for all $X \in {}_R({}^H_H\mathcal{YD}(\mathcal{C}))$, the Yetter-Drinfeld morphism $c^{\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C}),R)}_{V,X} : V\otimes X \to X\otimes V$ is left R-linear by Proposition 3.4.5. By Lemma 3.8.5(2), the Yetter-Drinfeld morphism

$$c^{\mathcal{YD}(\mathcal{C},R\#H)}_{F_2(V),F_1(X)} : V\otimes X \to X\otimes V$$

is left $R\#H$-linear, where V is the left $R\#H$-comodule $F_2(V)$ and the left $R\#H$-module $F_1(V)$. By Proposition 3.8.4(1), we can choose X such that $F_1(X)$ is the regular representation of $R\#H$. Hence $F(V)$ is an object in ${}^{R\#H}_{R\#H}\mathcal{YD}(\mathcal{C})$ by Proposition 3.4.5.

(2) Conversely, let $(V,\lambda^{R\#H},\delta^{R\#H}) \in {}^{R\#H}_{R\#H}\mathcal{YD}(\mathcal{C})$. Define $\lambda^H,\delta^H,\lambda^R$ and δ^R as in Lemma 3.8.6. Then by Lemma 3.8.6,

$$((V,\lambda^H,\delta^H),\lambda^R) \in {}_R({}^H_H\mathcal{YD}(\mathcal{C})), \qquad ((V,\lambda^H,\delta^H),\delta^R) \in {}^R({}^H_H\mathcal{YD}(\mathcal{C})).$$

Let $X \in {}_R({}^H_H\mathcal{YD}(\mathcal{C}))$. Then the map $c^{\mathcal{YD}(\mathcal{C},R\#H)}_{F_2(V),F_1(X)}$ in Lemma 3.8.5(2)(b) is left $R\#H$-linear by Proposition 3.4.5, since $V \in {}^{R\#H}_{R\#H}\mathcal{YD}(\mathcal{C})$. Hence by Lemma 3.8.5, the map $c^{\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C}),R)}_{V,X}$ is left R-linear, and it follows that

$$((V,\lambda^H,\delta^H),\lambda^R,\delta^R) \in {}^R_R\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C}))$$

by Proposition 3.4.5.

It is clear (using Lemma 3.8.2) that the inverse functor of F is given by the construction of $((V,\lambda^H,\delta^H),\lambda^R,\delta^R)$.

(3) The functor F is strict monoidal, since by Proposition 3.8.4, both functors F_1 and F_2 are strict monoidal. Moreover, F is prebraided, that is, for all V,W in ${}^R_R\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C}))$, the diagram

$$\begin{tikzcd} F(V)\otimes F(W) \arrow[r, "="] \arrow[d, "c^{\mathcal{YD}}_{F(V),F(W)}"'] & F(V\otimes W) \arrow[d, "F(c^{\mathcal{YD}}_{V,W})"'] \\ F(W)\otimes F(V) \arrow[r, "="] & F(W\otimes V) \end{tikzcd}$$

is commutative, where $c^{\mathcal{YD}}_{F(V),F(W)}$ and $c^{\mathcal{YD}}_{V,W}$ are the braidings in ${}^{R\#H}_{R\#H}\mathcal{YD}(\mathcal{C})$ and ${}^R_R\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C}))$, respectively. This is a special case of Lemma 3.8.5(1). □

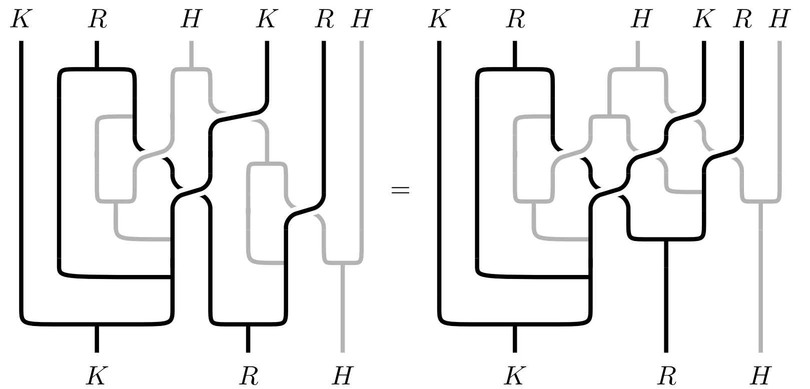
We next prove transitivity of bosonization in the following sense.

PROPOSITION 3.8.8. *Let R be a bialgebra in ${}^H_H\mathcal{YD}(\mathcal{C})$ with bosonization $R\#H$ in $\mathcal{C}$, and K a bialgebra in ${}^R_R\mathcal{YD}({}^H_H\mathcal{YD}(\mathcal{C}))$ with bosonization $K\#R$ in ${}^H_H\mathcal{YD}(\mathcal{C})$. Then the identity map*

$$(K\#R)\#H \to F(K)\#(R\#H)$$

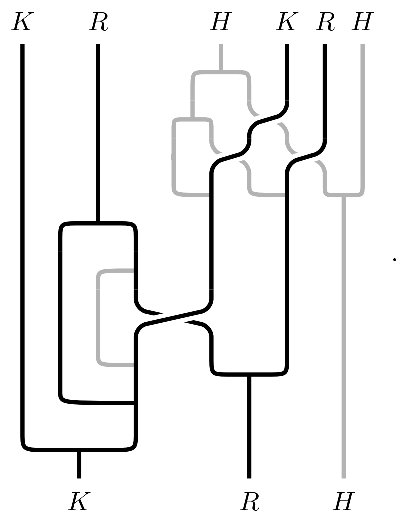
of $K\otimes R\otimes H$ is an isomorphism of bialgebras in $\mathcal{C}$.

Proof. The multiplication of $F(K)\#(R\#H)$ is defined by



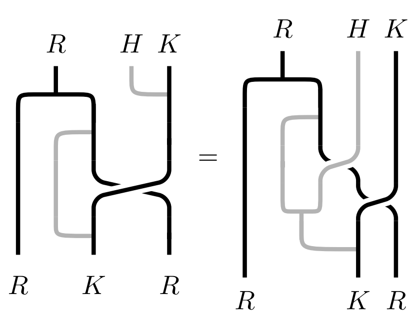
To prove the equality, we move the second comultiplication of H across the braiding and then use coassociativity.

The multiplication of $(K\#R)\#H$ is defined by



Note that

(3.8.3)



by moving the H-action of K across the braiding and then using associativity of the H-action of K. If we modify the third picture with (3.8.3), we obtain the second picture. We have shown that the identity is an algebra morphism.

It follows by duality that the identity is a coalgebra morphism. □

We finally show that the bosonization of a Hopf algebra is a Hopf algebra. We first characterize the antipode of a Hopf algebra in ${}_H^H\mathcal{YD}(\mathcal{C})$.

PROPOSITION 3.8.9. *Let C be a coalgebra, A an algebra and R a bialgebra in ${}_H^H\mathcal{YD}(\mathcal{C})$.*

(1) *Let $f \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ be a convolution invertible map which is a morphism in ${}_H^H\mathcal{YD}(\mathcal{C})$. Then f^{-1} is a morphism in ${}_H^H\mathcal{YD}(\mathcal{C})$.*
(2) *Suppose that there is a morphism $\mathcal{S} : R \to R$ in $\mathcal{C}$ which is convolution inverse to id_R. Then $\mathcal{S}$ is a morphism in ${}_H^H\mathcal{YD}(\mathcal{C})$, and R is a Hopf algebra in ${}_H^H\mathcal{YD}(\mathcal{C})$.*

PROOF. It is easy to see, that Proposition 1.2.11 holds for braided monoidal categories instead of vector spaces. This version of Proposition 1.2.11(2) implies (1), since $\Phi(f)$, hence also $\Phi(f)^{-1}$ and f^{-1} are morphisms in ${}_H^H\mathcal{YD}(\mathcal{C})$. Finally, (2) follows from (1). □

THEOREM 3.8.10. *Let R be a Hopf algebra in ${}_H^H\mathcal{YD}(\mathcal{C})$. Then the bosonization $R\#H$ of R is a Hopf algebra in $\mathcal{C}$. The antipode of $R\#H$ is the composition*

$$R \otimes H \xrightarrow{\delta\otimes\mathrm{id}_H} H \otimes R \otimes H \xrightarrow{\mathrm{id}_R\otimes c_{R,H}} H \otimes H \otimes R \xrightarrow{\mathcal{S}_H\mu_H\otimes\mathcal{S}_R} H \otimes R$$

$$= H \otimes R \xrightarrow{\Delta_H\otimes\mathrm{id}_R} H \otimes H \otimes R \xrightarrow{\mathrm{id}\otimes c_{H,R}} H \otimes R \otimes H \xrightarrow{\lambda\otimes\mathrm{id}_H} R \otimes H,$$

or equally, the convolution product $(\eta_R\varepsilon_R \otimes \mathcal{S}_H) * (\mathcal{S}_R \otimes \eta_H\varepsilon_H)$.

PROOF. By Proposition 3.8.4(2), $R\#H$ is a bialgebra in $\mathcal{C}$.

(a) The first claimed expression for the antipode of $R\#H$ can be rewritten as

$$\mathcal{S}_{R\#H} = c^{\mathcal{YD}}_{(H,\Delta),(R,\lambda)}(\mathcal{S}_H \otimes \mathcal{S}_R)c^{\mathcal{YD}}_{(R,\delta),(H,\mu)}. \tag{3.8.4}$$

(b) Equations (3.6.2) and (3.6.3) imply that

$$\mu_{R\#H}(\mathrm{id}_R \otimes \eta_H \otimes \eta_R \otimes \mathrm{id}_H) = \mathrm{id}_{R\#H}, \tag{3.8.5}$$

$$\mu_{R\#H}(\eta_R \otimes \mathrm{id}_H \otimes \mathrm{id}_R \otimes \eta_H) = c^{\mathcal{YD}}_{(H,\Delta),(R,\lambda)}, \tag{3.8.6}$$

$$(\mathrm{id}_R \otimes \varepsilon_H \otimes \varepsilon_R \otimes \mathrm{id}_H)\Delta_{R\#H} = \mathrm{id}_{R\#H}, \tag{3.8.7}$$

$$(\varepsilon_R \otimes \mathrm{id}_H \otimes \mathrm{id}_R \otimes \varepsilon_H)\Delta_{R\#H} = c^{\mathcal{YD}}_{(R,\delta),(H,\mu)}. \tag{3.8.8}$$

In particular, from Equations (3.8.4), (3.8.6) and (3.8.8) we obtain that

$$\begin{aligned}\mathcal{S}_{R\#H} &= \mu_{R\#H}(\eta_R\varepsilon_R \otimes \mathcal{S}_H \otimes \mathcal{S}_R \otimes \eta_H\varepsilon_H)\Delta_{R\#H}\\ &= (\eta_R\varepsilon_R \otimes \mathcal{S}_H) * (\mathcal{S}_R \otimes \eta_H\varepsilon_H).\end{aligned}$$

We proved that the two claimed formulas define the same morphism.

(c) The morphism $\eta_R\varepsilon_R \otimes \mathcal{S}_H$ is convolution invertible in $\mathrm{Hom}_{\mathcal{C}}(R\#H, R\#H)$ with convolution inverse $\eta_R\varepsilon_R\otimes\mathrm{id}_H$. Similarly, $\mathcal{S}_R\otimes\eta_H\varepsilon_H$ is convolution invertible in $\mathrm{Hom}_{\mathcal{C}}(R\#H, R\#H)$ with convolution inverse $\mathrm{id}_R \otimes \eta_H\varepsilon_H$. By (b), $\mathcal{S}_{R\#H}$ is the convolution inverse of $(\mathrm{id}_R \otimes \eta_H\varepsilon_H) * (\eta_R\varepsilon_R \otimes \mathrm{id}_H)$ in $\mathrm{Hom}_{\mathcal{C}}(R\#H, R\#H)$. The latter is equal to $\mathrm{id}_{R\#H}$ because of (3.8.5) and (3.8.7). Thus $\mathcal{S}_{R\#H}$ is the antipode of $R\#H$. □

COROLLARY 3.8.11. *Let R be a Hopf algebra in ${}_H^H\mathcal{YD}(\mathcal{C})$. The following are equivalent.*

(1) *The antipode of the bosonization $R\#H$ is an isomorphism in $\mathcal{C}$.*
(2) *The antipodes of H and of R are isomorphisms in $\mathcal{C}$.*

PROOF. We write $A = R\#H$. By Theorem 3.8.10, see also (3.8.4), the antipode of $R\#H$ is $\mathcal{S}_A = c^{\mathcal{YD}}_{(H,\Delta),(R,\lambda)}(\mathcal{S}_H \otimes \mathcal{S}_R)c^{\mathcal{YD}}_{(R,\delta),(H,\mu)}$.

(a) Assume that the antipode $\mathcal{S}_H$ of H is an isomorphism. Then the Yetter-Drinfeld maps $c^{\mathcal{YD}}_{(R,\delta),(H,\mu)}$ and $c^{\mathcal{YD}}_{(H,\Delta),(R,\lambda)}$ are isomorphisms by Proposition 3.4.8. Hence $\mathcal{S}_A$ is an isomorphism if and only if $\mathcal{S}_R$ is an isomorphism.

(b) Assume that $\mathcal{S}_A$ is an isomorphism. By Lemma 3.8.2 and Lemma 3.2.11,

$$\pi\mathcal{S}_A^{-1}\gamma\mathcal{S}_H = \pi\mathcal{S}_A^{-1}\mathcal{S}_{R\#H}\gamma = \mathrm{id}_H, \qquad \mathcal{S}_H\pi\mathcal{S}_A^{-1}\gamma = \mathrm{id}_H.$$

Hence $\mathcal{S}_H$ is an isomorphism. □

3.9. Characterization of smash products and coproducts

Let $\mathcal{C}$ be a braided strict monoidal category, and H a Hopf algebra in $\mathcal{C}$.

Let R be a left H-module algebra. We have seen in Proposition 3.6.5 that the smash product algebra $R\#H$ is a right H-comodule algebra with a right H-colinear algebra morphism $\gamma = \eta \otimes \mathrm{id}_H : H \to R\#H$, since $\eta : \Bbbk \to A$ is a left H-module algebra map. In this section we will show that a right H-comodule algebra with such a morphism γ is a smash product.

The next lemma follows easily from the definitions.

LEMMA 3.9.1. *Let X, Y be algebras and $f : X \to Y$, $g : X \to Y$ algebra morphisms in $\mathcal{C}$. Let (K, i) be an equalizer of (f, g). Then there is exactly one algebra structure (K, μ_K, η_K) on K such that $i : K \to X$ is an algebra morphism.*

With the next Theorem we generalize our result on smash product algebras in Theorem 2.6.23. Recall the notion of the left adjoint action and left coadjoint coaction in Definitions 3.7.3 and 3.7.8.

THEOREM 3.9.2. *Let A be a right H-comodule algebra in $\mathcal{C}$ with comodule structure δ_A. Assume that there is an algebra morphism $\gamma : H \to A$ which is right H-colinear, where H is an H-comodule via Δ.*

Assume that an equalizer $(R, \iota : R \to A)$ of $(\delta_A, \mathrm{id}_A \otimes \eta_H)$ exists in $\mathcal{C}$. Then R has a uniquely determined algebra structure such that ι is an algebra morphism. There are uniquely determined morphisms $\vartheta : A \to R$ and $\lambda_R : H \otimes R \to R$ with

$$\iota\vartheta = \big(A \xrightarrow{\delta_A} A \otimes H \xrightarrow{\mathrm{id}\otimes\gamma\mathcal{S}_H} A \otimes A \xrightarrow{\mu_A} A\big),$$

$$\iota\lambda_R = \big(H \otimes R \xrightarrow{\mathrm{id}\otimes\iota} H \otimes A \xrightarrow{\mathrm{ad}_\gamma} A\big),$$

and the following hold.

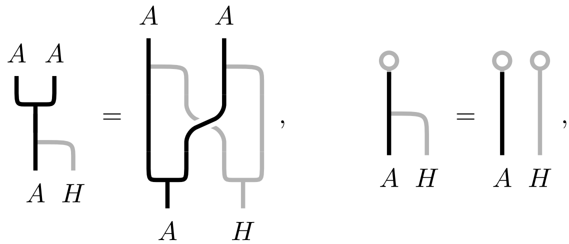
(1) $\vartheta\iota = \mathrm{id}_R$.
(2) *ϑ is right H-linear, where A is a right H-module by $\mu_A(\mathrm{id}_A \otimes \gamma)$ and R is the trivial H-module defined via ε.*
(3) $A \otimes H \underset{\mathrm{id}\otimes\varepsilon}{\overset{\mu_A(\mathrm{id}_A\otimes\gamma)}{\rightrightarrows}} A \xrightarrow{\vartheta} R$ *is a coequalizer diagram.*
(4) *(R, λ_R) is a left H-module algebra, $\lambda_R = \vartheta\mathrm{ad}_\gamma(\mathrm{id} \otimes \iota)$, and ι is left H-linear, where R and A are left H-modules by λ_R and ad_γ.*
(5) *$\vartheta : A \to R$ is left H-linear, where A and R are left H-modules with module structures $\mu_A(\gamma \otimes \mathrm{id}_A)$ and λ_R, respectively, and $\lambda_R = \vartheta\mu_A(\gamma \otimes \iota)$.*

(6) $\Phi = \big(R\#H \xrightarrow{\iota\otimes\gamma} A\otimes A \xrightarrow{\mu_A} A\big)$ *is a right H-colinear algebra isomorphism with inverse* $\Psi = \big(A \xrightarrow{\delta_A} A\otimes H \xrightarrow{\vartheta\otimes\mathrm{id}_H} R\#H\big)$.

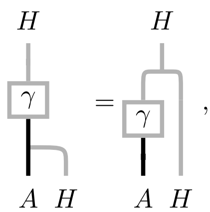
PROOF. Let $\delta_A =$ 

. Note that

(3.9.1)

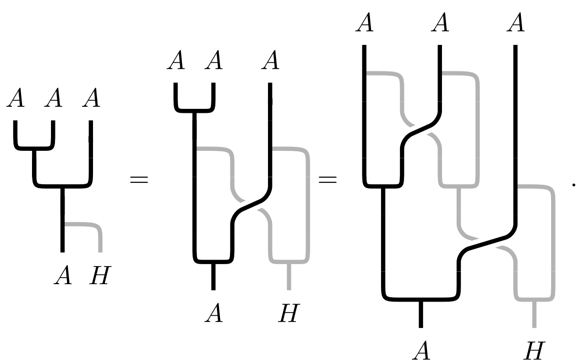


(3.9.2)



since μ_A, η_A and γ are H-colinear. By colinearity of μ_A,

(3.9.3)

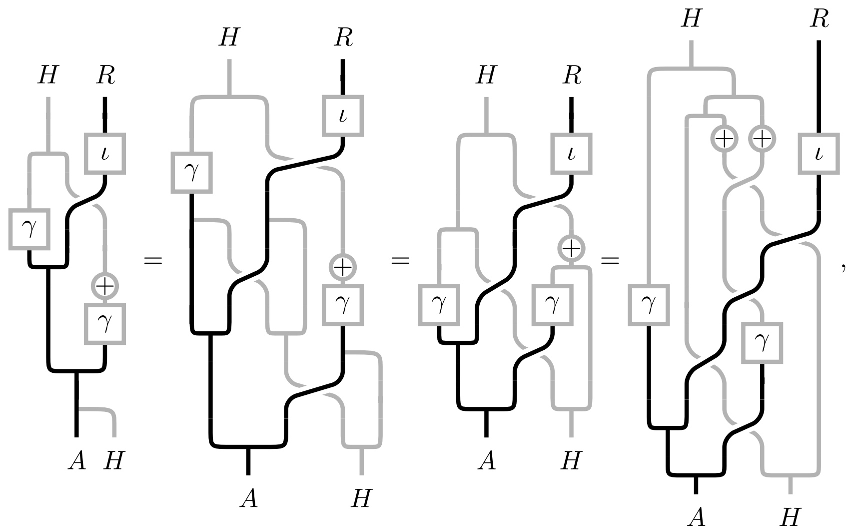


(1), (2), (3) Since δ_A and $\mathrm{id}\otimes\eta_H$ are algebra morphisms, by Lemma 3.9.1, R has a uniquely determined algebra structure such that ι is an algebra morphism. Let $\lambda_A = \mu_A(\mathrm{id}_A\otimes\gamma) : A\otimes H \to A$. Then (A, λ_A, δ_A) is a right H-Hopf module. By the version of Theorem 3.5.14 for right Hopf modules, ϑ exists and is uniquely determined, and (1), (2), (3) hold.

(4) We next prove existence and uniqueness of λ_R, that is, the diagram

$$H\otimes R \xrightarrow{\mathrm{id}\otimes\iota} H\otimes A \xrightarrow{\mathrm{ad}_\gamma} A \underset{\mathrm{id}_A\otimes\eta_H}{\overset{\delta_A}{\rightrightarrows}} A\otimes H \tag{3.9.4}$$

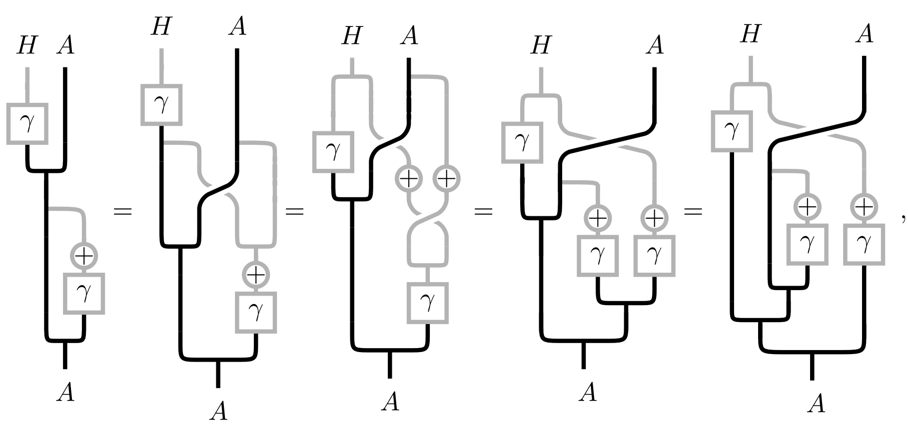
commutes. We first compute $\delta_A \mathrm{ad}_\gamma(\mathrm{id}_H \otimes \iota)$.



where the first equation follows from (3.9.3), the second from colinearity of γ, and the equality $\delta_A \iota = (\mathrm{id}_A \otimes \eta_H)\iota$, and the third from functoriality of the braiding (3.2.13) with $h = \Delta_H \mathcal{S}_H$, then from the rules of the antipode (3.2.26) and from coassociativity. The inner part of the last picture cancels because of the axiom of the antipode and functoriality of the braiding. The resulting picture is the second morphism in (3.9.4), $(\mathrm{id}_A \otimes \eta_H)\mathrm{ad}_\gamma(\mathrm{id}_H \otimes \iota)$.

Since ι is a monomorphism, it follows from Proposition 3.7.2(2) that R is a left H-module algebra with H-action λ_R. By definition of λ_R, ι is left H-linear. The formula for λ_R follows from (1).

(5) Let $\Theta = \big(A \xrightarrow{\delta_A} A \otimes H \xrightarrow{\mathrm{id}\otimes\gamma\mathcal{S}_H} A \otimes A \xrightarrow{\mu_A} A\big)$. We will show that $\Theta : (A, \mu_A(\gamma \otimes \mathrm{id}_A)) \to (A, \mathrm{ad}_\gamma)$ is left H-linear. Then (5) follows, since ι is a monomorphism, and the formula for λ_R follows from (2) and (4). In order to prove that $\Theta\mu_A(\gamma \otimes \mathrm{id}_A) = \mathrm{ad}_\gamma(\mathrm{id} \otimes \Theta)$, we begin with the left-hand side.



where the first equality follows from H-colinearity of μ_A (3.9.1), the second from colinearity of γ (3.9.2) and the rules of the antipode (3.2.26), the third from functoriality of the braiding (3.2.12) and since γ is an algebra morphism, and the fourth from associativity of A. It follows from functoriality of the braiding (3.2.12) (move Θ in the middle to the other side of the braiding) that the last picture is equal to $\mathrm{ad}_\gamma(\mathrm{id} \otimes \Theta)$.

(6) By Theorem 3.5.14 for right Hopf modules, Φ and Ψ are inverse isomorphisms. It is easy to see from colinearity of μ_A, (3.9.1), and since $\delta_A\iota = (\mathrm{id}_A \otimes \eta)\iota$ that Φ is right H-colinear. Proposition 3.7.2(1)(a) implies that Φ is an algebra morphism. $\square$

COROLLARY 3.9.3. *Let R be a left H-module algebra in $\mathcal{C}$ with module structure λ. Let $A = R\#H$ with H-comodule algebra structure δ_A, and*

$$\gamma = \eta \otimes \mathrm{id}_H : H \to R\#H, \quad \iota = \mathrm{id}_R \otimes \eta : R \to R\#H.$$

(1) *(R, ι) is an equalizer of $(\delta_A, \mathrm{id}_A \otimes \eta_H)$ and $\lambda_R = \lambda$.*
(2) *Assume that the antipode $\mathcal{S}_H$ is an isomorphism in $\mathcal{C}$. Then the morphism*

$$H \otimes R \xrightarrow{\gamma \otimes \iota} A \otimes A \xrightarrow{\mu_A} A$$

is an isomorphism in $\mathcal{C}$.

PROOF. (1) The first claim follows from the axioms for the unit and counit of the bialgebra H. The second holds by Theorem 3.9.2(5).

(2) We view $V = A$ as an H-bimodule via γ as in Definition 3.7.3. Then by Proposition 3.7.2(1)(b),

$$\begin{aligned}&\mu_A(\gamma \otimes \mathrm{id}_A)(\mathcal{S}_H \otimes \mathrm{ad}_\gamma)(\Delta_H \otimes \mathrm{id}_A)(\mathrm{id}_H \otimes \iota)\\ &\qquad = \mu_A(\mathrm{id}_A \otimes \gamma)(\mathrm{id}_A \otimes \mathcal{S}_H)c_{H,A}(\mathrm{id}_H \otimes \iota).\end{aligned}$$

This equality and the definition of λ_R in Theorem 3.9.2 imply that the compositions

$$H \otimes R \xrightarrow{\Delta_H \otimes \mathrm{id}_R} H \otimes H \otimes R \xrightarrow{\mathcal{S}_H \otimes \lambda_R} H \otimes R \xrightarrow{\gamma \otimes \iota} A \otimes A \xrightarrow{\mu_A} A,$$
$$H \otimes R \xrightarrow{c_{H,R}} R \otimes H \xrightarrow{\mathrm{id}_R \otimes \mathcal{S}_H} R \otimes H \xrightarrow{\iota \otimes \gamma} A \otimes A \xrightarrow{\mu_A} A$$

coincide. The second morphism is an isomorphism, since $\mathcal{S}_H$ is. Moreover, the morphism $(\mathrm{id}_H \otimes \lambda_R)(\Delta_H \otimes \mathrm{id}_R) : H \otimes R \to H \otimes R$ is an isomorphism with inverse $(\mathrm{id}_H \otimes \lambda_R)(\mathrm{id}_H \otimes \mathcal{S}_H \otimes \mathrm{id}_R)(\Delta_H \otimes \mathrm{id}_R)$, and the claim follows. $\square$

We note the dual results. They follow from Lemma 3.9.1 and Theorem 3.9.2 for the dual category $\mathcal{C}^{\mathrm{op}}$.

LEMMA 3.9.4. *Let X, Y be coalgebras and $f : X \to Y$, $g : X \to Y$ coalgebra morphisms in $\mathcal{C}$. Let*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{p} Q$$

be a coequalizer diagram. Then there is exactly one coalgebra structure $(Q, \Delta_Q, \varepsilon_Q)$ on Q such that $p : Y \to Q$ is a coalgebra morphism.

THEOREM 3.9.5. *Let C be a right H-module coalgebra in $\mathcal{C}$ with module structure λ_C. Assume that there is a coalgebra morphism $\pi : C \to H$ which is right H-linear, where H is a right H-module via μ, and that the coequalizer $(Q, \vartheta : C \to Q)$ of $(\lambda_C, \mathrm{id}_C \otimes \varepsilon)$ exists.*

Then Q has a uniquely determined coalgebra structure such that ϑ is a coalgebra morphism in $\mathcal{C}$, and there are uniquely determined morphisms $\delta_Q : Q \to H \otimes Q$ and $\iota : Q \to C$ with

$$\delta_Q\vartheta = \big(C \xrightarrow{\mathrm{coad}_\pi} H \otimes C \xrightarrow{\mathrm{id}\otimes\vartheta} H \otimes Q\big),$$
$$\iota\vartheta = (C \xrightarrow{\Delta} C \otimes C \xrightarrow{\mathrm{id}_C\otimes\mathcal{S}_H\pi} C \otimes H \xrightarrow{\lambda_C} C), \textit{and}$$

(1) $\vartheta\iota = \mathrm{id}_Q$.
(2) *ι is right H-colinear, where Q and C are right H-comodules via η and by $(\mathrm{id}_C \otimes \pi)\Delta$.*
(3) *(Q, δ_Q) is a left H-comodule coalgebra, ϑ is left H-colinear, where the H-comodule structures of C and Q are coad_π and δ_Q, respectively. Moreover, $\delta_Q = (\mathrm{id}_H \otimes \vartheta)\mathrm{coad}_\pi\iota$.*
(4) *ι is left H-colinear, where Q and C are left H-comodules by δ_Q and by $(\pi \otimes \mathrm{id}_C)\Delta$, respectively. Moreover, $\delta_Q = (\pi \otimes \vartheta)\Delta_C\iota$.*
(5) *$\Phi = \big(C \xrightarrow{\Delta} C\otimes C \xrightarrow{\vartheta\otimes\pi} Q\#H\big)$ is a right H-linear coalgebra isomorphism with inverse $\Psi = \big(Q\#H \xrightarrow{\iota\otimes\mathrm{id}_H} C \otimes H \xrightarrow{\lambda_C} C\big)$.*

3.10. Hopf algebra triples

Let $\mathcal{C}$ be a strict monoidal braided category. Let H be a Hopf algebra in $\mathcal{C}$ whose antipode is an isomorphism. In this section we study Hopf algebra triples in $\mathcal{C}$, see Definition 3.10.1. We will see that Hopf algebras in the category ${}^H_H\mathcal{YD}(\mathcal{C})$ arise naturally from Hopf algebra triples in $\mathcal{C}$.

DEFINITION 3.10.1. A **Hopf algebra triple over** H in $\mathcal{C}$ is a triple (A, π, γ), where $\pi : A \to H$, $\gamma : H \to A$ are morphisms of Hopf algebras in $\mathcal{C}$ such that $\pi\gamma = \mathrm{id}_H$. A **morphism** $\Phi : (A, \pi, \gamma) \to (A', \pi', \gamma')$ of Hopf algebra triples over H is a morphism $\Phi : A \to A'$ of Hopf algebras in $\mathcal{C}$ with $\pi'\Phi = \pi$ and $\Phi\gamma = \gamma'$.

If (A, π, γ) is a Hopf algebra triple over H, let

$$\delta_A = (\mathrm{id}_A \otimes \pi)\Delta_A : A \to A \otimes H, \qquad \lambda_A = \mu_A(\mathrm{id}_A \otimes \gamma) : A \otimes H \to A,$$
$$\varTheta_A = \mu_A(\mathrm{id} \otimes \gamma\pi\mathcal{S}_A)\Delta_A : A \to A, \qquad \varSigma_A = \mu_A(\gamma\pi \otimes \mathcal{S}_A)\Delta_A : A \to A.$$

REMARK 3.10.2. Let (A, π, γ) be a Hopf algebra triple over H. By definition, $\varTheta_A = \mathrm{id}_A * \gamma\pi\mathcal{S}_A$ and $\varSigma_A = \gamma\pi * \mathcal{S}_A$ in the convolution monoid $\mathrm{Hom}_{\mathcal{C}}(A, A)$. Hence

$$\varTheta_A * \gamma\pi = \mathrm{id}_A, \quad \varTheta_A * \varSigma_A = \eta_A\varepsilon_A = \varSigma_A * \varTheta_A. \tag{3.10.1}$$

REMARK 3.10.3. Let G, H be groups and $\pi : G \to H$, $\gamma : H \to G$ be group homomorphisms with $\pi\gamma = \mathrm{id}_H$. This situation is described by a **semidirect product of groups**. The group H acts on $\ker(\pi)$ by

$$\varphi : H \times \ker(\pi) \mapsto \ker(\pi),\ (h, x) \mapsto \gamma(h)x\gamma(h)^{-1}.$$

Let $\ker(\pi) \times_\varphi H$ be the corresponding semidirect product. Then

$$\ker(\pi) \times_\varphi H \xrightarrow{\cong} G,\ (g, h) \mapsto g\gamma(h),$$

is an isomorphism of groups. For Hopf algebras we have to replace the kernel of π by the right (or left) coinvariant elements which is a Yetter-Drinfeld Hopf algebra, and the object which generalizes the semidirect product of groups will be a smash product and a smash coproduct at the same time.

THEOREM 3.10.4. *Let (A,π,γ) be a Hopf algebra triple over H in $\mathcal{C}$ with morphisms $\delta_A, \lambda_A, \Theta_A$ and Σ_A introduced in Definition* 3.10.1. *Assume that an equalizer $(R, \iota : R \to A)$ of the pair $(\delta_A, \mathrm{id}_A \otimes \eta_H)$ exists. There is a uniquely determined morphism $\vartheta : A \to R$ with $\Theta_A = \iota\vartheta$; $(A, \vartheta : A \to R)$ is a coequalizer of $(\lambda_A, \mathrm{id}_A \otimes \varepsilon_H)$, and $\vartheta\iota = \mathrm{id}_R$. There are uniquely determined morphisms $\lambda_R : H \otimes R \to R$, $\delta_R : R \to H \otimes R$ with $\iota\lambda_R = \mathrm{ad}_A(\gamma \otimes \iota)$, $\delta_R\vartheta = (\pi \otimes \vartheta)\mathrm{coad}_A$. Let $\mathcal{S}_R = \lambda_R(\mathrm{id}_H \otimes \vartheta\mathcal{S}_A\iota)\delta_R : R \to R$. Then*

(1) *(R, λ_R, δ_R) is an object in ${}^H_H\mathcal{YD}(\mathcal{C})$, and a Hopf algebra in ${}^H_H\mathcal{YD}(\mathcal{C})$ with antipode $\mathcal{S}_R$, where*
   (a) *$\iota : (R, \mu_R, \eta_R) \to (A, \mu_A, \eta_A)$ is an algebra morphism in $\mathcal{C}$,*
   (b) *$\vartheta : (A, \Delta_A, \varepsilon_A) \to (R, \Delta_R, \varepsilon_R)$ is a coalgebra morphism in $\mathcal{C}$,*
   (c) *$\iota\mathcal{S}_R = \Sigma_A\iota$,*
   (d) *ι is a morphism in ${}^H\mathcal{C}$, and ϑ is a morphism in ${}_H\mathcal{C}$, where A and R are left H-comodules by $(\pi \otimes \mathrm{id}_A)\Delta_A$ and δ_R, respectively, and left H-modules by $\mu_A(\gamma \otimes \mathrm{id}_A)$ and λ_R, respectively.*

(2) *$\Phi = \big(R\#H \xrightarrow{\iota\otimes\gamma} A \otimes A \xrightarrow{\mu_A} A\big)$, is an isomorphism of algebras and coalgebras in $\mathcal{C}$ with inverse $\Psi = \big(A \xrightarrow{\Delta_A} A \otimes A \xrightarrow{\vartheta\otimes\pi} R\#H\big)$, where $R\#H$ is the bosonization of R.*

The situation of Theorem 3.10.4 is described in the diagram

$$\begin{tikzcd}
 & & H \arrow[dl, "\gamma"'] \arrow[d, "="] \\
R \arrow[r, "\iota"] \arrow[d, "="'] & A \arrow[r, "\pi"] \arrow[dl, "\vartheta"] & H \\
R & &
\end{tikzcd}$$

PROOF. Note that (A, δ_A) is a right H-comodule algebra, since δ_A is an algebra morphism. It follows from $\pi\gamma = \mathrm{id}_H$ that γ is a right H-colinear algebra morphism. Hence by Theorem 3.9.2, ϑ is well-defined, $\vartheta\iota = \mathrm{id}_R$, and $(A, \vartheta : A \to R)$ is a coequalizer of $(\lambda_A, \mathrm{id}_A \otimes \varepsilon_H)$.

Since (A, λ_A) is a right H-module coalgebra, and π is a right H-linear coalgebra morphism, Theorem 3.9.5 applies.

From both theorems we conclude the existence of $\vartheta, \lambda_R, \delta_R$, and of well-defined algebra and coalgebra structures $\mu_R, \eta_R, \Delta_R, \varepsilon_R$ satisfying (1)(a) and (1)(b), that (R, λ_R) is a left H-module algebra, (R, δ_R) is a left H-comodule coalgebra, and that Φ and Ψ in (2) are inverse isomorphisms of algebras and coalgebras in $\mathcal{C}$. Moreover, $\iota : (R, \lambda_R) \to (A, \mathrm{ad}_\gamma)$ and $\vartheta : (A, \mu_A(\gamma \otimes \mathrm{id}_A)) \to (R, \lambda_R)$ are morphisms in ${}_H\mathcal{C}$ by Theorem 3.9.2. By Theorem 3.9.5,

$$\vartheta : (A, \mathrm{coad}_\pi) \to (R, \delta_R) \text{ and } \iota : (R, \delta_R) \to (A, (\pi \otimes \mathrm{id}_A)\Delta_A)$$

are morphisms in ${}^H\mathcal{C}$.

It follows from Theorem 3.9.2(2) that also $\vartheta : (A, \mathrm{ad}_\gamma) \to (R, \lambda_R)$ is a morphism in ${}_H\mathcal{C}$, and from Theorem 3.9.5(2) that $\iota : (R, \delta_R) \to (A, \mathrm{coad}_\pi)$ is a morphism in ${}^H\mathcal{C}$.

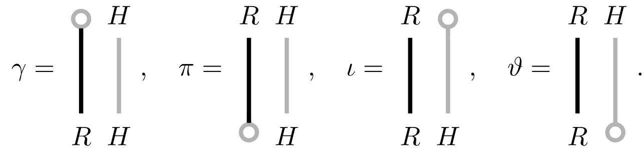
We next prove that R is a coalgebra in ${}_H\mathcal{C}$. Since $(A, \mu_A(\gamma \otimes \mathrm{id}_A))$ is a coalgebra in ${}_H\mathcal{C}$ and ϑ is a coalgebra morphism, $\Delta_R\vartheta = (\vartheta \otimes \vartheta)\Delta_A$ and $\varepsilon_R\vartheta = \varepsilon_A$ are morphisms in ${}_H\mathcal{C}$. Hence Δ_R and ε_R are morphisms in ${}_H\mathcal{C}$, since ϑ is, and since

$\vartheta\iota = \mathrm{id}_R$. Similarly it follows that R is an algebra in ${}^H\mathcal{C}$. It remains to prove

(I) (R, λ_R, δ_R) is a Yetter-Drinfeld module in ${}^H_H\mathcal{YD}(\mathcal{C})$.
(II) Δ_R is an algebra morphisms in ${}^H_H\mathcal{YD}(\mathcal{C})$.
(III) $\mathcal{S}_R$ is the antipode of R satisfying (1)(c).

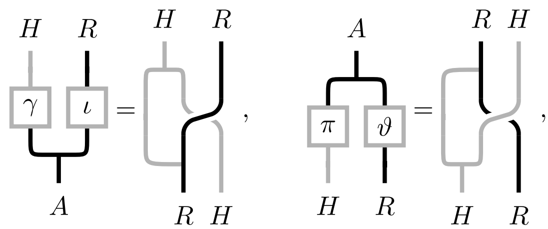
By (2), we may assume that $R\#H = A$ is a Hopf algebra, where

(3.10.2)

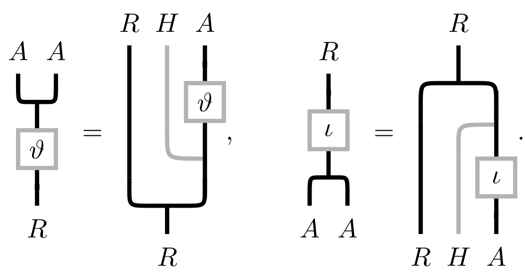


We denote action and coaction of R by $\lambda_R =$ , $\delta_R =$ . The next rules follow from the definition of μ_A and Δ_A and (3.10.2).

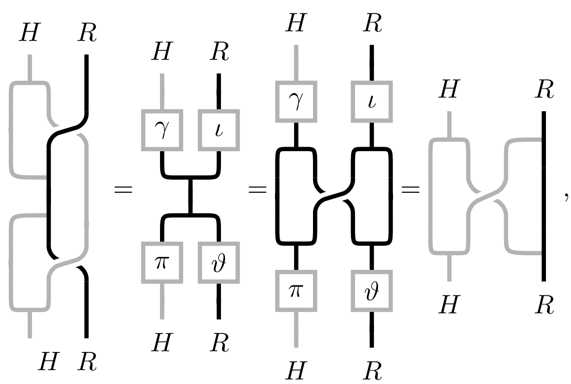
(3.10.3)



(3.10.4)



We prove (I).



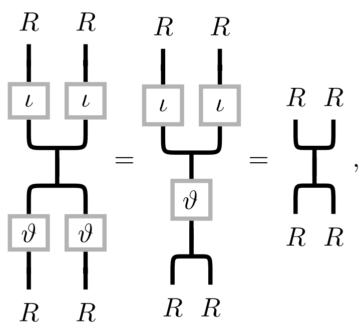
where the first equality follows from (3.10.3) and the second, since A is a bialgebra. To prove the third equality we move γ and π to the right, since γ and π are morphisms of coalgebras and of algebras, and since $\vartheta\iota = \mathrm{id}_R$, and then use (3.10.4) to identify λ_R and δ_R.

To prove (II), we note that

(3.10.5) $$\mu_R = \vartheta\mu_A(\iota \otimes \iota),$$

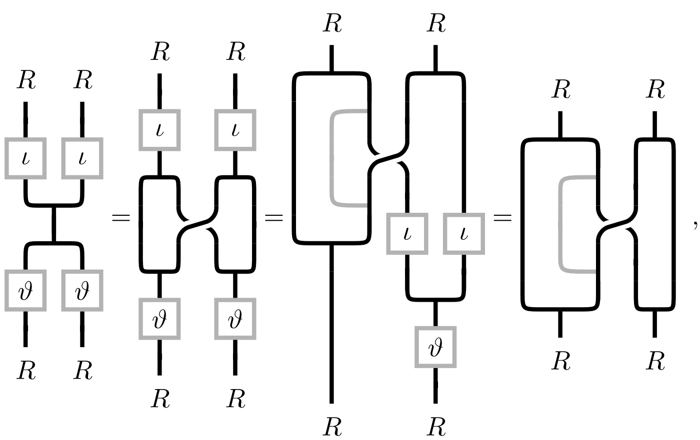
since ι is an algebra morphism with $\vartheta\iota = \mathrm{id}_R$. Hence

(3.10.6)



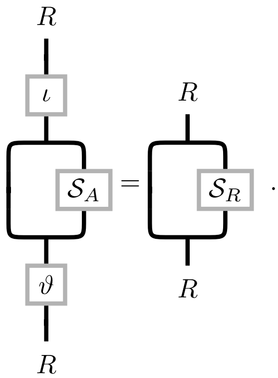
where the first equality holds, since ϑ is a coalgebra morphism, and the second by (3.10.5). On the other hand,

(3.10.7)



where the first equality holds, since A is a bialgebra, the second follows from (3.10.4) for the morphisms ι and the first θ, and the third from (3.10.5).

(III) By (3.10.4) and the definition of $\mathcal{S}_R$,



Hence $\mathrm{id}_R * \mathcal{S}_R = \eta_R \varepsilon_R$ in $\mathrm{Hom}_{\mathcal{C}}(R, R)$, since $\mathcal{S}_A$ is the antipode of A.

Since $\vartheta\iota = \mathrm{id}_R$, the map $\Pi : \mathrm{Hom}_{\mathcal{C}}(R, R) \to \mathrm{Hom}_{\mathcal{C}}(A, A)$, $f \mapsto \iota f \vartheta$, is an injective monoid morphism with respect to composition. Since ϑ is a morphism of coalgebras, and ι is a morphism of algebras, Π is a monoid morphism with respect to convolution.

By (3.10.1), $\varSigma_A$ is $*$-inverse to $\varTheta_A = \iota\vartheta = \Pi(\mathrm{id}_R)$. Since $\mathcal{S}_R$ is right $*$-inverse to id_R, it follows that $\varSigma_A = \Pi(\mathcal{S}_R)$, and that $\mathcal{S}_R$ is left $*$-inverse to id_R. Hence $\mathcal{S}_R$ is the antipode of R by Proposition 3.8.9. Note that (1)(c) follows from $\varSigma_A = \Pi(\mathcal{S}_R)$.

Since Π is a monoid morphism with respect to composition,

$$\varSigma_A = \varTheta_A \varSigma_A = \varSigma_A \varTheta_A, \tag{3.10.8}$$

follows from $\mathcal{S}_R = \mathrm{id}_R \mathcal{S}_R = \mathcal{S}_R \mathrm{id}_R$. □

Suppose that in Theorem 3.10.4 the antipode of A is an isomorphism. Then $(A^{\mathrm{cop}}, \pi : A^{\mathrm{cop}} \to H^{\mathrm{cop}}, \gamma : H^{\mathrm{cop}} \to A^{\mathrm{cop}})$ is a Hopf algebra triple over H^{cop} in $\overline{\mathcal{C}}$. Assume that $(L, \iota_L : L \to A^{\mathrm{cop}})$ is an equalizer of $(\delta_{A^{\mathrm{cop}}}, \mathrm{id} \otimes \eta)$ in $\mathcal{C}$.

We want to compare R and L. Recall that R and L are Hopf algebras in ${}^H_H\mathcal{YD}(\mathcal{C})$ and in ${}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$, respectively.

Lemma 3.10.5. *Let (A, π, γ) be a Hopf algebra triple over H, and assume that the antipode of A is an isomorphism. Let $\varTheta_A, \varSigma_A$ be defined by (A, π, γ) and $\delta_{A^{\mathrm{cop}}}, \varTheta_{A^{\mathrm{cop}}}$ by $(A^{\mathrm{cop}}, \pi, \gamma)$.*

(1) $\mathcal{S}_A \varTheta_{A^{\mathrm{cop}}} = \varSigma_A$,
(2) $\varTheta_{A^{\mathrm{cop}}} \varTheta_A = \varTheta_{A^{\mathrm{cop}}}$,
(3) $\varTheta_A \varTheta_{A^{\mathrm{cop}}} = \varTheta_A$.
(4) *Assume that the equalizer (L, ι_L) of $(\delta_{A^{\mathrm{cop}}}, \mathrm{id} \otimes \eta)$ exists. Then (L, ι_L) is an equalizer of $(\delta^l_A, \eta \otimes \mathrm{id})$, where $\delta^l_A = (\pi \otimes \mathrm{id}_A)\Delta_A$.*

Proof. (1) By definition of $\varTheta_{A^{\mathrm{cop}}}$,

$$\begin{aligned}
\mathcal{S}_A \varTheta_{A^{\mathrm{cop}}} &= \mathcal{S}_A \mu_A (\mathrm{id}_A \otimes \gamma\pi\mathcal{S}_A^{-1}) \overline{c}_{A,A} \Delta_A \\
&= \mu_A c_{A,A} (\mathcal{S}_A \otimes \mathcal{S}_A)(\mathrm{id}_A \otimes \gamma\pi\mathcal{S}_A^{-1}) \overline{c}_{A,A} \Delta_A \\
&= \mu_A c_{A,A} \overline{c}_{A,A} (\gamma\pi \otimes \mathcal{S}_A) \Delta_A \\
&= \varSigma_A,
\end{aligned}$$

where the second equality follows from the rule for the antipode (3.2.26), and the third from functoriality of the braiding.

(2) By (1), and since $\Sigma_A \Theta_A = \Sigma_A$ by (3.10.8),

$$\mathcal{S}_A \Theta_{A^{\mathrm{cop}}} \Theta_A = \Sigma_A \Theta_A = \Sigma_A = \mathcal{S}_A \Theta_{A^{\mathrm{cop}}},$$

and (2) follows, since $\mathcal{S}_A$ is an isomorphism.

(3) follows from (2) replacing (A, π, γ) by $(A^{\mathrm{cop}}, \pi, \gamma)$.

(4) Note that $\delta_{A^{\mathrm{cop}}} = \overline{c}_{H,A} \delta^l_A$, and $\mathrm{id} \otimes \eta = \overline{c}_{H,A} \eta \otimes \mathrm{id}$. □

THEOREM 3.10.6. *Assume the situation of Theorem* 3.10.4, *and assume that the antipode of A is an isomorphism. Let $\delta_{A^{\mathrm{cop}}}$ be defined by the Hopf algebra triple $(A^{\mathrm{cop}}, \pi, \gamma)$ in $\overline{\mathcal{C}}$. Assume that the equalizer (L, ι_L) of $(\delta_{A^{\mathrm{cop}}}, \mathrm{id} \otimes \eta)$ exists, and let $\vartheta_L : A^{\mathrm{cop}} \to L$ be defined by $(A^{\mathrm{cop}}, \pi, \gamma)$. Then the morphism $T = \vartheta \iota_L : L \to R$ is an isomorphism in $\mathcal{C}$ with $T^{-1} = \vartheta_L \iota$ and $\iota_L T^{-1} = \mathcal{S}_A^{-1} \iota \mathcal{S}_R$, and an isomorphism*

$$T : L \to (F, \varphi)(R^{\mathrm{cop}})$$

of Hopf algebras in ${}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$, where $(F, \varphi) : \overline{{}^H_H\mathcal{YD}(\mathcal{C})} \to {}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$ is the braided monoidal isomorphism in Corollary 3.4.17.

PROOF. We denote the braiding and the inverse braiding of ${}^H_H\mathcal{YD}(\mathcal{C})$ by $c^{\mathcal{YD}}$ and $\overline{c}^{\mathcal{YD}}$. Let $R' = (F, \varphi)(R^{\mathrm{cop}})$, and

$$\lambda'_R = \lambda_R, \quad \delta'_R = (\mathcal{S}_H^{-1} \otimes \mathrm{id}_R) \overline{c}^2_{H,R} \delta_R,$$
$$\mu'_R = \mu_R \overline{c}^{\mathcal{YD}}_{R,R} c_{R,R}, \quad \Delta'_R = \overline{c}_{R,R} \Delta_R.$$

Then by Corollary 3.4.17, $R' = (R, \lambda'_R, \delta'_R)$ as an object in ${}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$, and the Hopf algebra structure is given by the multiplication μ'_R and the comultiplication Δ'_R.

(1) We first show that T is an isomorphism in $\mathcal{C}$ with $T^{-1} = \vartheta_L \iota$ and with $\iota_L T^{-1} = \mathcal{S}_A^{-1} \iota \mathcal{S}_R$.

By Theorem 3.9.2, $\Theta_A = \iota \vartheta$ and $\Theta_{A^{\mathrm{cop}}} = \iota_L \vartheta_L$. Hence by Lemma 3.10.5(2) and (3),

$$\vartheta_L \iota \vartheta = \vartheta_L, \quad \vartheta \iota_L \vartheta_L = \vartheta, \tag{3.10.9}$$

since ι_L and ι are monomorphisms. Let $T' = \vartheta_L \iota : R \to L$. Then by (3.10.9),

$$T'T = \vartheta_L \iota \vartheta \iota_L = \vartheta_L \iota_L = \mathrm{id}_L, \quad TT' = \vartheta \iota_L \vartheta_L \iota = \vartheta \iota = \mathrm{id}_R,$$

and $T' = T^{-1}$.

By Lemma 3.10.5(1), $\mathcal{S}_A \iota_L T^{-1} = \mathcal{S}_A \iota_L \vartheta_L \iota = \mathcal{S}_A \Theta_{A^{\mathrm{cop}}} \iota = \Sigma_A \iota$. Hence the formula for $\iota_L T^{-1}$ follows from Theorem 3.10.4(1)(c).

(2) We want to show that T^{-1} is an isomorphism of Hopf algebras, that is, the following equations hold.

(a) $\lambda_L (\mathrm{id}_H \otimes T^{-1}) = T^{-1} \lambda_R$,
(b) $\delta_L T^{-1} = (\mathrm{id}_H \otimes T^{-1}) \delta'_R$,
(c) $\mu_L (T^{-1} \otimes T^{-1}) = T^{-1} \mu'_R$,
(d) $\Delta_L T^{-1} = (T^{-1} \otimes T^{-1}) \Delta'_R$.

(a) To prove that $T^{-1} : R' \to L$ is left H-linear, we recall that

$$H \otimes R \xrightarrow{\lambda_R} R, \quad H \otimes L \xrightarrow{\lambda_L} L$$

are the left H-module structures of R and of L, satisfying

$$\iota \lambda_R = \mathrm{ad}_A (\gamma \otimes \iota), \tag{3.10.10}$$

$$\iota_L \lambda_L = \mathrm{ad}_{A^{\mathrm{cop}}} (\gamma \otimes \iota_L), \tag{3.10.11}$$

by Theorem 3.10.4. Note that $\mathrm{ad}_A(\gamma \otimes \mathrm{id}_A) = \mathrm{ad}_\gamma$. By Lemma 3.7.4,

$$\mathrm{ad}_{A^{\mathrm{cop}}}(\mathrm{id} \otimes \mathcal{S}_A^{-1}) = \mathcal{S}_A^{-1}\mathrm{ad}_A. \tag{3.10.12}$$

Consider the following diagram.

$$\begin{tikzcd}
H \otimes R \arrow[r, "\lambda_R"] \arrow[d, "\mathrm{id}\otimes T^{-1}"'] & R \arrow[d, "T^{-1}"] \\
H \otimes L \arrow[r, "\lambda_L"] \arrow[d, "\gamma\otimes\iota_L"'] & L \arrow[d, "\iota_L"] \\
A \otimes A \arrow[r, "\mathrm{ad}_{A^{\mathrm{cop}}}"] & A
\end{tikzcd}$$

We want to show that the upper square commutes. The lower square commutes by (3.10.11). Since ι_L is injective, it is enough to prove commutativity of the large diagram. Since R is a Hopf algebra in the Yetter-Drinfeld category, the antipode $\mathcal{S}_R$ is H-linear, that is,

$$\mathcal{S}_R\lambda_R = \lambda_R(\mathrm{id} \otimes \mathcal{S}_R).$$

By (1), $\iota_L T^{-1} = \mathcal{S}_A^{-1}\iota\mathcal{S}_R$. Hence it remains to prove that

$$\mathcal{S}_A^{-1}\iota\lambda_R = \mathrm{ad}_{A^{\mathrm{cop}}}(\gamma \otimes \mathcal{S}_A^{-1}\iota).$$

This follows from (3.10.11) and (3.10.12), since

$$\begin{aligned}
\mathcal{S}_A^{-1}\iota\lambda_R &= \mathcal{S}_A^{-1}\mathrm{ad}_A(\gamma \otimes \iota) \\
&= \mathrm{ad}_{A^{\mathrm{cop}}}(\mathrm{id} \otimes \mathcal{S}_A^{-1})(\gamma \otimes \iota).
\end{aligned}$$

(b) The equations

$$(\mathrm{id}_H \otimes \iota)\delta_R = (\pi \otimes \mathrm{id}_A)\Delta_A\iota, \tag{3.10.13}$$

$$(\mathrm{id}_H \otimes \iota_L)\delta_L = (\pi \otimes \mathrm{id}_A)\overline{c}_{A,A}\Delta_A\iota_L. \tag{3.10.14}$$

follow from (3.10.4). We note that

$$\delta'_R\mathcal{S}_R = (\mathrm{id}_H \otimes \mathcal{S}_R)\delta'_R, \tag{3.10.15}$$

since the antipode $\mathcal{S}_R$ is left H-colinear with respect to δ_R, and since $\mathcal{S}_R$ commutes with the braiding.

Since $\mathrm{id}_H \otimes \iota_L$ is a monomorphism, (b) follows from the equality

$$(\mathrm{id}_H \otimes \iota_L)\delta_L T^{-1} = (\mathrm{id}_H \otimes \iota_L T^{-1})\delta'_R. \tag{3.10.16}$$

To prove (3.10.16), we begin to compute the left-hand side.

$$\begin{aligned}
(\mathrm{id}_H \otimes \iota_L)\delta_L T^{-1} &= (\pi \otimes \mathrm{id}_A)\overline{c}_{A,A}\Delta_A\mathcal{S}_A^{-1}\iota\mathcal{S}_R \\
&= (\mathcal{S}_H^{-1} \otimes \mathcal{S}_A^{-1})\overline{c}^2_{H,A}(\pi \otimes \mathrm{id}_A)\Delta_A\iota\mathcal{S}_R \\
&= (\mathcal{S}_H^{-1} \otimes \mathcal{S}_A^{-1})\overline{c}^2_{H,A}(\mathrm{id}_H \otimes \iota)(\mathcal{S}_H \otimes \mathrm{id}_R)c^2_{H,R}\delta'_R\mathcal{S}_R \\
&= (\mathrm{id}_H \otimes \mathcal{S}_A^{-1})(\mathrm{id}_H \otimes \iota)\delta'_R\mathcal{S}_R \\
&= (\mathrm{id}_H \otimes \iota_L T^{-1})\delta'_R,
\end{aligned}$$

where the first equality follows from (3.10.14) and then from (1), the second from the rules of the antipode and functoriality of the braiding, the third from (3.10.13), and since $\delta_R = (\mathcal{S}_H \otimes \mathrm{id}_R)c^2_{H,R}\delta'_R$ by the definition of δ'_R, the fourth from functoriality of the braiding, and the last from (3.10.15) and (1).

(c) The claim follows from the commutativity of the large diagram

$$\begin{tikzcd}
R \otimes R \arrow[r, "T^{-1}\otimes T^{-1}"] \arrow[d, "\mu'_R"] & L \otimes L \arrow[r, "\iota_L \otimes \iota_L"] \arrow[d, "\mu_L"] & A \otimes A \arrow[d, "\mu_A"] \\
R \arrow[r, "T^{-1}"] & L \arrow[r, "\iota_L"] & A
\end{tikzcd}$$

since ι_L is an algebra morphism, and the right square commutes. By the equation $\iota_L T^{-1} = \mathcal{S}_A^{-1}\iota\mathcal{S}_R$ in (1), the rules of the antipode, and since ι is an algebra morphism,

$$\begin{aligned}
\mu_A(\iota_L T^{-1} \otimes \iota_L T^{-1}) &= \mu_A(\mathcal{S}_A^{-1} \otimes \mathcal{S}_A^{-1})(\iota \otimes \iota)(\mathcal{S}_R \otimes \mathcal{S}_R)\\
&= \mathcal{S}_A^{-1}\iota\mu_R c_{R,R}(\mathcal{S}_R \otimes \mathcal{S}_R).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\iota_L T^{-1}\mu'_R &= \mathcal{S}_A^{-1}\iota\mathcal{S}_R\mu_R\overline{c}^{\mathcal{YD}}_{R,R}c_{R,R}\\
&= \mathcal{S}_A^{-1}\iota\mu_R c^{\mathcal{YD}}_{R,R}(\mathcal{S}_R \otimes \mathcal{S}_R)\overline{c}^{\mathcal{YD}}_{R,R}c_{R,R}\\
&= \mathcal{S}_A^{-1}\iota\mu_R c_{R,R}(\mathcal{S}_R \otimes \mathcal{S}_R),
\end{aligned}$$

where the first equality follows from (1) and the definition of μ'_R, the second from the rules of the antipode, and the last since $\mathcal{S}_R \otimes \mathcal{S}_R$ commutes with $c^{\mathcal{YD}}_{R,R}$ and with $c_{R,R}$.

(d) By Theorem 3.10.4(1)(b), $\vartheta_L : A^{\mathrm{cop}} \to L$ is a coalgebra morphism. Hence $\Delta_L\vartheta_L = (\vartheta_L \otimes \vartheta_L)\overline{c}_{A,A}\Delta_A$, and

$$\Delta_L T^{-1} = (\vartheta_L \otimes \vartheta_L)\overline{c}_{A,A}\Delta_A\iota. \tag{3.10.17}$$

We claim that the following diagram commutes.

$$\begin{tikzcd}
R \arrow[r, "T^{-1}"] \arrow[d, "\Delta_R"] & L \arrow[dd, "\Delta_L"] \\
R \otimes R \arrow[d, "\overline{c}_{R,R}"] & \\
R \otimes R \arrow[r, "T^{-1}\otimes T^{-1}"] & L \otimes L
\end{tikzcd}$$

Note that $\Delta_R = (\vartheta \otimes \vartheta)\Delta_A\iota$, since ϑ is a coalgebra morphism with $\vartheta\iota = \mathrm{id}_R$. Hence

$$\begin{aligned}
(T^{-1} \otimes T^{-1})\overline{c}_{R,R}\Delta_R &= (\vartheta_L \otimes \vartheta_L)\overline{c}_{A,A}(\iota \otimes \iota)\Delta_R\\
&= (\vartheta_L \otimes \vartheta_L)\overline{c}_{A,A}(\iota\vartheta \otimes \iota\vartheta)\Delta_A\iota\\
&= (\vartheta_L \otimes \vartheta_L)(\iota\vartheta \otimes \iota\vartheta)\overline{c}_{A,A}\Delta_A\iota.
\end{aligned}$$

Since by (3.10.9), $\vartheta_L\iota\vartheta = \vartheta_L$, we have shown that

$$(T^{-1} \otimes T^{-1})\overline{c}_{R,R}\Delta_R = (\vartheta_L \otimes \vartheta_L)\overline{c}_{A,A}\Delta_A\iota.$$

Thus the diagram commutes by (3.10.17). □

3.11. Notes

For monoidal and braided monoidal categories, we refer to the books [**Kas95**] and [**EG$^+$15**], see also [**ML98**], and [**Maj95**] for background information.

Important sources for our exposition of the theory are the fundamental and concise paper [**Bes97**], and [**BLS15**].

We thank Simon Lentner for sending us the macros for the graphical calculus from [**BLS15**].

3.1. Monoidal categories were introduced in [**Bén63**] by Bénabou already in 1963.

3.2, 3.3. Braided monoidal categories were introduced by Joyal and Street in 1986, see [**JS93**], [**JS91**].

Hopf algebras in braided monoidal categories using the graphical calculus were studied early by Majid, see the survey article [**Maj94**]. In the graphical calculus we follow the conventions of [**Tak99**] and [**Shi19**].

3.4. Yetter-Drinfeld modules in the category of vector spaces were introduced by Yetter in [**Yet90**] under the name of crossed bimodules, and in braided monoidal categories in [**Bes97**]. We often use the characterization of Yetter-Drinfeld modules which we have introduced in Proposition 3.4.5. For another proof of Theorem 3.4.15 see the sketch in [**Bes97**, Corollary 3.5.5], and [**BLS15**, Theorem 3.16]. Theorem 3.4.16 and Corollary 3.4.17 seem to be new. We will need them in Section 3.10.

3.5. Here, we follow the exposition in [**Tak99**].

3.6. The generalized smash product algebra of Definition 3.6.1 was introduced by Takeuchi for $\mathcal{C} = \mathcal{M}_{\Bbbk}$ in [**Tak80**, Section 8].

3.7. Let H be a Hopf algebra in the braided monoidal category $\mathcal{C} = \mathcal{M}_{\Bbbk}$. We denote by ${}^H_H\mathcal{C}^H_H = {}^H({}_H\mathcal{C}_H)^H$ the category of H-bicomodules in the category of H-bimodules, or Hopf bimodules over H. By Woronowicz [**Wor89**], ${}^H_H\mathcal{C}^H_H$ is a braided monoidal category. Let $V \in {}^H_H\mathcal{C}^H_H$. Then $V \in {}^H({}_H\mathcal{C}_H)$, and by Proposition 3.7.5, $(V, \mathrm{ad}, \delta) \in {}^H_H\mathcal{YD}(\mathcal{C})$. It follows that $V^{\mathrm{co}\,H}$ is a subobject of V in ${}^H_H\mathcal{YD}(\mathcal{C})$, and

$$ {}^H_H\mathcal{C}^H_H \to {}^H_H\mathcal{YD}(\mathcal{C}),\ V \mapsto V^{\mathrm{co}\,H}. $$

is a strict monoidal functor, and an equivalence, see [**Ros98**, Proposition 4], [**Sch94**], [**AD95**, Appendix]. The equivalence between Hopf bimodules and Yetter-Drinfeld modules was shown in the general case of braided monoidal categories $\mathcal{C}$ in [**BD98**].

3.8. Radford's biproduct (where $\mathcal{C} = \mathcal{M}_{\Bbbk}$) was introduced in [**Rad85**] in 1985 when Yetter-Drinfeld modules had not yet been defined. Majid observed in [**Maj93**] that the condition in [**Rad85**] for the biproduct can be expressed by the notion of a Hopf algebra in ${}^H_H\mathcal{YD}$. It is shown in a short sketch in [**Bes97**, Theorem 4.1.2] that the bosonization of a Hopf algebra in ${}^H_H\mathcal{YD}(\mathcal{C})$ is a Hopf algebra in $\mathcal{C}$. In our proof we tried to avoid checking huge pictures (see Proposition 3.8.4).

Theorem 3.8.7 is stated in [**Bes97**, Proposition 4.2.3] with a sketch of a proof.

3.9. See [**AV00**] for the more general case of crossed products and crossed coproducts.

3.10. The name Hopf algebra triple was coined by Takeuchi. Radford proved Theorem 3.10.4 for $\mathcal{C} = \mathcal{M}_{\Bbbk}$, and Bespalov proved the general case. His proof was not published, it only appeared in the preprint version of [**Bes97**]. A proof of the general case also follows from [**BD98**], where the theorem was shown by replacing the Yetter-Drinfeld category by the equivalent category of Hopf bimodules. An outline of the proof of Theorem 3.10.4 was given in [**AV00**].

Theorem 3.10.6 seems to be new. It is needed in Section 12.3.

CHAPTER 4

Yetter-Drinfeld modules over Hopf algebras

As a special case of the theory in Chapter 3 we study Yetter-Drinfeld modules over (usual) Hopf algebras. As an application of Section 3.5 we prove that finite-dimensional Yetter-Drinfeld Hopf algebras are Frobenius algebras.

Throughout the chapter let H denote a Hopf algebra with bijective antipode.

4.1. The braided monoidal category of Yetter-Drinfeld modules

After the introduction of Yetter-Drinfeld modules over groups in Section 1.4 and Yetter-Drinfeld modules in braided strict monoidal categories in Section 3.4, here we discuss the category ${}^H_H\mathcal{YD}(\mathcal{C})$ of Yetter-Drinfeld modules over the Hopf algebra H in the braided monoidal category $\mathcal{C} = \mathcal{M}_{\Bbbk}$ of vector spaces with the flip map as the braiding.

Let V be a left H-module and a left H-comodule with left action and left coaction

$$\begin{aligned} \lambda : H \otimes V \to V,\ \ & h \otimes x \mapsto h \cdot x = hx, \\ \delta : V \to H \otimes V,\ \ & x \mapsto x_{(-1)} \otimes x_{(0)}. \end{aligned}$$

Then (V, λ, δ) is a **(left) Yetter-Drinfeld module** over H if

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)} \tag{4.1.1}$$

for all $h \in H$ and $v \in V$.

We write ${}^H_H\mathcal{YD} = {}^H_H\mathcal{YD}(\mathcal{M}_{\Bbbk})$ for the category of Yetter-Drinfeld modules over H. Objects of ${}^H_H\mathcal{YD}$ are the left Yetter-Drinfeld modules over H, morphisms in ${}^H_H\mathcal{YD}$ are the H-linear and H-colinear maps. The full subcategory of ${}^H_H\mathcal{YD}$ consisting of finite-dimensional Yetter-Drinfeld modules is denoted by ${}^H_H\mathcal{YD}^{\mathrm{fd}}$.

We have seen in Section 3.4 that ${}^H_H\mathcal{YD}$ is a braided monoidal category with the following monoidal and braided structure. Let $V, W \in {}^H_H\mathcal{YD}$. The tensor product of vector spaces $V \otimes W$ becomes a Yetter-Drinfeld module with the usual diagonal action and coaction, where for all $h \in H$, $v \in V$ and $w \in W$,

$$\begin{aligned} h \cdot (v \otimes w) &= h_{(1)} \cdot v \otimes h_{(2)} \cdot w, \\ \delta(v \otimes w) &= v_{(-1)} w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}. \end{aligned}$$

The unit object is the field $\Bbbk$ with the trivial H-module and H-comodule structure, where $h \cdot 1 = \varepsilon(h)$ for all $h \in H$, and $\delta(1) = 1 \otimes 1$. The associativity and unit constraints are the same as for vector spaces. The braiding in ${}^H_H\mathcal{YD}$ and its inverse are defined by

$$\begin{aligned} c_{V,W} : V \otimes W \to W \otimes V,\ \ & v \otimes w \mapsto v_{(-1)} \cdot w \otimes v_{(0)}, \\ c^{-1}_{V,W} : W \otimes V \to V \otimes W,\ \ & w \otimes v \mapsto v_{(0)} \otimes \mathcal{S}^{-1}(v_{(-1)}) \cdot w. \end{aligned}$$

Yetter-Drinfeld modules can be viewed as a special case of the construction of the Drinfeld center of any (strict) monoidal category.

DEFINITION 4.1.1. Let $(\mathcal{C}, \otimes, I)$ be a strict monoidal category. The **left Drinfeld center** $\mathcal{Z}_l(\mathcal{C})$ of $\mathcal{C}$ is a braided monoidal category defined as follows. Objects of $\mathcal{Z}_l(\mathcal{C})$ are pairs (V, γ), where $V \in \mathcal{C}$, and

$$\gamma = (\gamma_X : V \otimes X \to X \otimes V)_{X \in \mathcal{C}}$$

is a natural isomorphism such that for all $X, Y \in \mathcal{C}$ the diagram

$$\begin{tikzcd} V \otimes X \otimes Y \arrow[rr, "\gamma_{X \otimes Y}"] \arrow[dr, "\gamma_X \otimes \mathrm{id}"'] & & X \otimes Y \otimes V \\ & X \otimes V \otimes Y \arrow[ur, "\mathrm{id} \otimes \gamma_Y"'] & \end{tikzcd} \tag{4.1.2}$$

commutes. Note that the definition implies that

$$\gamma_I = \mathrm{id}_V$$

for all $(V, \gamma) \in \mathcal{Z}_l(\mathcal{C})$.

A **morphism** $f : (V, \gamma) \to (W, \lambda)$ between objects (V, γ) and (W, λ) in $\mathcal{Z}_l(\mathcal{C})$ is a morphism $f : V \to W$ in $\mathcal{C}$ such that for all $X \in \mathcal{C}$ the diagram

$$\begin{tikzcd} V \otimes X \arrow[r, "\gamma_X"] \arrow[d, "f \otimes \mathrm{id}"'] & X \otimes V \arrow[d, "\mathrm{id} \otimes f"] \\ W \otimes X \arrow[r, "\lambda_X"] & X \otimes W \end{tikzcd}$$

commutes. Composition of morphisms is given by the composition of morphisms in $\mathcal{C}$.

For objects $(V, \gamma), (W, \lambda)$ in $\mathcal{Z}_l(\mathcal{C})$ the **tensor product** is defined by

$$(V, \gamma) \otimes (W, \lambda) = (V \otimes W, \sigma),$$

such that for all $X \in \mathcal{C}$, the diagram

$$\begin{tikzcd} V \otimes W \otimes X \arrow[rr, "\sigma_X"] \arrow[dr, "\mathrm{id} \otimes \lambda_X"'] & & X \otimes V \otimes W \\ & V \otimes X \otimes W \arrow[ur, "\gamma_X \otimes \mathrm{id}"'] & \end{tikzcd} \tag{4.1.3}$$

commutes. The pair (I, id), where $\mathrm{id}_X = \mathrm{id}_{I \otimes X}$ for all $X \in \mathcal{C}$, is the unit in $\mathcal{Z}_l(\mathcal{C})$.

The **braiding** is defined by

$$\gamma_W : (V, \gamma) \otimes (W, \lambda) \to (W, \lambda) \otimes (V, \gamma).$$

The **right Drinfeld center** $\mathcal{Z}_r(\mathcal{C})$ is defined similarly the objects being pairs (V, γ), where $\gamma = (\gamma_X : X \otimes V \to V \otimes X)_{X \in \mathcal{C}}$ is a natural isomorphism.

It is not difficult to see that the centers $\mathcal{Z}_l(\mathcal{C})$ and $\mathcal{Z}_r(\mathcal{C})$ are braided monoidal categories. For a proof, see [**Kas95**, Theorem XIII.4.2]. Note that

$$\mathcal{Z}_r(\mathcal{C}) \cong \overline{\mathcal{Z}_l(\mathcal{C})}, \ (V, \gamma) \mapsto (V, \gamma^{-1}),$$

is a braided monoidal isomorphism.

A monoidal isomorphism $(F,\varphi):\mathcal{C}\to\mathcal{D}$ between strict monoidal categories defines in the natural way a braided monoidal isomorphism between the left centers of $\mathcal{C}$ and $\mathcal{D}$, and similarly for the right centers. For all objects $(V,\gamma)\in\mathcal{Z}_l(\mathcal{C})$ let

$$F^{\mathcal{Z}_l}(V,\gamma)=(F(V),\widetilde{\gamma}),$$

where for all $X\in\mathcal{C}$, the isomorphism $\widetilde{\gamma}_{F(X)}$ is defined by the commutative diagram

$$\begin{CD} F(V)\otimes F(X) @>{\widetilde{\gamma}_{F(X)}}>> F(X)\otimes F(V)\\ @V{\varphi_{V,X}}VV @VV{\varphi_{X,V}}V\\ F(V\otimes X) @>{F(\gamma_X)}>> F(X\otimes V). \end{CD}$$

In other words, if $G:\mathcal{D}\to\mathcal{C}$ is the inverse functor of F, for all $Y\in\mathcal{D}$,

$$\widetilde{\gamma}_Y=\varphi^{-1}_{G(Y),V}F(\gamma_{G(Y)})\varphi_{V,G(Y)}.$$

For morphisms f in $\mathcal{Z}_l(\mathcal{C})$ we define $F^{\mathcal{Z}_l}(f)=F(f)$. For objects (V,γ) and (W,λ) in $\mathcal{Z}_l(\mathcal{C})$ let

$$\varphi^{\mathcal{Z}_l}_{(V,\gamma),(W,\lambda)}=\varphi_{V,W}.$$

We omit the somewhat tedious proof of the next lemma.

LEMMA 4.1.2. *Let $(F,\varphi):\mathcal{C}\to\mathcal{D}$ be a monoidal isomorphism between strict monoidal categories $\mathcal{C}$ and $\mathcal{D}$. Then*

$$(F^{\mathcal{Z}_l},\varphi^{\mathcal{Z}_l}):\mathcal{Z}_l(\mathcal{C})\to\mathcal{Z}_l(\mathcal{D})$$

is a well-defined braided monoidal isomorphism.

THEOREM 4.1.3. *The functor ${}^H_H\mathcal{YD}\to\mathcal{Z}_l({}_H\mathcal{M})$, mapping $M\in{}^H_H\mathcal{YD}$ to (M,γ), where for all $X\in{}_H\mathcal{M}$, $\gamma_X=c_{M,X}:M\otimes X\to X\otimes M$, and where morphisms f are mapped to f, is a strict isomorphism of braided strict monoidal categories.*

PROOF. Let $F:{}^H_H\mathcal{YD}\to\mathcal{Z}_l({}_H\mathcal{M})$ denote the functor of the theorem. It is clear from Propositions 3.4.5 and 3.4.6 that F is well-defined, strict monoidal, and braided.

We construct the inverse functor. Let $(M,\gamma)\in\mathcal{Z}_l({}_H\mathcal{M})$. We define

$$\delta=(M\xrightarrow{\mathrm{id}\otimes\eta}M\otimes H\xrightarrow{\gamma_H}H\otimes M). \tag{4.1.4}$$

Since Δ is unitary, the following diagram commutes.

$$\begin{CD} M @>{\mathrm{id}\otimes\eta}>> M\otimes H @>{\gamma_H}>> H\otimes M\\ @. @V{\mathrm{id}\otimes\Delta}VV @VV{\mathrm{id}\otimes\mathrm{id}\otimes\eta}V\\ @. M\otimes H\otimes H @>{\gamma_H\otimes\mathrm{id}}>> H\otimes M\otimes H \end{CD}$$

Hence

$$\begin{aligned}(\mathrm{id}\otimes\delta)\delta&=(\mathrm{id}\otimes\gamma_H)(\mathrm{id}\otimes\mathrm{id}\otimes\eta)\gamma_H(\mathrm{id}\otimes\eta)\\&=(\mathrm{id}\otimes\gamma_H)(\gamma_H\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)(\mathrm{id}\otimes\eta)\\&=\gamma_{H\otimes H}(\mathrm{id}\otimes\Delta)(\mathrm{id}\otimes\eta)\\&=(\Delta\otimes\mathrm{id})\gamma_H(\mathrm{id}\otimes\eta)\\&=(\Delta\otimes\mathrm{id})\delta,\end{aligned}$$

where the third equality holds by (4.1.2), and the fourth, since γ is a natural transformation. Thus δ is coassociative.

Note that $(\varepsilon\otimes\mathrm{id})\gamma_H = \mathrm{id}\otimes\varepsilon$, since γ is a natural transformation, and $\varepsilon : H \to \Bbbk$ is left H-linear, where $\Bbbk$ has the trivial H-module structure. Hence $(\varepsilon \otimes \mathrm{id})\delta = \mathrm{id}$. We have shown that (M, δ) is a left H-comodule.

We claim that $\gamma_H = c^{\mathcal{YD}}_{M,H}$, where $H \in {}_H\mathcal{M}$ via left multiplication. For any $h \in H$, right multiplication r_h by h is an endomorphism of H. Since γ_H is a natural transformation, it follows that

$$\begin{aligned}\gamma_H(m \otimes h) &= \gamma_H(\mathrm{id}_M \otimes r_h)(m \otimes 1)\\ &= (r_h \otimes \mathrm{id}_M)\gamma_H(m \otimes 1) = m_{(-1)}h \otimes m_{(0)} = c_{M,H}(m \otimes h)\end{aligned}$$

for all $h \in H$, where $m_{(-1)} \otimes m_{(0)} = \delta(m)$.

Proposition 3.4.5 then implies that M is an object in ${}^H_H\mathcal{YD}$.

The inverse functor $G : \mathcal{Z}_l({}_H\mathcal{M}) \to {}^H_H\mathcal{YD}$ is now defined as follows. For all objects $(M, \gamma) \in \mathcal{Z}_l({}_H\mathcal{M})$ let $G(M, \gamma) = (M, \lambda, \delta)$, where $\lambda : H \otimes M \to M$ is the given H-module structure on M, and $\delta : M \to H \otimes M$ is defined by (4.1.4). Let $f : (M, \gamma) \to (M', \gamma')$ be a morphism in $\mathcal{Z}_l({}_H\mathcal{M})$, that is, $f : M \to M'$ is H-linear, and for all $X \in {}_H\mathcal{M}$, $(\mathrm{id} \otimes f)\gamma_X = \gamma'_X(f \otimes \mathrm{id})$. Then the diagram

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[d, "f"] & M \otimes H \arrow[r, "\gamma_H"] \arrow[d, "f\otimes\mathrm{id}"] & H \otimes M \arrow[d, "\mathrm{id}\otimes f"] \\ M' \arrow[r, "\mathrm{id}\otimes\eta"] & M' \otimes H \arrow[r, "\gamma'_H"] & H \otimes M' \end{tikzcd}$$

commutes. Hence f is H-colinear by definition of the comodule structures of M and M', and $G(f) = f : G(M, \gamma) \to G(M', \gamma')$ is a morphism in ${}^H_H\mathcal{YD}$.

It is clear from the construction of G that $FG = \mathrm{id}$, $GF = \mathrm{id}$. $\square$

REMARK 4.1.4. Theorem 4.1.3 does not generalize directly to ${}^H_H\mathcal{YD}(\mathcal{C})$ for any braided strict monoidal category $\mathcal{C}$, see also the notes at the end of Chapter 4. Indeed, when proving that $\gamma_H = c^{\mathcal{YD}}_{M,H}$ for the construction of the inverse functor, it is used that H is a set and that for any $h \in H$ there is a morphism r_h sending 1 to h.

REMARK 4.1.5. Assume that H is finite-dimensional. Then the Drinfeld double $D(H)$ of H is a Hopf algebra by Remark 2.8.9. The monoidal category ${}_{D(H)}\mathcal{M}$ of left $D(H)$-modules is braided and as such it is equivalent to $\mathcal{Z}_r({}_H\mathcal{M})$. For a proof we refer to [**Kas95**, Theorem XIII.5.1]. Hence ${}_{D(H)}\mathcal{M} \cong \overline{\mathcal{Z}_l({}_H\mathcal{M})} \cong {}^H_H\mathcal{YD}$.

THEOREM 4.1.6. *The functor* ${}^H_H\mathcal{YD} \to \mathcal{Z}_r({}^H\mathcal{M})$*, mapping* $M \in {}^H_H\mathcal{YD}$ *to* (M, γ)*, where for all* $X \in {}^H\mathcal{M}$*,* $\gamma_X = c_{X,M} : X \otimes M \to M \otimes X$*, and where morphisms* f *are mapped to* f*, is a strict isomorphism of braided strict monoidal categories.*

PROOF. We dualize the proof of Theorem 4.1.3 using condition (4) in Proposition 3.4.5. The inverse functor $G : \mathcal{Z}_r({}^H\mathcal{M}) \to {}^H_H\mathcal{YD}$ is defined as follows. For all objects $(M, \gamma) \in \mathcal{Z}_r({}^H\mathcal{M})$ let $G(M, \gamma) = (M, \lambda, \delta)$, where $\delta : M \to H \otimes M$ is the given H-comodule structure on M, and the left H-module structure $\lambda : H\otimes M \to M$ is defined by

$$\lambda = (H \otimes M \xrightarrow{\gamma_H} M \otimes H \xrightarrow{\mathrm{id}\otimes\varepsilon} M).$$

The rest follows along the lines in the proof of Theorem 4.1.3. $\square$

REMARK 4.1.7. Since ${}^H_H\mathcal{YD}$ is a braided strict monoidal category, algebras, coalgebras, bialgebras and Hopf algebras in ${}^H_H\mathcal{YD}$ are defined in Chapter 3.

A Yetter-Drinfeld module $R \in {}^H_H\mathcal{YD}$ is an **algebra** in ${}^H_H\mathcal{YD}$ if R is an algebra such that the structure maps $\mu : R \otimes R \to R$ and $\eta : \Bbbk \to R$ are left H-linear and left H-colinear, that is, R with the given action and coaction of H is a left H-module algebra and a left H-comodule algebra.

An object $C \in {}^H_H\mathcal{YD}$ is a **coalgebra** in ${}^H_H\mathcal{YD}$ if C is a coalgebra such that the structure maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ are left H-linear and left H-colinear, that is, C is an H-module coalgebra and an H-comodule coalgebra. We usually will denote the comultiplication of a braided coalgebra C in a Sweedler notation by

$$\Delta_C : C \to C \otimes C,\ c \mapsto c^{(1)} \otimes c^{(2)}.$$

Let R and S be algebras in ${}^H_H\mathcal{YD}$. The tensor product $R \otimes S$ in ${}^H_H\mathcal{YD}$ is an algebra in ${}^H_H\mathcal{YD}$ with unit $1_R \otimes 1_S$ and the braided multiplication

$$(4.1.5) \qquad (r \otimes s)(x \otimes y) = r(s_{(-1)} \cdot x) \otimes s_{(0)}y \quad \text{for all } r, x \in R,\ s, y \in S.$$

Let C, D be coalgebras in ${}^H_H\mathcal{YD}$. The tensor product $C \otimes D$ in ${}^H_H\mathcal{YD}$ is a coalgebra in ${}^H_H\mathcal{YD}$ with counit $\varepsilon_C \otimes \varepsilon_D$ and the braided comultiplication

$$(4.1.6) \qquad \Delta(c \otimes d) = c^{(1)} \otimes {c^{(2)}}_{(-1)} \cdot d^{(1)} \otimes {c^{(2)}}_{(0)} \otimes d^{(2)} \quad \text{for all } c \in C,\ d \in D.$$

A **bialgebra** in ${}^H_H\mathcal{YD}$ is an algebra and a coalgebra R in ${}^H_H\mathcal{YD}$ such that the comultiplication $\Delta : R \to R \otimes R$, $x \mapsto x^{(1)} \otimes x^{(2)}$, and the counit $\varepsilon : R \to \Bbbk$ are algebra maps, where $R \otimes R$ is the braided tensor product of R with R. In particular, $\Delta(xy) = \Delta(x)\Delta(y)$ for all $x, y \in R$, that is,

$$(4.1.7) \qquad \Delta(xy) = x^{(1)}({x^{(2)}}_{(-1)} \cdot y^{(1)}) \otimes {x^{(2)}}_{(0)} y^{(2)}.$$

A **Hopf algebra** R in ${}^H_H\mathcal{YD}$ is a bialgebra in ${}^H_H\mathcal{YD}$ such that there is a map $\mathcal{S} : R \to R$ of Yetter-Drinfeld modules which is the convolution inverse of id_R. Let $\mathcal{S} : R \to R$ be a linear map which is convolution inverse to id_R. Then $\mathcal{S}$ is a morphism in ${}^H_H\mathcal{YD}$ by Proposition 3.8.9.

LEMMA 4.1.8. *Let R be a bialgebra in ${}^H_H\mathcal{YD}$. If $I \subseteq R$ is a coideal and a subobject in ${}^H_H\mathcal{YD}$, then RI, IR, and $(I) = RIR$ are coideals of R and subobjects in ${}^H_H\mathcal{YD}$. In particular, $R/(I)$ is a quotient bialgebra in ${}^H_H\mathcal{YD}$.*

PROOF. Let $r \in R$ and $x \in I$. Then

$$\Delta(rx) = \Delta(r)\Delta(x) \in (R \otimes R)(I \otimes R + R \otimes I) \subseteq RI \otimes R + R \otimes RI$$

by (4.1.7). Thus RI is a coideal. In the same way it follows that IR and (I) are coideals. It is clear that RI, IR and (I) are subobjects in ${}^H_H\mathcal{YD}$. □

For any bialgebra R in ${}^H_H\mathcal{YD}$, the space of **primitive elements** of R is

$$P(R) = \{x \in R \mid \Delta(x) = 1 \otimes x + x \otimes 1\}.$$

LEMMA 4.1.9. *Let R be a bialgebra in ${}^H_H\mathcal{YD}$. Then $P(R) \subseteq R$ is a subobject in ${}^H_H\mathcal{YD}$.*

PROOF. The map $R \to R \otimes R$, $x \mapsto \Delta(x) - 1 \otimes x - x \otimes 1$, is a map of Yetter-Drinfeld modules. and its kernel is $P(R)$. □

As an application of Theorem 4.1.6 we obtain a braided monoidal isomorphism between Yetter-Drinfeld modules over H and over a two-cocycle deformation of H, see Theorem 2.8.2.

DEFINITION 4.1.10. Let $\sigma : H \otimes H \to \Bbbk$ be a normalized two-cocycle. For all $M \in {}^H_H\mathcal{YD}$ with module structure $\lambda : H \otimes M \to M$, $h \otimes m \mapsto hm$, and comodule structure $\delta : M \to H \otimes M$, let $\delta_\sigma = \delta : M \to H_\sigma \otimes M$, and define $\lambda_\sigma : H_\sigma \otimes M \to M$, $h \otimes m \mapsto h \cdot_\sigma m$ for all $h \in H$, $m \in M$ by

$$h \cdot_\sigma m = \sigma(h_{(1)}, m_{(-2)})\sigma^{-1}(h_{(2)}m_{(-1)}\mathcal{S}(h_{(4)}), h_{(5)})h_{(3)}m_{(0)}.$$

THEOREM 4.1.11. *Let $\sigma : H \otimes H \to \Bbbk$ be a normalized two-cocycle. The functor*

$$F_\sigma : {}^H_H\mathcal{YD} \to {}^{H_\sigma}_{H_\sigma}\mathcal{YD}, \quad (M, \lambda, \delta) \mapsto (M, \lambda_\sigma, \delta_\sigma),$$

mapping morphisms f to f is an isomorphism of categories.

For all $M, N \in {}^H_H\mathcal{YD}$ let

$$(\varphi_\sigma)_{M,N} : F_\sigma(M) \otimes F_\sigma(N) \to F_\sigma(M \otimes N),\ x \otimes y \mapsto \sigma(x_{(-1)}, y_{(-1)})x_{(0)} \otimes y_{(0)},$$

and $\varphi_\sigma = ((\varphi_\sigma)_{M,N})_{M,N \in {}^H_H\mathcal{YD}}$. Then

$$(F_\sigma, \varphi_\sigma) : {}^H_H\mathcal{YD} \to {}^{H_\sigma}_{H_\sigma}\mathcal{YD}$$

is a braided monoidal isomorphism.

PROOF. We define $(F_\sigma, \varphi_\sigma)$ by the following commutative diagram of braided monoidal isomorphisms.

$$\begin{tikzcd} {}^H_H\mathcal{YD} \arrow[r] \arrow[d, "(F_\sigma,\varphi_\sigma)"'] & \mathcal{Z}_r({}^H\mathcal{M}) \arrow[d] \\ {}^{H_\sigma}_{H_\sigma}\mathcal{YD} \arrow[r] & \mathcal{Z}_r({}^{H_\sigma}\mathcal{M}) \end{tikzcd} \tag{4.1.8}$$

Here, the horizontal arrows are the strict monoidal isomorphisms of Theorem 4.1.6 for H and H_σ, and the right vertical arrow is the braided monoidal isomorphism in Lemma 4.1.2 induced from the monoidal isomorphism (F, φ_σ) in Proposition 3.1.10.

Let $M \in {}^H_H\mathcal{YD}$. The image of M in $\mathcal{Z}_r({}^H\mathcal{M})$ is $(M, c_{-,M})$. According to Lemma 4.1.2, $(M, c_{-,M})$ is mapped onto $(F_\sigma(M), \widetilde{\gamma})$ in $\mathcal{Z}_r({}^{H_\sigma}\mathcal{M})$, where for all X in ${}^H\mathcal{M}$, $\widetilde{\gamma}_{F_\sigma(X)}$ is defined by the commutative diagram

$$\begin{tikzcd} F_\sigma(X) \otimes F_\sigma(M) \arrow[r, "\widetilde{\gamma}_{F_\sigma(X)}"] \arrow[d, "\varphi_{X,M}"'] & F_\sigma(M) \otimes F_\sigma(X) \\ F_\sigma(X \otimes M) \arrow[r, "F_\sigma(c_{X,M})"] & F_\sigma(M \otimes X) \arrow[u, "(\varphi_{M,X})^{-1}"'] \end{tikzcd} .$$

To compute the H-action $\cdot_\sigma$ on M, let $X = H$, $h \in H$, and $m \in M$. Then

$$\begin{aligned} \widetilde{\gamma}_{F_\sigma(H)}(h \otimes m) &= (\varphi_{M,H})^{-1}\big(\sigma(h_{(1)}, m_{(-1)})h_{(2)}m_{(0)} \otimes h_{(3)}\big) \\ &= \sigma(h_{(1)}, m_{(-1)})\sigma^{-1}(h_{(2)}m_{(-1)}\mathcal{S}(h_{(4)}), h_{(5)})h_{(3)}m_{(0)} \otimes h_{(6)}. \end{aligned}$$

Hence $(\mathrm{id} \otimes \varepsilon)(\widetilde{\gamma}_{F_\sigma(H)}(h \otimes m)) = h \cdot_\sigma m$ in Definition 4.1.10. We have computed $F_\sigma(M)$.

It is easy to see from (4.1.8) that φ_σ is the monoidal structure of Proposition 3.1.10. □

COROLLARY 4.1.12. *Let $\sigma : H \otimes H \to \Bbbk$ be a normalized two-cocycle. Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ with multiplication and comultiplication denoted by $R \otimes R \to R$, $x \otimes y \mapsto xy$, $\Delta : R \to R \otimes R$, $x \mapsto x^{(1)} \otimes x^{(2)}$. Then $F_\sigma(R)$ is a Hopf algebra in ${}^{H_\sigma}_{H_\sigma}\mathcal{YD}$ with multiplication and comultiplication*

$$R \otimes R \to R, \quad x \otimes y \mapsto \sigma(x_{(-1)}, y_{(-1)}) x_{(0)} y_{(0)},$$
$$R \to R \otimes R, \quad x \mapsto \sigma^{-1}({x^{(1)}}_{(-1)}, {x^{(2)}}_{(-1)}) {x^{(1)}}_{(0)} \otimes {x^{(2)}}_{(0)},$$

and the same unit, counit and antipode as R.

PROOF. This follows from Theorem 4.1.11 and Remark 3.2.13. $\square$

For the following corollary we will use the two-cocycles of free abelian groups discussed in Remark 2.7.4.

DEFINITION 4.1.13. Let $\mathbf{q} = (q_{ij})_{1 \le i \le \theta}$ and $\mathbf{p} = (p_{ij})_{1 \le i \le \theta}$ be matrices with non-zero entries in $\Bbbk^\times$. The matrices $\mathbf{q}$, $\mathbf{p}$ are called **twist-equivalent**, if for all $i, j \in \{1, \dots, \theta\}$,

$$q_{ij} q_{ji} = p_{ij} p_{ji}, \quad q_{ii} = p_{ii}.$$

COROLLARY 4.1.14. *Let $\theta \ge 1$, $\mathbb{I} = \{1, \dots, \theta\}$, and let G be a free abelian group with basis $(g_i)_{i \in \mathbb{I}}$. Let $V \in {}^G_G\mathcal{YD}$ with basis $(x_i)_{i \in \mathbb{I}}$, and $W \in {}^G_G\mathcal{YD}$ with basis $(y_i)_{i \in \mathbb{I}}$, and assume that $x_i \in V_{g_i}^{\chi_i}$ and $y_i \in W_{g_i}^{\eta_i}$ for all $i \in \mathbb{I}$, where for all $i \in \mathbb{I}$, χ_i and η_i are characters of G, that is, elements of $\widehat{G} = \mathrm{Hom}(G, \Bbbk^\times)$. For all $i, j \in \mathbb{I}$ define $q_{ij} = \chi_j(g_i)$, $p_{ij} = \eta_j(g_i)$, and assume that the braiding matrices $(q_{ij})_{i,j \in \mathbb{I}}$ and $(p_{ij})_{i,j \in \mathbb{I}}$ are twist-equivalent. Then there is a normalized two-cocycle $\sigma : \Bbbk G \otimes \Bbbk G \to \Bbbk$ such that*

1. *$\psi : V \xrightarrow{\cong} F_\sigma(W)$, $x_i \mapsto y_i$ for all $i \in \mathbb{I}$, is an isomorphism in ${}^G_G\mathcal{YD}$.*
2. *There is a uniquely determined map $\Psi : \mathcal{B}(V) \to F_\sigma(\mathcal{B}(W))$ of $\mathbb{N}_0$-graded Hopf algebras in ${}^G_G\mathcal{YD}$ such that ψ is the restriction of Ψ to V.*

PROOF. (1) Note that $(\Bbbk G)_\sigma = \Bbbk G$ for any two-cocycle, since the group algebra is cocommutative. By Theorem 4.1.11 and Remark 2.7.4, we have to find non-zero elements $\sigma_{ij} \in \Bbbk$, $i, j \in \mathbb{I}$, such that for all $i, j \in \mathbb{I}$,

$$q_{ij} = \sigma_{ij} \sigma_{ji}^{-1} p_{ij}.$$

These equations are satisfied by defining $\sigma_{ij} = \begin{cases} q_{ij} p_{ij}^{-1}, & \text{if } i \le j, \\ 1, & \text{if } i > j. \end{cases}$.

(2) Since $(F_\sigma, \varphi_\sigma)$ is a braided monoidal isomorphism by Theorem 4.1.11, $F_\sigma(\mathcal{B}(W))$ is a Nichols algebra of $F_\sigma(W)$. Let $\pi : F_\sigma(\mathcal{B}(W)) \to \mathcal{B}(F_\sigma(W))$ be the isomorphism of Theorem 1.6.18 such that the restriction of π to $F_\sigma(W)$ is the identity. Then let Ψ be the composition of $\mathcal{B}(\psi)$ and π^{-1}. $\square$

4.2. Duality of Yetter-Drinfeld modules

By Example 3.5.3, the category $\mathcal{M}_\Bbbk^{\mathrm{fd}}$ of finite-dimensional vector spaces over $\Bbbk$ is a monoidal category with left duality in the standard way. For all $V \in \mathcal{M}_\Bbbk^{\mathrm{fd}}$, $V^* = \mathrm{Hom}(V, \Bbbk)$ is the dual space, and evaluation and coevaluation maps ev_V and

coev_V are defined by

$$\mathrm{ev}_V : V^* \otimes V \to \Bbbk, \quad f \otimes v \mapsto f(v), \tag{4.2.1}$$

$$\mathrm{coev}_V : \Bbbk \to V \otimes V^*, \quad 1 \mapsto \sum_{i=1}^{n} v_i \otimes f_i, \tag{4.2.2}$$

where $v_1, \dots, v_n \in V$ and $f_1, \dots, f_n \in V^*$ are dual bases, that is, $f_i(v_j) = \delta_{ij}$ for all $1 \leq i, j \leq n = \dim V$, or for all $v \in V$,

$$\sum_{i=1}^{n} v_i f_i(v) = v. \tag{4.2.3}$$

We are going to define a Yetter-Drinfeld structure on the dual vector space of a finite-dimensional Yetter-Drinfeld module. Before that we consider bilinear forms of Yetter-Drinfeld modules which are invariant under the action and coaction of H.

LEMMA 4.2.1. *Let $\langle\ ,\ \rangle : X \times Y \to \Bbbk$ be a bilinear form of vector spaces.*

(1) *If $X, Y \in {}_H\mathcal{M}_\Bbbk$, then the following are equivalent.*
   (a) *The form $\langle\ ,\ \rangle$ is left H-linear.*
   (b) *For all $x \in X$, $y \in Y$, and $h \in H$, $\langle h \cdot x, y\rangle = \langle x, \mathcal{S}(h) \cdot y\rangle$.*
(2) *If $X, Y \in {}^H\mathcal{M}_\Bbbk$, then the following are equivalent.*
   (a) *The form $\langle\ ,\ \rangle$ is left H-colinear.*
   (b) *For all $x \in X$ and $y \in Y$, $\mathcal{S}(x_{(-1)})\langle x_{(0)}, y\rangle = y_{(-1)}\langle x, y_{(0)}\rangle$.*

PROOF. (1) (a) $\Rightarrow$ (b): If the form is H-linear, then for all $x \in X$, $y \in Y$, and $h \in H$, $\langle h_{(1)} \cdot x, h_{(2)} \cdot y\rangle = \varepsilon(h)\langle x, y\rangle$. Hence

$$\langle h \cdot x, y\rangle = \langle h_{(1)} \cdot x, h_{(2)}\mathcal{S}(h_{(3)}) \cdot y\rangle = \varepsilon(h_{(1)})\langle x, \mathcal{S}(h_{(2)}) \cdot y\rangle = \langle x, \mathcal{S}(h) \cdot y\rangle.$$

(b) $\Rightarrow$ (a): Assume (b). Then for all $x \in X$, $y \in Y$, and $h \in H$,

$$\langle h_{(1)} \cdot x, h_{(2)} \cdot y\rangle = \langle x, \mathcal{S}(h_{(1)})h_{(2)} \cdot y\rangle = \varepsilon(h)\langle x, y\rangle.$$

(2) is shown similarly to (1). $\square$

Lemma 4.2.1, when applied to evaluation of functions, shows how a natural Yetter-Drinfeld module structures on the dual vector space V^* of any $V \in {}^H_H\mathcal{YD}^{\mathrm{fd}}$ can be defined.

LEMMA 4.2.2. *Let $V \in {}^H_H\mathcal{YD}^{\mathrm{fd}}$.*

(1) *V^* is an object in ${}^H_H\mathcal{YD}^{\mathrm{fd}}$ with action and coaction of H defined for all $h \in H$, $v \in V$ and $f \in V^*$ by*

$$(h \cdot f)(v) = f(\mathcal{S}(h) \cdot v), \qquad f_{(-1)}f_{(0)}(v) = \mathcal{S}^{-1}(v_{(-1)})f(v_{(0)}).$$

(2) *The maps $\mathrm{ev}_V : V^* \otimes V \to \Bbbk$ and $\mathrm{coev}_V : \Bbbk \to V \otimes V^*$ defined in* (4.2.1) *and* (4.2.2) *are morphisms in ${}^H_H\mathcal{YD}^{\mathrm{fd}}$, and $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$ is a left dual of V in the sense of Definition* 3.5.1.

PROOF. (1) It is clear by Proposition 2.2.2 that V^* is a left H-module and a left H-comodule, since the antipode of H is an algebra and coalgebra anti-homomorphism. We check the Yetter-Drinfeld property. Let $v \in V$, $f \in V^*$, and $h \in H$. Then

$$\begin{aligned} h_{(1)} f_{(-1)} \mathcal{S}(h_{(3)})(h_{(2)} \cdot f_{(0)})(v) &= h_{(1)} f_{(-1)} \mathcal{S}(h_{(3)}) f_{(0)}(\mathcal{S}(h_{(2)}) \cdot v) \\ &= h_{(1)} \mathcal{S}^{-1}((\mathcal{S}(h_{(2)}) \cdot v)_{(-1)}) \mathcal{S}(h_{(3)}) f((\mathcal{S}(h_{(2)}) \cdot v)_{(0)}) \\ &= h_{(1)} \mathcal{S}^{-1}(\mathcal{S}(h_{(4)}) v_{(-1)} \mathcal{S}^2(h_{(2)})) \mathcal{S}(h_{(5)}) f(\mathcal{S}(h_{(3)}) \cdot v_{(0)}) \\ &= \mathcal{S}^{-1}(v_{(-1)})(h \cdot f)(v_{(0)}) \\ &= (h \cdot f)_{(-1)}(h \cdot f)_{(0)}(v). \end{aligned}$$

(2) By Lemma 4.2.1, ev_V is left H-linear and H-colinear. We show that coev_V is left H-linear and left H-colinear, that is

$$\sum_{i=1}^{n} h_{(1)} \cdot v_i \otimes h_{(2)} \cdot f_i = \varepsilon(h) \sum_{i=1}^{n} v_i \otimes f_i \text{ for all } h \in H, \tag{4.2.4}$$

$$\sum_{i=1}^{n} v_{i(-1)} f_{i(-1)} \otimes v_{i(0)} \otimes f_{i(0)} = 1 \otimes \sum_{i=1}^{n} v_i \otimes f_i. \tag{4.2.5}$$

Both equations follow by evaluating both sides at $v \in V$, and from (4.2.3). For (4.2.5) note that $\sum_{i=1}^{n} v_{i(-1)} f_i(v) \otimes v_{i(0)} = v_{(-1)} \otimes v_{(0)}$.

The triple $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$ is a left dual of V by Example 3.5.3. □

Definition 4.2.3. The Yetter-Drinfeld module V^* in Lemma 4.2.2 is called the **(left) dual** of V.

Remark 4.2.4. By Lemma 4.2.2, the braided monoidal category ${}^H_H\mathcal{YD}^{\mathrm{fd}}$ is rigid. Let $V, W \in {}^H_H\mathcal{YD}$, and $f : V \to W$ a morphism of Yetter-Drinfeld modules. Then $f^* : W^* \to V^*$ defined in Remark 3.5.2(3) is the dual map $\mathrm{Hom}(f, \mathrm{id})$. The canonical map

$$V^* \oplus W^* \to (V \oplus W)^*,\ f + g \mapsto \big(v + w \mapsto f(v) + g(w)\big), \tag{4.2.6}$$

is an isomorphism in ${}^H_H\mathcal{YD}$. The isomorphisms

$$\begin{aligned} \varphi_{V,W} &: V^* \otimes W^* \to (V \otimes W)^*, \\ \psi_V &: V \to V^{**}, \end{aligned}$$

of Theorem 3.5.8 are given explicitly by

$$\varphi_{V,W}(f \otimes g)(v \otimes w) = f(v_{(0)}) g(v_{(-1)} \cdot w), \tag{4.2.7}$$

$$\psi_V(v)(f) = f(\mathcal{S}_H(v_{(-1)}) \cdot v_{(0)}), \tag{4.2.8}$$

for all $v \in V$, $w \in W$, and $f \in V^*$, $g \in W^*$.

Corollary 4.2.5. *The functor*

$$(\ )^* : ({}^H_H\mathcal{YD}^{\mathrm{fd}})^{\mathrm{op}} \to {}^H_H\mathcal{YD}^{\mathrm{fd}},\ V \mapsto V^*,\ \textit{the left dual of } V,$$

with $f^* = \mathrm{Hom}(f, \mathrm{id})$ *for morphisms* f*, is an equivalence, and*

$$(\psi_V : V \to V^{**})_{V \in {}^H_H\mathcal{YD}^{\mathrm{fd}}}$$

defined in (4.2.8) *is a natural isomorphism.*

Define $\varphi = (\varphi_{V,W})_{V,W \in {}^H_H\mathcal{YD}^{\mathrm{fd}}}$ *by* (4.2.7)*. Then*

$$((\ )^*, \varphi_0, \varphi) : ({}^H_H\mathcal{YD}^{\mathrm{fd}})^{\mathrm{op}} \to {}^H_H\mathcal{YD}^{\mathrm{fd}}$$

is a braided monoidal equivalence, where $\varphi_0 : \Bbbk \to \Bbbk^*$, $1 \mapsto \mathrm{id}_{\Bbbk}$.

Proof. This follows from Theorem 3.5.8, Lemma 4.2.2, and Remark 4.2.4. $\square$

Corollary 4.2.6. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}^{\mathrm{fd}}$, and R^* its left dual. Then R^* is a Hopf algebra in ${}^H_H\mathcal{YD}^{\mathrm{fd}}$ with unit ε_R^*, counit η_R^*, antipode $\mathcal{S}_R^*$, and multiplication and comultiplication are defined for all $f,g \in R^*$ and $x,y \in H$ by*

$$(fg)(x) = f((x^{(1)})_{(0)})g((x^{(1)})_{(-1)} \cdot x^{(2)}), \quad f^{(1)}(x_{(0)})f^{(2)}(x_{(-1)} \cdot y) = f(xy),$$

where $\Delta_R(x) = x^{(1)} \otimes x^{(2)}$, $\mu_R(x \otimes y) = xy$.

Proof. By Corollary 4.2.5 and Remark 3.5.9, R^* is a Hopf algebra with multiplication $\Delta_R^*\varphi_{R,R}$, comultiplication $\varphi_{R,R}^{-1}\mu_R^*$, unit ε_R^*, counit η^*, and antipode $\mathcal{S}_R^*$. Hence the corollary follows from the formula for $\varphi_{R,R}$ in (4.2.7). $\square$

Note that the dual Hopf algebra R^* in Corollary 4.2.6 is the dual Hopf algebra of Proposition 2.2.19 when H is the trivial Hopf algebra $\Bbbk$.

Remark 4.2.7. Let Γ be a set. A Γ-graded object in ${}^H_H\mathcal{YD}$ is a pair $(V,\mathcal{V})$, where $V \in {}^H_H\mathcal{YD}$, and $\mathcal{V} = (V(\gamma))_{\gamma\in\Gamma}$ is a family $V(\gamma) \subseteq V$, $\gamma \in \Gamma$, of subobjects in ${}^H_H\mathcal{YD}$ with $V = \bigoplus_{\gamma\in\Gamma} V(\gamma)$. Let $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ be the **category of Γ-graded Yetter-Drinfeld modules over** H with graded maps in ${}^H_H\mathcal{YD}$ as morphisms.

If Γ is a monoid, then $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ is monoidal. The tensor product of graded objects V,W in ${}^H_H\mathcal{YD}$ is the tensor product $V \otimes W$ in ${}^H_H\mathcal{YD}$ with diagonal grading in Definition 1.2.7. The unit object is the trivial object $\Bbbk$ in ${}^H_H\mathcal{YD}$ with grading given by $\Bbbk(e_\Gamma) = \Bbbk$.

If Γ is an abelian monoid, then the braiding map $c_{V,W} : V \otimes W \to W \otimes V$ in ${}^H_H\mathcal{YD}$ of Γ-graded objects in ${}^H_H\mathcal{YD}$ is a morphism in $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$. Hence the category $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ is braided monoidal with braiding c.

Let Γ be an abelian monoid. A bialgebra R in $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ is a bialgebra in ${}^H_H\mathcal{YD}$ and a Γ-graded object in ${}^H_H\mathcal{YD}$ such that $\mu_R, \eta_R, \Delta_R, \varepsilon_R$ are Γ-graded. A Hopf algebra R in $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ is a bialgebra in $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ and a Hopf algebra in ${}^H_H\mathcal{YD}$ whose antipode is Γ-graded.

Corollary 4.2.8. *Let R be a bialgebra in* $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ *such that* id_R *is convolution invertible in* $\mathrm{Hom}(R,R)$. *Then R is a Hopf algebra in* $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$.

Proof. Let $\mathcal{S}_R$ be convolution inverse to id_R. As noted in Section 4.1, $\mathcal{S}_R$ is a morphism in ${}^H_H\mathcal{YD}$ by Proposition 3.8.9 which follows from a version of Proposition 1.2.11. Similarly, $\mathcal{S}_R$ is Γ-graded by Proposition 1.2.11. $\square$

Let $\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}$ denote the full subcategory of $\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}$ of **locally finite graded Yetter-Drinfeld modules** $(V,(V(n))_{n\in\mathbb{N}_0})$, where $V(n)$ is finite-dimensional for all $n \in \mathbb{N}_0$. Note that $\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}$ is a braided monoidal subcategory, since the tensor product of locally finite $\mathbb{N}_0$-graded Yetter-Drinfeld modules is locally finite.

The duality of finite-dimensional Yetter-Drinfeld modules extends to a duality of $\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}$. Define a contravariant functor

$$(\ )^{*\mathrm{gr}} : \mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}} \to \mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}} \tag{4.2.9}$$

on objects by $(V,(V(n))_{n\geq 0})^{*\mathrm{gr}} = (\bigoplus_{n\geq 0} V(n)^*, (V(n)^*)_{n\geq 0})$, and on morphisms $f : (V,(V(n))_{n\geq 0}) \to (W,(W(n))_{n\geq 0})$ by $f^{*\mathrm{gr}} = \bigoplus_{n\geq 0}(f|V(n))^*$. For all objects $V, W \in \mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}$, define the morphism of graded Yetter-Drinfeld modules

$$\varphi_{V,W} = \bigoplus_{n\in\mathbb{N}_0} \varphi(n)_{V,W} : V^{*\mathrm{gr}} \otimes W^{*\mathrm{gr}} \to (V\otimes W)^{*\mathrm{gr}} \tag{4.2.10}$$

by $\varphi(n)_{V,W} = \bigoplus_{a+b=n} \varphi_{V(a),W(b)}$ for all $n \in \mathbb{N}_0$, where

$$\bigoplus_{a+b=n} \varphi_{V(a),W(b)} : \bigoplus_{a+b=n} V(a)^* \otimes W(b)^* \to \bigoplus_{a+b=n} (V(a)\otimes W(b))^*$$

is viewed as a map to $\left(\bigoplus_{a+b=n} V(a)\otimes W(b)\right)^*$ by the isomorphism (4.2.6). For all $V \in \mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}$ let

$$\psi_V = \bigoplus_{n\in\mathbb{N}_0} \psi_{V(n)} : V \to V^{*\mathrm{gr}\,*\mathrm{gr}}. \tag{4.2.11}$$

Let $\varphi_0 : \Bbbk \to D(\Bbbk)$ be defined by the isomorphism $\Bbbk \to \Bbbk^*$, $1 \mapsto \mathrm{id}_\Bbbk$, in degree zero.

COROLLARY 4.2.9. *Let* $\varphi = (\varphi_{V,W})_{V,W\in\mathbb{N}_0\text{-Gr}({}^H_H\mathcal{YD})^{\mathrm{lf}}}$. *Then*

$$((\ )^{*\mathrm{gr}}, \varphi_0, \varphi) : (\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}})^{\mathrm{op}} \to \mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}$$

is a braided monoidal equivalence, and

$$\psi = (\psi_V)_{V\in\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}} : \mathrm{id}_{\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}} \to (\ )^{*\mathrm{gr}\,*\mathrm{gr}}$$

is a natural isomorphism.

PROOF. This is a formal extension of Corollary 4.2.5. □

An $\mathbb{N}_0$-graded coalgebra C is strictly graded, see Definition 1.3.9, if $C(0)$ is one-dimensional, and $C(1) = P(C)$. We say that an $\mathbb{N}_0$-graded algebra A is **generated in degree one**, if $A(0)$ is one-dimensional, and A is generated as an algebra by $A(1)$.

COROLLARY 4.2.10. *Let C be a coalgebra and A an algebra in* $\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}$.

(1) *The following are equivalent.*
   (a) *C is strictly graded.*
   (b) *The algebra $C^{*\mathrm{gr}}$ is generated in degree one.*
(2) *The following are equivalent.*
   (a) *A is generated in degree one.*
   (b) *The coalgebra $A^{*\mathrm{gr}}$ is strictly graded.*

PROOF. (1) Let $B = C^{*\mathrm{gr}}$. Since $((\ )^{*\mathrm{gr}}, \varphi_0, \varphi)$ is a braided monoidal equivalence by Corollary 4.2.9, B is an algebra in $\mathbb{N}_0\text{-Gr}\,{}^H_H\mathcal{YD}^{\mathrm{lf}}$. For all $n \geq 1$ let

$$\Delta^n : C \to C^{\otimes n}, \qquad \mu^n : B^{\otimes n} \to B$$

be the n-fold comultiplication of C and the n-fold multiplication of B defined inductively by $\Delta^n = (\mathrm{id}_C \otimes \Delta^{n-1})\Delta$, $\mu^n = \mu(\mathrm{id}_B \otimes \mu^{n-1})$, and $\Delta^1 = \mathrm{id}_C$, $\mu^1 = \mathrm{id}_B$. We define the isomorphisms $\varphi^n_C : (C^{*\mathrm{gr}})^{\otimes n} \to (C^{\otimes n})^{*\mathrm{gr}}$ inductively by

$$\varphi^2_C = \varphi_{C,C}, \quad \varphi^n_C = \varphi_{C,C^{\otimes(n-1)}}(\mathrm{id}_{C^{*\mathrm{gr}}} \otimes \varphi^{n-1}_C),\ n \geq 3.$$

In the same way we define isomorphisms $\varphi^n_{C(1)} : (C(1)^*)^{\otimes n} \to (C(1)^{\otimes n})^*$. By the definition of $\Delta_{1^n} : C(n) \to C(1)^{\otimes n}$ in Definition 1.3.12, the restriction of μ^n to the subspace $(C(1)^*)^{\otimes n}$ is equal to the composition

$$(C(1)^*)^{\otimes n} \xrightarrow{\varphi^n_{C(1)}} (C(1)^{\otimes n})^* \xrightarrow{(\Delta_{1^n})^*} C(n)^* \subseteq C^{*\mathrm{gr}}.$$

By Proposition 1.3.14, C is strictly graded if and only if $C(0)$ is one-dimensional, and Δ_{1^n} is injective for all $n \geq 2$. Hence the equivalence of (a) and (b) follows, since the maps $\varphi^n_{C(1)}$, $n \geq 2$, are isomorphisms.

(2) is shown dually to (1). □

Braidings of Yetter-Drinfeld modules are very important examples of braidings of vector spaces. We define a property of braidings which characterizes braidings of Yetter-Drinfeld modules over some Hopf algebra with bijective antipode.

DEFINITION 4.2.11. Let (V, c) be a finite-dimensional braided vector space. Then (V, c) is called **rigid** if the composition $c^\flat$ of the three maps

$$\begin{aligned} V^* \otimes V &\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{coev}_V} V^* \otimes V \otimes V \otimes V^* \\ &\xrightarrow{\mathrm{id}\otimes c\otimes\mathrm{id}} V^* \otimes V \otimes V \otimes V^* \xrightarrow{\mathrm{ev}_V\otimes\mathrm{id}\otimes\mathrm{id}} V \otimes V^* \end{aligned}$$

is bijective.

EXAMPLE 4.2.12. Let V be a vector space of finite dimension at least two. Let $c = \mathrm{id}_{V\otimes V} \in \mathrm{Aut}(V \otimes V)$ be the identity map. Then (V, c) is a (non-interesting) braided vector space which is not rigid by Definition 4.2.11.

PROPOSITION 4.2.13. *Let $V \in {}^H_H\mathcal{YD}^{\mathrm{fd}}$ and let $c = c_{V,V}$. Then $c^\flat = c^{-1}_{V,V^*}$. In particular, (V, c) is rigid.*

PROOF. Let $v \in V$, $f \in V^*$, and let $v_1, \dots, v_n \in V$ and $f_1, \dots, f_n \in V^*$ be dual bases. Then

$$\begin{aligned} c^\flat(f \otimes v) &= \sum_{i=1}^n f(v_{(-1)} \cdot v_i)v_{(0)} \otimes f_i = \sum_{i=1}^n (\mathcal{S}^{-1}(v_{(-1)}) \cdot f)(v_i)v_{(0)} \otimes f_i \\ &= v_{(0)} \otimes \sum_{i=1}^n (\mathcal{S}^{-1}(v_{(-1)}) \cdot f)(v_i)f_i \\ &= v_{(0)} \otimes \mathcal{S}^{-1}(v_{(-1)}) \cdot f = c^{-1}_{V,V^*}(f \otimes v). \end{aligned}$$

This proves the claim. □

4.3. Hopf algebra triples and bosonization

Let R be an Hopf algebra in ${}^H_H\mathcal{YD}$, and a coalgebra in ${}^H\mathcal{M}$. We denote the H-action, H-coaction, comultiplication and counit of R by

$$\lambda_R : H \otimes R \to R,\ h \otimes r \mapsto h \cdot r, \qquad \delta_R : R \to H \otimes R,\ r \mapsto r_{(-1)} \otimes r_{(0)},$$
$$\Delta_R : R \to R \otimes R,\ r \mapsto r^{(1)} \otimes r^{(2)}, \qquad \varepsilon_R : R \to \Bbbk.$$

Recall that in the smash product algebra $R\#H$ and the smash coproduct coalgebra $R\#H$,

$$(r\#g)(s\#h) = r(g_{(1)} \cdot s)\#g_{(2)}h, \tag{4.3.1}$$

$$\Delta_{R\#H}(r\#h) = r^{(1)}\#r^{(2)}{}_{(-1)}h_{(1)} \otimes r^{(2)}{}_{(0)}\#h_{(2)}, \tag{4.3.2}$$

$$\varepsilon_{R\#H}(r\#h) = \varepsilon_R(r)\varepsilon(h) \tag{4.3.3}$$

for all $r, s \in R$, $g, h \in H$. The element $1\#1$ is the unit element in the algebra $R\#H$.

We reformulate Theorem 3.10.4 for the category $\mathcal{C} = {}^H_H\mathcal{YD}$ in the following more direct way.

COROLLARY 4.3.1. *Let (A, π, γ) be a Hopf algebra triple over the Hopf algebra H. Let $R = A^{\mathrm{co}\,H} = \{a \in A \mid a_{(1)} \otimes \pi(a_{(2)}) = a \otimes 1\}$. The antipodes of A and H are denoted by $\mathcal{S}$. Let*

$$\vartheta : A \to R, \quad a \mapsto a_{(1)}\gamma\pi\mathcal{S}(a_{(2)}).$$

Then R is a left coideal subalgebra of A, ϑ is a well-defined left R-linear map with $\vartheta|R = \mathrm{id}_R$, and the following hold.

1. *R is an object in ${}^H_H\mathcal{YD}$ with H-action $\cdot = \lambda_R : H \otimes R \to R$ and H-coaction δ_R, where for all $r \in R$, $h \in H$,*
   (a) $h \cdot r = \gamma(h_{(1)})r\gamma(\mathcal{S}(h_{(2)}))$,
   (b) $\delta_R(r) = \pi(r_{(1)}) \otimes r_{(2)}$.
2. *For all $a \in A$, $h \in H$,*
   (a) $\vartheta(a\gamma(h)) = \vartheta(a)\varepsilon(h)$,
   (b) $\vartheta(\gamma(h)a) = h \cdot \vartheta(a)$.
3. *R is a Hopf algebra in ${}^H_H\mathcal{YD}$, where R is a subalgebra of A, the map $\vartheta : A \to R$ is a coalgebra morphism which induces a coalgebra isomorphism $A/A\gamma(H)^+ \cong R$, and comultiplication Δ_R, counit ε_R and antipode $\mathcal{S}_R$ are defined for all $h \in H$, $r \in R$ by*
   (a) $\Delta_R(r) = \vartheta(r_{(1)}) \otimes r_{(2)}, \quad \varepsilon_R(r) = \varepsilon_A(r)$,
   (b) $\mathcal{S}_R(r) = \gamma\pi(r_{(1)})\mathcal{S}(r_{(2)})$.
4. *$\Phi : R\#H \to A$, $r\#h \mapsto r\gamma(h)$, is an isomorphism of algebras and coalgebras with inverse $\Psi : A \to R\#H$, $a \mapsto \vartheta(a_{(1)})\#\pi(a_{(2)})$, where $R\#H$ is the bosonization of R.*

EXAMPLE 4.3.2. Let q be a primitive n-th root of unity with $n \geq 2$, and let

$$T_{q,n} = \Bbbk\langle g, x \mid g^n = 1, x^n = 0, gx = qxg\rangle$$

be the Taft Hopf algebra of Example 2.4.10 with $\Delta(x) = g \otimes x + x \otimes 1$ and group-like element g. Let G be the cyclic group of order n generated by g. Define Hopf algebra maps $\pi : T_{q,n} \to \Bbbk G$ and $\gamma : \Bbbk G \to T_{q,n}$ by

$$\pi(g) = g, \ \pi(x) = 0 \text{ and } \gamma(g) = g.$$

Then $\pi\gamma = \mathrm{id}_{\Bbbk G}$, and Corollary 4.3.1 applies. Since $x \in R$, it follows from Lemma 2.6.25 that $R = \Bbbk[x]$. Then R is a Hopf algebra in ${}^G_G\mathcal{YD}$, where

$$\begin{aligned} g \cdot x &= gxg^{-1} = qx, \\ \delta_R(x) &= g \otimes x, \\ \Delta_R(x) &= 1 \otimes x + x \otimes 1. \end{aligned}$$

COROLLARY 4.3.3. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ with bosonization $R\#H$. Define*

$$\pi_R = \varepsilon_R \otimes \mathrm{id}_H : R\#H \to H, \quad \gamma_R = \eta_R \otimes \mathrm{id}_H : H \to R\#H.$$

Then $(R\#H, \pi_R, \gamma_R)$ is a Hopf algebra triple over H, and

$$\iota : R \to (R\#H)^{\mathrm{co}\,H},\ r \mapsto r\#1,$$

is an isomorphism of Hopf algebras in ${}^H_H\mathcal{YD}$.

PROOF. By Theorem 3.8.10, $R\#H$ is a Hopf algebra. Thus $(R\#H, \pi_R, \gamma_R)$ is a Hopf algebra triple over H by Lemma 3.8.2(1). It is clear that ι is an algebra isomorphism in ${}^H_H\mathcal{YD}$. The map ι is a coalgebra homomorphism since for all $r \in R$,

$$\vartheta(r_{(1)}) \otimes r_{(2)} = \vartheta(r^{(1)} r^{(2)}{}_{(-1)}) \otimes r^{(2)}{}_{(0)} = r^{(1)} \otimes r^{(2)},$$

where we used the definition of the comultiplication of $(R\#H)^{\mathrm{co}\,H}$ and rules for ϑ in Corollary 4.3.1. □

REMARK 4.3.4. By Corollary 4.3.3 and Propositions 3.6.5 and 3.6.9, there is a unique functor from the category of Hopf algebras in ${}^H_H\mathcal{YD}$ to the category of Hopf algebra triples over H mapping a Hopf algebra R in ${}^H_H\mathcal{YD}$ to $(R\#H, \pi_R, \gamma_R)$ and a Hopf algebra morphism φ to $\varphi \otimes \mathrm{id}_H$. Corollaries 4.3.1 and 4.3.3 imply that this functor is an equivalence.

We recall the following convention for a smash product algebra $R\#H$. For all $r \in R$, $h \in H$ we write $r\#h = rh$, that is, we identify $r\#1$ with r and $1\#h$ with h.

COROLLARY 4.3.5. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ with antipode $\mathcal{S}_R$. Let $A = R\#H$ be the bosonization of R. We denote the antipodes of A and of H by $\mathcal{S}$.*

(1) *For all $r \in R$ and $h \in H$,*

$$\mathcal{S}(rh) = \mathcal{S}(h)\mathcal{S}(r_{(-1)})\mathcal{S}_R(r_{(0)}),$$

(2) *The map $R \to R$, $r \mapsto \mathcal{S}^2(r)$, is a well-defined algebra and coalgebra map, and for all $h \in H$, $r \in R$,*
   (a) $\mathcal{S}^2(r) = \mathcal{S}_R^2(\mathcal{S}(r_{(-1)}) \cdot r_{(0)})$,
   (b) $\mathcal{S}^2(h \cdot r) = \mathcal{S}^2(h) \cdot \mathcal{S}^2(r)$,
   (c) $\delta_R(\mathcal{S}^2(r)) = \mathcal{S}^2(r_{(-1)}) \otimes \mathcal{S}^2(r_{(0)})$.

PROOF. (1) is a special case of Theorem 3.8.10.
(2) Let $r \in R$. Using the formula for $\mathcal{S}$ in (1) we compute

$$\begin{aligned}
\mathcal{S}^2(r) &= \mathcal{S}(\mathcal{S}(r_{(-1)})\mathcal{S}_R(r_{(0)})) \\
&= \mathcal{S}(\mathcal{S}_R(r_{(0)}))\mathcal{S}^2(r_{(-1)}) \\
&= \mathcal{S}(r_{(-1)})\mathcal{S}_R^2(r_{(0)})\mathcal{S}^2(r_{(-2)}) \\
&= \mathcal{S}(r_{(-1)}) \cdot \mathcal{S}_R^2(r_{(0)}) \\
&= \mathcal{S}_R^2(\mathcal{S}(r_{(-1)}) \cdot r_{(0)}).
\end{aligned}$$

Then (b) and (c) follow from (a) and the Yetter-Drinfeld condition. The restriction of $\mathcal{S}^2$ is a coalgebra morphism, since by the definition of Δ_R,

$$\begin{aligned}\Delta_R(\mathcal{S}^2(r)) &= \mathcal{S}^2(r_{(1)})\mathcal{S}\pi(\mathcal{S}^2(r_{(2)})) \otimes \mathcal{S}^2(r_{(3)})\\ &= \mathcal{S}^2(r_{(1)}\pi\mathcal{S}(r_{(2)})) \otimes \mathcal{S}^2(r_{(3)})\\ &= \mathcal{S}^2(r^{(1)}) \otimes \mathcal{S}^2(r^{(2)}).\end{aligned}$$

□

The theory of bosonization and Hopf algebra triples in Chapter 3 can also be applied to graded Yetter-Drinfeld modules in ${}^H_H\mathcal{YD}$. We mention some results in this context which we derive from the non-graded theory.

Corollary 4.3.6. *Let R be an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Then $R\#H$ is an $\mathbb{N}_0$-graded Hopf algebra, where the grading is defined by*

$$(R\#H)(n) = R(n)\#H \text{ for all } n \geq 0.$$

Proof. This follows from Theorem 3.8.10, and from the explicit formulas for the multiplication and comultiplication of $R\#H$. □

The special class of Hopf algebra triples of the following corollary is important for this book.

Corollary 4.3.7. *Let A be an $\mathbb{N}_0$-graded Hopf algebra such that $H = A(0)$ is a Hopf algebra with bijective antipode. Let $\pi : A \to H$ be the canonical projection with $\pi(x) = 0$ for all $x \in A(n)$, $n \geq 1$, and $\pi|H = \mathrm{id}_H$. Let $R = A^{\mathrm{co}\,H}$ with respect to π. Then R is an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ with grading $R(n) = R \cap A(n)$ for all $n \geq 0$, $R(0) = \Bbbk 1$, and*

$$R\#H \to A, \quad r\#h \mapsto rh,$$

is an isomorphism of $\mathbb{N}_0$-graded Hopf algebras, where the grading of $R\#H$ is defined by $(R\#H)(n) = R(n) \otimes H$ for all $n \geq 0$.

Proof. It is clear from the definition that $R(0) = \Bbbk 1$. By definition, R is the kernel of the graded map $A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id}\otimes(\pi-\varepsilon)} A \otimes H$, where H is trivially graded. Hence R is an $\mathbb{N}_0$-graded object in ${}^H_H\mathcal{YD}$ by Corollary 4.3.1(2). The map $\vartheta : A \to R$ is $\mathbb{N}_0$-graded, since the antipode of A is graded by Corollary 1.2.27. Hence Δ_R is graded by Corollary 4.3.1(3), and R is a graded coalgebra. It is clear that R is a graded algebra. By Corollary 4.3.1 and Corollary 4.2.8, R is a graded Hopf algebra in ${}^H_H\mathcal{YD}$, and $\Phi : R\#H \to A$, $r\#h \mapsto rh$, is a Hopf algebra isomorphism, which is graded. □

Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ with antipode $\mathcal{S}_R$.

We recall the braided, strict monoidal isomorphism

$$F : {}^R_R\mathcal{YD}({}^H_H\mathcal{YD}) \xrightarrow{\cong} {}^{R\#H}_{R\#H}\mathcal{YD} \tag{4.3.4}$$

of Theorem 3.8.7. For any Hopf algebra K in ${}^R_R\mathcal{YD}({}^H_H\mathcal{YD})$, the image $F(K)$ is a Hopf algebra in ${}^{R\#H}_{R\#H}\mathcal{YD}$. By Remark 4.3.4,

$$(F(K)\#(R\#H), \pi_{F(K)}, \gamma_{F(K)}) \text{ and } (R\#H, \pi_R, \gamma_R)$$

are Hopf algebra triples over $R\#H$ and over H, respectively.

Corollary 4.3.8. *Let K be a Hopf algebra in ${}^R_R\mathcal{YD}({}^H_H\mathcal{YD})$.*

(1) *The identity map*

$$(K\#R)\#H \xrightarrow{\cong} F(K)\#(R\#H),\ x\otimes r\otimes h \mapsto x\otimes r\otimes h,$$

is an isomorphism of Hopf algebras between the bosonizations of $K\#R$ and $F(K)$.

(2) *The map*

$$K\#R \xrightarrow{\cong} (F(K)\#(R\#H))^{\mathrm{co}\,H},\ x\#r \mapsto x\#r\#1,$$

is an isomorphism of Hopf algebras in ${}^H_H\mathcal{YD}$.
(Here, $(F(K)\#(R\#H))^{\mathrm{co}\,H}$ is defined with respect to the Hopf algebra triple $(F(K)\#(R\#H), \pi_R\pi_{F(K)}, \gamma_{F(K)}\gamma_R)$ over H.)

Proof. (1) is a special case of Theorem 3.8.7, and (2) follows from (1) and Corollary 4.3.3. □

Proposition 4.3.9. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ and (P,π,γ) a Hopf algebra triple in ${}^H_H\mathcal{YD}$ over R. Then $(P\#H, \pi\otimes \mathrm{id}_H, \gamma\otimes\mathrm{id}_H)$ is a Hopf algebra triple over $R\#H$. Let $P^{\mathrm{co}\,R}$ and $(P\#H)^{\mathrm{co}\,R\#H}$ be the sets of right coinvariant elements. Then the embedding $P\to P\#H$, $p\mapsto p\otimes 1$, induces an isomorphism*

$$\iota_1 : F(P^{\mathrm{co}\,R}) \xrightarrow{\cong} (P\#H)^{\mathrm{co}\,(R\#H)},\ x\mapsto x\otimes 1,$$

of Hopf algebras in ${}^{R\#H}_{R\#H}\mathcal{YD}$.

Proof. The first claim follows from Remark 4.3.4.

Let $K = P^{\mathrm{co}\,R}$. By Corollary 4.3.8(1), and Theorem 3.10.4 for the triple (P,π,γ),

$$F(K)\#(R\#H) \to (K\#R)\#H,\ x\otimes r\otimes h\mapsto x\otimes r\otimes h,$$
$$(K\#R)\#H \to P\#H,\ x\otimes r\otimes h \mapsto x\gamma(r)\otimes h,$$

are isomorphisms of Hopf algebras. Hence the composition

$$\Phi : F(K)\#(R\#H) \to P\#H,\ x\otimes r\otimes h\mapsto x\gamma(r)\otimes h,$$

is an isomorphism of Hopf algebras. Since Φ is an isomorphism of Hopf algebra triples $(F(K), \pi_{F(K)}, \gamma_{F(K)})$ and $(P\#\mathrm{id}_H, \pi\#\mathrm{id}_H, \gamma\#H)$, the restriction of Φ to the coinvariant elements defines the isomorphism of Hopf algebras in ${}^{R\#H}_{R\#H}\mathcal{YD}$ in the proposition. □

We close this section with some useful formulas on the adjoint action.

Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$, and $\mathrm{ad}_R : R\otimes R\to R$ the **braided adjoint action** in Definition 3.7.3. Then

$$\mathrm{ad}_R = (R\otimes R \xrightarrow{\Delta_R\otimes\mathrm{id}_R} R\otimes R\otimes R \xrightarrow{\mathrm{id}_R\otimes c_{R,R}} R\otimes R\otimes R$$
$$\xrightarrow{\mathrm{id}_R\otimes\mathrm{id}_R\otimes\mathcal{S}_R} R\otimes R\otimes R \xrightarrow{\mu_R(\mathrm{id}_R\otimes\mu_R)} R),$$

that is for all $x,y\in R$,

$$\mathrm{ad}_R(x\otimes y) = x^{(1)}({x^{(2)}}_{(-1)}\cdot y)\mathcal{S}_R({x^{(2)}}_{(0)}).$$

We also write $\mathrm{ad}_R = \mathrm{ad}_c = \mathrm{ad}$, and $\mathrm{ad}\,x(y) = \mathrm{ad}(x\otimes y)$.

Example 4.3.10. Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$, and $x,y\in R$. If x is primitive, then $\mathrm{ad}_c x(y) = xy - (x_{(-1)}\cdot y)x_{(0)}$ is the **braided commutator** of x and y.

LEMMA 4.3.11. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$, and $x, y \in R$. Then*

$$\mathrm{ad}_R x(y) = x_{(1)} y \mathcal{S}(x_{(2)}),$$

where $x_{(1)} y \mathcal{S}(x_{(2)}) = \mathrm{ad}_A x(y)$ is the adjoint action of x on y in the bosonization $A = R\#H$.

PROOF. By Corollary 4.3.5(1), $\mathcal{S}_R(r) = r_{(-1)}\mathcal{S}(r_{(0)})$ for all $r \in R$. Hence

$$\begin{aligned}
\mathrm{ad}_R x(y) &= x^{(1)}(x^{(2)}{}_{(-1)} \cdot y)\mathcal{S}_R(x^{(2)}{}_{(0)}) \\
&= x^{(1)}x^{(2)}{}_{(-3)} y \mathcal{S}(x^{(2)}{}_{(-2)}) x^{(2)}{}_{(-1)} \mathcal{S}(x^{(2)}{}_{(0)}) \\
&= x^{(1)}x^{(2)}{}_{(-1)} y \mathcal{S}(x^{(2)}{}_{(0)}) \\
&= x_{(1)} y \mathcal{S}(x_{(2)}).
\end{aligned}$$

□

PROPOSITION 4.3.12. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$, and let $q, r, s \in \Bbbk$ and $g, h \in G(H)$ with $gh = hg$. Let $x, y \in P(R)$, and assume that*

$$\delta_R(x) = g \otimes x, \qquad \delta_R(y) = h \otimes y, \qquad g \cdot x = qx, \qquad g \cdot y = ry, \qquad h \cdot x = sx.$$

Let $A = R\#H$ be the bosonization of R. Then for all $m \in \mathbb{N}_0$,

(1) $(\mathrm{ad}_R x)^m(y) = \displaystyle\sum_{k=0}^{m} (-1)^k r^k q^{k(k-1)/2} \binom{m}{k}_q x^{m-k} y x^k,$

(2) $$\Delta_A((\mathrm{ad}_R x)^m(y)) = (\mathrm{ad}_R x)^m(y) \otimes 1 + \sum_{k=0}^{m} \binom{m}{k}_q \Big(\prod_{l=k}^{m-1}(1 - q^l rs)\Big) x^{m-k} g^k h \otimes (\mathrm{ad}_R x)^k(y).$$

(3) $$\Delta_R((\mathrm{ad}_R x)^m(y)) = (\mathrm{ad}_R x)^m(y) \otimes 1 + \sum_{k=0}^{m} \binom{m}{k}_q \Big(\prod_{l=k}^{m-1}(1 - q^l rs)\Big) x^{m-k} \otimes (\mathrm{ad}_R x)^k(y).$$

PROOF. (1) Note that for all $a \in R$, $(\mathrm{ad}_R x)(a) = xa - (g \cdot a)x = (F + G)(a)$, where $F, G \in \mathrm{Hom}(R, R)$ with $F(a) = xa$, $G(a) = -(g \cdot a)x$ for all $a \in R$. Then in $\mathrm{Hom}(R, R)$, $GF = qFG$. Hence (1) follows from the q-binomial formula in Proposition 1.9.5.

(2) By definition of Δ_A in (4.3.2), $x \in P_{g,1}(A)$, $y \in P_{h,1}(A)$. By (1),

$$(\mathrm{ad}_R x)^n(y) = x^n \triangleright y \quad \text{in Proposition 2.4.3.}$$

Hence (2) follows from Proposition 2.4.3(1) and Lemma 4.3.11.

(3) Let $\vartheta = \mathrm{id}_R \otimes \varepsilon : A \to R$. The formula in (3) follows by applying $\vartheta \otimes \mathrm{id}$ to (2), since for all $r \in R$, $\Delta_R(r) = (\vartheta \otimes \mathrm{id})\Delta_A(r)$. □

4.4. Finite-dimensional Yetter-Drinfeld Hopf algebras are Frobenius algebras

In 1969, Larson and Sweedler proved in their pioneering paper [**LS69**] that an arbitrary finite-dimensional Hopf algebra is a Frobenius algebra. Extending their ideas we next show that finite-dimensional Hopf algebras in ${}^H_H\mathcal{YD}$ are Frobenius. We first discuss Frobenius algebras.

The dual vector space $A^* = \mathrm{Hom}(A, \Bbbk)$ of an algebra A is an A-bimodule by

$$(af)(x) = f(xa), \quad (fa)(x) = f(ax)$$

for all $a, x \in A$ and $f \in A^*$.

Lemma 4.4.1. *Let A be a finite-dimensional algebra, and $f \in A^*$. Then the following are equivalent.*

(1) *The left A-module A^* is free with basis f.*
(2) *The right A-module A^* is free with basis f.*

Proof. Let $\mathrm{can} : A \to A^{**}$, $a \mapsto (\varphi \mapsto \varphi(a))$, be the canonical isomorphism. Let $F : A \to A^*$, $a \mapsto af$. Then for all $a \in A$, $F^*\mathrm{can}(a) = fa$, and the claim follows. □

Definition 4.4.2. A finite-dimensional algebra A is a **Frobenius algebra** if $A \cong A^*$ as a left (or by Lemma 4.4.1 equivalently right) A-module. A basis f of A^* as a left or right A-module is called a **Frobenius element**.

Example 4.4.3. Let G be a finite group. Define $f \in (\Bbbk G)^*$ by

$$f(g) = \begin{cases} 1, & \text{if } g = 1, \\ 0, & \text{if } g \neq 1. \end{cases}$$

Then the elements $g^{-1}f$, $g \in G$, form the dual basis of the basis G of the group algebra. Thus $\Bbbk G$ is a Frobenius algebra with Frobenius element f.

Definition 4.4.4. Let A be an augmented algebra, that is an algebra together with an algebra map $\varepsilon : A \to \Bbbk$. An element $\Lambda \in A$ is called a **left integral** of A if $a\Lambda = \varepsilon(a)\Lambda$ for all $a \in A$. It is called a **right integral** of A if $\Lambda a = \varepsilon(a)\Lambda$ for all $a \in A$. We denote by $I_l(A)$ and $I_r(A)$ the set of left and right integrals of A, respectively.

Let C be a coalgebra with a distinguished group-like element 1_C. We denote by $I_l(C^*)$ and $I_r(C^*)$ the sets of left and right integrals of C^*, respectively, with respect to the algebra map $\varepsilon : C^* \to \Bbbk$, $f \mapsto f(1_C)$.

Lemma 4.4.5. *Let C be a coalgebra with a distinguished group-like element $1_C \in C$, and let $\lambda \in C^*$. Then $\lambda \in I_r(C^*)$ if and only if for all $c \in C$,*

$$\lambda(c_{(1)})c_{(2)} = \lambda(c)1_C.$$

Proof. By definition, $\lambda \in I_r(C^*)$ if and only if for all $f \in C^*$, $c \in C$,

$$\lambda(c_{(1)})f(c_{(2)}) = \lambda(c)f(1_C) \text{ or } f(\lambda(c_{(1)})c_{(2)}) = f(\lambda(c)1_C),$$

that is, if and only if for all $c \in C$, $\lambda(c_{(1)})c_{(2)} = \lambda(c)1_C$. □

If A is an algebra and $X \subseteq A$ is a subspace, then we denote the left and right annihilators of X by

$$l(X) = \{a \in A \mid ax = 0 \text{ for all } x \in X\},$$
$$r(X) = \{a \in A \mid xa = 0 \text{ for all } x \in X\}.$$

Lemma 4.4.6. *Let A be a Frobenius algebra with Frobenius element f.*

(1) *For all right ideals I of A and all left ideals J of A,*

$$\dim l(I) = \dim A/I, \quad \dim r(J) = \dim A/J.$$

(2) *Let* $\varepsilon : A \to \Bbbk$ *be an augmentation of* A. *Then* $I_l(A)$ *and* $I_r(A)$ *are one-dimensional, and* $f(I_l(A)) \neq 0$, $f(I_r(A)) \neq 0$.

PROOF. (1) The assumptions imply that the maps $l(I) \to (A/I)^*$, $a \mapsto fa$, and $r(J) \to (A/J)^*$, $a \mapsto af$, are bijective.

(2) follows from (1), since $I_l(A) = r(A^+)$ and $I_r(A) = l(A^+)$. Note that for $\Lambda \in I_r(A)$, $\Gamma \in I_l(A)$, $\Lambda, \Gamma \neq 0$ implies that $f(\Lambda) \neq 0$, $f(\Gamma) \neq 0$. □

Frobenius algebras can be described by various equivalent conditions. In this context the notion of a Casimir element is useful. If A is an algebra, and x_i, y_i, $1 \leq i \leq n$, are elements in A, then $\sum_{i=1}^n x_i \otimes y_i \in A \otimes A$ is called a **Casimir element** of A if for all $x \in A$,

$$\sum_{i=1}^n xx_i \otimes y_i = \sum_{i=1}^n x_i \otimes y_i x.$$

LEMMA 4.4.7. *Let* A *be an algebra,* $x_i, y_i \in A$, $1 \leq i \leq n$, *and assume that* $\sum_{i=1}^n x_i \otimes y_i$ *is a Casimir element of* A. *Then*

$$\Delta : A \to A \otimes A,\ x \mapsto \sum_{i=1}^n xx_i \otimes y_i = \sum_{i=1}^n x_i \otimes y_i x,$$

is coassociative and left and right A*-linear, where the* A*-module structures of* $A \otimes A$ *are defined by the multiplication in* A.

PROOF. For all $x \in A$,

$$(\Delta \otimes \mathrm{id}_A)\Delta(x) = \sum_{i=1}^n \Delta(xx_i) \otimes y_i = \sum_{1 \leq i,j \leq n} xx_i x_j \otimes y_j \otimes y_i,$$

$$(\mathrm{id}_A \otimes \Delta)\Delta(x) = \sum_{i=1}^n xx_i \otimes \Delta(y_i) = \sum_{1 \leq i,j \leq n} xx_i \otimes y_i x_j \otimes y_j,$$

and equality follows, since $\sum_{i=1}^n x_i \otimes y_i x_j = \sum_{i=1}^n x_j x_i \otimes y_i$ for all $1 \leq j \leq n$. □

PROPOSITION 4.4.8. *Let* A *be a finite-dimensional algebra, and* $f : A \to \Bbbk$ *a linear map. Define*

$$F : A \otimes A \to \mathrm{Hom}(A, A),\ x \otimes y \mapsto (a \mapsto xf(ya)).$$

The following are equivalent.

(1) A *is a Frobenius algebra with Frobenius element* f.
(2) F *is bijective.*
(3) *There are an integer* $n \geq 1$ *and* $x_i, y_i \in A$ *for all* $1 \leq i \leq n$ *such that for all* $x \in A$,
  (a) $x = \sum_{i=1}^n x_i f(y_i x)$,
  (b) $x = \sum_{i=1}^n f(xx_i) y_i$.
(4) *There is a linear map* $\Delta : A \to A \otimes A$ *such that*
  (a) (A, Δ, f) *is a coalgebra.*
  (b) *The map* $\Delta : A \to A \otimes A$ *is left and right* A*-linear, where the* A*-module structures of* $A \otimes A$ *are defined by the multiplication in* A.

Proof. (1) $\Leftrightarrow$ (2) The map F is the composition of

$$A \otimes A \to A \otimes A^*,\ x \otimes y \mapsto x \otimes fy,$$

and the isomorphism $A \otimes A^* \to \mathrm{Hom}(A, A)$, $x \otimes \varphi \mapsto (a \mapsto x\varphi(a))$.

(2) $\Rightarrow$ (3) Choose $x_i, y_i \in A$, $1 \le i \le n$, with $F(\sum_{i=1}^n x_i \otimes y_i) = \mathrm{id}_A$. By definition of F, equation (a) follows. Hence for all $x, y \in A$

$$f\Big(\sum_{i=1}^n f(xx_i)y_i y\Big) = \sum_{i=1}^n f(xx_i)f(y_i y) = f\Big(x\sum_{i=1}^n x_i f(y_i y)\Big) = f(xy).$$

We have shown that $f\sum_{i=1}^n f(xx_i)y_i = fx$ for all $x \in A$. Since A is a Frobenius algebra with Frobenius element f, the second equation (b) follows.

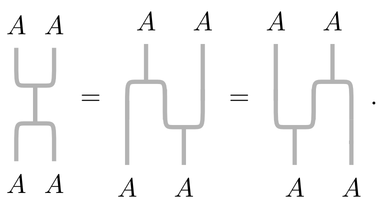
(3) $\Rightarrow$ (1) Let $x \in A$ with $xf = 0$. Then $x = 0$ by (3)(a).

(2) $\Rightarrow$ (4) Choose $x_i, y_i \in A, 1 \le i \le n$ with $F(\sum_{i=1}^n x_i \otimes y_i) = \mathrm{id}_A$. By definition and injectivity of F, $\sum_{i=1}^n x_i \otimes y_i$ is a Casimir element of A. By Lemma 4.4.7, it defines a left and right A-linear coassociative map $\Delta : A \to A \otimes A$. By equations (3)(a) and (3)(b), f is a counit for Δ.

(4) $\Rightarrow$ (3) Choose $x_i, y_i \in A$, $1 \le i \le n$, with $\Delta(1) = \sum_{i=1}^n x_i \otimes y_i$. Then (3) follows using (4)(b) and that f is a counit. □

Let A be an algebra and a coalgebra. By Proposition 4.4.8, A is a Frobenius algebra if $\Delta : A \to A \otimes A$ is a map of (A, A)-bimodules. This last condition is equivalent to the commutativity of two diagrams. Note that condition (4) in Proposition 4.4.8 implies that A is finite-dimensional. Hence Frobenius algebras can be defined in monoidal categories.

Definition 4.4.9. Let $\mathcal{C}$ be a strict monoidal category. A **Frobenius algebra in** $\mathcal{C}$ is a quintuple $(A, \mu.\eta, \Delta, \varepsilon)$, where A is an object in $\mathcal{C}$, (A, μ, η) is an algebra and (A, Δ, ε) is a coalgebra in $\mathcal{C}$ such that



Recall that H is a Hopf algebra with bijective antipode. The next theorem says that a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$ is a Frobenius algebra, that is, a Frobenius algebra in the category of vector spaces. In general it is not a Frobenius algebra in ${}^H_H\mathcal{YD}$, since the Frobenius element f is not a morphism of Yetter-Drinfeld modules (see Example 4.4.15). The Hopf algebra H acts on f by a character which in general is not trivial.

We recall some notation from Section 3.5. Let R be a Hopf algebra in $\mathcal{C} = {}^H_H\mathcal{YD}$.

Let $V \in \mathcal{C}$ be finite-dimensional. By Lemma 4.2.2, $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$ is a left dual of V, where V^* as an object in $\mathcal{C}$ is defined in Lemma 4.2.2 with evaluation and coevaluation maps as for vector spaces.

Let $(V, \delta) \in \mathcal{C}^R$, and $(V, \lambda) \in {}_R\mathcal{C}$. Then $(V^*, \lambda_r) \in \mathcal{C}_R$ and $(V^*, \delta_l) \in {}^R\mathcal{C}$ by Lemma 3.5.10. If we use the notation

$$\lambda(r \otimes v) = rv,\ \lambda_r(f \otimes r) = fr, \quad \delta(v) = v_{[0]} \otimes v_{[1]},\ \delta_l(f) = f_{[-1]} \otimes f_{[0]}$$

for all $r \in R$, $f \in V^*$, $v \in V$, then

$$f_{[-1]}f_{[0]}(v) = f(v_{[0]})v_{[1]}, \quad fr(v) = f(rv).$$

In this notation, the left R-module structure λ_{r+} is defined by

$$\lambda_{r+} : R \otimes V^* \to V^*,\ r \otimes f \mapsto (r_{(-1)} \cdot f)\mathcal{S}_R(r_{(0)}).$$

If (V, λ, δ) is a Hopf module in ${}_R\mathcal{C}^R$, then by Theorem 3.5.14, $(V^*, \lambda_{r+}, \delta_l)$ is a Hopf module in ${}^R_R\mathcal{C}$.

Integrals in the dual algebra R^* of the coalgebra R are defined with respect to the augmentation $R^* \to \Bbbk$, $f \mapsto f(1)$. Note that R^* has two algebra structures. The dual vector space R^* is an algebra by the dual algebra structure of the coalgebra R and by the algebra structure of the dual braided Hopf algebra. For clarity we denote the dual braided Hopf algebra by $R^{*\mathrm{br}}$.

LEMMA 4.4.10. *For any finite-dimensional Hopf algebra R in ${}^H_H\mathcal{YD}$, the algebra structure of $(R^{\mathrm{copop}})^{*\mathrm{br}}$ is $R^{*\mathrm{op}}$, where R^* is the dual algebra of the coalgebra R.*

PROOF. The algebra structure of $(R^{\mathrm{copop}})^{*\mathrm{br}}$ is defined as the composition

$$R^* \otimes R^* \xrightarrow{\varphi_{R,R}} (R \otimes R)^* \xrightarrow{(\overline{c}_{R,R})^*} (R \otimes R)^* \xrightarrow{\Delta_R^*} R^*.$$

Let can be the isomorphism

$$\mathrm{can} : R^* \otimes R^* \to (R \otimes R)^*, \quad f \otimes g \mapsto (x \otimes y \mapsto f(x)g(y)),$$

and $\tau : R \otimes R \to R \otimes R$ the flip map. By (4.2.7),

$$\varphi_{R,R} = \big(R^* \otimes R^* \xrightarrow{\mathrm{can}} (R \otimes R)^* \xrightarrow{\tau^*} (R \otimes R)^* \xrightarrow{(c_{R,R})^*} (R \otimes R)^*\big).$$

Hence the multiplication of $(R^{\mathrm{copop}})^{*\mathrm{br}}$ is

$$R^* \otimes R^* \xrightarrow{\mathrm{can}} (R \otimes R)^* \xrightarrow{\tau} (R \otimes R)^* \xrightarrow{\Delta_R^*} R^*.$$

□

THEOREM 4.4.11. *Let R be a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$.*

(1) *The antipode of R is bijective.*
(2) *Both the algebra R and the dual algebra R^* of the coalgebra R are Frobenius algebras. Non-zero elements in $I_r(R^*)$ are Frobenius elements of R.*

PROOF. (a) Multiplication and comultiplication define R as a Hopf module in ${}_R\mathcal{C}^R$. By Theorem 3.5.12, R^* is a Hopf module in ${}^R_R\mathcal{C}$. Hence by Theorem 3.5.14, the multiplication map

$$R \otimes {}^{\mathrm{co}\,R}R^* \to R^*$$

is bijective. Thus ${}^{\mathrm{co}\,R}R^*$ is a one-dimensional object in ${}^H_H\mathcal{YD}$. Let $0 \neq \lambda \in {}^{\mathrm{co}\,R}R^*$ and let χ be the character of H given by $h \cdot \lambda = \chi(h)\lambda$ for all $h \in H$. If the left R-module structure on R^* is denoted by $R \otimes R^* \to R^*$, $r \otimes f \mapsto r \circ f$, then for all $r, x \in R$,

$$(r \circ \lambda)(x) = (r_{(-1)} \cdot \lambda)(\mathcal{S}_R(r_{(0)})x) = \lambda(\mathcal{S}_R(\chi(r_{(-1)})r_{(0)})x).$$

Hence the composition

$$R \xrightarrow{\varphi} R \xrightarrow{\mathcal{S}_R} R \xrightarrow{F} R^*$$

is bijective, where $\varphi(r) = \chi(r_{(-1)})r_{(0)}$, $F(r) = \lambda r$ for all $r \in R$. Therefore $\mathcal{S}_R$ is bijective. Moreover, R is a Frobenius algebra with Frobenius element λ. By definition of the left R-comodule structure of R^*,

$$^{\mathrm{co}\,R}R^* = \{\lambda \in R^* \mid \lambda(x^{(1)})x^{(2)} = \lambda(x)1 \text{ for all } x \in R\},$$

where we write $\Delta(x) = x^{(1)} \otimes x^{(2)}$ for all $x \in R$. Hence $^{\mathrm{co}\,R}R^* = I_r(R^*)$ by Lemma 4.4.5.

(b) To prove the remaining claim that the dual algebra of the coalgebra R is Frobenius, we apply (a) to $(R^{\mathrm{copop}})^{*\mathrm{br}}$. Hence $(R^{\mathrm{copop}})^{*\mathrm{br}}$ is a Frobenius algebra, and the dual algebra R^* is Frobenius by Lemmas 4.4.10 and 4.4.1. □

Let us say that a one-dimensional Yetter-Drinfeld module $\Bbbk x \in {}^H_H\mathcal{YD}$ is given by (g, χ) with $g \in G(H)$, $\chi \in \mathrm{Alg}(H, \Bbbk)$, if action and coaction of H have the form

$$h \cdot x = \chi(h)x, \quad \delta(x) = g \otimes x.$$

Corollary 4.4.12. *Let R be a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$.*

(1) *$I_l(R), I_R(R) \subseteq R$ and $I_l(R^*), I_r(R^*) \subseteq R^*$ are one-dimensional subobjects in ${}^H_H\mathcal{YD}$.*
(2) *$\mathcal{S}_R(I_l(R)) = I_r(R)$.*
(3) *There are $g \in G(H)$ and $\chi \in \mathrm{Alg}(H, \Bbbk)$ such that the Yetter-Drinfeld structures of $I_r(R)$ and $I_l(R)$ are given by (g, χ), and the Yetter-Drinfeld structures of $I_l(R^*)$ and $I_r(R^*)$ are given by (g^{-1}, χ^{-1}).*

Proof. By Theorem 4.4.11, $I_l((R^{\mathrm{copop}})^{*\mathrm{br}}) = I_r(R^*)$ is a one-dimensional Yetter-Drinfeld module. By the self-duality of finite-dimensional Hopf algebras in ${}^H_H\mathcal{YD}$, $I_l(R)$ is an object in ${}^H_H\mathcal{YD}$. Let Γ be a basis of $I_l(R)$, and χ a character of H with $h \cdot \Gamma = \chi(h)\Gamma$ for all $h \in H$. Then for all $x \in R$,

$$\varepsilon(x)\mathcal{S}_R(\Gamma) = \mathcal{S}_R(x\Gamma) = \mathcal{S}_R(x_{(-1)} \cdot \Gamma)\mathcal{S}_R(x_{(0)}) = \mathcal{S}_R(\Gamma)\mathcal{S}_R(\chi(x_{(-1)})x_{(0)}).$$

Hence $\mathcal{S}_R(\Gamma)$ is a right integral, since $\varepsilon(\mathcal{S}_R(\chi(x_{(-1)})x_{(0)})) = \varepsilon(x)$. We have shown that $\mathcal{S}_R$ induces an isomorphism $I_l(R) \cong I_r(R)$ of Yetter-Drinfeld modules. Then also $I_l((R^{\mathrm{copop}})^{*\mathrm{br}}) = I_r(R^*)$ and $I_r((R^{\mathrm{copop}})^{*\mathrm{br}}) = I_l(R^*)$ are isomorphic objects in ${}^H_H\mathcal{YD}$.

Let the Yetter-Drinfeld modules $I_l(R), I_r(R)$ be given by (g, χ), and $I_l(R^*)$, $I_r(R^*)$ by (g', χ'). If $0 \neq \Lambda \in I_r(R)$, $0 \neq \lambda \in I_r(R^*)$, then for all $h \in H$,

$$\chi'(h)\lambda(\Lambda) = (h \cdot \lambda)(\Lambda) = (\chi\mathcal{S}_H)(h)\lambda(\Lambda),$$
$$g'\lambda(\Lambda) = \lambda_{(-1)}\lambda_{(0)}(\Lambda) = \mathcal{S}_H^{-1}(g)\lambda(\Lambda),$$

and $\chi' = \chi^{-1}$, $g' = g^{-1}$, since $\lambda(\Lambda) \neq 0$ by Lemma 4.4.6. □

We apply the previous theorem to a special situation. The assumptions of the next theorem in particular hold for any finite-dimensional Nichols algebra in ${}^H_H\mathcal{YD}$.

Theorem 4.4.13. *Let $R = \bigoplus_{n \geq 0} R(n)$ be a finite-dimensional $\mathbb{N}_0$-graded connected Hopf algebra in ${}^H_H\mathcal{YD}$. Let $N \geq 0$ be the largest $n \geq 0$ with $R(n) \neq 0$. Then $R(N)$ is one-dimensional. Let $0 \neq \Lambda \in R(N)$, and define $\lambda : R \to \Bbbk$ by $\mathrm{pr}_N(r) = \lambda(r)\Lambda$ for all $r \in R$.*

(1) *Let $x_1, \dots, x_t$ be a basis of $R(1)$, and assume that R is generated as an algebra by $R(1)$, that is, R is pre-Nichols. Let $x_{i_1} \cdots x_{i_l}$ be a non-zero monomial in $x_1, \dots, x_t$ of maximal length. Then $l = N$, $x_{i_1} \cdots x_{i_l}$ is a basis of $R(N)$, and $R(n) \neq 0$ for all $0 \leq n \leq N$.*

(2) *R is a local algebra with maximal ideal $R^+ = \bigoplus_{i=1}^N R(i)$.*
(3) *Λ is a basis of $I_r(R) = I_l(R)$, λ is a basis of $I_r(R^*) = I_l(R^*)$, and R is a Frobenius algebra with Frobenius element λ.*
(4) *Let $0 \leq n \leq N$. The map $R(n) \times R(N-n) \to \Bbbk$, $(x, y) \mapsto \lambda(xy)$, is a non-degenerate bilinear form, and $\dim R(n) = \dim R(N-n)$.*

PROOF. We may assume that $N \geq 1$. Let $1 \leq n \leq N$, and $x \in R(n)$. Then

$$x\Lambda = 0 = \Lambda x = \varepsilon(x)\Lambda,$$

since R is an $\mathbb{N}_0$-graded algebra, and $R(N+n) = 0$. Thus Λ is a non-zero left and right integral of R. Hence Λ is a basis of $R(N)$, since R is a Frobenius algebra by Theorem 4.4.11, and its space of left or right integrals is one-dimensional by Lemma 4.4.6. Since R is an $\mathbb{N}_0$-graded coalgebra, by Lemma 1.3.6,

$$\Delta(x) \in 1 \otimes x + x \otimes 1 + \bigoplus_{i=1}^{n-1} R(i) \otimes R(n-i).$$

Hence

$$\lambda(x^{(1)})x^{(2)} = \lambda(x)1 = x^{(1)}\lambda(x^{(2)}),$$

and λ is a non-zero left and right integral of R^*. Again by Theorem 4.4.11, $I_r(R^*)$ and $I_l(R^*)$ are both one-dimensional with basis λ. We have proved (3), and (1) is now obvious. (2) holds for any finite-dimensional $\mathbb{N}_0$-graded algebra with one-dimensional degree 0 part, since R^+ is nilpotent.

By Theorem 4.4.11, λ is a Frobenius element of R. Hence the multiplication maps $R \to R^*$, $x \mapsto \lambda x$, and $R \to R^*$, $x \mapsto x\lambda$, are bijective. They induce injections $R(n) \to R(N-n)^*$, $x \mapsto \lambda x$, and $R(N-n) \to R(n)^*$, $y \mapsto y\lambda$, which proves (4). □

COROLLARY 4.4.14. *Let R and S be finite-dimensional $\mathbb{N}_0$-graded connected Hopf algebras in ${}^H_H\mathcal{YD}$. Let $\pi : R \to S$ be a surjective $\mathbb{N}_0$-graded algebra homomorphism, and assume that $\pi(R(N)) \neq 0$, where $N \geq 0$ is the largest $n \geq 0$ with $R(n) \neq 0$. Then π is bijective.*

PROOF. Since π is surjective and $\mathbb{N}_0$-graded, the top-degree of S is N. By Theorem 4.4.13, $R(N)$ and $S(N)$ are one-dimensional. Let Λ_R be a basis of $R(N)$. Then $\Lambda_S = \pi(\Lambda_R)$ is a basis of $S(N)$. We denote the integrals of R^* and S^* defined by Λ_R and Λ_S in Theorem 4.4.13 by λ_R and λ_S. Let $F_R : R \to R^*$, $r \mapsto \lambda_R r$, and $F_S : S \to S^*$, $s \mapsto \lambda_S s$ be the induced isomorphisms. Since $\lambda_R = \lambda_S \pi$, we obtain that

$$F_R = (R \xrightarrow{\pi} S \xrightarrow{F_S} S^* \xrightarrow{\pi^*} R^*).$$

Hence π is bijective. □

EXAMPLE 4.4.15. Let $m \geq 2$, and q a primitive m-th root of unity. Let $G = \langle g \rangle$ be the cyclic group of order m, and $T_{q,m}$ the Taft Hopf algebra in Examples 2.4.10 and 4.3.2 with projection $\pi : T_{q,m} \to \Bbbk G$ and $R = T_{q,m}^{\mathrm{co}\,\Bbbk G}$. Then $R = \Bbbk\langle x \mid x^m = 0\rangle$ is an m-dimensional Hopf algebra in ${}^G_G\mathcal{YD}$ with integral x^{m-1}. The G-action is defined by $g \cdot x = qx$. The linear map

$$\lambda : R \to \Bbbk, \quad \lambda(x^i) = \delta_{i,m-1}, \quad 0 \leq i \leq m-1,$$

is an integral in R^* and a Frobenius element. Note that λ is not a morphism in ${}^G_G\mathcal{YD}$, since $\lambda(g \cdot x^{m-1}) = q^{-1} \neq 1$.

Finally, motivated by Theorem 4.4.13, we discuss a more general class of Frobenius algebras.

DEFINITION 4.4.16. Let $R = \bigoplus_{n\geq 0} R(n)$ be a finite-dimensional $\mathbb{N}_0$-graded algebra with multiplication μ and unit η. A **PBW deformation of** R is an associative algebra (R, ν, η), such that for all $k, l \geq 0$,

$$(\nu - \mu)(R(k) \otimes R(l)) \subseteq \bigoplus_{i=0}^{k+l-1} R(i).$$

REMARK 4.4.17. Traditionally one defines a PBW deformation of a finite-dimensional $\mathbb{N}_0$-graded algebra R as an $\mathbb{N}_0$-filtered algebra A such that $\operatorname{gr} A \cong R$. It is easy to see that the two definitions are equivalent.

PROPOSITION 4.4.18. *Let $R = \bigoplus_{n\geq 0} R(n)$ be a finite-dimensional $\mathbb{N}_0$-graded algebra. Let $N \in \mathbb{N}$ and let $\lambda : R \to \Bbbk$ be a linear map with $\lambda(R(n)) = 0$ for all $n \neq N$. Assume that for any $n \geq 0$ the bilinear form*

$$R(n) \times R(N-n), \quad (x, y) \mapsto \lambda(xy),$$

is non-degenerate. Then $R(N) \neq 0$, $R(n) = 0$ for all $n > N$, and any PBW deformation of R is a Frobenius algebra with Frobenius element λ.

PROOF. For any $n > N$, $\lambda(xy) = 0$ for all $(x, y) \in R(n) \times R(N-n)$, since $R(N-n) = 0$. Thus $R(n) = 0$ by the non-degeneracy of the bilinear form. For a similar reason, $R(N) \neq 0$ since $R(0) \neq 0$.

Let (R, ν, η) be a PBW deformation of R and let $x \in R$ be non-zero. Let $n \leq N$ be such that $x \in \bigoplus_{i=0}^{n} R(i)$, $x \notin \bigoplus_{i=0}^{n-1} R(i)$. Then, by assumption, there exists $y \in R(N-n)$ with $\lambda(\nu(x \otimes y)) = \lambda(xy) \neq 0$. Therefore $\lambda x \neq 0$, that is, (R, ν, η) is a Frobenius algebra with Frobenius element λ. $\square$

COROLLARY 4.4.19. *Let $R = \bigoplus_{n\geq 0} R(n)$ be a finite-dimensional $\mathbb{N}_0$-graded connected Hopf algebra in ${}^H_H\mathcal{YD}$. Then any PBW deformation of R is a Frobenius algebra.*

PROOF. This follows from Theorem 4.4.13 and Proposition 4.4.18. $\square$

EXAMPLE 4.4.20. A standard example of a non-trivial PBW deformation is the Clifford algebra

$$\mathrm{Cl}(V, q) = T(V)/(v^2 - q(v) \mid v \in V)$$

of a quadratic form q on a finite-dimensional vector space V. Indeed, one can show that $\operatorname{gr}(\mathrm{Cl}(V, q))$ is isomorphic to the exterior algebra of V. Thus $\mathrm{Cl}(V, q)$ is a Frobenius algebra by Corollary 4.4.19.

4.5. Induction and restriction functors for Yetter-Drinfeld modules

In the following Propositions 4.5.1, 4.5.2 and Corollaries 4.5.3, 4.5.5 we assume that K, H are Hopf algebras with bijective antipodes, and $\varphi : K \to H$ is a map of Hopf algebras.

For Yetter-Drinfeld modules $V \in {}^K_K\mathcal{YD}$ and $W \in {}^H_H\mathcal{YD}$, we define

$$\mathrm{Hom}_K^H(V, W) = \{f \mid f : V \to W \text{ left } K\text{-linear and left } H\text{-colinear}\},$$

where W is a K-module by $\lambda_W(\varphi \otimes \mathrm{id}_W)$ and V is an H-comodule by $(\varphi \otimes \mathrm{id})\delta_V$.

PROPOSITION 4.5.1. *Let H be the right K-module with right module structure $H \otimes K \to H$, $h \otimes k \mapsto h\varphi(k)$.*

(1) *Let $V \in {}^K_K\mathcal{YD}$. The induced module $H \otimes_K V$ is an object in ${}^H_H\mathcal{YD}$ with left action $\cdot$ and left coaction δ, where*

$$h \cdot (h' \otimes v) = hh' \otimes v, \quad \delta(h \otimes v) = h_{(1)}\varphi(v_{(-1)})\mathcal{S}(h_{(3)}) \otimes (h_{(2)} \otimes v_{(0)})$$

for all $h, h' \in H$, $v \in V$.

(2) *The induced module construction in (1) defines a functor*

$$\varphi_* : {}^K_K\mathcal{YD} \to {}^H_H\mathcal{YD}, \quad V \mapsto H \otimes_K V,$$

mapping morphisms $f : V \to V'$ onto $\mathrm{id}_H \otimes_K f$.

(3) *Let $V \in {}^K_K\mathcal{YD}$, $W \in {}^H_H\mathcal{YD}$. The maps*

$$\mathrm{Hom}^H_K(V, W) \xrightarrow{\Phi} \mathrm{Hom}_{{}^H_H\mathcal{YD}}(H \otimes_K V, W), \ f \mapsto (h \otimes v \mapsto hf(v)),$$

$$\mathrm{Hom}_{{}^H_H\mathcal{YD}}(H \otimes_K V, W) \xrightarrow{\Psi} \mathrm{Hom}^H_K(V, W), \ F \mapsto F(\eta \otimes \mathrm{id}_V),$$

are inverse bijections.

PROOF. (1) Clearly, $(V, (\varphi \otimes \mathrm{id})\delta_V, \mathrm{id}_V \otimes \eta_H)$ is an H-bicomodule. By Remark 3.7.10, $(H \otimes V, \mu, \mathrm{coad}) \in {}^H_H\mathcal{YD}$. The map $\delta : H \otimes_K V \to H \otimes (H \otimes_K V)$ is well-defined since

$$\begin{aligned}\delta(h\varphi(k) \otimes v) =& h_{(1)}\varphi(k_{(1)})\varphi(v_{(-1)})\mathcal{S}(\varphi(k_{(3)}))\mathcal{S}(h_{(3)}) \otimes (h_{(2)}\varphi(k_{(2)}) \otimes v_{(0)}) \\ =& h_{(1)}\varphi((kv)_{(-1)})\mathcal{S}(h_{(3)}) \otimes (h_{(2)} \otimes (kv)_{(0)}) \\ =& \delta(h \otimes kv)\end{aligned}$$

for all $h \in H$, $k \in K$, $v \in V$. Thus the Yetter-Drinfeld structure of $H \otimes V$ induces the claimed Yetter-Drinfeld structure of $H \otimes_K V$.

(2) Let $V, V' \in {}^K_K\mathcal{YD}$, and $f : V \to V'$ a morphism in ${}^K_K\mathcal{YD}$. Then the map $\mathrm{id}_H \otimes f : H \otimes_K V \to H \otimes_K V'$ is left H-linear. It is left H-colinear, since coad in the proof of (1) is left H-colinear.

(3) Let $f \in \mathrm{Hom}^H_K(V, W)$, and $F = \Phi(f)$. Then F is a well-defined left H-linear map, since f is K-linear. To see that F is H-colinear, let $h \in H$, $v \in V$. Then $\delta_W(f(v)) = \varphi(v_{(-1)}) \otimes f(v_{(0)})$, since f is H-colinear. Hence

$$\begin{aligned}\delta_W(F(h \otimes v)) = \delta_W(hf(v)) &= h_{(1)}\varphi(v_{(-1)})\mathcal{S}(h_{(3)}) \otimes h_{(2)}f(v_{(0)}) \\ &= (\mathrm{id} \otimes F)\delta(h \otimes v).\end{aligned}$$

The map $\eta \otimes \mathrm{id}_V : V \to H \otimes_K V$, $v \mapsto 1 \otimes v$, is K-linear and H-colinear. Hence Ψ is well-defined, and Φ and Ψ are inverse bijections. □

We note that the construction of $M(g, V)$ in Definition 1.4.15 is a special case of the induction functor in Proposition 4.5.1. Let G be a group, $g \in G$, and $\varphi : \Bbbk G^g \to \Bbbk G$ the Hopf algebra map induced by the inclusion of the centralizer G^g into G. Any left $\Bbbk G^g$-module V is an object in ${}^{G^g}_{G^g}\mathcal{YD}$ with coaction $\delta : V \to \Bbbk G^g \otimes V$, $v \mapsto g \otimes v$, and the given G^g-action. Then $\varphi_*(V) = M(g, V) \in {}^G_G\mathcal{YD}$.

The cotensor product, see Definition 2.2.9, defines a restriction functor.

PROPOSITION 4.5.2. *Let K be the right H-comodule with comodule structure $(\mathrm{id}_K \otimes \varphi)\Delta_K : K \to K \otimes H$.*

(1) *Let $W \in {}^H_H\mathcal{YD}$. The cotensor product $K\square_H W$ is a subobject in ${}^K_K\mathcal{YD}$ of $K \otimes W$ with K-action $\cdot$ and K-coaction δ, where*

$$x \cdot (y \otimes w) = x_{(1)} y \mathcal{S}(x_{(3)}) \otimes \varphi(x_{(2)})w, \quad \delta(x \otimes w) = x_{(1)} \otimes x_{(2)} \otimes w$$

for all $x, y \in K$, $w \in W$.

(2) *The cotensor product in (1) defines a functor*

$$\varphi^* : {}^H_H\mathcal{YD} \to {}^K_K\mathcal{YD}, \quad W \mapsto K\square_H W,$$

where morphisms $f : W \to W'$ are mapped onto $\mathrm{id}_K \square f$.

(3) *The maps*

$$\mathrm{Hom}^H_K(V, W) \xrightarrow{\Phi} \mathrm{Hom}_{{}^K_K\mathcal{YD}}(V, K\square_H W),\ f \mapsto (v \mapsto v_{(-1)} \otimes f(v_{(0)})),$$

$$\mathrm{Hom}_{{}^K_K\mathcal{YD}}(V, K\square_H W) \xrightarrow{\Psi} \mathrm{Hom}^H_K(V, W),\ F \mapsto (\varepsilon \otimes \mathrm{id}_W)F,$$

are inverse bijections.

Proof. (1) Consider W as a trivial right K-module and a left K-module via $\lambda_W(\varphi \otimes \mathrm{id}_W)$. By Remark 3.7.10, the triple $(K \otimes W, \mathrm{ad}, \Delta_K \otimes \mathrm{id}_W)$ is an object in ${}^K_K\mathcal{YD}$. Moreover, $H \otimes W$ is a left K-module via the action

$$k(h \otimes w) = \varphi(k_{(1)}) h \mathcal{S}(\varphi(k_{(3)})) \otimes \varphi(k_{(2)})w$$

for $k \in K$, $h \in H$, $w \in W$, and hence $(K \otimes H \otimes W, \mathrm{ad}, \Delta_K \otimes \mathrm{id}_{H\otimes W}) \in {}^K_K\mathcal{YD}$. The H-coaction $\delta_W : W \to H \otimes W$ of W is a K-bimodule map by the Yetter-Drinfeld condition for W, and hence $\mathrm{id} \otimes \delta_W : K \otimes W \to K \otimes H \otimes W$ is a morphism in ${}^K_K\mathcal{YD}$. Let

$$\delta' = (\mathrm{id} \otimes \varphi \otimes \mathrm{id}_W)(\Delta_K \otimes \mathrm{id}_W) : K \otimes W \to K \otimes H \otimes W.$$

Then δ' is left K-linear and left K-colinear by construction, and we conclude that $K\square_H W = \ker(\delta' - \mathrm{id} \otimes \delta_W)$ is a Yetter-Drinfeld submodule of $K \otimes W$.

(2) Let $f : W \to W'$ be a morphism in ${}^H_H\mathcal{YD}$. Then $\mathrm{id}_K \otimes f : K \otimes W \to K \otimes W'$ is a morphism in ${}^K_K\mathcal{YD}$. The following diagram commutes.

$$\begin{tikzcd}
K \otimes W \arrow[r, "\delta' - \mathrm{id}\otimes\delta_W"] \arrow[d, "\mathrm{id}_K\otimes f"'] & K \otimes H \otimes W \arrow[d, "\mathrm{id}_K\otimes\mathrm{id}_H\otimes f"] \\
K \otimes W' \arrow[r, "\delta' - \mathrm{id}\otimes\delta_{W'}"] & K \otimes H \otimes W'
\end{tikzcd}$$

Hence f induces a morphism $K\square_H f : K\square_H W \to K\square_H W'$ in ${}^K_K\mathcal{YD}$.

(3) Let $f \in \mathrm{Hom}^H_K(V, W)$, and $F = \Phi(f)$. Then $F(v) \in K\square_H W$ for all $v \in V$, since f is H-colinear. Hence F is a well-defined K-colinear map. To see that F is K-linear, let $x \in K$, $v \in V$. Then

$$F(xv) = x_{(1)} v_{(-1)} \mathcal{S}(x_{(3)}) \otimes f(x_{(2)} v_{(0)}) = x \cdot F(v),$$

since f is K-linear.

The map $\varepsilon \otimes \mathrm{id}_W : K\square_H W \to W$, $\sum_{i=1}^n x_i \otimes w_i \mapsto \sum_{i=1}^n \varepsilon(x_i) w_i$, is left K-linear and left H-colinear, where W is a left K-module by restriction via φ, and $K\square_H W$ is a left H-comodule by$(\varphi \otimes \mathrm{id})\delta_{K\square_H W}$. Hence Ψ is well-defined, and Φ and Ψ are inverse bijections. $\square$

Corollary 4.5.3. *The functor φ_* is left adjoint to φ^*.*

Proof. This follows from Propositions 4.5.1(3) and 4.5.2(3). $\square$

REMARK 4.5.4. Propositions 4.5.1(3) and 4.5.2(3) show that the forgetful functor ${}^H_H\mathcal{YD} \to {}^H\mathcal{M}$ has the left adjoint functor $V \mapsto H \otimes V$, and that the forgetful functor ${}^K_K\mathcal{YD} \to {}_K\mathcal{M}$ has the right adjoint functor $W \mapsto K \otimes W$.

We need the following special cases of the induction and restriction functors.

COROLLARY 4.5.5. (1) *Assume that φ is surjective. Let $V = (V, \lambda, \delta)$ be an object in ${}^K_K\mathcal{YD}$. Assume that $\lambda = \lambda'(\varphi \otimes \mathrm{id}_V)$, where V is a left H-module by $\lambda' : H \otimes V \to V$. Then*

$$\varphi_*(V) \cong (V, \lambda', (\varphi \otimes \mathrm{id}_V)\delta) \quad \text{in } {}^H_H\mathcal{YD},$$

and the Yetter-Drinfeld modules V and $(V, \lambda', (\varphi \otimes \mathrm{id}_V)\delta)$ have the same braiding map.

(2) *Assume that φ is injective. Let $V = (V, \lambda, \delta)$ be an object in ${}^H_H\mathcal{YD}$. Assume that $\delta = (\varphi \otimes \mathrm{id}_V)\delta'$, where V is a left K-comodule by $\delta' : V \to K \otimes V$. Then*

$$\varphi^*(V) \cong (V, \lambda(\varphi \otimes \mathrm{id}_V), \delta') \quad \text{in } {}^K_K\mathcal{YD},$$

and the Yetter-Drinfeld modules V and $(V, \lambda(\varphi \otimes \mathrm{id}_V), \delta')$ have the same braiding map.

PROOF. (1) The map $\varphi_*(V) = H \otimes_K V \to V$, $h \otimes v \mapsto \lambda'(h \otimes v)$, is an isomorphism in ${}^H_H\mathcal{YD}$, since φ is surjective. Its inverse is the map $V \to H \otimes_K V$, $v \mapsto 1 \otimes v$. The braiding of $(V, \lambda', (\varphi \otimes \mathrm{id}_V)\delta)$ is defined by

$$c(v \otimes w) = \varphi(v_{(-1)}) \cdot w \otimes v_{(0)} = c_{V,V}(v \otimes w)$$

for all $v, w \in V$.

(2) The map $K \Box_H V \to (V, \lambda(\varphi \otimes \mathrm{id}_V), \delta')$ induced by $\varepsilon \otimes \mathrm{id}_V$ is an isomorphism in ${}^K_K\mathcal{YD}$, since φ is injective. Its inverse is the map given by $v \mapsto v_{(-1)} \otimes v_{(0)}$, where $V \xrightarrow{\lambda'} K \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$ denotes the K-comodule structure. The braiding of $(V, \lambda(\varphi \otimes \mathrm{id}_V), \delta')$ is defined by

$$c(v \otimes w) = \varphi(v_{(-1)}) \cdot w \otimes v_{(0)} = c_{V,V}(v \otimes w)$$

for all $v, w \in V$. □

Let G be a group. The braided vector space $(V, c_{V,V})$ of a Yetter-Drinfeld module $V \in {}^G_G\mathcal{YD}$ does not determine the Yetter-Drinfeld module V nor the group G uniquely. We first want to decide when two Yetter-Drinfeld modules over groups have isomorphic braidings.

A left G-module V is called a **faithful** G-module, if the identity element is the only element $g \in G$ such that $g \cdot v = v$ for all $v \in V$. If V is a faithful G-module, we can identify G with a subgroup of $\mathrm{Aut}(V)$, and the action of G on V with the application of automorphisms to elements of V.

DEFINITION 4.5.6. Let G be a group and $V \in {}^G_G\mathcal{YD}$. Then V is called an **essential Yetter-Drinfeld module** over G if V is a faithful G-module, and the group G is generated by the elements $g \in G$ with $V_g \neq 0$.

COROLLARY 4.5.7. *Let G be a group and $V \in {}^G_G\mathcal{YD}$ with representation*

$$\rho : G \to \mathrm{Aut}(V), \quad g \mapsto (v \mapsto g \cdot v).$$

Let $G(V) = \rho(G_1)$, where $G_1 \subseteq G$ is the subgroup of G generated by all $g \in G$ with $V_g \neq 0$. Let $\widetilde{V} = V$ as a vector space with $G(V)$-action and $G(V)$-grading given by

$$\rho(g) \cdot v = \rho(g)(v), \quad \widetilde{V}_{\rho(g)} = \bigoplus_{h \in G_1, \rho(h) = \rho(g)} V_h$$

for all $g \in G_1$ and $v \in V$.

(1) *$\widetilde{V} \in {}^{G(V)}_{G(V)}\mathcal{YD}$ is an essential Yetter-Drinfeld module, and*

$$(V, c_{V,V}) = (\widetilde{V}, c_{\widetilde{V},\widetilde{V}}).$$

(2) *A direct sum decomposition $V = \bigoplus_{i \in I} V_i$ of V in ${}^G_G\mathcal{YD}$ is a direct sum decomposition $\widetilde{V} = \bigoplus_{i \in I} V_i$ of $\widetilde{V}$ in ${}^{G(V)}_{G(V)}\mathcal{YD}$.*

Proof. (1) The vector space V is a Yetter-Drinfeld module over G_1 by Corollary 4.5.5(2), where φ is the inclusion map $G_1 \subseteq G$. Then V is a Yetter-Drinfeld module over $\rho(G_1) = G(V)$ by Corollary 4.5.5(1), where φ is the surjective map $G_1 \to G(V)$, $g \mapsto \rho(g)$. Hence $\widetilde{V} \in {}^{G(V)}_{G(V)}\mathcal{YD}$ with the same braiding as V, and it is an essential Yetter-Drinfeld module by construction.

(2) is obvious from the definition of $\widetilde{V}$. □

If G, H are groups, and $\varphi : G \to H$ is an isomorphism of groups, we denote the induced category equivalence between the categories of Yetter-Drinfeld modules by

$$\mathcal{YD}(\varphi) : {}^G_G\mathcal{YD} \to {}^H_H\mathcal{YD}, \quad (V, \lambda, \delta) \mapsto (V, \lambda(\varphi^{-1} \otimes \mathrm{id}_V), (\varphi \otimes \mathrm{id}_V)\delta).$$

Note that $\varphi_* \cong \mathcal{YD}(\varphi)$ by Corollary 4.5.5.

In the next proposition we formulate a criterion to decide when Yetter-Drinfeld modules over groups have isomorphic braidings. Recall that braided vector spaces (V, c) and (W, d) are isomorphic, if there is a linear isomorphism $f : V \to W$ such that $d(f \otimes f) = (f \otimes f)c$. We then write $(V, c) \cong (W, d)$.

Proposition 4.5.8. *Let G, H be groups, and let $V \in {}^G_G\mathcal{YD}$ and $W \in {}^H_H\mathcal{YD}$. Define $\widetilde{V} \in {}^{G(V)}_{G(V)}\mathcal{YD}$ and $\widetilde{W} \in {}^{G(W)}_{G(W)}\mathcal{YD}$ as in Corollary 4.5.7. Then the following are equivalent:*

(1) *The braided vector spaces $(V, c_{V,V})$ and $(W, c_{W,W})$ are isomorphic.*
(2) *There is a group isomorphism $\varphi : G(V) \to G(W)$ such that*

$$\mathcal{YD}(\varphi)(\widetilde{V}) \cong \widetilde{W} \quad \text{in } {}^{G(W)}_{G(W)}\mathcal{YD}.$$

(3) *There is a linear isomorphism $f : V \to W$ such that*

$$\widetilde{W}_{fgf^{-1}} = f(\widetilde{V}_g) \quad \text{for all } g \in G(V).$$

Proof. By Corollary 4.5.7 we may assume that

$$G = G(V), V = \widetilde{V} \in {}^{G(V)}_{G(V)}\mathcal{YD}, \text{ and } H = G(W), W = \widetilde{W} \in {}^{G(W)}_{G(W)}\mathcal{YD}.$$

(1) ⇒ (2): By definition there is a linear isomorphism $f : V \to W$ with

$$(f \otimes f)c_{V,V} = c_{W,W}(f \otimes f).$$

We denote by $\Phi : \mathrm{Aut}(V) \to \mathrm{Aut}(W)$, $\Phi(g) = fgf^{-1}$ for all $g \in \mathrm{Aut}(V)$, the induced group isomorphism. Let $g \in G$ and $0 \neq v \in V_g$. Then there are elements

$h_i \in H$, $0 \neq w_i \in W_{h_i}$ for all $1 \leq i \leq n$, $n \geq 1$, with $f(v) = \sum_{i=1}^n w_i$ and $h_i \neq h_j$ for all $i \neq j$. Hence for all $v' \in V$,

$$\begin{aligned}(f \otimes f)c_{V,V}(v \otimes v') &= f(g \cdot v') \otimes f(v) = \sum_{i=1}^n f(g \cdot v') \otimes w_i \\ &= c_{W,W}(f \otimes f)(v \otimes v') = \sum_{i=1}^n c_{W,W}(w_i \otimes f(v')) = \sum_{i=1}^n h_i \cdot f(v') \otimes w_i.\end{aligned}$$

Hence $n = 1$, and $h_1 = \Phi(g)$. We conclude that $f(V_g) \subseteq W_{\Phi(g)}$. Let X be the set of all $g \in G$ with $V_g \neq 0$. It follows that

$$W = f(V) = \bigoplus_{g \in X} f(V_g) \subseteq \bigoplus_{g \in X} W_{\Phi(g)},$$

and therefore, $f(V_g) = W_{\Phi(g)}$ for all $g \in G$, and $W_{\Phi(g)} = 0$ for all $g \in G \setminus X$. Hence $\Phi(G) = H$, since by assumption G is generated by X, and H is generated by $\{h \in H \mid W_h \neq 0\}$.

This proves (2), since $\varphi : G \to H$, $g \mapsto \Phi(g)$, is an isomorphism of groups, and $f : \mathcal{YD}(\varphi)(V) \to W$ is an isomorphism of Yetter-Drinfeld modules over H.

(2) $\Rightarrow$ (3): Let $f : \mathcal{YD}(\varphi)(V) \to W$ be an isomorphism of Yetter-Drinfeld modules over H. Then for all $v \in \mathcal{YD}(\varphi)(V)$ and $g \in G$,

$$\varphi(g) \cdot v = g(v), \quad f(\varphi(g) \cdot v) = \varphi(g)(f(v)),$$

since f is an H-linear map. Hence $\varphi(g) = fgf^{-1}$. Since f is an H-graded map, and $\mathcal{YD}(\varphi)(V)_{\varphi(g)} = V_g$, (3) follows.

(3) $\Rightarrow$ (1): Let $g \in G, v \in V_g$ and $v' \in V$. Then by (3), $f(v) \in W_{fgf^{-1}}$, and hence

$$\begin{aligned}c_{W,W}(f \otimes f)(v \otimes v') &= c_{W,W}(f(v) \otimes f(v')) = fgf^{-1}(f(v')) \otimes f(v) \\ &= f(g(v')) \otimes f(v) = (f \otimes f)c_{V,V}(v \otimes v').\end{aligned}$$

This proves the Proposition. □

We now consider Yetter-Drinfeld modules over groups with diagonal braidings. It is clear from the definition that finite direct sums of one-dimensional Yetter-Drinfeld modules have diagonal braiding.

PROPOSITION 4.5.9. *Let $n \in \mathbb{N}_0$ and let (V, c) and (W, d) be n-dimensional braided vector spaces. Let $x_1, \dots, x_n$ be a basis of V, $y_1, \dots, y_n$ a basis of W and $q_{ij}, p_{ij} \in \Bbbk$ for all $1 \leq i, j \leq n$ such that*

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad d(y_i \otimes y_j) = p_{ij} y_j \otimes y_i$$

for all $1 \leq i, j \leq n$. Then the following are equivalent:

(1) *The braided vector spaces (V, c) and (W, d) are isomorphic.*
(2) *There is a permutation $\sigma \in \mathbb{S}_n$ such that*

$$q_{ij} = p_{\sigma(i)\sigma(j)} \quad \text{for all } 1 \leq i, j \leq n.$$

PROOF. Let G be a free abelian group with basis $(g_i)_{1 \leq i \leq n}$. Define characters χ_i, η_i of G by $\chi_j(g_i) = q_{ij}$, $\eta_j(g_i) = p_{ij}$ for all $1 \leq i, j \leq n$. Let $V \in {}^G_G\mathcal{YD}$ and $W \in {}^G_G\mathcal{YD}$ with $x_i \in V_{g_i}^{\chi_i}$, $y_i \in W_{g_i}^{\eta_i}$ for all $1 \leq i \leq n$. Then $c = c_{V,V}$, $d = c_{W,W}$.

By definition, $\widetilde{V} \in {}^{G(V)}_{G(V)}\mathcal{YD}$ is the direct sum of the one-dimensional Yetter-Drinfeld modules $\Bbbk x_i$, $1 \leq i \leq n$, over $G(V)$, and $\widetilde{W} \in {}^{G(W)}_{G(W)}\mathcal{YD}$ is the direct sum of the one-dimensional Yetter-Drinfeld modules $\Bbbk y_i$, $1 \leq i \leq n$, over $G(W)$.

Clearly, (2) implies (1). Assume now (1). By Proposition 4.5.8,

$$\mathcal{YD}(\varphi)(\widetilde{V}) \cong \widetilde{W} \quad \text{in } {}^{G(W)}_{G(W)}\mathcal{YD},$$

where $\varphi : G(V) \to G(W)$ is an isomorphism of groups. Then $\mathcal{YD}(\varphi)(\widetilde{V})$ is the direct sum of the one-dimensional Yetter-Drinfeld modules $\Bbbk x_i$ over $G(W)$, $1 \leq i \leq n$. By Krull-Schmidt there is a permutation $\sigma \in \mathbb{S}_n$ such that $\Bbbk x_i \cong \Bbbk y_{\sigma(i)}$ as Yetter-Drinfeld modules over $G(W)$ for all $1 \leq i \leq n$. This proves (2), since the braidings of V and $\mathcal{YD}(\varphi)(\widetilde{V})$ and of W and $\widetilde{W}$ coincide. □

COROLLARY 4.5.10. *Let G be a group and $V \in {}^G_G\mathcal{YD}^{\mathrm{fd}}$ with representation $\rho : G \to \mathrm{Aut}(V)$. Let $G_1 \subseteq G$ be the subgroup generated by all $g \in G$ with $V_g \neq 0$. Then the following are equivalent.*

(1) *The braided vector space $(V, c_{V,V})$ is of diagonal type.*
(2) *V is a direct sum of one-dimensional G_1-modules.*

Assume that $\rho(G_1)$ is finite, $\Bbbk$ is algebraically closed and $\mathrm{char}(\Bbbk)$ *does not divide the order of $\rho(G_1)$. Then* (1) *and* (2) *are equivalent to*

(3) *$\rho(G_1)$ is abelian.*

PROOF. Assume (1). We prove (2). By assumption, there is a basis $x_1, \dots, x_n$ of V and scalars $q_{ij} \in \Bbbk^\times$ for $1 \leq i, j \leq n$ with

$$c_{V,V}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$$

for all $1 \leq i, j \leq n$. Let H be a free abelian group with basis $(g_i)_{1 \leq i \leq n}$ and characters $(\chi_i)_{1 \leq i \leq n}$ of H with $\chi_j(g_i) = q_{ij}$ for all i, j. Let $W \in {}^H_H\mathcal{YD}$ with basis $(y_i)_{1 \leq i \leq n}$ and $y_i \in W^{\chi_i}_{g_i}$ for all i. Then $(V, c_{V,V}) \cong (W, c_{W,W})$. Hence by Proposition 4.5.8(2), $\widetilde{V}$ is a direct sum of one-dimensional Yetter-Drinfeld modules in ${}^{G(V)}_{G(V)}\mathcal{YD}$. This implies (2), since $G(V) = \rho(G_1)$.

Assume (2). Then $\rho(G_1)$ is abelian, hence (3) holds. Moreover, by Lemma 1.4.5, $\widetilde{V}$ is a direct sum of one-dimensional Yetter-Drinfeld modules in ${}^{G(V)}_{G(V)}\mathcal{YD}$. Thus $(V, c_{V,V}) = (\widetilde{V}, c_{\widetilde{V},\widetilde{V}})$ is of diagonal type, which proves (1).

Finally, (3) implies (1) by Proposition 1.4.6. □

COROLLARY 4.5.11. *Let G be a group and $V \in {}^G_G\mathcal{YD}^{\mathrm{fd}}$. Let $G_1 \subseteq G$ be the subgroup generated by all $g \in G$ with $V_g \neq 0$. Assume that $(V, c_{V,V})$ is of diagonal type, and that V is a faithful G_1-module. Then G_1 is abelian.*

PROOF. Let $\rho : G \to \mathrm{Aut}(V)$ be the representation of the G-module structure of V. By Corollary 4.5.10, $\rho(G_1)$ is abelian. Hence G_1 is abelian, since V is a faithful G_1-module. □

The assumption on the faithfulness of V in Corollary 4.5.11 can not be dropped, as Example 4.5.12 shows.

EXAMPLE 4.5.12. The dihedral group D_4 of order 8 is generated by two elements r, s with relations $r^4 = 1$, $s^2 = 1$, $sr = r^3 s$. Let $t_i = r^{i-1}s$ for all $i \in \mathbb{Z}$. Then for all $i, j \in \mathbb{Z}$, $t_i = t_j$ if and only if $i \equiv j \pmod 4$, and $t_i t_j t_i = t_{2i-j}$, $t_i^2 = 1$. The group D_4 is generated by t_1 and t_4, and the set $\{t_i \mid 1 \leq i \leq 4\}$ is stable under the adjoint action of D_4.

Let $\varepsilon_{r^i} = 1$ and $\varepsilon_{t_i} = -1$ for all $1 \leq i \leq 4$. Thus $D_4 \to \{1, -1\}$, $g \mapsto \varepsilon_g$, is a group homomorphism. We define a Yetter-Drinfeld module $V \in {}^{D_4}_{D_4}\mathcal{YD}$ with basis x_{t_i}, $1 \leq i \leq 4$, where the D_4-action and coaction is defined by

$$g \cdot x_{t_i} = \varepsilon_g x_{g t_i g^{-1}}, \quad \delta(x_{t_i}) = t_i \otimes x_{t_i}$$

for all $g \in D_4$, $1 \leq i \leq 4$. We set $x_i = x_{t_i}$ for all $i \in \mathbb{Z}$. Note that

$$t_i \cdot x_j = -x_{2i-j}, \quad r \cdot x_j = x_{j+2} \quad \text{for all } i, j \in \mathbb{Z}.$$

Let $\rho : D_4 \to \mathrm{Aut}(V)$ be the representation of the action of D_4 on V. Then $\rho(t_1) = \rho(t_3)$, $\rho(t_2) = \rho(t_4)$, and the automorphisms $\rho(t_1)$ and $\rho(t_4)$ commute. Hence $\rho(D_4)$ is abelian. Assume that the characteristic of $\Bbbk$ is not two, and let

$$y_1 = x_1 + x_3, \quad y_2 = x_2 - x_4, \quad y_3 = x_1 - x_3, \quad y_4 = x_2 + x_4.$$

Then $\widetilde{V} = \bigoplus_{i=1}^4 \Bbbk y_i$ is a direct sum of one-dimensional Yetter-Drinfeld modules over $\rho(D_4)$. Thus $(V, c_{V,V})$ is of diagonal type. Moreover, $\widetilde{V} = V_1 \oplus V_2$, where $V_1 = \Bbbk y_1 \oplus \Bbbk y_2$, $V_2 = \Bbbk y_3 \oplus \Bbbk y_4$, and $c^2 | V_2 \otimes V_1 = \mathrm{id}_{V_2 \otimes V_1}$. It follows from Proposition 1.10.12 that $\mathcal{B}(\widetilde{V})$ is isomorphic to $\mathcal{B}(V_1) \otimes \mathcal{B}(V_2)$. The braidings of V_1 and V_2 are of Cartan type with Cartan matrix A_2, see Definition 8.2.2. Then by Theorem 16.3.17, the Nichols algebras of V_i, $1 \leq i \leq 2$, have dimension 8, and $\dim \mathcal{B}(V) = 64$.

4.6. Notes

4.1. The Drinfeld center was introduced around 1990 independently by Drinfeld, Majid [**Maj91**] and Joyal and Street [**JS91**]. Theorem 4.1.3 is due to Drinfeld, see [**Maj94**], Example 1.3, where a proof is given from the point of view of Tannaka-Krein reconstruction theory.

Let H be a Hopf algebra in a braided monoidal category $(\mathcal{C}, c)$. The functor in Theorem 4.1.3 identifies ${}^H_H\mathcal{YD}(\mathcal{C})$ with a subcategory of the centre $\mathcal{Z}_l({}_H\mathcal{C})$ which is described in [**Bes97**], Proposition 3.6.1. The Hopf algebra H defines a Hopf algebra $\mathbb{H} = (H, c_{H,-})$ in $\mathcal{Z}_l(\mathcal{C})$. By [**BV13**], Remark 2.15, ${}^{\mathbb{H}}_{\mathbb{H}}\mathcal{YD}(\mathcal{Z}_l(\mathcal{C})) \cong \mathcal{Z}_l({}_H\mathcal{C})$ as braided categories.

Theorem 4.1.11 was shown in [**MO99**], Theorem 2.7, by direct computations in the category of two-sided Hopf modules which is equivalent to ${}^H_H\mathcal{YD}$.

The notion of a rigid braided vector space was introduced by Lyubashenko. Let (V, c) be a finite-dimensional braided vector space which is rigid. Following ideas of Lyubashenko, it was shown by Schauenburg (see the exposition by Takeuchi [**Tak00**]) that there is a coquasitriangular Hopf algebra (H, σ) and a right H-comodule structure on V such that c is the braiding arising from σ. Then V has the structure of a Yetter-Drinfeld module in $\mathcal{YD}^H_H$ such that $c = c_{V,V}$.

4.4. An early proof of Theorem 4.4.11(2) was given in [**FMS97**] using β-Frobenius extensions.

Let R be a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$. If $A \subseteq R$ is an H-stable subalgebra with $\Delta_R(A) \subseteq A \otimes R$, in particular, a right coideal subalgebra in ${}^H_H\mathcal{YD}$, then by [**ST16**], Theorem 5.3, A is a Frobenius algebra, R is free as a left and as a right A-module, and A is a direct summand in R as a left and as a right A-module.

This is a braided version of a fundamental result of Skryabin [**Skr07**]; its proof is based on [**Skr07**] and [**SVO06**].

Freeness of R over Hopf subalgebras in ${}^H_H\mathcal{YD}$ was shown earlier by Takeuchi, see [**Tak00**], and in [**Sch01**] extending the arguments in [**NZ89**].

Corollary 4.4.14 is taken from [**AGn03**], Theorem 6.4.

4.5. Example 4.5.12 is Example 6.5 in [**MS00**]. The Nichols algebra there was computed in a different way; the elements y_1, y_2, y_3, y_4 were proposed by Graña.

CHAPTER 5

Gradings and filtrations

Several objects in this book like algebras, coalgebras and Yetter-Drinfeld modules, admit a natural filtration or a grading by a monoid more general than the natural numbers. In particular, Nichols systems in Chapter 13 will be graded by $\mathbb{N}_0^\theta$ for some $\theta \geq 1$. In this chapter we discuss filtrations and gradings of this type.

Assuming standard results on the Jacobson radical of algebras we study the coradical filtration, and its associated graded coalgebra. We prove a weak version of the Theorem of Taft and Wilson which allows us to give a rather detailed description of the first part A_1 of the coradical filtration of a pointed Hopf algebra A with abelian group $G(A)$. This description is useful to determine the structure of A when $\operatorname{gr} A$ is given.

5.1. Gradings

Let Γ be a set.

Recall the definition of the category $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$ of Γ-graded vector spaces in Section 1.1. By Proposition 1.1.17, $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$ can be identified with the category of left (or right) comodules over $\Bbbk\Gamma$.

Let V be a Γ-graded vector space. A **graded subspace** $U \subseteq V$ is a subspace and a graded vector space $U = \bigoplus_{\alpha\in\Gamma} U(\alpha)$ satisfying the following equivalent conditions.

(1) $U(\alpha) = U \cap V(\alpha)$ for all $\alpha \in \Gamma$.
(2) $U(\alpha) \subseteq V(\alpha)$ for all $\alpha \in \Gamma$.

The intersection of a family of graded subspaces of V is a graded subspace. The category $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$ is abelian. Let X, Y be objects in $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$, $X' \subseteq X$ and $Y' \subseteq Y$ graded subobjects, and $f : X \to Y$ a graded map. For all $\gamma \in \Gamma$ let $f_\gamma : X(\gamma) \to Y(\gamma)$ be the restriction of f. Then

$$\ker(f) = \bigoplus_{\gamma\in\Gamma} \ker(f_\gamma),\ \operatorname{im}(f) = \bigoplus_{\gamma\in\Gamma} \operatorname{im}(f_\gamma),\ X/X' = \bigoplus_{\gamma\in\Gamma} X(\gamma)/X'(\gamma),$$
$$f^{-1}(Y') = \ker(X \xrightarrow{f} Y \to Y/Y')$$

are all graded.

Assume that Γ is a monoid with unit element e.

By Definition 1.2.7, $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$ is a monoidal category with diagonal grading on the tensor product $V \otimes W$ of Γ-graded vector spaces V, W. By Remark 1.2.8, the monoidal categories $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$ and ${}^{\Bbbk\Gamma}\mathcal{M}_{\Bbbk}$ can be identified.

A Γ-**graded algebra** A is an algebra in $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$, that is, A is an algebra (with unit $1_A = 1$), $A = \bigoplus_{\alpha\in\Gamma} A(\alpha)$ is Γ-graded such that

$$A(\beta)A(\gamma) \subseteq A(\beta\gamma) \quad \text{for all } \beta,\gamma \in \Gamma, \tag{5.1.1}$$

$$1_A \in A(e), \tag{5.1.2}$$

that is, the multiplication and unit maps are graded.

A Γ-**graded coalgebra** C is a coalgebra in the monoidal category $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$. Thus $C = \bigoplus_{\alpha\in\Gamma} C(\alpha)$ is a Γ-graded vector space and a coalgebra with comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to \Bbbk$ such that Δ and ε are graded, that is,

$$\Delta(C(\alpha)) \subseteq \bigoplus_{\beta\gamma=\alpha} C(\beta) \otimes C(\gamma) \quad \text{for all } \alpha \in \Gamma, \tag{5.1.3}$$

$$\varepsilon(C(\alpha)) = 0 \quad \text{for all } \alpha \in \Gamma \setminus \{e\}. \tag{5.1.4}$$

Note that (5.1.2) and (5.1.4) are redundant if the monoid Γ is **cancellative**, that is, if $\alpha,\beta,\gamma \in M$ with $\alpha\gamma = \beta\gamma$ or $\gamma\alpha = \gamma\beta$ implies that $\alpha = \beta$.

LEMMA 5.1.1. *Assume that Γ is cancellative.*

(1) *Let $C = \bigoplus_{\alpha\in\Gamma} C(\alpha)$ be a graded vector space and a coalgebra such that $\Delta(C(\alpha)) \subseteq \bigoplus_{\beta\gamma=\alpha} C(\beta) \otimes C(\gamma)$ for all $\alpha \in \Gamma$. Then for all $\alpha \neq e$, $\varepsilon(C(\alpha)) = 0$.*

(2) *Let $A = \bigoplus_{\alpha\in\Gamma} A(\alpha)$ be a graded vector space and an algebra such that $A(\beta)A(\gamma) \subseteq A(\beta\gamma)$ for all $\beta,\gamma \in \Gamma$. Then $1_A \in A(e)$.*

PROOF. (1) We give an indirect proof. Let $x \in C(\alpha)$ with $\alpha \neq e$. Assume that $\varepsilon(x) \neq 0$. Since Δ is graded and Γ is cancellative, we can write

$$\Delta(x) = \sum_{i=1}^{n} x_i \otimes y_i,\ x_i \in C(\alpha_i), y_i \in C(\beta_i) \text{ for all } 1 \le i \le n,$$

where $\alpha_i,\beta_i \in \Gamma$, $\alpha_i\beta_i = \alpha$ for all i, and where $y_1,\dots,y_n$ are linearly independent. Since $\sum_{i=1}^{n} x_i\varepsilon(y_i) = x \in C(\alpha)$, there exists $j \in \{1,\dots,n\}$ such that $x_j \in C(\alpha)$ and $\varepsilon(x_j) \neq 0$. Hence $\alpha_j = \alpha$ and $\beta_j = e$, since Γ is cancellative. It follows that $x = \sum_{i=1}^{n} \varepsilon(x_i)y_i \notin C(\alpha)$, since $y_j \in C(e)$. Thus $\varepsilon(x) = 0$ for all $x \in C(\alpha)$ with $\alpha \neq e$.

(2) Let $1_A = \bigoplus_{\alpha\in\Gamma} a_\alpha$, where $a_\alpha \in A(\alpha)$ for all $\alpha \in \Gamma$. Let $1' = a_e$. Since Γ is cancellative, $x = 1_A x = 1'x$ for all $x \in A(\alpha)$ with $\alpha \in \Gamma$. Hence $1_A = 1' \in A(e)$ by uniqueness of the unit element of an algebra. □

Let A be a Γ-graded algebra. The multiplication map $\mu : A \otimes A \to A$ is determined by its components

$$\mu_{\beta,\gamma} : A(\beta) \otimes A(\gamma) \to A(\beta\gamma), \quad x \otimes y \mapsto xy, \tag{5.1.5}$$

for all $x \in A(\beta)$, $y \in A(\gamma)$ and $\beta,\gamma \in \Gamma$.

Let $C = \bigoplus_{\alpha\in\Gamma} C(\alpha)$ be a Γ-graded coalgebra with graded projection maps $\pi_\alpha = \pi^C_\alpha : C \to C(\alpha)$ for all $\alpha \in \Gamma$. We write

$$\Delta_{\beta,\gamma} : C(\beta\gamma) \subseteq C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi_\beta\otimes\pi_\gamma} C(\beta) \otimes C(\gamma),\ \beta,\gamma \in \Gamma, \tag{5.1.6}$$

for the (β,γ)-th component of the comultiplication Δ.

A Γ-**graded left A-module** V is a left A-module in $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$, that is, a Γ-graded vector space and a left A-module V with graded structure map $A \otimes V \to V$.

A **Γ-graded left C-comodule** is a left C-comodule in $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$, that is, a Γ-graded vector space and a left C-comodule V with graded structure map $V \to C\otimes V$.

LEMMA 5.1.2. (1) *Let A be a Γ-graded algebra, and V a Γ-graded left A-module.*

(a) *If $U \subseteq V$ is a graded subspace and an A-submodule, then U is a Γ-graded A-module.*

(b) *If $U \subseteq V$ is a submodule, then $\bigoplus_{\gamma\in\Gamma} U \cap V(\gamma)$ is a graded submodule of V.*

(2) *Let C be a Γ-graded coalgebra, and V a Γ-graded left C-comodule.*

(a) *If $U \subseteq V$ is a graded subspace and a subcomodule, then U is a graded C-comodule.*

(b) *Assume that Γ is cancellative. If $U \subseteq V$ is a subcomodule, then $\bigoplus_{\gamma\in\Gamma} U \cap V(\gamma)$ is a graded subcomodule of V.*

PROOF. (1) is obvious.

(2)(a) Let $\delta : V \to C \otimes V$ be the comodule structure of V. For all $\alpha \in \Gamma$,

$$\delta(U(\alpha)) \subseteq (C \otimes U) \cap (C \otimes V)(\alpha) = (C \otimes U)(\alpha),$$

since $C \otimes U \subseteq C \otimes V$ is a graded subspace. Hence U is a graded C-comodule.

(2)(b) Let $U' = \bigoplus_{\gamma\in\Gamma} U \cap V(\gamma)$. We prove that U' is a subcomodule of V. Then the claim follows from (a).

Let $\alpha \in \Gamma$, and $u \in U \cap V(\alpha)$. Since V is a graded C-comodule, there are an integer $r \geq 1$, $\beta_i, \gamma_i \in \Gamma$ for all $1 \leq i \leq r$, such that $\beta_i\gamma_i = \alpha$ for all i, and $\delta(u) \in \bigoplus_{i=1}^{r} C(\beta_i) \otimes V(\gamma_i)$. Since Γ is cancellative, we may assume that $\beta_i \neq \beta_j$ for all $i \neq j$. Hence

$$\delta(u) \in (C \otimes U) \cap \bigoplus_{i=1}^{r} C(\beta_i) \otimes V(\gamma_i) = \bigoplus_{i=1}^{r} C(\beta_i) \otimes (U \cap V(\gamma_i)) \subseteq C \otimes U',$$

where the last equality follows by choosing bases in $C(\beta_i)$ for all i. $\square$

COROLLARY 5.1.3. *Let C be a Γ-graded coalgebra and A a Γ-graded algebra. Then $\mathrm{Hom}_{\mathrm{gr}}(C, A) \subseteq \mathrm{Hom}(C, A)$ is a subalgebra with respect to the convolution product. If $f \in \mathrm{Hom}_{\mathrm{gr}}(C, A)$ is invertible in $\mathrm{Hom}(C, A)$, then $f^{-1} \in \mathrm{Hom}_{\mathrm{gr}}(C, A)$.*

PROOF. This follows from Proposition 1.2.11(2), since the maps $\Phi(f)$ and $\Phi^{-1}(f)$ are both graded. $\square$

Assume that Γ is an abelian monoid with neutral element 0.

Then we define a braiding on the monoidal category $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$ by the usual flip map of vector spaces $V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$, for all $v \in V$, $w \in W$.

A **Γ-graded bialgebra** $(H, \mathcal{H})$ is a bialgebra in $\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$, that is, $(H, \mathcal{H})$ is a graded algebra and a graded coalgebra, and H is a bialgebra.

A **Γ-graded Hopf algebra** H is a graded bialgebra such that there exists a graded linear map $\mathcal{S} : H \to H$ which is convolution inverse to id_H.

COROLLARY 5.1.4. *Let H be a Γ-graded bialgebra, and assume that H is a Hopf algebra. Then the antipode of H is a graded map. Thus H is a graded Hopf algebra, and $H(0) \subseteq H$ is a Hopf subalgebra.*

PROOF. The antipode $\mathcal{S} = \mathrm{id}^{-1}$ is graded by Corollary 5.1.3. In particular, $H(0)$ is stable under $\mathcal{S}$. Hence $H(0) \subseteq H$ is a Hopf subalgebra. $\square$

REMARK 5.1.5. The preceding notions can be generalized. Replace the category $\mathcal{M}_{\Bbbk}$ by an abelian braided monoidal category $\mathcal{C}$ with arbitrary direct sums. If Γ is an abelian monoid, and $(V, \mathcal{V})$ and $(W, \mathcal{W})$ are Γ-graded objects in $\mathcal{C}$, then the braiding $c_{V,W} : V \otimes W \to W \otimes V$ is Γ-graded; for all $\beta, \gamma \in \Gamma$, $c_{V,W}$ induces an isomorphism $V(\beta) \otimes W(\gamma) \to W(\gamma) \otimes V(\beta)$, since the braiding is a functorial isomorphism. Thus the category $\Gamma\text{-Gr}\,\mathcal{C}$ of Γ-graded objects in $\mathcal{C}$ is braided monoidal. A special case is the category $\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ defined in Remark 4.2.7.

5.2. Filtrations and gradings by totally ordered abelian monoids

Let Γ be an abelian monoid with monoid structure $+$. The neutral element is denoted by 0. If $<$ is a total order on Γ, we define the following conditions for the pair $(\Gamma, <)$.

(M1) For any $\alpha \in \Gamma$ the set $\{\beta \in \Gamma \mid \beta < \alpha\}$ is finite.
(M2) For any $\alpha, \beta, \gamma \in \Gamma$ the relation $\alpha < \beta$ implies that $\alpha + \gamma < \beta + \gamma$.

EXAMPLE 5.2.1. Let θ be a positive integer and let $\Gamma = \mathbb{N}_0^\theta$. Write

$$\alpha < \beta \text{ for } \alpha = (a_1, \dots, a_\theta) \in \Gamma, \quad \beta = (b_1, \dots, b_\theta) \in \Gamma$$

if $\sum_{i=1}^\theta a_i < \sum_{i=1}^\theta b_i$ or if $\sum_{i=1}^\theta (a_i - b_i) = 0$ and there exists $1 \le i \le \theta$ such that $a_i < b_i$ and $a_j = b_j$ for all $1 \le j < i$. Then $(\Gamma, <)$ satisfies conditions (M1) and (M2). In particular, $\Gamma = \mathbb{N}_0$ with the natural ordering satisfies the conditions (M1) and (M2).

In the remainder of this section we assume a total ordering $<$ on the abelian monoid Γ satisfying (M1) and (M2).

A monoid M is called **positive** if 0 is its only unit.

LEMMA 5.2.2. *The monoid Γ satisfies the following.*

(1) *Let $\alpha \in \Gamma \setminus \{0\}$. Then $\alpha > 0$.*
(2) *The monoid Γ is torsion-free, cancellative, and positive.*

PROOF. (1) Assume that $\alpha < 0$. Then $\cdots < 3\alpha < 2\alpha < \alpha < 0$ by (M2), which is a contradiction to (M1).

(2) Let $\alpha \in \Gamma \setminus \{0\}$. Then $0 < \alpha < 2\alpha \cdots < (m-1)\alpha < m\alpha$ by (1) and by (M2). Thus $m\alpha \neq 0$ for all $m \ge 1$, and hence Γ is torsion-free.

Let $\alpha, \beta, \gamma \in \Gamma$ with $\alpha + \gamma = \beta + \gamma$. Then both $\alpha < \beta$ and $\beta < \alpha$ contradict to (M2). Hence $\alpha = \beta$, that is, Γ is cancellative.

Let $\alpha \in \Gamma$ be a unit. If $\alpha \neq 0$, then $\alpha > 0$ and $-\alpha > 0$ by (1), and hence $0 = \alpha + (-\alpha) > \alpha > 0$, a contradiction. Thus Γ is positive. □

Graded vector spaces often come from natural filtrations, and filtrations are a useful tool to study graded objects.

A **Γ-filtration of a vector space** V is a family $\mathcal{F}(V) = (F_\alpha(V))_{\alpha \in \Gamma}$ of subspaces of V such that

$$F_\alpha(V) \subseteq F_\beta(V) \quad \text{for all } \alpha, \beta \in \Gamma,\ \alpha < \beta,$$
$$V = \bigcup_{\alpha \in \Gamma} F_\alpha(V).$$

A **Γ-filtered vector space** is a pair $(V, \mathcal{F}(V))$, where V is a vector space and $\mathcal{F}(V)$ is a filtration of V.

Let Γ-Filt $\mathcal{M}_{\Bbbk}$ be the **category of Γ-filtered vector spaces**. Objects are the Γ-filtered vector spaces, and a morphism between filtered vector spaces $(V, \mathcal{F}(V))$ and $(W, \mathcal{F}(W))$ is a $\Bbbk$-linear map $f : V \to W$ which is **filtered**, that is,

$$f(F_\alpha(V)) \subseteq F_\alpha(W) \text{ for all } \alpha \in \Gamma.$$

We define

$$\mathrm{Hom}_{\mathrm{filt}}(V, W) = \{f \in \mathrm{Hom}(V, W) \mid f \text{ is filtered}\}.$$

The tensor product of $(V, \mathcal{F}(V))$ and $(W, \mathcal{F}(W))$ is the tensor product $V \otimes W$ of vector spaces with filtration defined by

$$F_\alpha(V \otimes W) = \sum_{\beta+\gamma \leq \alpha} F_\beta(V) \otimes F_\gamma(W) \text{ for all } \alpha \in \Gamma.$$

The category Γ-Filt $\mathcal{M}_{\Bbbk}$ is monoidal with this tensor product and unit object $\Bbbk$ with filtration $F_\alpha(\Bbbk) = \Bbbk$ for all $\alpha \in \Gamma$. Again the associativity and unit constraints are the same as for vector spaces, and Γ-Filt $\mathcal{M}_{\Bbbk}$ is braided monoidal with the flip of vector spaces as braiding.

REMARK 5.2.3. Filtered objects can be defined in more general categories than vector spaces. In particular, for a Hopf algebra H with bijective antipode, the **category Γ-Filt ${}^H_H\mathcal{YD}$ of Γ-filtered Yetter-Drinfeld modules over H** is braided monoidal with the monoidal structure and the braiding of ${}^H_H\mathcal{YD}$.

A filtered vector space V in Γ-Filt $\mathcal{M}_{\Bbbk}$ is called **locally finite** if $F_\alpha(V)$ is finite-dimensional for all $\alpha \in \Gamma$. We denote the full subcategory of Γ-Filt $\mathcal{M}_{\Bbbk}$ of locally finite vector spaces by Γ-Filt $\mathcal{M}_{\Bbbk}^{\mathrm{lf}}$.

A **coalgebra filtration** of a coalgebra C is a vector space filtration of C, $\mathcal{F}(C) = (F_\alpha(C))_{\alpha\in\Gamma}$, such that

$$\Delta(F_\alpha(C)) \subseteq \sum_{\beta+\gamma \leq \alpha} F_\beta(C) \otimes F_\gamma(C) \quad \text{for all } \alpha \in \Gamma. \tag{5.2.1}$$

A **filtered coalgebra** $(C, \mathcal{F}(C))$ is a coalgebra in the monoidal category Γ-Filt $\mathcal{M}_{\Bbbk}$, that is, a coalgebra C with a coalgebra filtration $\mathcal{F}(C)$. Note that the counit $\varepsilon : C \to \Bbbk$ is always a filtered map.

We want to prove two useful results about filtered coalgebras. We first look at their simple subcoalgebras.

PROPOSITION 5.2.4. *Let C be a coalgebra with a coalgebra filtration $\mathcal{F}(C)$. Then any simple subcoalgebra of C is contained in $F_0(C)$.*

PROOF. Let $D \subseteq C$ be a simple subcoalgebra. Since $F_0(C) \cap D$ is a subcoalgebra of C, it is enough to prove that $F_0(C) \cap D$ is non-zero. Let $\alpha \in \Gamma$ be minimal such that $F_\alpha(C) \cap D \neq 0$, and let $x \in F_\alpha(C) \cap D$ be a non-zero element. If $\Delta(x) \in F_0(C) \otimes D$, then $x = (\mathrm{id} \otimes \varepsilon)\Delta(x) \in F_0(C)$, and we are done. If $\Delta(x) \notin F_0(C) \otimes D$, then there exists $f \in C^* = \mathrm{Hom}(C, \Bbbk)$ such that $f(x_{(1)})x_{(2)} \neq 0$ and $f(F_0(C)) = 0$. Since $f(x_{(1)})x_{(2)} \in F_{<\alpha}(C) \cap D$, we obtain a contradiction to the minimality of α. $\square$

If C is a one-dimensional coalgebra, then there is a unique group-like element 1_C, and $C = \Bbbk 1_C$.

COROLLARY 5.2.5. *Let C be a coalgebra with coalgebra filtration $\mathcal{F}(C)$. If $F_0(C)$ is one-dimensional, then $F_0(C)$ is the unique simple subcoalgebra of C. The coalgebra C has a unique group-like element which spans $F_0(C)$.*

PROOF. The subcoalgebra $F_0(C)$ is one-dimensional, hence simple. Thus the claim follows from Proposition 5.2.4. □

COROLLARY 5.2.6. *Let C be a coalgebra with coalgebra filtration $\mathcal{F}(C)$, and $0 \neq V \in \mathcal{M}^C$ with comodule structure $\delta : V \to V \otimes C$. Then there is a non-zero element $v \in V$ with $\delta(v) \in V \otimes F_0(C)$.*

PROOF. By the Finiteness Theorem 2.1.3 for comodules, V contains a simple subcomodule $U \subseteq V$. By Proposition 2.2.13(2), there is a simple subcoalgebra $D \subseteq C$ with $\delta(U) \subseteq U \otimes D$. Hence the claim follows from Proposition 5.2.4. □

An **algebra filtration** of an algebra A is by definition a vector space filtration $\mathcal{F}(A) = (F_\alpha(A))_{\alpha\in\Gamma}$ of A such that

$$F_\alpha(A)F_\beta(A) \subseteq F_{\alpha+\beta}(A) \quad \text{for all } \alpha, \beta \in \Gamma, \tag{5.2.2}$$

$$1_A \in F_0(A). \tag{5.2.3}$$

A **filtered algebra** $(A, \mathcal{F}(A))$ is an algebra in $\Gamma\text{-Filt}\,\mathcal{M}_\Bbbk$, that is, an algebra A with an algebra filtration $\mathcal{F}(A)$.

EXAMPLE 5.2.7. Let A be an algebra and let X be a subset of A generating the algebra A. The standard $\mathbb{N}_0$-filtration $\mathcal{F}(A)$ of A defined by X is the algebra filtration

$$F_0(A) = \Bbbk 1_A, \quad F_n(A) = (X \cup \{1_A\})^n \quad \text{for all } n \geq 1,$$

where $(X \cup \{1_A\})^n \subseteq A$ is the subspace generated by all elements $a_1 \cdots a_n$ with $a_1, \ldots, a_n \in X \cup \{1_A\}$.

EXAMPLE 5.2.8. Let A and Γ be as in Example 5.2.7. Let I be an index set, and for all $i \in I$ let $\alpha_i \in \Gamma \setminus \{0\}$ and $X_i \subseteq X$ such that $X = \cup_{i\in I} X_i$. Then $\mathcal{F}(A)$ with

$$F_\alpha(A) = \sum_{\substack{n\geq 0, i_1,\ldots,i_n \in I,\\ \alpha_{i_1}+\cdots+\alpha_{i_n} \leq \alpha}} \Bbbk X_{i_1} \cdots X_{i_n}$$

for all $\alpha \in \Gamma$, where $\Bbbk X_{i_1} \cdots X_{i_n} = \Bbbk 1$ for $n = 0$, defines an algebra filtration of A by Γ.

A **filtered bialgebra** $(H, \mathcal{F}(H))$ is a bialgebra in $\Gamma\text{-Filt}\,\mathcal{M}_\Bbbk$, that is, $(H, \mathcal{F}(H))$ is a filtered coalgebra and a filtered algebra, and H is a bialgebra.

A **filtered Hopf algebra** $(H, \mathcal{F}(H))$ is a bialgebra in $\Gamma\text{-Filt}\,\mathcal{M}_\Bbbk$ such that there exists a filtered map $\mathcal{S} : H \to H$ which is convolution inverse to id_H.

Next we discuss convolution inverses of maps on coalgebras. Let C be a coalgebra and A an algebra. Recall that $\mathrm{Hom}(C, A)$ is an algebra, where the product is the convolution of maps and the unity is $\eta\varepsilon$. Define the iterations $\Delta^n : C \to C^{\otimes(n+1)}$, $n \geq 0$, of Δ inductively by

$$\Delta^0 = \mathrm{id} : C \to C, \quad \Delta^n = (\mathrm{id} \otimes \Delta^{n-1})\Delta \quad \text{for all } n \geq 1. \tag{5.2.4}$$

If $(C, \mathcal{F}(C))$ is a filtered coalgebra, then

$$\Delta^n(F_\alpha(C)) \subseteq \sum_{\alpha_0+\alpha_1+\cdots+\alpha_n=\alpha} F_{\alpha_0}(C) \otimes F_{\alpha_1}(C) \otimes \ldots \otimes F_{\alpha_n}(C)$$

for all $n \geq 0$ and $\alpha \in \Gamma$.

PROPOSITION 5.2.9. *(Takeuchi's Lemma) Let* $(C, \mathcal{F}(C))$ *be a filtered coalgebra,* A *an algebra, and* $f : C \to A$ *a linear map.*

(1) *Assume that* $f(F_0(C)) = 0$*. Then* $\eta\varepsilon - f$ *is invertible in* $\mathrm{Hom}(C, A)$ *with inverse*

$$(\eta\varepsilon - f)^{-1} = \sum_{n\geq 0} f^n.$$

(2) *Assume that the restriction* $f| : F_0(C) \to A$ *of* f *is invertible. Let* $g \in \mathrm{Hom}(C, A)$ *with* $g|F_0(C) = (f|F_0(C))^{-1}$*. Then* f *is invertible in* $\mathrm{Hom}(C, A)$ *with inverse*

$$f^{-1} = g \sum_{n\geq 0} (\eta\varepsilon - fg)^n.$$

(3) *Assume that* $F_0(C) = \Bbbk 1_C$ *is one-dimensional. If* $f(1_C) = 1_A$*, then* f *is invertible with inverse*

$$f^{-1} = \sum_{n\geq 0} (\eta\varepsilon - f)^n.$$

PROOF. (1) Let $\alpha \in \Gamma$. Then the set

$$\{(n, \alpha_1, \alpha_2, \ldots, \alpha_n) \mid n \geq 0, \alpha_1, \ldots, \alpha_n \in \Gamma \setminus \{0\}, \alpha_1 + \cdots + \alpha_n \leq \alpha\}$$

is finite. Since $f(F_0(C)) = 0$, there exists $N(\alpha) \in \mathbb{N}_0$ such that

$$f^n(F_\alpha(C)) \subseteq \sum_{\alpha_1+\cdots+\alpha_n\leq\alpha} f(F_{\alpha_1}(C)) \ldots f(F_{\alpha_n}(C)) = 0$$

for all $n > N(\alpha)$. Thus $\sum_{n\geq 0} f^n$ is a well-defined linear map, and

$$\Big(\sum_{n\geq 0} f^n\Big)(x) = \sum_{n=0}^{N(\alpha)} f^n(x)$$

for all $x \in F_\alpha(C)$. Let $x \in F_\alpha(C)$. Then

$$\begin{aligned}(\eta\varepsilon - f)\Big(\sum_{n\geq 0} f^n\Big)(x) &= (\varepsilon(x_{(1)})1_A - f(x_{(1)}))\Big(\sum_{n=0}^{N(\alpha)} f^n(x_{(2)})\Big)\\ &= \sum_{n=0}^{N(\alpha)} f^n(x) - \sum_{n=0}^{N(\alpha)} f^{n+1}(x)\\ &= \eta\varepsilon(x).\end{aligned}$$

Thus $(\eta\varepsilon - f)\big(\sum_{n\geq 0} f^n\big) = \eta\varepsilon$, and $\big(\sum_{n\geq 0} f^n\big)(\eta\varepsilon - f) = \eta\varepsilon$ by a similar calculation.

(2) By assumption $(\eta\varepsilon - fg)(F_0(C)) = 0$ and $(\eta\varepsilon - gf)(F_0(C)) = 0$. Hence

$$fg \sum_{n\geq 0} (\eta\varepsilon - fg)^n = \eta\varepsilon, \qquad \sum_{n\geq 0} (\eta\varepsilon - gf)^n gf = \eta\varepsilon$$

by (1). This proves (2).

(3) follows from (2) with $g = \eta\varepsilon$. □

COROLLARY 5.2.10. *Let C be a filtered coalgebra and A a filtered algebra. Then $\mathrm{Hom}_{\mathrm{filt}}(C,A) \subseteq \mathrm{Hom}(C,A)$ is a subalgebra. If $f \in \mathrm{Hom}_{\mathrm{filt}}(C,A)$ is invertible in $\mathrm{Hom}(C,A)$ with inverse f^{-1} and the filtrations of C and A are locally finite, then $f^{-1} \in \mathrm{Hom}_{\mathrm{filt}}(C,A)$.*

PROOF. It is clear from the definitions that $\mathrm{Hom}_{\mathrm{filt}}(C,A) \subseteq \mathrm{Hom}(C,A)$ is a subalgebra. Let $f \in \mathrm{Hom}_{\mathrm{filt}}(C,A)$ be invertible in $\mathrm{Hom}(C,A)$. Then $\Phi(f)$ in Proposition 1.2.11 is a filtered endomorphism of $C \otimes A$, and $\Phi(f)$ is invertible by Proposition 1.2.11(2). If the filtrations of C and A are locally finite, the filtration of $A \otimes C$ is locally finite, and then $\Phi(f)^{-1}$ is filtered. In this case $f^{-1} \in \mathrm{Hom}_{\mathrm{filt}}(C,A)$ by Proposition 1.2.11(2). □

COROLLARY 5.2.11. (1) *Let $(H, \mathcal{F}(H))$ be a filtered bialgebra, such that the filtration is locally finite. Assume that H is a Hopf algebra with antipode $\mathcal{S}$. Then $\mathcal{S}(F_\alpha(H)) \subseteq F_\alpha(H)$ for all $\alpha \in \Gamma$. Thus H is a filtered Hopf algebra, and $F_0(H) \subseteq H$ is a Hopf subalgebra.*

(2) *Let H be a bialgebra with a coalgebra filtration $\mathcal{F}(H)$. If $F_0(H)$ is one-dimensional, then H is a Hopf algebra with antipode*

$$\mathcal{S} = \sum_{n \geq 0} (\eta\varepsilon - \mathrm{id})^n.$$

If $F_0(H) \subseteq H$ is a subbialgebra and a Hopf algebra, then H is a Hopf algebra. If $F_0(H)$ is a Hopf algebra with bijective antipode, then the antipode of H is bijective.

PROOF. (1) The antipode is filtered by Corollary 5.2.10. In particular, $F_0(H)$ is a Hopf subalgebra of H.

(2) Assume that $F_0(H) \subseteq H$ is a subbialgebra and a Hopf algebra with antipode $\mathcal{S}_{F_0(H)}$. Then the restriction of $\mathrm{id} : H \to H$ to $F_0(H)$ is invertible. Hence id_H is invertible by Proposition 5.2.9(2), and H is a Hopf algebra. If in addition the antipode of $F_0(H)$ is bijective, then the dual algebra $F_0(H)^{\mathrm{op}}$ also is a Hopf algebra with antipode $\overline{\mathcal{S}_{F_0(H)}}$, where $\overline{\mathcal{S}_{F_0(H)}}$ is the linear inverse of $\mathcal{S}_{F_0(H)}$. The dual algebra H^{op} is a bialgebra with the same coalgebra filtration $(F_\alpha(H)^{\mathrm{op}})_{\alpha\in\Gamma}$, and $F_0(H)^{\mathrm{op}}$ is a Hopf subalgebra of H^{op}. Hence H^{op} is a Hopf algebra by the argument we have just shown. Thus the antipode of H is bijective.

If $F_0(H)$ is one-dimensional, then the formula for the antipode follows from Proposition 5.2.9(3) with $f = \mathrm{id}$. □

COROLLARY 5.2.12. *Let $H = \bigoplus_{\alpha\in\Gamma} H(\alpha)$ be a bialgebra and a graded coalgebra. If $H(0) = \Bbbk 1_H$, then H is a Hopf algebra with antipode*

$$\mathcal{S} = \sum_{n \geq 0} (\eta\varepsilon - \mathrm{id})^n.$$

If $H(0) \subseteq H$ is a subbialgebra and a Hopf algebra, then H is a Hopf algebra. If $H(0)$ is a Hopf algebra with bijective antipode, then the antipode of H is bijective.

PROOF. This follows from Corollary 5.2.11(2), where we use the coalgebra filtration associated to the grading of H. □

PROPOSITION 5.2.13. *Let H be a Hopf algebra with bijective antipode, and R be a bialgebra in ${}^H_H\mathcal{YD}$ with an $\mathbb{N}_0$-coalgebra filtration $(R_n)_{n\geq 0}$, and $R_0 = \Bbbk 1$. Then R is a Hopf algebra in ${}^H_H\mathcal{YD}$ with bijective antipode.*

PROOF. By (4.3.2) the filtration

$$H \subseteq R_1\#H \subseteq R_2\#H \subseteq \cdots \subseteq R\#H$$

is a coalgebra filtration of the bosonization $R\#H$. By Proposition 3.8.4(1), $R\#H$ is a bialgebra. Since the antipode of H is bijective, $R\#H$ is a Hopf algebra with bijective antipode by Corollary 5.2.11(2). By Proposition 5.2.9(3), id_R is convolution invertible, hence R is a Hopf algebra in ${}^H_H\mathcal{YD}$ by Proposition 3.8.9. Then the antipode of R is bijective by Corollary 3.8.11. $\square$

Graded and filtered vector spaces are related by the functors

$$\mathrm{gr}\ : \Gamma\text{-Filt}\,\mathcal{M}_\Bbbk \to \Gamma\text{-Gr}\,\mathcal{M}_\Bbbk, \quad \mathrm{filt} : \Gamma\text{-Gr}\,\mathcal{M}_\Bbbk \to \Gamma\text{-Filt}\,\mathcal{M}_\Bbbk.$$

For a filtered vector space V with filtration $\mathcal{F}(V)$ let

$$F_{<\alpha}(V) = \begin{cases} 0 & \text{if } \alpha = 0, \\ F_\beta(V) & \text{if } \alpha \neq 0, \text{ where } \beta = \max\{\gamma \in \Gamma \mid \gamma < \alpha\}. \end{cases}$$

Then we define

$$\mathrm{gr}\, V = \bigoplus_{\alpha \in \Gamma} F_\alpha(V)/F_{<\alpha}(V).$$

For all $\alpha \in \Gamma$ let

$$p_\alpha^V = p_\alpha : F_\alpha(V) \to F_\alpha(V)/F_{<\alpha}(V) = (\mathrm{gr}\, V)(\alpha)$$

be the canonical epimorphism. If $f : (V, \mathcal{F}(V)) \to (W, \mathcal{F}(W))$ is a morphism, the induced map $\mathrm{gr}\, f$ is defined by

$$\mathrm{gr}\, f : \mathrm{gr}\, V \to \mathrm{gr}\, W, \quad p_\alpha(v) \mapsto p_\alpha(f(v)) \quad \text{for all } v \in F_\alpha(V),\ \alpha \in \Gamma.$$

For a graded vector space V with gradation $\mathcal{V}$ we define $\mathrm{filt}(V, \mathcal{V}) = V$ with filtration

$$F_\alpha(V) = \bigoplus_{\beta \leq \alpha} V(\beta)$$

for all $\alpha \in \Gamma$. If $f : (V, \mathcal{V}) \to (W, \mathcal{W})$ is a morphism in $\mathrm{Gr}\,\mathcal{M}_\Bbbk^\Gamma$, then we define $\mathrm{filt}(f) = f : V \to W$.

Note that $\mathrm{gr}\,\mathrm{filt} \cong \mathrm{id}_{\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk}$. Usually information is lost by applying the functor gr. But in some cases properties of the filtered object can be derived from the associated graded object. A first example of this type is given in the next lemma.

LEMMA 5.2.14. *Let $f : (V, \mathcal{F}(V)) \to (W, \mathcal{F}(W))$ be a morphism of filtered vector spaces. If $\mathrm{gr}\, f$ is surjective, then f is surjective. If $\mathrm{gr}\, f$ is injective, then f is injective.*

PROOF. Assume that f is surjective. We show by induction that the restriction $f_\alpha : F_\alpha(V) \to F_\alpha(W)$ of f is surjective. This is true for $\alpha = 0$, since $f_0 = (\mathrm{gr}\, f)(0)$. Let $\alpha \in \Gamma$, $\beta = \max\{\gamma \in \Gamma \mid \gamma < \alpha\}$, and assume that f_β is surjective. Then f_α is surjective, since f_β and the quotient map $(\mathrm{gr}\, f)(\alpha)$ are. The second claim one proves analogously. $\square$

The functor filt is a braided strict monoidal functor, that is, filt maps the unit object of $\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk$ to the unit object of $\Gamma\text{-Filt}\,\mathcal{M}_\Bbbk$, and if V and W are graded vector spaces, then $\mathrm{filt}(V) \otimes \mathrm{filt}(W) = \mathrm{filt}(V \otimes W)$. To enlarge gr to a monoidal functor we need some linear algebra lemmas.

REMARK 5.2.15. Let I be an index set, and $(I_j)_{j\in J}$ a family of subsets of I. Let $(V_i)_{i\in I}$ be a family of vector spaces. Then, by the definition of the direct sum,

$$\bigcap_{j\in J}\bigoplus_{i\in I_j} V_i = \bigoplus_{i\in\bigcap_{j\in J} I_j} V_i.$$

The next lemma essentially shows that gr is a monoidal functor.

LEMMA 5.2.16. *Let V and W be vector spaces with Γ-filtrations $\mathcal{F}(V)$ and $\mathcal{F}(W)$. Then for all $\alpha\in\Gamma$,*

$$\bigcap_{\beta+\gamma\geq\alpha}(V\otimes F_{<\gamma}(W)+F_{<\beta}(V)\otimes W) = \sum_{\beta+\gamma<\alpha} F_\beta(V)\otimes F_\gamma(W).$$

PROOF. Choose subspaces $X_\beta\subseteq V$ and $Y_\beta\subseteq W$ for all $\beta\in\Gamma$ such that

$$F_\beta(V)=F_{<\beta}(V)\oplus X_\beta, \qquad F_\beta(W)=F_{<\beta}(W)\oplus Y_\beta$$

for all $\beta\in\Gamma$. Then $F_0(V)=X_0$ and $F_0(W)=Y_0$. Let $\alpha\in\Gamma$. Then

$$\bigcap_{\beta+\gamma\geq\alpha}(V\otimes F_{<\gamma}(W)+F_{<\beta}(V)\otimes W) = \bigcap_{\beta+\gamma\geq\alpha}\ \bigoplus_{\substack{\beta'<\beta\\ \text{or }\gamma'<\gamma}} X_{\beta'}\otimes Y_{\gamma'},$$

$$\sum_{\beta+\gamma<\alpha} F_\beta(V)\otimes F_\gamma(W) = \bigoplus_{\beta'+\gamma'<\alpha} X_{\beta'}\otimes Y_{\gamma'}.$$

Clearly,

$$\beta'+\gamma'\geq\alpha \iff \exists\beta,\gamma\in\Gamma:\beta+\gamma\geq\alpha, \beta'\geq\beta,\gamma'\geq\gamma$$

for all $\beta',\gamma'\in\Gamma$. Hence the lemma follows from Remark 5.2.15. □

PROPOSITION 5.2.17. *The functor* gr $:\Gamma\text{-Filt}\,\mathcal{M}_{\Bbbk}\to\Gamma\text{-Gr}\,\mathcal{M}_{\Bbbk}$ *maps the unit object to the unit object. For all $V,W\in\Gamma\text{-Filt}\,\mathcal{M}_{\Bbbk}$ there is a graded linear isomorphism*

$$\varphi_{V,W}:\mathrm{gr}\,(V\otimes W)\to\mathrm{gr}\,V\otimes\mathrm{gr}\,W$$

such that for all $\alpha,\beta,\gamma\in\Gamma$ with $\beta+\gamma=\alpha$ and all $v\in F_\beta(V)$, $w\in F_\gamma(W)$,

$$\varphi_{V,W}(\alpha)(p_\alpha(v\otimes w))=p_\beta(v)\otimes p_\gamma(w).$$

The family $\varphi=(\varphi_{V,W})_{V,W\in\mathrm{Gr}\,\mathcal{M}_{\Bbbk}^\Gamma}$ is a natural isomorphism of bifunctors, and $(\mathrm{gr}\,,\varphi^{-1})$ is a braided monoidal functor.

PROOF. For all $\alpha\in\Gamma$ let

$$q^V_\alpha=q_\alpha:V\to V/F_{<\alpha}(V),\quad q^W_\alpha=q_\alpha:W\to W/F_{<\alpha}(W)$$

be the canonical epimorphisms. Define

$$f_\alpha:V\otimes W\to\bigoplus_{\beta+\gamma=\alpha} V/F_{<\beta}(V)\otimes W/F_{<\gamma}(W)$$

by $f_\alpha(v\otimes w)=\sum_{\beta+\gamma=\alpha}q_\beta(v)\otimes q_\gamma(w)$ for all $v\in V$, $w\in W$.

Let $\beta,\beta',\gamma,\gamma'\in\Gamma$ with $\beta+\gamma\geq\alpha$, $\beta'+\gamma'=\alpha$, and let $v\in F_{\beta'}(V)$, $w\in F_{\gamma'}(W)$. If $\beta+\gamma>\alpha$, then $\beta>\beta'$ or $\gamma>\gamma'$. In this case

$$F_\alpha(V\otimes W)\subseteq V\otimes F_{<\gamma}(W)+F_{<\beta}(V)\otimes W.$$

If $\beta+\gamma=\alpha$, then $\beta>\beta'$ or $\gamma>\gamma'$ or $\beta=\beta'$, $\gamma=\gamma'$, and hence

$$f_\alpha(v\otimes w)=q_{\beta'}(v)\otimes q_{\gamma'}(w).$$

Hence

$$\begin{aligned}\ker(f_\alpha|F_\alpha(V\otimes W)) &= \bigcap_{\beta+\gamma\geq\alpha}(V\otimes F_{<\gamma}(W)+F_{<\beta}(V)\otimes W)\\ &= \sum_{\beta+\gamma<\alpha}F_\beta(V)\otimes F_\gamma(W)\end{aligned}$$

by Lemma 1.1.11 and Lemma 5.2.16, and f_α induces an isomorphism

$$\varphi_{V,W}(\alpha):\mathrm{gr}\,(V\otimes W)(\alpha)\to(\mathrm{gr}\,V\otimes\mathrm{gr}\,W)(\alpha).$$

The remaining claims of the proposition are easy to check. □

The functor $\mathrm{gr}\;:\Gamma\text{-Filt}\,{}^H_H\mathcal{YD}\to\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$ is defined in the obvious way for filtered objects in ${}^H_H\mathcal{YD}$, H a Hopf algebra with bijective antipode, instead of vector spaces. For filtered Yetter-Drinfeld modules V,W, the isomorphism $\varphi_{V,W}$ in Proposition 5.2.17 is an isomorphism of graded Yetter-Drinfeld modules.

COROLLARY 5.2.18. *Let H be a Hopf algebra with bijective antipode.*

$$(\mathrm{gr}\,,\varphi^{-1}):\Gamma\text{-Filt}\,{}^H_H\mathcal{YD}\to\Gamma\text{-Gr}\,{}^H_H\mathcal{YD}$$

is a braided monoidal functor.

PROOF. Follow the proof of Proposition 5.2.17. □

REMARK 5.2.19. The braided monoidal functor$(\mathrm{gr}\,,\varphi^{-1})$ of Proposition 5.2.17 preserves filtered algebras, coalgebras, bialgebras, and Hopf algebras. We describe these constructions explicitly.

(1) Let C be a coalgebra with coalgebra filtration $\mathcal{F}(C)=(F_\alpha(C))_{\alpha\in\Gamma}$. Then $\mathrm{gr}\,(C)=\bigoplus_{\alpha\in\Gamma}F_\alpha(C)/F_{<\alpha}(C)$ is a graded coalgebra.

The counit of $\mathrm{gr}\,(C)$ is given for $\alpha\in\Gamma$, $x\in F_\alpha(C)$ and $\overline{x}\in F_\alpha(C)/F_{<\alpha}(C)$ by $\varepsilon(\overline{x})=\varepsilon(x)$ if $\alpha=0$, and $\varepsilon(\overline{x})=0$ if $\alpha\neq0$.

The comultiplication $\varDelta$ on $F_\alpha(C)/F_{<\alpha}(C)$, where $\alpha\in\Gamma$, is defined in the following way: Let $x\in F_\alpha(C)$ and $\overline{x}\in F_\alpha(C)/F_{<\alpha}(C)$. We can write

$$\varDelta(x)=\sum_{\beta+\gamma=\alpha}\sum_{l\in L_\beta}y_l\otimes z_l,$$

where the L_β with $\beta\in\Gamma$ are disjoint finite index sets, and $y_l\in F_\beta(C)$, $z_l\in F_{\alpha-\beta}(C)$ for all $\beta\in\Gamma$ and $l\in L_\beta$. Then

$$\varDelta(\overline{x})=\sum_{\beta+\gamma=\alpha}\sum_{l\in L_\beta}\overline{y_l}\otimes\overline{z_l},$$

where $\overline{y_l}\in F_\beta(C)/F_{<\beta}(C)$ and $\overline{z_l}\in F_\gamma(C)/F_{<\gamma}(C)$ for all $\beta,\gamma\in\Gamma$ with $\beta+\gamma=\alpha$ and all $l\in L_\beta$.

(2) Let $(A,\mathcal{F}(A))$ be a filtered algebra. Then $\mathrm{gr}\,(A)=\bigoplus_{\alpha\in\Gamma}A(\alpha)$ is a graded algebra with unit element $1\in F_0(A)=\mathrm{gr}\,(A)(0)$ and multiplication defined for all $\beta,\gamma\in\Gamma$ by

$$F_\beta(A)/F_{<\beta}(A)\otimes F_\gamma(A)/F_{<\gamma}(A)\to F_{\beta+\gamma}(A)/F_{<\beta+\gamma}(A),\;\;\overline{x}\otimes\overline{y}\mapsto\overline{xy}.$$

(3) Let $(H,\mathcal{F}(H))$ be a filtered bialgebra. Then $\mathrm{gr}\,(H)$ is a graded bialgebra with coalgebra and algebra structure described in (1) and (2), respectively. If $(H,\mathcal{F}(H))$ is a filtered Hopf algebra with antipode $\mathcal{S}$, then $\mathrm{gr}\,(H)$ is a graded Hopf algebra with antipode $\mathcal{S}$, where $\mathcal{S}(\overline{x})=\overline{\mathcal{S}(x)}$ for all $\alpha\in\Gamma$, $x\in F_\alpha(H)$ and $\overline{x},\overline{\mathcal{S}(x)}\in F_\alpha(H)/F_{<\alpha}(H)$.

EXAMPLE 5.2.20. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra of dimension m with basis $x_1, \dots, x_m$. Let $(U_n(\mathfrak{g}))_{n \geq 0}$ be the standard algebra filtration defined by the generating set $\mathfrak{g}$, that is, $U_0(\mathfrak{g}) = \Bbbk 1$ and $U_n(\mathfrak{g}) = \sum_{k=0}^n \mathfrak{g}^k$ for all $n \geq 1$. By the coproduct formula in Example 1.2.24, $U(\mathfrak{g})$ is a filtered bialgebra with the standard filtration, where the elements of $\mathfrak{g}$ are primitive. Then by the theorem of Poincaré, Birkhoff and Witt, $\operatorname{gr} U(\mathfrak{g})$ is a commutative polynomial algebra in the variables $\overline{x_1}, \dots, \overline{x_m} \in (\operatorname{gr} U(\mathfrak{g}))(1) = (\mathfrak{g} + \Bbbk 1)/\Bbbk 1$, with $\Delta(\overline{x_i}) = 1 \otimes \overline{x_i} + \overline{x_i} \otimes 1$ for all i.

Let $\theta \geq 1$ and $\Gamma = \mathbb{N}_0^\theta$ the totally ordered abelian monoid defined in Example 5.2.1. Let $\alpha_1, \dots, \alpha_\theta$ be the standard basis of $\mathbb{Z}^\theta$. We describe a general method to construct $\mathbb{N}_0^\theta$-graded Hopf algebras in ${}^H_H\mathcal{YD}$.

PROPOSITION 5.2.21. *Let H be a Hopf algebra with bijective antipode. Let $\theta \geq 1$, $\Gamma = \mathbb{N}_0^\theta$, and $I = \{1, \dots, \theta\}$. Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ and $(M_i)_{i \in I}$ a family of subobjects of R in ${}^H_H\mathcal{YD}$. Assume that the algebra R is generated by $\sum_{i \in I} M_i = \bigoplus_{i \in I} M_i$, and that $M_i \subseteq P(R)$ for all $i \in I$. For all $\alpha \in \Gamma$ define*

$$F_\alpha(R) = \sum_{\substack{n \geq 0, i_1, \dots, i_n \in I, \\ \alpha_{i_1} + \dots + \alpha_{i_n} \leq \alpha}} M_{i_1} \cdots M_{i_n}, \quad \operatorname{gr}(R)(\alpha) = F_\alpha(R)/F_{<\alpha}(R),$$

where $M_{i_1} \cdots M_{i_n} = \Bbbk$, if $n = 0$. Then

(1) *$F_\alpha(R) \subseteq R$ is a subobject in ${}^H_H\mathcal{YD}$ for all $\alpha \in \Gamma$.*
(2) *$\mathcal{F}(R) = (F_\alpha(R))_{\alpha \in \Gamma}$ is an algebra and a coalgebra filtration of R.*
(3) *$\operatorname{gr}(R) = \bigoplus_{\alpha \in \Gamma} \operatorname{gr}(R)(\alpha)$ is a Γ-graded Hopf algebra in ${}^H_H\mathcal{YD}$ which is generated as an algebra by the subspaces $\operatorname{gr}(R)(\alpha_i)$, $i \in I$, and for all $i \in I$, $\operatorname{gr}(R)(\alpha_i) \cong M_i$ in ${}^H_H\mathcal{YD}$.*

PROOF. (1) and the first part of (2) are obvious. To prove that $\mathcal{F}(R)$ is a coalgebra filtration, we show by induction on $n \geq 1$ that for all $i_1, \dots, i_n \in I$, $x_{i_1} \in M_{i_1}, \dots, x_{i_n} \in M_{i_n}$, and $\alpha = \alpha_{i_1} + \dots + \alpha_{i_n}$,

$$\Delta_R(x_{i_1} \cdots x_{i_n}) \subseteq \sum_{\beta + \gamma \leq \alpha} F_\beta(R) \otimes F_\gamma(R).$$

This is clear for $n = 1$, since x_{i_1} is primitive. Assume by induction that

$$\Delta_R(x_{i_2} \cdots x_{i_n}) \subseteq \sum_{\beta + \gamma \leq \alpha'} F_\beta(R) \otimes F_\gamma(R),$$

where $\alpha' = \alpha_{i_2} + \dots + \alpha_{i_n}$. Then

$$\begin{aligned} \Delta_R(x_{i_1} \cdots x_{i_n}) &\in (x_{i_1} \otimes 1 + 1 \otimes x_{i_1}) \sum_{\beta + \gamma \leq \alpha'} F_\beta(R) \otimes F_\gamma(R) \\ &= \sum_{\beta + \gamma \leq \alpha'} (x_{i_1} F_\beta(R) \otimes F_\gamma(R) + x_{i_1 (-1)} \cdot F_\beta(R) \otimes x_{i_1 (0)} F_\gamma(R)), \end{aligned}$$

and the claim follows.

(3) follows from Corollary 5.2.18. Note that

$$F_{\alpha_i}(R) = \Bbbk + M_i + M_{i+1} + \dots + M_\theta, \quad F_{<\alpha_i}(R) = \Bbbk + M_{i+1} + \dots + M_\theta.$$

□

5.3. The coradical filtration

We use the notation of $U^\perp$ and $X^\perp$ from Remark 2.2.6, where V is a vector space, and $U \subseteq V$ and $X \subseteq V^*$ are subspaces. By definition, $X^\perp$ is the kernel of the map $\rho_X : V \to X^*$, $v \mapsto (f \mapsto f(v))$.

LEMMA 5.3.1. *Let V, W be vector spaces, and let $X \subseteq V^*$, $Y \subseteq W^*$ be subspaces. Identify $V^* \otimes W^*$ with a subspace of $(V \otimes W)^*$ via the canonical monomorphism of Lemma 2.2.3. Then $X \otimes Y \subseteq (V \otimes W)^*$, and*

$$(X \otimes Y)^\perp = X^\perp \otimes W + V \otimes Y^\perp$$

in $V \otimes W$.

PROOF. Note that

$$\rho_{X\otimes Y} = (V \otimes W \xrightarrow{\rho_X \otimes \rho_Y} X^* \otimes Y^* \subseteq (X \otimes Y)^*).$$

Hence $(X \otimes Y)^\perp = \ker(\rho_{X\otimes Y}) = X^\perp \otimes W + V \otimes Y^\perp$ by Lemma 1.1.11. □

LEMMA 5.3.2. *Let C be a coalgebra, and let I_n, $n \geq 1$, be ideals of C^* with*

$$\cdots \subseteq I_{n+1} \subseteq I_n \subseteq \cdots \subseteq I_1 \subseteq C^*.$$

Define

$$F_0(C) = I_1^\perp \subseteq F_1(C) = I_2^\perp \subseteq \cdots \subseteq F_n(C) = I_{n+1}^\perp \subseteq \cdots \subseteq C.$$

Assume that $I_i I_j \subseteq I_{i+j}$ for all $i, j \geq 1$. Then

$$\Delta(F_n(C)) \subseteq \sum_{i=0}^{n} F_i(C) \otimes F_{n-i}(C) \text{ for all } n \geq 0.$$

PROOF. Let $I_0 = C^*$. Then $I_i I_j \subseteq I_{i+j}$ for all $i, j \geq 0$ by assumption. Let $n \geq 0$, $0 \leq i \leq n+1$, $f \in I_i$, and $g \in I_{n+1-i}$, and $c \in F_n(C)$. Then $fg \in I_{n+1}$ and $0 = (fg)(c) = f(c_{(1)})g(c_{(2)})$. Hence

$$\Delta(c) \in (I_i \otimes I_{n+1-i})^\perp = I_i^\perp \otimes C + C \otimes I_{n+1-i}^\perp$$

by Lemma 5.3.1. Let $F_{-1}(C) = 0$. We conclude that

$$\Delta(F_n(C)) \subseteq \bigcap_{i=0}^{n+1} (F_{i-1}(C) \otimes C + C \otimes F_{n-i}(C)),$$

and hence $\Delta(F_n(C)) \subseteq \sum_{i=0}^{n} F_i(C) \otimes F_{n-i}(C)$ by Lemma 5.2.16. □

DEFINITION 5.3.3. Let C be a coalgebra. The **coradical** $\mathrm{Corad}(C)$ is the sum of all simple subcoalgebras of C. One says that C is **cosemisimple** if $C = \mathrm{Corad}(C)$.

PROPOSITION 5.3.4. *Let C be a coalgebra, and $(C_i)_{i\in I}$ a family of subcoalgebras of C.*

1. *Let $D \subseteq C$ be a simple subcoalgebra. If $D \subseteq \sum_{i\in I} C_i$, then $D \subseteq C_i$ for some $i \in I$.*
2. *Assume that $(C_i)_{i\in I}$ is a family of pairwise different simple subcoalgebras. Then $\sum_{i\in I} C_i = \bigoplus_{i\in I} C_i$.*
3. *Let $D \subseteq C$ be a subcoalgebra, and assume that $C = \bigoplus_{i\in I} C_i$. Then $D = \bigoplus_{i\in I}(D \cap C_i)$.*

Proof. (1) By Theorem 2.1.3, simple subcoalgebras are finite-dimensional. Hence we may assume that I is finite and C is finite-dimensional. Then it suffices to prove the claim for $I = \{1, 2\}$. So assume that $D \subseteq C_1 + C_2$ and $D \not\subseteq C_1$. Then there exist $f \in (C_1 + C_2)^*$, $d \in D$ such that $f|C_1 = 0$, $f(d) \neq 0$. Then $0 \neq d_{(1)} f(d_{(2)}) \in C_2$ since $\Delta(D) \subseteq C_1 \otimes C_1 + C_2 \otimes C_2$. Thus the coalgebra $D \cap C_2$ is non-zero and hence $D \subseteq C_2$ by simplicity of D.

(2) Assume that $\sum_{i \in I} C_i$ is not direct. Then there exists $j \in I$ such that $C_j \cap \sum_{i \in I \setminus \{j\}} C_i \neq 0$. Then $C_j \subseteq \sum_{i \in I \setminus \{j\}} C_i$ by simplicity of C_j. Hence $C_j \subset C_i$ for some $i \neq j$ by (1), a contradiction to the simplicity of C_i.

(3) Again we may assume that C is finite-dimensional. Then the claim follows by duality, since the ideals in a direct product of algebras are direct products of ideals. Here is a more direct argument. Let $x = \sum_{i \in I} x_i \in D$, where $x_i \in C_i$ for all $i \in I$ and I is finite. We have to show that $x_i \in D$ for all $i \in I$. Let $i \in I$ and let $f_i \in C^*$ such that $f_i|C_i = \varepsilon$ and $f_i|C_j = 0$ for all $j \neq i$. Then $x_i = x_{(1)} f_i(x_{(2)}) \in D$. □

The following corollary justifies the definition of cosemisimplicity.

Corollary 5.3.5. *Let C be a coalgebra, and $\mathcal{M}$ the set of its simple subcoalgebras.*

(1) $\mathrm{Corad}(C) = \bigoplus_{S \in \mathcal{M}} S$.
(2) *Let $D \subseteq C$ be a subcoalgebra. Then* $\mathrm{Corad}(D) = D \cap \mathrm{Corad}(C)$.

Proof. This is a consequence of Proposition 5.3.4(2) and (3). □

It follows from Proposition 5.3.4(2) that group-like elements in a coalgebra are linearly independent, since they span one-dimensional subcoalgebras. Thus we have given another proof of Proposition 1.1.6.

Remark 5.3.6. We recall some standard properties of the **Jacobson radical** $\mathrm{Rad}(R)$ of a ring R.

(1) [**Lam91**, (4.5)] By definition, $\mathrm{Rad}(R)$ is the intersection of the maximal left ideals of R. Then $\mathrm{Rad}(R)$ is the intersection of the maximal right ideals of R.
(2) [**Lam91**, Ex. 4.10] Let $\varphi : R \to S$ be a surjective ring homomorphism. Then preimages of maximal left ideals of S are maximal left ideals of R and hence $\varphi(\mathrm{Rad}(R)) \subseteq \mathrm{Rad}(S)$.
(3) [**Lam91**, (4.6),(4.14),(3.5)] Assume that R is a finite-dimensional algebra. Then $R/\mathrm{Rad}(R)$ is a semisimple algebra. Hence it follows from the theorem of Wedderburn-Artin that $\mathrm{Rad}(R)$ is the intersection of the maximal (two-sided) ideals of R. In particular, if R is finite-dimensional and simple, then $\mathrm{Rad}(R) = 0$.
(4) [**Lam91**, (4.5)] $\mathrm{Rad}(R)$ is the largest ideal I of R such that $1 - r$ is invertible for all $r \in I$.
(5) [**Lam91**, (4.12)] If R is a finite-dimensional algebra, then $\mathrm{Rad}(R)$ is the largest nilpotent ideal of R.

Proposition 5.3.7. *Let C be a coalgebra. Then* $\mathrm{Corad}(C)^\perp = \mathrm{Rad}(C^*)$.

Proof. Let $f \in \mathrm{Rad}(C^*)$. Let $D \subseteq C$ be a simple subcoalgebra. Then Corollary 2.2.8 implies that D^* is a simple algebra. By Remark 5.3.6(2), the image of f under the restriction map $C^* \to D^*$ is contained in $\mathrm{Rad}(D^*)$, and $\mathrm{Rad}(D^*) = 0$ by Remark 5.3.6(3). Hence $f \in \mathrm{Corad}(C)^\perp$.

Conversely, let $f \in \mathrm{Corad}(C)^\perp$. Since $\mathrm{Corad}(C)^\perp$ is an ideal of C^*, by Remark 5.3.6(4) it is enough to show that $\varepsilon - f$ is invertible in C^*. Let $D \subseteq C$ be a finite-dimensional subcoalgebra. It follows from Corollary 2.2.8 and Remark 5.3.6(3) that $\mathrm{Corad}(D)^\perp = \mathrm{Rad}(D^*)$. Hence the image f_D of f under the restriction map $C^* \to D^*$ is contained in $\mathrm{Corad}(D)^\perp = \mathrm{Rad}(D^*)$, and $\varepsilon - f_D$ is invertible by Remark 5.3.6(4). Then by Corollary 2.1.4, $\varepsilon - f$ is invertible in C^*. $\square$

THEOREM 5.3.8. *Let C be a coalgebra, $C_0 \subseteq C$ a subcoalgebra with canonical map $\pi : C \to C/C_0$ be the canonical map. Let $I = C_0^\perp$, $C_n = (I^{n+1})^\perp$ for all $n \geq 1$, and $\mathcal{F}(C) = (C_n)_{n\geq 0}$.*

(1) (a) *For all $n \geq 0$, $C_n \subseteq C_{n+1}$ and $\Delta(C_n) \subseteq \sum_{i=0}^n C_i \otimes C_{n-i}$.*
(b) *If $\mathrm{Corad}(C) \subseteq C_0$, then $\mathcal{F}(C)$ is a coalgebra filtration of C.*
(c) *For all $1 \leq i \leq n$, $C_n = \Delta^{-1}(C_{i-1} \otimes C + C \otimes C_{n-i})$.*
(d) *For all $n \geq 1$,*

$$C_n = \ker(C \xrightarrow{\Delta^n} C^{\otimes(n+1)} \xrightarrow{\pi^{\otimes(n+1)}} (C/C_0)^{\otimes(n+1)}).$$

(2) *Assume that C is a bialgebra, $C_0 \subseteq C$ is a subbialgebra, and assume that $\mathrm{Corad}(C) \subseteq C_0$. Then $\mathcal{F}(C)$ is a bialgebra filtration of C. If C_0 is a Hopf algebra, then C is a Hopf algebra, and $\mathcal{F}(C)$ is a Hopf algebra filtration of C.*

PROOF. (1)(a) By Remark 2.2.6(1), $C_0 = I^\perp$. Thus Lemma 5.3.2 yields that

$$\Delta(C_n) \subseteq \sum_{i=0}^n C_i \otimes C_{n-i} \quad \text{for all } n \geq 0.$$

(1)(b) Assume that $\mathrm{Corad}(C) \subseteq C_0$. By (1)(a) and by Theorem 2.1.3 it is enough to show that any finite-dimensional subcoalgebra $D \subseteq C$ is contained in C_n for some $n \geq 0$. For a finite-dimensional subcoalgebra D, the restriction map $\pi : C^* \to D^*$ is a surjective algebra map. Let $J = \pi(I)$. By Proposition 5.3.7,

$$I = C_0^\perp \subseteq \mathrm{Corad}(C)^\perp = \mathrm{Rad}(C^*),$$

and hence $J \subseteq \mathrm{Rad}(D^*)$. Moreover, for all $d \in D$ and $n \geq 0$, $d \in C_n$ if and only if $d \in (J^{n+1})^\perp$ in D. Since D^* is a finite-dimensional algebra, its radical is nilpotent by Remark 5.3.6(5). Hence $J^{n+1} = 0$ for some $n \geq 0$, and $D \subseteq C_n$.

(1)(c) Since $(I^i)^\perp \otimes C + C \otimes (I^{n+1-i})^\perp = (I^i \otimes I^{n+1-i})^\perp$ by Lemma 5.3.1, we conclude that $\Delta^{-1}(C_{i-1} \otimes C + C \otimes C_{n-i}) = (I^i I^{n+1-i})^\perp = C_n$.

(1)(d) By definition, $I = C_0^\perp = \mathrm{im}((C/C_0)^* \xrightarrow{\pi^*} C^*)$. Hence

$$\begin{aligned}(I^{n+1})^\perp &= \{x \in C \mid f_1\pi(x_{(1)}) \cdots f_{n+1}\pi(x_{(n+1)}) = 0, f_1, \dots, f_{n+1} \in (C/C_0)^*\} \\ &= \ker(C \xrightarrow{\Delta^n} C^{\otimes(n+1)} \xrightarrow{\pi^{\otimes(n+1)}} (C/C_0)^{\otimes(n+1)}).\end{aligned}$$

(2) To show that $C_iC_{n-i} \subseteq C_n$ for all for all $n \geq 0$, $0 \leq i \leq n$, we proceed by induction on n. If $n = 0$, then $C_0C_0 \subseteq C_0$ by assumption. Let $n \geq 1$. For all $0 \leq i \leq n$ and $f \in I$, $g \in I^n$ we have to show that $(fg)(C_iC_{n-i}) = 0$. By (1)(a),

$$\Delta(C_i) \subseteq \sum_{k=0}^i C_k \otimes C_{i-k}, \quad \Delta(C_{n-i}) \subseteq \sum_{l=0}^{n-i} C_l \otimes C_{n-i-l}.$$

Hence

$$(fg)(C_iC_{n-i}) \subseteq \sum_{\substack{0\le k\le i\\ 0\le l\le n-i}} f(C_kC_l)g(C_{i-k}C_{n-i-l}).$$

Let $0 \le k \le i$, $0 \le l \le n-i$. Then $f(C_kC_l)g(C_{i-k}C_{n-i-l}) = 0$. Indeed, if $k+l=0$, then $k=0$ and $l=0$ and $f(C_0C_0)=0$, since $f \in I$. If $k+l>0$, then $g(C_{i-k}C_{n-i-l})=0$ by induction, since $i-k+n-i-l = n-k-l < n$ and $g \in I^n$.

If C_0 is a Hopf algebra, then the restriction of the identity map id_C to C_0 is invertible, hence C has an antipode $\mathcal{S}$ by Proposition 5.2.9(2), and $\mathcal{S}(C_0) \subseteq C_0$. Since $\mathcal{S}$ is a coalgebra anti-homomorphism, (1)(b) implies that

$$\Delta(\mathcal{S}(C_n)) \subseteq \sum_{i+j=n} \mathcal{S}(C_i) \otimes \mathcal{S}(C_j) \subseteq \mathcal{S}(C_0) \otimes C + C \otimes \mathcal{S}(C_{n-1})$$

for all $n \ge 0$. Hence it follows from (1)(c) by induction on n that $\mathcal{S}$ is a filtered map. $\square$

Definition 5.3.9. Let C be a coalgebra. For all $n \ge 0$ let

$$C_n = (\mathrm{Rad}(C^*)^{n+1})^{\perp}.$$

Then $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C$ is called the **coradical filtration** of C. We define $\mathrm{gr}\, C = \bigoplus_{n\ge 0} C_n/C_{n-1}$.

The coradical filtration is a coalgebra filtration of C by Theorem 5.3.8 with $C_0 = \mathrm{Corad}(C)$, hence $C_0^{\perp} = \mathrm{Rad}(C^*)$ by Proposition 5.3.7. By Theorem 5.3.8(1), the coradical filtration can be defined inductively by

$$C_0 = \mathrm{Corad}(C), \quad C_n = \Delta^{-1}(C_0 \otimes C + C \otimes C_{n-1}) \tag{5.3.1}$$

for all $n \ge 1$.

Corollary 5.3.10. *Let A be an algebra, C a coalgebra with coradical C_0, and $f : C \to A$ a linear map. Then f is convolution invertible if its restriction to C_0 is convolution invertible in* $\mathrm{Hom}(C_0, A)$.

Proof. This follows from Proposition 5.2.9(2) and the existence of the coradical filtration. $\square$

Definition 5.3.11. An $\mathbb{N}_0$-graded coalgebra $C = \bigoplus_{n\ge 0} C(n)$ is called **coradically graded** if the coradical filtration $(C_n)_{n\ge 0}$ of C is given by

$$C_n = \bigoplus_{i=0}^{n} C(i)$$

for all $n \ge 0$.

Corollary 5.3.12. *Let A be a bialgebra such that $H =$ $\mathrm{Corad}(A)$ *is a subbialgebra of A. Then* $\mathrm{gr}\, A$ *with respect to the coradical filtration is a coradically graded bialgebra. If H is a Hopf algebra, then* $\mathrm{gr}\, A$ *is a Hopf algebra. If H is a Hopf algebra with bijective antipode, then* $\mathrm{gr}\, A$ *is a Hopf algebra with bijective antipode.*

Proof. By Theorem 5.3.8(2) with $C_0 = \mathrm{Corad}(A)$, the coradical filtration of A is a bialgebra filtration. Thus $\mathrm{gr}\, A$ is a bialgebra by Proposition 5.2.17. The remaining claims follow from Corollary 5.2.11 applied to $\mathrm{filt}(\mathrm{gr}\, A)$. $\square$

PROPOSITION 5.3.13. *Let* $C = \bigoplus_{n\geq 0} C(n)$ *be an* $\mathbb{N}_0$*-graded coalgebra. Assume that* $C(0)$ *is cosemisimple. Then the following are equivalent.*

(1) C *is coradically graded.*
(2) *For all* $n \geq 2$, $\Delta_{1,n-1} : C(n) \to C(1) \otimes C(n-1)$ *is injective.*

PROOF. We denote the coradical filtration of C by $(C_n)_{n\geq 0}$.

(1) $\Rightarrow$ (2): Let $0 \neq x \in C(n)$, $n \geq 2$. Then $x \notin C_{n-1} = \bigoplus_{i=0}^{n-1} C(i)$, since C is coradically graded. Hence $\Delta_{1,n-1}(x) \neq 0$ by (5.3.1), since

$$\Delta(x) \in \bigoplus_{i=0}^{n} C(i) \otimes C(n-i) \subseteq C_0 \otimes C + C(1) \otimes C(n-1) + C \otimes C_{n-2}.$$

(2) $\Rightarrow$ (1): The natural filtration

$$C(0) \subseteq C(0) \oplus C(1) \subseteq C(0) \oplus C(1) \oplus C(2) \subseteq \cdots$$

is a coalgebra filtration. Hence $C_0 \subseteq C(0)$ by Proposition 5.2.4. Since $C(0)$ is cosemisimple, it follows that $C_0 = C(0)$.

Let $n \geq 1$. The inclusion $C(n) \subseteq C_n$ follows easily by induction, since

$$\Delta(C(n)) \subseteq \bigoplus_{i=0}^{n} C(i) \otimes C(n-i) \subseteq C(0) \otimes C + C \otimes \Big(\bigoplus_{i=0}^{n-1} C(i)\Big).$$

Hence $\bigoplus_{i=0}^{n} C(i) \subseteq C_n$. We prove equality by induction on $n \geq 0$. Suppose there are integers $n \geq 1$, $m > n$ and elements $x_i \in C(i)$, $0 \leq i \leq m$, with $x = \sum_{i=0}^{m} x_i \in C_n$. Then $\Delta(x) \in C_0 \otimes C + C \otimes C_{n-1}$ by (5.3.1). By induction, $C_{n-1} = \bigoplus_{i=0}^{n-1} C(i)$. Hence $\Delta_{1,m-1}(x) = 0$. Then $\Delta_{1,m-1}(x_m) = 0$, and $x_m = 0$ by (2). □

Recall from Proposition 1.3.14 that (2) in Proposition 5.3.13 is equivalent to the injectivity of $\Delta_{i,j}$ for all $i, j \geq 0$.

COROLLARY 5.3.14. *Let* C *be a connected* $\mathbb{N}_0$*-graded coalgebra. Then the following are equivalent.*

(1) C *is strictly graded.*
(2) C *is coradically graded.*

PROOF. This follows from Proposition 5.3.13 and Proposition 1.3.14. □

PROPOSITION 5.3.15. *Let* C *be a coalgebra. Then* $\operatorname{gr} C$ *is coradically graded.*

PROOF. By definition, C_0 is cosemisimple. By Proposition 5.3.13 it is enough to prove that $\Delta_{1,n-1}$ for $\operatorname{gr} C$ is injective for all $n \geq 2$. We choose subspaces $X_n \subseteq C$, $n \geq 1$, with $C_n = C_{n-1} \oplus X_n$ for all $n \geq 1$. Then

$$C_1 \otimes C_{n-1} = C_0 \otimes C_{n-1} + X_1 \otimes X_{n-1} + X_1 \otimes C_{n-2}$$

for all $n \geq 2$. Hence, by (1.3.3),

$$\begin{aligned}\Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i} &\subseteq C_0 \otimes C_n + C_1 \otimes C_{n-1} + C \otimes C_{n-2}\\ &\subseteq C_0 \otimes C + X_1 \otimes X_{n-1} + C \otimes C_{n-2}.\end{aligned}$$

Since $\Delta^{-1}(C_0 \otimes C + C \otimes C_{n-2}) = C_{n-1}$, the map

$$\Delta' : C_n/C_{n-1} \to (X_1 \otimes X_{n-1} + C_0 \otimes C + C \otimes C_{n-2})/(C_0 \otimes C + C \otimes C_{n-2})$$

induced by Δ is injective. Thus $\Delta_{1,n-1}$ is injective. □

COROLLARY 5.3.16. *Let A be a Hopf algebra with coradical filtration $(A_n)_{n\geq 0}$ and let $H = A_0$. Assume that H is a Hopf subalgebra of A with bijective antipode. Let $\pi : \operatorname{gr} A \to H$ be the canonical graded projection, that is, $\pi(x) = 0$ for all $x \in \operatorname{gr} A(n)$, $n \geq 1$, and $\pi|H = \mathrm{id}_H$. Define $R = \operatorname{gr} A^{\mathrm{co}\, H}$ with respect to π, and $R(n) = R \cap \operatorname{gr} A(n)$ for $n \geq 0$.*

(1) *R is an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ with grading $(R(n))_{n\geq 0}$. The map*

$$R\#H \to \operatorname{gr} A,\ r\#h \mapsto rh,$$

is an isomorphism of $\mathbb{N}_0$-graded Hopf algebras, where the $\mathbb{N}_0$-grading of $R\#H$ is given by $(R\#H)(n) = R(n) \otimes H$ for all $n \geq 0$.

(2) *R is strictly graded.*

(3) *R is generated as an algebra by $R(1)$ if and only if A is generated as an algebra by A_1.*

PROOF. (1) follows from Corollary 4.3.7.

(2) By Corollary 5.3.12, $\operatorname{gr} A$ is a coradically graded Hopf algebra with bijective antipode. For all $n \geq 2$, let

$$\Delta^A_{1,n-1} : A(n) \to A(1) \otimes A(n-1),\ \Delta^R_{1,n-1} : R(n) \to R(1) \otimes R(n-1)$$

be the $(1, n-1)$-th component of the comultiplications of A and R. The maps $\Delta^A_{1,n-1}$ are injective by Proposition 5.3.13. Let $\varphi : A \otimes R \to A \otimes R$ be the isomorphism given by $\varphi(a \otimes x) = a\mathcal{S}(x_{(-1)}) \otimes x_{(0)}$ for all $a \in A$, $x \in R$. Then for all $x \in R$, $h \in H$,

$$\varphi\Delta_A(x) = \varphi(x^{(1)}x^{(2)}{}_{(-1)} \otimes x^{(2)}{}_{(0)}) = x^{(1)} \otimes x^{(2)}.$$

From these formulas it follows for all $n \geq 2$ that $\Delta^R_{1,n-1}$ is injective, since $\Delta^A_{1,n-1}$ is injective. Hence R is strictly graded by Proposition 5.3.13.

(3) follows from (1), since A is generated by A_1 if and only if $\operatorname{gr} A$ is generated by $A_0 \oplus A_1/A_0$. □

REMARK 5.3.17. A Hopf algebra H is called cosemisimple if it is cosemisimple as a coalgebra. It is known that the antipode of a cosemisimple Hopf algebra is bijective, see [**Lar71**, Thm. 3.3]. Therefore in Corollary 5.3.16 the assumption on the bijectivity of the antipode of the Hopf algebra H can be dropped by Corollary 5.3.5. A similar remark applies to Corollary 5.3.12 and to Proposition 5.3.18.

PROPOSITION 5.3.18. *Let H be a cosemisimple Hopf algebra with bijective antipode. Let R be an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$, and assume that R is strictly graded. Then the $\mathbb{N}_0$-graded Hopf algebra $R\#H$ is coradically graded and has a bijective antipode.*

PROOF. Let $A = R\#H$. By Corollary 4.3.5 and the definition of the multiplication, comultiplication and antipode of A, A is an $\mathbb{N}_0$-graded Hopf algebra with $A(0) = 1\#H$. Hence $A(0)$ is cosemisimple. By Corollary 5.2.12, A is a Hopf algebra with bijective antipode. By the definition of $\varDelta_A$ and the rule for ϑ in Corollary 4.3.1(2)(a),

$$\begin{aligned}(\vartheta \otimes \mathrm{id}_A)\varDelta_A(xh) &= \vartheta(x^{(1)} x^{(2)}{}_{(-1)} h_{(1)}) \otimes x^{(2)}{}_{(0)} h_{(2)} \\ &= x^{(1)} \otimes x^{(2)} h\end{aligned}$$

for all $x \in R$, $h \in H$. Thus for all $n \geq 2$, $\varDelta^A_{1,n-1}$ is injective, since $\varDelta^R_{1,n-1}$ is injective. Hence A is coradically graded by Proposition 5.3.13. □

5.4. Pointed coalgebras

By Definition 1.3.3, a coalgebra C is called pointed if every simple subcoalgebra of C is one-dimensional. A bialgebra or a Hopf algebra is pointed if its underlying coalgebra is pointed.

If C is pointed, then

$$G(C) \to \{D \subseteq C \mid D \text{ is a simple subcoalgebra}\},\ g \mapsto \Bbbk g,$$

is bijective.

PROPOSITION 5.4.1. *Let C be a coalgebra. Then the following are equivalent.*

(1) *C is pointed.*
(2) $\mathrm{Corad}(C) = \bigoplus_{g\in G(C)} \Bbbk g$.
(3) *Any simple right C-comodule is one-dimensional.*
(4) *Any simple left C-comodule is one-dimensional.*

PROOF. (1) implies (2) by Proposition 5.3.4(2), and (2) implies (1) by Proposition 5.3.4(1). We prove that (1) and (3) are equivalent.

Assume (1) and let (V, δ) be a simple right C-comodule. Then $C(V)$ is simple by Proposition 2.2.13(1), and $C(V) = \Bbbk g$ for some $g \in G(C)$ by (1). Thus $\delta(v) = v \otimes g$ for all $v \in V$, hence (3) holds.

Assume now (3) and let D be a simple subcoalgebra of C. Let V be a simple right subcomodule of D. By (3), there exist $0 \neq v \in V$ and $d \in D$ with $\Delta(v) = v\otimes d$, $\varepsilon(d) = 1$. Since $v \in D$, it follows from the axiom of the counit that $v = \varepsilon(v)d$. Hence $D = \Bbbk d$ since D is a simple coalgebra, that is, (1) holds.

The equivalence of (3) and (4) follows from the category isomorphism between right comodules over C and left comodules over C^{cop} and from the equivalence of (1) and (3). □

If C is pointed, then it follows from Proposition 5.4.1 that there is a bijection from $G(C)$ to the set of isomorphism classes of simple left (respectively right) C-comodules mapping a group-like element g to the isomorphism class of a simple one-dimensional comodule V with basis v and $\delta_V(v) = g\otimes v$ (respectively $\delta_V(v) = v\otimes g$).

PROPOSITION 5.4.2. (1) *Let C be a coalgebra with a coalgebra filtration $\mathcal{F}(C) = (F_n(C))_{n\geq 0}$. If $F_0(C)$ is a pointed coalgebra, then C is pointed, and* $\mathrm{Corad}(C) \subseteq F_0(C)$.
(2) *A connected $\mathbb{N}_0$-graded coalgebra is pointed.*
(3) *Let C, D be coalgebras, $f : C \to D$ a coalgebra map, and assume that C is pointed. Then f is a filtered map with respect to the coradical filtrations.*
(4) *Let C, D be coalgebras, and $\pi : C \to D$ a surjective coalgebra homomorphism. Then* $\mathrm{Corad}(D) \subseteq \pi(\mathrm{Corad}(C))$. *If C is pointed, then D is pointed, and $G(C) \to G(D)$, $g \mapsto \pi(g)$, is surjective.*

PROOF. (1) follows from Proposition 5.2.4, and (2) is a special case of (1).

(3) Since C is pointed, $f(C_0)$ is a sum of one- or zero-dimensional subcoalgebras of D, hence contained in D_0. Using the inductive definition of the coradical filtration, it follows easily by induction on n that $f(C_n) \subseteq D_n$ for all $n \geq 0$.

(4) Let $(C_n)_{n\geq 0}$, $C_0 = \operatorname{Corad}(C)$, be the coradical filtration of C.

Then $\mathcal{F}(D) = (F_n(D))_{n\geq 0}$, $F_n(D) = \pi(C_n)$ for all $n \geq 0$, is a coalgebra filtration of D. Hence $D_0 = \operatorname{Corad}(D) \subseteq \pi(C_0)$ by Proposition 5.2.4. The rest is clear. □

COROLLARY 5.4.3. (1) *A pointed bialgebra H is a Hopf algebra if and only if $G(H)$ is a group (under multiplication in H).*

(2) *Let H be a pointed Hopf algebra with antipode $\mathcal{S}$. Then $\mathcal{S}$ is bijective, and $\mathcal{S}(I) = I$ for any Hopf ideal $I \subseteq H$.*

PROOF. (1) If H is a Hopf algebra, then the monoid $G(H)$ is a group by Proposition 2.4.1. Let H be a pointed bialgebra with coradical filtration $(H_n)_{n\geq 0}$. Then $H_0 = \Bbbk G(H)$ by Proposition 5.4.1. Hence, by Corollary 5.2.11, if $G(H)$ is a group then H is a Hopf algebra with bijective antipode.

(2) By Proposition 5.4.2, H/I is a pointed Hopf algebra with antipode induced by the antipode of H. By the proof of (1), the antipodes of H and of H/I are bijective. This implies that $\mathcal{S}(I) = I$. □

COROLLARY 5.4.4. *Let H be a bialgebra, J an index set, and for all $j \in J$, $x_j \in H$, $g_j, h_j \in G(H)$ with $g_j^{-1}, h_j^{-1} \in G(H)$ and*

$$\Delta(x_j) = g_j \otimes x_j + x_j \otimes h_j.$$

Let G be a subgroup of $G(H)$ containing all g_j, h_j with $j \in J$. Assume that H is generated as an algebra by G and by the elements x_j, $j \in J$. Then H is a pointed Hopf algebra, and $G = G(H)$.

PROOF. Let $X = \{x_j \mid j \in J\}$. For all $n \geq 0$, let $F_n(H)$ be the $\Bbbk$-span of all monomials $a_1 a_2 \cdots a_m$, where $m \geq 0$, $a_i \in X \cup G$ for all $1 \leq i \leq m$, and such that $a_i \in X$ for at most n indices i. Then $(F_n(H))_{n\geq 0}$ is a coalgebra filtration of H with $F_0(H) = \Bbbk G$. Hence, by Propositions 5.4.2(1) and 5.4.1, H is pointed, $G = G(H)$, and H is a Hopf algebra by Corollary 5.4.3. □

Corollary 5.4.4 shows that for any Lie algebra $\mathfrak{g}$, the universal enveloping algebra $U(\mathfrak{g})$ is a pointed Hopf algebra, and 1 is the only group-like element of $U(\mathfrak{g})$. We will use the same argument for the deformed universal enveloping algebras in Chapter 8.

We extend Proposition 1.3.10 from strictly graded coalgebras to pointed coalgebras. By a theorem of Heyneman and Radford the next theorem holds for arbitrary coalgebras. We will only need the pointed version.

THEOREM 5.4.5. *Let C, D be coalgebras, and $f : C \to D$ a coalgebra map. Assume that C is pointed and the restriction of f to C_1 (defined by the coradical filtration) is injective. Then $\operatorname{gr} f : \operatorname{gr} C \to \operatorname{gr} D$ and f are injective.*

PROOF. Since C is pointed, f induces a coalgebra map $\operatorname{gr} f : \operatorname{gr} C \to \operatorname{gr} D$ by Proposition 5.4.2. By Lemma 5.2.14, it is enough to show that $\operatorname{gr} f$ is injective. Corollary 5.3.5(2) implies that $(\operatorname{gr} f)_1 : (\operatorname{gr} C)_1 \to (\operatorname{gr} D)_1$ is injective. Hence we can assume that C, D are $\mathbb{N}_0$-graded coalgebras, f is graded, and C is coradically graded. In this case the theorem follows easily by induction from Proposition 5.3.13(2). □

COROLLARY 5.4.6. *Let C be a pointed coalgebra. Let $(C_n)_{n\geq 0}$ and $((\operatorname{gr} C)_n)_{n\geq 0}$ be the coradical filtrations of C and $\operatorname{gr} C$, respectively. Then for all $n \geq 0$, the inclusion $C_n \subseteq C$ defines an injective coalgebra map $\operatorname{gr} C_n \to (\operatorname{gr} C)_n = \oplus_{k=0}^n (\operatorname{gr} C)(k)$.*

PROOF. By Theorem 5.4.5, the induced map $\operatorname{gr} C_n \to \operatorname{gr} C$ is injective. For all $k > n$, the map $(\operatorname{gr} C_n)(k) \to (\operatorname{gr} C)(k)$ is zero. Hence $(\operatorname{gr} C_n)(k) = 0$ for all $k > n$, and the corollary follows from Proposition 5.3.15. □

We next give a short proof of a weak version of the Taft-Wilson theorem. This weak version is enough to prove Corollary 5.4.9 and 5.4.16 below which are useful to lift information of $\operatorname{gr} A$ to A for pointed Hopf algebras A with abelian group $G(A)$.

THEOREM 5.4.7. *Let A be a pointed Hopf algebra, and let $(A_n)_{n\geq 0}$ be its coradical filtration.*

(1) *For all $n \geq 1$, $A_n = \sum_{g,h\in G(A)} A_n(g,h)$, where for all $g, h \in G(H)$,*

$$A_n(g,h) = \{x \in A_n \mid \Delta(x) = g \otimes x + x \otimes h + u \text{ with } u \in A_{n-1} \otimes A_{n-1}\}$$

(2) *$A_1 = \Bbbk G(A) + \sum_{g,h\in G(A)} P_{g,h}(A)$.*

PROOF. Let $\pi : \operatorname{gr} A \to A(0)$ be the projection onto elements of degree 0, and $R = (\operatorname{gr} A)^{\operatorname{co} H}$ with respect to π. Let $G = G(A)$. By Proposition 5.3.15 and Corollary 5.3.16, $\operatorname{gr} A$ is coradically graded, R is strictly graded, and the multiplication map $R \# \Bbbk G \to \operatorname{gr} A$ is a graded isomorphism.

(a) We first prove the theorem for R, that is,

(1)' for all $n \geq 1$,

$$R_n = \{x \in R_n \mid \Delta_R(x) = 1 \otimes x + x \otimes 1 + u, \text{ where } u \in R_{n-1} \otimes R_{n-1}\},$$

(2)' $R_1 = \Bbbk 1 \oplus P(R)$.

By Corollary 5.3.14, R is coradically graded. Hence (2)' follows immediately. To prove (1)', let $x \in R(n)$, $n \geq 1$. Then by Lemma 1.3.6(2),

$$\Delta_R(x) \in 1 \otimes x + x \otimes 1 + \bigoplus_{i=1}^{n-1} R(i) \otimes R(n-i).$$

This proves (1)', since R is coradically graded.

(b) Now we prove the theorem for $\operatorname{gr} A$.

(1) Let $n \geq 1$ and $x \in (\operatorname{gr} A)_n$. To prove that $x \in \sum_{g,h\in G} A_n(g,h)$, it suffices to assume that $x \in (\operatorname{gr} A)(n)$, $x = r\#h$, where $h \in H$, $r \in R(n)$ with $\delta(r) = g \otimes r$, $g \in G$. Here $\delta : R \to H \otimes R$ is the H-coaction of R. Then

$$\Delta_R(r) \in 1 \otimes r + r \otimes 1 + \oplus_{i=1}^{n-1} R(i) \otimes R(n-i),$$
$$\Delta_{\operatorname{gr} A}(x) \in gh \otimes x + x \otimes h + \oplus_{i=1}^{n-1} R(i)\#\Bbbk G \otimes R(n-i)\#\Bbbk G.$$

Hence $x \in (\operatorname{gr} A)_n(gh, h)$, and (1) follows.

If $n = 1$, then $\Delta_{\operatorname{gr} A}(x) = gh \otimes x + x \otimes h$. This proves (2).

(c) Now we prove the theorem for A.

(1) Let $x \in A_n$, $n \geq 1$, and $\overline{x}$ the residue class of x in A_n/A_{n-1}. By (b), we can assume that

$$\Delta_{\operatorname{gr} A}(\overline{x}) \in g \otimes \overline{x} + \overline{x} \otimes h + \oplus_{i=1}^{n-1} (\operatorname{gr} A)(i) \otimes (\operatorname{gr} A)(n-i).$$

Hence there are $a, b \in A_{n-1}$ and $v \in A_{n-1} \otimes A_{n-1}$ with

$$\begin{aligned}\Delta(x) &= g \otimes (x+a) + (x+b) \otimes h + v \\ &= g \otimes x + x \otimes h + (g \otimes a + b \otimes h + v) \\ &\in g \otimes x + x \otimes h + A_{n-1} \otimes A_{n-1}.\end{aligned}$$

(2) For all $g, h \in G$, let

$$A_{g,h} = \{x \in A \mid \Delta(x) = g \otimes x + x \otimes h + u, \text{ where } u \in \Bbbk G \otimes \Bbbk G\}.$$

By (1) we know that $A_1 = \sum_{g,h \in G} A_{g,h}$. So we have to show that

$$A_{g,h} = P_{g,h}(A) + \Bbbk G.$$

The inclusion $\supseteq$ is trivial. To prove the other inclusion, let $x \in A_{g,h}$, and let $u \in \Bbbk G \otimes \Bbbk G$ with

$$\Delta(x) = g \otimes x + x \otimes h + u.$$

It follows from coassociativity of Δ that

$$u \otimes h + (\Delta \otimes \mathrm{id})(u) = g \otimes u + (\mathrm{id} \otimes \Delta)(u). \tag{5.4.1}$$

Let $u = \sum_{a,b \in G} \alpha_{a,b} a \otimes b$, where $\alpha_{a,b} \in \Bbbk$ for all $a, b \in G$. By subtracting $\sum_{a \in G} \alpha_{a,a} a$ from x, we may assume that $\alpha_{a,a} = 0$ for all $a \in G \setminus \{g, h\}$. Now we express all terms in (5.4.1) as a linear combination of monomials $a \otimes b \otimes c$ with $a, b, c \in G$. For any $b \in G$ with $g \neq b \neq h$, by looking at the coefficients of $g \otimes b \otimes b$ and $b \otimes b \otimes h$ in (5.4.1) it follows that $\alpha_{g,b} = \alpha_{b,h} = 0$. Then for any $a, b \in G$ with $g \neq a \neq b \neq h$, by looking at the coefficient of $a \otimes b \otimes b$ we obtain that $\alpha_{a,b} = 0$. Finally, if $g \neq h$, then by looking at the coefficients of $g \otimes g \otimes g$ and $h \otimes h \otimes h$ we get $\alpha_{g,g} = \alpha_{h,h} = 0$. It follows that $u = \alpha_{g,h} g \otimes h$. Then $x + \alpha_{g,h} h \in P_{g,h}(A)$, which proves (2). $\square$

Let A be a pointed Hopf algebra, and $G = G(A)$. Note that the coradical filtration $(A_n)_{n \geq 0}$ of A is stable under the adjoint action of G, since the subspaces $A_n \subseteq A$, $n \geq 0$, are left and right $\Bbbk G$-submodules of A by restriction. This follows from their inductive definition in (5.3.1). Assume that G is abelian. Then G acts on $P_{g,h}(A)$ by the adjoint action. For all $g, h \in G$, and $\chi \in \widehat{G}$, let

$$P^{\chi}_{g,h}(A) = \{a \in P_{g,h}(A) \mid uau^{-1} = \chi(u)a \text{ for all } u \in G\}.$$

LEMMA 5.4.8. *Let A be a finite-dimensional pointed Hopf algebra. Assume that $G = G(A)$ is abelian, and* $\mathrm{char}(\Bbbk) = 0$. *Then for all $g, h \in G$, $P^{\varepsilon}_{g,h}(A) \subseteq \Bbbk G$.*

PROOF. We may assume that $h = 1$, since $P^{\varepsilon}_{gh,h}(A) = P^{\varepsilon}_{g,1}(A)h$. Choose $a \in P^{\varepsilon}_{g,1}(A)$ with canonical image $\overline{a}$ in A_1/A_0. Since $\Delta_{\mathrm{gr}\, A}(\overline{a}) = g \otimes \overline{a} + \overline{a} \otimes 1$, we see that $\overline{a} \in V = R(1) \in {}^G_G\mathcal{YD}$, where the Yetter-Drinfeld structure is given by

$$\delta(\overline{a}) = g \otimes \overline{a}, \quad u \cdot \overline{a} = u\overline{a}u^{-1} = \overline{a} \text{ for all } u \in G,$$

since $a \in P^{\varepsilon}_{g,1}(A)$. Thus $\overline{a} \in V^{\varepsilon}_g$. Now finite-dimensionality of A implies finite-dimensionality of $\mathcal{B}(V)$ by Example 1.10.2. Therefore $\overline{a} = 0$ and $a \in \Bbbk G$. $\square$

COROLLARY 5.4.9. *Let A be a finite-dimensional pointed Hopf algebra. Assume that $G = G(A)$ is abelian, and that $\Bbbk$ is algebraically closed, and* $\operatorname{char}(\Bbbk) = 0$. *Let $(A_n)_{n\geq 0}$ be the coradical filtration of A. Then*

$$A_1 = A_0 \oplus \bigoplus_{\substack{(g,h,\chi)\\ g,h\in G, \varepsilon\neq\chi\in\widehat{G}}} P^{\chi}_{g,h}(A),$$

and for all $g, h \in G$, $\varepsilon \neq \chi \in \widehat{G}$, the canonical map $A_1 \to A_1/A_0$ induces an isomorphism $P^{\chi}_{g,h}(A) \xrightarrow{\cong} P^{\chi}_{g,h}(\operatorname{gr} A)$.

PROOF. (1) By Lemma 5.4.8, Theorem 5.4.7, and Proposition 1.4.6, it follows that

$$A_1 = A_0 \oplus \bigoplus_{\varepsilon\neq\chi\in\widehat{G}} \sum_{g,h\in G} P^{\chi}_{g,h}(A). \tag{5.4.2}$$

Let $\varepsilon \neq \chi \in \widehat{G}$. To prove that the sum $\sum_{g,h\in G} P^{\chi}_{g,h}(A)$ is direct, let for all $g, h \in G$, $a_{g,h} \in P^{\chi}_{g,h}$, and assume that $\sum_{g,h\in G} a_{g,h} = 0$. Then

$$0 = \Delta_A(\sum_{g,h\in G} a_{g,h}) = \sum_{g\in G}(g \otimes \sum_{h\in G} a_{g,h}) + \sum_{h\in G}(\sum_{g\in G} a_{g,h} \otimes h).$$

Since $A_0 \cap \sum_{g,h\in G} P^{\chi}_{g,h}(A) = 0$ by (5.4.2), we obtain that $\sum_{h\in G} a_{g,h} = 0$ for all $g \in G$, hence

$$0 = \Delta_A(\sum_{h\in G} a_{g,h}) = g \otimes \sum_{h\in G} a_{g,h} + \sum_{h\in G} a_{g,h} \otimes h = \sum_{h\in G} a_{g,h} \otimes h,$$

and $a_{g,h} = 0$ for all $g, h \in G$.

(2) By (1), the canonical map $A_1 \to A_1/A_0$ induces an isomorphism

$$\bigoplus_{\substack{(g,h,\chi)\\ g,h\in G, \varepsilon\neq\chi\in\widehat{G}}} P^{\chi}_{g,h}(A) \to A_1/A_0 = \bigoplus_{(g,h,\chi)\} g,h\in G, \varepsilon\neq\chi\in\widehat{G}} P^{\chi}_{g,h}(\operatorname{gr} A),$$

Since $\operatorname{gr} A$ is coradically graded by Proposition 5.3.15, the equality follows from (1) for $\operatorname{gr} A$ instead of A. For all $g, h \in G$, $\varepsilon \neq \chi \in \widehat{G}$, the canonical map induces a linear map $P^{\chi}_{g,h}(A) \to P^{\chi}_{g,h}(\operatorname{gr} A)$. Since the direct sum of these maps is an isomorphism, the maps $P^{\chi}_{g,h}(A) \to P^{\chi}_{g,h}(\operatorname{gr} A)$ are bijective for all g, h, χ. □

To describe a decomposition of A_1 as in Proposition 5.4.9 for certain infinite-dimensional Hopf algebras, we need some standard results on locally finite representations of abelian groups.

DEFINITION 5.4.10. Let G be an abelian group, and V a $\Bbbk G$-module. For all $\chi \in \widehat{G}$, we define

$$V^{(\chi)} = \{v \in V \mid \text{ for all } g \in G, (g - \chi(g))^s v = 0 \text{ for some } s \geq 1\}.$$

Recall that $V^{\chi} = \{v \in V \mid gv = \chi(g)v \text{ for all } g \in G\}$.

LEMMA 5.4.11. *Let G be an abelian group, V a $\Bbbk G$-module, $S, T \subseteq \widehat{G}$ subsets, and $\chi \in \widehat{G}$.*

(1) *$V^{\chi} \subseteq V^{(\chi)} \subseteq V$ are $\Bbbk G$-submodules, and $(V^{(\chi)})^{(\chi)} = V^{(\chi)}$.*
(2) *Let $\mu, \nu \in \widehat{G}$ with $\mu \neq \nu$. Then $V^{(\mu)} \cap V^{(\nu)} = 0$ and $(V^{(\mu)})^{(\nu)} = 0$.*

(3) *Let $(V_i)_{i\in I}$ be a family of $\Bbbk G$-modules. Then $(\bigoplus_{i\in I} V_i)^{(\chi)} = \bigoplus_{i\in I} V_i^{(\chi)}$.*

(4) *Let V, W be $\Bbbk G$-modules, and assume that*

$$\bigoplus_{\chi\in S} V^{(\chi)} \cong \bigoplus_{\chi\in T} W^{(\chi)} \text{ as } \Bbbk G\text{-modules,}$$

where for all $\chi \in S$, $V^{(\chi)} \neq 0$, and for all $\chi \in T$, $W^{(\chi)} \neq 0$. Then $S = T$, and $V^{(\chi)} \cong W^{(\chi)}$ as $\Bbbk G$-modules for all $\chi \in S$.

PROOF. (1) is obvious, since G is abelian.

(2) Let $x \in V^{(\mu)} \cap V^{(\nu)}$. For all $g \in G$, there is an integer $s \geq 1$ with

$$(g - \mu(g))^s x = 0, (g - \nu(g))^s x = 0, \text{ and}$$
$$(\nu(g) - \mu(g))^{2s} x = ((g - \mu(g)) - (g - \nu(g)))^{2s} x = 0.$$

Thus $x = 0$, and therefore $V^{(\mu)} \cap V^{(\nu)} = 0$. Hence $(V^{(\mu)})^{(\nu)} \subseteq V^{(\mu)} \cap V^{(\nu)} = 0$.

(3) is obvious, and (4) follows from (2) and (3). □

From now on we assume in this section that $\Bbbk$ is **algebraically closed**.

LEMMA 5.4.12. *Let G be an abelian group, and V a finite-dimensional $\Bbbk G$-module with representation $\rho : G \to \operatorname{Aut}(V)$. Then there is a basis of V such that for all $g \in G$, the representing matrix of $\rho(g)$ is upper triangular.*

PROOF. Let $g \in G$. Since $\Bbbk$ is algebraically closed, there is an eigenvalue λ of $\rho(g)$. Let $V_{g,\lambda} = \{v \in V \mid gv = \lambda v\}$. Since G is abelian, $V_{g,\lambda}$ is a G-subspace of V. If $V_{g,\lambda} = V$ for all g, λ, the lemma is obvious. Hence we may assume that $V_{g,\lambda} \subsetneq V$ for some g, λ. By induction on $\dim V$, there is a non-zero element $v_1 \in V$ such that $\Bbbk v_1$ is G-invariant. Again by induction there are elements $v_2, \dots, v_n \in V$ such that their residue classes are a basis as claimed in the lemma for $V/\Bbbk v_1$. Then the basis $v_1, \dots, v_n$ of V has the required property. □

PROPOSITION 5.4.13. *Let G be an abelian group, and V a locally finite $\Bbbk G$-module. Then $V = \bigoplus_{\chi\in\widehat{G}} V^{(\chi)}$.*

PROOF. We can assume that V is finite-dimensional. We prove the proposition by induction on the dimension of V. Let $\dim V = n \geq 1$, and assume the theorem holds for $\Bbbk G$-modules of dimension $< n$. Let $\rho : \Bbbk G \to \operatorname{End}(V)$ be the representation of G.

(1) Assume that for all $g \in G$, $\rho(g)$ has exactly one eigenvalue $\chi(g)$. By Lemma 5.4.12, there is a basis of V such that for all $g \in G$, the representing matrix $(a_{ij}(g))_{1\leq i,j\leq n}$ of $\rho(g)$ with respect to this basis is upper triagonal. Hence for all $g \in G$, $1 \leq i \leq n$, $a_{ii}(g) = \chi(g)$. This implies that $\chi(gh) = \chi(g)\chi(h)$ for all $g, h \in G$, hence $\chi \in \widehat{G}$. Moreover, $V = V^{(\chi)}$, since $\rho(g) - \chi(g)$ is nilpotent for all $g \in G$. For all $\mu \in \widehat{G}$, $\mu \neq \chi$, $V^{(\mu)} = 0$ by Lemma 5.4.11(2).

(2) Now we assume that there is an element $g \in G$ such that $\rho(g)$ has at least two eigenvalues. Let $V = \bigoplus_{i=1}^n V_i$ be the decomposition of V into generalized eigenspaces of $\rho(g)$

$$V_i = \{v \in V \mid (g - \lambda_i)^s v = 0 \text{ for some } s \geq 1\}$$

with eigenvalue λ_i, $1 \leq i \leq n$. Then $n \geq 2$. Since G is abelian, $V_i \subsetneq V$ is a $\Bbbk G$-submodule for all $1 \leq i \leq n$. By induction, the claim holds for all V_i, $1 \leq i \leq n$. The claim for V follows from Lemma 5.4.11(3). □

COROLLARY 5.4.14. *Let G be an abelian group, V a locally finite $\Bbbk G$-module, $U \subseteq V$ a $\Bbbk G$-submodule, and $S \subseteq \widehat{G}$ a subset.*

(1) *If $V = \bigoplus_{\chi \in S} V^{(\chi)}$, then $U = \bigoplus_{\chi \in S} U^{(\chi)}$.*
(2) *If $V = \bigoplus_{\chi \in S} V^{\chi}$, then $U = \bigoplus_{\chi \in S} U^{\chi}$.*

PROOF. By Proposition 5.4.13, $U = \bigoplus_{\chi \in \widehat{G}} U^{(\chi)}$. Hence (1) and (2) follow from Lemma 5.4.11. $\square$

COROLLARY 5.4.15. *Let G be an abelian group, V a locally finite $\Bbbk G$-module, $U \subseteq V$ a $\Bbbk G$-submodule, and $S, T \subseteq \widehat{G}$ disjoint subsets such that*

$$U = \bigoplus_{\chi \in S} U^{\chi}, \quad V/U = \bigoplus_{\chi \in T} (V/U)^{\chi}.$$

Then

$$V = U \oplus \bigoplus_{\chi \in T} V^{\chi}, \quad U = \bigoplus_{\chi \in S} V^{\chi}.$$

PROOF. By Proposition 5.4.13, $V = \bigoplus_{\chi \in S} V^{(\chi)} \oplus \bigoplus_{\chi \in \widehat{G} \setminus S} V^{(\chi)}$, and

$$\bigoplus_{\chi \in T} (V/U)^{\chi} = V/U \cong \bigoplus_{\chi \in S} V^{(\chi)}/U^{\chi} \oplus \bigoplus_{\chi \in \widehat{G} \setminus S} V^{(\chi)},$$

since by Lemma 5.4.11(2),(3), for all $\chi \in T$, $(V/U)^{(\chi)} = (V/U)^{\chi}$, and for all $\chi \in \widehat{G}$, $U^{(\chi)} = U^{\chi}$, if $\chi \in S$, and $U^{(\chi)} = 0$, if $\chi \notin S$. Since for all $\chi \in T$, $(V/U)^{\chi} = 0$ implies that $V^{\chi} = 0$, we may assume that $(V/U)^{\chi} \neq 0$ for all $\chi \in T$. We conclude from Lemma 5.4.11(4) that $V^{(\chi)} = U^{\chi}$ for all $\chi \in S$, $V^{(\chi)} = V^{\chi}$ for all $\chi \in T$, and $V^{(\chi)} = 0$ for all $\chi \in \widehat{G} \setminus (S \cup T)$. This proves the claim. $\square$

PROPOSITION 5.4.16. *Let $\Bbbk$ be algebraically closed, A a pointed Hopf algebra with coradical filtration $(A_n)_{n \geq 0}$, and abelian group $G = G(A)$. Let $R = (\operatorname{gr} A)^{\operatorname{co} \Bbbk G}$ with respect to the projection of $\operatorname{gr} A$ onto degree 0. Assume that $V = R(1) \in {}^{G}_{G}\mathcal{YD}$ is finite-dimensional. Then the following hold.*

(1) *A_1 is a locally finite $\Bbbk G$-module under the adjoint action.*
(2) *Assume that $V = \bigoplus_{\varepsilon \neq \chi} V^{\chi}$. Then*

$$A_1 = A_0 \oplus \bigoplus_{\substack{(g,h,\chi) \\ g,h \in G, \varepsilon \neq \chi \in \widehat{G}}} P^{\chi}_{g,h}(A),$$

and for all $g, h \in G$, $\varepsilon \neq \chi \in \widehat{G}$, the canonical map $A_1 \to A_1/A_0$ induces an isomorphism $P^{\chi}_{g,h}(A) \xrightarrow{\cong} P^{\chi}_{g,h}(\operatorname{gr} A)$.

PROOF. (1) The coradical filtration is stable under the adjoint action of G. By Theorem 5.4.7,

$$A_1 = A_0 + \sum_{g,h \in G(A)} P_{g,h}(A). \tag{5.4.3}$$

By Corollary 5.3.16, multiplication defines an isomorphism $\operatorname{gr} A \cong R \# \Bbbk G$ of $\mathbb{N}_0$-graded Hopf algebras, hence as $\Bbbk G$-modules under the adjoint action. In particular,

$$A_1/A_0 \cong V \# \Bbbk G \text{ as } \Bbbk G\text{-modules}, \tag{5.4.4}$$

where $g \cdot (v \otimes h) = g \cdot v \otimes h$ for all $g, h \in G$, $v \in V$. Hence A_1/A_0 is a locally finite $\Bbbk G$-module.

Let $g, h \in G$. Then $P_{g,h}(A)/\Bbbk(g-h)$ is embedded into A_1/A_0 as a $\Bbbk G$-module. Hence $P_{g,h}(A)/\Bbbk(g-h)$ and $P_{g,h}(A)$ are locally finite.

Then it follows from (5.4.3) that A_1 is locally finite.

(2) By (5.4.4), $A_1/A_0 = \bigoplus_{\varepsilon \neq \chi \in \widehat{G}} (A_1/A_0)^\chi$. Hence

$$A_1 = A_0 \oplus \bigoplus_{\varepsilon \neq \chi \in \widehat{G}} (A_1)^\chi, \quad A_0 = (A_1)^\varepsilon,$$

by Corollary 5.4.15 with $S = \{\varepsilon\}$, $T = \widehat{G} \setminus \{\varepsilon\}$. Then by Corollary 5.4.14(2) and (5.4.3), for all $\varepsilon \neq \chi \in \widehat{G}$, $(A_1)^\chi = \sum_{g,h \in G} P^\chi_{g,h}(A)$. The claim in (2) now follows by the same argument as in part (2) of the proof of Corollary 5.4.9. $\square$

5.5. Graded Yetter-Drinfeld modules

Let Γ be an abelian monoid, and H a Γ-graded Hopf algebra with bijective antipode.

The category $\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk$ is braided monoidal, where the braiding is the flip mapping (see Section 5.1), and H is a Hopf algebra in $\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk$. We study the Yetter-Drinfeld category ${}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$ defined in Section 3.4. An object V in this category is an object V in ${}^H_H\mathcal{YD}$ such that $V = \bigoplus_{\alpha \in \Gamma} V(\alpha)$ is a graded vector space, and the module and comodule structure maps $H \otimes V \to V$ and $V \to H \otimes V$ are graded.

If H is trivially graded, that is, $H(0) = H$ and $H(\alpha) = 0$ for all non-zero $\alpha \in \Gamma$, then an object in ${}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$ is an object in ${}^H_H\mathcal{YD}$ which is a graded vector space $V = \bigoplus_{\alpha \in \Gamma} V(\alpha)$ such that $V(\alpha) \subseteq V$ are subobjects in ${}^H_H\mathcal{YD}$ for all $\alpha \in \Gamma$, that is, $V \in \Gamma\text{-Gr}{}^H_H\mathcal{YD}$.

Lemma 5.5.1. *Let $V \in {}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$.*

(1) *Let $U \subseteq V$ be a Γ-graded subspace and a submodule and subcomodule. Then U is a subobject of V in ${}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$.*
(2) *If $U \subseteq V$ is a Γ-graded H-subcomodule, then HU is the smallest subobject of V in ${}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$ containing U.*
(3) *Assume that Γ is cancellative. If $U \subseteq V$ is a Γ-graded H-submodule, then UH^* is the smallest subobject of V in ${}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$ which contains U. Here, UH^* is the smallest H-subcomodule of V containing U.*

Proof. (1) follows from Lemma 5.1.2(1)(a) and (2)(a).

(2) Since U is a graded vector space, HU is a graded H-submodule of V. Let $\delta : V \to H \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$, be the comodule structure of V. Then for all $h \in H$ and $u \in U$, $\delta(hu) = h_{(1)}u_{(-1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)}u_{(0)}$. Hence HU is an H-subcomodule of V, and the claim follows from (1).

(3) By definition, V is a right H^*-module with $vf = f(v_{(-1)})v_{(0)}$ for all $v \in V$, $f \in H^*$. By Corollary 2.2.18, UH^* is the smallest subcomodule of V containing U. By Lemma 5.1.2(2)(b), $\bigoplus_{\alpha \in \Gamma}(UH^*) \cap V(\alpha)$ is a graded subcomodule of V, and $U \subseteq \bigoplus_{\alpha \in \Gamma}(UH^*) \cap V(\alpha)$, since U is graded. Hence $\bigoplus_{\alpha \in \Gamma}(UH^*) \cap V(\alpha) = UH^*$. For all $h \in H$, $u \in U$ and $f \in H^*$,

$$h(uf) = f(u_{(-1)})hu_{(0)} = (h_{(2)}u)(h_{(3)}f\mathcal{S}(h_{(1)})).$$

Hence UH^* is a left H-submodule of V, since $U \subseteq V$ is an H-submodule. The claim follows from (1). $\square$

The category ${}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$ is a braided monoidal category with monoidal structure and braiding as in ${}^H_H\mathcal{YD}$.

Let $\mathcal{C} = {}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$. Algebras, coalgebras, bialgebras, and Hopf algebras in $\mathcal{C}$ are called **Γ-graded algebras, coalgebras, bialgebras, and Hopf algebras in ${}^H_H\mathcal{YD}$**, respectively.

LEMMA 5.5.2. *Let R be a Γ-graded bialgebra in ${}^H_H\mathcal{YD}$, and*

$$P(R) = \{x \in R \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

Then $P(R)$ is a Γ-graded subobject of R in ${}^H_H\mathcal{YD}$.

PROOF. The maps $R \xrightarrow{\Delta} R\otimes R$ and $R \to R\otimes R$, $r \mapsto r\otimes 1 + 1\otimes r$, are Γ-graded maps in ${}^H_H\mathcal{YD}$. $\square$

Graded objects in $\mathcal{C}$ are defined as in Remark 5.1.5. The category $\mathbb{N}_0\text{-Gr}\,\mathcal{C}$ is monoidal with $(V \otimes W)(n) = \bigoplus_{i+j=n} V(i) \otimes W(j)$ for all $n \geq 0$, where $V, W \in \mathcal{C}$. It is braided with the braiding of $\mathcal{C}$, since the braiding $c_{V,W} : V \otimes W \to W \otimes V$ in $\mathcal{C}$ of graded objects is graded.

We construct the tensor algebra as an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$.

Let $V \in \mathcal{C}$. The **tensor algebra**

$$T(V) = \bigoplus_{n\geq 0} T^n(V), \;\; T^0(V) = \Bbbk,\, T^n(V) = V^{\otimes n} \text{ for all } n > 0,$$

is an $\mathbb{N}_0$-graded algebra with multiplication given by concatenation, that is, for all $i, j \geq 0$,

$$\mu_{i,j} = \mathrm{id} : T^i(V) \otimes T^j(V) \to T^{i+j}(V).$$

Then $T(V)$ is an $\mathbb{N}_0$-graded algebra in the monoidal category $\mathcal{C}$, where $T^n(V)$ is the n-fold tensor product of graded Yetter-Drinfeld modules for all $n \geq 0$. Thus action and coaction of H are defined by

$$h \cdot (v_1 \otimes \cdots \otimes v_n) = h_{(1)}v_1 \otimes \cdots \otimes h_{(n)}v_n$$
$$\delta(v_1 \otimes \cdots \otimes v_n) = v_{1(-1)} \cdots v_{n(-1)} \otimes v_{1(0)} \otimes \cdots \otimes v_{n(0)}$$

for all $h \in H$, $v_1, \dots, v_n \in V$, $n \geq 0$.

EXAMPLE 5.5.3. Let $\Gamma = \mathbb{N}_0^\theta$ as in Example 5.2.1, and let $\alpha_1, \dots, \alpha_\theta$ be the standard basis of $\mathbb{Z}^\theta$. Assume that $V = \bigoplus_{i=1}^\theta V_i$ is a direct sum decomposition in ${}^H_H\mathcal{YD}$. We define an $\mathbb{N}_0^\theta$-grading on V by setting

$$V(\alpha_i) = V_i \text{ for all } i, \text{ and } V(\alpha) = 0 \text{ for all } \alpha \in \mathbb{N}_0^\theta \setminus \{\alpha_1, \dots, \alpha_\theta\}.$$

Note that for all $n_1, \dots, n_\theta \in \mathbb{N}_0$,

$$T(V)\Big(\sum_{i=1}^\theta n_i\alpha_i\Big) \subset T^n(V), \text{ where } n = \sum_{i=1}^\theta n_i.$$

The tensor algebra has the usual universal property.

LEMMA 5.5.4. *Let V be an object and R an algebra in $\mathcal{C}$. For any morphism $f : V \to R$ in $\mathcal{C}$ there is exactly one morphism $\varphi : T(V) \to R$ of algebras in $\mathcal{C}$ such that $\varphi(v) = f(v)$ for all $v \in V$.*

PROOF. Let $\varphi(v_1 \otimes \cdots \otimes v_n) = f(v_1) \cdots f(v_n)$ for all $v_1, \ldots, v_n \in V$, $n \geq 0$. $\square$

PROPOSITION 5.5.5. *Let $V \in \mathcal{C}$.*

(1) *There exists a uniquely determined map $\Delta : T(V) \to T(V) \underline{\otimes} T(V)$ of algebras in $\mathcal{C}$ such that*

$$\Delta(v) = 1 \otimes v + v \otimes 1 \quad \textit{for all } v \in V.$$

The algebra $T(V)$ is an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$ with comultiplication Δ and counit $\pi_0^{T(V)} : T(V) \to \Bbbk$, and $\mathcal{S}(v) = -v$ for all $v \in V$, where $\mathcal{S}$ is the antipode of $T(V)$.

(2) *Let R be a bialgebra in $\mathcal{C}$ and $f : V \to P(R)$ a homomorphism in $\mathcal{C}$. Then there is exactly one map $\varphi : T(V) \to R$ of bialgebras in $\mathcal{C}$ such that $\varphi(v) = f(v)$ for all $v \in V$.*

PROOF. (1) It is clear from the universal property of the tensor algebra that Δ exists and is uniquely determined. We show that $T(V)$ with comultiplication Δ and counit $\varepsilon = \pi_0^{T(V)} : T(V) \to \Bbbk$ becomes an $\mathbb{N}_0$-graded bialgebra in $\mathcal{C}$. It is easy to see by induction that the comultiplication is $\mathbb{N}_0$-graded, since for all i, j the braiding of $T(V)$ maps $T^i(V) \otimes T^j(V)$ onto $T^j(V) \otimes T^i(V)$. By the universal property of the tensor algebra it is enough to check the axioms of coassociativity and counitarity on elements of V which is obvious, since the elements of V are primitive. Finally $T(V)$ has an antipode by Proposition 5.2.9. Then for all $v \in V$, $\mathcal{S}(v) = -v$, since v is primitive and $\varepsilon(v) = 0$ by definition.

(2) By the universal property of the tensor algebra there is exactly one map $\varphi : T(V) \to R$ of algebras in $\mathcal{C}$ with $\varphi|V = f$. By the same argument as before, φ is a coalgebra map, since it is enough to check the equalities $\Delta\varphi = (\varphi \otimes \varphi)\Delta$ and $\varepsilon\varphi = \varepsilon$ on elements of V. $\square$

We formulate a graded version of Corollary 4.3.3. Let H be a Γ-graded Hopf algebra. The category of **graded Hopf algebra triples over** H is defined as follows. The objects of this category are triples (A, π, γ), where A is a Γ-graded Hopf algebra, and $\pi : A \to H$, $\gamma : H \to A$ are Γ-graded Hopf algebra homomorphisms with $\pi\gamma = \mathrm{id}_H$. A morphism between graded triples (A, π, γ), (A', π', γ') is a Γ-graded Hopf algebra homomorphism $\Phi : A \to A'$ with $\pi'\Phi = \pi$ and $\Phi\gamma = \gamma'$.

Recall that $\mathcal{C} = {}^H_H\mathcal{YD}(\Gamma\text{-Gr}\,\mathcal{M}_\Bbbk)$.

THEOREM 5.5.6. *Let H be a Γ-graded Hopf algebra with bijective antipode.*

(1) *Let R be a Hopf algebra in $\mathcal{C}$. Then $(R\#H, \pi_R, \gamma_R)$ is a graded Hopf algebra triple over H, where the grading of $R\#H$ is the tensor product grading. Moreover, $(R\#H)^{\mathrm{co}\,H}$ is a graded subspace of $R\#H$, and*

$$R \to (R\#H)^{\mathrm{co}\,H},\ r \mapsto r\#1,$$

is an isomorphism of Hopf algebras in $\mathcal{C}$.

(2) *Let A be a Γ-graded Hopf algebra, and $\pi : A \to H$ and $\gamma : H \to A$ graded Hopf algebra homomorphisms with $\pi\gamma = \mathrm{id}_H$, and define the Hopf algebra $R\#H$ with $R = A^{\mathrm{co}\,H}$. Then R is a Hopf algebra in $\mathcal{C}$ with induced grading $R(\alpha) = R \cap A(\alpha)$ for all $\alpha \in \Gamma$, and*

$$\Phi : R\#H \to A,\ r\#h \mapsto r\gamma(h),$$

is a graded Hopf algebra isomorphism with $\pi_R = \pi\Phi$ and $\Phi\gamma_R = \gamma$.

PROOF. Adapt the proof of Corollary 4.3.3 replacing $\mathcal{M}_\Bbbk$ by ${}^{\Bbbk\Gamma}\mathcal{M}$. $\square$

5.6. Notes

5.2. A variant of Proposition 5.2.9 was formulated first in [**Tak71**, Lemma 14] for the coradical filtration.

A version of Lemma 5.2.16 was already used in [**Swe69**, Lemma 9.1.5].

5.3. We present the classical theory of the coradical filtration. See [**Swe69**], [**Mon93**], [**Rad12**] for a slightly different exposition without using properties of the Jacobson radical.

5.4. For a proof of the general case of the Theorem of Heyneman and Radford [**HR74**] see [**Mon93**, Theorem 5.3.1], and in generalized form [**Rad12**, Theorem 4.7.4].

Our proof of Theorem 5.4.7 follows [**AS00b**]. The proof of Corollary 5.4.16 is inspired by [**AS04**, Lemma 4.4].

Let C be a pointed coalgebra, $G = G(C)$, and $(C_n)_{n\geq 0}$ the coradical filtration of C. For all $g,h \in G$, let $P'_{g,h}(C) \subseteq P_{g,h}(C)$ be a vector subspace such that $P_{g,h}(C) = \Bbbk(g-h) \oplus P'_{g,h}(C)$. The Theorem of Taft and Wilson [**TW74**] says that Theorem 5.4.7(1) holds for C, and $C_1 = C_0 \bigoplus_{g,h\in G} P'_{g,h}(C)$. See [**Mon93**, Theorem 5.4.1] and [**Rad12**, Theorem 4.3.2] for a proof. In the situation of Proposition 5.4.16 we have shown that $P'_{g,h}(A) = \bigoplus_{\varepsilon\neq\chi} P^{\chi}_{g,h}$ is a possible choice for the Theorem of Taft and Wilson.

In Lemma 5.4.11, Proposition 5.4.13 and Corollary 5.4.15 we prove some standard results on locally finite representations of abelian groups following the presentation of Dixmier in [**Dix96**], Theorem 1.3.19, for nilpotent Lie algebras.

CHAPTER 6

Braided structures

In Chapter 1 we defined the Nichols algebra of a braided vector space where the braiding comes from a Yetter-Drinfeld module structure. It is possible to develop the basic theory of braided Hopf algebras and Nichols algebras for arbitrary braided vector spaces. This will be done in the following two chapters.

In Section 6.3 we study quotient theory of pointed braided Hopf algebras, in particular of pointed Hopf algebras where the braiding is the twist map. In Corollary 6.3.10 we describe the Hilbert series of a quotient; in Section 7.1, this leads to a formula which compares the Hilbert series of the Nichols algebra with the Hilbert series of the tensor algebra.

However, more sophisticated tools like Cartan graphs and root systems can not be discussed in this context, and therefore in later chapters we will turn back again to categories of Yetter-Drinfeld modules.

6.1. Braided vector spaces

Let (V,c) be a braided vector space. Recall from Definition 1.7.9 that we have defined linear maps $c_{m,n} \in \mathrm{Aut}(V^{\otimes m} \otimes V^{\otimes n})$ for all $m,n \geq 0$. In particular, by Corollary 1.7.10,

$$c_{1,n} = c_n c_{n-1} \cdots c_1, \tag{6.1.1}$$

$$c_{n,1} = c_1 c_2 \cdots c_n. \tag{6.1.2}$$

If V is an object of a braided strict monoidal category, the braid group acts on tensor powers of V as in Lemma 1.7.5.

Lemma 6.1.1. *Let V be an object in a braided strict monoidal category with braiding $c = c_{V,V} : V \otimes V \to V \otimes V$. Then for all $m,n \geq 1$,*

$$c_{V^{\otimes m}, V^{\otimes n}} = c_{m,n}.$$

Proof. See the proof of Lemma 1.7.11. $\square$

Definition 6.1.2. Let (V,c) be a braided vector space, and $m,n \geq 0$. A linear map $f : V^{\otimes m} \to V^{\otimes n}$ **commutes with the braiding of** V if

$$(f \otimes \mathrm{id}_V)c_{1,m} = c_{1,n}(\mathrm{id}_V \otimes f), \ (\mathrm{id}_V \otimes f)c_{m,1} = c_{n,1}(f \otimes \mathrm{id}_V), \tag{6.1.3}$$

that is, if the diagrams

$$\begin{tikzcd} V \otimes V^{\otimes m} \arrow[r, "c_{1,m}"] \arrow[d, "\mathrm{id}_V \otimes f"] & V^{\otimes m} \otimes V \arrow[d, "f \otimes \mathrm{id}_V"] \\ V \otimes V^{\otimes n} \arrow[r, "c_{1,n}"] & V^{\otimes n} \otimes V \end{tikzcd} \qquad \begin{tikzcd} V^{\otimes m} \otimes V \arrow[r, "c_{m,1}"] \arrow[d, "f \otimes \mathrm{id}_V"] & V \otimes V^{\otimes m} \arrow[d, "\mathrm{id}_V \otimes f"] \\ V^{\otimes n} \otimes V \arrow[r, "c_{n,1}"] & V \otimes V^{\otimes n} \end{tikzcd}$$

commute.

Let $V \in {}^H_H\mathcal{YD}$ be a Yetter-Drinfeld module over some Hopf algebra H with bijective antipode. Then any linear map $f : V^{\otimes m} \to V^{\otimes n}$ which is a morphism of Yetter-Drinfeld modules commutes with the braiding, since the braiding is a functorial isomorphism. Thus equations (6.1.3) are a substitute for the functoriality of the braiding.

Equations (6.1.3) can be described by the pictures (3.2.12) and (3.2.13), where $h = f$, and $X_i = V = Y_j$ for all i, j.

LEMMA 6.1.3. *Let (V, c) be a braided vector space.*

(1) *The set of linear maps between tensor powers of V which commute with the braiding of V is closed under composition, addition, scalar multiplication and tensor products.*
(2) *All left multiplications with elements of $\Bbbk\mathbb{B}_n$ on $V^{\otimes n}$, $n \geq 1$, commute with the braiding of V.*
(3) *If $f : V^{\otimes m} \to V^{\otimes n}$, $m, n \geq 0$, is a linear map commuting with the braiding of V, then the following diagrams commute for all $r \geq 0$:*

$$\begin{tikzcd} V^{\otimes r} \otimes V^{\otimes m} \arrow[r, "c_{r,m}"] \arrow[d, "\mathrm{id}_{V^{\otimes r}} \otimes f"] & V^{\otimes m} \otimes V^{\otimes r} \arrow[d, "f \otimes \mathrm{id}_{V^{\otimes r}}"] \\ V^{\otimes r} \otimes V^{\otimes n} \arrow[r, "c_{r,n}"] & V^{\otimes n} \otimes V^{\otimes r} \end{tikzcd} \qquad \begin{tikzcd} V^{\otimes m} \otimes V^{\otimes r} \arrow[r, "c_{m,r}"] \arrow[d, "f \otimes \mathrm{id}_{V^{\otimes r}}"] & V^{\otimes r} \otimes V^{\otimes m} \arrow[d, "\mathrm{id}_{V^{\otimes r}} \otimes f"] \\ V^{\otimes n} \otimes V^{\otimes r} \arrow[r, "c_{n,r}"] & V^{\otimes r} \otimes V^{\otimes n} \end{tikzcd}$$

(4) *If $f : V^{\otimes p} \to V^{\otimes q}$, $g : V^{\otimes r} \to V^{\otimes s}$, $p, q, r, s \geq 0$, are linear maps commuting with the braiding of V, then the following diagram commutes:*

$$\begin{tikzcd} V^{\otimes p} \otimes V^{\otimes r} \arrow[r, "c_{p,r}"] \arrow[d, "f \otimes g"] & V^{\otimes r} \otimes V^{\otimes p} \arrow[d, "g \otimes f"] \\ V^{\otimes q} \otimes V^{\otimes s} \arrow[r, "c_{q,s}"] & V^{\otimes s} \otimes V^{\otimes q} \end{tikzcd}$$

PROOF. (1) is obvious for composition, addition and scalar multiplication of linear maps, and follows for tensor products from Corollary 1.7.10(4),(5).

(2) follows from (1), since the equation $c_1c_2c_1 = c_2c_1c_2$ implies that c and hence each $c_i \in \mathrm{End}(V^{\otimes n})$, $1 \leq i \leq n-1$, commutes with the braiding.

(3) We prove the commutativity of the first diagram by induction on r. The commutativity of the second diagram follows in the same way. For $r = 0$ the first diagram is trivially commutative, and for $r = 1$ it is commutative, since f commutes with the braiding. In the diagram

$$\begin{tikzcd} V \otimes V^{\otimes r} \otimes V^{\otimes m} \arrow[r, "\mathrm{id}_V \otimes c_{r,m}"] \arrow[d, "\mathrm{id}_V \otimes \mathrm{id}_{V^{\otimes r}} \otimes f"] & V \otimes V^{\otimes m} \otimes V^{\otimes r} \arrow[r, "c_{1,m}"] \arrow[d, "\mathrm{id}_V \otimes f \otimes \mathrm{id}_{V^{\otimes r}}"] & V^{\otimes m} \otimes V \otimes V^{\otimes r} \arrow[d, "f \otimes \mathrm{id}_V \otimes \mathrm{id}_{V^{\otimes r}}"] \\ V \otimes V^{\otimes r} \otimes V^{\otimes n} \arrow[r, "\mathrm{id}_V \otimes c_{r,n}"] & V \otimes V^{\otimes n} \otimes V^{\otimes r} \arrow[r, "c_{1,n}"] & V^{\otimes n} \otimes V \otimes V^{\otimes r} \end{tikzcd}$$

the first square commutes by induction, and the second square commutes, since f commutes with the braiding. The claim follows since by Corollary 1.7.10(4), the composition of the upper and lower horizontal maps is $c_{r+1,m}$ and $c_{r+1,n}$, respectively.

(4) follows from (3) by using that $f \otimes g = (\mathrm{id} \otimes g)(f \otimes \mathrm{id})$. $\square$

REMARK 6.1.4. Let (V, c) be a braided vector space. For clarity, we denote by $V^{\otimes n}, n \geq 0$, a vector space satisfying the universal property with respect to multilinear maps. Let $\mathcal{C}(V)$ be the strict monoidal category with objects $V^{\otimes n}$, $n \geq 0$, and linear maps as morphisms. The monoidal structure is the functor

$$\mathcal{C}(V) \times \mathcal{C}(V) \to \mathcal{C}(V), \quad (V^{\otimes m}, V^{\otimes n}) \mapsto V^{\otimes (m+n)},$$

where morphism (f, g) are mapped onto $f \otimes g$. Let $\mathcal{C}(V, c)$ be the strict monoidal subcategory of $\mathcal{C}(V)$ with the same objects $V^{\otimes m}$, $m \geq 0$, and where the morphisms are the linear maps $f : V^{\otimes m} \to V^{\otimes n}$, $m, n \geq 0$, which commute with c. By Lemma 6.1.3(1), $\mathcal{C}(V, c)$ is a monoidal subcategory of $\mathcal{C}(V)$. For all $m, n \geq 0$, let

$$c_{(V^{\otimes m}, V^{\otimes n})} = c_{m,n} : V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}.$$

By Lemma 6.1.3(4), $(c_{X,Y})_{X,Y \in \mathcal{C}(V,c)}$ is a natural isomorphism. Hence $\mathcal{C}(V, c)$ is a braided strict monoidal category by Corollary 1.7.10(4) and (5).

DEFINITION 6.1.5. Let (V, c) be a braided vector space, and $U \subseteq V$ a subspace. Then

(1) U is a **categorical subspace** of V if

$$c(U \otimes V) = V \otimes U \text{ and } c(V \otimes U) = U \otimes V,$$

(2) U is a **braided subspace** of V if $c(U \otimes U) = U \otimes U$,
(3) V/U is a **braided quotient space** of V if

$$c(U \otimes V + V \otimes U) = U \otimes V + V \otimes U.$$

A subspace $U \subseteq V$ of a Yetter-Drinfeld module $V \in {}^H_H\mathcal{YD}$ with braiding $c_{V,V}$ is categorical if it is a subobject in ${}^H_H\mathcal{YD}$.

REMARK 6.1.6. Let (V, c) be a braided vector space.

A subspace $U \subseteq V$ is categorical if and only if c induces bijections

$$\overline{c} : V/U \otimes V \to V \otimes V/U, \ \ \overline{c} : V \otimes V/U \to V/U \otimes V.$$

If U_1 and U_2 are categorical subspaces of V, then $U_1 \cap U_2 \subseteq V$ is categorical, and $c(U_1 \otimes U_2) = U_2 \otimes U_1$.

If $U \subseteq V$ is a categorical subspace, then U is a braided subspace, and V/U is a braided quotient space.

If $U \subseteq V$ is a subspace, then V/U is a braided quotient space if and only if there exists a (uniquely determined) braiding

$$\overline{c} : V/U \otimes V/U \to V/U \otimes V/U$$

such that the quotient map $\pi : V \to V/U$ is a map of braided vector spaces.

LEMMA 6.1.7. *Let (V, c) be a braided vector space, and $f : V^{\otimes m} \to V^{\otimes n}$ with $m, n \geq 0$ be a linear map commuting with the braiding of V. Then* $\ker(f) \subseteq V^{\otimes m}$ *and* $\mathrm{im}(f) \subseteq V^{\otimes n}$ *are categorical subspaces.*

PROOF. By taking the kernels of the vertical maps in the commutative diagrams in Lemma 6.1.3(3) with $r = m$, we see that

$$c_{m,m}(V^{\otimes m} \otimes \ker(f)) = \ker(f) \otimes V^{\otimes m}, \ c_{m,m}(\ker(f) \otimes V^{\otimes m}) = V^{\otimes m} \otimes \ker(f).$$

By taking the images of the same vertical maps with $r = n$ we see that $\mathrm{im}(f)$ is a categorical subspace of $V^{\otimes n}$. $\square$

DEFINITION 6.1.8. Let Γ be a set. A **Γ-graded braided vector space** is a braided vector space (V,c) which is a Γ-graded vector space $V=\bigoplus_{\gamma\in\Gamma}V(\gamma)$ such that $c(V(\gamma)\otimes V(\lambda))=V(\lambda)\otimes V(\gamma)$ for all $\gamma,\lambda\in\Gamma$.

LEMMA 6.1.9. *Let Γ be a set, (V,c) a Γ-graded braided vector space, and $\gamma\in\Gamma$.*

(1) *$V(\gamma)\subseteq V$ is a categorical subspace.*

(2) *The linear map $V\xrightarrow{\pi_\gamma}V(\gamma)\subseteq V$ commutes with c, where π_γ is the projection map.*

PROOF. (1) is obvious.

(2) The diagrams in Definition 6.1.2 with $f=(V\xrightarrow{\pi_\gamma}V(\gamma)\subseteq V)$ commute, since by (1) they commute on $V\otimes V(\lambda)$ and $V(\lambda)\otimes V$ for all $\lambda\in\Gamma$. □

COROLLARY 6.1.10. *Let (V,c) be a braided vector space. Then $(T(V),c^{T(V)})$ is an $\mathbb{N}_0$-graded braided vector space, where by definition for all $m,n\geq 0$, the restriction of $c^{T(V)}$ to $V^{\otimes m}\otimes V^{\otimes n}$ is $c_{m,n}$.*

PROOF. We have to show that for all $r,s,t\geq 0$,

$$c_{s,t}c_{r,t}{}^{\uparrow s}c_{r,s}=c_{r,s}{}^{\uparrow t}c_{r,t}c_{s,t}{}^{\uparrow r}. \tag{6.1.4}$$

By Lemma 6.1.3(2), $c_{s,t}$ commutes with the braiding of V. Hence the first diagram in Lemma 6.1.3(3) with $f=c_{s,t}$ commutes. This proves (6.1.4), since by Corollary 1.7.10(5), $c_{r,t}{}^{\uparrow s}c_{r,s}=c_{r,s+t}=c_{r,s}{}^{\uparrow t}c_{r,t}$. □

6.2. Braided algebras, coalgebras and bialgebras

We discuss algebra and coalgebra structures on a braided vector space.

Recall from Remark 6.1.4 the definition of the braided strict monoidal category $\mathcal{C}(V,c)$ for a braided vector space (V,c). The results of Chapter 3 apply to $\mathcal{C}(V,c)$.

DEFINITION 6.2.1. A **braided algebra** is a quadruple $A=(A,\mu,\eta,c)$ such that (A,c) is a braided vector space and (A,μ,η) is an algebra in $\mathcal{C}(A,c)$ (that is, (A,μ,η) is an algebra and μ and η commute with c). A **braided coalgebra** is a quadruple $C=(C,\Delta,\varepsilon,c)$ such that (C,c) is a braided vector space and (C,Δ,ε) is a coalgebra in $\mathcal{C}(C,c)$.

A **homomorphism** or a **map of braided algebras (coalgebras)** $A\to B$ is a braided linear map $(A,c)\to(B,d)$ which is also an algebra (coalgebra) map.

REMARK 6.2.2. Let (A,μ,η,c) be a braided algebra. Then (A,μ,η) is an algebra in the category $\mathcal{C}(A,c)$ by definition.

(1) By Proposition 3.2.4, for any three algebras B,C,D in $\mathcal{C}(A,c)$, the tensor product of B and C, denoted by $B\underline{\otimes}C$, is an algebra in $\mathcal{C}(A,c)$, and the algebra structures on $(B\underline{\otimes}C)\underline{\otimes}D$ and on $B\underline{\otimes}(C\underline{\otimes}D)$ coincide. In particular, for any $m\geq 1$ the m-fold tensor product $(A^{\otimes m},\mu_{A^{\otimes m}},\eta_{A^{\otimes m}})$ of the algebra A is uniquely determined as an algebra in $\mathcal{C}(A,c)$.

A similar remark holds for braided coalgebras using Proposition 3.2.5.

(2) By Lemma 6.1.3, compositions and tensor products of algebra morphisms in $\mathcal{C}(A,c)$ are algebra morphisms in $\mathcal{C}(A,c)$. (They commute with c by definition of a morphism in $\mathcal{C}(A,c)$.)

PROPOSITION 6.2.3. *Let $\varphi:A\to B$ be a map of braided (co)algebras. Then for any $m\geq 1$, $\varphi^{\otimes m}:A^{\otimes m}\to B^{\otimes m}$ is a map of (co)algebras.*

PROOF. Assume that $A = (A, \mu_A, \eta_A, c)$ and $B = (B, \mu_B, \eta_B, d)$ are braided algebras. We prove the claim by induction on m. For $m = 1$ the claim is trivial. Assume that $m \geq 2$. Then

$$\begin{aligned}\varphi^{\otimes m}\mu_{A^{\otimes m}} &= \varphi^{\otimes m}(\mu_{A^{\otimes m-1}} \otimes \mu_A)c^{\uparrow m-1}_{1,m-1} \\ &= (\mu_{B^{\otimes m-1}} \otimes \mu_B)\varphi^{\otimes 2m}c^{\uparrow m-1}_{1,m-1} \\ &= (\mu_{B^{\otimes m-1}} \otimes \mu_B)d^{\uparrow m-1}_{1,m-1}\varphi^{\otimes 2m} = \mu_{B^{\otimes m}}(\varphi^{\otimes m} \otimes \varphi^{\otimes m}),\end{aligned}$$

where the first equation holds by definition of the tensor product, the second follows from induction hypothesis and since φ is an algebra map, the third follows since φ is a braided linear map, and the last one holds again by definition of $\mu_{B^{\otimes m}}$. Similarly, $\varphi^{\otimes m}\eta_{A^{\otimes m}} = \eta_{B^{\otimes m}}$. Hence $\varphi^{\otimes m}$ is an algebra map.

For coalgebra maps the proof is analogous. □

DEFINITION 6.2.4. Let $A = (A, \mu, \eta, \Delta, \varepsilon, c)$ be a 6-tuple such that (A, μ, η, c) is a braided algebra and $(A, \Delta, \varepsilon, c)$ is a braided coalgebra. Then A is a **braided bialgebra** if $\Delta : A \to A\underline{\otimes}A$ and $\varepsilon : A \to \Bbbk$ are algebra maps.

A **braided Hopf algebra** is a braided bialgebra with an antipode, that is, a convolution inverse of the identity map.

A **homomorphism** or a **map of braided bialgebras (respectively Hopf algebras)** is a homomorphism of braided algebras and of braided coalgebras.

A braided bialgebra A is a bialgebra in $\mathcal{C}(A, c)$.

Since the antipode $\mathcal{S}$ of a braided Hopf algebra A is convolution inverse to the identity, $\mathcal{S}$ is a unitary and augmented map, that is, $\mathcal{S}(1) = 1$, and $\varepsilon(\mathcal{S}(x)) = \varepsilon(x)$ for all $x \in A$.

LEMMA 6.2.5. *Let A be a braided algebra, coalgebra or bialgebra, and I an ideal, coideal or bi-ideal of A such that A/I is a braided quotient space. Then A/I is a braided algebra, coalgebra or bialgebra, such that the quotient map $A \to A/I$ is a homomorphism of braided algebras, coalgebras or bialgebras.*

PROOF. Obviously, the structure maps of A/I commute with the quotient braiding of A/I. □

PROPOSITION 6.2.6. *Let A be a braided Hopf algebra with antipode $\mathcal{S}$ and braiding c.*

(1) *$\mathcal{S}$ commutes with c, in particular, $(\mathcal{S} \otimes \mathcal{S})c = c(\mathcal{S} \otimes \mathcal{S})$.*
(2) *$\mathcal{S}\mu = \mu c(\mathcal{S} \otimes \mathcal{S})$.*
(3) *$\Delta\mathcal{S} = (\mathcal{S} \otimes \mathcal{S})c\Delta$.*

PROOF. (1) Since $\mathcal{S}$ is the convolution inverse of the identity in $\mathrm{Hom}(A, A)$, by Proposition 1.2.19, $\mathcal{S}$ is the composition of the maps

$$A \xrightarrow{\eta \otimes \mathrm{id}_A} A \otimes A \xrightarrow{\mathcal{G}^{-1}} A \otimes A \xrightarrow{\mathrm{id}_A \otimes \varepsilon} A, \tag{6.2.1}$$

where $\mathcal{G}$ is the isomorphism $\mathcal{G} = (\mu \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta)$. Hence $\mathcal{S}$ commutes with the braiding of A, since $\eta \otimes \mathrm{id}_A$, $\mathcal{G}^{-1}$ and $\mathrm{id}_A \otimes \varepsilon$ all commute with the braiding of A.

(2) and (3) follow from Proposition 3.2.12 and (1). □

DEFINITION 6.2.7. Let $A = (A, \mu, \eta, \mathit{\Delta}, \varepsilon, c)$ be a braided bialgebra. Let

$$A^{\mathrm{op}} = (A, \mu c^{-1}, \eta, \mathit{\Delta}, \varepsilon, c^{-1}),$$
$$A^{\mathrm{cop}} = (A, \mu, \eta, c^{-1}\mathit{\Delta}, \varepsilon, c^{-1}).$$

PROPOSITION 6.2.8. *Let H be a braided bialgebra.*

(1) *H^{op} and H^{cop} are braided bialgebras.*
(2) *If H is a braided Hopf algebra, then the following are equivalent.*
   (a) *The antipode of H is bijective.*
   (b) *H^{op} is a braided Hopf algebra.*
   (c) *H^{cop} is a braided Hopf algebra.*
(3) *If H is a braided Hopf algebra with bijective antipode, then*
   (a) *H^{op} and H^{cop} are braided Hopf algebras with antipode $\mathcal{S}^{-1}$.*
   (b) *$\mathcal{S} : H^{\mathrm{op}} \to H^{\mathrm{cop}}$ is an isomorphism of braided Hopf algebras.*

PROOF. (1) follows from Proposition 3.2.15. Since $\mathcal{S}$ commutes with the braiding by Proposition 6.2.6, (2) and (3) follow from Proposition 3.2.15 and Corollary 3.2.16. □

REMARK 6.2.9. The crucial axiom for a braided bialgebra is the equality

$$\mathit{\Delta}\mu = (\mu \otimes \mu)c_2(\mathit{\Delta} \otimes \mathit{\Delta}), \tag{6.2.2}$$

which can be written as

$$\mathit{\Delta}(xy) = x^{(1)}c(x^{(2)} \otimes y^{(1)})y^{(2)} \tag{6.2.3}$$

for all $x, y \in A$, where $A \otimes A$ is viewed as a left and a right A-module by multiplication on the left and the right tensorand, respectively.

LEMMA 6.2.10. (1) *Let A be a braided algebra, and $V_1, \dots, V_n$, $n \geq 2$, categorical subspaces. Then $V_1 \cdots V_n$ is a categorical subspace of A.*
(2) *Let A be a braided bialgebra, and I a categorical coideal of A. Then AI, IA and AIA are categorical coideals of A.*
(3) *Let C be a braided coalgebra, and assume that the braided vector space C is $\mathbb{N}_0$-graded. Then for all $n \geq 1$, $I_C(n) = \ker(\Delta_{1^n}) \subseteq C$ is a categorical subspace.*

PROOF. (1) By induction, it is enough to consider the case when $n = 2$. Since $\mu : A \otimes A \to A$ commutes with the braiding of A, the image of the categorical subspace $V_1 \otimes V_2$ is categorical.

(2) follows easily form (6.2.3) and (1).

(3) By Lemma 6.1.9, the map $f = (C \xrightarrow{\pi_n} C(n) \subseteq C)$ commutes with c, and $C(n) \subseteq C$ is categorical. Hence it follows from Lemma 6.1.3 and Lemma 6.1.7 that the map $f^{\otimes n}\Delta^{\otimes(n-1)} : C \to C^{\otimes n}$ commutes with c, and that the subspace $\ker(\Delta_{1^n}) = C(n) \cap \ker(f^{\otimes n}\Delta^{\otimes(n-1)})$ of C is categorical. □

PROPOSITION 6.2.11. (1) *Let A be a braided Hopf algebra. Then the braiding c of A is determined by the multiplication, the comultiplication, and the antipode of A. More precisely,*

$$c(x \otimes y) = \mathcal{S}(x^{(1)})\mathit{\Delta}(x^{(2)}y^{(1)})\mathcal{S}(y^{(2)}) \tag{6.2.4}$$

for all $x, y \in A$.

(2) *Let A, B be braided Hopf algebras with antipodes $\mathcal{S}_A$, $\mathcal{S}_B$. Let $\varphi : A \to B$ be a morphism of algebras and of coalgebras. Then $\mathcal{S}_B\varphi = \varphi\mathcal{S}_A$, and φ is a map of braided vector spaces.*

PROOF. (1) The formula for the braiding follows from (6.2.3).

(2) The equality $\mathcal{S}_B\varphi = \varphi\mathcal{S}_A$ is shown as for usual Hopf algebras in Proposition 1.2.17(2). Since φ commutes with the multiplication, the comultiplication and the antipodes of A and B, it is braided linear by (1). □

We note an application of (6.2.4) to the group-like elements $G(A)$ of a braided Hopf algebra.

PROPOSITION 6.2.12. *Let A be a braided Hopf algebra. Then the following are equivalent:*

(1) *$G(A)$ is multiplicatively closed.*
(2) *$G(A)$ is a subgroup of the group of invertible elements of A.*
(3) *For all $g, h \in G(A)$, $c(g \otimes h) = h \otimes g$.*

PROOF. (1) $\Rightarrow$ (3). Let $g, h \in G(A)$. Then $gh \in G(A)$, and by (6.2.4),

$$c(g \otimes h) = g^{-1}(gh \otimes gh)h^{-1} = h \otimes g.$$

(3) $\Rightarrow$ (2). Let $g, h \in G(A)$. Then $\Delta(gh) = gc(g \otimes h)h = gh \otimes gh$ by (6.2.3). By Proposition 6.2.6(3), $\Delta(\mathcal{S}(g)) = (\mathcal{S} \otimes \mathcal{S})(g \otimes g) = \mathcal{S}(g) \otimes \mathcal{S}(g)$. Hence $g^{-1} \in G(A)$, since $\mathcal{S}(g) = g^{-1}$.

(2) $\Rightarrow$ (1) is trivial. □

PROPOSITION 6.2.13. *Let A be a braided pointed Hopf algebra with antipode $\mathcal{S}$ and braiding c.*

(1) *The following are equivalent.*
   (a) *$\mathcal{S}$ is bijective.*
   (b) *Every group-like element in A is invertible in A^{op}.*
(2) *Assume that for all $g \in G(A)$,*

$$c(g \otimes g^{-1}) = g^{-1} \otimes g,\ c(g^{-1} \otimes g) = g \otimes g^{-1}.$$

Then $\mathcal{S}$ is bijective.

PROOF. (1) (a) $\Rightarrow$ (b). Since $\mathcal{S}$ is bijective, $\mathcal{S} : A^{\mathrm{op}} \to A$ is an algebra isomorphism by Proposition 6.2.8(3)(b). Let $g \in G(A)$. Then $\mathcal{S}(g)$ is invertible in A with inverse g, and hence g is invertible in A^{op}.

(b) $\Rightarrow$ (a). By (b), the inclusion map $\mathrm{Corad}(A^{\mathrm{op}}) = \Bbbk G(A) \to A^{\mathrm{op}}$ has a convolution inverse, which maps a group-like element of A to its inverse in A^{op}. Hence the braided bialgebra A^{op} has an antipode by Proposition 5.2.9, and $\mathcal{S}$ is bijective by Proposition 6.2.8(2).

(2) Every element $g \in G(A)$ is invertible in A^{op}, since $\mu c^{-1}(g \otimes g^{-1}) = 1$ and $\mu c^{-1}(g^{-1} \otimes g) = 1$ by the assumption in (2). Hence $\mathcal{S}$ is bijective by (1). □

DEFINITION 6.2.14. A braided algebra is called **braided commutative**, if $\mu c = \mu$. A braided coalgebra is called **braided cocommutative**, if $c\Delta = \Delta$.

As a corollary of Proposition 6.2.8 and (6.2.4), we now can see that a braided Hopf algebra with a general braiding is usually neither braided commutative nor braided cocommutative.

COROLLARY 6.2.15. *Let A be a braided Hopf algebra with braiding c. If A is braided commutative or braided cocommutative, then $c^2 = \mathrm{id}_{A\otimes A}$.*

PROOF. Assume that A is braided commutative. Then the bialgebra A^{op} in Proposition 6.2.8 is a Hopf algebra with antipode $\mathcal{S}$. Hence $c(x\otimes y) = c^{-1}(x\otimes y)$ for all $x, y \in A$ by (6.2.4) for A and A^{op}. If A is cocommutative, then A^{cop} is a Hopf algebra with antipode $\mathcal{S}$, and again we obtain $c = c^{-1}$. □

DEFINITION 6.2.16. Let A be a braided algebra with braiding c. Let $x, y \in A$. The **braided commutator** of x, y is the element

$$[x, y]_c = xy - \mu c(x \otimes y).$$

PROPOSITION 6.2.17. *Let A be a braided bialgebra.*

(1) *Let $x, y \in A$. Then $\Delta[x, y]_c = [\Delta(x), \Delta(y)]_c$, where the braided commutator on the right-hand side is taken in $A\underline{\otimes}A$.*
(2) *Let $x, y \in P(A)$. Then*

$$\Delta[x, y]_c = [x, y]_c \otimes 1 + 1 \otimes [x, y]_c + (\mathrm{id}_{A\otimes A} - c^2)(x \otimes y).$$

PROOF. (1) The formula follows from

$$\Delta\mu c = \mu_{A\underline{\otimes}A}(\Delta\otimes\Delta)c = \mu_{A\underline{\otimes}A}c_{2,2}(\Delta\otimes\Delta),$$

where the first equality holds since Δ is an algebra map, and the second follows from Lemma 6.1.3(4), since Δ commutes with the braiding.

(2) By (1),

$$\begin{aligned}\Delta[x, y]_c &= [\Delta(x), \Delta(y)]_c = [1\otimes x + x\otimes 1, 1\otimes y + y\otimes 1]_c\\ &= [1\otimes x, 1\otimes y]_c + [1\otimes x, y\otimes 1]_c + [x\otimes 1, 1\otimes y]_c + [x\otimes 1, y\otimes 1]_c.\end{aligned}$$

The maps $A \to A\underline{\otimes}A$, $x \mapsto 1\otimes x$, and $A \to A\underline{\otimes}A$, $x \mapsto x\otimes 1$, are braided algebra morphisms. Hence

$$[x, y]_c \otimes 1 = [x\otimes 1, y\otimes 1]_c, \quad 1\otimes [x, y]_c = [1\otimes x, 1\otimes y]_c,$$

and (2) follows from

$$[1\otimes x, y\otimes 1]_c = 0, \tag{6.2.5}$$

$$[x\otimes 1, 1\otimes y]_c = x\otimes y - c^2(x\otimes y). \tag{6.2.6}$$

Recall that the braiding of $A\underline{\otimes}A$ is $c_{2,2} = c_2c_1c_3c_2$ by Corollary 1.7.10. Hence $\mu_{A\underline{\otimes}A}c_{2,2}(x\otimes 1\otimes 1\otimes y) = \mu_{A\underline{\otimes}A}(1\otimes c(x\otimes y)\otimes 1) = c^2(x\otimes y)$, and (6.2.6) follows.

To prove (6.2.5), let $c(x\otimes y) = \sum_{i=1}^n y_i\otimes x_i$, where $x_i, y_i \in A$ for all i. Then $\mu_{A\underline{\otimes}A}c_{2,2}(1\otimes x\otimes y\otimes 1) = \mu_{A\underline{\otimes}A}(\sum_{i=1}^n y_i\otimes 1\otimes 1\otimes x_i) = c(x\otimes y)$, which implies (6.2.5). □

6.3. The fundamental theorem for pointed braided Hopf algebras

We define left and right coideal subalgebras of a braided Hopf algebra A as in Chapter 1. A **left coideal subalgebra** K of A is a subalgebra such that $\Delta(K) \subseteq A\otimes K$. Similarly, a **right coideal subalgebra** K of A is a subalgebra such that $\Delta(K) \subseteq K\otimes A$.

Let K be a left coideal subalgebra with $c(K\otimes A) \subseteq A\otimes K$. Then $A\otimes K \subseteq A\underline{\otimes}A$ is a subalgebra, and $A\otimes K$ is an $(A\otimes K, K)$-bimodule, where $A\otimes K$ is a right K-module by multiplication on the right tensorand. Hence for any left K-module

V with structure map λ_V, $A \otimes K \otimes_K V \cong A \otimes V$ is an $A \otimes K$-module, hence a left K-module by restriction via Δ with the **braided diagonal action**

$$K \otimes A \otimes V \xrightarrow{\Delta \otimes \mathrm{id}_{A\otimes V}} A \otimes K \otimes A \otimes V \xrightarrow{\mathrm{id}_A \otimes c \otimes \mathrm{id}_V} A \otimes A \otimes K \otimes V \xrightarrow{\mu \otimes \lambda_V} A \otimes V.$$

If $K \subseteq A$ is a right coideal subalgebra with $c(A \otimes K) \subseteq K \otimes A$, and V is a right K-module, then $V \otimes A$ is a right K-module in the same way by the braided diagonal action.

The following type of Hopf modules for braided Hopf algebras is an important tool in this section.

DEFINITION 6.3.1. Let A be a braided Hopf algebra with braiding c.

(1) Let $K \subseteq A$ be a left coideal subalgebra with $c(K \otimes A) \subseteq A \otimes K$. A **left Hopf module** $V \in {}^A_K\mathcal{M}$ is a left K-module V and a left A-comodule such that the comodule structure map $\delta_V : V \to A \otimes V$ is left K-linear, where $A \otimes V$ is a left K-module by the braided diagonal action.

(2) Let $K \subseteq A$ be a right coideal subalgebra with $c(A \otimes K) \subseteq K \otimes A$. A **right Hopf module** $V \in \mathcal{M}^A_K$ is a right K-module V and a right A-comodule such that the comodule structure map $\delta_V : V \to V \otimes A$ is right K-linear, where $V \otimes A$ is a right K-module by the braided diagonal action.

Hopf modules in ${}^A_K\mathcal{M}$ and $\mathcal{M}^A_K$, respectively, form an abelian category, where morphisms are left A-colinear left K-linear and right A-colinear right K-linear maps, respectively. In particular, A is an object in ${}^A_K\mathcal{M}$, and in $\mathcal{M}^A_K$, where the A-comodule structure is given by the comultiplication of A, and the K-module structure by restriction of the multiplication in A. More generally, if $K \subseteq K' \subseteq A$ are left or right coideal subalgebras, then $K' \subseteq A$ is a subobject in ${}^A_K\mathcal{M}$ or in $\mathcal{M}^A_K$.

For a braided Hopf algebra A with braiding c we introduce the following notation.

$$\mathfrak{S}(A) = \{K \mid K \text{ is a left coideal subalgebra of } A,\, c(K \otimes A) = A \otimes K\},$$

$$\mathfrak{Q}(A) = \{I \mid I \text{ is a coideal and right ideal of } A,\, c(I \otimes A) = A \otimes I\}.$$

For a coideal $I \subseteq A$, we define $A^{\mathrm{co}\, A/I} = \{x \in A \mid x^{(1)} \otimes \overline{x^{(2)}} = x \otimes \overline{1}\}$.

The next theorem is the fundamental theorem for braided pointed Hopf algebras.

THEOREM 6.3.2. *Let A be a braided pointed Hopf algebra with braiding c. Assume that $c(a \otimes g) = g \otimes a$ for all $a \in A$, $g \in G(A)$.*

(1) *The maps*

$$\{K \in \mathfrak{S}(A) \mid G(A) \cap K \textit{ is a group}\} \leftrightarrows \mathfrak{Q}(A),\ K \mapsto K^+A,\ I \mapsto A^{\mathrm{co}\, A/I},$$

are mutually inverse bijections.

(2) *Let $K \in \mathfrak{S}(A)$ and assume that $G(A) \cap K$ is a group. Then Hopf modules in ${}^A_K\mathcal{M}$ and in $\mathcal{M}^{A^{\mathrm{cop}}}_{K^{\mathrm{cop}}}$ are free over K. In particular, any left coideal subalgebra $K \subseteq K' \subseteq A$ is free as a left and as a right K-module, and $K \subseteq K'$ is a direct summand as a left and as a right K-module.*

(3) *Let $I \in \mathfrak{Q}(A)$, and define $K = A^{\mathrm{co}\, A/I}$. Then there is a left K-linear and right A/I-colinear isomorphism $A \cong K \otimes A/I$.*

In (3), the module and comodule structures are the standard ones: A is a left K-module by restriction, a right A/I-comodule by $x \mapsto x^{(1)} \otimes \overline{x^{(2)}}$, and $K \otimes A/I$ is a left K-module by multiplication on the first tensorand, and a right A/I-comodule by comultiplication on the second tensorand.

We note that in Theorem 6.3.2, $G(A)$ is a group under multiplication by Proposition 6.2.12. Thus if $K \in \mathfrak{S}(A)$, then $G(A) \cap K$ is a group if and only if for any $g \in G(A) \cap K$, the inverse g^{-1} is in K. If all elements of $G(A)$ have finite order, this last condition is always guaranteed. But if $g \in G(A)$ is an element of infinite order, then the condition fails for the left and right coideal subalgebra $\Bbbk[g] \subseteq A$.

Before we prove the theorem, we need some preparations.

LEMMA 6.3.3. *Let A be a braided Hopf algebra with braiding c.*

(1) *Let $K \in \mathfrak{S}(A)$. Then $K^+A \in \mathfrak{Q}(A)$.*
(2) *Let $I \in \mathfrak{Q}(A)$. Then $K := A^{\mathrm{co}\, A/I} \in \mathfrak{S}(A)$, and $g^{-1} \in K$ for all elements $g \in G(A) \cap K$.*

PROOF. (1) Since the augmentation map of A commutes with the braiding, and $c(K \otimes A) = A \otimes K$ by assumption, it follows that $c(K^+ \otimes A) = A \otimes K^+$. Since the multiplication of A commutes with c and with c^{-1}, we obtain from this equality that $c(K^+A \otimes A) \subseteq A \otimes K^+A$ and $c^{-1}(A \otimes K^+A) \subseteq K^+A \otimes A$. Thus $c(K^+A \otimes A) = A \otimes K^+A$.

By Lemma 1.1.14, K^+ is a coideal of A. Hence K^+A is a coideal of A, since $c(K^+ \otimes A) \subseteq A \otimes K^+$.

(2) Let $\pi : A \to A/I$ be the canonical map. By Lemma 2.5.6, K is a left coideal of A. To see that $K \subseteq A$ is a subalgebra, note that $A \otimes I$ is an $A\underline{\otimes}A$-submodule, and $A \otimes A/I$ is an $A\underline{\otimes}A$-quotient module of $A \otimes A$ as a right $A\underline{\otimes}A$-module. Here, the assumption $c(I \otimes A) \subseteq A \otimes I$ is used. Let $x \in K$ and $y \in A$. Then

$$\begin{aligned}
(xy)^{(1)} \otimes \pi((xy)^{(2)}) &= (\mathrm{id}_A \otimes \pi)((x^{(1)} \otimes x^{(2)})(y^{(1)} \otimes y^{(2)})) \\
&= (x^{(1)} \otimes \pi(x^{(2)}))(y^{(1)} \otimes y^{(2)}) \\
&= (x \otimes \pi(1))(y^{(1)} \otimes y^{(2)}) \\
&= xy^{(1)} \otimes \pi(y^{(2)}).
\end{aligned}$$

Thus the map $(\mathrm{id}_A \otimes \pi)\Delta : A \to A \otimes A/I$ is left K-linear, where $A \otimes A/I$ is a left K-module by multiplication on the first tensorand. In particular, if $x, y \in K$, then $xy \in K$. Since $c(I \otimes A) = A \otimes I$, the braiding of A induces an isomorphism $\overline{c} : A/I \otimes A \to A \otimes A/I$ such that the diagrams

$$\begin{tikzcd}
A \otimes A \arrow[r, "c"] \arrow[d, "\pi \otimes \mathrm{id}_A"'] & A \otimes A \arrow[d, "\mathrm{id}_A \otimes \pi"] \\
A/I \otimes A \arrow[r, "\overline{c}"] & A \otimes A/I
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "c_{2,1}"] \arrow[d, "\mathrm{id}_A \otimes \pi \otimes \mathrm{id}_A"'] & A \otimes A \otimes A \arrow[d, "\mathrm{id}_A \otimes \mathrm{id}_A \otimes \pi"] \\
A \otimes A/I \otimes A \arrow[r, "\overline{c}_{2,1}"] & A \otimes A \otimes A/I
\end{tikzcd}$$

commute, where $\overline{c}_{2,1} = (c \otimes \mathrm{id}_{A/I})(\mathrm{id}_A \otimes \overline{c})$. Let

$$\varphi : A \to A \otimes A/I,\ x \mapsto (\mathrm{id}_A \otimes \pi)\Delta(x) - x \otimes \pi(1),$$

thus $K = \ker(\varphi)$. Since the comultiplication of A commutes with the braiding we obtain a commutative diagram

$$\begin{tikzcd}
A \otimes A \arrow[rr, "c"] \arrow[d, "\varphi\otimes \mathrm{id}_A"'] & & A \otimes A \arrow[d, "\mathrm{id}_A \otimes \varphi"] \\
A \otimes A/I \otimes A \arrow[rr, "\overline{c}_{2,1}"] & & A \otimes A \otimes A/I
\end{tikzcd}$$

and it follows that $c(K \otimes A) = A \otimes K$.

Finally, let $g \in G(A) \cap K$. Then $g \otimes \pi(g) = g \otimes \pi(1)$. By multiplying with $g^{-2} \otimes g^{-1}$ from the right, we see that $g^{-1} \in K$. $\square$

The next lemma is the braided version of Proposition 1.2.19 (for left coideal subalgebras).

LEMMA 6.3.4. *Let A be a braided Hopf algebra with braiding c, and $K \subseteq A$ a left coideal subalgebra with $c(K \otimes A) \subseteq A \otimes K$. Let $\overline{A} = A/K^+A$. Then the canonical map*

$$\mathrm{can} : A \otimes_K A \to A \otimes \overline{A},\ x \otimes y \mapsto xy^{(1)} \otimes \overline{y^{(2)}},$$

is bijective.

PROOF. For any right A-module X, the maps

$$\Phi_X : X \otimes A \to X \otimes A,\ x \otimes a \mapsto xa^{(1)} \otimes a^{(2)},$$
$$\Phi_X^{-1} : X \otimes A \to X \otimes A,\ x \otimes a \mapsto x\mathcal{S}(a^{(1)}) \otimes a^{(2)},$$

are inverse bijections. In particular, the restriction of Φ_A induces a bijection

$$\Phi : A \otimes K \to A \otimes K.$$

Clearly, there is a unique right A-module structure on $A \otimes K$ given by

$$A \otimes K \otimes A \xrightarrow{\mathrm{id}_A \otimes c} A \otimes A \otimes K \xrightarrow{\mu_A \otimes \mathrm{id}_K} A \otimes K.$$

Then

$$\Psi : A \otimes K \otimes A \xrightarrow{\Phi \otimes \mathrm{id}_A} A \otimes K \otimes A \xrightarrow{\Phi_{A\otimes K}} A \otimes K \otimes A$$

is bijective. Let $\mu_1 : A \otimes K \to A$ and $\mu_2 : K \otimes A \to A$ be the restrictions of the multiplication map. The square in the following diagram is commutative, since $\Delta : A \to A \underline{\otimes} A$ is a braided algebra map, and ε commutes with the braiding of A.

$$\begin{tikzcd}
A \otimes K \otimes A \arrow[r, "\mu_1 \otimes \mathrm{id}_A - \mathrm{id}_A \otimes \mu_2"] \arrow[d, "\Psi"'] & A \otimes A \arrow[r, "\mathrm{can}"] \arrow[d, "\Phi_A"'] & A \otimes_K A \arrow[r] & 0 \\
A \otimes K \otimes A \arrow[r, "\mathrm{id}_A \otimes \varepsilon \otimes \mathrm{id}_A - \mathrm{id}_A \otimes \mu_2"] & A \otimes A \arrow[r, "\mathrm{can}"] & A \otimes A/K^+A \arrow[r] & 0
\end{tikzcd}$$

Since both rows are exact, Φ induces the isomorphism can. $\square$

LEMMA 6.3.5. *Let $B \subseteq A$ be a ring extension, and assume that B is a direct summand of A as a left or as a right B-module. Then the sequence*

$$0 \to B \subseteq A \xrightarrow{i_1 - i_2} A \otimes_B A$$

is exact, where $i_1(x) = x \otimes 1$, $i_2(x) = 1 \otimes x$ for all $x \in A$.

Proof. Let $f : A \to B$ be a left or right B-linear map such that $f|B = \mathrm{id}_B$. If $x \in A$ with $x \otimes 1 = 1 \otimes x$ in $A \otimes_B A$, then $x = f(x) \in B$. □

Proposition 6.3.6. *Let A be a braided Hopf algebra with braiding c, and let $K \subseteq A$ be a left coideal subalgebra with $c(K \otimes A) \subseteq A \otimes K$ (a right coideal subalgebra with $c(A \otimes K) \subseteq K \otimes A$, respectively).*

(1) *Assume that any non-zero Hopf module V in ${}_K^A\mathcal{M}$ ($\mathcal{M}_K^A$, respectively) contains a non-zero Hopf submodule which is K-free. Then any Hopf module in ${}_K^A\mathcal{M}$ ($\mathcal{M}_K^A$, respectively) is K-free.*

(2) *Assume that A is pointed and*
   (a) *for all $g \in G(A) \cap K$, $g^{-1} \in K$,*
   (b) *for all $g \in G(A)$ and $a \in K$, $c(a \otimes g) = g \otimes a$ ($c(g \otimes a) = a \otimes g$, respectively).*

   Then any Hopf module in ${}_K^A\mathcal{M}$ ($\mathcal{M}_K^A$, respectively) is K-free.

Proof. We only prove the version for left coideal subalgebras.

(1) This is a standard application of Zorn's Lemma. Let V be a non-zero Hopf module in ${}_K^A\mathcal{M}$ and let S be the set of K-linearly independent subsets X of V such that $\sum_{x \in X} Kx$ is a Hopf module in ${}_K^A\mathcal{M}$. The set S is partially ordered by inclusion. Clearly, $\emptyset \in S$ and the union of any totally ordered subset of S is an element of S. Hence by Zorn's Lemma there is a maximal element X of S. Let $U = \sum_{x \in X} Kx$ and assume that $U \neq V$. Then V/U is a non-zero Hopf module in ${}_K^A\mathcal{M}$. By assumption there is a Hopf submodule V' of V strictly containing U such that V'/U is K-free. This is a contradiction to the maximality of X. Hence $U = V$ and V is K-free.

(2) Let $0 \neq V \in {}_K^A\mathcal{M}$. Then there is a simple A-subcomodule $W \subseteq V$. Since A is pointed, W is one-dimensional with basis element v such that $\delta_V(v) = g \otimes v$, where $g \in G(A)$. Then $\delta_V(Kv) \subseteq \Delta(K)\delta_V(v) \subseteq A \otimes Kv$, since K is a left coideal of A and $c(K \otimes A) \subseteq A \otimes K$. We will show that Kv is K-free. Then the claim follows from (1), since $Kv \in {}_K^A\mathcal{M}$. The map $\varphi : K \to Kv$, $x \mapsto xv$, is a left K-linear epimorphism. Note that by (b), for all $x \in K$,

$$\delta_V(xv) = (x^{(1)} \otimes x^{(2)})(g \otimes v) = x^{(1)}g \otimes x^{(2)}v.$$

Hence the kernel of φ is a left coideal of A, since for all $x \in \ker(\varphi)$,

$$0 = \delta_V(xv) = x^{(1)}g \otimes x^{(2)}v, \quad \text{hence } 0 = x^{(1)} \otimes x^{(2)}v.$$

Assume that $\ker(\varphi) \neq 0$. Since A is pointed, $\ker(\varphi)$ contains a simple left A-subcomodule of the form $\Bbbk a$, $0 \neq a \in K$, $\Delta(a) = h \otimes a$, $h \in G(A)$. Then $a = h\varepsilon(a)$, hence $h \in \ker(\varphi) \subseteq K$. By (a), $h^{-1} \in K$. Since $\ker(\varphi)$ is a left ideal of K, we obtain the contradiction $0 \neq h^{-1}a = \varepsilon(a) \in \ker(\varphi)$. □

Definition 6.3.7. Let C be a coalgebra, and $V \in \mathcal{M}^C$. Then V is an **injective C-comodule** if for all $U, W \in \mathcal{M}^C$, for all injective C-colinear maps $i : U \to W$, and for all C-colinear maps $f : U \to V$ there is a C-colinear map $g : W \to V$ with $f = gi$.

Recall from Lemma 1.2.10 that the functor $X \mapsto (X \otimes C, \mathrm{id}_X \otimes \Delta)$ is right adjoint to the forgetful functor $\mathcal{M}^C \to \mathcal{M}_\Bbbk$.

Proposition 6.3.8. *Let C be a coalgebra, and $V \in \mathcal{M}^C$. The following are equivalent.*

(1) *V is an injective C-comodule.*
(2) *There is a vector space X such that V is isomorphic to a direct summand of $X \otimes C$ as a right C-comodule.*

PROOF. (1) $\Rightarrow$ (2). The comodule structure map $\delta : V \to V \otimes C$ is injective and right C-colinear. By (1), there is a C-colinear map $g : V \otimes C \to V$ with $g\delta = \mathrm{id}_V$, and $V \otimes C = \delta(V) \oplus \ker(g)$.

(2) $\Rightarrow$ (1). Since a direct summand of an injective comodule is injective, it is enough to show that $X \otimes C$ is injective for any vector space X. Let $i : U \to W$ be an injective C-colinear map, and $f : U \to X \otimes C$ a C-colinear map. By Lemma 1.2.10, there is a linear map $g : U \to X$ such that $f(u) = g(u_{(0)}) \otimes u_{(1)}$ for all $u \in U$. Choose a linear map $g_1 : W \to X$ with $g = g_1 i$. Then $g_2 : W \to X \otimes C$, $w \mapsto g_1(w_{(0)}) \otimes w_{(1)}$ is C-colinear, and $f = g_2 i$. □

PROOF OF THEOREM 6.3.2. We first prove (2). Let $K \subseteq K' \subseteq A$ be left coideal subalgebras. Hopf modules in ${}^{A}_{K}\mathcal{M}$ are K-free by Proposition 6.3.6. Recall that A is a Hopf module in ${}^{A}_{K}\mathcal{M}$, and $K \subseteq K' \subseteq A$ are Hopf submodules. Hence K' and K'/K are K-free by Proposition 6.3.6. Thus the inclusion $K \subseteq K'$ of left K-modules splits.

By Proposition 6.2.13, the antipode $\mathcal{S}$ of A is bijective. Hence A^{cop} is a braided Hopf algebra with braiding c^{-1}, and $\mathcal{S} : A \to A^{\mathrm{cop}}$ is an isomorphism of coalgebras by Proposition 6.2.8. Thus A^{cop} is a pointed coalgebra. Since $K \in \mathfrak{S}(A)$, $c^{-1}\Delta(K) \subseteq c^{-1}(A \otimes K) = K \otimes A$. Hence $K^{\mathrm{cop}} \subseteq A^{\mathrm{cop}}$ is a right coideal subalgebra. By definition,

$$G(A^{\mathrm{cop}}) = \{g \in A \mid \Delta(g) = c(g \otimes g)\}.$$

Hence $G(A^{\mathrm{cop}}) = G(A)$. It is now clear that the assumptions of Proposition 6.3.6 are satisfied for the right coideal subalgebra K^{cop} of A^{cop}. Hence Hopf modules in $\mathcal{M}^{A^{\mathrm{cop}}}_{K^{\mathrm{cop}}}$ are free over K. Note that A^{cop} is a Hopf module in $\mathcal{M}^{A^{\mathrm{cop}}}_{K^{\mathrm{cop}}}$, and $K \subseteq K'$ are Hopf submodules. Thus K' is free as a right K-module, and $K \subseteq K'$ is a direct summand as a right K-module.

(1) Both maps are well-defined by Lemma 6.3.3, and the claim follows from

(a) Let $K \in \mathfrak{S}(A)$, and define $I = K^+A$. Then $K = A^{\mathrm{co}\,A/I}$.
(b) Let $I \in \mathfrak{Q}(A)$, and define $K = A^{\mathrm{co}\,A/I}$. Then $I = K^+A$.

Proof of (a). The following diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\iota_2"] \arrow[d, "\iota_1"] & A \arrow[r, "i_1 - i_2"] \arrow[d, "="] & A \otimes_K A \arrow[d, "\mathrm{can}"] \\
0 \arrow[r] & A^{\mathrm{co}\,A/I} \arrow[r] & A \arrow[r] & A \otimes A/I
\end{tikzcd}$$

is commutative with exact rows, where ι_1 and ι_2 are the inclusion maps, and the lower sequence is the defining sequence of $A^{\mathrm{co}\,A/I}$. The upper sequence is exact by Lemma 6.3.5 and (2). Hence $K = A^{\mathrm{co}\,A/I}$.

(b) Let $\pi : A \to A/I$ be the quotient map. By definition, for all $x \in K$, $x^{(1)} \otimes \pi(x^{(2)}) = x \otimes \pi(1)$, hence $\pi(x) = \varepsilon(x)\pi(1)$. Thus $K^+A \subseteq I$. By Lemma 6.3.4 it suffices to show that the composition

$$\Phi : A \otimes_K A \xrightarrow{\mathrm{can}} A \otimes A/K^+A \to A \otimes A/I,\ x \otimes y \mapsto xy^{(1)} \otimes \pi(y^{(2)}),$$

is bijective.

We have seen in the proof of Lemma 6.3.3(2) that the A/I-comodule structure map $A \to A \otimes A/I$, $x \mapsto x^{(1)} \otimes \pi(x^{(2)})$, is left K-linear, where $A \otimes A/I$ is a left K-module by multiplication on the first tensorand. Hence $A \otimes_K A$ is a right A/I-comodule with structure map

$$A \otimes_K A \to A \otimes_K A \otimes A/I,\ x \otimes y \mapsto x \otimes y^{(1)} \otimes \pi(y^{(2)}),$$

and Φ is a surjective right A/I-colinear map, where $A \otimes A/I$ is an A/I-comodule with coaction $\mathrm{id}_A \otimes \Delta_{A/I}$. We have to show that Φ is injective. Since by Proposition 5.4.2(2), A/I is pointed and $G(A) \to G(A/I)$ is surjective, it remains to show by Proposition 2.2.14 that for all $g \in G(A)$ the induced map

$$\Phi(\Bbbk\pi(g)) : (A \otimes_K A)(\Bbbk\pi(g)) \to A \otimes (A/I)(\Bbbk\pi(g))$$

is bijective. Since A is free as a right K-module by (2), it follows that

$$(A \otimes_K A)(\Bbbk\pi(g)) \cong A \otimes_K A(\Bbbk\pi(g)).$$

It is clear that $(A/I)(\Bbbk\pi(g)) = \Bbbk\pi(g)$. Moreover, $A(\Bbbk\pi(g)) = Kg$, since for all $x \in A$,

$$x \in A(\Bbbk\pi(g)) \iff x^{(1)} \otimes \pi(x^{(2)}) = x \otimes \pi(g) \iff xg^{-1} \in K,$$

where the last equivalence follows from the assumption that $c(a \otimes g^{-1}) = g^{-1} \otimes a$ for all $a \in A$. Thus we are reduced to show that

$$A \otimes_K Kg \to A \otimes \pi(g),\ x \otimes ag \mapsto x(ag)^{(1)} \otimes \pi((ag)^{(2)}) = xag \otimes \pi(g),$$

is bijective. But this is obvious since the multiplication map $A \otimes_K Kg \to A$ is bijective with inverse $x \mapsto xg^{-1} \otimes g$.

(3) Let $\overline{A} = A/I$. Since K is a direct summand of the left K-module A by (2), it follows from Lemma 6.3.4 that A is a direct summand if the right $\overline{A}$-comodule $A \otimes \overline{A}$. Hence A is an injective $\overline{A}$-comodule by Proposition 6.3.8.

Since A is pointed, the map $G(A) \to G(\overline{A})$, $g \mapsto \overline{g}$, is surjective by Proposition 5.4.2. Choose a map $\gamma : G(\overline{A}) \to G(A)$ with $\overline{\gamma(\overline{g})} = \overline{g}$ for all $\overline{g} \in G(\overline{A})$. Then the linear map $f : \Bbbk G(\overline{A}) \to A$, $\overline{g} \mapsto \gamma(\overline{g})$, is right $\overline{A}$-colinear. Note that f is convolution invertible, since γ maps group-like elements to invertible elements in A. Since A is injective as a right $\overline{A}$-comodule, f can be extended to a right $\overline{A}$-colinear map $h : \overline{A} \to A$, which is convolution invertible by Corollary 5.3.10.

Define

$$\Phi(h) : A \otimes \overline{A} \to A \otimes \overline{A},\ x \otimes \overline{y} \mapsto xh(\overline{y^{(1)}}) \otimes \overline{y^{(2)}},$$
$$\Psi(h) : A \otimes \overline{A} \to A \otimes_K A,\ x\overline{y} \mapsto x \otimes h(\overline{y}).$$

Then $\Phi(h)$ is bijective by Proposition 1.2.11(2), since γ is invertible. Since h is right $\overline{A}$-colinear, $\mathrm{can}\Psi(h) = \Phi(h)$. Hence $\Psi(h)$ is bijective by (1). Thus the map $A \otimes_K K \otimes \overline{A} \to A \otimes_K A$, $a \otimes x \otimes \overline{y} \mapsto a \otimes xh(\overline{y})$, is bijective. Since K is a right K-direct summand of A by (2), the induced map $K \otimes \overline{A} \to A$ is bijective. $\square$

We next prove a graded version of the decomposition in Theorem 6.3.2(3).

LEMMA 6.3.9. *Let C be an $\mathbb{N}_0$-graded coalgebra, X, Y and E $\mathbb{N}_0$-graded right C-comodules, and $i : X \to Y$ an injective $\mathbb{N}_0$-graded right C-colinear map. If E is an injective C-comodule, and $f : X \to E$ is an $\mathbb{N}_0$-graded right C-colinear map, then there is an $\mathbb{N}_0$-graded right C-colinear map $g : Y \to E$ with $gi = f$.*

PROOF. Since E is an injective comodule, there is a right C-colinear map $\widetilde{g} : Y \to E$ with $\widetilde{g}i = f$. Define $g : Y \to E$ by $g(y) = \widetilde{g}(y)(n)$ for all $y \in Y(n)$, $n \geq 0$. Then g is a graded map with $gi = f$, and g is right C-colinear, since for all $y \in Y(n)$, $n \geq 0$, the homogeneous part of degree n of $\delta(\widetilde{g}(y)) = (\widetilde{g} \otimes \mathrm{id}_C)(\delta(y))$ is $\delta(g(y)) = (g \otimes \mathrm{id}_C)(\delta(y))$. □

For any $\mathbb{N}_0$-graded vector space $V = \bigoplus_{n\geq 0} V(n)$ such that $V(n)$ is finite-dimensional for all $n \geq 0$, we denote by

$$\mathcal{H}_V = \mathcal{H}_V(t) = \sum_{n\geq 0} \dim V(n)t^n$$

the **Hilbert series** of V.

COROLLARY 6.3.10. *Let A be a braided Hopf algebra with an $\mathbb{N}_0$-grading as a vector space such that A is a connected $\mathbb{N}_0$-graded algebra and coalgebra. Let $I \subseteq A$ be an $\mathbb{N}_0$-graded coideal and right ideal with $c(I \otimes A) = A \otimes I$, and define $K = A^{\mathrm{co}\, A/I}$.*

(1) *There is an $\mathbb{N}_0$-graded left K-linear and right A/I-colinear isomorphism $A \cong K \otimes A/I$, where $K \subseteq A$ is an $\mathbb{N}_0$-graded subalgebra.*
(2) *If $A(n)$ is finite-dimensional for all $n \geq 1$, then $\mathcal{H}_A = \mathcal{H}_K\mathcal{H}_{A/I}$.*

PROOF. Since A is connected, it is pointed by Proposition 5.4.2, and Theorem 6.3.2 applies. It follows easily from the definition that $K \subseteq A$ is an $\mathbb{N}_0$-graded subalgebra. Let $\pi : A \to A/I$ be the quotient map. Note that $I(0) = 0$ since I is a coideal. Hence $A/I = \bigoplus_{n\geq 0} A(n)/I(n)$ is an $\mathbb{N}_0$-graded coalgebra with $(A/I)(0) = A(0) = \Bbbk 1$. The map $f : (A/I)(0) = A(0) \subseteq A$ is $\mathbb{N}_0$-graded and right A/I-colinear, where A is a right A/I-comodule with coaction $(\mathrm{id}_A \otimes \pi)\Delta$. By Theorem 6.3.2(3), A is an injective right A/I-comodule. Hence by Lemma 6.3.9, f can be extended to an $\mathbb{N}_0$-graded right A/I-colinear map $\gamma : A/I \to A$. By Corollary 5.3.10, γ is convolution invertible. Then we have shown in the proof of Theorem 6.3.2(3), that the map

$$K \otimes A/I \to A, \quad x \otimes \overline{y} \mapsto x\gamma(\overline{y}),$$

is bijective. This proves (1), and (2) is an immediate consequence of (1). □

6.4. The braided tensor algebra

We now introduce graded braided structures, but at this moment we will study only $\mathbb{N}_0$-gradings.

DEFINITION 6.4.1. Let Γ be an abelian monoid. A **Γ-graded braided algebra (coalgebra, respectively)** is a braided algebra (coalgebra, respectively) A with a Γ-grading as a vector space such that A is a Γ-graded braided vector space and a Γ-graded algebra (coalgebra, respectively). A **Γ-graded braided bialgebra** is a braided bialgebra with a Γ-grading as a vector space such that A is a Γ-graded braided vector space and Γ-graded as an algebra and as a coalgebra. A **Γ-graded braided Hopf algebra** is a Γ-graded braided bialgebra with an antipode.

PROPOSITION 6.4.2. *Let $A = \bigoplus_{n\geq 0} A(n)$ be an $\mathbb{N}_0$-graded braided bialgebra.*

(1) *Assume that the subbialgebra $A(0)$ is a braided Hopf algebra. Then A is a braided Hopf algebra.*
(2) *If the antipode of $A(0)$ in (1) is bijective, then the antipode of A is bijective.*

PROOF. (1) We apply Proposition 5.2.9(2) to the $\mathbb{N}_0$-filtered coalgebra with filtration $A(n) = \bigoplus_{i \le n} A(i), n \ge 0$. The restriction of the id_A to $A(0)$ is invertible, since $A(0)$ has an antipode. Hence A has an antipode.

(2) The braided bialgebra A^{op} is an $\mathbb{N}_0$-filtered bialgebra, and $(A^{\mathrm{op}})(0)$ is the braided bialgebra $A(0)^{\mathrm{op}}$. Hence the antipode of A is bijective by (1) and Proposition 6.2.8(2). $\square$

Recall that by Corollary 6.1.10 the tensor algebra $T(V)$ is an $\mathbb{N}_0$-graded braided vector space with braiding given for all $m, n \ge 0$ by

$$c_{m,n} : V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}.$$

For all $n \ge 1$, we denote the n-fold multiplication map of an algebra A with multiplication μ by $\mu^n : A^{\otimes(n+1)} \to A$. Thus μ^1 is the multiplication of A, and $\mu^n = \mu(\mathrm{id}_A \otimes \mu^{n-1})$. We set $\mu^0 = \mathrm{id}_A$. If A is a braided algebra, it follows by induction on n that μ^n commutes with the braiding of A for all $n \ge 0$.

PROPOSITION 6.4.3. *Let (V, c) be a braided vector space, and A a braided algebra.*

(1) *The tensor algebra $T(V)$ is an $\mathbb{N}_0$-graded braided algebra.*
(2) *For every map of braided vector spaces $f : V \to A$ there is a unique morphism of braided algebras $\varphi : T(V) \to A$ such that $\varphi|V = f$. If A is an $\mathbb{N}_0$-graded braided algebra, and $f(V) \subseteq A(1)$, then φ is $\mathbb{N}_0$-graded.*

PROOF. (1) It is clear that the unit map $\eta : \Bbbk \to T(V)$ commutes with the braiding of $T(V)$. The identity maps $V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes m+n}$, $m, n \ge 0$, are the components of the multiplication of the tensor algebra $T(V)$, hence they commute with the braiding by Corollary 1.7.10(5). Thus $T(V)$ is an $\mathbb{N}_0$-graded braided algebra.

(2) We have to show that the algebra map $\varphi : T(V) \to A$ determined by $\varphi|V = f$ is a map of braided vector spaces.

For all $m, n \ge 1$ the following diagram commutes, where c denotes the braiding of V and A, respectively.

$$\begin{CD}
V^{\otimes m} \otimes V^{\otimes n} @>{f^{\otimes m} \otimes f^{\otimes n}}>> A^{\otimes m} \otimes A^{\otimes n} @>{\mu^{m-1} \otimes \mu^{n-1}}>> A \otimes A \\
@V{c_{m,n}}VV @V{c_{m,n}}VV @VV{c}V \\
V^{\otimes n} \otimes V^{\otimes m} @>{f^{\otimes n} \otimes f^{\otimes m}}>> A^{\otimes n} \otimes A^{\otimes m} @>{\mu^{n-1} \otimes \mu^{m-1}}>> A \otimes A
\end{CD}$$

This is clear for the left square, since f is a map of braided vector spaces. The right square commutes by Lemma 6.1.3(4), since μ^{m-1} and μ^{n-1} commute with the braiding of A. The commutativity of the outer square implies that $\varphi : T(V) \to A$ is a map of braided vector spaces. $\square$

LEMMA 6.4.4. *Let A be a braided bialgebra. Then the map*

$$f : A \to A \underline{\otimes} A, \quad a \mapsto a \otimes 1 + 1 \otimes a,$$

commutes with the braiding of A, and $P(A) = \{a \in A \mid \Delta(a) = a \otimes 1 + 1 \otimes a\}$ is a categorical subspace of A.

PROOF. By assumption, $\eta : \Bbbk \to A$ and $\Delta : A \to A \underline{\otimes} A$ commute with the braiding of A. Hence by Lemma 6.1.3(1), $f = \eta \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \eta$ and $\Delta - f$ commute with the braiding of A. Then $P(A) = \ker(\Delta - f) \subseteq A$ is a categorical subspace by Lemma 6.1.7. □

LEMMA 6.4.5. *Let (C, c) be a braided vector space, and let $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ be linear maps. Assume that*

(1) *ε commutes with the braiding of C,*
(2) *$(\varepsilon \otimes \mathrm{id}_C)\Delta = \mathrm{id}_C$,*
(3) *Δ is a braided linear map.*

Then Δ commutes with the braiding of C.

PROOF. The diagram

$$\begin{tikzcd}
C \otimes C \arrow[r, "c"] \arrow[d, "\Delta\otimes\Delta"] & C \otimes C \arrow[d, "\Delta\otimes\Delta"] \\
C^{\otimes 2} \otimes C^{\otimes 2} \arrow[r, "c_{2,2}"] \arrow[d, "\varepsilon\otimes\mathrm{id}_C\otimes\mathrm{id}_{C^{\otimes 2}}"] & C^{\otimes 2} \otimes C^{\otimes 2} \arrow[d, "\mathrm{id}_{C^{\otimes 2}}\otimes\varepsilon\otimes\mathrm{id}_C"] \\
C \otimes C^{\otimes 2} \arrow[r, "c_{1,2}"] & C^{\otimes 2} \otimes C
\end{tikzcd}$$

commutes, since the upper part commutes by (3), and the lower part by (1) and Lemma 6.1.7. Hence $(\Delta \otimes \mathrm{id}_C)c = c_{1,2}(\mathrm{id}_C \otimes \Delta)$ by (2), and similarly one proves that $(\mathrm{id}_C \otimes \Delta)c = c_{2,1}(\Delta \otimes \mathrm{id}_C)$. Thus Δ commutes with the braiding of C. □

PROPOSITION 6.4.6. *Let (V, c) be a braided vector space. The tensor algebra $T(V)$ is an $\mathbb{N}_0$-graded braided Hopf algebra with comultiplication Δ and counit ε given by $\Delta(v) = v \otimes 1 + 1 \otimes v$, $\varepsilon(v) = 0$, for all $v \in V$.*

PROOF. (1) By Proposition 6.4.3(1) and Remark 6.2.2(1), $T(V)\underline{\otimes}T(V)$ and $T(V)$ are algebras in $\mathcal{C}(T(V), c^{T(V)})$. By the universal property of the tensor algebra there are algebra maps

$$\Delta : T(V) \to T(V)\underline{\otimes}T(V), \quad \varepsilon : T(V) \to \Bbbk,$$

determined by $\Delta(x) = x\otimes 1 + 1\otimes x$, $\varepsilon(x) = 0$, $x \in V$. To see that $T(V)$ is a braided bialgebra, it remains to prove

(a) ε commutes with the braiding of $T(V)$, and $(\mathrm{id}_{T(V)} \otimes \varepsilon)\Delta = \mathrm{id}_{T(V)}$, $(\varepsilon \otimes \mathrm{id}_{T(V)})\Delta = \mathrm{id}_{T(V)}$,
(b) Δ commutes with the braiding of $T(V)$, and Δ is coassociative.

(a) Since $\varepsilon(V^{\otimes n}) = 0$ for all $n \geq 0$, it is easy to see that ε commutes with the braiding. Hence by Remark 6.2.2(2), $\mathrm{id}_{T(V)} \otimes \varepsilon$ and $\varepsilon \otimes \mathrm{id}_{T(V)}$ are algebra morphisms in $\mathcal{C}(T(V), c^{T(V)})$. Then the equations in (a) follow from the universal property of the tensor algebra.

(b) The linear map $f : T(V) \to T(V)\underline{\otimes}T(V)$, $a \mapsto a\otimes 1 + 1\otimes a$, is a morphism of braided vector spaces by Lemma 6.4.4 and Lemma 6.1.3(4). Hence the restriction of f to V is braided, and by Proposition 6.4.3(2), $\Delta : T(V) \to T(V)\underline{\otimes}T(V)$ is braided. By Lemma 6.4.5 and (a), Δ commutes with the braiding of $T(V)$.

We can now prove coassociativity. The maps

$$\mathrm{id}_{T(V)} \otimes \Delta : T(V)\underline{\otimes}T(V) \to T(V)\underline{\otimes}(T(V)\underline{\otimes}T(V)),$$
$$\Delta \otimes \mathrm{id}_{T(V)} : T(V)\underline{\otimes}T(V) \to (T(V)\underline{\otimes}T(V))\underline{\otimes}T(V)$$

are algebra maps by Remark 6.2.2(2). By Remark 6.2.2(1),

$$T(V)\underline{\otimes}(T(V)\underline{\otimes}T(V)) = (T(V)\underline{\otimes}T(V))\underline{\otimes}T(V) \quad \text{as algebras.}$$

Hence the diagram

$$\begin{tikzcd} T(V) \arrow[r, "\Delta"] \arrow[d, "\Delta"] & T(V)\underline{\otimes}T(V) \arrow[d, "\mathrm{id}_{T(V)}\otimes\Delta"] \\ T(V)\underline{\otimes}T(V) \arrow[r, "\Delta\otimes\mathrm{id}_{T(V)}"] & T(V)\underline{\otimes}T(V)\underline{\otimes}T(V) \end{tikzcd}$$

commutes, since it commutes on V.

Finally, $T(V)$ has an antipode by Proposition 5.2.9(3). □

PROPOSITION 6.4.7. *Let V be a braided vector space, and A a braided bialgebra. For every map of braided vector spaces $f : V \to P(A)$, there is a unique morphism of braided bialgebras $\varphi : T(V) \to A$ such that $\varphi|V = f$. If A is a connected $\mathbb{N}_0$-graded bialgebra, and* $\mathrm{im}(f) \subseteq A(1)$*, then φ is $\mathbb{N}_0$-graded.*

PROOF. Recall that $P(A) \subseteq A$ is a categorical, hence a braided subspace by Lemma 6.4.4. By Proposition 6.4.3, there is a uniquely determined map of braided algebras $\varphi : T(V) \to A$ extending f. It remains to show that φ is a coalgebra map, that is, the diagrams

$$\begin{tikzcd} T(V) \arrow[r, "\varphi"] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ T(V)\underline{\otimes}T(V) \arrow[r, "\varphi\otimes\varphi"] & A\underline{\otimes}A \end{tikzcd} \qquad \begin{tikzcd} T(V) \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon"'] & & A \arrow[dl, "\varepsilon"] \\ & \Bbbk & \end{tikzcd}$$

commute. By Proposition 6.2.3, $\varphi \otimes \varphi$ is an algebra map. Hence all maps in the diagrams are algebra maps, and it is enough to prove commutativity on the generators in V. It is clear from the assumption on f that both diagrams commute on elements of V. □

REMARK 6.4.8. By Proposition 6.4.7, any morphism $f : V \to W$ of braided vector spaces defines a morphism $T(f) : T(V) \to T(W)$ of $\mathbb{N}_0$-graded braided Hopf algebras determined by $T(f)|V = f$. Thus the tensor algebra construction is a functor from braided vector spaces to $\mathbb{N}_0$-graded braided Hopf algebras.

By Proposition 6.4.6, the tensor algebra of a braided vector space is an $\mathbb{N}_0$-graded coalgebra. In the next theorem we compute the components of its comultiplication (see Definitions 1.2.26(1) and 1.3.12).

THEOREM 6.4.9. *Let (V, c) be a braided vector space, and $n \geq 2$. The comultiplication of $T(V)$ is denoted by Δ.*

(1) *For all $1 \leq i \leq n-1$, $\Delta_{i,n-i} = S_{i,n-i}$ in* $\mathrm{End}(V^{\otimes n})$.
(2) $\Delta_{1^n} = S_n$ *in* $\mathrm{End}(V^{\otimes n})$.

PROOF. See the proofs of Theorem 1.3.12 and Corollary 1.9.7. □

Finally we note a useful property of $\mathbb{N}_0$-graded braided coalgebras.

PROPOSITION 6.4.10. *Let C be an $\mathbb{N}_0$-graded braided coalgebra which is a strictly graded coalgebra. Then $C\underline{\otimes}C$ is strictly graded.*

PROOF. This follows from Proposition 1.3.17. □

6.5. Notes

6.1, 6.2. The definitions of maps commuting with the braiding and of braided algebras, coalgebras, and Hopf algebras are taken form [**Tak00**] and [**Tak05**], see also [**HH92**].

Proposition 6.2.11 and Corollary 6.2.15 are observed in [**Sch98**].

6.3. The non-braided version of Theorem 6.3.2 is a result of [**Mas91**]. We follow the proof sketched in the end of [**Sch90**]. The freeness of Hopf modules is shown by an argument in [**Rad78**].

Theorem 6.3.2 for a connected Hopf algebra in the braided category ${}^H_H\mathcal{YD}$ is shown in [**AA$^+$14**], Proposition 3.6.

6.4. In [**Kha15**], Section 6.2, another proof of Proposition 6.4.6 is given by explicit calculations in the group algebra of the braid group.

We want to mention the construction of an $\mathbb{N}_0$-graded braided Hopf algebra which is dual to $T(V)$ in Section 6.4, see [**Ros95**], [**Sch96**], [**Tak05**], or [**Kha15**], Chapter 6.

Let (V,c) a braided vector space, and $T(V)=\bigoplus_{n\geq 0} V^{\otimes n}$ be the braided vector space with braiding $c^{T(V)}$ defined in Corollary 6.1.10. $T(V)$ is a $\mathbb{N}_0$-graded coalgebra with comultiplication given by

$$\Delta(v_1\otimes\cdots\otimes v_n)=\sum_{i=0}^{n}(v_1\otimes\cdots\otimes v_i)\otimes(v_{i+1}\otimes\cdots\otimes v_n)$$

for all $n\geq 0$, $v_1,\dots,v_n\in V$. We define another algebra structure on $T(V)$ by

$$(v_1\otimes\cdots\otimes v_i)\cdot(v_{i+1}\otimes\cdots\otimes v_n)=\sum_{w\in\mathbb{S}_{i,n-i}} c_w(v_1\otimes\cdots\otimes v_n)$$

for all $n\geq 0$, $v_1,\dots,v_n\in V$. Recall from Definition 1.8.1 that $\mathbb{S}_{i,n-i}$ denotes the set of all i-shuffles in $\mathbb{S}_n$. Then $T(V)$ with multiplication and comultiplication just defined and braiding $c^{T(V)}$ is an $\mathbb{N}_0$-graded braided Hopf algebra called the **shuffle algebra** of (V,c) and denoted by $Sh(V)$.

$Sh(V)$ is Sweedler's shuffle algebra in [**Swe69**], Chapter XII, when c is the twist map. If V is finite-dimensional, then $Sh(V)$ is isomorphic to the graded dual of $T(V)$.

Let $S: T(V)\to Sh(V)$ be the algebra morphism with $S(v)=v$ for all $v\in V$. Then S is a morphism of $\mathbb{N}_0$-graded braided Hopf algebras, and for all $n\geq 0$, the n-th component of S is the braided symmetrizer map

$$S_n=\sum_{w\in\mathbb{S}_n} c_w : V^{\otimes n}\to V^{\otimes n}.$$

CHAPTER 7

Nichols algebras

In this short chapter we first define and characterize the Nichols algebra of a braided vector space, and of Yetter-Drinfeld modules over any Hopf algebra with bijective antipode. We proceed exactly as we did for Yetter-Drinfeld modules over groups in Chapter 1.

In Section 7.2 we introduce the important non-degenerate duality pairing of Nichols algebras. This is the starting point of the theory of reflections of Nichols algebras in Part 3 of the book. In the last section we define differential operators for Nichols algebras. In the case of Yetter-Drinfeld modules over groups they are skew derivations which form a very efficient tool for computations, for example to decide whether an element of the Nichols algebra is non-zero.

7.1. The Nichols algebra of a braided vector space and of a Yetter-Drinfeld module

In Section 6.4 we have defined the tensor algebra $T(V)$ of a braided vector space (V,c) as an $\mathbb{N}_0$-graded braided Hopf algebra. In this section we define a basic universal quotient Hopf algebra of $T(V)$. Recall the definition of the maps Δ_{1^n} in Definition 1.3.12.

Definition 7.1.1. Let (V,c) be a braided vector space. Then

$$\mathcal{B}(V,c)=\mathcal{B}(V)=T(V)/\bigoplus_{n\geq 2}\ker(\Delta_{1^n}^{T(V)})$$

is called the **Nichols algebra of** (V,c). Let

$$I(V,c)=I(V)=\bigoplus_{n\geq 2}\ker(\Delta_{1^n}^{T(V)}).$$

As a vector space, $\mathcal{B}(V)=\bigoplus_{n\geq 0}\mathcal{B}^n(V)$ is $\mathbb{N}_0$-graded, where

$$\mathcal{B}^0(V)=\Bbbk 1,\quad \mathcal{B}^1(V)=V, \text{ and } \mathcal{B}^n(V)=V^{\otimes n}/\ker(\Delta_{1^n}^{T(V)}) \text{ for all } n\geq 2.$$

Theorem 7.1.2. *Let (V,c) be a braided vector space.*

(1) (a) *$I(V)$ is the largest coideal of $T(V)$ contained in $\bigoplus_{n\geq 2}V^{\otimes n}$.*
  (b) *$I(V)$ is the only coideal I of $T(V)$ contained in $\bigoplus_{n\geq 2}V^{\otimes n}$ such that $P(T(V)/I)=V$.*

(2) *$I(V)$ is a categorical subpace of $T(V)$, and $\mathcal{B}(V)$ is an $\mathbb{N}_0$-graded braided graded Hopf algebra quotient of $T(V)$. As a coalgebra $\mathcal{B}(V)$ is strictly graded, and as an algebra it is generated by $\mathcal{B}^1(V)=V$.*

(3) *For all $n\geq 2$ let $S_n: V^{\otimes n}\to V^{\otimes n}$ be the braided symmetrizer map. Then*

$$\mathcal{B}(V)=\Bbbk 1\oplus V\oplus\bigoplus_{n\geq 2}V^{\otimes n}/\ker(S_n).$$

Proof. (1) is a special case of Theorem 1.3.16.

(2) The subspace $I(V) \subseteq T(V)$ is categorical by Lemmas 6.1.3(2) and 6.1.7. Thus Lemma 6.2.10 implies that the ideal of $T(V)$ generated by $I(V)$ is a coideal. Hence $I(V)$ is an ideal of $T(V)$ by (1)(a). The coalgebra $\mathcal{B}(V)$ is strictly graded by (1). Hence $T(V)/I(V)$ is a braided quotient bialgebra of $T(V)$ by Lemma 6.2.5, and the quotient $T(V)/I(V)$ is an $\mathbb{N}_0$-graded braided vector space by Definition 6.1.8. Finally, $\mathcal{B}(V)$ has an antipode by Proposition 5.2.9(3).

(3) follows from Theorem 6.4.9, since by definition

$$\mathcal{B}(V) = \Bbbk \oplus V \oplus \bigoplus_{n\geq 2} V^{\otimes n}/\ker(\Delta_{1^n}).$$

□

The following rather pathological example shows some phenomena, which are out of the scope of the current developments.

Example 7.1.3. Let V be a vector space and let $c = \mathrm{id}_{V\otimes V}$. Then (V,c) is a braided vector space, and for all $n \geq 2$, $S_n = n!\mathrm{id}_{V^{\otimes n}} : V^{\otimes n} \to V^{\otimes n}$. Thus

$$\mathcal{B}(V) = \begin{cases} T(V) & \text{if } \mathrm{char}(\Bbbk) = 0, \\ T(V)/(V^p) & \text{if } \mathrm{char}(\Bbbk) = p > 0. \end{cases}$$

If $\mathrm{char}(\Bbbk) = p > 0$ and V is finite-dimensional, then the Nichols algebra $\mathcal{B}(V)$ is a finite-dimensional $\mathbb{N}_0$-graded braided Hopf algebra. By Lemma 4.4.6, $\mathcal{B}(V)$ is not a Frobenius algebra if $\dim V \geq 2$, since

$$\mathcal{B}(V) = \Bbbk \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes (p-1)},$$

and the space $V^{\otimes(p-1)}$ of left and right integrals is not one-dimensional.

Remark 7.1.4. By Proposition 6.4.7, any morphism $f : V \to W$ of braided vector spaces induces a morphism $T(f) : T(V) \to T(W)$ of $\mathbb{N}_0$-graded braided bialgebras. Since $T(f)$ is an $\mathbb{N}_0$-graded coalgebra map, it maps $I(V)$ to $I(W)$. Hence f defines a morphism $\mathcal{B}(f) : \mathcal{B}(V) \to \mathcal{B}(W)$ of $\mathbb{N}_0$-graded braided Hopf algebras determined by $\mathcal{B}(f)|V = f$. Thus the construction of the Nichols algebra is a functor from braided vector spaces to $\mathbb{N}_0$-graded braided Hopf algebras.

Lemma 7.1.5. *Let (V,c) be a braided vector space, and $U \subseteq V$ a braided subspace. Then the inclusion map defines an injective map $\mathcal{B}(U) \to \mathcal{B}(V)$ of $\mathbb{N}_0$-graded braided Hopf algebras.*

Proof. Since $c(U \otimes U) = U \otimes U$, $T(U) \subseteq T(V)$ is an $\mathbb{N}_0$-graded braided subcoalgebra, and it follows from the definition that $I(U) = I(V) \cap T(U)$. □

Definition 7.1.6. Let (V,c) be a braided vector space. An $\mathbb{N}_0$-graded connected braided Hopf algebra R is a **pre-Nichols algebra of** V, if

(N1) $R(1) \cong V$ as braided vector spaces, where the braiding of $R(1)$ is induced by the braiding of R,

(N2) R is generated as an algebra by $R(1)$.

A pre-Nichols algebra of V is a **Nichols algebra of** V, if

(N3) R is strictly graded, that is, $P(R) = R(1)$.

Theorem 7.1.7. *Let (V,c) be a braided vector space.*

(1) *$\mathcal{B}(V)$ is a Nichols algebra of V.*

(2) *Let R be a pre-Nichols algebra of V and $f : R(1) \xrightarrow{\cong} V$ an isomorphism of braided vector spaces.*
   (a) *There is exactly one morphism $\pi : R \to \mathcal{B}(V)$ of $\mathbb{N}_0$-graded braided Hopf algebras such that f is the restriction of π to $R(1)$, and π is surjective.*
   (b) *π is bijective if and only if R is a Nichols algebra of V.*

PROOF. (1) is shown in Theorem 7.1.2, and (2) follows as in the proof of Theorem 1.6.18. □

COROLLARY 7.1.8. *Let A be a braided bialgebra. Assume that $A = \bigoplus_{n\geq 0} A(n)$ is an $\mathbb{N}_0$-graded vector space such that A is a connected $\mathbb{N}_0$-graded braided algebra with $A(1) = P(A)$. Let $V \subseteq A(1)$ be a categorical subspace of A. Then the subalgebra $\Bbbk[V]$ generated by V is a subcoalgebra of A, and an $\mathbb{N}_0$-graded braided Hopf algebra isomorphic to $\mathcal{B}(V)$.*

PROOF. By Lemma 6.2.10, the subspaces $V^n \subseteq A$, $n \geq 0$, are categorical. The subalgebra $B = \Bbbk[V]$ is an $\mathbb{N}_0$-graded braided algebra with $B(n) = V^n$ for all $n \geq 1$.

Since $V \subseteq P(A)$, $\varepsilon(v) = 0$ for all $v \in V$. Hence $\varepsilon(v) = 0$ for all $v \in V^n$ with $n \geq 1$. We show by induction on n that $\Delta(B(n)) \subseteq \bigoplus_{i=0}^n B(i) \otimes B(n-i)$ for all $n \geq 1$. This is clear for $n = 0$. Let $x \in B(1)$, $y \in B(n)$, $n \geq 0$, and assume that

$$\Delta(y) = y^{(1)} \otimes y^{(2)} \in \bigoplus_{i=0}^{n} B(i) \otimes B(n-i).$$

Then

$$\begin{aligned}\Delta(xy) &= (x \otimes 1 + 1 \otimes x)(y^{(1)} \otimes y^{(2)})\\ &= x\, y^{(1)} \otimes y^{(2)} + c(x \otimes y^{(1)})y^{(2)}\\ &\in \sum_{i=0}^{n} B(1)B(i) \otimes B(n-i) + \sum_{i=0}^{n} c(B(1) \otimes B(i))B(n-i).\end{aligned}$$

Hence $\Delta(xy) \in \bigoplus_{i=0}^{n+1} B(i) \otimes B(n+1-i)$, since $c(B(1) \otimes B(i)) = B(i) \otimes B(1)$ for all i.

We have shown that $\Bbbk[V]$ is an $\mathbb{N}_0$-graded braided bialgebra. It is strictly graded as a coalgebra, since $P(A) = A(1)$. Hence $\Bbbk[V] \cong \mathcal{B}(V)$ by Theorem 7.1.7(2). □

COROLLARY 7.1.9. *Let (V, c) be a braided vector space, and let $\mathcal{S}$ be the antipode of $\mathcal{B}(V, c)$.*

(1) *$\mathcal{S}$ is bijective, and for all $x \in V$, $\mathcal{S}(x) = -x$.*
(2) *$\mathcal{B}(V, c)^{\mathrm{cop}} = \mathcal{B}(V, c^{-1})$, and $I(V, c) = I(V, c^{-1})$.*
(3) *$\mathcal{S} : \mathcal{B}(V)^{\mathrm{op}} \to \mathcal{B}(V)^{\mathrm{cop}}$ is an isomorphism of $\mathbb{N}_0$-graded braided Hopf algebras.*

PROOF. (1) and (3) follow from Propositions 6.2.13 and 6.2.8(2)(c). For (2) note that $\mathcal{B}(V, c)^{\mathrm{cop}}$ is a pre-Nichols algebra of (V, c^{-1}), and that $P(\mathcal{B}(V, c)^{\mathrm{cop}}) = V$. Hence $\mathcal{B}(V, c)^{\mathrm{cop}} = \mathcal{B}(V, c^{-1})$ by Theorem 7.1.7(2), and $\mathcal{B}(V, c) = \mathcal{B}(V, c^{-1})$ as algebras. □

REMARK 7.1.10. Let (V, c) be a braided vector space. The defining ideal $I(V)$ of the Nichols algebra is an $\mathbb{N}_0$-graded and categorical subspace and a coideal of

$T(V)$. Hence for all $N \geq 2$,

$$I_N(V) := \bigoplus_{2 \leq n \leq N} \ker(S_n)$$

is an $\mathbb{N}_0$-graded coideal of $T(V)$, and a categorical subspace. The two-sided ideals $(I_N(V))$ of $T(V)$ generated by $I_N(V)$ are coideals and categorical subspaces by Lemma 6.2.10. Hence the quotients $T(V)/(I_N(V))$ are pre-Nichols algebras of V.

We apply Theorem 6.3.2 to the Hopf algebra quotient $\mathcal{B}(V)$ of the tensor algebra of a braided vector space. In particular, it turns out that the Hilbert series of $\mathcal{B}(V)$ only depends on the dimensions of the kernels of all the maps $S_{n-1,1} : V^{\otimes n} \to V^{\otimes n}$, $n \geq 2$.

PROPOSITION 7.1.11. *Let (V, c) be a braided vector space, $\pi : T(V) \to \mathcal{B}(V)$ the canonical surjection, and $K = T(V)^{\mathrm{co}\,\mathcal{B}(V)}$, where $T(V)$ is a right $\mathcal{B}(V)$-comodule by $(\mathrm{id}_{T(V)} \otimes \pi)\Delta$.*

(1) *$K \subseteq T(V)$ is an $\mathbb{N}_0$-graded left coideal subalgebra, and there is an $\mathbb{N}_0$-graded, left K-linear and right $\mathcal{B}(V)$-colinear isomorphism*

$$T(V) \cong K \otimes \mathcal{B}(V).$$

(2) *For all $n \geq 2$, $K(n) = \ker(S_{n-1,1} : V^{\otimes n} \to V^{\otimes n})$, and $K(n)$ contains all primitive elements of $T(V)$ of degree n. As a right ideal of $T(V)$, the defining ideal $I(V)$ of $\mathcal{B}(V)$ is generated by $\bigoplus_{n \geq 2} K(n)$.*

PROOF. (1) By Theorem 7.1.2(2), $I(V)$ is a categorical $\mathbb{N}_0$-graded coideal and ideal of $T(V)$. Thus K is a left coideal subalgebra of $T(V)$ by Theorem 6.3.2, and the remaining claim is a special case of Corollary 6.3.10.

(2) By definition, $K = \bigoplus_{n \geq 0} K(n)$, where for all $n \geq 0$,

$$K(n) = \{x \in V^{\otimes n} \mid x^{(1)} \otimes \pi(x^{(2)}) = x \otimes 1\}.$$

In particular, $K(0) = \Bbbk 1$, and $K(1) = 0$. Recall that for $x \in V^{\otimes n}$ and $n \geq 2$, $\Delta(x) = 1 \otimes x + x \otimes 1 + \sum_{i=1}^{n-1} \Delta_{i,n-i}(x)$, where $\Delta_{i,n-i} = S_{i,n-i}$ by Corollary 1.8.4. Hence

$$\begin{aligned} x \in K(n) \iff & 1 \otimes \pi_n(x) + \sum_{i=1}^{n-1} (\mathrm{id}_{V^{\otimes i}} \otimes \pi_{n-i})(S_{i,n-i}(x)) = 0 \\ \iff & \pi_n(x) = 0, \text{ and for all } 1 \leq i \leq n-1, \\ & (\mathrm{id}_{V^{\otimes i}} \otimes \pi_{n-i})(S_{i,n-i}(x)) = 0. \end{aligned}$$

Since $\ker(\pi_m) = \ker(S_m)$ for all $m \geq 1$, we conclude that

$$\begin{aligned} K(n) &= \{x \in V^{\otimes n} \mid S_n(x) = 0, {S_{n-i}}^{\uparrow i} S_{i,n-i}(x) = 0 \text{ for all } 1 \leq i \leq n-1\} \\ &= \ker(S_{n-1,1}), \end{aligned}$$

where the last equality holds by Corollary 1.8.8(3) and (4).

Primitive elements x of $T(V)$ of degree n are contained in $I(V)$, hence in $K = T(V)^{\mathrm{co}\,T(V)/I(V)}$ by definition. Finally, $I(V) = K^+T(V)$ follows from Theorem 6.3.2. $\square$

COROLLARY 7.1.12. *Let (V,c) be a finite-dimensional braided vector space, and $d_n = \dim \ker(S_{n-1,1} : V^{\otimes n} \to V^{\otimes n})$ for all $n \geq 2$. Then*

$$\mathcal{H}_{T(V)}(t) = \mathcal{H}_{\mathcal{B}(V)}(t)\Big(1 + \sum_{n\geq 2} d_n t^n\Big).$$

PROOF. This follows from Proposition 7.1.11, and $K(0) = \Bbbk 1, K(1) = 0$. □

We now extend the definition of the Nichols algebra of a braided vector space in the obvious way to Yetter-Drinfeld modules. The Nichols algebra becomes a Hopf algebra in the braided category ${}^H_H\mathcal{YD}$ and not just a braided Hopf algebra. Thus we extend Section 1.6 from $\mathcal{C} = {}^G_G\mathcal{YD}$, G a group, to $\mathcal{C} = {}^H_H\mathcal{YD}$, H a Hopf algebra with bijective antipode.

DEFINITION 7.1.13. Let H be a Hopf algebra with bijective antipode, and $V \in {}^H_H\mathcal{YD}$. Then

$$\mathcal{B}(V) = T(V)/\bigoplus_{n\geq 2} \ker(\varDelta^{T(V)}_{1^n})$$

is called the **Nichols algebra** of V.

An $\mathbb{N}_0$-graded connected Hopf algebra R in ${}^H_H\mathcal{YD}$ is a **pre-Nichols algebra of** V, if

(N1) $R(1) \cong V$ in ${}^H_H\mathcal{YD}$,
(N2) R is generated as an algebra by $R(1)$.

A pre-Nichols algebra of V is a **Nichols algebra of** V, if

(N3) R is strictly graded, that is, $P(R) = R(1)$.

THEOREM 7.1.14. *Let $V \in {}^H_H\mathcal{YD}$.*

(1) *$\mathcal{B}(V)$ is a Nichols algebra of V.*
(2) *Let R be a pre-Nichols algebra of V and $f : R(1) \xrightarrow{\cong} V$ an isomorphism in ${}^H_H\mathcal{YD}$.*
    (a) *There is exactly one morphism $\pi : R \to \mathcal{B}(V)$ of $\mathbb{N}_0$-graded Hopf algebras in ${}^H_H\mathcal{YD}$ such that f is the restriction of π to $R(1)$, and π is surjective.*
    (b) *π is bijective if and only if R is a Nichols algebra of V.*

PROOF. See the proof of Theorem 7.1.7 or 1.6.18. □

Direct sum decompositions of Yetter-Drinfeld modules give rise to very important gradings of the Nichols algebra.

COROLLARY 7.1.15. *Let Γ be an abelian monoid, H a Γ-graded Hopf algebra with bijective antipode, and let V, W be Γ-graded objects in ${}^H_H\mathcal{YD}$.*

(1) *The Nichols algebra $\mathcal{B}(V)$ is a Γ-graded Hopf algebra quotient of $T(V)$ in ${}^H_H\mathcal{YD}$, where $\mathcal{B}(V)(\gamma) \cap V = V(\gamma)$ for all $\gamma \in \Gamma$, and $\mathcal{B}(V) = \bigoplus_{n\geq 0} B^n(V)$ is a decomposition into Γ-graded subobjects in ${}^H_H\mathcal{YD}$.*
(2) *Let $f : V \to W$ be a morphism of Γ-graded objects in ${}^H_H\mathcal{YD}$. Then there is a unique morphism $\mathcal{B}(f) : \mathcal{B}(V) \to \mathcal{B}(W)$ of Γ-graded Hopf algebras in ${}^H_H\mathcal{YD}$ such that $\mathcal{B}(f)|V = f$. If f is injective (surjective, bijective) then $\mathcal{B}(f)$ is injective (surjective, bijective, respectively).*

PROOF. (1) By Proposition 5.5.5, the tensor algebra $T(V)$ is a Γ-graded Hopf algebra in ${}_H^H\mathcal{YD}$. Hence for all $n \geq 2$, the map

$$\Delta_{1^n} : V^{\otimes n} \subseteq T(V) \xrightarrow{\Delta^{n-1}} T(V)^{\otimes n} \xrightarrow{\pi_1^{\otimes n}} T^1(V)^{\otimes n}$$

is a Γ-graded map of Yetter-Drinfeld modules, and $I(V)(n) = \ker(\Delta_{1^n})$ and $\mathcal{B}^n(V)$ are Γ-graded objects in ${}_H^H\mathcal{YD}$.

(2) The uniqueness of $\mathcal{B}(f)$ is clear. The existence of $\mathcal{B}(f)$ as a morphism of $\mathbb{N}_0$-graded Hopf algebras in ${}_H^H\mathcal{YD}$ follows by the argument in Remark 7.1.4. The morphism $\mathcal{B}(f)$ restricted to $\mathcal{B}(V)(n)$, where $n \in \mathbb{N}_0$, is induced by $f^{\otimes n}$ and hence it is Γ-graded. Indeed,

$$\mathcal{B}(f)(V(\gamma_1)\cdots V(\gamma_n)) = f(V(\gamma_1))\cdots f(V(\gamma_n)) \subseteq \mathcal{B}(W)(\gamma)$$

for all $n \in \mathbb{N}_0$ and $\gamma, \gamma_1, \dots, \gamma_n \in \Gamma$ with $\gamma = \gamma_1 + \cdots + \gamma_n$. The claim on the surjectivity of $\mathcal{B}(f)$ is obvious. The injectivity of $\mathcal{B}(f)$ for an injective f follows from the equations

$$\Delta_{1^n} f^{\otimes n} = f^{\otimes n}\Delta_{1^n}$$

for all $n \in \mathbb{N}_0$. $\square$

REMARK 7.1.16. In Corollary 7.1.15(2), $\ker(f)$ is clearly contained in $\ker(\mathcal{B}(f))$. In general, however, $\ker(\mathcal{B}(f))$ is larger than the ideal generated by $\ker(f)$. Indeed, assume that $\Bbbk$ has characteristic 2. Let $g \in \mathbb{Z}$ and $V = V(1,2) \in {}_{\mathbb{Z}}^{\mathbb{Z}}\mathcal{YD}$ as in Example 1.4.19. Then $V = V_g$ and there is a basis v_1, v_2 of V with $g \cdot v_1 = v_1$, $g \cdot v_2 = v_1 + v_2$. Let $W = \Bbbk w \in {}_{\mathbb{Z}}^{\mathbb{Z}}\mathcal{YD}$ with $\delta(w) = g \otimes w$, $g \cdot w = w$. Then there is a unique morphism $f : V \to W$ of Yetter-Drinfeld modules with $f(v_2) = w$ and $\ker(f) = \Bbbk v_1$. Moreover, $\mathcal{B}(W)(2) = 0$ and

$$\mathcal{B}(V)(2) = V^{\otimes 2}/\mathrm{span}_{\Bbbk}\{v_1^2\}.$$

Hence $v_2^2 \in \ker(\mathcal{B}(f))$ but $v_2^2 \notin (v_1)$.

Nichols algebras of Yetter-Drinfeld modules play an important role in the classification theory of Hopf algebras. They appear naturally as subalgebras of graded Hopf algebras associated to the coradical filtration.

COROLLARY 7.1.17. *Let A be a Hopf algebra, and assume that its coradical $H = \mathrm{Corad}(A)$ is a Hopf subalgebra of A with bijective antipode. Let $\mathrm{gr}\,A$ be the $\mathbb{N}_0$-graded Hopf algebra associated to the coradical filtration of A, and let $\pi : \mathrm{gr}\,A \to H$ be the projection onto elements of degree 0. Then*

$$R = A^{\mathrm{co}\,H} = \{x \in A \mid x_{(1)} \otimes \pi(x_{(2)}) = x \otimes 1\}$$

is a Hopf algebra in ${}_H^H\mathcal{YD}$. The space $V = P(R)$ of primitive elements in R is an object in ${}_H^H\mathcal{YD}$, the subalgebra of R generated by V is isomorphic to $\mathcal{B}(V)$ as an $\mathbb{N}_0$-graded Hopf algebra in ${}_H^H\mathcal{YD}$ with grading $R(n) = R \cap \mathrm{gr}\,A(n)$ for all $n \geq 0$, and

$$\mathcal{B}(V)\#H \subseteq R\#H \cong \mathrm{gr}\,A$$

is a Hopf subalgebra.

Recall from Remark 5.3.17 that the assumption on the bijectivity of the antipode of H can be dropped.

PROOF. By Corollary 5.3.16, R is a strictly $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$, and $R\#H \cong \operatorname{gr} A$. Hence the subalgebra $\Bbbk[V] \subseteq R$ generated by V is a pre-Nichols algebra of V. It is strictly $\mathbb{N}_0$-graded as a subcoalgebra of the strictly graded coalgebra R. By Theorem 7.1.14, $\Bbbk[V] \cong \mathcal{B}(V)$. □

7.2. Duality of Nichols algebras

DEFINITION 7.2.1. Let X, Y be vector spaces, and let $\langle\ ,\ \rangle : X \otimes Y \to \Bbbk$ be a bilinear form. The **extended form of** $\langle\ ,\ \rangle$ is the unique bilinear form

$$(\ ,\ ) : T(X) \otimes T(Y) \to \Bbbk$$

on the tensor algebras such that

$$(1,1) = 1, \tag{7.2.1}$$

$$(T^n(X), T^m(Y)) = 0 \quad \text{for all } n \neq m, \tag{7.2.2}$$

$$(x_n \cdots x_2 x_1, y_1 y_2 \cdots y_n) = \prod_{i=1}^{n} \langle x_i, y_i \rangle \tag{7.2.3}$$

for all $n \geq 1$, $1 \leq i \leq n$, $x_i \in X$, $y_i \in Y$.

Rule (7.2.3) is the natural choice from a categorical point of view, since it makes sense in any monoidal category instead of vector spaces. Recall that a bilinear form $\langle\ ,\ \rangle : X \otimes Y \to \Bbbk$ is non-degenerate if the induced maps $X \to Y^*$, $x \mapsto (y \mapsto \langle x, y \rangle)$, and $Y \to X^*$, $y \mapsto (x \mapsto \langle x, y \rangle)$, are injective.

LEMMA 7.2.2. *Let X, Y be vector spaces, and $\langle\ ,\ \rangle : X \otimes Y \to \Bbbk$ a bilinear form. Let $(\ ,\ ) : T(X) \otimes T(Y) \to \Bbbk$ be the extended form of $\langle\ ,\ \rangle$.*

(1) *If $\langle\ ,\ \rangle$ is non-degenerate, then the extended form is non-degenerate.*

(2) *If $\langle\ ,\ \rangle$ is non-degenerate, X, Y are finite-dimensional, and (Y, d) (respectively, (X, c)) is a braided vector space, then (X, c) (respectively, (Y, d)) is a braided vector space, where the braiding of X (respectively, Y) is uniquely determined by the equation*

$$(c(x), y) = (x, d(y))$$

for all $x \in T^2(X)$, $y \in T^2(Y)$.

(3) *Let (X, c) and (Y, d) be braided vector spaces, and assume that*

$$(c(x), y) = (x, d(y))$$

for all $x \in T^2(X)$, $y \in T^2(Y)$. Then

$$(S_n(x), y) = (x, S_n(y))$$

for all $x \in T^n(X)$, $y \in T^n(Y)$, $n \geq 1$.

PROOF. (1) We show that $T^n(X) \to T^n(Y)^*$, $x \mapsto (y \mapsto (x, y))$, is injective for all $n \geq 2$. Let $x \in T^n(X)$ and suppose that $(x, y) = 0$ for all $y \in T^n(Y)$. Write $x = \sum_{i=1}^{r} x_i \otimes x'_i$, where $x_1, \ldots, x_r$ are linearly independent in X, and $x'_i \in T^{n-1}(X)$ for all i. Then for all $y' \in T^{n-1}(Y)$ and $y \in Y$,

$$0 = (x, y' \otimes y) = \sum_{i=1}^{r} (x_i, y)(x'_i, y') = \left\langle \sum_{i=1}^{r} (x'_i, y') x_i, y \right\rangle.$$

Hence $(x'_i, y') = 0$ for all i, and the claim follows by induction. The injectivity of $T^n(Y) \to T^n(X)^*$ follows in the same way.

(2) We assume that (Y, d) is a braided vector space (the case when (X, c) is braided is treated in the same way). Since the extended form $(\ ,\ )$ of $\langle\ ,\ \rangle$ is non-degenerate and X, Y are finite-dimensional, the map

$$T^n(X) \xrightarrow{\cong} T^n(Y)^*,\ \ x \mapsto (y \mapsto (x, y)),$$

is an isomorphism for all $n \geq 0$. In particular, if $n = 2$, we can define for each $x \in T^2(X)$ a unique element $c(x) \in T^2(X)$ such that $(c(x), y) = (x, d(y))$ for all $y \in T^2(Y)$. Then $c : X \otimes X \to X \otimes X$ is an isomorphism. We have to check the braid relation $c_1c_2c_1 = c_2c_1c_2$ on $X^{\otimes 3}$. Note that by construction,

$$(c_i(x), y) = (x, d_{n-i}(y)) \tag{7.2.4}$$

for all $x \in T^n(X)$, $y \in T^n(Y)$ and $1 \leq i \leq n-1$.

In particular for all $x \in T^3(X)$, $y \in T^3(Y)$,

$$(c_1c_2c_1(x), y) = (x, d_2d_1d_2(y)) = (x, d_1d_2d_1(y)) = (c_2c_1c_2(x), y).$$

Hence $c_1c_2c_1(x) = c_2c_1c_2(x)$ for all $x \in T^3(X)$ by non-degeneracy of $(\ ,\ )$.

(3) It follows from the assumption in (3) that the braidings of X and Y satisfy (7.2.4). Hence by Remark 1.8.6, for any $w \in \mathbb{S}_n$ with reduced decomposition $(i_1, \ldots, i_t)$,

$$(c_w(x), y) = (c_{i_1} \cdots c_{i_t}(x), y) = (x, d_{n-i_t} \cdots d_{n-i_1}(y)) = (x, d_{w_0ww_0}(y))$$

for all $x \in T^n(X)$, $y \in T^n(y)$. Then (3) follows from the definition of S_n. $\square$

THEOREM 7.2.3. *Let (X, c), (Y, d) be braided vector spaces, $\langle\ ,\ \rangle : X \otimes Y \to \Bbbk$ a bilinear form, and $(\ ,\ )$ the extended form of $\langle\ ,\ \rangle$. If $(c(x), y) = (x, d(y))$ for all $x \in T^2(X)$ and $y \in T^2(Y)$, then there exists a unique bilinear form*

$$\langle\ ,\ \rangle : T(X) \otimes T(Y) \to \Bbbk$$

extending the given form on $X \otimes Y$ such that

$$\langle 1, 1\rangle = 1, \tag{7.2.5}$$

$$\langle T^n(X), T^m(Y)\rangle = 0 \quad \textit{for all } n \neq m, \tag{7.2.6}$$

and for all $w, x \in T(X)$ and $y, z \in T(Y)$

$$\langle wx, y\rangle = \langle w, y^{(2)}\rangle\langle x, y^{(1)}\rangle, \tag{7.2.7}$$

$$\langle x, yz\rangle = \langle x^{(2)}, y\rangle\langle x^{(1)}, z\rangle. \tag{7.2.8}$$

If the form $\langle\ ,\ \rangle : X \otimes Y \to \Bbbk$ is non-degenerate, then the defining ideals of the Nichols algebras of X and Y are given by

$$I(X) = T(Y)^{\perp} = \{x \in T(X) \mid \langle x, T(Y)\rangle = 0\}, \tag{7.2.9}$$

$$I(Y) = T(X)^{\perp} = \{y \in T(Y) \mid \langle T(X), y\rangle = 0\}. \tag{7.2.10}$$

PROOF. We define a bilinear form $\langle\ ,\ \rangle : T(X) \otimes T(Y) \to \Bbbk$ by (7.2.5), (7.2.6), and

$$\langle x, y\rangle = (x, S_n(y)) = (S_n(x), y)$$

for all $x \in T^n(X)$, $y \in T^n(Y)$, $n > 0$, where we have used Lemma 7.2.2(3). To prove (7.2.7), let $n \geq 1$, $1 \leq i \leq n-1$, and $w \in T^{n-i}(X)$, $x \in T^i(X)$, $y \in T^n(Y)$.

Then, by Lemma 7.2.2(3),

$$\begin{aligned}
\langle wx, y\rangle &= (wx, S_n(y)) \\
&= (wx, (S_i \otimes S_{n-i})S_{i,n-i}(y)) && \text{(by (1.8.10))} \\
&= (S_{n-i}(w) \otimes S_i(x), S_{i,n-i}(y)) && \text{(by Lemma 7.2.2(3))} \\
&= (S_{n-i}(w) \otimes S_i(x), \Delta_{i,n-i}(y)) && \text{(by Theorem 1.9.1)} \\
&= (S_{n-i}(w), y^{(2)})(S_i(x), y^{(1)}) && \text{(by (7.2.6))} \\
&= \langle w, y^{(2)}\rangle\langle x, y^{(1)}\rangle.
\end{aligned}$$

Equation (7.2.8) is proved in the same way, beginning with

$$\langle x, yz\rangle = (S_n(x), yz), \text{ if } x \in T^n(X).$$

The uniqueness of the form $\langle\ ,\ \rangle$ on the tensor algebras is clear by induction using (7.2.7).

If $\langle\ ,\ \rangle : X \otimes Y \to \Bbbk$ is non-degenerate, then the extended form $(\ ,\ )$ is non-degenerate by Lemma 7.2.2(1). Hence, for all $n \geq 2$,

$$\begin{aligned}
\{x \in T^n(X) \mid \langle x, T^n(Y)\rangle = 0\} &= \{x \in T^n(X) \mid (S_n(x), T^n(Y)) = 0\} \\
&= \ker(S_n) \\
&= I(X)(n),
\end{aligned}$$

where the last equality holds by Corollary 1.9.7. Thus $I(X) = T(Y)^\perp$ by (7.2.6), and $I(Y) = T(X)^\perp$ is shown in the same way. $\square$

Definition 7.2.4. Let (V, c) be a finite-dimensional braided vector space, and let $(\ ,\ ) : T(V^*) \otimes T(V) \to \Bbbk$ be the form of Definition 7.2.1 extending the evaluation $V^* \otimes V \to \Bbbk$. The **dual braiding** $c : V^* \otimes V^* \to V^* \otimes V^*$ is defined by

$$(c(f \otimes g), v \otimes w) = (f \otimes g, c(v \otimes w)) \tag{7.2.11}$$

for all $f, g \in V^*$ and $v, w \in V$.

Note that the dual braiding of a finite-dimensional braided vector space is a well-defined braiding by Lemma 7.2.2(2). We finally can formulate the very useful duality property of Nichols algebras.

Corollary 7.2.5. *Let (V, c) be a finite-dimensional braided vector space, and let $\mathcal{B}(V)$ and $\mathcal{B}(V^*)$ be the Nichols algebras of V and V^* with respect to c and to the dual braiding, respectively. Then there is a unique non-degenerate bilinear form $\langle\ ,\ \rangle : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to \Bbbk$ extending the evaluation map $\langle\ ,\ \rangle : V^* \otimes V \to \Bbbk$ such that*

$$\langle 1, 1\rangle = 1, \tag{7.2.12}$$

$$\langle \mathcal{B}^n(V^*), \mathcal{B}^m(V)\rangle = 0 \quad \textit{for all } n \neq m, \tag{7.2.13}$$

and for all $f, g \in \mathcal{B}(V^*)$ and $v, w \in \mathcal{B}(V)$

$$\langle fg, v\rangle = \langle f, v^{(2)}\rangle\langle g, v^{(1)}\rangle, \tag{7.2.14}$$

$$\langle f, vw\rangle = \langle f^{(2)}, v\rangle\langle f^{(1)}, w\rangle. \tag{7.2.15}$$

PROOF. We apply Theorem 7.2.3 to the evaluation form $\langle\ ,\ \rangle : V^* \otimes V \to \Bbbk$. By dividing out the radicals $I(V^*) = T(V)^\perp$ and $I(V) = T(V^*)^\perp$ of the form in $\langle\ ,\ \rangle : T(V^*) \otimes T(V) \to \Bbbk$ in Theorem 7.2.3, we get a non-degenerate form on the Nichols algebras satisfying all the claims. The uniqueness of the form is clear by (7.2.14). □

REMARK 7.2.6. We note that the form in Corollary 7.2.5 is defined explicitly as follows. Let $(\ ,\ ) : T(V^*) \otimes T(V) \to \Bbbk$ be the extended form of the evaluation $\langle\ ,\ \rangle : V^* \otimes V \to \Bbbk$. Then $\langle\ ,\ \rangle : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to \Bbbk$ is defined by

$$\langle f_n \cdots f_1, v_1 \cdots v_n\rangle = (f_n \otimes \cdots \otimes f_1, S_n(v_1 \otimes \cdots \otimes v_n))$$

for all $f_i \in V^*$, $v_i \in V$, $1 \leq i \leq n$, $n \geq 2$, where

$$S_n(v_1 \otimes \cdots \otimes v_n) = \Delta_{1^n}(v_1 \otimes \cdots \otimes v_n)$$

is defined with respect to the tensor algebra $T(V)$.

Let H be a Hopf algebra with bijective antipode. We apply the results in this section to Yetter-Drinfeld modules.

LEMMA 7.2.7. (1) *Let $X, Y \in {}^H_H\mathcal{YD}$, and let $\langle\ ,\ \rangle : X \otimes Y \to \Bbbk$ be a bilinear form with extended form $(\ ,\ )$. If $\langle\ ,\ \rangle$ is a morphism in ${}^H_H\mathcal{YD}$, then $(\ ,\ )$ is a morphism in ${}^H_H\mathcal{YD}$, and*

$$(c(x), y) = (x, c(y))$$

for all $x \in T^2(X)$, $y \in T^2(Y)$.

(2) *Let $V \in {}^H_H\mathcal{YD}$ be finite-dimensional. Then the dual braiding of the Yetter-Drinfeld braiding $c_{V,V}$ is the braiding of the (left) dual V^* in ${}^H_H\mathcal{YD}$.*

PROOF. (1) We show by induction that the extended form restricted to the subspace $T^n(X) \otimes T^n(Y)$ for $n \geq 1$ is H-linear and H-colinear. Let $n \geq 2$, $h \in H$, and $x \in X$, $y \in Y$, $u \in T^{n-1}(X)$, $v \in T^{n-1}(Y)$. Then

$$\begin{aligned}
\big(h_{(1)} \cdot (u \otimes x), h_{(2)} \cdot (y \otimes v)\big) &= (h_{(1)} \cdot u \otimes h_{(2)}x, h_{(3)} \cdot y \otimes h_{(4)} \cdot v)\\
&= (h_{(1)} \cdot u, h_{(4)} \cdot v)\langle h_{(2)}x, h_{(3)} \cdot y\rangle\\
&= (h_{(1)} \cdot u, h_{(3)} \cdot v)\varepsilon(h_{(2)})\langle x, y\rangle\\
&= (h_{(1)} \cdot u, h_{(2)} \cdot v)\langle x, y\rangle\\
&= \varepsilon(h)(u, v)\langle x, y\rangle = \varepsilon(h)(u \otimes x, y \otimes v),
\end{aligned}$$

where the second last equation follows from induction hypothesis. In a similar way one proves that the extended form of $\langle\ ,\ \rangle$ is H-colinear.

To show that the braidings are adjoint under the form $(\ ,\ )$, let $x, x' \in X$ and $y, y' \in Y$. Then

$$\begin{aligned}
(c(x \otimes x'), y \otimes y') &= (x_{(-1)} \cdot x' \otimes x_{(0)}, y \otimes y')\\
&= \langle x_{(-1)} \cdot x', y'\rangle\langle x_{(0)}, y\rangle\\
&= \langle \mathcal{S}^{-1}(y_{(-1)}) \cdot x', y'\rangle\langle x, y_{(0)}\rangle \quad \text{(by Lemma 4.2.1(2))}\\
&= \langle x', y_{(-1)} \cdot y'\rangle\langle x, y_{(0)}\rangle \quad \text{(by Lemma 4.2.1(1))}\\
&= (x \otimes x', y_{(-1)} \cdot y' \otimes y_{(0)})\\
&= (x \otimes x', c(y \otimes y')).
\end{aligned}$$

(2) follows from (1), since the evaluation map $V^* \otimes V \to \Bbbk$ is a morphism in ${}^H_H\mathcal{YD}$ by Lemma 4.2.1. $\square$

COROLLARY 7.2.8. *Let $V \in {}^H_H\mathcal{YD}$ be finite-dimensional. Then there is a unique non-degenerate bilinear form $\langle\ ,\ \rangle : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to \Bbbk$ extending the evaluation map $\langle\ ,\ \rangle : V^* \otimes V \to \Bbbk$ satisfying (7.2.12)–(7.2.15), and for all $h \in H$, $v \in \mathcal{B}(V)$, $f \in \mathcal{B}(V^*)$,*

$$\langle h \cdot f, v\rangle = \langle f, \mathcal{S}(h) \cdot v\rangle, \tag{7.2.16}$$

$$f_{(-1)}\langle f_{(0)}, v\rangle = \mathcal{S}^{-1}(v_{(-1)})\langle f, v_{(0)}\rangle \tag{7.2.17}$$

PROOF. By Lemma 7.2.7 we can apply Corollary 7.2.5 to V with the Yetter-Drinfeld braiding. This proves the first part of the claim. By the proof of Theorem 7.2.3 with $X = V^*$, $Y = V$, the form $\langle\ ,\ \rangle$ on the Nichols algebras is induced from the form $\langle\ ,\ \rangle : T(V^*) \otimes T(V) \to \Bbbk$, defined by (7.2.5), (7.2.6), and

$$\langle f, v\rangle = (f, S_n(v)) = (S_n(f), v)$$

for all $f \in T^n(V^*)$, $v \in T^n(V)$, $n > 0$. Here, $(\ ,\ )$ is the extended form of the evaluation form. The form $\langle\ ,\ \rangle : T(V^*) \otimes T(V) \to \Bbbk$ is a morphism in ${}^H_H\mathcal{YD}$ since the maps S_n and by Lemma 7.2.7 the extended form $(\ ,\ )$ are morphisms in ${}^H_H\mathcal{YD}$. Hence the induced form on the Nichols algebras is a morphism in ${}^H_H\mathcal{YD}$. Thus (7.2.16) and (7.2.17) follow from Lemma 4.2.1. $\square$

7.3. Differential operators for Nichols algebras

Differential operators for braided Hopf algebras can be defined as linear maps in the general context of graded coalgebras. We restrict ourselves here to the discussion of first order differential operators.

DEFINITION 7.3.1. Let $C = \bigoplus_{n\geq 0} C(n)$ be an $\mathbb{N}_0$-graded coalgebra with projection maps $\pi_n : C \to C(n)$, $n \geq 0$. We write $\Delta(x) = x^{(1)} \otimes x^{(2)}$ for the comultiplication of $x \in C$. For any linear form $f : C(1) \to \Bbbk$ we define linear maps by

$$\partial^l_f : C \to C,\ x \mapsto f\pi_1(x^{(1)})x^{(2)}, \qquad \partial^r_f : C \to C,\ x \mapsto x^{(1)} f\pi_1(x^{(2)}).$$

Thus $\partial^l_f(C(n)) \subseteq C(n-1)$, $\partial^r_f(C(n)) \subseteq C(n-1)$ for all $n \geq 1$, $\partial^l_f(C(0)) = 0$, $\partial^r_f(C(0)) = 0$, and for all $x \in C(n)$, $n \geq 1$,

$$\partial^l_f(x) = (f \otimes \mathrm{id})\Delta_{1,n-1}(x), \qquad \partial^r_f(x) = (\mathrm{id} \otimes f)\Delta_{n-1,1}(x).$$

REMARK 7.3.2. Let C be an $\mathbb{N}_0$-graded and connected coalgebra. Recall from Section 1.3 that $I_C = \bigoplus_{n\geq 2} \ker(\Delta_{1^n})$ is the largest coideal of C contained in $\bigoplus_{n\geq 2} C(n)$, and $\mathcal{B}(C) = C/I_C$ is the universal strictly graded quotient coalgebra of C with $C(1) = \mathcal{B}(C)(1)$.

We note some immediate consequences of Definition 7.3.1.

(1) The following diagrams commute for all $f \in C(1)^*$

$$\begin{tikzcd} C \arrow[r, "\partial^r_f"] \arrow[d, "\pi"] & C \arrow[d, "\pi"] \\ \mathcal{B}(C) \arrow[r, "\partial^r_f"] & \mathcal{B}(C) \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\partial^l_f"] \arrow[d, "\pi"] & C \arrow[d, "\pi"] \\ \mathcal{B}(C) \arrow[r, "\partial^l_f"] & \mathcal{B}(C), \end{tikzcd}$$

where we have used the same notation for ∂^r_f and ∂^l_f for the coalgebras C and $\mathcal{B}(C)$, respectively.

(2) For all $f, g \in C(1)^*$ and $x \in C$,

$$\Delta(\partial^r_f(x)) = x^{(1)} \otimes \partial^r_f(x^{(2)}), \tag{7.3.1}$$

$$\Delta(\partial^l_f(x)) = \partial^l_f(x^{(1)}) \otimes x^{(2)}, \tag{7.3.2}$$

$$\partial^r_f \partial^l_g = \partial^l_g \partial^r_f. \tag{7.3.3}$$

As always we denote the kernel of the counit ε by $C^+ = \bigoplus_{n \geq 1} C(n)$. Let $\pi : C \to \mathcal{B}(C)$ be the canonical surjection of coalgebras.

For $\partial = \partial^r$ or $\partial = \partial^l$, a subspace $I \subseteq C$ is called **∂-invariant**, if $\partial_f(I) \subseteq I$ for all $f \in C(1)^*$.

We formulate the next proposition for ∂^r. There is also a ∂^l-version of the proposition which is proved in the same way.

PROPOSITION 7.3.3. *Let C be an $\mathbb{N}_0$-graded connected coalgebra.*

1. *Assume that C is strictly graded.*
   (a) *If $x \in C^+$ and $\partial^r_f(x) = 0$ for all $f \in C(1)^*$, then $x = 0$.*
   (b) *Any ∂^r-invariant subspace of C^+ is zero.*
2. *I_C is the largest ∂^r-invariant subspace of C^+.*

PROOF. (1) (a) Let $x = \sum_{i=1}^n x_i$, $x_i \in C(i)$ for all $1 \leq i \leq n$, and assume that $\partial^r_f(x) = 0$ for all $f \in C(1)^*$. Then for all $1 \leq i \leq n$, $\partial^r_f(x_i) = 0$ for all $f \in C(1)^*$, hence $\Delta_{i-1,1}(x_i) = 0$. By Proposition 1.3.14, $\Delta_{i-1,1}$ is injective for all $i \geq 2$, and $\Delta_{0,1}$ is bijective by definition. Thus $x = 0$.

(b) Let $I \subset C^+$ be a ∂^r-invariant subspace. Assume that $I \neq 0$. Let $n \geq 1$ and $x = \sum_{i=1}^n x_i \in I$ with $x_i \in C(i)$ for all $1 \leq i \leq n$, $x_n \neq 0$, and $I \cap \sum_{i=1}^{n-1} C(i) = 0$. Then $\partial^r_f(x) \in I \cap \sum_{i=1}^{n-1} C(i)$ for all $f \in C(1)^*$ by the ∂^r-invariance of I and since $I \subseteq C^+$. Hence $\partial^r_f(x) = 0$ for all $f \in C(1)^*$ by assumption on n. This contradicts (a). Hence $I = 0$.

(2) By Lemma 1.3.13(1b), for all $n \geq 2$, $\Delta_{1^n} = (\Delta_{1^{n-1}} \otimes \Delta_{1^1})\Delta_{n-1,1}$. Hence for all $x \in \ker(\Delta_{1^n})$, $\Delta_{n-1,1}(x) \in \ker(\Delta_{1^{n-1}}) \otimes C(1)$, since Δ_{1^1} is the identity. This proves that I_C is ∂^r-invariant.

Let $I \subseteq C^+$ be a ∂^r-invariant subspace. Then $\pi(I) \subseteq \mathcal{B}(C)^+$ is a subspace with $\partial^r_f(\pi(I)) = \pi\partial^r_f(I) \subseteq \pi(I)$ for all $f \in C(1)^*$ by Remark 7.3.2. Hence $\pi(I) = 0$ by (1)(b). □

Proposition 7.3.3 is very useful if we want to know whether a given element $x \in \mathcal{B}(C)(n)$, $n \geq 2$, is non-zero. If $x \neq 0$, then there are linear forms $f_1, \ldots, f_n$ in $\mathcal{B}(C)(1)^*$ such that $\partial^r_{f_1} \cdots \partial^r_{f_n}(x) \neq 0$ in $\Bbbk$.

PROPOSITION 7.3.4. *Let (V, c) be a braided vector space. Then the defining ideal $I(V) \subseteq T(V)$ of the Nichols algebra of V is generated as a $T(V)$-module, in particular as an ideal of $T(V)$, by*

$$\bigcup_{n \geq 2} \{x \in T^n(V) \mid \partial^r_f(x) = 0 \textit{ for all } f \in V^*\}.$$

PROOF. For all $n \geq 2$ and $f \in V^*$, $\partial^r_f \mid T^n(V) = (\mathrm{id}_{T^{n-1}(V)} \otimes f)\Delta_{n-1,1}$. Thus

$$T^n(V) \cap \bigcap_{f \in V^*} \ker(\partial^r_f) = \ker(\Delta_{n-1,1})$$

for all $n \geq 2$. Since $\Delta_{n-1,1} = S_{n-1,1}$ by Theorem 1.9.1, the claim follows from Proposition 7.1.11(2). $\square$

Let H be a Hopf algebra with bijective antipode. The maps ∂^r_f and ∂^l_f for graded Yetter-Drinfeld Hopf algebras are skew derivations in the sense of the next lemma.

LEMMA 7.3.5. *Let R be an $\mathbb{N}_0$-graded connected Hopf algebra in ${}^H_H\mathcal{YD}$, and assume that $V = R(1)$ is finite-dimensional. Then for all $f \in V^*$ and $x, y \in R$,*

(1) $\partial^r_f(xy) = x\partial^r_f(y) + \partial^r_{f_{(0)}}(x)\mathcal{S}(f_{(-1)}) \cdot y$,
(2) $\partial^l_f(xy) = x_{(0)}\partial^l_{\mathcal{S}^{-1}(x_{(-1)})\cdot f}(y) + \partial^l_f(x)y$.

If $x \in V$, then $\partial^r_f(x) = f(x) = \partial^l_f(x)$.

PROOF. (1) Let $x, y \in R(n)$, $n \geq 1$. Since R is a graded connected coalgebra, we can write

$$\Delta(x) \in x \otimes 1 + \sum_{l\in L} a_l \otimes x_l + \sum_{i\geq 2} R \otimes R(i),$$
$$\Delta(y) \in y \otimes 1 + \sum_{l\in L} b_l \otimes x_l + \sum_{i\geq 2} R \otimes R(i),$$

where L is a finite index set, and $a_l, b_l \in R(n-1)$, $x_l \in R(1)$ for all $l \in L$. Hence by multiplying $\Delta(x)\Delta(y) = \Delta(xy)$ we obtain

$$\Delta(xy) \in xy \otimes 1 + \sum_{l\in L} xb_l \otimes x_l + \sum_{l\in L}(a_l \otimes x_l)(y \otimes 1) + \sum_{i\geq 2} R \otimes R(i).$$

For all $l \in L$, $(a_l \otimes x_l)(y \otimes 1) = a_l(x_{l(-1)} \cdot y) \otimes x_{l(0)}$, hence

$$(\mathrm{id} \otimes f)((a_l \otimes x_l)(y \otimes 1)) = a_l(x_{l(-1)} \cdot y)f(x_{l(0)}) = a_l(\mathcal{S}(f_{(-1)}) \cdot y)f_{(0)}(x_l)$$

by definition of the H-coaction of V^* in Lemma 4.2.2. Thus

$$\begin{aligned}\partial^r_f(xy) &= \sum_{l\in L} xb_l f(x_l) + \sum_{l\in L} a_l(\mathcal{S}(f_{(-1)}) \cdot y)f_{(0)}(x_l)\\ &= x\partial^r_f(y) + \partial^r_{f_{(0)}}(x)\mathcal{S}(f_{(-1)}) \cdot y.\end{aligned}$$

(2) This proof is very similar to the proof of (1). We write

$$\Delta(x) \in 1 \otimes x + \sum_{l\in L} x_l \otimes a_l + \sum_{i\geq 2} R(i) \otimes R,$$
$$\Delta(y) \in 1 \otimes y + \sum_{l\in L} x_l \otimes b_l + \sum_{i\geq 2} R(i) \otimes R,$$

where $x_l \in R(1), a_l, b_l \in R(n-1)$ for all $l \in L$.

Finally, $\partial^r_f(x) = f(x) = \partial^l_f(x)$ for $x \in V$ follows by definition. $\square$

REMARK 7.3.6. Let R be a pre-Nichols algebra of a finite-dimensional Yetter-Drinfeld module $V \in {}^H_H\mathcal{YD}$. Then the function

$$\partial^r : V^* \to \mathrm{Hom}(R, R), \quad f \mapsto \partial^r_f,$$

is uniquely determined by the rules in Lemma 7.3.5. In other words, if

$$d : V^* \to \mathrm{Hom}(R, R), \quad f \mapsto d_f,$$

is a linear map satisfying

(1) $d_f(xy) = xd_f(y) + d_{f_{(0)}}(x)\mathcal{S}(f_{(-1)}) \cdot y$ for all $f \in V^*$, $x, y \in R$,
(2) $d_f(x) = f(x)$ for all $f \in V^*$, $x \in V$,

then $d_f = \partial^r_f$ for all $f \in V^*$. For the proof note that $d_f(1) = 0$ by (1), and if $d_f(x) = \partial^r_f(x)$ and $d_f(y) = \partial^r_f(y)$, then by (1), $d_f(xy) = \partial^r_f(xy)$.

A similar uniqueness property holds for ∂^l.

We now consider the case when $H = \Bbbk G$ is a group algebra. We show that then the maps ∂^r_f are skew derivations.

DEFINITION 7.3.7. Let G be a group and $V \in {}^G_G\mathcal{YD}$ a finite-dimensional Yetter-Drinfeld module over the group algebra of G. We choose a basis $x_1, \dots, x_\theta$ of G-homogeneous elements, and for all $1 \le i \le \theta$ let $g_i \in G$ with $\delta(x_i) = g_i \otimes x_i$. Let $f_1, \dots, f_\theta$ be the dual basis of $(x_i)_{1 \le i \le \theta}$ in V^*. Let R be a pre-Nichols algebra of V. We define

$$\partial^r_i = \partial^r_{f_i} : R \to R,\ 1 \le i \le \theta.$$

COROLLARY 7.3.8. *Assume the situation of Definition* 7.3.7.

(1) *The linear maps* $\partial^r_i : R \to R$, $1 \le i \le \theta$, *are determined by*
  (a) $\partial^r_i(1) = 0$, $\partial^r_i(x_j) = \delta_{ij}$ *for all* $1 \le i, j \le \theta$,
  (b) $\partial^r_i(xy) = x\partial^r_i(y) + \partial^r_i(x)g_i \cdot y$ *for all* $1 \le i \le \theta$, $x, y \in R$.
(2) *If* $R = \mathcal{B}(V)$, *then for any non-zero element* $x \in \mathcal{B}(V)^+$, $\partial^r_i(x) \ne 0$ *for some* i.
(3) *If* $R = T(V)$, *then* $I(V)$ *is the largest subspace* $I \subseteq T(V)^+$ *such that* $\partial^r_i(I) \subseteq I$ *for all* $1 \le i \le \theta$. *As a right ideal,* $I(V)$ *is generated by* $\cup_{n\ge2}\{x \in T^n(V) \mid \partial^r_i(x) = 0$ *for all* $1 \le i \le \theta\}$.

PROOF. (1) follows from Lemma 7.3.5(1), since for all i, $\delta(f_i) = g_i^{-1} \otimes f_i$. If $f \in V^*$, $f = \sum_{i=1}^\theta \alpha_i f_i$ with scalars $\alpha_i \in \Bbbk$, then $\partial^r_f = \sum_{i=1}^\theta \alpha_i \partial^r_i$. Hence (2) and (3) follow from Propositions 7.3.3 and 7.3.4. □

EXAMPLE 7.3.9. In the situation of Definition 7.3.7, let $1 \le i \le \theta$, and assume that there is a scalar $q_i \in \Bbbk$ such that $g_i \cdot x_i = q_i x_i$. It is easy to check by induction that for all $t \ge 2$, $1 \le j \le \theta$,

$$\partial^r_j(x_i^t) = \delta_{ij}(t)_{q_i} x_i^{t-1}.$$

Hence, by Corollary 7.3.8, $x_i^t \in I_R$ if and only if $(s)_{q_i} = 0$ for some $2 \le s \le t$.

EXAMPLE 7.3.10. We go back to Example 1.10.3 and assume that $n = 3$. Then $\mathcal{O}_2 = \{(1\,2), (2\,3), (1\,3)\}$. Let $g_1 = (1\,2)$, $g_2 = (2\,3)$, $g_3 = (1\,3)$, and let V_3 be the Yetter-Drinfeld module over $\mathbb{S}_3$ with basis x_t, $t \in \mathcal{O}_2$, and

$$\delta(x_t) = t \otimes x_t, \quad s \cdot x_t = -x_{sts},$$

for all $s, t \in \mathcal{O}_2$. Let $a = x_{(1\,2)}$, $b = x_{(2\,3)}$, $c = x_{(1\,3)}$. Then the following quadratic relations hold in $\mathcal{B}(V_3)$.

$$a^2 = 0,\ b^2 = 0,\ c^2 = 0, \tag{7.3.4}$$

$$ab + bc + ca = 0, \tag{7.3.5}$$

$$ba + ac + cb = 0. \tag{7.3.6}$$

Indeed, it is easily checked that the skew derivations $\partial^r_{x_t^*}$ with $t \in \mathcal{O}_2$ annihilate the left-hand sides of the relations. Thus the claim follows from Corollary 7.3.8(2). Multiplying (7.3.5) with a on the right and b on the left gives the equations

$$aba + bca = 0, \quad bab + bca = 0.$$

Hence

$$aba = bab. \tag{7.3.7}$$

Let $\Lambda = abac$. It is easy to check that Λ is a right integral of $\mathcal{B}(V_3)$, that is, $\Lambda a = 0$, $\Lambda b = 0$, $\Lambda c = 0$. To see that Λ is non-zero, we compute derivations.

$$\begin{aligned} \partial^r_{c^*}(abac) &= aba, \\ \partial^r_{b^*}(aba) &= a\partial^r_{b^*}(ba) = a(g_2 \cdot a) = -ac, \\ \partial^r_{c^*}(ac) &= a. \end{aligned}$$

Hence $\partial^r_{a^*}\partial^r_{c^*}\partial^r_{b^*}\partial^r_{c^*}(\Lambda) = -1$.

By choosing the ordering $a < b < c$ of the generators and by writing relations (7.3.4)–(7.3.6) and (7.3.7) as

$$\begin{gathered} a^2 = 0, \quad b^2 = 0, \quad c^2 = 0, \\ ca = -ac - bc, \quad cb = -ac - ba, \\ bab = aba, \end{gathered}$$

we conclude that the monomials

$$1, a, b, c, ab, ac, ba, bc, aba, abc, bac, abac \tag{7.3.8}$$

span the vector space $\mathcal{B}(V_3)$. Since Λ is a non-zero integral in $\mathcal{B}(V_3)$, the relations in (7.3.4)–(7.3.6) generate the ideal $I(V_3)$ by Corollary 4.4.14 for $S = \mathcal{B}(V_3)$.

The monomials in (7.3.8) are non-zero since $\Lambda \neq 0$ and $\partial^r_{b^*}\partial^r_{c^*}(abc) \neq 0$. Finally, the monomials of degree two are linearly independent by definition, and those of degree three because no two of them have the same $\mathbb{S}_3$-degree. Thus (7.3.8) is a basis of $\mathcal{B}(V_3)$ (which proves in a second way that the ideal $I(V_3)$ is generated by (7.3.4)–(7.3.6)). Thus the Hilbert series of $\mathcal{B}(V_3)$ is

$$\mathcal{H}_{\mathcal{B}(V_3)}(t) = 1 + 3t + 4t^2 + 3t^3 + t^4 = (1+t)^2(1+t+t^2).$$

7.4. Notes

7.1. The denotation Nichols algebra and pre-Nichols algebra appeared first in [**AS00a**] and [**Mas08**], respectively.

7.2. We extend the description of Nichols algebras via bilinear forms in [**AGn99**] to Nichols algebras of braided vector spaces.

7.3. Already Nichols [**Nic78**] used "twisted derivations" in the context of Nichols algebras.

CHAPTER 8

Quantized enveloping algebras and generalizations

Quantized enveloping algebras are non-commutative and non-cocommutative Hopf algebra analogues of enveloping algebras of finite-dimensional complex semisimple Lie algebras or of Kac-Moody algebras. They enjoy great attention far beyond the theory of Hopf algebras. Our intention with this chapter is to study quantized enveloping algebras and related Hopf algebras using standard tools in the theory of pointed Hopf algebras. Structural results related to root systems will be discussed in Chapter 16.

Let $n \in \mathbb{N}$ and let $A = (a_{ij})_{i,j\in\{1,\dots,n\}}$ be a symmetrizable Cartan matrix. Let $D = (d_i)_{1\le i\le n}$ be a family of positive integers such that $(d_i a_{ij})_{i,j\in\{1,\dots,n\}}$ is symmetric. For any non-negative integers m, r with $m \ge r$ let

$$[m]_v = \frac{v^m - v^{-m}}{v - v^{-1}}, \quad [m]_v^! = \prod_{i=1}^{m} [i]_v, \quad \begin{bmatrix} m \\ r \end{bmatrix}_v = \frac{[m]_v^!}{[r]_v^! [m-r]_v^!}$$

in $\mathbb{Z}[v, v^{-1}]$. Note that

$$[m]_v^! = v^{-m(m-1)/2} (m)_{v^2}^!, \qquad \begin{bmatrix} m \\ r \end{bmatrix}_v = v^{r(r-m)} \binom{m}{r}_{v^2} \tag{8.0.1}$$

for all $m, r \in \mathbb{N}_0$ with $0 \le r \le m$.

Let $q \in \Bbbk^\times$. The ring homomorphism $\mathbb{Z}[v, v^{-1}] \to \Bbbk$, $v \mapsto q$, defines q-analogues of the above v-numbers, v-factorials and v-binomial coefficients. Assume $q^{2d_i} \ne 1$ for any $i \in \{1, \dots, n\}$. Let U_q denote the associative $\Bbbk$-algebra (depending on q, A, and D) given by generators E_i, F_i, K_i, K_i^{-1}, where $1 \le i \le n$, and relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$
$$K_i E_j = q^{d_i a_{ij}} E_j K_i, \quad K_i F_j = q^{-d_i a_{ij}} F_j K_i,$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}},$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} E_i^m E_j E_i^{1-a_{ij}-m} = 0, \quad (i \ne j)$$
$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} F_i^m F_j F_i^{1-a_{ij}-m} = 0 \quad (i \ne j)$$

with $i, j \in \{1, \dots, n\}$. The algebra U_q is called the **quantized enveloping algebra** of the Kac-Moody algebra associated to A. It is known to be a Hopf algebra with comultiplication Δ, counit ε and antipode $\mathcal{S}$ given by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1, \quad \varepsilon(K_i^{-1}) = 1,$$

$$\mathcal{S}(E_i) = -K_i^{-1}E_i, \quad \mathcal{S}(F_i) = -F_iK_i, \quad \mathcal{S}(K_i) = K_i^{-1}, \quad \mathcal{S}(K_i^{-1}) = K_i$$

for any $1 \leq i \leq n$. We give a proof of this fact in Corollary 8.1.7, where U_q is presented as a quotient of Drinfeld's quantum double of two Hopf algebras.

In Section 8.3 we study Hopf algebras constructed from a Yetter-Drinfeld datum and a linking. In Section 8.4 we specialize this construction to perfect linkings and relate the obtained Hopf algebras to quantized enveloping algebras.

If q is not a root of 1, then the subalgebra U_q^+ of U generated by E_i, $1 \leq i \leq n$, is a Nichols algebra of diagonal type. This will be shown in Chapter 16 in the case when A is of finite type.

8.1. Construction of the Hopf algebra U_q

We will construct U_q as a quotient Hopf algebra of the Drinfeld double with respect to a skew pairing of certain infinite-dimensional Hopf algebras. See Section 2.8 for the general theory of the Drinfeld double.

PROPOSITION 8.1.1. *Let A, U be bialgebras with an invertible skew pairing τ of A and U. Let σ be the associated two-cocycle of $A \otimes U$. Assume that A and U are given by generators $(a_i)_{i\in I_A}$ and $(x_k)_{k\in I_U}$, respectively, and relations $r_j((a_i)_{i\in I_A})$, $j \in J_A$, and $s_j((x_k)_{k\in I_U})$, $j \in J_U$, respectively. Assume moreover that the following hold.*

1. *For any $i \in I_A$, a_i is group-like or $(1, a_l)$-primitive for some $l \in I_A$ with group-like a_l, and*
2. *for any $k \in I_U$, x_k is group-like or $(x_l, 1)$-primitive for some $l \in I_U$ with group-like x_l.*

Then $(A \otimes U)_\sigma$ can be presented by generators $\bar{a}_i = a_i \otimes 1$, $\bar{x}_k = 1 \otimes x_k$ with $i \in I_A$, $k \in I_U$, and relations $r_j((\bar{a}_i)_{i\in I_A})$, $j \in J_A$, $s_j(\bar{x}_k)_{k\in I_U})$, $j \in J_U$, and

$$\bar{x}_k\bar{a}_i = \bar{a}_i\bar{x}_k \tag{8.1.1}$$

if a_i and x_k are group-like,

$$\bar{x}_k\bar{a}_i = \tau(a_i \otimes x_l)\bar{a}_i\bar{x}_k + \tau(a_i \otimes x_k)(\bar{a}_i - \bar{a}_i\bar{x}_l) \tag{8.1.2}$$

if a_i, x_l are group-like and x_k is $(x_l, 1)$-primitive,

$$\bar{x}_k\bar{a}_i = \tau^{-1}(a_l \otimes x_k)\bar{a}_i\bar{x}_k + \tau^{-1}(a_i \otimes x_k)(\bar{x}_k - \bar{a}_l\bar{x}_k) \tag{8.1.3}$$

if a_l, x_k are group-like and a_i is $(1, a_l)$-primitive, and

$$\begin{aligned}\bar{x}_k\bar{a}_i = &\tau(a_i \otimes x_k)\bar{a}_l + \tau^{-1}(a_i \otimes x_k)\bar{x}_m \\ &+ (\bar{a}_i + \tau(a_i \otimes x_m)\bar{a}_l)(\bar{x}_k + \tau^{-1}(a_l \otimes x_k)\bar{x}_m))\end{aligned} \tag{8.1.4}$$

if a_l, x_m are group-like, a_i is $(1, a_l)$-primitive, and x_k is $(x_m, 1)$-primitive.

PROOF. By Corollary 2.8.8, the elements $\bar{a}_i$ with $i \in I_A$ and $\bar{x}_i$ with $i \in I_U$ generate $(A \otimes U)_\sigma$ as an algebra, and $r_j((\bar{a}_i)_{i \in I_A})$, $j \in J_A$, and $s_j((\bar{x}_k)_{k \in I_U})$, $j \in J_U$, are relations of $(A \otimes U)_\sigma$. We check that in $(A \otimes U)_\sigma$ Equations (8.1.1)–(8.1.4) hold. Let $i \in I_A$ and $k \in I_U$.

Assume that a_i and x_k are group-like. Then, by Corollary 2.8.8,

$$\bar{x}_k \bar{a}_i = \tau(a_i \otimes x_k)\bar{a}_i \bar{x}_k \tau^{-1}(a_i \otimes x_k) = \bar{a}_i \bar{x}_k,$$

since $\tau(a_i \otimes x_k)\tau^{-1}(a_i \otimes x_k) = \varepsilon(a_i)\varepsilon(x_k) = 1$. This proves (8.1.1).

Assume that x_k is group-like and a_i is $(1, g)$-primitive for some $g \in G(A)$. Then, by Corollary 2.8.8,

$$\begin{aligned}
\bar{x}_k \bar{a}_i &= \tau(a_{i(1)} \otimes x_k)(a_{i(2)} \otimes 1)\bar{x}_k \tau^{-1}(a_{i(3)} \otimes x_k) \\
&= \bar{x}_k \tau^{-1}(a_i \otimes x_k) + \bar{a}_i \bar{x}_k \tau^{-1}(g \otimes x_k) + \tau(a_i \otimes x_k)(g \otimes 1)\bar{x}_k \tau^{-1}(g \otimes x_k).
\end{aligned}$$

Now observe that

$$0 = \tau\tau^{-1}(a_i \otimes x_k) = \tau(a_i \otimes x_k)\tau^{-1}(g \otimes x_k) + \tau(1 \otimes x_k)\tau^{-1}(a_i \otimes x_k).$$

From this we conclude (8.1.3). The proofs of (8.1.2) and (8.1.4) are analogous.

Let A' and U' be the subalgebras of $(A \otimes U)_\sigma$ spanned by the elements $a \otimes 1$, $a \in A$, and $1 \otimes x$, $x \in U$, respectively. Then A' is canonically isomorphic to A, U' is canonically isomorphic to U, and the multiplication map $A' \otimes U' \to (A \otimes U)_\sigma$ is bijective by Corollary 2.8.8(1). Thus the proposition follows from the above and from Lemma 2.8.10 for the algebra $C = (A \otimes U)_\sigma$ and its subalgebras A' and U'. $\square$

An important class of examples of quantum doubles is given by the (multi-parameter versions of) quantized enveloping algebras of complex semi-simple Lie algebras, or, more generally, of symmetrizable Kac-Moody algebras. We introduce these examples in several steps.

Let $n \in \mathbb{N}$. Let $A = (a_{ij})_{i,j \in \{1,\dots,n\}}$ be a symmetrizable Cartan matrix and $D = \mathrm{diag}(d_1, \dots, d_n)$ a diagonal matrix with positive integer entries such that DA is symmetric.

EXAMPLE 8.1.2. Let G be an abelian group, $g_1, \dots, g_n \in G$, $\chi_1, \dots, \chi_n \in \widehat{G}$, and $(q_{ij})_{i,j \in \{1,\dots,n\}}$ the family of non-zero scalars in $\Bbbk$ such that

$$\chi_j(g_i) = q_{ij}$$

for all $i, j \in \{1, \dots, n\}$. Let V be an n-dimensional vector space over $\Bbbk$ with basis $E_1, \dots, E_n$. By Example 1.5.3, V has the structure of a Yetter-Drinfeld module over $\Bbbk G$, where

$$g \cdot E_i = \chi_i(g)E_i, \quad \delta_V(E_i) = g_i \otimes E_i$$

for all $1 \leq i \leq n$ and $g \in G$. Then $T(V)$ is a Hopf algebra in ${}^G_G\mathcal{YD}$ by Proposition 1.6.13 and $A = T(V)\#\Bbbk G$ is a Hopf algebra by Corollary 4.3.5.

Let Y be a $\Bbbk G$-submodule of $T(V)$ spanned by skew-primitive elements of A. Let (Y) be the ideal of $T(V)$ generated by Y. Then $\big(T(V)/(Y)\big)\#\Bbbk G$ is a Hopf algebra by Proposition 2.4.4. This fact will be used in the proof of the next proposition.

PROPOSITION 8.1.3. *Let $q \in \Bbbk^\times$. Let $U_q^{\geq 0}$ denote the $\Bbbk$-algebra (depending on A and D) given by generators E_i, K_i, K_i^{-1}, where $1 \leq i \leq n$, and relations*

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$K_iE_j = q^{d_ia_{ij}}E_jK_i,$$
$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} E_i^m E_j E_i^{1-a_{ij}-m} = 0 \quad (i \neq j)$$

with $i, j \in \{1, \dots, n\}$.

(1) *The algebra $U_q^{\geq 0}$ is a Hopf algebra with comultiplication Δ, counit ε and antipode $\mathcal{S}$ given by*

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \Delta(K_i) = K_i \otimes K_i,\ \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$
$$\varepsilon(E_i) = 0, \quad \varepsilon(K_i) = 1, \quad \varepsilon(K_i^{-1}) = 1,$$
$$\mathcal{S}(E_i) = -K_i^{-1}E_i, \quad \mathcal{S}(K_i) = K_i^{-1}, \quad \mathcal{S}(K_i^{-1}) = K_i$$

for any $1 \leq i \leq n$.

(2) *Let U_q^+ and U_q^0 be the subalgebras of $U_q^{\geq 0}$ generated by $E_1, \dots, E_n$ and $K_1, \dots, K_n, K_1^{-1}, \dots, K_n^{-1}$, respectively. Then the multiplication map*

$$U_q^+ \otimes U_q^0 \to U_q^{\geq 0}$$

is bijective.

We write $K_\mu = K_1^{m_1} \cdots K_n^{m_n}$ for any $\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n$.

PROOF. Let $G = \mathbb{Z}^n$, V an n-dimensional vector space over $\Bbbk$ with basis $E_1, \dots, E_n$, and $g_i = K_i$ for all $1 \leq i \leq n$, where $K_1, \dots, K_n$ are the standard generators of $\mathbb{Z}^n$. Let $\chi_j \in \widehat{G}$ such that $\chi_j(g_i) = q^{d_ia_{ij}}$ for all $i, j \in \{1, 2, \dots, n\}$. Then $U_q^{\geq 0}$ is a quotient algebra of the Hopf algebra $T(V)\#\Bbbk\mathbb{Z}^n$ by Example 8.1.2. Let $i, j \in \{1, \dots, n\}$ with $i \neq j$, and let $q' = q^{2d_i}$ and $r = q^{d_ia_{ij}}$. Then

$$K_iE_iK_i^{-1} = q'E_i, \quad K_iE_jK_i^{-1} = rE_j, \quad K_jE_iK_j^{-1} = rE_i$$

and

$$q'^{-a_{ij}}r^2 = q^{-2d_ia_{ij}+2d_ia_{ij}} = 1.$$

Therefore $E_i^{1-a_{ij}} \triangleright E_j \in P_{K_i^{1-a_{ij}},1}$ by Proposition 2.4.3(2). By (8.0.1),

$$\begin{aligned}
E_i^{1-a_{ij}} \triangleright E_j &= \sum_{m=0}^{1-a_{ij}} (-q^{d_ia_{ij}})^m (q^{2d_i})^{m(m-1)/2} \binom{1-a_{ij}}{m}_{q^{2d_i}} E_i^{1-a_{ij}-m} E_j E_i^m \\
&= \sum_{m=0}^{1-a_{ij}} (-1)^m q^{d_i(a_{ij}m + m(m-1) + m(1-a_{ij}-m))} \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} E_i^{1-a_{ij}-m} E_j E_i^m \\
&= \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} E_i^{1-a_{ij}-m} E_j E_i^m \\
&= (-1)^{1-a_{ij}} \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} E_i^m E_j E_i^{1-a_{ij}-m}.
\end{aligned}$$

Therefore $U_q^{\geq 0}$ is isomorphic to the Hopf algebra $\big(T(V)/(Y)\big)\#\Bbbk\mathbb{Z}^n$, where (Y) is the ideal of $T(V)$ generated by all $E_i^{1-a_{ij}} \triangleright E_j$ with $i \neq j$. $\square$

REMARK 8.1.4. Let $\lambda_1, \dots, \lambda_n \in \Bbbk^\times$. Then there is a unique Hopf algebra automorphism φ_λ of $U_q^{\geq 0}$, as defined in Proposition 8.1.3, with

$$\varphi_\lambda(E_i) = \lambda_i E_i, \quad \varphi_\lambda(K_i) = K_i, \quad \varphi_\lambda(K_i^{-1}) = K_i^{-1}$$

for any $1 \leq i \leq n$. The linear map φ_λ also defines Hopf algebra automorphisms of $U_q^{\geq 0\,\mathrm{op}}$, $U_q^{\geq 0\,\mathrm{cop}}$, and $U_q^{\geq 0\,\mathrm{op\,cop}}$. All of these automorphisms will be denoted by φ_λ.

Recall the notion of the dual Hopf algebra from Definition 2.3.8.

LEMMA 8.1.5. *Let $q \in \Bbbk^\times$ and let $U_q^{\geq 0}$ be as in Proposition* 8.1.3*. For any $1 \leq i \leq n$ let $k_i^+, k_i^-, e_i \in (U_q^{\geq 0})^*$ such that*

$$\begin{aligned} k_i^{\pm}(EK_\mu) =& \varepsilon(E) q^{\pm \sum_{j=1}^n d_i a_{ij} m_j}, \\ e_i(E_{i_1} \cdots E_{i_r} K_\mu) =& \delta_{r,1}\delta_{i_1,i} \end{aligned}$$

for any $\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n$, $r \in \mathbb{N}_0$, $i_1, \dots, i_r \in \{1, \dots, n\}$ and $E \in U_q^+$.

1. *The functionals k_i^+, k_i^-, e_i for $1 \leq i \leq n$ are contained in the dual Hopf algebra of $U_q^{\geq 0}$.*
2. *There is a Hopf algebra homomorphism from $U_q^{\geq 0}$ to its dual which maps K_i, K_i^{-1} and E_i to k_i^+, k_i^- and e_i, respectively.*

PROOF. The Hopf algebra $U_q^{\geq 0}$ is $\mathbb{N}_0$-graded with

$$\deg E_i = 1, \qquad \deg K_i = \deg K_i^{-1} = 0$$

for any $1 \leq i \leq n$. Therefore $k_i^{\pm}$ and e_i, where $1 \leq i \leq n$, are well-defined. The defining relations of $U_q^{\geq 0}$ imply that

$$k_i^+(E'E'') = k_i^+(E')k_i^+(E''), \quad k_i^-(E'E'') = k_i^-(E')k_i^-(E'')$$

and that

$$e_i(E'E'') = e_i(E')\varepsilon(E'') + k_i^+(E')e_i(E'')$$

for all $E', E'' \in U_q^{\geq 0}$. (The only non-trivial case to check for the latter equation is when $E' = K_\mu$ and $E'' = E_i K_\nu$, where $\mu, \nu \in \mathbb{Z}^n$. In this case one needs that the matrix DA is symmetric.) Therefore for any $1 \leq i \leq n$ the elements $k_i^{\pm}$ are group-like and the elements e_i are (k_i^+, ε)-primitive in the dual Hopf algebra of $U_q^{\geq 0}$ by Corollary 2.3.12. This proves (1).

Let $i \in \{1, \dots, n\}$, $E \in U^+$, and $\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n$. Since

$$\Delta(E) \in U_q^{\geq 0} \otimes U_q^+,$$

we obtain that

$$\begin{aligned} k_i^+ k_i^-(EK_\mu) =& k_i^+(E_{(1)}K_\mu) k_i^-(E_{(2)}K_\mu) \\ =& k_i^+(E_{(1)}K_\mu)\varepsilon(E_{(2)}) q^{-\sum_{j=1}^n d_i a_{ij} m_j} \\ =& \varepsilon(E). \end{aligned}$$

Hence $k_i^+ k_i^- = \varepsilon$. Similarly, $k_i^- k_i^+ = \varepsilon$.

Let $r \in \mathbb{N}_0$, $i, j, i_1, \dots, i_r \in \{1, \dots, n\}$, and $\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n$. Then

$$\begin{aligned} k_i^{\pm} e_j (E_{i_1} \cdots E_{i_r} K_\mu) =& k_i^{\pm}((E_{i_1} \cdots E_{i_r})_{(1)} K_\mu) e_j ((E_{i_1} \cdots E_{i_r})_{(2)} K_\mu) \\ =& k_i^{\pm}(K_{i_1} \cdots K_{i_r} K_\mu) e_j (E_{i_1} \cdots E_{i_r} K_\mu) \\ =& \delta_{r,1} \delta_{i_1,j} q^{\pm d_i a_{ij}} q^{\pm \sum_{l=1}^n d_i a_{il} m_l}. \end{aligned}$$

Similarly,

$$\begin{aligned} e_j k_i^{\pm} (E_{i_1} \cdots E_{i_r} K_\mu) =& e_j ((E_{i_1} \cdots E_{i_r})_{(1)} K_\mu) k_i^{\pm}((E_{i_1} \cdots E_{i_r})_{(2)} K_\mu) \\ =& e_j (E_{i_1} \cdots E_{i_r} K_\mu) k_i^{\pm}(K_\mu) \\ =& \delta_{r,1} \delta_{i_1,j} q^{\pm \sum_{l=1}^n d_i a_{il} m_l}. \end{aligned}$$

Therefore $k_i^{\pm} e_j = q^{\pm d_i a_{ij}} e_j k_i^{\pm}$.

By the argument in the proof of Proposition 8.1.3,

$$e_i^{1-a_{ij}} \triangleright e_j = (-1)^{1-a_{ij}} \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} e_i^m e_j e_i^{1-a_{ij}-m}$$

is $(k_i^{1-a_{ij}} k_j, \varepsilon)$-primitive. Note that any monomial $e_{i_1} \cdots e_{i_r} \in (U_q^{\geq 0})^*$ with $r \geq 2$, $i_1, \dots, i_r \in \{1, \dots, n\}$ (and hence $e_i^{1-a_{ij}} \triangleright e_j$) vanishes on E_k and on K_μ for any $k \in \{1, \dots, n\}$ and $\mu \in \mathbb{Z}^n$. Since $e_i^{1-a_{ij}} \triangleright e_j$ is skew-primitive, it vanishes on $U^{\geq 0}$. This implies (2). $\square$

Proposition 8.1.6. *Let $q \in \Bbbk^\times$ and let $U_q^{\geq 0}$ be as in Proposition 8.1.3. Let $U_q^{\leq 0} = (U_q^{\geq 0})^{\mathrm{cop}}$. We write F_i, L_i, L_i^{-1}, $1 \leq i \leq n$, for the generators of $U_q^{\leq 0}$ corresponding to E_i, K_i, K_i^{-1}, respectively, and U_q^- for the subalgebra of $U_q^{\leq 0}$ generated by F_i, $1 \leq i \leq n$. Let $(\lambda_i)_{1 \leq i \leq n} \in (\Bbbk^\times)^n$. Then there is a unique skew pairing $\tau : U_q^{\leq 0} \otimes U_q^{\geq 0} \to \Bbbk$ such that for all $1 \leq i, j \leq n$,*

$$\begin{aligned} &\tau(F_i \otimes E_j) = \delta_{ij} \lambda_i, && \tau(F_i \otimes K_j) = 0, \\ &\tau(L_i \otimes E_j) = 0, && \tau(L_i \otimes K_j) = q^{d_i a_{ij}}. \end{aligned}$$

The corresponding Drinfeld's quantum double of $U_q^{\leq 0}$ and $U_q^{\geq 0}$ is isomorphic to the Hopf algebra given by generators $E_i, F_i, K_i, K_i^{-1}, L_i, L_i^{-1}$ and relations

$$\begin{gathered} K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\ L_i L_j = L_j L_i, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad K_i L_j = L_j K_i, \\ K_i E_j = q^{d_i a_{ij}} E_j K_i, \quad K_i F_j = q^{-d_i a_{ij}} F_j K_i, \\ L_i E_j = q^{-d_i a_{ij}} E_j L_i, \quad L_i F_j = q^{d_i a_{ij}} F_j L_i, \\ E_i F_j - F_j E_i = \delta_{ij} \lambda_i (L_i - K_i), \\ \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} E_i^m E_j E_i^{1-a_{ij}-m} = 0, \quad (i \neq j) \\ \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q^{d_i}} F_i^m F_j F_i^{1-a_{ij}-m} = 0 \quad (i \neq j) \end{gathered}$$

with $i,j\in\{1,\dots,n\}$, where the Hopf algebra structure is given by

$$\Delta(E_i)=E_i\otimes 1+K_i\otimes E_i,\quad \Delta(F_i)=F_i\otimes L_i+1\otimes F_i,$$
$$\Delta(K_i)=K_i\otimes K_i,\quad \Delta(K_i^{-1})=K_i^{-1}\otimes K_i^{-1},$$
$$\Delta(L_i)=L_i\otimes L_i,\quad \Delta(L_i^{-1})=L_i^{-1}\otimes L_i^{-1},$$

$$\varepsilon(E_i)=0,\quad \varepsilon(F_i)=0,\quad \varepsilon(K_i)=1,\quad \varepsilon(L_i)=1,$$

$$\mathcal{S}(E_i)=-K_i^{-1}E_i,\quad \mathcal{S}(F_i)=-F_iL_i^{-1},\quad \mathcal{S}(K_i)=K_i^{-1},\quad \mathcal{S}(L_i)=L_i^{-1}$$

for all $1\le i\le n$.

PROOF. The relations $K_iK_i^{-1}=K_i^{-1}K_i=1$ and $L_iL_i^{-1}=L_i^{-1}L_i=1$ for $1\le i\le n$ imply that any skew pairing of $U_q^{\le 0}$ and $U_q^{\ge 0}$ with the required properties satisfies additionally the equations

$$\tau(F_i\otimes K_j^{-1})=0,\quad \tau(L_i^{-1}\otimes E_j)=0,$$
$$\tau(L_i^{-1}\otimes K_j)=\tau(L_i\otimes K_j^{-1})=q^{-d_ia_{ij}},\quad \tau(L_i^{-1}\otimes K_j^{-1})=q^{d_ia_{ij}}$$

for all $i,j\in\{1,\dots,n\}$. For example,

$$\begin{aligned}0=\varepsilon(F_i)&=\tau(F_i\otimes 1)=\tau(F_i\otimes K_jK_j^{-1})\\&=\tau(F_i\otimes K_j^{-1})\tau(L_i\otimes K_j)+\tau(1\otimes K_j^{-1})\tau(F_i\otimes K_j)\end{aligned}$$

and hence $\tau(F_i\otimes K_j^{-1})q^{d_ia_{ij}}=0$ for any $i,j\in\{1,\dots,n\}$. The span of $1,E_i,K_i,K_i^{-1}$, $1\le i\le n$, and $1,F_i,L_i,L_i^{-1}$, $1\le i\le n$, is a subcoalgebra of $U_q^{\ge 0}$ and $U_q^{\le 0}$, respectively, and generates $U_q^{\ge 0}$ and $U_q^{\le 0}$ as algebra, respectively. Therefore the uniqueness of the skew pairing follows from Definition 2.8.4.

Let φ be the Hopf algebra homomorphism from $U_q^{\ge 0}$ to its own dual described in Lemma 8.1.5. Let φ_λ be the Hopf algebra automorphism of $U_q^{\ge 0}$ defined in Remark 8.1.4. Then the composed map $\varphi\circ\varphi_\lambda$ defines a Hopf algebra homomorphism from $U_q^{\le 0\,\mathrm{cop}}=U_q^{\ge 0}$ to the dual of $U_q^{\ge 0}$, and maps the generators L_i, L_i^{-1} and F_i to k_i^+, k_i^-, and e_i, respectively, for any $1\le i\le n$. The skew pairing of $U_q^{\le 0}$ and $U_q^{\ge 0}$ defined by $\varphi\circ\varphi_\lambda$, as explained in Remark 2.8.5, satisfies all properties of τ. This proves the existence of τ.

The claim on the presentation of Drinfeld's quantum double by generators and relations follows from Propositions 8.1.1 and 8.1.3 by inserting appropriate values of τ. By definition, the coalgebra structure of Drinfeld's quantum double coincides with the coalgebra structure of $U_q^{\le 0}\otimes U_q^{\ge 0}$. This also implies the formulas for the antipode. □

The definition of the quantized enveloping algebra of a symmetrizable Kac-Moody algebra and Proposition 8.1.6 immediately imply the following.

COROLLARY 8.1.7. *Let $n\in\mathbb{N}$ and let $A=(a_{ij})_{i,j\in\{1,\dots,n\}}$ be a symmetrizable Cartan matrix. Let $D=(d_i)_{1\le i\le n}$ be a family of positive integers such that $(d_ia_{ij})_{i,j\in\{1,\dots,n\}}$ is symmetric. Let $q\in\Bbbk^\times$. Assume that $q^{2d_i}\ne 1$ for any $1\le i\le n$. Let τ be the skew pairing of $U_q^{\le 0}$ and $U_q^{\ge 0}$ in Proposition 8.1.6 with parameters $\lambda_i=(q^{-d_i}-q^{d_i})^{-1}$, $1\le i\le n$. The quantized enveloping algebra U_q of the Kac-Moody algebra associated to A is isomorphic to the quotient Hopf algebra of the Drinfeld double of $U_q^{\le 0}$ and $U_q^{\ge 0}$ corresponding to τ by the two-sided ideal generated by K_iL_i-1, $1\le i\le n$.*

8.2. YD-data and linking

DEFINITION 8.2.1. Let I be a finite set, $q_{ij} \in \Bbbk^\times$ for all $i, j \in I$, and let $\boldsymbol{q} = (q_{ij})_{i,j\in I}$. We say that $\boldsymbol{q}$ is

(1) **generic**, if q_{ii} is not a root of unity for all $i \in I$,
(2) **quasi-generic**, if
  (a) $\mathrm{char}(\Bbbk) > 0$ and $\boldsymbol{q}$ is generic, or
  (b) $\mathrm{char}(\Bbbk) = 0$ and for all $i \in I$, q_{ii} is not a root of unity or $q_{ii} = 1$,
(3) of **(finite) Cartan type** if there is a (finite) Cartan matrix $(a_{ij})_{i,j\in I}$ such that for all $i, j \in I$,

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \text{ where } 0 \leq -a_{ij} < \mathrm{ord}(q_{ii}) \text{ if } i \neq j, \tag{8.2.1}$$

and $1 \leq \mathrm{ord}(q_{ii}) \leq \infty$.

DEFINITION 8.2.2. A **YD-datum** $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ consists of an abelian group G, a finite non-empty set I, and for all $i \in I$, elements $g_i \in G$, and characters χ_i in $\widehat{G} = \mathrm{Hom}(G, \Bbbk^\times)$. We define the **braiding matrix** $(q_{ij})_{i,j\in I}$ of $\mathcal{D}$ by

$$q_{ij} = \chi_j(g_i) \text{ for all } i, j \in I. \tag{8.2.2}$$

A YD-datum is called **generic**, **quasi-generic**, and of **(finite) Cartan type**, respectively, if its braiding matrix $\boldsymbol{q}$ is.

Let $\boldsymbol{q} = (q_{ij})_{i,j\in I}$ be a matrix of non-zero elements in $\Bbbk$. Assume that for all $i \neq j$ in I there are $m_{ij} \in \mathbb{N}_0$ with

$$q_{ij}q_{ji} = q_{ii}^{-m_{ij}} \quad \text{for all } i, j \in I,\ i \neq j.$$

We choose $0 \leq m_{ij} < \mathrm{ord}(q_{ii})$ for all $i \neq j$ in I. Then $\boldsymbol{q}$ is of Cartan type with Cartan matrix $A = (a_{ij})_{i,j\in I}$, where $a_{ii} = 2$ for all i, and $a_{ij} = -m_{ij}$ for all $i \neq j$.

Note that the Cartan matrix of a YD-datum of Cartan type is uniquely determined.

EXAMPLE 8.2.3. Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable Cartan matrix, and $(d_i)_{i\in I}$ positive integers with $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$. Let $0 \neq q \in \Bbbk$, and assume that for all $i, j \in I$, $0 \leq -a_{ij} < \mathrm{ord}(q^{2d_i}) \leq \infty$. Define

$$q_{ij} = q^{d_i a_{ij}} \text{ for all } i, j \in I.$$

Then the matrix $(q_{ij})_{i,j\in I}$ is of Cartan type with Cartan matrix A. This is the braiding which appeared in Proposition 8.1.3.

Let $\sim$ be the usual equivalence relation on the index set I of a Cartan matrix $(a_{ij})_{i,j\in I}$: For all $i, j \in I$, $i \sim j$ if and only if there are elements $i_1, \dots, i_t \in I$, $t \geq 2$, with $i_1 = i$, $i_t = j$, $a_{i_l,i_{l+1}} \neq 0$ for all $1 \leq l < t$. The set of equivalence classes of I with respect to $\sim$, also called **connected components**, will be denoted by $\mathcal{X}$.

LEMMA 8.2.4. *Let $\boldsymbol{q} = (q_{ij})_{i,j\in I}$ be a matrix of non-zero elements in $\Bbbk$. Assume that $\boldsymbol{q}$ is of finite Cartan type with Cartan matrix $A = (a_{ij})_{i,j\in I}$. Then there are integers $d_i \in \{1, 2, 3\}$, for all $i \in I$, and for each connected component J of I with respect to $\sim$ there exists $q_J \in \Bbbk^\times$ such that*

$$q_{ii} = q_J^{d_i} \text{ for all } i \in J. \tag{8.2.3}$$

PROOF. By Theorem 1.10.18, there exist unique integers d_i, $i \in I$, such that

$$d_i a_{ij} = d_j a_{ji} \quad \text{for all } i, j \in I, \tag{8.2.4}$$

and for each equivalence class J in I with respect to $\sim$, $\{d_j \mid j \in J\}$ is one of the sets $\{1\}$, $\{1,2\}$, $\{1,3\}$.

Let $i_1, i_2 \in I$ which belong to the same equivalence class J. Assume that $d_{i_1} = d_{i_2}$. Then there is $k \geq 1$ and a sequence $j_1, \dots, j_k \in J$ such that $i_1 = j_1$, $i_2 = j_k$, and

$$d_{j_l} = d_{j_1}, \quad a_{j_l j_{l+1}} = a_{j_{l+1} j_l} = -1$$

for all $1 \leq l < k$. Therefore

$$q_{j_l j_l}^{-1} = q_{j_l j_{l+1}} q_{j_{l+1} j_l} = q_{j_{l+1} j_{l+1}}^{-1}$$

for all $1 \leq l < k$ and hence $q_{i_1 i_1} = q_{i_2 i_2}$.

Assume now that $d_{i_1} = 1$, $d_{i_2} > 1$, and $a_{i_1 i_2} \neq 0$. Then $a_{i_1 i_2} = -d_{i_2}$, $a_{i_2 i_1} = -1$ because of (8.2.4), and

$$q_{i_2 i_2}^{-1} = q_{i_2 i_1} q_{i_1 i_2} = q_{i_1 i_1}^{-d_{i_2}}.$$

This implies that for each component J of I there exists $q_J \in \Bbbk^\times$ (namely, $q_J = q_{jj}$ with $d_j = 1$) such that $q_{ii} = q_J^{d_i}$ for all $i \in J$. $\square$

REMARK 8.2.5. More complicated situations may appear if A is not of finite type. For example, assume that $\dim V = 2$, $q \in \Bbbk^\times$ with $q^2 \neq 1$, and

$$\boldsymbol{q} = \begin{pmatrix} q & q^{-1} \\ q^{-1} & -q \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Then A is symmetric, but the diagonal entries of $\boldsymbol{q}$ do not coincide.

REMARK 8.2.6. Let $\boldsymbol{q} = (q_{ij})_{i,j \in I}$ be a matrix of non-zero elements in $\Bbbk$. Assume that $\boldsymbol{q}$ is of Cartan type with Cartan matrix $A = (a_{ij})_{i,j \in I}$. For all $i \in I$ and $J \in \mathcal{X}$, let d_i be a positive integer and $q_J \in \Bbbk^\times$. Assume that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$, and

$$q_{ii} = q_J^{2d_i} \text{ for all } i \in J.$$

Define

$$p_{ij} = \begin{cases} q_J^{d_i a_{ij}} & \text{for all } J \in \mathcal{X},\ i, j \in J, \\ 1 & \text{for all } i, j \in I,\ i \not\sim j. \end{cases}$$

Let $\mathbf{p} = (p_{ij})_{i,j \in I}$. Then the matrices $\boldsymbol{q}$ and $\mathbf{p}$ are twist-equivalent. By Corollary 4.1.14, the Nichols algebras of braided vector spaces of diagonal type with braiding matrices $\mathbf{q}$ and $\mathbf{p}$ are very similar. The only difference between $\mathbf{p}$ and the matrix in Example 8.2.3 is that the elements q_J can vary for different components J. Note that by Lemma 8.2.4, the assumptions in this remark are satisfied for matrices $\boldsymbol{q}$ of finite Cartan type (if the elements q_J have square roots in $\Bbbk$).

DEFINITION 8.2.7. Let $\mathcal{D} = \mathcal{D}(G, (g_i)_{i \in I}, (\chi_i)_{i \in I})$ be a YD-datum of Cartan type with braiding matrix $(q_{ij})_{i,j \in I}$.

(1) Let $i, j \in I$. The pair (i, j) is called **linkable**, and i is called linkable to j, if

$$i \not\sim j,\ g_i g_j \neq 1, \text{ and } \chi_i \chi_j = 1.$$

(2) A matrix $\lambda = (\lambda_{ij})_{i,j \in I, i \not\sim j}$ of elements in $\Bbbk$ is called a **linking parameter** for $\mathcal{D}$, if for all $i, j \in I$, $i \not\sim j$,

(a) if $\lambda_{ij} \neq 0$, then (i,j) is linkable,
(b) $\lambda_{ij} = -q_{ij}\lambda_{ji}$.

(3) If $\lambda = (\lambda_{ij})_{i,j\in I, i\not\sim j}$ is a linking parameter for $\mathcal{D}$, then a pair (i,j) of elements in $I, i \not\sim j$, is called **linked** if $\lambda_{ij} \neq 0$.

If $i \not\sim j$, then $a_{ij} = 0$, hence $q_{ij}q_{ji} = 1$, and $\lambda_{ij} = -q_{ij}\lambda_{ji}$ implies $\lambda_{ji} = -q_{ji}\lambda_{ij}$. Note that (j,i) is linked, if (i,j) is.

LEMMA 8.2.8. *Let $\mathcal{D}$ be a YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in I}$ and braiding matrix $(q_{ij})_{i,j\in I}$.*

(1) *Let $i,j,k,l \in I$.*
  (a) *If (i,k) is linkable, then $a_{ik} = 0$, and $q_{ii} = q_{kk}^{-1} = q_{ki} = q_{ik}^{-1}$.*
  (b) *If (i,k) and (j,l) are linkable, then $q_{ii}^{a_{ij}} = q_{ii}^{a_{kl}}$.*
  (c) *If (i,k) and (j,l) are linkable, and $i \not\sim l$, then $q_{ij} = q_{lk}^{-1}$.*

(2) *Let i be any vertex of I. Assume that for all $k \in I$,*
  (a) *q_{kk} is not a root of one, or*
  (b) *the order of q_{kk} is finite and for all $l \in I$, $l \neq k$, $\operatorname{ord}(q_{kk})$ does not divide $2 - a_{kl}$.*

  Then i is linkable to at most one $k \in I$.

PROOF. (1) (a) follows easily from the definition of linkable pairs.

(b) We first note that $q_{ij} = q_{il}^{-1}$, $q_{ji} = q_{jk}^{-1}$, $q_{kl} = q_{kj}^{-1}$, $q_{lk} = q_{li}^{-1}$, since by assumption $\chi_i\chi_k = 1$, $\chi_j\chi_l = 1$. Hence

$$(q_{ij}q_{ji})(q_{kl}q_{lk}) = (q_{il}q_{li})^{-1}(q_{jk}q_{kj})^{-1}. \tag{8.2.5}$$

If $i \sim l$ or $j \sim k$, then $i \not\sim j$ and $k \not\sim l$ since by assumption $i \not\sim k$ and $j \not\sim l$; then the left-hand of (8.2.5) is equal to 1. And if $i \not\sim l$ and $j \not\sim k$, then the right-hand of (8.2.5) is equal to 1. Therefore $q_{ii}^{a_{ij}} q_{kk}^{a_{kl}} = 1$ by (8.2.1). Then the claim follows from (a).

(c) We have noted in the proof of (b) that $q_{ij} = q_{il}^{-1}$, $q_{lk} = q_{li}^{-1}$. The assumption $i \not\sim l$ implies $a_{il} = 0$, hence $q_{li} = q_{il}^{-1}$, and (c) follows.

(2) Let $k,l \in I$ such that $k \neq l$ and both i,k and i,l are linkable. Then by (1)(b) and (a), $q_{kk}^{a_{ii}} = q_{kk}^{a_{kl}}$, and (2) follows, since $a_{ii} = 2$. $\square$

REMARK 8.2.9. There are examples of YD-data with vertices linkable to several vertices. E. g. if $q_{ij} = -1$ for all $i,j \in I$, $g_ig_j \neq 1$ for all $i \neq j$, and G is generated by g_i, $i \in I$, then each pair (i,j) with $i \neq j$ is linkable.

DEFINITION 8.2.10. Let $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ be a YD-datum of Cartan type, and $\lambda = (\lambda_{ij})_{i,j\in I, i\not\sim j}$ a linking parameter for $\mathcal{D}$. Let $\mathcal{X}$ be the set of connected components of I with respect to $\sim$. The **linking graph** of $(\mathcal{D}, \lambda)$ is the graph with set of vertices $\mathcal{X}$, where there is an edge between $J_1, J_2 \in \mathcal{X}$ if and only if there are elements $i \in J_1$ and $j \in J_2$ with $\lambda_{ij} \neq 0$.

Recall that a graph is called **bipartite** if the set of its vertices V can be written as the disjoint union of non-empty subsets V^+ and V^- such that there is no edge between vertices in V^+ and no edge between vertices in V^-.

LEMMA 8.2.11. *Let $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ be a YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in I}$, and λ a linking parameter for $\mathcal{D}$. Assume that any element of I is linked with at most one element of I.*

Let $(J_l)_{1\leq l\leq n}$ be a circle of length $n\geq 3$ in the linking graph of $\mathcal{D}$, that is, $J_1,\dots,J_n\in\mathcal{X}$, $J_k\neq J_l$ for all k,l, and for all $1\leq l\leq n$ there are $i_l,j_l\in J_l$ such that

$$(j_1,i_2),(j_2,i_3),\dots,(j_{n-1},i_n),(j_n,i_1) \text{ are linked.}$$

For each l there exist $i_1(l),i_2(l),\dots,i_{p(l)}(l)\in J_l$, where $p(l)\geq 2$, such that $a_{i_p(l)\,i_{p+1}(l)}<0$ for all $1\leq p\leq p(l)-1$, and $i_1(l)=i_l\neq j_l=i_{p(l)}(l)$. Let

$$a_l=\prod_{p=1}^{p(l)-1} a_{i_p(l)\,i_{p+1}(l)},\quad b_l=\prod_{p=1}^{p(l)-1} a_{i_{p+1}(l)\,i_p(l)}.$$

Then

$$(q_{i_1i_1})^{a_1\cdots a_n}=(q_{i_1i_1})^{(-1)^n b_1\cdots b_n}. \tag{8.2.6}$$

Proof. The elements $i_1(l),i_2(l),\dots,i_{p(l)}(l)$ exist, since $i_l\sim j_l$. Note that $i_l\neq j_l$, since any element of I is linked with at most one element in I. The Cartan condition implies $q_{i_li_l}^{a_l}=q_{j_lj_l}^{b_l}$ for all l. Let $i_{n+1}=i_1$. Hence for all $1\leq l\leq n$, $q_{i_li_l}^{a_l}=q_{i_{l+1}i_{l+1}}^{-b_l}$, since (j_l,i_{l+1}) are linked, and (8.2.6) follows. □

Corollary 8.2.12. *Let $\mathcal{D}$ be a generic YD-datum of Cartan type, and let λ be a linking parameter for $\mathcal{D}$. Then the linking graph of $(\mathcal{D},\lambda)$ is bipartite.*

Proof. By a well-known result in graph theory, see [**Die18**], Section 1.6, a graph is bipartite if and only if it contains no odd cycle. Assume there is a cycle in the linking graph of $(\mathcal{D},\lambda)$ of length $n\geq 3$. By Lemma 8.2.8(2), since $\mathcal{D}$ is generic, any element of I is linkable to at most one element of I. Hence we can apply Lemma 8.2.11. Then (8.2.6) implies that n is even, since for all l, the non-zero integers a_l and b_l have the same sign. □

Remark 8.2.13. The conditions in Lemma 8.2.8 and (8.2.6) in Lemma 8.2.11 imply that the linking graph of $(\mathcal{D},\lambda)$ is bipartite in many other cases than the generic one. For example, the linking graph is bipartite in the simply laced case, when the values of the Cartan matrix are 0, 2 or -1, and when the order of q_{ii} is >3 for all $i\in I$.

In the generic case the equality (8.2.6) implies that $a_1\cdots a_n=b_1\cdots b_n$ for cycles of even length n. This gives further restrictions when the a_i are not all equal to ± 1.

For the definition of the diagram of a YD-datum and linking parameter we will use the notion of the Dynkin diagram of a Cartan matrix.

Definition 8.2.14. The **Dynkin diagram** of a Cartan matrix $A=(a_{ij})_{i,j\in I}$ is a graph with vertex set I as follows. For $i\neq j$ with $a_{ij}a_{ji}\leq 4$ and $|a_{ij}|\geq|a_{ji}|$, the vertices i and j are connected by $|a_{ij}|$ lines, and these lines are equipped with an arrow pointing towards i if $|a_{ij}|>1$. If $a_{ij}a_{ji}>4$, the vertices i and j are connected by a bold-faced line equipped with the ordered pair $(|a_{ij}|,|a_{ji}|)$.

Definition 8.2.15. Let $\mathcal{D}$ be a YD-datum of Cartan type A and λ a linking parameter. The **diagram of** $(\mathcal{D},\lambda)$ is the Dynkin diagram of A together with dotted edges between linked pairs of vertices.

The next proposition describes a large class of possible diagrams $(\mathcal{D}, \lambda)$ when the linking graph is bipartite.

If $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ is a YD-datum, and $J \subseteq I$ is a subset, we define

$$\mathcal{D}(J) = \mathcal{D}(G, (g_i)_{i\in J}, (\chi_i)_{i\in J}).$$

PROPOSITION 8.2.16. *Let $A = (a_{ij})_{i,j\in I}$ be a Cartan matrix, and $\mathcal{X}$ the set of connected components of I with respect to $\sim$. Let*

$$\mathcal{X}^+, \mathcal{X}^- \subseteq \mathcal{X} \text{ with } \mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-,\ \mathcal{X}^+ \cap \mathcal{X}^- = \emptyset,$$
$$I^+ = \bigcup_{J\in\mathcal{X}^+} J, \quad I^- = \bigcup_{J\in\mathcal{X}^-} J.$$

Let $l : I_l^+ \to I_l^-$ be a bijective map between subsets $I_l^+ \subseteq I^+$ and $I_l^- \subseteq I^-$. Let G be a free abelian group with basis $(g_i)_{i\in I}$, and

$$\mathcal{D}_1 = \mathcal{D}(G, (g_i)_{i\in I^+}, (\mu_i)_{i\in I^+}), \quad \mathcal{D}_2 = \mathcal{D}(G, (g_i)_{i\in I^-}, (\nu_i)_{i\in I^-}),$$

YD-data of Cartan type $(a_{ij})_{i,j\in I^+}$ and $(a_{ij})_{i,j\in I^-}$. Then the following are equivalent.

(1) *There is a YD-datum $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ of Cartan type with Cartan matrix A, and a linking parameter λ of $\mathcal{D}$ such that*
 (a) *$\mathcal{D}(I^+) = \mathcal{D}_1$, $\mathcal{D}(I^-) = \mathcal{D}_2$, and*
 (b) *the dotted lines in the diagram of $(\mathcal{D}, \lambda)$ are the lines between i and $l(i)$ for all $i \in I_l^+$.*

(2) *For all $i, j \in I_l^+$, $\mu_j(g_i) = \nu_{l(i)}(g_{l(j)})^{-1}$.*

PROOF. (1) $\Rightarrow$ (2) follows from Lemma 8.2.8(1)(c).

(2) $\Rightarrow$ (1) For all $i \in I^+$ we define a character χ_i of G as follows.

(α) Let $\chi_i(g_j) = \mu_i(g_j)$ for all $j \in I^+$.
(β) If $i \in I_l^+$, let $\chi_i(g_j) = \nu_{l(i)}^{-1}(g_j)$ for all $j \in I^-$.
(γ) If $i \notin I_l^+$, let $\chi_i(g_{l(k)}) = \mu_k(g_i)$ for all $k \in I_l^+$.
(δ) If $i \notin I_l^+$, $j \in I^-$, $j \notin I_l^-$, let $\chi_i(g_j)$ be an arbitrary element in $\Bbbk$.

Then we define for all $i \in I^-$ a character χ_i of G by

$$\chi_i(g_j) = \begin{cases} \nu_i(g_j) & \text{for all } j \in I^-, \\ \chi_j(g_i)^{-1} & \text{for all } j \in I^+. \end{cases} \tag{8.2.7}$$

It follows that $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ is a YD-datum of Cartan type with Cartan matrix A.

We claim that

$$\chi_i = \chi_{l(i)}^{-1} \text{ for all } i \in I_l^+. \tag{8.2.8}$$

Let $j \in I^-$. Then $\chi_i(g_j) = \nu_{l(i)}^{-1}(g_j)$ by (β), and $\chi_{l(i)}(g_j) = \nu_{l(i)}(g_j)$ by (8.2.7). Let $j \in I_l^+$. Then

$$\chi_{l(i)}(g_j) = \chi_j(g_{l(i)})^{-1} = \nu_{l(j)}(g_{l(i)}) = \mu_i(g_j)^{-1} = \chi_i(g_j)^{-1},$$

where the first equality follows from (8.2.7), the second from (β), the third from (2), and the last from (α).

Let $j \in I^+$ with $j \notin I_l^+$. Then

$$\chi_{l(i)}(g_j) = \chi_j^{-1}(g_{l(i)}) = \mu_i^{-1}(g_j) = \chi_i^{-1}(g_j),$$

where the first equality follows from by (8.2.7), the second from (γ), and the last from (α).

This proves (8.2.8). Finally, let $(l_i)_{i \in I_l^+}$ be a family of non-zero scalars, and for all $i, j \in I, i \not\sim j$, let

$$\lambda_{ij} = \begin{cases} l_i & \text{if } i \in I_l^+,\ j = l(i), \\ -q_{l(j)j} l_j & \text{if } j \in I_l^+,\ i = l(j), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda = (\lambda_{ij})_{i,j \in I, i \not\sim j}$ is a linking parameter for $\mathcal{D}$, and the linked pairs of $(\mathcal{D}, \lambda)$ are $\{(i, l(i)), (l(i), i) \mid i \in I_l^+\}$. $\square$

COROLLARY 8.2.17. *Under the assumptions of Proposition* 8.2.16 *let A be symmetrizable, and $(d_i)_{i \in I}$ a family of positive integers with $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$. For all $i \in I$ we denote the connected component of I with respect to $\sim$ containing i by $I(i)$. Let $(t_J)_{J \in \mathcal{X}}$ be a family of non-zero integers and $0 \neq q \in \Bbbk$ not a root of* 1. *Define YD-data $\mathcal{D}_1$ and $\mathcal{D}_2$ with characters μ_i, $i \in I^+$, and ν_k, $k \in I^-$, such that*

$$\mu_i(g_j) = q^{d_i a_{ij} t_{J(i)}}, \quad \nu_k(g_j) = q^{d_k a_{kj} t_{J(k)}} \quad \text{for all } j \in I.$$

Assume that for all $i, j \in I_l^+$,

$$a_{ij} = a_{l(i)l(j)}, \quad d_i t_{J(i)} = -d_{l(i)} t_{J(l(i))}. \tag{8.2.9}$$

Then there is a YD-datum $\mathcal{D} = \mathcal{D}(G, (g_i)_{i \in I}, (\chi_i)_{i \in I})$ of Cartan type with Cartan matrix A, and a linking parameter λ of $\mathcal{D}$ such that

(1) *$\mathcal{D}(I^+) = \mathcal{D}_1$, $\mathcal{D}(I^-) = \mathcal{D}_2$, and*
(2) *the dotted lines in the diagram of $(\mathcal{D}, \lambda)$ are the lines between i and $l(i)$ for all $i \in I_l^+$.*

PROOF. The matrices $(\mu_j(g_i))_{i,j \in I^+}$ and $(\nu_j(g_k))_{k,j \in I^-}$ are of Cartan type with Cartan matrices $(a_{ij})_{i,j \in I^+}$ and $(a_{kj})_{k,j \in I^-}$, respectively. Indeed, by definition of a Cartan matrix, if $a_{ij} = 0$, then $a_{ji} = 0$, and $J(i) = J(j)$ otherwise; moreover, q is not a root of 1. Condition (2) in Proposition 8.2.16 follows from (8.2.9). Hence the corollary follows from Proposition 8.2.16. $\square$

The diagonal elements of the braiding matrix of $\mathcal{D}$ in the last corollary satisfy an additional condition: For all $J \in \mathcal{X}$ there is an element $q_J \in \Bbbk^\times$ (namely $q_J = q^{2t_J}$) such that $q_{ii} = q_J^{d_i}$ for all $i \in J$. By Lemma 8.2.4, this condition always holds in the case of finite Cartan matrices.

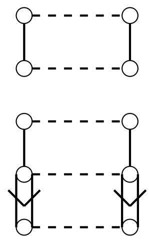
COROLLARY 8.2.18. *Under the assumptions of Proposition* 8.2.16 *let A be simply laced, that is, $a_{ij} \in \{0, -1\}$ for all $i, j \in I$, $i \neq j$. The following are equivalent.*

(1) *There is a generic YD-datum $\mathcal{D}$ of Cartan type with Cartan matrix A, and a linking parameter λ for $\mathcal{D}$ such that the dotted lines in the diagram of $(\mathcal{D}, \lambda)$ are the lines between i and $l(i)$ for all $i \in I_l^+$.*
(2) *For all $i, j \in I_l^+$, $a_{ij} = a_{l(i)l(j)}$.*

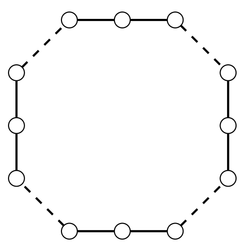
PROOF. (1) $\Rightarrow$ (2) follows from Lemma 8.2.8(1)(b).

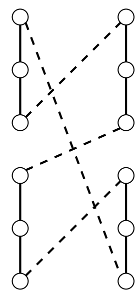
(2) $\Rightarrow$ (1) follows from Corollary 8.2.17 with $t_{J(i)} = 1$, $t_{J(l(i))} = -1$ for all $i \in I_l^+$. $\square$

EXAMPLES 8.2.19. (1) The diagram of a quantum group with perfect linking, see Section 8.4, is the most well-known example of a diagram with non-trivial linking parameter.



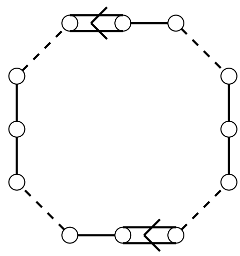
(2) Here is an example of four copies of A_3 linked in a circle. The linking graph is bipartite. In the second picture of the same graph, the decomposition of the set of connected components $\mathcal{X} = \mathcal{X}^- \cup \mathcal{X}^+$ is shown.

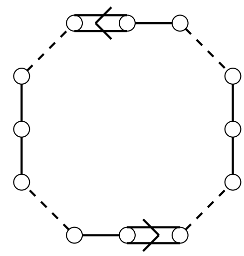




It follows immediately from Corollary 8.2.18 that this diagram is the diagram of some $(\mathcal{D}, \lambda)$.

(3) In the next two diagrams there are two double arrows with different directions. The first diagram can be realized as the diagram of some $(\mathcal{D}, \lambda)$ by Corollary 8.2.17. But for the second diagram the integers t_J do not exist. In fact this diagram cannot be realized as the diagram of some $(\mathcal{D}, \lambda)$ when $\mathcal{D}$ is generic. This follows from Lemma 8.2.11.





8.3. The Hopf algebra $U(\mathcal{D}, \lambda)$

Let $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ be a YD-datum. The Yetter-Drinfeld module $X \in {}^G_G\mathcal{YD}$ defined by $\mathcal{D}$ is a vector space with basis $(x_i)_{i\in I}$ and G-coaction and

G-action given by

$$\delta(x_i) = g_i \otimes x_i, \quad gx_i = \chi_i(g)x_i,$$

for all $i \in I$, $g \in G$. The braiding $c = c_{X,X}$ is the diagonal braiding with

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i,\ q_{ij} = \chi_j(g_i),$$

for all $i, j \in I$.

We identify the tensor algebra $T(X)$ with the free algebra in the generators x_i, $i \in I$. Recall that $T(X)$ is a Hopf algebra in ${}^G_G\mathcal{YD}$. The bosonization $T(X)\#\Bbbk G$ is a Hopf algebra. We identify elements $x \in T(X)$ with $x \otimes 1$ in $T(X)\#\Bbbk G$, and $g \in G$ with $1 \otimes g$. By Lemma 4.3.11, the braided adjoint action $\operatorname{ad} x(y) \in T(X)$ of elements $x, y \in T(X)$ can be identified with the adjoint action $\operatorname{ad} x(y)$ of the Hopf algebra $T(X)\#\Bbbk G$.

Let $i, j_1, \dots, j_t \in I$, $t \geq 1$, and $y = x_{j_1} \cdots x_{j_t} \in T(X)$. Then

$$\operatorname{ad} x_i(y) = x_i y - q_{ij_1} \cdots q_{ij_t} y x_i.$$

Definition 8.3.1. Let $\mathcal{D} = \mathcal{D}(G, (g_i)_{i \in I}, (\chi_i)_{i \in I})$ be a YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j \in I}$, and $\lambda = (\lambda_{ij})_{i,j \in I, i \not\sim j}$ a linking parameter for $\mathcal{D}$. Let $X \in {}^G_G\mathcal{YD}$ be defined by $\mathcal{D}$ with basis $(x_i)_{i \in I}$. Let

$$U(\mathcal{D}) = T(X)/((\operatorname{ad} x_i)^{1-a_{ij}}(x_j) \mid i, j \in I, i \neq j), \tag{8.3.1}$$

$$U(\mathcal{D},\lambda) = (T(X)\#\Bbbk G)/I(\mathcal{D},\lambda), \tag{8.3.2}$$

where $I(\mathcal{D},\lambda)$ is the ideal generated by the elements

$$(\operatorname{ad} x_i)^{1-a_{ij}}(x_j) \text{ for all } i, j \in I,\ i \sim j,\ i \neq j, \tag{8.3.3}$$

$$x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(g_i g_j - 1) \text{ for all } i, j \in I,\ i \not\sim j. \tag{8.3.4}$$

We denote the images of x_i, $i \in I$, in $U(\mathcal{D})$ and $U(\mathcal{D},\lambda)$ again by x_i, and the images of $g \in G$ in $U(\mathcal{D},\lambda)$ by g. For each pair $(i,j) \in I \times I$, $i \not\sim j$,

$$-q_{ji}(x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(g_i g_j - 1)) = x_j x_i - q_{ji} x_i x_j - \lambda_{ji}(g_j g_i - 1).$$

Hence in (8.3.4) we can omit one of the relations for the pair (i,j) and the pair (j,i).

Proposition 8.3.2. *Let $\mathcal{D}$ be a YD-datum of Cartan type, and λ a linking parameter for $\mathcal{D}$.*

1. *$U(\mathcal{D})$ is a quotient Hopf algebra of $T(X)$ in ${}^G_G\mathcal{YD}$.*
2. *$U(\mathcal{D},\lambda)$ is a quotient Hopf algebra of $T(X)\#\Bbbk G$.*
3. *Let I_λ be the ideal in $U(\mathcal{D})\#\Bbbk G$ generated by the images of the elements in* (8.3.4). *Then*
$$U(\mathcal{D},\lambda) \cong (U(\mathcal{D})\#\Bbbk G)/I_\lambda.$$
4. *$U(\mathcal{D},0) \cong U(\mathcal{D})\#\Bbbk G$.*

Proof. (1) It follows from Proposition 4.3.12 that for all $i, j \in I$, $i \neq j$, $(\operatorname{ad} x_i)^{1-a_{ij}}(x_j)$ is primitive in $T(X)$. Hence the elements in (8.3.3) and in (8.3.4) are skew-primitive in $T(X)\#\Bbbk G$. This implies (1) and (2). (3) is clear from the definition of $U(\mathcal{D},\lambda)$, and (4) follows from (3), since

$$(\operatorname{ad} x_i)^{1-a_{ij}}(x_j) = x_i x_j - q_{ij} x_j x_i \quad \text{for all } i, j \in I,\ i \not\sim j.$$

$\square$

Our next goal is to prove that $U(\mathcal{D},\lambda)$ is isomorphic to a quotient Hopf algebra by central group-like elements of a quantum double of two smash products of the form $U(\mathcal{D}')\#\Bbbk G'$. To prove this decomposition we have to assume that the linking graph is bipartite.

We begin with some general observations on bosonizations.

Let G be a monoid, $H=\Bbbk G$ the monoid algebra, R a left H-module algebra, and T an algebra. Let $(g_k)_{k\in K}$ be generators of the monoid G, and $(r_l)_{l\in L}$ generators of the algebra R. Let $\varphi_1: R\to T$, $\varphi_2: H\to T$ be algebra maps satisfying the commutation rule

$$\varphi_2(g_k)\varphi_1(r_l)=\varphi_1(g_k\cdot r_l)\varphi_2(g_k) \tag{8.3.5}$$

for all $k\in K$ and $l\in L$. Then

$$\varphi: R\#H\to T,\ r\#h\mapsto \varphi_1(r)\varphi_2(h),$$

is an algebra map, and any algebra map $R\#H\to T$ has this form.

LEMMA 8.3.3. *Let G be a group, $R=\bigoplus_{n\geq \mathbb{N}_0} R(n)$ be an $\mathbb{N}_0$-graded Hopf algebra in ${}^G_G\mathcal{YD}$ with $R(0)=\Bbbk 1$, and $U=R\#\Bbbk G$ the bosonization.*

(1) *$\widehat{G}\to \mathrm{Alg}(U,\Bbbk)=G(U^0)$, $\chi\mapsto\widetilde{\chi}=\varepsilon\otimes\chi$, is a well-defined group homomorphism.*

(2) *Let $\chi\in\widehat{G}$. Assume that $f: R(1)\to {}_{\chi^{-1}}\Bbbk$ is a G-linear map, where ${}_{\chi^{-1}}\Bbbk$ is the G-module $\Bbbk$ with G-action $g\cdot 1=\chi^{-1}(g)$ for all $g\in G$. Then $f\pi_1\otimes\chi\in P_{1,\widetilde{\chi}}(U^0)$, where $\pi_1: R\to R(1)$ is the projection.*

PROOF. (1) Let $\chi\in\widehat{G}$. The function $\widetilde{\chi}=\varepsilon\otimes\chi: R\#\Bbbk G\to\Bbbk$ is an algebra map, since $\varepsilon: R\to\Bbbk$ is a G-linear algebra map.

Let $\chi_1,\chi_2\in\widehat{G}$. Then $\widetilde{\chi_1}*\widetilde{\chi_2}=\widetilde{\chi_1\chi_2}$, since for all $r\in R$, $g\in G$,

$$\begin{aligned}\widetilde{\chi_1}*\widetilde{\chi_2}(r\#g)&=\widetilde{\chi_1}(r^{(1)}\#r^{(2)}{}_{(-1)}g)\widetilde{\chi_2}(r^{(2)}{}_{(0)}\#g)\\&=\varepsilon(r^{(1)})\chi_1(r^{(2)}{}_{(-1)})\chi_1(g)\varepsilon(r^{(2)}{}_{(0)})\chi_2(g)\\&=\widetilde{\chi_1\chi_2}(r\#g).\end{aligned}$$

(2) Let $\delta=f\pi_1\otimes\chi$. By Lemma 2.3.11 we have to show that $\rho: U\to M_2(\Bbbk)$, $u\mapsto\begin{pmatrix}\varepsilon(u)&\delta(u)\\0&\widetilde{\chi}(u)\end{pmatrix}$, is an algebra map. It is clear that the restrictions of ρ to R and to $\Bbbk G$ are algebra maps. The commutation relations (8.3.5) are equivalent to the G-linearity of f. $\square$

In the remainder of this section let

- $\mathcal{D}=\mathcal{D}(G,(g_i)_{i\in I},(\chi_i)_{i\in I})$ be a YD-datum of Cartan type,
- $(q_{ij})_{i,j\in I}$ the braiding matrix of $\mathcal{D}$,
- $\lambda=(\lambda_{ij})_{i,j\in I,i\not\sim j}$ a linking parameter for $\mathcal{D}$, such that
- the linking graph of $(\mathcal{D},\lambda)$ is bipartite.

We choose non-empty subsets $\mathcal{X}^+$ and $\mathcal{X}^-$ of the set $\mathcal{X}$ of connected components of I with $\mathcal{X}=\mathcal{X}^+\cup\mathcal{X}^-$, $\mathcal{X}^+\cap\mathcal{X}^-=\emptyset$, and

$$I^+=\bigcup_{J\in\mathcal{X}^+}J,\quad I^-=\bigcup_{J\in\mathcal{X}^-}J, \tag{8.3.6}$$

such that $\lambda_{ij}=0$ whenever $i,j\in I^+$ or $i,j\in I^-$.

REMARK 8.3.4. The relations of $U(\mathcal{D}, \lambda)$ can be regrouped as follows.

$$U(\mathcal{D}, \lambda) = (T(X)\#\Bbbk G)/I,$$

where the ideal I is generated by the elements

$$(\operatorname{ad} x_i)^{1-a_{ij}}(x_j) \text{ for all } i, j \in I^+,\ i \neq j, \tag{8.3.7}$$

$$(\operatorname{ad} x_i)^{1-a_{ij}}(x_j) \text{ for all } i, j \in I^-,\ i \neq j, \tag{8.3.8}$$

$$x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(g_i g_j - 1) \text{ for all } i \in I^-,\ j \in I^+. \tag{8.3.9}$$

Let F be a free abelian group with basis $(e_j)_{j\in I^+}$, and define characters $(\eta_j)_{j\in I^+}$ of F by

$$\eta_j(e_k) = \chi_j(g_k) = q_{kj} \text{ for all } k \in I^+. \tag{8.3.10}$$

Let

$$\mathcal{D}^+ = \mathcal{D}(G, (g_i)_{i\in I^+}, (\chi_i)_{i\in I^+}), \quad \mathcal{D}^- = \mathcal{D}(G, (g_i)_{i\in I^-}, (\chi_i)_{i\in I^-}), \tag{8.3.11}$$

be the restrictions of $\mathcal{D}$ to I^+ and I^-, and

$$\mathcal{D}^{(+)} = \mathcal{D}(F, (e_j)_{j\in I^+}, (\eta_j)_{j\in I^+}). \tag{8.3.12}$$

Let X, X^+, X^- be objects in ${}^G_G\mathcal{YD}$, and $X^{(+)} \in {}^F_F\mathcal{YD}$ with basis $(x_i)_{i\in I}$, $(v_i)_{i\in I^+}$, $(u_i)_{i\in I^-}$, and $(a_j)_{j\in I^+}$, respectively, where

$$x_i \in X^{\chi_i}_{g_i} \text{ for all } i \in I,$$
$$v_i \in (X^+)^{\chi_i}_{g_i} \text{ for all } i \in I^+, \quad u_i \in (X^-)^{\chi_i}_{g_i} \text{ for all } i \in I^-,$$
$$a_j \in (X^{(+)})^{\eta_j}_{e_j} \text{ for all } j \in I^+.$$

Then $\mathcal{D}^+$ and $\mathcal{D}^-$ are of Cartan type $(a_{jk})_{j,k\in I^+}$ and $(a_{ij})_{i,j\in I^-}$, respectively. By (8.3.10), $\mathcal{D}^+$ and $\mathcal{D}^{(+)}$ have the same braiding matrix, and $\mathcal{D}^{(+)}$ is of Cartan type $(a_{jk})_{j,k\in I^+}$.

Let $U(\mathcal{D}, \lambda)$ be defined with respect to X. Finally, let $U(\mathcal{D}^{(+)})$ and $U(\mathcal{D}^-)$ be the pre-Nichols algebras defined in Definition 8.3.1 with respect to $X^{(+)}$ and X^-, respectively. Let

$$A = U(\mathcal{D}^{(+)})\#\Bbbk F \quad \text{and} \quad U = U(\mathcal{D}^-)\#\Bbbk G$$

be the bosonizations. To define a quantum double of A and U, we first construct a Hopf algebra homomorphism $\varphi : A \to (U^0)^{\mathrm{cop}}$.

LEMMA 8.3.5. *For all $j \in I^+$, let $\gamma_j = \varepsilon \otimes \chi_j \in U^0$, $\delta_j = f_j\pi_1 \otimes \chi_j \in U^0$, where $f_j : U(\mathcal{D}^-)(1) = X^- \to \Bbbk$ is the linear map defined by $f_j(u_i) = -\lambda_{ij}$ for all $i \in I^-$. Then*

$$\varphi : A \to (U^0)^{\mathrm{cop}},\ a_j \mapsto \delta_j,\ e_j \mapsto \gamma_j,\ \textit{for all } j \in I^+,$$

defines a Hopf algebra map. Let

$$\tau : A \otimes U \to \Bbbk,\ a \otimes u \mapsto \varphi(a)(u),$$

be the skew pairing defined by φ. Then for all $j \in I^+$, $i \in I^-$, $g \in G$,

$$\tau(e_j \otimes g) = \chi_j(g), \qquad \tau^{\pm 1}(e_j \otimes u_i) = 0, \qquad \tau^{\pm 1}(a_j \otimes g) = 0,$$
$$\tau(a_j \otimes u_i) = -\lambda_{ij}, \qquad \tau^{-1}(a_j \otimes u_i) = q_{ij}^{-1}\lambda_{ij}.$$

PROOF. Let $j \in I^+$, $i \in I^-$, and $g \in G$. Then

$$f_j(g \cdot u_i) = \chi_i(g) f_j(u_i) = \chi_j^{-1}(g) f_j(u_i),$$

since $\chi_i \chi_j = \varepsilon$ if $\lambda_{ij} \neq 0$. Hence $f_j : X^- \to {}_{\chi_j^{-1}}\Bbbk$ is G-linear, and by Lemma 8.3.3(2) it follows that $\delta_j : U \to \Bbbk$ is an (ε, γ_j)-derivation.

We claim that

$$\widetilde{\varphi} : T(X^{(+)}) \# \Bbbk F \to (U^0)^{\mathrm{cop}},\ a_j \mapsto \delta_j,\ e_j \mapsto \gamma_j \text{ for all } j \in I^+$$

defines an algebra map. We only have to check the commutation relations (8.3.5), that is,

$$\gamma_k^{\pm 1} * \delta_j = \eta_j^{\pm 1}(e_k) \delta_j * \gamma_k^{\pm 1} \tag{8.3.13}$$

for all $j, k \in I^+$. Let $i \in I^-$, $g \in G$. Then

$$\Delta_U(u_i g) = u_i g \otimes 1 \# g + g_i g \otimes u_i g.$$

Hence for all $j, k \in I^+$,

$$\begin{aligned} (\gamma_k^{\pm 1} * \delta_j)(u_i g) &= \chi_k^{\pm 1}(g_i g) f_j(u_i) \chi_j(g), \\ \eta_j^{\pm 1}(e_k)(\delta_j * \gamma_k^{\pm 1})(u_i g) &= \chi_j^{\pm 1}(g_k) f_j(u_i) \chi_j(g) \chi_k^{\pm 1}(g). \end{aligned}$$

Note that $a_{ik} = 0$, since $i \in I^-$, $k \in I^+$. If $\lambda_{ij} \neq 0$, then $\chi_i \chi_j = 1$, hence

$$\chi_j(g_k) = \chi_i^{-1}(g_k) = q_{ki}^{-1} = q_{ik} = \chi_k(g_i),$$

and the equations (8.3.13) follow.

Since $\widetilde{\varphi}$ is an algebra map, it is clear from its definition that $\widetilde{\varphi}$ is a map of Hopf algebras. Therefore, for all $j, k \in I^+$, $j \neq k$, the elements

$$\widetilde{\varphi}((\operatorname{ad} a_j)^{1-a_{jk}}(a_k)) = (\operatorname{ad} \delta_j)^{1-a_{jk}}(\delta_k)$$

are skew derivations of U. Note that a skew derivation $\delta : U \to \Bbbk$ is 0, if it vanishes on algebra generators of U. Hence the elements $(\operatorname{ad} \delta_j)^{1-a_{jk}}(\delta_k)$ are 0, since by Lemma 4.3.11 and Proposition 4.3.12, they are linear combinations of monomials $\delta_{j_1} \cdots \delta_{j_t}$, $j_1, \dots j_t \in I^+$, $t \geq 2$, and any product $\delta_{j_1}\delta_{j_2}$ vanishes on G and on all elements u_i, $i \in I^-$.

We have shown that $\widetilde{\varphi}$ factorizes over A. This proves the existence of φ. The values of τ in the lemma are easy to check. □

We call the two-cocycle σ for $A \otimes U$ associated to the skew pairing τ defined in Lemma 8.3.5 the two-cocycle defined by φ. Let $\cdot_\sigma$ be the multiplication of $(A \otimes U)_\sigma$. Then by Corollary 2.8.8 and Lemma 2.8.6, for all $a, b \in A$, $u, v \in U$,

$$(a \otimes u) \cdot_\sigma (b \otimes v) = \tau(b_{(1)} \otimes u_{(1)}) a b_{(2)} \otimes u_{(2)} v \tau^{-1}(b_{(3)} \otimes u_{(3)}), \tag{8.3.14}$$

$$\tau^{-1}(a \otimes u) = \tau(\mathcal{S}(a) \otimes u) = \tau(a \otimes \mathcal{S}^{-1}(u)). \tag{8.3.15}$$

In particular,

$$(a \otimes 1) \cdot_\sigma (b \otimes v) = ab \otimes v, \qquad (a \otimes u) \cdot_\sigma (1 \otimes v) = a \otimes uv \tag{8.3.16}$$

for all $a, b \in A$, $u, v \in U$, and

$$(a \otimes u) \cdot_\sigma (b \otimes v) = ab \otimes uv, \qquad \text{if } a, b \in G(A),\ u, v \in G(U). \tag{8.3.17}$$

Recall that for all $j \in I^+$, $i \in I^-$,

$$\Delta^2(a_j) = e_j \otimes e_j \otimes a_j + e_j \otimes a_j \otimes 1 + a_j \otimes 1 \otimes 1,$$
$$\Delta^2(u_i) = g_i \otimes g_i \otimes u_i + g_i \otimes u_i \otimes 1 + u_i \otimes 1 \otimes 1.$$

THEOREM 8.3.6. *Let $\varphi : A \to (U^0)^{\mathrm{cop}}$ be the Hopf algebra homomorphism of Lemma 8.3.5, and σ the two-cocycle defined by φ. Then for all $j \in I^+$, the elements $e_j \otimes g_j^{-1}$ are central group-like elements of $(A \otimes U)_\sigma$, and there is an isomorphism of Hopf algebras*

$$\Phi : U(\mathcal{D}, \lambda) \to (A \otimes U)_\sigma/(e_j \otimes g_j^{-1} - 1 \otimes 1 \mid j \in I^+),$$

mapping x_j with $j \in I^+$, x_i with $i \in I^-$, and $g \in G$ onto the residue classes of $a_j \otimes 1$, $1 \otimes u_i$, and $1 \otimes g$, respectively.

PROOF. (1) We show that Φ is a well-defined Hopf algebra map. Using the formulas (8.3.14), (8.3.15), it is easy to check that the group-like elements $e_j \otimes g_j^{-1}$, $j \in I^+$, are central, since they commute with $a_k \otimes 1$ and $1 \otimes u_i$ for all $k \in I^+$, $i \in I^-$. The elements $e_j \otimes g_j^{-1} - 1 \otimes 1$, $j \in I^+$, generate a Hopf ideal by Proposition 2.4.4.

There is a well-defined algebra homomorphism

$$\widetilde{\Phi} : T(X)\#\Bbbk G \to (A \otimes U)_\sigma,\ x_j \mapsto a_j \otimes 1,\ x_i \mapsto 1 \otimes u_i,\ g \mapsto 1 \otimes g,$$

for all $j \in I^+$, $i \in I^-$, $g \in G$. This follows from the commutation rules

$$(1 \otimes g) \cdot_\sigma (1 \otimes u_i) = \chi_i(g)(1 \otimes u_i) \cdot_\sigma (1 \otimes g),$$
$$(1 \otimes g) \cdot_\sigma (a_j \otimes 1) = \chi_j(g)(a_j \otimes 1) \cdot_\sigma (1 \otimes g),$$

for all $j \in I^+$, $i \in I^-$, $g \in G$.

Next we show that the relations of $U(\mathcal{D}, \lambda)$ are preserved under $\widetilde{\Phi}$ modulo the ideal in $(A \otimes U)_\sigma$ generated by the elements $e_j \otimes g_j^{-1} - 1 \otimes 1$, $j \in I^+$. The Serre relations (8.3.3) are already zero in $(A \otimes U)_\sigma$. It is enough to check the linking relations (8.3.4)

$$x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(g_i g_j - 1) \text{ for all } j \in I^+,\ i \in I^-.$$

Let $j \in I^+$, $i \in I^-$. Then $\widetilde{\Phi}(x_j x_i) = (a_j \otimes 1) \cdot_\sigma (1 \otimes u_i) = a_j \otimes u_i$. We compute $\widetilde{\Phi}(x_i x_j) = (1 \otimes u_i) \cdot_\sigma (a_j \otimes 1)$:

$$\begin{aligned}
(1 \otimes u_i) \cdot_\sigma (a_j \otimes 1) &= \tau(e_j \otimes u_{i(1)}) e_j \otimes u_{i(2)} \tau^{-1}(a_j \otimes u_{i(3)}) \\
&\quad + \tau(e_j \otimes u_{i(1)}) a_j \otimes u_{i(2)} \tau^{-1}(1 \otimes u_{i(3)}) \\
&\quad + \tau(a_j \otimes u_{i(1)}) 1 \otimes u_{i(2)} \tau^{-1}(1 \otimes u_{i(3)}) \\
&= \tau(e_j \otimes g_i) e_j \otimes g_i \tau^{-1}(a_j \otimes u_i) + \tau(e_j \otimes g_i) a_j \otimes u_i + \tau(a_j \otimes u_i) 1 \otimes 1 \\
&= \lambda_{ij} e_j \otimes g_i + q_{ij} a_j \otimes u_i - \lambda_{ij} 1 \otimes 1,
\end{aligned}$$

where we used the values of τ in Lemma 8.3.5. We obtain that

$$\begin{aligned}
\widetilde{\Phi}(x_i x_j - q_{ij} x_j x_i) &= \lambda_{ij}(e_j \otimes g_i - 1 \otimes 1) \\
&\equiv \lambda_{ij}(1 \otimes g_i g_j - 1 \otimes 1) \quad \mathrm{mod}\ (e_k \otimes g_k^{-1} - 1 \otimes 1 \mid k \in I^+) \\
&= \widetilde{\Phi}(\lambda_{ij}(g_i g_j - 1)).
\end{aligned}$$

Thus Φ is a well-defined algebra map. Then it follows easily from the construction of Φ that it is a map of Hopf algebras.

(2) To construct the inverse of Φ, we define Hopf algebra maps

$$\psi^{(+)}: A \to U(\mathcal{D},\lambda),\ a_j \mapsto x_j,\ e_j \mapsto g_j \text{ for all } j \in I^+,$$
$$\psi^-: U \to U(\mathcal{D},\lambda),\ u_i \mapsto x_i,\ g \mapsto g, \text{ for all } i \in I^-,\ g \in G.$$

Note that $\psi^{(+)}$ and ψ^- are well-defined algebra maps, since $\lambda_{ij} = 0$, if i, j are both in I^+ or both in I^-.

We want to show that

$$\widetilde{\Psi}: (A \otimes U)_\sigma \to U(\mathcal{D},\lambda),\ a \otimes x \mapsto \psi^{(+)}(a)\psi^-(x),$$

is a Hopf algebra map. Let $\mathcal{P}$ be the set of all pairs (a, x), $a \in A$, $x \in U$, satisfying

$$\psi^-(x_{(1)})\psi^{(+)}(a_{(1)})\tau(a_{(2)} \otimes x_{(2)}) = \tau(a_{(1)} \otimes x_{(1)})\psi^{(+)}(a_{(2)})\psi^-(x_{(2)}).$$

By Proposition 2.8.11, it suffices to show that the pairs

$$(e_j^{\pm 1}, g), (e_j^{\pm 1}, u_i), (a_j, g), (a_j, u_i) \text{ for all } j \in I^+,\ i \in I^-,\ g \in G,$$

are elements of $\mathcal{P}$. This can be checked case by case using the values of τ in Lemma 8.3.5. In the proof of the last case the linking relations are required.

For all $j \in I^+$, $\widetilde{\Psi}(e_j \otimes g_j^{-1}) = g_j g_j^{-1} = 1$. Hence the Hopf algebra map $\widetilde{\Psi}$ defines a Hopf algebra map

$$(A \otimes U)_\sigma/(e_j \otimes g_j^{-1} - 1 \otimes 1 \mid j \in I^+) \to U(\mathcal{D},\lambda),$$

mapping the residue class of $a \otimes x$ onto $\psi^{(+)}(a)\psi^-(x)$ for all $a \in A$, $x \in U$.

It is clear that this map is inverse to Φ. $\square$

In the next theorem we will show that the Hopf algebra $U(\mathcal{D},\lambda)$ is a two-cocycle deformation of $U(\mathcal{D},0)$. We first prove a general lemma on two-cocycle deformations.

LEMMA 8.3.7. *Let H be a bialgebra, $M \subset G(H)$ a subset, and σ_1, σ_0 two-cocycles for H. Assume that M is central in H_{σ_0}, and that*

$$\sigma_1(g \otimes x) = \sigma_0(g \otimes x), \quad \sigma_1(x \otimes g) = \sigma_0(x \otimes g)$$

for all $g \in M$, $x \in H$. Let $\rho = \sigma_1 * \sigma_0^{-1}$. Then ρ is a two-cocycle for H_{σ_0}, $(H_{\sigma_0})_\rho = H_{\sigma_1}$, and

(1) *$M \subseteq H_{\sigma_1}$ is a central subset, and $g \cdot_{\sigma_0} x = g \cdot_{\sigma_1} x$, $x \cdot_{\sigma_0} g = x \cdot_{\sigma_1} g$ for all $g \in M$, $x \in H$.*
(2) *$\overline{H_{\sigma_i}} = H_{\sigma_i}/(g - 1 \mid g \in M)$, $i \in \{0,1\}$, is a quotient bialgebra of H_{σ_i}, and ρ induces a two-cocycle*

$$\overline{\rho}: \overline{H_{\sigma_0}} \otimes \overline{H_{\sigma_0}} \to \Bbbk,\ \overline{x} \otimes \overline{y} \mapsto \rho(x \otimes y),$$

for $\overline{H_{\sigma_0}}$.
(3) *$(\overline{H_{\sigma_0}})_{\overline{\rho}} = \overline{H_{\sigma_1}}$ as Hopf algebras.*

PROOF. By Remark 2.8.3, ρ is a two-cocycle for H_{σ_0}, and $(H_{\sigma_0})_\rho = H_{\sigma_1}$.

The assumptions on σ_1 and σ_0 imply that

$$\sigma_1^{-1}(g \otimes x) = \sigma_0^{-1}(g \otimes x), \quad \sigma_1^{-1}(x \otimes g) = \sigma_0^{-1}(x \otimes g)$$

for all $g \in M$, $x \in H$. Hence (1) and the equality as coalgebras in (3) follow. By Proposition 2.4.4, $H_{\sigma_i}/(g - 1 \mid g \in M)$, $i \in \{0,1\}$, is a quotient bialgebra of H_{σ_i}.

To prove (2), it is enough to show that the linear maps $\overline{\rho^{\pm 1}}$ are well-defined. Indeed, M is central in H_{σ_0}, and for all $g \in M$, $x, y \in H$,

$$\rho^{\pm 1}(g \otimes x) = \varepsilon(x) = \rho^{\pm 1}(x \otimes g).$$

Hence

$$\rho^{\pm 1}(g \cdot_{\sigma_0} x \otimes y) = \rho^{\pm 1}(x \otimes y) = \rho^{\pm 1}(x \otimes g \cdot_{\sigma_0} y)$$

by the two-cocycle conditions (2.7.1) on ρ for H_{σ_0} and on ρ^{-1} for $H_{\sigma_0}^{\mathrm{cop}}$ for the triples (g, x, y) and (x, y, g).

(3) now follows, since for all $x, y \in H$,

$$\overline{x} \cdot_{\overline{\rho}} \overline{y} = \overline{\rho(x_{(1)} \otimes y_{(1)}) x_{(2)} \cdot_{\sigma_0} y_{(2)} \rho^{-1}(x_{(3)} \otimes y_{(3)})} = \overline{x \cdot_{\sigma_1} y},$$

since $(H_{\sigma_0})_\rho = H_{\sigma_1}$. $\square$

Recall the definition of $\mathcal{D}^+$ and $\mathcal{D}^-$ in (8.3.11). We define maps of Hopf algebras in ${}^G_G\mathcal{YD}$ by

$$\varphi^+ : U(\mathcal{D}^+) \to U(\mathcal{D}),\ v_j \mapsto x_j \text{ for all } j \in I^+,$$
$$\varphi^- : U(\mathcal{D}^-) \to U(\mathcal{D}),\ u_i \mapsto x_i \text{ for all } i \in I^-.$$

LEMMA 8.3.8. (1) *The maps*

$$\varphi^{\mp} : U(\mathcal{D}^-) \otimes U(\mathcal{D}^+) \to U(\mathcal{D}),\ u \otimes v \mapsto \varphi^-(u)\varphi^+(v),$$
$$\varphi^{\pm} : U(\mathcal{D}^+) \otimes U(\mathcal{D}^-) \to U(\mathcal{D}),\ v \otimes u \mapsto \varphi^+(v)\varphi^-(u),$$

are isomorphisms of Hopf algebras in ${}^G_G\mathcal{YD}$.

(2) *The maps*

$$(U(\mathcal{D}^-) \otimes U(\mathcal{D}^+))\#\Bbbk G \xrightarrow{\varphi^{\mp}\otimes \mathrm{id}} U(\mathcal{D})\#\Bbbk G \cong U(\mathcal{D}, 0),$$
$$(U(\mathcal{D}^+) \otimes U(\mathcal{D}^-))\#\Bbbk G \xrightarrow{\varphi^{\pm}\otimes \mathrm{id}} U(\mathcal{D})\#\Bbbk G \cong U(\mathcal{D}, 0),$$

are isomorphisms of Hopf algebras.

PROOF. (1) We prove the lemma for $\varphi^{\pm}$. The result for $\varphi^{\mp}$ follows by changing +,– into –,+.

The algebra map

$$\mathcal{F} : U(\mathcal{D}) \to U(\mathcal{D}^+) \otimes U(\mathcal{D}^-),\ x_j \mapsto v_j \otimes 1, x_i \mapsto 1 \otimes u_i \text{ for all } j \in I^+, i \in I^-,$$

is well-defined, since for all $i \in I^-$, $j \in I^+$, $a_{ij} = a_{ji} = 0$ and $q_{ij}q_{ji} = 1$, hence

$$(\mathrm{ad}\, x_i)(x_j) = x_i x_j - q_{ij} x_j x_i = 0 \text{ in } U(\mathcal{D}),$$
$$\mathcal{F}(x_i)\mathcal{F}(x_j) = (1 \otimes u_i)(v_j \otimes 1) = q_{ij}\mathcal{F}(x_j)\mathcal{F}(x_i) \text{ in } U(\mathcal{D}^+) \otimes U(\mathcal{D}^-).$$

By construction, $\varphi^{\pm}$ is the composition

$$U(\mathcal{D}^+) \otimes U(\mathcal{D}^-) \xrightarrow{\varphi^+ \otimes \varphi^-} U(\mathcal{D}) \otimes U(\mathcal{D}) \xrightarrow{\mu_{U(\mathcal{D})}} U(\mathcal{D}).$$

Hence $\varphi^{\pm}$ is a coalgebra map in ${}^G_G\mathcal{YD}$.

To prove that $\varphi^{\pm}$ is an algebra map, it is enough to prove the following.

$$\varphi^{\pm}((1 \otimes u)(v \otimes 1)) = \varphi^{\pm}(1 \otimes u)\varphi^{\pm}(v \otimes 1)$$

for all

$$u = u_{\underline{i}} = u_{i_1} \cdots u_{i_s},\ \underline{i} = (i_1, \dots, i_s) \in (I^-)^s,$$
$$v = v_{\underline{j}} = v_{j_1} \cdots v_{j_t},\ \underline{j} = (j_1 \dots, j_t) \in (I^+)^t,\ s, t \geq 1.$$

Let $q_{\underline{i}\underline{j}} = \prod_{1 \leq k \leq s, 1 \leq l \leq t} q_{i_k j_l}$. Then

$$(1 \otimes u_{\underline{i}})(v_{\underline{j}} \otimes 1) = q_{\underline{i}\underline{j}} v_{\underline{j}} \otimes u_{\underline{i}}.$$

Hence $\varphi^{\pm}((1 \otimes u)(v \otimes 1)) = q_{\underline{i}\underline{j}} x_{\underline{j}} x_{\underline{i}} = x_{\underline{i}} x_{\underline{j}} = \varphi^{\pm}(1 \otimes u)\varphi^{\pm}(v \otimes 1)$.

It is obvious that the algebra maps $\mathcal{F}$ and $\varphi^{\pm}$ are inverse isomorphisms.

(2) follows from (1). □

As in part (2) of the proof of Theorem 8.3.6, there are Hopf algebra maps

$$\psi^+ : U(\mathcal{D}^+)\#\Bbbk G \to U(\mathcal{D}, \lambda),\ v_j \mapsto x_j, g \mapsto g \text{ for all } j \in I^+,\ g \in G,$$
$$\psi^- : U(\mathcal{D}^-)\#\Bbbk G \to U(\mathcal{D}, \lambda),\ u_i \mapsto x_i,\ g \mapsto g \text{ for all } i \in I^-,\ g \in G,$$

THEOREM 8.3.9. *There are two-cocycles ν for $(U(\mathcal{D}^-) \otimes U(\mathcal{D}^+))\#\Bbbk G$ and ν' for $(U(\mathcal{D}^+) \otimes U(\mathcal{D}^-))\#\Bbbk G$ such that*

$$\Psi : ((U(\mathcal{D}^-) \otimes U(\mathcal{D}^+))\#\Bbbk G)_\nu \to U(\mathcal{D}, \lambda),\ u \otimes v \otimes g \mapsto \psi^-(u)\psi^+(v)g,$$
$$\Psi' : ((U(\mathcal{D}^+) \otimes U(\mathcal{D}^-))\#\Bbbk G)_{\nu'} \to U(\mathcal{D}, \lambda),\ v \otimes u \otimes g \mapsto \psi^+(v)\psi^-(u)g,$$

are isomorphisms of Hopf algebras.

PROOF. Let $H = A \otimes U$, $M = \{e_j \otimes g_j^{-1} \mid j \in I^+\}$. Let σ_0 be the two-cocycle of A corresponding to the algebra map φ_0 defined in Lemma 8.3.5, where λ is replaced by 0 (and same I^+, I^-). Let τ_0 be the skew pairing defined by φ_0. Lemma 8.3.8 can be applied with $\sigma_1 = \sigma$ as above and σ_0, since for all $j \in I^+$, $a \in A$, $u \in U$,

$$\tau(a \otimes g_j^{-1}) = \tau_0(a \otimes g_j^{-1}),\ \tau(e_j \otimes u) = \tau_0(e_j \otimes u)$$

do not depend on λ, hence

$$\sigma(e_j \otimes g_j^{-1} \otimes a \otimes u) = \tau(a \otimes g_j^{-1})\varepsilon(u) = \sigma_0(e_j \otimes g_j^{-1} \otimes a \otimes u),$$
$$\sigma(a \otimes u \otimes e_j \otimes g_j^{-1}) = \varepsilon(a)\tau(e_j \otimes u) = \sigma_0(a \otimes u \otimes e_j \otimes g_j^{-1}).$$

Let Φ_0 be the isomorphism Φ of Theorem 8.3.6 with λ replaced by 0. Hence by Lemma 8.3.7 and Theorem 8.3.6, the composition

$$(U(\mathcal{D}, 0))_{\widetilde{\rho}} \xrightarrow{\Phi_0} ((A \otimes U)_{\sigma_0}/(g - 1 \mid g \in M))_{\overline{\rho}} = (A \otimes U)_\sigma/(g - 1 \mid g \in M)$$
$$\xrightarrow{\Phi^{-1}} U(\mathcal{D}, \lambda)$$

is an isomorphism of Hopf algebras, where $\rho = \sigma * \sigma_0^{-1}$, and where $\widetilde{\rho}$ is the two-cocycle for $U(\mathcal{D}, 0)$ defined from $\overline{\rho}$ by transport of structure with respect to the isomorphism Φ_0.

We compute the linear isomorphism $\Phi^{-1}\Phi_0 : U(\mathcal{D}, 0) \to U(\mathcal{D}, \lambda)$. Let $n \geq 1$. As in the proof of Lemma 8.3.8 let

$$a_{\underline{j}} = a_{j_1} \cdots a_{j_n} \in A,\ u_{\underline{i}} = u_{i_1} \cdots u_{i_n} \in U \text{ for all } \underline{j} \in (I^+)^n, \underline{i} \in (I^-)^n.$$

As before, we denote the images of x_i, $i \in I$, and $g \in G$ in $U(\mathcal{D}, 0)$ and in $U(\mathcal{D}, \lambda)$ again by x_i and g. We write $x_{\underline{i}} = x_{i_1} \cdots x_{i_n}$ for all $\underline{i} \in I^n$ in $U(\mathcal{D}, 0)$ and in $U(\mathcal{D}, \lambda)$. Then for all $\underline{i} \in (I^-)^s, \underline{j} \in (I^+)^t$, $s, t \geq 1$, and $g \in G$,

$$\Phi^{-1}\Phi_0(x_{\underline{j}}x_{\underline{i}}g) = \Phi^{-1}(\overline{(a_{\underline{j}} \otimes u_{\underline{i}}g)}) = x_{\underline{j}}x_{\underline{i}}g.$$

Hence Ψ' is the composition

$$(U(\mathcal{D}^+) \otimes U(\mathcal{D}^-))\#\Bbbk G \xrightarrow{\varphi^{\pm}\otimes \mathrm{id}} U(\mathcal{D})\#\Bbbk G \cong U(\mathcal{D}, 0) \xrightarrow{\Phi^{-1}\Phi_0} U(\mathcal{D}, \lambda),$$

where the first map is the isomorphism of Lemma 8.3.8(2). The two-cocyle ν' is now defined by transport of structure with respect to the Hopf algebra isomorphism $(U(\mathcal{D}^+) \otimes U(\mathcal{D}^-))\#\Bbbk G \cong U(\mathcal{D}, 0)$ and the two-cocycle $\widetilde{\rho}$.

We have shown the theorem for Ψ'. The claim for Ψ follows by changing +,– to –,+. $\square$

8.4. Perfect linkings and multiparameter quantum groups

In this section we single out an important subclass of the Hopf algebras $U(\mathcal{D}, \lambda)$ with bipartite linking graph.

DEFINITION 8.4.1. A **reduced YD-datum**

$$\mathcal{D}_{\mathrm{red}} = \mathcal{D}_{\mathrm{red}}(G, (L_i)_{i\in I}, (K_i)_{i\in I}, (\chi_i)_{i\in I})$$

consists of an abelian group G, a finite, non-empty set I, $L_i \in G$, $K_i \in G$, and $\chi_i \in \widehat{G}$ for all $i \in I$ satisfying

$$\chi_j(K_i) = \chi_i(L_j) \text{ for all } i, j \in I, \tag{8.4.1}$$

$$K_iL_i \neq 1 \text{ for all } i \in I. \tag{8.4.2}$$

For all $i, j \in I$, let $q_{ij} = \chi_j(K_i)$.

Let $\mathcal{D}_{\mathrm{red}}$ be a reduced YD-datum. $\mathcal{D}_{\mathrm{red}}$ is called of **(finite) Cartan type** if the braiding matrix $(q_{ij})_{i,j\in I}$ is; in this case, the Cartan matrix of $\mathcal{D}_{\mathrm{red}}$ is the Cartan matrix of $(q_{ij})_{i,j\in I}$. $\mathcal{D}_{\mathrm{red}}$ is called **generic** and **quasi-generic**, respectively, if the braiding matrix $(q_{ij})_{i,j\in I}$ is.

A **linking parameter** ℓ for a reduced YD-datum over the index set I is a family $\ell = (\ell_i)_{i\in I}$ of non-zero elements in $\Bbbk$.

For simplicity, for the index set I we take

$$\mathbb{I} = \{1, \ldots, \theta\}, \text{ where } \theta \geq 1 \text{ is a natural number.}$$

DEFINITION 8.4.2. Let $\mathcal{D}_{\mathrm{red}} = \mathcal{D}_{\mathrm{red}}(G, (L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ be a reduced YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$. Let $X \in {}^G_G\mathcal{YD}$ with basis $x_1, \ldots, x_\theta, y_1, \ldots, y_\theta$, and $x_i \in X^{\chi_i}_{K_i}$, $y_i \in X^{\chi_i^{-1}}_{L_i}$ for all $i \in \mathbb{I}$. Let $\ell = (\ell_i)_{i\in\mathbb{I}}$ be a linking parameter for $\mathcal{D}_{\mathrm{red}}$. We define $U(\mathcal{D}_{\mathrm{red}}, \ell)$ as the quotient Hopf algebra of the smash product $T(X)\#\Bbbk G$ modulo the ideal generated by

$$(\mathrm{ad}\, x_i)^{1-a_{ij}}(x_j) \text{ for all } i, j \in \mathbb{I},\ i \neq j, \tag{8.4.3}$$

$$(\mathrm{ad}\, y_i)^{1-a_{ij}}(y_j) \text{ for all } i, j \in \mathbb{I},\ i \neq j, \tag{8.4.4}$$

$$x_iy_j - \chi_j^{-1}(K_i)y_jx_i - \delta_{ij}\ell_i(K_iL_i - 1) \text{ for all } i, j \in \mathbb{I}. \tag{8.4.5}$$

Note that $U(\mathcal{D}_{\rm red}, \ell)$ is a Hopf algebra, since the elements (8.4.3), (8.4.4), (8.4.5) are skew-primitive by Proposition 4.3.12.

To see that the quantized enveloping algebras U_q of Kac-Moody algebras and their multiparameter versions are special cases of $U(\mathcal{D}_{\rm red}, \ell)$, we introduce slightly different generators of $U(\mathcal{D}_{\rm red}, \ell)$.

If H is a Hopf algebra with antipode $\mathcal{S}$, we denote the left and right adjoint actions ad_l and ad_r by

$$(\mathrm{ad}_l x)(y) = x_{(1)} y \mathcal{S}(x_{(2)}), \quad (\mathrm{ad}_r x)(y) = \mathcal{S}(x_{(1)}) y x_{(2)} \text{ for all } x, y \in H.$$

In Example 2.6.3, we wrote $\mathrm{ad} = \mathrm{ad}_l$.

LEMMA 8.4.3. *Let $\mathcal{D}_{\rm red} = \mathcal{D}_{\rm red}(G, (L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ be a reduced YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$, braiding matrix $(q_{ij})_{i,j\in\mathbb{I}}$, and linking parameter $\ell = (\ell_i)_{i\in\mathbb{I}}$. We denote by ad_l and ad_r the adjoint actions of the Hopf algebra $H = \Bbbk\langle x_1, \dots, x_\theta, y_1, \dots, y_\theta\rangle \# \Bbbk G$. Define*

$$e_i = x_i, \quad f_i = y_i L_i^{-1}, \text{ for all } i \in \mathbb{I}.$$

Then for all $i, j \in \mathbb{I}$,

$$\Delta(e_i) = K_i \otimes e_i + e_i \otimes 1, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes L_i^{-1}, \tag{8.4.6}$$

$$(\mathrm{ad}_l e_i)^{1-a_{ij}}(e_j) = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k e_i^{1-a_{ij}-k} e_j e_i^k, \tag{8.4.7}$$

$$(\mathrm{ad}_r f_i)^{1-a_{ij}}(f_j) = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k f_i^k f_j f_i^{1-a_{ij}-k}, \tag{8.4.8}$$

$$\mathcal{S}((\mathrm{ad}_l y_i)^n(y_j)) = (-1)^{n+1} q_{ii}^n q_{jj} (\mathrm{ad}_r f_i)^n(f_j), \tag{8.4.9}$$

$$\begin{aligned}(e_i f_j - f_j e_i - \delta_{ij}\ell_i(K_i - L_i^{-1}))L_j &= \\ x_i y_j - \chi_j^{-1}(K_i) y_j x_i &- \delta_{ij}\ell_i(K_i L_i - 1).\end{aligned} \tag{8.4.10}$$

PROOF. (8.4.6) is obvious.

Let $i \in \mathbb{I}$. Note that $\mathcal{S}(e_i) = -K_i^{-1} e_i$ and $\mathcal{S}(f_i) = -f_i L_i$. For all $a \in H$, let L_a and R_a in $\mathrm{End}(H)$ be the left and right muliplication with a. Let σ, τ be the inner automorphisms of H given by $\sigma(x) = K_i x K_i^{-1}, \tau(x) = L_i x L_i^{-1}$ for all $x \in H$. Then

$$\begin{aligned}\mathrm{ad}_l e_i &= A + B, \quad \text{where } A = L_{e_i},\ B = -R_{e_i}\sigma,\ BA = q_{ii} AB,\\ \mathrm{ad}_r f_i &= C + D, \quad \text{where } C = R_{f_i},\ D = -L_{f_i}\tau,\ CD = q_{ii} DC.\end{aligned}$$

By the q-binomial formula in Proposition 1.9.5,

$$(\mathrm{ad}_l e_i)^n = \sum_{k=0}^{n} \binom{n}{k}_{q_{ii}} A^{n-k} B^k, \quad (\mathrm{ad}_r f_i)^n = \sum_{k=0}^{n} \binom{n}{k}_{q_{ii}} D^k C^{n-k}$$

for all n. Since for all $k \geq 0$,

$$B^k = (-1)^k q_{ii}^{\frac{k(k-1)}{2}} R_{e_i^k} \sigma^k, \quad D^k = (-1)^k q_{ii}^{-\frac{k(k-1)}{2}} L_{f_i^k} \tau^k,$$

we obtain for all $j \in \mathbb{I}$,

$$(\mathrm{ad}_l e_i)^n(e_j) = \sum_{i=0}^{n} (-1)^k \binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k e_i^{n-k} e_j e_i^k,$$

$$(\mathrm{ad}_r f_i)^n(f_j) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q_{ii}} q_{ii}^{-\frac{k(k-1)}{2}} q_{ii}^{-k(n-k)} q_{ji}^{-k} f_i^k f_j f_i^{n-k}.$$

This proves (8.4.7), and (8.4.8) follows, since $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$.

Recall that $\Delta(y_i) = L_i \otimes y_i + y_i \otimes 1$, $\Delta(f_i) = 1 \otimes f_i + f_i \otimes L_i^{-1}$, and hence $\mathcal{S}(y_i) = -L_i^{-1} y_i = -q_{ii} y_i L_i^{-1} = -q_{ii} f_i$. The formula

$$\mathcal{S}((\mathrm{ad}_l y_i)^n(x)) = (-1)^n q_{ii}^n (\mathrm{ad}_r f_i)^n(\mathcal{S}(x)) \text{ for all } x \in H, n \geq 1,$$

is shown by induction on n. (8.4.9) follows with $x = y_j$, and (8.4.10) is obvious. □

DEFINITION 8.4.4. Let $\mathcal{D}_{\mathrm{red}} = \mathcal{D}_{\mathrm{red}}(G, (L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ be a reduced YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$, braiding matrix $(q_{ij})_{i,j\in\mathbb{I}}$, and linking parameter $\ell = (\ell_i)_{i\in\mathbb{I}}$. Let

$$\mathbf{H} = \Bbbk\langle E_1, \dots, E_\theta, F_1, \dots, F_\theta\rangle \# \Bbbk G$$

be the smash product algebra, where $\Bbbk\langle E_1, \dots, E_\theta, F_1, \dots, F_\theta\rangle$ is the free algebra with 2θ generators and G-action defined by

$$g \cdot E_i = \chi_i(g) E_i, \quad g \cdot F_i = \chi_i^{-1}(g) F_i \quad \text{for all } i \in \mathbb{I},\ g \in G. \tag{8.4.11}$$

Let $\mathbf{U}(\mathcal{D}_{\mathrm{red}}, \ell)$ be the quotient algebra of $\mathbf{H}$ modulo the ideal generated by

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k E_i^{1-a_{ij}-k} E_j E_i^k, \tag{8.4.12}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k F_i^k F_j F_i^{1-a_{ij}-k}, \tag{8.4.13}$$

for all $i, j \in \mathbb{I}$, $i \neq j$, and

$$E_i F_j - F_j E_i - \delta_{ij} \ell_i (K_i - L_i^{-1}) \text{ for all } i, j \in \mathbb{I}. \tag{8.4.14}$$

We denote the images of E_i, F_i, $i \in \mathbb{I}$, and $g \in G$ in $\mathbf{U}(\mathcal{D}_{\mathrm{red}}, \ell)$ again by E_i, F_i, g. For all $i \in \mathbb{I}$, $g \in G$, let

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \tag{8.4.15}$$

$$\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \qquad \varepsilon(E_i) = 0, \tag{8.4.16}$$

$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes L_i^{-1}, \qquad \varepsilon(F_i) = 0. \tag{8.4.17}$$

PROPOSITION 8.4.5. *Let $\mathcal{D}_{\mathrm{red}} = \mathcal{D}_{\mathrm{red}}(G, (L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ be a reduced YD-datum of Cartan type. Then*

1. *$\mathbf{U}(\mathcal{D}_{\mathrm{red}}, \ell)$ is a Hopf algebra with Δ, ε given by* (8.4.15)–(8.4.17).
2. *The map $U(\mathcal{D}_{\mathrm{red}}, \ell) \to \mathbf{U}(\mathcal{D}_{\mathrm{red}}, \ell)$, $g \mapsto g$, $x_i \mapsto E_i$, $y_i \mapsto F_i L_i$ for all $g \in G$, $i \in \mathbb{I}$, is a Hopf algebra isomorphism.*

PROOF. Let $H = \Bbbk\langle x_1, \dots, x_\theta, y_1, \dots, y_\theta\rangle \# \Bbbk G$. Then

$$\varphi : H \to \mathbf{H}, \quad g \mapsto g,\ x_i \mapsto E_i,\ y_i \mapsto F_i L_i \quad \text{for all } g \in G,\ i \in \mathbb{I},$$

is an algebra isomorphism. By definition, $U(\mathcal{D}_{\text{red}}, \ell) = H/I$, where I is the ideal generated by the elements (8.4.3), (8.4.4) and (8.4.5). The bosonization H is a pointed Hopf algebra by Corollary 5.4.4, and $I \subseteq H$ is a Hopf ideal. As in Lemma 8.4.3, we set $e_i = x_i$, $f_i = y_i L_i^{-1}$ for all $i \in \mathbb{I}$. Let I' be the ideal of H generated by

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k e_i^{1-a_{ij}-k} e_j e_i^k, \tag{8.4.18}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k f_i^k f_j f_i^{1-a_{ij}-k}, \tag{8.4.19}$$

for all $i, j \in \mathbb{I}$, $i \neq j$, and

$$e_i f_j - f_j e_i - \delta_{ij}\ell_i(K_i - L_i^{-1}) \text{ for all } i, j \in \mathbb{I}. \tag{8.4.20}$$

The generators of I' are skew-primitive by Lemma 8.4.3, since the antipode preserves skew-primitive elements. Hence I' is a Hopf ideal. By Corollary 5.4.3, the antipode $\mathcal{S}$ of H is bijective, and $\mathcal{S}(I') = I'$. Then it follows from Lemma 8.4.3 that $I' \subseteq I$, since $\mathcal{S}(I) \subseteq I$, and $I \subseteq I'$, since $\mathcal{S}^{-1}(I') \subseteq I'$. Thus φ induces an isomorphism of Hopf algebras. We have shown (1) and (2). $\square$

EXAMPLE 8.4.6. Let $A = (a_{ij})_{i,j\in\mathbb{I}}$ be a symmetrizable Cartan matrix, and $(d_i)_{i\in\mathbb{I}}$ a family of positive integers such that the matrix $(d_i a_{ij})_{i,j\in\mathbb{I}}$ is symmetric. Let $0 \neq q \in \Bbbk$ such that $q^{2d_i} \neq 1$ for all $i \in \mathbb{I}$. Let G be a free abelian group of rank θ with basis $(K_i)_{i\in\mathbb{I}}$, and $L_i = K_i$ for all $i \in \mathbb{I}$. Define characters χ_i, $i \in \mathbb{I}$, of G by

$$\chi_j(K_i) = q^{d_i a_{ij}} \text{ for all } i, j \in \mathbb{I}.$$

Then $\mathcal{D}_{\text{red}} = \mathcal{D}_{\text{red}}(G, (L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ is a reduced datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$. Let U_q be the quantized enveloping algebra of the Kac-Moody algebra associated to A. Then the Serre relations (8.4.12) and (8.4.13) are the Serre relations of U_q, and $\mathbf{U}(\mathcal{D}_{\text{red}}, \ell) = U_q$, where $\ell_i = (q^{d_i} - q^{-d_i})^{-1}$ for all $i \in \mathbb{I}$.

EXAMPLE 8.4.7. Let $\mathcal{D}_{\text{red}} = \mathcal{D}_{\text{red}}(G, (L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ be a reduced YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$, braiding matrix $(q_{ij})_{i,j\in\mathbb{I}}$, and linking parameter $\ell = (\ell_i)_{i\in\mathbb{I}}$. Let $\mathcal{X}$ be the set of connected components of $\mathbb{I}$ with respect to $(a_{ij})_{i,j\in\mathbb{I}}$. Assume that there are positive integers $(d_i)_{i\in\mathbb{I}}$ and non-zero elements $(q_J)_{J\in\mathcal{X}}$ in $\Bbbk$, $q_J^{2d_i} \neq 1$ for all J, $i \in J$, such that

$$d_i a_{ij} = d_j a_{ji} \quad \text{for all } i, j \in \mathbb{I},$$
$$q_{ii} = q_J^{2d_i} \quad \text{for all } J \in \mathcal{X},\ i \in J.$$

Note that by Lemma 8.2.4 this assumption in particular holds when $(a_{ij})_{i,j\in\mathbb{I}}$ is of finite type (and the elements q_J in Lemma 8.2.4 have a square root). Then the Serre relations (8.4.12) and (8.4.13) have the form

$$\sum_{k=0}^{1-a_{ij}} (-p_{ij})^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_J^{d_i}} E_i^{1-a_{ij}-k} E_j E_i^k = 0,$$
$$\sum_{k=0}^{1-a_{ij}} (-p_{ij})^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_J^{d_i}} F_i^k E_j E_i^{1-a_{ij}-k} = 0,$$

where $p_{ij} = q_{ij} q_J^{-d_i a_{ij}}$ for all $J \in \mathcal{X}$, $i, j \in J$, $i \neq j$.

EXAMPLE 8.4.8. Let $n \geq 1$ be a natural number, and $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$ be the basis of a free abelian group G of rank $2(n+1)$. Fix $r, s \in \Bbbk^\times$ with $r \neq s$. Define characters $\chi_i \in \widehat{G}$ and $K_i, L_i \in G$ for all $1 \leq i \leq n$ by

$$\chi_i(a_j) = r^{\delta_{i,j} - \delta_{i+1,j}}, \chi_i(b_j) = s^{\delta_{i,j} - \delta_{i+1,j}} \text{ for all } 1 \leq j \leq n+1,$$
$$K_i = a_i b_{i+1},\ L_i = (a_{i+1} b_i)^{-1}.$$

Then $\mathcal{D}_{\text{red}} = \mathcal{D}_{\text{red}}(G, (L_i)_{1 \leq i \leq n}, (K_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n})$ is a reduced YD-datum of finite Cartan type A_n with braiding matrix $(q_{ij})_{1 \leq i,j \leq n}$, and for all $1 \leq i, j \leq n$, $q_{ii} = rs^{-1}$, $q_{i,i+1} = s$, $q_{i+1,i} = r^{-1}$, if $1 \leq i < n$, and $q_{i,j} = 1$, if $|i - j| > 1$.

The Serre relations of $\mathbf{U}(\mathcal{D}_{\text{red}}, \ell)$ for $E_1, \dots, E_n$ are

$$E_i E_j - E_j E_i = 0, \text{ if } |i - j| > 1,$$
$$E_i^2 E_{i+1} - (r+s) E_i E_{i+1} E_i + rs E_{i+1} E_i^2 = 0, \text{ if } 1 \leq i < n,$$
$$E_{i+1}^2 E_i - (r^{-1} + s^{-1}) E_{i+1} E_i E_{i+1} + r^{-1} s^{-1} E_i E_{i+1}^2 = 0, \text{ if } 1 \leq i < n.$$

Let $l_i = (r-s)^{-1}$, and $\ell = (l_i)_{1 \leq i \leq n}$. Then $\mathbf{U}(\mathcal{D}_{\text{red}}, \ell) = U_{r,s}(\mathfrak{gl}_{n+1})$ is the two-parameter deformation defined in [**BW04**].

Let $\mathcal{D}_{\text{red}}$ be a reduced YD-datum with Cartan matrix A and a linking parameter ℓ. It is easy to see that $(\mathcal{D}_{\text{red}}, \ell)$ can be identified with $(\mathcal{D}, \lambda)$, and $U(\mathcal{D}_{\text{red}}, \ell)$ with $U(\mathcal{D}, \lambda)$ for some λ, where the Cartan matrix of $\mathcal{D}$ is a block matrix with 2 copies of A in the diagonal, and such that each point of the Dynkin diagram of $\mathcal{D}$ is linked with its copy.

Hence we can apply the results of the previous section to reduced data.

LEMMA 8.4.9. *Let $\mathcal{D}_{\text{red}} = \mathcal{D}_{\text{red}}(G, (L_i)_{i \in \mathbb{I}}, (K_i)_{i \in \mathbb{I}}, (\chi_i)_{i \in \mathbb{I}})$ be a reduced YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j \in \mathbb{I}}$. Let $\ell = (\ell_i)_{i \in \mathbb{I}}$ be a linking parameter of $\mathcal{D}_{\text{red}}$, $\widetilde{\mathbb{I}} = \{1, \dots, 2\theta\}$, and define*

$$(g_1, \dots, g_{2\theta}) = (K_1, \dots, K_\theta, L_1, \dots, L_\theta),$$
$$(\eta_1, \dots, \eta_{2\theta}) = (\chi_1, \dots, \chi_\theta, \chi_1^{-1}, \dots, \chi_\theta^{-1}),$$
$$\widetilde{a}_{ij} = a_{ij} = \widetilde{a}_{\theta+i,\theta+j}, \quad \widetilde{a}_{i,\theta+j} = 0 = \widetilde{a}_{\theta+i,j} \textit{ for all } i, j \in \mathbb{I},$$
$$\lambda_{ij} = \begin{cases} \ell_i & \textit{if } i \in \mathbb{I},\ j = \theta + i, \\ -q_{jj} \ell_j & \textit{if } j \in \mathbb{I},\ i = \theta + j, \\ 0 & \textit{otherwise} \end{cases} \qquad \textit{for all } i, j \in \widetilde{\mathbb{I}},\ i \not\approx j,$$

where $\approx$ is the equivalence relation of $\widetilde{\mathbb{I}}$ with respect to the Cartan matrix $(\widetilde{a}_{i,j})_{i,j \in \widetilde{\mathbb{I}}}$.

1. *$\mathcal{D} = \mathcal{D}(G, (g_i)_{i \in \widetilde{\mathbb{I}}}, (\eta_i)_{i \in \widetilde{\mathbb{I}}})$ is a YD-datum of Cartan type with Cartan matrix $(\widetilde{a}_{ij})_{i,j \in \widetilde{\mathbb{I}}}$ and linking parameter $\lambda = (\lambda_{i,j})_{i,j \in \widetilde{\mathbb{I}}, i \not\approx j}$.*
2. *$U(\mathcal{D}_{\text{red}}, \ell) = U(\mathcal{D}, \lambda)$.*
3. *The linking graph of $(\mathcal{D}, \lambda)$ is bipartite. The Dynkin diagram of $\mathcal{D}$ consists of two copies of the Dynkin diagram of $(a_{ij})_{i,j \in \mathbb{I}}$, and each vertex is linked with its copy. A decomposition of $\widetilde{\mathbb{I}}$ as in* (8.3.6) *is*

$$\widetilde{\mathbb{I}}^+ = \{1, \dots, \theta\} = \mathbb{I}, \quad \widetilde{\mathbb{I}}^- = \{\theta + 1, \dots, 2\theta\}.$$

PROOF. Let $\widetilde{\mathcal{X}}$ be the set of connected components of $\widetilde{\mathbb{I}}$ with respect to the Cartan matrix $(\widetilde{a}_{ij})_{i,j \in \widetilde{\mathbb{I}}}$, and $\widetilde{\mathcal{X}}^+$ the set of connected components of $\mathbb{I}$ with respect to the Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$. For all $J \in \widetilde{\mathcal{X}}^+$, let

$$J' = \{\theta + j \mid j \in J\}.$$

Let $\widetilde{\mathcal{X}}^- = \{J' \mid J \in \widetilde{\mathcal{X}}^+\}$. Then

$$\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}^+ \cup \widetilde{\mathcal{X}}^-,\ \widetilde{\mathcal{X}}^+ \cap \widetilde{\mathcal{X}}^- = \emptyset, \quad \widetilde{\mathbb{I}}^+ = \bigcup_{J\in\widetilde{\mathbb{I}}^+} J,\ \widetilde{\mathbb{I}}^- = \bigcup_{J\in\widetilde{\mathbb{I}}^-} J'.$$

By (8.4.1), $(\widetilde{a}_{ij})_{i,j\in\widetilde{\mathbb{I}}}$ is the Cartan matrix of $\mathcal{D}$, and (1) and (2) are easy to check.

For all $i \in \mathbb{I}$, the vertices $i, \theta + i$ are linked, and $\lambda_{ij} = 0$ for all $i, j \in \widetilde{\mathbb{I}}^+$ and for all $i, j \in \widetilde{\mathbb{I}}^-$. This proves (3). □

DEFINITION 8.4.10. A linking parameter of a YD-datum $\mathcal{D}$ of Cartan type is called **perfect** if any vertex is linked.

See 8.2.19(1) for an example of a perfect linking.

PROPOSITION 8.4.11. *Let $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ be a YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in I}$, braiding matrix $(q_{ij})_{i,j\in I}$, and linking parameter $\lambda = (\lambda_{ij})_{i,j\in I, i\not\sim j}$. Assume that for all $i, j \in I$, $i \neq j$, $\operatorname{ord}(q_{ii})$ does not divide $2 - a_{ij}$ (this holds in particular, if q_{ii} is not a root of one). Then the following are equivalent.*

(1) *The linking parameter λ is perfect.*
(2) *There are a reduced YD-datum $\mathcal{D}_{\mathrm{red}}$ of Cartan type and a linking parameter ℓ for $\mathcal{D}_{\mathrm{red}}$ such that*

$$U(\mathcal{D}, \lambda) \cong U(\mathcal{D}_{\mathrm{red}}, \ell)$$

as Hopf algebras, and up to renumbering of the vertices, $(\mathcal{D}, \lambda)$ is constructed from $(\mathcal{D}_{\mathrm{red}}, \ell)$ as in Lemma 8.4.9.

PROOF. (1) ⇒ (2). Let $\mathcal{X}$ be the set of connected components of I. It follows from Lemma 8.2.8 that for each $i \in I$, there is exactly one $i' \in I$ such that (i, i') is linked; moreover, $a_{ij} = a_{i'j'}$. Hence vertices $i, j \in I$ are in the same connected component of I if and only if i' and j' are in the same connected component. Thus the involution $I \to I$, $i \mapsto i'$, induces the involution $\mathcal{X} \to \mathcal{X}$, $J \mapsto J' = \{j' \mid j \in J\}$. By definition of a linking parameter, this involution has no fixed point. Thus the linking graph is bipartite, since there is an edge between $J_1, J_2 \in \mathcal{X}$ if and only if $J_2 = J_1'$.

After renumbering we may assume that

$$I = I^+ \cup I^-,\ I^+ \cap I^- = \emptyset,\ I^+ = \{1, \dots, \theta\},\ I^- = \{\theta + 1, \dots, 2\theta\},$$

where $|I| = 2\theta$, and $i' = \theta + i$ for all $i \in I^+$. Then for all $i \in I^+$, $\chi_{\theta+i} = \chi_i^{-1}$, and $\lambda_{\theta+i,i} \neq 0$, since $(i, \theta + i)$ is linked.

Define $\mathcal{D}_{\mathrm{red}}(G, (L_i)_{i\in I^+}, (K_i)_{i\in I^+}, (\chi_i)_{i\in I^+})$ and $\ell = (\ell_i)_{i\in I^+}$ by

$$(g_1, \dots, g_{2\theta}) = (K_1, \dots, K_\theta, L_1, \dots, L_\theta),$$
$$\ell_i = \lambda_{i,\theta+i} \text{ for all } i \in I^+.$$

Then $\mathcal{D}_{\mathrm{red}}$ is a reduced YD-datum of Cartan type with linking parameter ℓ, and (2) follows from Lemma 8.4.9.

(2) ⇒ (1) is obvious. □

Let $\mathcal{D}_{\text{red}} = \mathcal{D}_{\text{red}}(G, (L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ be a reduced YD-datum of Cartan type with Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$ with linking parameter ℓ. Then

$$\mathcal{D}^+ = \mathcal{D}(G, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}}), \quad \mathcal{D}^- = \mathcal{D}(G, (L_i)_{i\in\mathbb{I}}, (\chi_i^{-1})_{i\in\mathbb{I}}),$$

are YD-data of Cartan type with the same Cartan matrix. Let X^+, X^- in ${}_G^G\mathcal{YD}$ be defined by $\mathcal{D}^+$ and $\mathcal{D}^-$ with bases $(x_i)_{i\in\mathbb{I}}$ of X^+ and $(y_i)_{i\in\mathbb{I}}$ of X^-, that is,

$$x_i \in (X^+)^{\chi_i}_{K_i},\ y_i \in (X^-)^{\chi_i^{-1}}_{L_i} \text{ for all } i \in \mathbb{I}.$$

Note that $X = X^+ \oplus X^-$ in ${}_G^G\mathcal{YD}$, where X is the Yetter-Drinfeld module in Definition 8.4.2. We define $U(\mathcal{D}^+)$ and $U(\mathcal{D}^-)$ with respect to X^+ and X^-. By abuse of notation, the images of x_i and y_i in $U(\mathcal{D}^+)$ and $U(\mathcal{D}^-)$ and in $U(\mathcal{D}_{\text{red}}, \ell)$ are again denoted by x_i and y_i. Then there are Hopf algebra maps

$$\psi^- : U(\mathcal{D}^-)\#\Bbbk G \to U(\mathcal{D}_{\text{red}}, \ell),\ y_i \mapsto y_i, g \mapsto g, \text{ for all } i \in \mathbb{I},\ g \in G,$$
$$\psi^+ : U(\mathcal{D}^+)\#\Bbbk G \to U(\mathcal{D}_{\text{red}}, \ell),\ x_i \mapsto x_i, g \mapsto g, \text{ for all } i \in \mathbb{I},\ g \in G.$$

COROLLARY 8.4.12. *There are two-cocycles ν for $(U(\mathcal{D}^-) \otimes U(\mathcal{D}^+))\#\Bbbk G$ and ν' for $(U(\mathcal{D}^+) \otimes U(\mathcal{D}^-))\#\Bbbk G$ such that*

$$(U(\mathcal{D}^-) \otimes U(\mathcal{D}^+) \otimes \Bbbk G)_\nu \to U(\mathcal{D}_{\text{red}}, \ell),\ y \otimes x \otimes g \mapsto \psi^-(y)\psi^+(x)g,$$
$$(U(\mathcal{D}^+) \otimes U(\mathcal{D}^-) \otimes \Bbbk G)_{\nu'} \to U(\mathcal{D}_{\text{red}}, \ell),\ x \otimes y \otimes g \mapsto \psi^+(x)\psi^-(y)g,$$

are isomorphisms of Hopf algebras.

PROOF. This follows from Lemma 8.4.9 and Theorem 8.3.9. □

8.5. Notes

8.1. The Hopf algebras U_q were defined by Jimbo [**Jim85**] and Drinfeld [**Dri87**].

8.2. Our definition of the Dynkin diagram of a Cartan matrix follows [**Kac90**, § 4.7].

Linking was first defined in [**AS02**].

See [**Did02**] for more information about the possible diagrams of $(\mathcal{D}, \lambda)$ and a different approach not assuming that the linking graph is bipartite. Corollary 8.2.12 was shown in [**RS08b**, Lemma 4.2], where the linking graph was introduced.

8.3. Theorem 8.3.6 is Theorem 4.4 in [**RS08b**] (for finite Cartan type); the strategy of the proof comes from the proof of Theorem 5.17 in [**AS02**]. There are various versions of Theorem 8.3.6 where the Serre relations in the definition of $U(\mathcal{D}, \lambda)$ are replaced by the relations of a Nichols algebra [**RS08a**, Theorem 8.3], or a pre-Nichols algebra [**Mas08**, Theorem 4.3, Theorem 5.3].

The two-cocycles in Theorem 8.3.9 are derived from the isomorphism in Theorem 8.3.6 as in the proof of [**Did05**, Theorem 1], where Didt showed that the finite-dimensional pointed Hopf algebras $A = u(\mathcal{D}, \lambda, \mu)$ in [**AS10**] with $\mu = 0$ are two-cocycle deformations of $\operatorname{gr} A$; see also [**KS05**, Section 4].

We will see in Theorem 16.5.5 that the pointed Hopf algebra $A = U(\mathcal{D}, \lambda)$, $\mathcal{D}$ a generic YD-datum of finite Cartan type, is a two-cocycle deformation of $\operatorname{gr} A$.

As culmination of a series of papers of various authors, it was shown by Angiono and Iglesias, see [**AI18**], [**AGI19**], that any finite-dimensional pointed Hopf algebra

A with abelian group $G(A)$ over an algebraically closed field of characteristic 0 is a two-cocycle deformation of $\operatorname{gr} A$.

Let $\mathcal{D}$ be a generic YD-datum of finite Cartan type, and $0 \neq \lambda$ a linking parameter for $\mathcal{D}$. In [**RS08b**, Theorem 4.6] a bijection is constructed between the isomorphism classes of finite-dimensional irreducible $U(\mathcal{D}, \lambda)$-modules and dominant characters of G. In the case of $U(\mathcal{D}_{\text{red}}, \ell)$, a character χ of G is dominant if there are natural numbers $m_i \geq 0$, $1 \leq i \leq \theta$, such that $\chi(K_i L_i) = q_{ii}^{m_i}$ for all i.

8.4. Reduced YD-data were introduced in [**RS08b**].

An early example of a two-parameter quantum group was given by Takeuchi in [**Tak90**], which is up to notation the Hopf algebra in Example 8.4.8 (with opposite comultiplication).

In [**PHR10**] a general class of multi-parameter quantum groups $U_{\boldsymbol{q}}(\mathfrak{g}_A)$, A a symmetrizable Cartan matrix, was defined. In [**PHR10**], Remark 9, it is noted that many multi-parameter quantum groups which appeared before are of the form $U_{\boldsymbol{q}}(\mathfrak{g}_A)$. The Hopf algebras $U_{\boldsymbol{q}}(\mathfrak{g}_A)$ are special cases of our Hopf algebras $\mathbf{U}(\mathcal{D}_{\text{red}}, \ell)$ with some restrictions on the field and the group.

The representation theory of $U(\mathcal{D}_{\text{red}}, \ell)$ was studied in [**ARS10**]. Assume that $\mathcal{D}_{\text{red}} = \mathcal{D}_{\text{red}}(G, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta})$ is a generic reduced YD-datum of Cartan type, and that $\mathcal{D}_{\text{red}}$ is regular, that is, the characters $\chi_1, \dots, \chi_\theta$ are $\mathbb{Z}$-linearly independent in $\widehat{G}$. Then the representation theory in Sections 3.4, 3.5, 6.1 and 6.2 of [**Lus93**] (irreducible highest weight modules, integrable modules, quantum Casimir operator, complete reducibility theorems) can be extended to $U(\mathcal{D}_{\text{red}}, \ell)$, where ℓ is a linking parameter of $\mathcal{D}_{\text{red}}$.

Let $\mathcal{D}$ be a generic YD-datum of finite Cartan type with abelian group G, and λ a linking datum for $\mathcal{D}$. The representation theory above is used to prove the following characterization of perfect linking parameters. Let A be an algebra, and $B \subseteq A$ a subalgebra. A is called reductive, if any finite-dimensional left A-module is semisimple; A is called B-reductive, if any finite-dimensional left A-module which is B-semisimple when restricted to B, is A-semisimple. By Theorem 5.3 in [**ARS10**], the following are equivalent.

(1) $U(\mathcal{D}, \lambda)$ is $\Bbbk G$-reductive.
(2) The linking parameter λ is perfect.

Thus by Proposition 8.4.11, $U(\mathcal{D}, \lambda) \cong U(\mathcal{D}_{\text{red}}, \ell)$, where $\mathcal{D}_{\text{red}}$ is a generic reduced YD-datum of finite Cartan type, and ℓ is a linking parameter for $\mathcal{D}_{\text{red}}$. Let G^2 be the subgroup of G generated by the products $K_i L_i$, $1 \leq i \leq \theta$. By Theorem 5.3 in [**ARS10**], the following are equivalent.

(1) $U(\mathcal{D}, \lambda)$ is reductive.
(2) The linking parameter is perfect, and G/G^2 is finite.

Part 2

Cartan graphs, Weyl groupoids, and root systems

CHAPTER 9

Cartan graphs and Weyl groupoids

The (generalized) Cartan matrix and its associated Weyl group as well as the root system are among the most fundamental invariants of a semi-simple Lie algebra and more generally of a Kac-Moody algebra. Important classes of Nichols algebras appear to have fundamental invariants of the same significance. However, instead of one Cartan matrix one has to consider a family of Cartan matrices with a natural relationship among them. This leads us to the notions in the title of this part of the book.

Because of their outstanding role, Weyl groups are studied in many different generalities and have several interpretations. Similarly, several other explanations and interpretations of Weyl groupoids and related structures are available in the literature. We chose a presentation of the topic which is most suitable to explain in Part 3 the deep interrelation between a Nichols algebra, its root system and its Weyl groupoid.

Nevertheless, the subject discussed in Part 2 is independent of the notion of a Nichols algebra.

9.1. Axioms and examples

The most natural way to extend the notion of the Weyl group of one Cartan matrix to the situation of a family of Cartan matrices seems to be the Weyl groupoid of a semi-Cartan graph to be introduced in this chapter. However, this definition is much too general to be useful. We define a Cartan graph as a semi-Cartan graph satisfying two additional conditions which allow to show that the Weyl groupoid is a Coxeter groupoid.

We fix once and for all the notation $(\alpha_i)_{i\in I}$ for the **standard basis** of $\mathbb{Z}^I$ for any finite set I.

DEFINITION 9.1.1. Let I be non-empty finite set, $\mathcal{X}$ a non-empty set, and $r : I \times \mathcal{X} \to \mathcal{X}$, $A : I \times I \times \mathcal{X} \to \mathbb{Z}$ maps. For all $i, j \in I$ and $X \in \mathcal{X}$ we write $r_i(X) = r(i, X)$, $a^X_{ij} = A(i, j, X)$, and $A^X = (a^X_{ij})_{i,j\in I} \in \mathbb{Z}^{I\times I}$. The quadruple $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is called a **semi-Cartan graph** if for all $X \in \mathcal{X}$, the matrix A^X is a Cartan matrix in the sense of Definition 1.10.17, and if the following hold.

(CG1) For all $i \in I$, $r_i^2 = \mathrm{id}_{\mathcal{X}}$.

(CG2) For all $i, j \in I$ and $X \in \mathcal{X}$, $a^X_{ij} = a^{r_i(X)}_{ij}$, that is, A^X and $A^{r_i(X)}$ have the same i-th row.

The cardinality of I is called the **rank of** $\mathcal{G}$. The elements of $\mathcal{X}$ are called the **points of** $\mathcal{G}$, and the elements of I the **labels of** $\mathcal{G}$. For any $X \in \mathcal{X}$ and any $i \in I$ let

$$s^X_i \in \mathrm{Aut}(\mathbb{Z}^I), \quad s^X_i(\alpha_j) = \alpha_j - a^X_{ij}\alpha_i \text{ for all } j \in I.$$

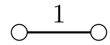


FIGURE 9.1.1. Exchange graph with two points

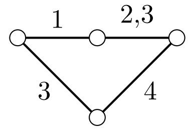


FIGURE 9.1.2. Exchange graph with four points

As in (CG1) and (CG2), typically we will view r as a family $(r_i)_{i\in I}$ of permutations of $\mathcal{X}$, and A as a family $(A^X)_{X\in\mathcal{X}}$ of matrices. Note that $s_i^X(\alpha_i) = -\alpha_i$, and $(s_i^X)^2 = \mathrm{id}$ in Definition 9.1.1, since $a_{ii}^X = 2$. (CG2) says that $s_i^X = s_i^{r_i(X)}$ for all $X \in \mathcal{X}$, $i \in I$.

EXAMPLE 9.1.2. Let $I = \{1,2\}$, $\mathcal{X} = \{X_1, X_2\}$, $r_1 : \mathcal{X} \to \mathcal{X}$ the non-trivial permutation and $r_2 : \mathcal{X} \to \mathcal{X}$ the identity. Let

$$A^{X_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad A^{X_2} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Then $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is a semi-Cartan graph.

DEFINITION 9.1.3. Let I, $\mathcal{X}$ and r be as in Definition 9.1.1, and assume that (CG1) holds. The **exchange graph of** $(\mathcal{X}, r, I)$ is a labeled non-oriented graph with vertices corresponding to the elements of $\mathcal{X}$, and edges marked by elements $i \in I$, where two vertices X, Y are connected by an edge i if and only if $X \neq Y$ and $r_i(X) = Y$ (and $r_i(Y) = X$). The exchange graph of a semi-Cartan graph $\mathcal{G}(I, \mathcal{X}, r, A)$ is the exchange graph of $(\mathcal{X}, r, I)$.

The exchange graph of $(\mathcal{X}, r, I)$ may have multiple edges (with different labels) but no loops. Any two edges with a vertex in common have different labels. Conversely, labeled graphs with these two properties describe triples $(\mathcal{X}, r, I)$ satisfying (CG1). For simplicity, instead of several edges with different labels, we display only one edge with several labels. The exchange graph of Example 9.1.2 is displayed in Figure 9.1.1.

EXAMPLE 9.1.4. Let $I = \{1, 2, 3, 4, 5\}$, and $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$ a set of four points. Let $\sigma_1 = (1\,2)$, $\sigma_2 = (2\,3)$, $\sigma_3 = (2\,3)(1\,4)$, $\sigma_4 = (3\,4)$. Let $r_i(X_j) = X_{\sigma_i(j)}$ for all $1 \le i, j \le 4$, and $r_5 = \mathrm{id}_{\mathcal{X}}$. Then (CG1) is satisfied. The exchange graph of $(\mathcal{X}, r, I)$ is shown in Figure 9.1.2.

A semi-Cartan graph can be viewed as a labeled exchange graph, where any vertex $X \in \mathcal{X}$ has A^X as a label.

The semi-Cartan graph $\mathcal{G}$ in Example 9.1.2 as a labeled graph is drawn in Figure 9.1.3. In a short-hand notation we used the Cartan matrices A^X as placeholders for the vertices $X \in \mathcal{X}$. This describes $\mathcal{G}$ completely.

Given $(\mathcal{X}, r, I)$ satisfying (CG1), there is always a family A of Cartan matrices such that $\mathcal{G}(I, \mathcal{X}, r, A)$ is a semi-Cartan graph. For any Cartan matrix A we may simply define $A^X = A$ for all $X \in \mathcal{X}$.

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

FIGURE 9.1.3. The semi-Cartan graph in Example 9.1.2

DEFINITION 9.1.5. A semi-Cartan graph $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is called **standard** if $A^X = A^Y$ for any two points $X, Y \in \mathcal{X}$.

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ and $\mathcal{G}' = \mathcal{G}(J, \mathcal{Y}, t, B)$ be semi-Cartan graphs. A **morphism** $(\beta, \gamma) : \mathcal{G} \to \mathcal{G}'$ of semi-Cartan graphs is a pair (β, γ), where $\beta : I \to J$, $\gamma : \mathcal{X} \to \mathcal{Y}$ are maps such that for all $i, j \in I$ and $X \in \mathcal{X}$,

$$\gamma(r_i(X)) = t_{\beta(i)}(\gamma(X)), \quad a_{ij}^X = b_{\beta(i)\beta(j)}^{\gamma(X)},$$

that is, the diagrams

$$\begin{tikzcd} I \times \mathcal{X} \arrow[r, "\beta\times\gamma"] \arrow[d, "r"] & J \times \mathcal{Y} \arrow[d, "t"] \\ \mathcal{X} \arrow[r, "\gamma"] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} I \times I \times \mathcal{X} \arrow[rr, "\beta\times\beta\times\gamma"] \arrow[dr, "A"'] & & J \times J \times \mathcal{Y} \arrow[dl, "B"] \\ & \mathbb{Z} & \end{tikzcd}$$

commute.

Semi-Cartan graphs together with their morphisms form a category, where composition of morphisms is defined by composition of maps. Thus a morphisms (β, γ) is an isomorphism if and only if both maps β and γ are bijective.

DEFINITION 9.1.6. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph. If $\mathcal{Y} \subseteq \mathcal{X}$ is a non-empty subset such that $r_i(Y) \in \mathcal{Y}$ for all $i \in I$ and $Y \in \mathcal{Y}$, then the quadruple $\mathcal{G}' = \mathcal{G}(I, \mathcal{Y}, r|(I \times \mathcal{Y}), A|(I \times I \times \mathcal{Y}))$ is called the **semi-Cartan subgraph** of $\mathcal{G}$ with point set $\mathcal{Y}$. Then $(\mathrm{id}, \gamma) : \mathcal{G}' \to \mathcal{G}$ is a morphism, where $\gamma : \mathcal{Y} \to \mathcal{X}$ is the inclusion.

We say that $\mathcal{G}$ is **connected** if there is no proper non-empty subset $\mathcal{Y} \subseteq \mathcal{X}$ such that $r_i(Y) \in \mathcal{Y}$ for all $i \in I$, $Y \in \mathcal{Y}$, that is, if $\mathcal{G}$ is the only semi-Cartan subgraph of $\mathcal{G}$.

For any point $X \in \mathcal{X}$ of a semi-Cartan graph $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$, the semi-Cartan subgraph with point set

$$\{r_{i_1} \cdots r_{i_k}(X) \mid k \geq 0, i_1, \ldots, i_k \in I\}$$

is the only connected semi-Cartan subgraph containing X. It is called the **connected component** of $\mathcal{G}$ containing X. The set $\mathcal{X}$ is the disjoint union of non-empty subsets $\mathcal{X}_l \subseteq \mathcal{X}$, l in some index set L, such that the semi-Cartan subgraphs with sets of points $\mathcal{X}_l$, $l \in L$, are the connected components of $\mathcal{G}$.

The connected components of a semi-Cartan graph are given by the connected components of its exchange graph.

EXAMPLE 9.1.7. Let $\mathcal{G}$ be a semi-Cartan graph with set of labels $I = \{1, 2\}$ of two elements. Then the connected components of the exchange graph of $\mathcal{G}$ are either chains as in Figure 9.1.4 or cycles as in Figure 9.1.5.

DEFINITION 9.1.8. Let $\mathcal{X}$ be a set and M a monoid. We denote by $\mathcal{D}(\mathcal{X}, M)$ the category with objects $\mathrm{Ob}\mathcal{D}(\mathcal{X}, M) = \mathcal{X}$, and morphisms

$$\mathrm{Hom}(X, Y) = \{(Y, f, X) \mid f \in M\} \text{ for all } X, Y \in \mathcal{X},$$

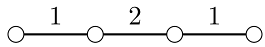


FIGURE 9.1.4. Chain with two labels

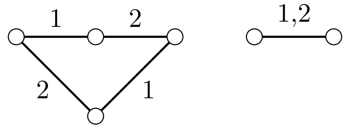


FIGURE 9.1.5. Cycles with two labels

where composition of morphism is defined by

$$(Z, g, Y) \circ (Y, f, X) = (Z, gf, X) \text{ for all } X, Y, Z \in \mathcal{X},\ f, g \in M.$$

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph, and $\mathrm{End}(\mathbb{Z}^I)$ the monoid (with respect to composition) of endomorphisms of the group $\mathbb{Z}^I$. We call the smallest subcategory of $\mathcal{D}(\mathcal{X}, \mathrm{End}(\mathbb{Z}^I))$ which contains all morphisms $(r_i(X), s_i^X, X)$ with $i \in I$, $X \in \mathcal{X}$ the **Weyl groupoid of** $\mathcal{G}$. We write $\mathcal{W}(\mathcal{G})$ for this subcategory. The morphisms $(r_i(X), s_i^X, X)$ are usually abbreviated by s_i^X, or by s_i, if no confusion is likely.

REMARK 9.1.9. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph, $X \in \mathcal{X}$, and $i \in I$. Then $s_i^X = s_i^{r_i(X)} \in \mathrm{Aut}(\mathbb{Z}^I)$ by (CG2), and $(s_i^X)^2 = \mathrm{id}_{\mathbb{Z}^I}$. Therefore the morphisms $s_i^X : X \to r_i(X)$ and $s_i^{r_i(X)} : r_i(X) \to r_i^2(X) = X$ by (CG1) are inverse isomorphisms. Consequently, all morphisms of $\mathcal{W}(\mathcal{G})$ are invertible, and hence $\mathcal{W}(\mathcal{G})$ is a groupoid. Recall that a **groupoid** is a category where each morphism is an isomorphism.

Let $X \in \mathcal{X}$. The set $\mathrm{Hom}(X, X)$ then forms a group, which is called the **automorphism group of** X. It is denoted by $\mathrm{Aut}(X)$. As in any groupoid, if $Y \in \mathcal{X}$, and $w : X \to Y$ is a morphism, then

$$\mathrm{Aut}(X) \cong \mathrm{Aut}(Y), \quad v \mapsto wvw^{-1},$$

is a group isomorphism, and $\mathrm{Hom}(Y, X) = \mathrm{Aut}(X)w^{-1}$.

For any morphism $w = (Y, f, X)$ in $\mathcal{W}(\mathcal{G})$ with $X, Y \in \mathcal{X}$, $f \in \mathrm{Aut}(\mathbb{Z}^I)$ we define

$$\det(w) = \det(f) \quad \text{and} \quad w(\alpha) = f(\alpha) \text{ for all } \alpha \in \mathbb{Z}^I.$$

We call $F(w) = f$ the **linear function of** w. In fact, F can (and should) be viewed as a functor $F : \mathcal{W}(\mathcal{G}) \to \mathbb{Z}^I$.

The morphisms of $\mathcal{W}(\mathcal{G})$ ending in $X \in \mathcal{X}$ are the triples

$$\begin{aligned} w &= (X, s_{i_1}^{r_{i_1}(X)} s_{i_2}^{r_{i_2} r_{i_1}(X)} \cdots s_{i_m}^{r_{i_m} \cdots r_{i_1}(X)}, r_{i_m} \cdots r_{i_1}(X)) \\ &= (r_{i_m} \cdots r_{i_1}(X) \xrightarrow{s_{i_m}^{r_{i_m} \cdots r_{i_1}(X)}} r_{i_{m-1}} \cdots r_{i_1}(X) \to \cdots \to r_{i_1}(X) \xrightarrow{s_{i_1}^{r_{i_1}(X)}} X) \end{aligned}$$

with $m \geq 0$ and $i_1, \ldots, i_m \in I$. We also write $w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}$. Note that

$$\det(w) = (-1)^m, \text{ if } w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}. \tag{9.1.1}$$

The semi-Cartan graph $\mathcal{G}$ is connected if and only if the groupoid $\mathcal{W}(\mathcal{G})$ is connected, that is, if for any two points X, Y of $\mathcal{G}$ there is a morphism from X to Y in $\mathcal{W}(\mathcal{G})$.

DEFINITION 9.1.10. A semi-Cartan graph $\mathcal{G}$ is called **simply connected** if for any two points X, Y of $\mathcal{G}$ there is at most one morphism from X to Y in $\mathcal{W}(\mathcal{G})$.

EXAMPLE 9.1.11. Let $\mathcal{G}$ be the semi-Cartan graph in Example 9.1.2. We compute $\mathcal{W}(\mathcal{G})$. The Weyl groupoid is generated by the morphisms

$$s = s_1^{X_2} : X_2 \to X_1,\ t = s_1^{X_1} : X_1 \to X_2,$$
$$u = s_2^{X_1} : X_1 \to X_1,\ v = s_2^{X_2} : X_2 \to X_2.$$

Then s and t are inverse isomorphisms in $\mathcal{W}(\mathcal{G})$, and u, v are self-inverse. Hence the automorphism group $\mathrm{Aut}(X_1)$ is generated by u and

$$w = svt = (X_1 \xrightarrow{t} X_2 \xrightarrow{v} X_2 \xrightarrow{s} X_1).$$

The matrices of u and w (with respect to α_1, α_2) are

$$A = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$$

and have order two. Since $-(AB)^2 = \mathrm{id}_{\mathbb{Z}^2}$, the matrix AB has order four, and the group generated by A, B is the dihedral group of order eight with generators A and AB. Thus

$$\mathrm{Aut}(X_1) = \{u^k (uw)^l \mid 0 \le k \le 1, 0 \le l \le 3\}$$

is the dihedral group of order eight, $\mathrm{Aut}(X_2) = t\mathrm{Aut}(X_1)s \cong \mathrm{Aut}(X_1)$, and

$$\mathrm{Hom}(X_2, X_1) = \mathrm{Aut}(X_1)s,\ \mathrm{Hom}(X_1, X_2) = t\mathrm{Aut}(X_1).$$

DEFINITION 9.1.12. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph, X, Y points of $\mathcal{G}$, and $w \in \mathrm{Hom}(Y, X)$. We call

$$\ell(w) = \min\{k \mid w = \mathrm{id}_X s_{i_1} \cdots s_{i_k}, k \ge 0, i_1, \dots, i_k \in I\}$$

the **length of** w. If $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$, where $i_1, \dots, i_l \in I$ and $l = \ell(w)$, then $(i_1, \dots, i_l)$ is called a **reduced decomposition of** w.

LEMMA 9.1.13. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph. Then for all $X, Y, Z \in \mathcal{X}$, $w \in \mathrm{Hom}(X, Y)$, $w' \in \mathrm{Hom}(Y, Z)$, $k \ge 0$, and $i_1, \dots, i_k \in I$,*

(1) $|\ell(w) - \ell(w')| \le \ell(w'w) \le \ell(w') + \ell(w)$, $\ell(w^{-1}) = \ell(w)$,
(2) $\ell(w'w) \equiv \ell(w') + \ell(w) \mod 2$,
(3) $\ell(s_i w), \ell(w s_i) \in \{\ell(w) + 1, \ell(w) - 1\}$ *for all* $i \in I$,
(4) $k - \ell(\mathrm{id}_X s_{i_1} \cdots s_{i_k})$ *is a non-negative even integer.*

PROOF. (1) It follows from the definition of the Weyl groupoid that

$$\ell(w^{-1}) = \ell(w) \text{ and } \ell(w'w) \le \ell(w) + \ell(w').$$

Then $\ell(w) \le \ell(w'^{-1}) + \ell(w'w)$ and hence $\ell(w) - \ell(w') \le \ell(w'w)$. Similarly it follows that $\ell(w') - \ell(w) \le \ell(w'w)$.

(2) and (4) follow from (9.1.1), and (3) follows from (1) and (2). □

For any category $\mathcal{D}$ and any object X of $\mathcal{D}$ let

$$\mathrm{Hom}(\mathcal{D}, X) = \bigcup_{Y \in \mathcal{D}} \mathrm{Hom}(Y, X).$$

DEFINITION 9.1.14. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph. For all $X \in \mathcal{X}$, the set

$$\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{w(\alpha_i) \in \mathbb{Z}^I \mid w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X), i \in I\}$$

is called the **set of real roots of $\mathcal{G}$ at** X. The real roots α_i, $i \in I$, are called **simple**. The elements of

$$\boldsymbol{\Delta}_+^{X\,\mathrm{re}} = \boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \mathbb{N}_0^I \text{ and } \boldsymbol{\Delta}_-^{X\,\mathrm{re}} = \boldsymbol{\Delta}^{X\,\mathrm{re}} \cap -\mathbb{N}_0^I$$

are called **positive** and **negative**, respectively.

The semi-Cartan graph $\mathcal{G}$ is called **finite**, if $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is finite for all $X \in \mathcal{X}$.

For any $X \in \mathcal{X}$ and $i, j \in I$ let

$$m_{ij}^X = |\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|.$$

We say that $\mathcal{G}$ is a **Cartan graph** if the following hold.

(CG3) For all $X \in \mathcal{X}$, the set $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ consists of positive and negative roots.

(CG4) Let $X \in \mathcal{X}$, and $i, j \in I$. If $m_{ij}^X < \infty$, then $(r_i r_j)^{m_{ij}^X}(X) = X$.

EXAMPLE 9.1.15. We continue with the notation of Example 9.1.11, and compute the real roots of the semi-Cartan graph in Example 9.1.2. The matrix of s is $C = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$. We know from Example 9.1.11 that the matrices of the linear functions of the morphisms in $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X_1)$ are $\pm\mathrm{id}_{\mathbb{Z}^2}$, $\pm A$, $\pm B$, $\pm AB$, $\pm C$, $\pm AC$, $\pm BC$, $\pm ABC$, since $(AB)^2 = -\mathrm{id}_{\mathbb{Z}^2}$. Hence

$$\begin{aligned} \boldsymbol{\Delta}^{X_1\,\mathrm{re}} &= \{\pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3, \pm 1^2 2^3, \pm 1^3 2^4, \pm 1^3 2^5\}, \\ t(\boldsymbol{\Delta}^{X_1\,\mathrm{re}}) = \boldsymbol{\Delta}^{X_2\,\mathrm{re}} &= \{\pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3, \pm 12^4, \pm 1^2 2^3, \pm 1^2 2^5\}, \end{aligned}$$

where we abbreviate $a\alpha_1 + b\alpha_2$ by $1^a 2^b$ for all $a, b \in \mathbb{N}$. Thus $m_{ij}^X = 8$ for all $X \in \mathcal{X}$, $i, j \in I$, $i \neq j$, and $\mathcal{G}$ is a finite Cartan graph, despite of the fact that in one of its points the Cartan matrix is not of finite type.

Axiom (CG3) does not follow from (CG1) and (CG2), and Axiom (CG4) does not follow from (CG1)–(CG3). This is shown for finite semi-Cartan graphs by Examples 9.2.3 and 9.1.26, respectively, below. Example 9.2.3 also shows that in a semi-Cartan graph Axiom (CG3) can be satisfied in all points of a connected component but in one. In Example 9.1.15 we have seen a finite Cartan graph with a point X such that the Cartan matrix A^X is not of finite type. However, we will show in Theorem 10.2.18 that a finite semi-Cartan graph always has a point X with Cartan matrix A^X of finite type.

Axiom (CG3) is very strong and crucial. In Corollary 9.2.22 we will show that in a Cartan graph for any point X and labels i, j with $m_{ij}^X < \infty$, the Coxeter relation $\mathrm{id}_X(s_i s_j)^{m_{ij}^X} = \mathrm{id}_X$ holds. Since $\mathrm{id}_X(s_i s_j)^{m_{ij}^X}$ is a morphism from $(r_j r_i)^{m_{ij}^X}(X)$ to X, (CG4) is a necessary condition for the Coxeter relations to be satisfied.

We will give an equivalent definition of a Cartan graph in Section 9.2.

REMARK 9.1.16. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph.

(1) Let $X, Y \in \mathcal{X}$ and $w \in \mathrm{Hom}(Y, X)$. Then the map

$$w : \boldsymbol{\Delta}^{Y\,\mathrm{re}} \to \boldsymbol{\Delta}^{X\,\mathrm{re}}, \ \alpha \mapsto w(\alpha),$$

is bijective, and $\boldsymbol{\Delta}^{X\,\mathrm{re}} = -\boldsymbol{\Delta}^{X\,\mathrm{re}}$, since $us_i(\alpha_i) = -u(\alpha_i)$ for all $u \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ and all $i \in I$. Thus a connected semi-Cartan graph is finite if $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is finite for at least one point $X \in \mathcal{X}$.

(2) Let $X \in \mathcal{X}$ and $\alpha \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$. Then the only multiples of α which are real roots at X are $\pm\alpha$. Indeed, if $m\alpha = w(\alpha_i)$, where $m \in \mathbb{Z}$, $i \in I$, and $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$, then $\alpha_i = mw^{-1}(\alpha)$ with $w^{-1}(\alpha) \in \mathbb{Z}^I$, hence $m = \pm 1$.

(3) If $\mathcal{G}$ is finite, then $m_{ij}^X < \infty$ for all $i, j \in I$ and $X \in \mathcal{X}$. If $\mathcal{G}$ is a connected finite Cartan graph, we will show in Corollary 9.3.12 that $\mathcal{G}$ is finite in the strongest sense, that is, $\mathcal{X}$ is finite, and the sets $\mathrm{Hom}(X, Y)$ are finite for all $X, Y \in \mathcal{X}$.

(4) The part of Axiom (CG4) with $i = j$ is redundant by (CG1).

Example 9.1.17. Let $n \in \mathbb{N}$, $I = \{1, \dots, n\}$, $\mathcal{X} = \{X\}$, and let $A = A^X$ be a Cartan matrix. Then $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r = \mathrm{id}_{\mathcal{X}}, A)$ is a semi-Cartan graph. The Weyl groupoid of $\mathcal{G}$ has only one object and is just the Weyl group of the Cartan matrix A in the sense of Kac, see [**Kac90**, Ch. 3]. By the general theory of Kac-Moody algebras, $\mathcal{G}$ is a Cartan graph, and $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is the set of real roots of the Kac-Moody algebra $\mathfrak{g}(A)$. It is known that $\mathcal{G}$ is finite if and only if A is of finite type.

In the rest of the section let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph.

As a first approximation towards Remark 9.1.16(3), already now we can prove the following finiteness result.

Lemma 9.1.18. *Assume that $\mathcal{G}$ is finite. Let X be a point of $\mathcal{G}$, and let*

$$\mathcal{Y} = \{r_{i_1} \cdots r_{i_k}(X) \mid k \geq 0, i_1, \dots, i_k \in I\}.$$

Then $\bigcup_{Y \in \mathcal{Y}} \boldsymbol{\Delta}^{Y\,\mathrm{re}}$ is a finite set, and $\mathrm{Hom}(Y, X)$ *is a finite set for any point Y of $\mathcal{G}$.*

Proof. Let $Y \in \mathcal{X}$. If $Y \notin \mathcal{Y}$, then $\mathrm{Hom}(Y, X)$ is empty by definition. Assume that $Y \in \mathcal{Y}$, and let $w = (X, f, Y) \in \mathrm{Hom}(Y, X)$. Then for all $i \in I$, $f(\alpha_i) \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$. Since $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is finite, and the linear function $F(w) = f$ is uniquely determined by the family $(f(\alpha_i))_{i \in I}$, the set $\mathrm{Hom}(Y, X)$ is finite. By the same reason,

$$\mathcal{F} = \cup_{Y \in \mathcal{Y}} \{F(w) \mid w \in \mathrm{Hom}(Y, X)\}$$

is a finite set. Since $\boldsymbol{\Delta}^{Y\,\mathrm{re}} = f^{-1}(\boldsymbol{\Delta}^{X\,\mathrm{re}})$ for all $f = F(w)$, $w \in \mathrm{Hom}(Y, X)$ and $Y \in \mathcal{Y}$, the finiteness of $\mathcal{F}$ implies that $\bigcup_{Y \in \mathcal{Y}} \boldsymbol{\Delta}^{Y\,\mathrm{re}}$ is finite. $\square$

We collect first consequences of Axiom (CG3) or of weak versions of it. Thus we assume that $\boldsymbol{\Delta}^{X\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ for certain points $X \in \mathcal{X}$.

The observation in the next lemma is very useful.

Lemma 9.1.19. *Let $X \in \mathcal{X}$, $i \in I$, and assume that*

$$\boldsymbol{\Delta}^{X\,\mathrm{re}}, \boldsymbol{\Delta}^{r_i(X)\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I.$$

(1) *The map s_i^X maps $\pm\alpha_i$ to $\mp\alpha_i$, and it induces bijections*

$$s_i^X : \boldsymbol{\Delta}_+^{X\,\mathrm{re}} \setminus \{\alpha_i\} \to \boldsymbol{\Delta}_+^{r_i(X)\,\mathrm{re}} \setminus \{\alpha_i\},$$
$$s_i^X : \boldsymbol{\Delta}_-^{X\,\mathrm{re}} \setminus \{-\alpha_i\} \to \boldsymbol{\Delta}_-^{r_i(X)\,\mathrm{re}} \setminus \{-\alpha_i\}.$$

(2) $m_{ij}^X = m_{ij}^{r_i(X)}$.

Proof. (1) Note that $s_i^X(\alpha) \in \alpha + \mathbb{Z}\alpha_i$ for all $\alpha \in \mathbb{Z}^I$. By Remark 9.1.16(2), $m\alpha_i \notin \boldsymbol{\Delta}^{X\,\mathrm{re}}$ for any $m \in \mathbb{Z} \setminus \{1, -1\}$. Moreover, $\boldsymbol{\Delta}^{r_i(X)\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ by assumption. Hence both maps in the claim are well-defined. Their inverses are induced by $s_i^{r_i(X)}$, and they are well-defined again by Remark 9.1.16(2) and since $\boldsymbol{\Delta}^{X\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$.

(2) follows from (1). □

As for Weyl groups, we associate another natural number $N(w)$ to any morphism w in the Weyl groupoid of $\mathcal{G}$, counting the number of positive real roots made negative by w^{-1}. It is one of the goals of Section 9.3 to prove for Cartan graphs the equality $N(w) = \ell(w)$.

Definition 9.1.20. Let $X, Y \in \mathcal{X}$ and $w \in \mathrm{Hom}(Y, X)$. We define

$$\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \{\alpha \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}} \mid w^{-1}(\alpha) \in -\mathbb{N}_0^I\},$$
$$N(w) = |\boldsymbol{\Delta}^{X\,\mathrm{re}}(w)|.$$

Lemma 9.1.21. *Let $X, Y \in \mathcal{X}$, $i \in I$, and $w \in \mathrm{Hom}(Y, X)$.*

(1) *$N(w) = N(w^{-1})$.*

(2) *Assume that $\boldsymbol{\Delta}^{Y\,\mathrm{re}}, \boldsymbol{\Delta}^{r_i(Y)\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$.*

   (a) *If $w(\alpha_i) \in \mathbb{N}_0^I$, then $N(ws_i) = N(w) + 1$, and*
   $$\boldsymbol{\Delta}^{X\,\mathrm{re}}(ws_i) = \boldsymbol{\Delta}^{X\,\mathrm{re}}(w) \cup \{w(\alpha_i)\}.$$
   (b) *If $w(\alpha_i) \in -\mathbb{N}_0^I$, then $N(ws_i) = N(w) - 1$, and*
   $$\boldsymbol{\Delta}^{X\,\mathrm{re}}(ws_i) = \boldsymbol{\Delta}^{X\,\mathrm{re}}(w) \setminus \{-w(\alpha_i)\}.$$

(3) *If $\boldsymbol{\Delta}^{Z\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ for all $Z \in \mathcal{X}$, then $N(w) \le \ell(w)$.*

Proof. (1) $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) \to \boldsymbol{\Delta}^{Y\,\mathrm{re}}(w^{-1})$, $\alpha \mapsto -w^{-1}(\alpha)$, is bijective.

(2) Note that $\boldsymbol{\Delta}^{X\,\mathrm{re}}(ws_i) = \{\alpha \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}} \mid s_i^Y(w^{-1}(\alpha)) \in -\mathbb{N}_0^I\}$. Hence (a) and (b) follow from Lemma 9.1.19(1).

(3) follows from (2) by induction on $\ell(w)$, since $N(\mathrm{id}_X) = 0$. □

In typical situations, the subsets $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w)$ of $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ have a characteristic property.

Theorem 9.1.22. *Assume that (CG3) holds. Then for any $X \in \mathcal{X}$ and any finite subset R of $\boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ the following are equivalent.*

(1) *There exists $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ such that $R = \boldsymbol{\Delta}^{X\,\mathrm{re}}(w)$.*

(2) *For any $k, l \ge 0$ and any $\beta_1, \dots, \beta_k \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}} \setminus R$ and $\gamma_1, \dots, \gamma_l \in R$, $\sum_{i=1}^k \beta_i - \sum_{j=1}^l \gamma_j \in \mathbb{Z}^I \setminus R$.*

Proof. Assume (1). Let $k, l \ge 0$ and let $\beta_1, \dots, \beta_k \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}} \setminus R, \gamma_1, \dots, \gamma_l \in R$. Then $w^{-1}(\beta_i)$ and $w^{-1}(-\gamma_j)$ are positive for any $1 \le i \le k$ and any $1 \le j \le l$. Hence $w^{-1}(\beta)$ with $\beta = \sum_{i=1}^k \beta_i - \sum_{j=1}^l \gamma_j$ is a sum of positive roots. In particular, $\beta \notin R$, which proves (2).

Assume now (2). We prove (1) by induction on $|R|$. For $R = \emptyset$ the claim holds since $\boldsymbol{\Delta}^{X\,\mathrm{re}}(\mathrm{id}_X) = \emptyset$.

Let $X \in \mathcal{X}$, $R \subseteq \boldsymbol{\Delta}^{X\,\mathrm{re}}$, and $m = |R|$. Assume that $m \ge 1$ and that the claim holds for subsets of real roots (at any point) with $m - 1$ elements. Since

$R \neq \emptyset$ and any element of R is a sum of simple roots, (2) with $l = 0$ and $\beta_1, \dots, \beta_k$ simple implies that there exists $i_0 \in I$ such that $\alpha_{i_0} \in R$. Let $Y = r_{i_0}(X)$ and $R' = s_{i_0}^X(R \setminus \{\alpha_{i_0}\})$. Then $|R'| = m - 1$ and $R' \subseteq \mathbf{\Delta}_+^{Y\,\mathrm{re}}$ by Lemma 9.1.19(1). By assumption,

$$\sum_{i=1}^k \beta_i - \sum_{j=1}^l \gamma_j - n\alpha_{i_0} \in \mathbb{Z}^I \setminus R$$

for any $k, l, n \geq 0$, $\beta_1, \dots, \beta_k \in \mathbf{\Delta}_+^{X\,\mathrm{re}} \setminus R$, and $\gamma_1, \dots, \gamma_l \in R \setminus \{\alpha_{i_0}\}$. Thus

$$\sum_{i=1}^k s_{i_0}^X(\beta_i) + n\alpha_{i_0} - \sum_{j=1}^l s_{i_0}^X(\gamma_j) \in \mathbb{Z}^I \setminus s_{i_0}^X(R)$$

for any $k, l \geq 0$, $\beta_1, \dots, \beta_k \in \mathbf{\Delta}_+^{X\,\mathrm{re}} \setminus R$, $\gamma_1, \dots, \gamma_l \in R \setminus \{\alpha_{i_0}\}$, and $n \geq 0$. By induction hypothesis, there exists $w' \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), Y)$ with $R' = \mathbf{\Delta}^{Y\,\mathrm{re}}(w')$. Hence $R = \mathbf{\Delta}^{X\,\mathrm{re}}(s_{i_0}^Y w')$, and the proof is completed. □

At this place we also add a related general lemma which will lead to strong restrictions on some elements of the Weyl groupoid.

LEMMA 9.1.23. *Let I be a non-empty finite set and $J \subseteq I$. Let $w \in \mathrm{Aut}(\mathbb{Z}^I)$ such that $w(\alpha_j) \in -\sum_{k \in J} \mathbb{N}_0 \alpha_k$ for any $j \in J$. Assume that $w^{-1}(\alpha_j) \in \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ for any $j \in J$. Then there is permutation σ of J such that $w(\alpha_j) = -\alpha_{\sigma(j)}$ for all $j \in J$.*

PROOF. Since $w \in \mathrm{Aut}(\mathbb{Z}^I)$ and since $w(\alpha_j) \in \sum_{k \in J} \mathbb{Z}\alpha_k$, the elements $w^{-1}(\alpha_j)$ with $j \in J$ form a basis of $\sum_{k \in J} \mathbb{Z}\alpha_k$. Let $j \in J$. Then, by assumption, there exists $(\lambda_k)_{k \in J} \in \mathbb{Z}^J$ such that $w^{-1}(\alpha_j) = \sum_{k \in J} \lambda_k \alpha_k$ and either $\lambda_k \geq 0$ for all $k \in J$ or $\lambda_k \leq 0$ for all $k \in J$. Hence

$$\alpha_j = ww^{-1}(\alpha_j) = \sum_{k \in J} \lambda_k w(\alpha_k).$$

Since $w(\alpha_k) \in -\mathbb{N}_0^I$ for all $k \in J$, it follows that $\lambda_k = -1$, $w(\alpha_k) = -\alpha_j$ for some $k \in J$, and that $\lambda_l = 0$ for all $l \in J \setminus \{k\}$. This implies the claim. □

At the end of this section we discuss some more examples.

EXAMPLE 9.1.24. We can easily define two semi-Cartan graphs of rank one. The first is $\mathcal{G}(\{1\}, \{X\}, r, (2))$ with $r_1(X) = X$, and the second semi-Cartan graph is $\mathcal{G}(\{1\}, \{X, Y\}, r, (A^X = A^Y = (2)))$ with $r_1(X) = Y$, $r_1(Y) = X$. Conversely, any connected semi-Cartan graph of rank one is isomorphic to one of these semi-Cartan graphs. The situation is more complicated for connected semi-Cartan graphs of higher rank.

Now we give some non-trivial examples of different type. Some of them are Cartan graphs, others are not. In Section 10.3 we will give a classification of all finite connected simply connected Cartan graphs of rank two up to isomorphism. This classification provides us with yet another class of non-trivial examples.

First we slightly generalize Example 9.1.17.

EXAMPLE 9.1.25. Let A be a Cartan matrix and let $\mathcal{G}$ be a standard Cartan graph, such that the Cartan matrix of any point of $\mathcal{G}$ is A. Then by Proposition 9.3.15, the set of real roots $\mathbf{\Delta}^{X\,\mathrm{re}}$ at any point X coincides with the set of real

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

FIGURE 9.1.6. The semi-Cartan graph in Example 9.1.26

roots of the Kac-Moody algebra $\mathfrak{g}(A)$. Hence $\mathcal{G}$ is finite if and only if A is of finite type.

It is easy to describe all connected standard Cartan graphs. Indeed, let A be a Cartan matrix over a finite index set I, let W be the corresponding Weyl group, and let U be a subgroup of W. Let $\mathcal{X} = W/U = \{wU \,|\, w \in W\}$ and let $r_i : \mathcal{X} \to \mathcal{X}$, $wU \mapsto s_i wU$, for all $i \in I$. Let $A^X = A$ for all $X \in \mathcal{X}$. Then (CG4) holds for $\mathcal{G}_U = \mathcal{G}(I, \mathcal{X}, r, A)$ since $(s_i s_j)^{m_{ij}} = \mathrm{id}$ for all $i, j \in I$, where $m_{ij} = |\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$. Hence $\mathcal{G}_U$ is a standard Cartan graph. Since W is generated by simple reflections, $\mathcal{G}_U$ is connected.

On the other hand, let $\mathcal{G}$ be a standard Cartan graph with Cartan matrix $A = (a_{ij})_{i,j\in I}$, then for any point X of $\mathcal{G}$ the group $\mathrm{Hom}(X, X)$ naturally identifies with a subgroup U of the Weyl group W of A. (The assumption (CG4) ensures that $\mathrm{id}_X(s_i s_j)^{m_{ij}^X} \in \mathrm{Hom}(X, X)$ for all points X and all $i, j \in I$.) Now it is easy to see that if $\mathcal{G}$ is connected, then $\mathcal{G}$ is isomorphic to the standard Cartan graph described in the previous paragraph.

Let U, U' be subgroups of W. Assume that $\mathcal{G}_U$ and $\mathcal{G}_{U'}$ are isomorphic. Then there exists a permutation β of I such that $a_{ij} = a_{\beta(i)\beta(j)}$ for all $i, j \in I$. Moreover, there is a bijection $\gamma : W/U \to W/U'$ of left cosets such that

$$\gamma(s_{i_1} \cdots s_{i_k} U) = s_{\beta(i_1)} \cdots s_{\beta(i_k)} \gamma(U) \tag{9.1.2}$$

for all $k \geq 0$, $i_1, \dots, i_k \in I$. Since W is a Coxeter group and the Coxeter relations are obtained from the entries of A, the permutation β induces a group isomorphism $\beta^* : W \to W$ such that $\beta^*(s_i) = s_{\beta(i)}$ for all $i \in I$. Therefore, by (9.1.2) there exists $w' \in W$ such that $\beta^*(w)w'U' = w'U'$ for all $w \in U$, that is, U' is conjugate to $\beta^*(U)$ in W.

Conversely, assume that there is a permutation β of I such that $a_{ij} = a_{\beta(i)\beta(j)}$ for all $i, j \in I$, and that $U' = w'^{-1}\beta^*(U)w'$ for some $w' \in W$. Then the map $\gamma : W/U \to W/U'$, $wU \mapsto \beta^*(w)w'U'$, is a well-defined bijection and fulfills (9.1.2). Hence $\mathcal{G}_U$ and $\mathcal{G}_{U'}$ are isomorphic via (β, γ).

EXAMPLE 9.1.26. Let $I = \{1, 2\}$, $\mathcal{X} = \{X_1, X_2, X_3\}$, and $r_1, r_2 : \mathcal{X} \to \mathcal{X}$ the permutations $r_1(X_i) = X_{\sigma(i)}$, $r_2(X_i) = X_{\pi(i)}$, where $\sigma, \pi \in \mathbb{S}_3$, $\sigma = (1\,2)$, $\pi = (2\,3)$. For all $1 \leq i \leq 3$ let $A^{X_i} = A \in \mathbb{Z}^{2\times 2}$ with $a_{11} = a_{22} = 2$, $a_{12} = a_{21} = 0$. Then $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is a standard semi-Cartan graph. We display it in Figure 9.1.6. Moreover, $s_i^X(\alpha_j) = \alpha_j$ for all $X \in \mathcal{X}$, $i, j \in I$ with $i \neq j$. This implies that $\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{\pm\alpha_1, \pm\alpha_2\}$ and $m_{ij}^X = 2$ for all $X \in \mathcal{X}$ and $i \neq j$. In particular, $\mathcal{G}$ is finite, but it does not satisfy (CG4), since $r_1 r_2$ is defined by the permutation $(1\,2\,3)$ of order 3. Thus $\mathcal{G}$ is not a Cartan graph.

An example of a finite semi-Cartan graph not satisfying (CG3) will be given in Example 9.2.3.

Now we give two non-standard examples of rank three with two points.

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \overset{3}{\text{———}} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \overset{3}{\text{———}} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

FIGURE 9.1.7. The semi-Cartan graph in Example 9.1.29

EXAMPLE 9.1.27. Let $I = \{1,2,3\}$, $\mathcal{X} = \{X, Y\}$, and r_1 the non-trivial permutation of $\mathcal{X}$, and $r_2 = r_3 = \mathrm{id}_{\mathcal{X}}$. Let

$$A^X = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad A^Y = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Then $\mathcal{G}(I, \mathcal{X}, r, A)$ is a finite Cartan graph. Indeed, one checks that

$$\begin{aligned} \boldsymbol{\Delta}^{X\,\mathrm{re}} = \{ & \pm 1, \pm 1^3 2^4 3^2, \pm 1^2 2^3 3, \pm 1^2 2^3 3^2, \pm 1^2 2^2 3, \\ & \pm 12, \pm 12^2, \pm 12^2 3, \pm 12^2 3^2, \pm 123, \pm 2, \pm 23, \pm 3\}, \\ \boldsymbol{\Delta}^{Y\,\mathrm{re}} = \{ & \pm 1, \pm 12, \pm 12^2, \pm 12^4 3^2, \pm 12^3 3, \pm 12^3 3^2, \\ & \pm 12^2 3, \pm 12^2 3^2, \pm 123, \pm 2, \pm 2^2 3, \pm 23, \pm 3\}, \end{aligned}$$

where $a\alpha_1 + b\alpha_2 + c\alpha_3 \in \mathbb{N}_0^3$ is abbreviated by $1^a 2^b 3^c$. Moreover, $m_{12}^X = 4$ and $m_{13}^X = 2$. Note that the third rows of the two Cartan matrices A^X and A^Y coincide, which has to be the case according to Lemma 9.3.3 with $i = 1$, $j = 3$. Further, the Cartan matrix A^Y is not of finite type.

EXAMPLE 9.1.28. Let $I = \{1,2,3\}$, $\mathcal{X} = \{X, Y\}$, and r_1 the non-trivial permutation of $\mathcal{X}$, and $r_2 = r_3 = \mathrm{id}_{\mathcal{X}}$. Let

$$A^X = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad A^Y = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Then $\mathcal{G}(I, \mathcal{X}, r, A)$ is a finite Cartan graph. Indeed, one checks that

$$\begin{aligned} \boldsymbol{\Delta}^{X\,\mathrm{re}} = \{ & \pm 1, \pm 1^4 2^3 3, \pm 1^4 2^3 3^2, \pm 1^4 2^2 3, \pm 1^3 2^2 3, \\ & \pm 1^2 2, \pm 1^2 2^2 3, \pm 1^2 23, \pm 12, \pm 123, \pm 2, \pm 23, \pm 3\}, \\ \boldsymbol{\Delta}^{Y\,\mathrm{re}} = \{ & \pm 1, \pm 1^2 2, \pm 1^2 2^3 3, \pm 1^2 2^3 3^2, \pm 1^2 2^2 3, \\ & \pm 1^2 23, \pm 12, \pm 12^2 3, \pm 123, \pm 2, \pm 2^2 3, \pm 23, \pm 3\}, \end{aligned}$$

where $a\alpha_1 + b\alpha_2 + c\alpha_3 \in \mathbb{N}_0^3$ is abbreviated by $1^a 2^b 3^c$.

We now give an example of a semi-Cartan graph of rank three, which shows that the claims of Lemma 9.3.1 and Proposition 9.4.18 below do not hold for all semi-Cartan graphs.

EXAMPLE 9.1.29. Let $I = \{1,2,3\}$ and let $\mathcal{G}$ be the semi-Cartan graph in Figure 9.1.7. Then $\alpha_1 + \alpha_2 = s_3 s_2 s_3 s_2(\alpha_1)$ is a real root at the last point X. In particular, $a_{12}^X = a_{21}^X = 0$, but $m_{12}^X \neq 2$, and hence Lemma 9.3.1 does not hold for all semi-Cartan graphs. Similarly, $\mathrm{id}_X s_{i_1} \cdots s_{i_k}(\alpha_j) \in \{\pm\alpha_1, \pm\alpha_2\}$ for any $k \geq 0$ and $i_1, \ldots, i_k, j \in \{1, 2\}$. Hence Proposition 9.4.18 does not hold if we omit the assumption on Axiom (CG3).

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

FIGURE 9.2.1. The semi-Cartan graph in Example 9.2.3

9.2. Reduced sequences and positivity of roots

The aim of this section is to prove with Corollary 9.2.20 that Axioms (CG3) and (CG4) for a semi-Cartan graph are equivalent to two other axioms (CG3'), (CG4') based on reduced sequences, see below. This characterization of a Cartan graph will in particular be essential in Section 14.5. With Proposition 9.2.25 we give a characterization of the finiteness of some semi-Cartan graphs.

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph.

DEFINITION 9.2.1. Let $X \in \mathcal{X}$, $l \geq 0$, and $\kappa = (i_1, \dots, i_l) \in I^l$.

(1) For all $1 \leq k \leq l$ let

$$\beta_k^{X,\kappa} = \mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}),$$

and let

$$\Lambda^X(\kappa) = \{\beta_k^{X,\kappa} \mid 1 \leq k \leq l\}.$$

(2) We say that κ is **X-reduced** if for any $1 \leq k < l$,

$$\alpha_{i_k} \notin \Lambda^{r_{i_k} \cdots r_{i_1}(X)}(i_{k+1}, \dots, i_l).$$

The integer l is called the **length of** κ.

The definition immediately implies the following lemma.

LEMMA 9.2.2. *Let $X \in \mathcal{X}$, $l \geq 1$, and $i_1, \dots, i_l \in I$.*

(1) $\Lambda^X(i_1, \dots, i_l) = \{\alpha_{i_1}\} \cup s_{i_1}^{r_{i_1}(X)}(\Lambda^{r_{i_1}(X)}(i_2, \dots, i_l))$.
(2) *The following are equivalent.*
   (a) $(i_1, \dots, i_l)$ *is X-reduced.*
   (b) $(i_2, \dots, i_l)$ *is $r_{i_1}(X)$-reduced and* $\alpha_{i_1} \notin \Lambda^{r_{i_1}(X)}(i_2, \dots, i_l)$.

In order to point out some pitfalls, let us discuss first an example of a finite semi-Cartan graph which violates several positivity and length constraints.

EXAMPLE 9.2.3. Let $I = \{1, 2\}$, $\mathcal{X} = \{X_1, X_2, X_3\}$, and $r_1, r_2 : \mathcal{X} \to \mathcal{X}$ the permutations $r_1(X_i) = X_{\sigma(i)}$, $r_2(X_i) = X_{\pi(i)}$, where $\sigma, \pi \in \mathbb{S}_3$, $\sigma = (1\,2)$, $\pi = (2\,3)$. Let

$$A^{X_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad A^{X_2} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \quad A^{X_3} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Then $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is the semi-Cartan graph in Figure 9.2.1.

Direct calculation shows that the real roots of $\mathcal{G}$ are

$$\begin{aligned}\mathbf{\Delta}^{X_1\,\mathrm{re}} &= \{\pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3, \pm 1^2 2^3,\\ &\quad \pm 1^3 2^4, \pm 1^3 2^5, \pm 1^4 2^5, \pm 1^4 2^7, \pm 1^5 2^7, \pm 1^5 2^8\},\\ \mathbf{\Delta}^{X_2\,\mathrm{re}} &= \{\pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3, \pm 12^4,\\ &\quad \pm 12^5, \pm 1^2 2^3, \pm 1^2 2^5, \pm 1^2 2^7, \pm 1^3 2^7, \pm 1^3 2^8\},\\ \mathbf{\Delta}^{X_3\,\mathrm{re}} &= \{\pm 12^{-1}, \pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3,\\ &\quad \pm 12^4, \pm 1^2 2, \pm 1^2 2^3, \pm 1^2 2^5, \pm 1^3 2^4, \pm 1^3 2^5\},\end{aligned}$$

where we abbreviate $a\alpha_1 + b\alpha_2$ with $a, b \in \mathbb{N}_0$ by $1^a 2^b$, and $\alpha_1 - \alpha_2$ by 12^{-1}. In particular, $\mathcal{G}$ is finite but it does not satisfy (CG3) since $\alpha_1 - \alpha_2 \in \mathbf{\Delta}^{X_3\,\mathrm{re}}$. Thus $\mathcal{G}$ is not a Cartan graph.

It is instructive to look at X_i-reduced sequences for $1 \le i \le 3$. By direct calculations one obtains that

$$\begin{aligned}\Lambda^{X_1}(1,2,1,2,1,2,1,2,1) &= \{1, 12, 1^3 2^4, 1^2 2^3, 1^5 2^8, 1^3 2^5, 1^4 2^7, 12^2, 2\}\\ &\ne \mathbf{\Delta}_+^{X_1\,\mathrm{re}},\end{aligned}$$

but

$$\sigma(\alpha_1) = -\alpha_2, \quad \sigma(\alpha_2) = -\alpha_1 - \alpha_2$$

for $\sigma = \mathrm{id}_{X_1} s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1$. Hence $(1,2,1,2,1,2,1,2,1,i)$ is not X_1-reduced for any $i \in \{1,2\}$ by Lemma 9.2.5 below. Further, one can show that there is no X_1-reduced sequence of length ten. On the other hand,

$$\Lambda^{X_2}(1,2,1,2,1,2,1,2,1,2) = \{1, 12, 1^2 2^3, 12^2, 1^2 2^5, 12^3, 1^2 2^7, 12^4, 12^5, 2\}$$

and hence there is an X_2-reduced sequence of length ten. One can also check that the longest X_2-reduced sequence starting with 2 has length eight, and that

$$\Lambda^{X_3}(1,2,1,2,1,2,1,2,1) = \{1, 12, 1^3 2^4, 1^2 2^3, 1^3 2^5, 12^2, 12^3, 2, 1^{-1} 2\}.$$

Note that in the latter set there is a root which is neither positive nor negative, but the sequence is X_3-reduced.

For Cartan graphs we will prove in Theorem 9.3.5 that a sequence $(i_1, \dots, i_l)$ is X-reduced if and only if it is a reduced decomposition of $\mathrm{id}_X s_{i_1} \cdots s_{i_l}$ in the Weyl groupoid of the Cartan graph.

The definition of $\Lambda^X(\kappa)$ is compatible with reversing κ.

LEMMA 9.2.4. *Let $X \in \mathcal{X}$, $l \ge 1$, $\kappa = (i_1, \dots, i_l) \in I^l$, and $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$, $Y = r_{i_l} \cdots r_{i_1}(X)$, $\kappa' = (i_l, \dots, i_1)$. Then*

$$\beta_k^{Y,\kappa'} = -w^{-1}\big(\beta_{l+1-k}^{X,\kappa}\big)$$

for any $1 \le k \le l$.

PROOF. Let $1 \le k \le l$. Then

$$\begin{aligned}w\big(\beta_k^{Y,\kappa'}\big) &= \mathrm{id}_X s_{i_1} \cdots s_{i_l} s_{i_l} \cdots s_{i_{l+2-k}}(\alpha_{i_{l+1-k}})\\ &= \mathrm{id}_X s_{i_1} \cdots s_{i_{l+1-k}}(\alpha_{i_{l+1-k}}) = -\beta_{l+1-k}^{X,\kappa}.\end{aligned}$$

This proves the lemma. □

We continue with equivalent conditions for X-reducedness.

LEMMA 9.2.5. *Let $X \in \mathcal{X}$, $l \geq 0$, and $\kappa = (i_1, \dots, i_l) \in I^l$. Then the following are equivalent.*

(1) *κ is X-reduced.*
(2) *$\beta_p^{X,\kappa} \neq -\beta_q^{X,\kappa}$ for all $1 \leq p < q \leq l$.*
(3) *$(i_l, \dots, i_1)$ is $r_{i_l} \cdots r_{i_1}(X)$-reduced.*

In particular, if $\Lambda^X(\kappa) \subseteq \mathbb{N}_0^I$ then κ is X-reduced.

PROOF. Let $1 \leq p < q \leq l$. By definition,

$$\beta_p^{X,\kappa} = \mathrm{id}_X s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) = -\mathrm{id}_X s_{i_1} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_p}),$$
$$\beta_q^{X,\kappa} = \mathrm{id}_X s_{i_1} \cdots s_{i_p} s_{i_{p+1}} \cdots s_{i_{q-1}}(\alpha_{i_q}).$$

Hence $\beta_p^{X,\kappa} = -\beta_q^{X,\kappa}$ if and only if $\alpha_{i_p} = \mathrm{id}_{r_{i_p} \cdots r_{i_1}(X)} s_{i_{p+1}} \cdots s_{i_{q-1}}(\alpha_{i_q})$. This proves the equivalence of (1) and (2).

Let $Y = r_{i_l} \cdots r_{i_1}(X)$ and $\kappa' = (i_l, \dots, i_1)$. By the previous paragraph, (3) holds if and only if $\beta_p^{Y,\kappa'} \neq -\beta_q^{Y,\kappa'}$ for all $1 \leq p < q \leq l$. This is equivalent to (2) because of Lemma 9.2.4. □

REMARK 9.2.6. (1) Let $X \in \mathcal{X}$, $l \geq 2$, and $\kappa = (i_1, \dots, i_l) \in I^l$. If $i_j = i_{j+1}$ for some $1 \leq j < l$, then $\alpha_{i_j} \in \Lambda^{r_{i_j} \cdots r_{i_1}(X)}(i_{j+1}, \dots, i_l)$, and hence κ is not X-reduced.

(2) Let $X \in \mathcal{X}$, $l \geq 2$, $1 \leq j < l$, and $\kappa = (i_1, \dots, i_l) \in I^l$. Then $\beta_j^{X,\kappa} = \beta_{j+1}^{X,\kappa}$ if and only if $s_{i_j}^Y(\alpha_{i_j}) = \alpha_{i_{j+1}}$, where $Y = r_{i_{j-1}} \cdots r_{i_1}(X)$. This is impossible since $s_{i_j}^Y(\alpha_{i_j}) = -\alpha_{i_j} \notin \mathbb{N}_0^I$. Similarly, $\beta_j^{X,\kappa} = -\beta_{j+1}^{X,\kappa}$ if and only if $i_j = i_{j+1}$.

Further equivalences to X-reducedness hold under additional conditions.

LEMMA 9.2.7. *Let $X \in \mathcal{X}$, $l \geq 0$, and $\kappa = (i_1, \dots, i_l) \in I^l$.*

(1) *Assume that $\boldsymbol{\Delta}^{Y\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ for any $Y = r_{i_k} \cdots r_{i_1}(X)$ with $0 \leq k < l$. Then the following are equivalent.*
   (a) *κ is X-reduced.*
   (b) *$\beta_1^{X,\kappa}, \dots, \beta_l^{X,\kappa}$ are pairwise distinct elements in $\mathbb{N}_0^I$.*
   (c) *$\Lambda^X(\kappa) \subseteq \mathbb{N}_0^I$.*
(2) *Let $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$. Assume $\boldsymbol{\Delta}^{Y\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ for any $Y = r_{i_k} \cdots r_{i_1}(X)$ with $0 \leq k \leq l$. If κ is X-reduced then*

$$\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \Lambda^X(\kappa).$$

PROOF. (1) Assume (a). We prove (b) by induction on l. If $l \leq 1$, (b) is trivial. Assume that $l > 1$. Let $Z = r_{i_1}(X)$. Then $\kappa' = (i_2, \dots, i_l)$ is Z-reduced by Lemma 9.2.2. By induction hypothesis, $\beta_1^{Z,\kappa'}, \dots, \beta_{l-1}^{Z,\kappa'}$ are pairwise dictinct elements in $\mathbb{N}_0^I$, and by assumption (1)(a), they are contained in $\boldsymbol{\Delta}_+^{Z\,\mathrm{re}} \setminus \{\alpha_{i_1}\}$. Hence by Lemma 9.1.19(1),

$$\beta_1^{X,\kappa} = \alpha_{i_1},\ \beta_2^{X,\kappa} = s_{i_1}^Z\big(\beta_1^{Z,\kappa'}\big),\ \dots\ ,\beta_l^{X,\kappa} = s_{i_1}^Z\big(\beta_{l-1}^{Z,\kappa'}\big)$$

are pairwise distinct elements in $\mathbb{N}_0^I$.

Finally (b) implies (c), and (c) implies (a) because of Lemma 9.2.5.

(2) Again we proceed by induction on l. If $l = 0$, then $w = \mathrm{id}_X, \boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \emptyset$. Assume that $l > 0$ and that κ is X-reduced. Then $(i_1, \dots, i_{l-1})$ is X-reduced.

Let $w' = \mathrm{id}_X s_{i_1} \cdots s_{i_{l-1}}$. Then $w = w' s_{i_l}$, $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w') = \{\beta_1^{X,\kappa}, \dots, \beta_{l-1}^{X,\kappa}\}$, and $w'(\alpha_{i_l}) = \beta_l^{X,\kappa} \in \mathbb{N}_0^I$ by (1). Hence

$$\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \boldsymbol{\Delta}^{X\,\mathrm{re}}(w') \cup \{\beta_l^{X,\kappa}\} = \Lambda^X(\kappa)$$

by Lemma 9.1.21(2). □

LEMMA 9.2.8. *Let $X \in \mathcal{X}$, $l \geq 0$, and $\kappa = (i_1, \dots, i_l) \in I^l$. Assume that κ is X-reduced, $|\boldsymbol{\Delta}_+^{X\,\mathrm{re}}| > l$, and that $\boldsymbol{\Delta}^{r_{i_k} \cdots r_{i_1}(X)\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ for any $0 \leq k \leq l$. Then there exists $i \in I$ such that $(i, i_1, \dots, i_l)$ is $r_i(X)$-reduced.*

PROOF. Let $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$. Then $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) \neq \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ by Lemma 9.2.7(2), since $|\boldsymbol{\Delta}_+^{X\,\mathrm{re}}| > l$. Thus $\alpha_i \notin \boldsymbol{\Delta}^{X\,\mathrm{re}}(w)$ for some $i \in I$. Hence the claim follows from Lemma 9.2.2(2) and Lemma 9.2.7(2). □

REMARK 9.2.9. Let $i, j \in I$ with $i \neq j$ and $\kappa = (i_k)_{k \geq 1} = (i, j, i, \dots)$ with $i_k = i$ if k is odd and $i_k = j$ if k is even. Let $X \in \mathcal{X}$. By Remark 9.2.6(1), any X-reduced sequence with entries in $\{i, j\}$ and starting with i is a beginning of κ. Let κ_{ij}^X be the longest X-reduced beginning of κ, if it exists, and κ otherwise. We write $\overline{m}_{ij}^X$ for the length of κ_{ij}^X. Clearly, $\overline{m}_{ij}^X \geq 2$.

For any $X \in \mathcal{X}$ and any $i, j \in I$ with $i \neq j$ let

$$\tau(X, i, j) = (r_i(X), j, i), \quad \sigma(X, i, j) = (X, j, i). \tag{9.2.1}$$

Clearly, $\sigma^2(X, i, j) = (X, i, j)$ for any $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. Moreover, τ is invertible with inverse

$$\tau^{-1}(X, i, j) = (r_j(X), j, i) = \sigma\tau\sigma(X, i, j) \tag{9.2.2}$$

for any $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. Hence the permutation group of the set of triples (X, i, j) with $X \in \mathcal{X}$ and $i, j \in I$, $i \neq j$, generated by τ and σ, is generated by τ and σ as a monoid. In Proposition 9.2.14 we prove that $\overline{m}_{ij}^X$ is constant on the orbits of this group in an important special case.

With the help of the notation in Remark 9.2.9 we are in the position to introduce axioms characterizing Cartan graphs.

(CG3') For any $X \in \mathcal{X}$ and any X-reduced sequence κ, $\Lambda^X(\kappa) \subseteq \mathbb{N}_0^I$.
(CG4') For any $X \in \mathcal{X}$ and any $i, j \in I$ with $i \neq j$ and $\overline{m}_{ij}^X < \infty$, we have

$$(r_j r_i)^m(X) = X, \qquad \mathrm{id}_X (s_i s_j)^m(\alpha_k) = \alpha_k$$

for all $k \in I \setminus \{i, j\}$, where $m = \overline{m}_{ij}^X$.

For the proof of Theorem 9.2.18 and Corollary 9.2.20 below, which relate these axioms to those of a Cartan graph, we need some preparation.

LEMMA 9.2.10. *Let $X \in \mathcal{X}$, $l \geq 2$, $\kappa = (i_1, \dots, i_l) \in I^l$, and $1 \leq m < n \leq l$. Assume that $i_m = i_n$ and that $i_k \neq i_m$ for any $m < k < n$. Then there exist $a_{m+1}, \dots, a_{n-1} \in \mathbb{N}_0$ such that*

$$\beta_m^{X,\kappa} + \beta_n^{X,\kappa} = \sum_{k=m+1}^{n-1} a_k \beta_k^{X,\kappa}. \tag{9.2.3}$$

PROOF. Induction by $n-m$. If $n=m+1$ then the claim holds by Remark 9.2.6(1). If $n>m+1$, then

$$\begin{aligned}\beta_m^{X,\kappa}+\beta_n^{X,\kappa}&=\beta_m^{X,\kappa}+\mathrm{id}_X s_{i_1}\cdots s_{i_{n-1}}(\alpha_{i_n})\\&=\beta_m^{X,\kappa}+\mathrm{id}_X s_{i_1}\cdots s_{i_{n-2}}(\alpha_{i_n}+a_{n-1}\alpha_{i_{n-1}})\\&=\beta_m^{X,\kappa'}+\beta_{n-1}^{X,\kappa'}+a_{n-1}\beta_{n-1}^{X,\kappa},\end{aligned}$$

where $\kappa'=(i_1,\dots,i_{n-2},i_n)$ and $a_{n-1}\in\mathbb{N}_0$. Thus the claim follows from induction hypothesis, since $\beta_k^{X,\kappa'}=\beta_k^{X,\kappa}$ for any $m\le k<n-1$. $\square$

LEMMA 9.2.11. *Let $X\in\mathcal{X}$, $l\ge 3$, and $\kappa=(i_1,\dots,i_l)\in I^l$. Assume that $i_{l-2}=i_l$ and that $\beta_{l-1}^{X,\kappa}=\alpha_j$ for some $j\in I$. Then $\beta_{l-2}^{X,\kappa}\notin\mathbb{N}_0^I$ or $\beta_l^{X,\kappa}\notin\mathbb{N}_0^I$.*

PROOF. If $i_{l-1}=i_l$ then $\beta_l^{X,\kappa}=-\beta_{l-1}^{X,\kappa}=-\alpha_j$ and the claim is proven. Assume that $i_{l-1}\ne i_l$. Then, by Lemma 9.2.10, there exists $a\in\mathbb{N}_0$ such that $\beta_{l-2}^{X,\kappa}+\beta_l^{X,\kappa}=a\alpha_j$. Since $\beta_{l-1}^{X,\kappa}=\alpha_j$, Remarks 9.1.16(2) and 9.2.6(2) imply that $\beta_l^{X,\kappa}\notin\mathbb{N}_0\alpha_j$. This implies the claim. $\square$

LEMMA 9.2.12. *Assume that there exist a point $X\in\mathcal{X}$ and an X-reduced sequence κ such that $\Lambda^X(\kappa)$ contains an element in $-\mathbb{N}_0^I$. Then there exist a point $Y\in\mathcal{X}$ and a Y-reduced sequence κ' such that $\Lambda^Y(\kappa')$ contains an element in $\mathbb{Z}^I\setminus(\mathbb{N}_0^I\cup-\mathbb{N}_0^I)$.*

PROOF. Let $X\in\mathcal{X}$ and $\kappa=(i_1,\dots,i_l)\in I^l$ be an X-reduced sequence with $l\ge 1$. Assume that $\Lambda^Z(\kappa'')$ contains no elements in $-\mathbb{N}_0^I$ for any point Z and any Z-reduced sequence κ'' of length $<l$, and that $\beta_l^{X,\kappa}\in-\mathbb{N}_0^I$. Then $l>1$. Moreover, $\beta_l^{X,\kappa}\ne-\alpha_{i_1}$ since κ is X-reduced. Thus $\beta_l^{X,\kappa}\notin\mathbb{Z}\alpha_{i_1}$. Since $(i_2,\dots,i_l)$ is $r_{i_1}(X)$-reduced by Lemma 9.2.2(2), we conclude from our assumption on reduced sequences of length $<l$ that

$$\Lambda^{r_{i_1}(X)}(i_2,\dots,i_l)\ni s_{i_1}^X(\beta_l^{X,\kappa})=\beta_l^{X,\kappa}+b\alpha_{i_1},$$

where $b\in\mathbb{Z}$, is contained in $\mathbb{Z}^I\setminus(\mathbb{N}_0^I\cup-\mathbb{N}_0^I)$. $\square$

LEMMA 9.2.13. *Assume that $|I|\ge 2$. Let $Y\in\mathcal{X}$, $i,j\in I$ with $i\ne j$, and $\kappa=(j,i,j,i,\dots)=(j_1,\dots,j_m)$ be a Y-reduced beginning of κ_{ji}^Y with $m\ge 2$. If $\overline{m}_{ij}^{r_i(Y)}=m$, then $\beta_m^{Y,\kappa}=\alpha_i$.*

PROOF. Let $\kappa'=(i,j_1,\dots,j_m)$. Since $\overline{m}_{ij}^{r_i(Y)}\le m$, κ' is not $r_i(Y)$-reduced. Hence $\beta_p^{r_i(Y),\kappa'}=-\beta_q^{r_i(Y),\kappa'}$ for some $1\le p<q\le m+1$ by Lemma 9.2.5. Since $\beta_{k+1}^{r_i(Y),\kappa'}=s_i^Y(\beta_k^{Y,\kappa})$ for any $1\le k\le m$ and since κ is Y-reduced, we conclude that $p=1$. Then $q=m+1$, since $\overline{m}_{ij}^{r_i(Y)}=m$. Therefore $\alpha_i=-s_i^Y(\beta_m^{Y,\kappa})$, which implies that $\beta_m^{Y,\kappa}=\alpha_i$. $\square$

PROPOSITION 9.2.14. *Assume that $|I|\ge 2$ and that $\mathcal{G}$ satisfies (CG3'). Let $X\in\mathcal{X}$, $i,j\in I$ with $i\ne j$, and*

$$\mathcal{Y}=\{r_{i_1}\cdots r_{i_k}(X)\mid k\ge 0,i_1,\dots,i_k\in\{i,j\}\}.$$

Then $\overline{m}_{ij}^Y=\overline{m}_{ji}^Y=\overline{m}_{ij}^X$ for any $Y\in\mathcal{Y}$.

PROOF. If $\overline{m}_{ij}^Y = \overline{m}_{ji}^Y = \infty$ for any $Y \in \mathcal{Y}$ then we are done.

Assume that $\overline{m}_{ij}^X < \infty$ and that $\overline{m}_{ij}^Y, \overline{m}_{ji}^Y \geq \overline{m}_{ij}^X$ for any $Y \in \mathcal{Y}$. We prove that $\overline{m}_{i'j'}^Y = \overline{m}_{ij}^X$ for $(Y, i', j') = \tau(X, i, j)$ and for $(Y, i', j') = \sigma(X, i, j)$, where τ and σ are as in (9.2.1). This implies then the Proposition by Remark 9.2.9.

Let $m = \overline{m}_{ij}^X$, $Y = r_i(X)$, and $\kappa = (j, i, j, i, \dots) = (j_1, \dots, j_m)$. Then κ is Y-reduced since $\overline{m}_{ji}^Y \geq \overline{m}_{ij}^X$. In particular, $\beta_{m-1}^{Y,\kappa} \in \mathbb{N}_0^I$ by (CG3'). Moreover, $\beta_m^{Y,\kappa} = \alpha_i$ by Lemma 9.2.13. Then $\beta_{m+1}^{Y,\kappa} \notin \mathbb{N}_0^I$ by Lemma 9.2.11, and hence $\overline{m}_{ji}^Y = m$ by (CG3').

Let now $Z = r_{j_m} \cdots r_{j_1}(Y)$ and $\kappa' = (j_m, \dots, j_1, i)$. Then κ' is not Z-reduced by Lemma 9.2.5, since $(i, j_1, \dots, j_m)$ is not X-reduced because of $m = \overline{m}_{ij}^X$. On the other hand, $\overline{m}_{j_m j_{m-1}}^Z \geq \overline{m}_{ij}^X = m$. Thus $\overline{m}_{j_m j_{m-1}}^Z = m$. Since

$$\tau^{m+1}(Z, j_m, j_{m-1}) = \tau(Y, i, j) = (X, j, i) = \sigma(X, i, j),$$

the previous paragraph applied $m+1$ times implies that $\overline{m}_{ji}^X = m$. This proves the proposition. □

LEMMA 9.2.15. *Assume that $|I| \geq 2$. Let $X \in \mathcal{X}$, $i, j \in I$ with $i \neq j$, $\kappa = \kappa_{ij}^X$, and $m = \overline{m}_{ij}^X$. Assume that $m < \infty$.*

(1) *If $\mathcal{G}$ satisfies (CG3'), then $\beta_1^{X,\kappa} = \alpha_i$, $\beta_m^{X,\kappa} = \alpha_j$, and*
$$\mathrm{id}_X (s_i s_j)^m (\alpha_i) = \alpha_i, \quad \mathrm{id}_X (s_i s_j)^m (\alpha_j) = \alpha_j.$$
(2) *If $\mathcal{G}$ satisfies (CG3') and (CG4'), then $\mathrm{id}_X (s_i s_j)^m = \mathrm{id}_X$.*

PROOF. (1) Assume that $\mathcal{G}$ satisfies (CG3'). Then $\overline{m}_{ji}^{r_j(X)} = \overline{m}_{ij}^X$ by Proposition 9.2.14. Thus $\beta_m^{X,\kappa} = \alpha_j$ by Lemma 9.2.13, and $\beta_1^{X,\kappa} = \alpha_i$ by definition.

For any $1 \leq n \leq 2m$, let $i_n = i$ if n is odd and $i_n = j$ if n is even. Thus, by Proposition 9.2.14 and by the first part of the proof,

$$\begin{aligned}
\mathrm{id}_X (s_i s_j)^m (\alpha_i) &= \mathrm{id}_X s_{i_1} \cdots s_{i_m} s_{i_{m+1}} (\alpha_{i_{m+1}}) \\
&= -\mathrm{id}_X s_{i_1} \cdots s_{i_m} (\alpha_{i_{m+1}}) = -\mathrm{id}_X s_i (\alpha_i) = \alpha_i, \\
\mathrm{id}_X (s_i s_j)^m (\alpha_j) &= -\mathrm{id}_X s_{i_1} \cdots s_{i_{2m-1}} (\alpha_j) \\
&= -\mathrm{id}_X s_{i_1} \cdots s_{i_m} (\alpha_{i_m}) = \mathrm{id}_X s_{i_1} \cdots s_{i_{m-1}} (\alpha_{i_m}) = \alpha_j.
\end{aligned}$$

This proves (1). Now (2) follows from (1) trivially. □

The following proposition is a variant of the weak exchange condition for Weyl groups.

PROPOSITION 9.2.16. *Assume that the semi-Cartan graph $\mathcal{G}$ satisfies (CG3') and (CG4'). Let $X \in \mathcal{X}$, $l \geq 1$, $\kappa = (i_1, \dots, i_l) \in I^l$, and $i \in I$, such that κ is X-reduced and $\mathrm{id}_X s_{i_1} \cdots s_{i_l} (\alpha_i) \notin \mathbb{N}_0^I$. Then there exists an X-reduced sequence $(j_1, \dots, j_l) \in I^l$ such that $j_l = i$ and $\mathrm{id}_X s_{i_1} \cdots s_{i_l} = \mathrm{id}_X s_{j_1} \cdots s_{j_l}$.*

REMARK 9.2.17. In the classical situation of semisimple Lie algebras or of Kac-Moody algebras, $\mathcal{X}$ has only one point, and only one Cartan matrix is given. If the Cartan matrix is of finite type, then the Weyl group W is a group of orthogonal transformations of a euclidian space of dimension $|I|$, and the maps $s_i \in \mathrm{Aut}(\mathbb{Z}^I)$ are hyperplane reflections at the hyperplane orthogonal to α_i. Hence for any element $w \in W$, and $i, j \in I$ with $w(\alpha_i) = \alpha_j$, the conjugate transformation $w s_i w^{-1}$ is the hyperplane reflection at the hyperplane orthogonal to $w(\alpha_i)$, that is, $w s_i w^{-1} = s_j$.

This last relation is also true in the Kac-Moody case (see [**Kac90**], proof of Lemma 3.10), and it is an essential device in the study of the Weyl group. However, for Cartan graphs an analogous argument is not available. This is one of the main reasons why the proof of Proposition 9.2.16 and some other claims are different from the classical ones.

Proof of Proposition 9.2.16. If $l=1$, then $i_l=i$ since $s_{i_1}^{r_{i_1}(X)}(\alpha_i)\notin\mathbb{N}_0^I$. Generally, if $i_l=i$ then the Proposition holds with $(j_1,\dots,j_l)=\kappa$.

Assume now that $i_l\neq i$. Then $l\geq 2$. Let $\mathcal{M}$ be the set of pairs (κ',p'), where $\kappa'=(i'_1,\dots,i'_l)\in I^l$ is X-reduced and $0\leq p'<l$, such that $i'_l=i_l$, $i'_n\in\{i,i_l\}$ for any $p'<n\leq l$, and $\mathrm{id}_X s_{i_1}\cdots s_{i_l}=\mathrm{id}_X s_{i'_1}\cdots s_{i'_l}$. Then $\mathcal{M}\neq\emptyset$ since $(\kappa,l-1)\in\mathcal{M}$. Let $((k_1,\dots,k_l),p)\in\mathcal{M}$ with a smallest possible p. Then $\Lambda^X(k_1,\dots,k_l)\subseteq\mathbb{N}_0^I$ by (CG3'). In particular, $(k_1,\dots,k_p)$ is X-reduced by Lemma 9.2.5.

Let $j\in\{i,i_l\}$ and assume that $\mathrm{id}_X s_{k_1}\cdots s_{k_p}(\alpha_j)\notin\mathbb{N}_0^I$. Then $p\geq 1$. By induction hypothesis there exist $k'_1,\dots,k'_p\in I$ such that $(k'_1,\dots,k'_p)$ is X-reduced, $k'_p=j$, and $\mathrm{id}_X s_{k_1}\cdots s_{k_p}=\mathrm{id}_X s_{k'_1}\cdots s_{k'_p}$. Let

$$\kappa'=(k'_1,\dots,k'_p,k_{p+1},\dots,k_l).$$

Then

$$\Lambda^X(\kappa')=\Lambda^X(k'_1,\dots,k'_p)\cup\{\beta_n^{X,(k_1,\dots,k_l)}\mid p+1\leq n\leq l\}\subseteq\mathbb{N}_0^I,$$

and hence κ' is X-reduced by Lemma 9.2.5. Thus $(\kappa',p-1)\in\mathcal{M}$, which is a contradiction to the choice of $((k_1,\dots,k_l),p)$.

By the previous paragraph, $\mathrm{id}_X s_{k_1}\cdots s_{k_p}(\alpha_j)\in\mathbb{N}_0^I$ for any $j\in\{i,i_l\}$. Then

$$\mathrm{id}_X s_{k_1}\cdots s_{k_p}(a\alpha_i+b\alpha_{i_l})\in\mathbb{N}_0^I \tag{9.2.4}$$

for any $a,b\in\mathbb{N}_0$. Let $Y=r_{i_p}\cdots r_{i_1}(X)$. Then $(k_{p+1},\dots,k_l)$ is Y-reduced and

$$\mathrm{id}_Y s_{k_{p+1}}\cdots s_{k_l}(\alpha_i)\in\mathbb{Z}\alpha_i+\mathbb{Z}\alpha_{i_l}\setminus\mathbb{N}_0^I$$

because of (9.2.4) and since

$$\begin{aligned}\mathbb{N}_0^I\not\ni\mathrm{id}_X s_{i_1}\cdots s_{i_l}(\alpha_i)&=\mathrm{id}_X s_{k_1}\cdots s_{k_l}(\alpha_i)\\&=\mathrm{id}_X s_{k_1}\cdots s_{k_p}\big(\mathrm{id}_Y s_{k_{p+1}}\cdots s_{k_l}(\alpha_i)\big).\end{aligned}$$

Thus $(k_{p+1},\dots,k_l,i)$ is not Y-reduced by (CG3'), and then $l-p=\overline{m}^Y_{k_{p+1},k_{p+2}}$. Therefore

$$\mathrm{id}_Y s_{k_{p+1}}\cdots s_{k_l}=\mathrm{id}_Y s_{k_{p+2}}\cdots s_{k_l}s_{k_{l+1}}$$

by (CG4') and Lemma 9.2.15, where $k_l=i_l$ and $k_{l+1}=i$. Thus the proposition holds for $(j_1,\dots,j_l)=(k_1,\dots,k_p,k_{p+2},\dots,k_{l+1})$. □

Theorem 9.2.18. *Assume that the semi-Cartan graph $\mathcal{G}$ satisfies (CG3') and (CG4'). Then for any $X\in\mathcal{X}$, $\boldsymbol{\Delta}^{X\,\mathrm{re}}=\boldsymbol{\Delta}_+^{X\,\mathrm{re}}\cup-\boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ and*

$$\boldsymbol{\Delta}_+^{X\,\mathrm{re}}=\bigcup_\kappa\Lambda^X(\kappa),$$

where the union is taken over all X-reduced sequences κ.

Proof. Let $X\in\mathcal{X}$ and let $\alpha\in\boldsymbol{\Delta}^{X\,\mathrm{re}}$, $l\geq 0$, $\kappa=(i_1,\dots,i_l)\in I^l$, and $i\in I$ such that $\alpha=\mathrm{id}_X s_{i_1}\cdots s_{i_l}(\alpha_i)$. Let $\kappa'=(i_1,\dots,i_l,i)$. Assume that

$$\alpha\neq\mathrm{id}_X s_{j_1}\cdots s_{j_k}(\alpha_j)$$

for any $0\leq k<l$, $j\in I$, and $(j_1,\dots,j_k)\in I^k$.

Assume first that κ is X-reduced and $\alpha \in \mathbb{N}_0^I$. Then $\Lambda^X(\kappa') \subseteq \mathbb{N}_0^I$ by assumption, and hence κ' is X-reduced by Lemma 9.2.5. Moreover, $\alpha \in \Lambda^X(\kappa')$ by definition.

Assume now that κ is X-reduced and $\alpha \notin \mathbb{N}_0^I$. Then, by Proposition 9.2.16, there exists an X-reduced sequence $(j_1, \dots, j_l) \in I^l$ such that $j_l = i$ and

$$\mathrm{id}_X s_{i_1} \cdots s_{i_l} = \mathrm{id}_X s_{j_1} \cdots s_{j_l}.$$

Therefore

$$\alpha = \mathrm{id}_X s_{i_1} \cdots s_{i_l}(\alpha_i) = \mathrm{id}_X s_{j_1} \cdots s_{j_l}(\alpha_i) = -\mathrm{id}_X s_{j_1} \cdots s_{j_{l-1}}(\alpha_i).$$

In particular, $-\alpha \in \Lambda^X(j_1, \dots, j_l)$.

Finally, assume that κ is not X-reduced. Then, by Lemma 9.2.5 and by assumption there exists $2 \le k \le l$ such that $\beta_j^{X,\kappa} \in \mathbb{N}_0^I$ for all $1 \le j < k$ and $\beta_k^{X,\kappa} \notin \mathbb{N}_0^I$. Hence, by Proposition 9.2.16, there exist $j_1, \dots, j_{k-1} \in I$ such that $j_{k-1} = i_k$ and $\mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}} = \mathrm{id}_X s_{j_1} \cdots s_{j_{k-1}}$. We conclude that

$$\begin{aligned}\alpha = \mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}} s_{i_k} \cdots s_{i_l}(\alpha_i) &= \mathrm{id}_X s_{j_1} \cdots s_{j_{k-1}} s_{i_k} \cdots s_{i_l}(\alpha_i)\\ &= \mathrm{id}_X s_{j_1} \cdots s_{j_{k-2}} s_{i_{k+1}} \cdots s_{i_l}(\alpha_i),\end{aligned}$$

a contradiction to the assumption in the first paragraph of the proof. □

As a consequence of Theorem 9.2.18 we can relate the axioms of a Cartan graph to (CG3') and (CG4'). To do so, we need a lemma on $\mathrm{Aut}(\mathbb{Z}^I)$.

LEMMA 9.2.19. *Let J be a finite set, $i, j \in J$, and $w \in \mathrm{Aut}(\mathbb{Z}^J)$. Assume that $w(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$ for all $k \in J$ and $w(\alpha_j)$, $w^{-1}(\alpha_j) \in \mathbb{N}_0^J \cup -\mathbb{N}_0^J$. If $\det(w) = 1$ and $w(\alpha_i) = \alpha_i$, then $w(\alpha_j) = \alpha_j$. If additionally $w(\alpha_k)$, $w^{-1}(\alpha_k) \in \mathbb{N}_0^J \cup -\mathbb{N}_0^J$ for all $k \in J$, then $w = \mathrm{id}$.*

PROOF. If $i = j$ then the first claim clearly holds. So assume that $i \neq j$. By assumption, $w(\alpha_j) = a\alpha_i + b\alpha_j$ for some $a, b \in \mathbb{Z}$. Then $b = 1$ since $\det(w) = 1$, $w(\alpha_i) = \alpha_i$, and $w(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$ for any $k \in J \setminus \{i, j\}$. We conclude that $w^{-1}(\alpha_j) = -a\alpha_i + \alpha_j$. Therefore $a = 0$ since $w(\alpha_j), w^{-1}(\alpha_j) \in \mathbb{N}_0^J$. Hence the first claim holds if $i \neq j$.

The second claim holds by a similar argument. Let $k \in J \setminus \{i, j\}$. Since $w(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$, and $w(\alpha_k) \in \mathbb{N}_0^J \cup -\mathbb{N}_0^J$, there exist unique $a, b \in \mathbb{N}_0$ such that $w(\alpha_k) = \alpha_k + a\alpha_i + b\alpha_j$, where $b = 0$ if $i = j$. Since $w^{-1}(\alpha_k) = \alpha_k - a\alpha_i - b\alpha_j$ is contained in $\mathbb{N}_0^I$ by assumption, we get $a = b = 0$. Thus $w = \mathrm{id}$. □

COROLLARY 9.2.20. *For any semi-Cartan graph $\mathcal{G}$ the following are equivalent.*

(1) *$\mathcal{G}$ satisfies (CG3') and (CG4').*
(2) *$\mathcal{G}$ is a Cartan graph.*

Moreover, if $\mathcal{G}$ satisfies (CG3), then $m_{ij}^X = \overline{m}_{ij}^X$ for any point X and any two distinct labels i, j of $\mathcal{G}$.

PROOF. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$. Suppose that $\mathcal{G}$ satisfies (CG3). Then (CG3') holds because of Lemma 9.2.7(1). We prove first that $m_{ij}^X = \overline{m}_{ij}^X$ for any $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$ in this setting.

Let $X \in \mathcal{X}$ and $i, j \in I$. Assume that $i \neq j$. Then $m_{ij}^X \ge \overline{m}_{ij}^X$ because of Lemma 9.2.7(1). In particular, if $\overline{m}_{ij}^X = \infty$ then $m_{ij}^X = \infty$. Assume that $m = \overline{m}_{ij}^X < \infty$, and let $w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}$, where $(i_1, \dots, i_m) = \kappa_{ij}^X$. Since κ_{ij}^X is

X-reduced, $\Lambda^X(\kappa_{ij}^X) \subseteq \mathbb{N}_0^I$ by (CG3'). Thus $w(\alpha_i), w(\alpha_j) \notin \mathbb{N}_0^I$ by definition of κ_{ij}^X and by Lemma 9.2.5. Hence $w(\alpha_i), w(\alpha_j) \in -\mathbb{N}_0^I$ by (CG3), and therefore

$$\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) \subseteq \boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \Lambda^X(\kappa_{ij}^X)$$

by Lemma 9.2.7(2). Since $|\Lambda^X(\kappa_{ij}^X)| = \overline{m}_{ij}^X$ by Lemma 9.2.7(1), we conclude that $m_{ij}^X = \overline{m}_{ij}^X$.

Now we prove that (2) implies (1). Assume that $\mathcal{G}$ is a Cartan graph. Then, by the previous paragraph, (CG3') holds and $m_{ij}^X = \overline{m}_{ij}^X$ for any $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. Therefore (CG4') follows from (CG4), Lemma 9.2.15(1), and Lemma 9.2.19 with $w = \mathrm{id}_X(s_is_j)^m$ and $m = \overline{m}_{ij}^X$, since $\det(\mathrm{id}_X(s_is_j)^m) = 1$ by (9.1.1).

Assume now (1). Then (CG3) holds by Theorem 9.2.18. Hence $m_{ij}^X = \overline{m}_{ij}^X$ for any $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$ by the first paragraph of the proof. Moreover, (CG4) holds because of (CG4'). □

Definition 9.2.21. For any semi-Cartan graph $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ and for all $X \in \mathcal{X}$, $i, j \in I$ with $i \neq j$, and $k \in \mathbb{N}_0$ let

$$\mathrm{Prod}_{ij}^X(2k) = \mathrm{id}_X(s_is_j)^k, \quad \mathrm{Prod}_{ij}^X(2k+1) = \mathrm{id}_X(s_is_j)^k s_i$$

as morphisms in $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$.

Corollary 9.2.22. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph. Let $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. If m_{ij}^X is finite then*

(1) $\mathrm{id}_X(s_is_j)^{m_{ij}^X} = \mathrm{id}_X$,
(2) $\mathrm{Prod}_{ij}^X(m_{ij}^X) = \mathrm{Prod}_{ji}^X(m_{ij}^X)$.

The relations in (2) are called the **Coxeter relations**.

Proof. (1) follows from Corollary 9.2.20 and Lemma 9.2.15. (2) follows then from (1). □

In the next theorem we state a variant of the Coxeter relations in the Weyl groupoid of a semi-Cartan graph which is not necessarily Cartan. Recall that we denote by $F(w)$ the linear automorphism of a morphism w in the Weyl groupoid.

Theorem 9.2.23. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph. Let $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. Let $\mathcal{Y} \subseteq \mathcal{X}$ with $X \in \mathcal{Y}$ such that $r_i(\mathcal{Y}) \cup r_j(\mathcal{Y}) \subseteq \mathcal{Y}$, and assume that $\boldsymbol{\Delta}^{Y\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ for all $Y \in \mathcal{Y}$. If m_{ij}^X is finite then*

$$m_{ij}^X = \min\{n \geq 1 \,|\, F(\mathrm{id}_X(s_is_j)^n) = \mathrm{id}_{\mathbb{Z}^I}\}.$$

If m_{ij}^X is infinite, then for all $n \geq 1$, $F(\mathrm{id}_X(s_is_j)^n) \neq \mathrm{id}_{\mathbb{Z}^I}$.

Proof. We may assume that $\mathcal{G}$ is connected. Then, since $\mathcal{G}$ satisfies (CG3), $m_{ij}^X = \overline{m}_{ij}^X$ by Corollary 9.2.20. Moreover, (CG3') holds by Lemma 9.2.7(1).

Assume that m_{ij}^X is finite. Then $F(\mathrm{id}_X(s_is_j)^{m_{ij}^X}) = \mathrm{id}_{\mathbb{Z}^I}$ by Lemma 9.2.15 and Lemma 9.2.19. Now it suffices to prove that $N(\mathrm{id}_X(s_is_j)^n) > 0$ for all $1 \leq n < m_{ij}^X$. For the latter note that $N(\mathrm{Prod}_{ij}^X(\overline{m}_{ij}^X)) = m_{ij}^X$ by Lemma 9.2.7, and hence the claim follows from Lemma 9.1.21(2).

If $m_{ij}^X = \infty$, then κ_{ij}^X has infinite length and hence $\mathrm{id}_X(s_is_j)^n(\alpha_i) \neq \alpha_i$ for all $n \geq 1$ by Lemma 9.2.7(1). □

EXAMPLE 9.2.24. Here we discuss an example of a semi-Cartan graph satisfying (CG3') and the first condition in (CG4'), but not the second.

Let $I = \{1, 2, 3\}$ and $\mathcal{X} = \{1, 2, 3, 4\}$. Let r_1, r_2, r_3 be the permutations

$$r_1 = (1\,2)(3\,4), \quad r_2 = (2\,3), \quad r_3 = \mathrm{id}_{\mathcal{X}}$$

of $\mathcal{X}$. Then $r_i^2 = \mathrm{id}_{\mathcal{X}}$ for any $i \in I$. Moreover, let

$$A_1 = A_4 = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}, \quad A_m = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -p_m & 0 & 2 \end{pmatrix}$$

for $m \in \{2, 3\}$, where $2 \le p_2 < p_3$. Then $\mathcal{G}_{p_2,p_3} = \mathcal{G}(I, \mathcal{X}, r, A)$ is a semi-Cartan graph. Its exchange graph is displayed in Figure 9.1.4.

Let now

$$\begin{aligned} \Lambda_2 = \Lambda_3 &= \{a\alpha_1 + b\alpha_2 + c\alpha_3 \mid a, b, c \in \mathbb{N}_0,\ a < b + c\}, \\ P_1 &= \{a\alpha_1 + b\alpha_2 + c\alpha_3 \mid a, b, c \in \mathbb{N}_0,\ a > b + c\}, \\ P_2 &= \{a\alpha_1 + b\alpha_2 + c\alpha_3 \mid a, b, c \in \mathbb{N}_0,\ b > a + c\}, \\ P_3 &= \{a\alpha_1 + b\alpha_2 + c\alpha_3 \mid a, b, c \in \mathbb{N}_0,\ c > a + b\}, \end{aligned}$$

and

$$\Lambda_1 = \Lambda_4 = P_2 \cup P_3.$$

Then the following hold.

(1) $\overline{m}_{23}^X = \overline{m}_{32}^X = 2$ for $X \in \{2, 3\}$.

(2) For any $X \in \mathcal{X}$, a sequence $\kappa = (i_1, \dots, i_l) \in I^l$ with $l \ge 1$ is not X-reduced, if
  - (a) there exists $1 \le k < l$ such that $i_k = i_{k+1}$, or
  - (b) there exists $1 \le k \le l - 2$ such that $r_{i_{k-1}} \cdots r_{i_1}(X) \in \{2, 3\}$ and $(i_k, i_{k+1}, i_{k+2}) \in \{(2, 3, 2), (3, 2, 3)\}$.

  We denote by N_X the set of such sequences.

(3) For any $X \in \mathcal{X}$ and any sequence $\kappa \notin N_X$, $\Lambda^X(\kappa) \subseteq \Lambda_X \cup P_1$.

(4) $\overline{m}_{ij}^X = \infty$ whenever $X \in \{1, 4\}$ or $\{i, j\} \ne \{2, 3\}$.

(5) $\mathrm{id}_2 s_2 s_3(\alpha_1) = \alpha_1 + 2\alpha_2 + p_3\alpha_3 \ne \alpha_1 + 2\alpha_2 + p_2\alpha_3 = \mathrm{id}_2 s_3 s_2(\alpha_1)$.

The verification of these claims is straightforward except (3) and (4). Claim (3) can be obtained by showing the following by induction on l.

(3)(a) For any $X \in \{1, 4\}$ and any sequence $\kappa = (i_1, \dots, i_l) \notin N_X$ with $l \ge 1$, $\Lambda^X(\kappa) \subseteq P_{i_1}$.

(3)(b) For any $X \in \{2, 3\}$ and any sequence $\kappa = (i_1, \dots, i_l) \notin N_X$ with $l \ge 1$ and $i_1 = 1$, $\Lambda^X(\kappa) \subseteq P_1$.

(3)(c) For any $X \in \{2, 3\}$ and any sequence $\kappa = (i_1, \dots, i_l) \notin N_X$ with $l \ge 1$ and $i_1 \in \{2, 3\}$, $\Lambda^X(\kappa) \subseteq \Lambda_2$.

Then (4) follows from (3) and Lemma 9.2.5.

Now (2) and (3) imply that (CG3') holds, and (1) and (4) imply that the first condition of (CG4') holds. Finally, the second condition of (CG4') fails because of (5).

We close the section with a criterion for finiteness of a semi-Cartan graph in terms of reduced sequences.

PROPOSITION 9.2.25. *Assume that the semi-Cartan graph $\mathcal{G}$ is connected and satisfies (CG3). Let $X \in \mathcal{X}$. The following are equivalent.*

(1) *There exists $m \in \mathbb{N}_0$ such that for any $Y \in \mathcal{X}$, any Y-reduced sequence has length at most m.*
(2) *There exists $m \in \mathbb{N}_0$ such that any X-reduced sequence has length at most m.*
(3) *$\mathcal{G}$ is finite.*

PROOF. Clearly, (1) implies (2). Now assume (2). In order to prove (3), it suffices to show that $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is finite, since $\mathcal{G}$ is connected. Assume to the contrary that $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is infinite. Let $(i_1, \dots, i_l)$ be an X-reduced sequence of maximal length. Let $Y = r_{i_l} \cdots r_{i_1}(X)$. Then $(i_l, \dots, i_1)$ is Y-reduced by Lemma 9.2.5. Moreover, there exists $i \in I$ such that $(i, i_l, \dots, i_1)$ is $r_i(Y)$-reduced because of (CG3) and Lemma 9.2.8. Thus $(i_1, \dots, i_l, i)$ is X-reduced by Lemma 9.2.5, which contradicts the maximality assumption on l.

Finally, we prove that (3) implies (1). Since $\mathcal{G}$ is connected, Lemma 9.1.18 and (3) imply that $\cup_{Y \in \mathcal{X}} \boldsymbol{\Delta}^{Y\,\mathrm{re}}$ is a finite set. Let m be its cardinality. Then for any $Y \in \mathcal{X}$, any Y-reduced sequence has length at most m by the equivalence of Lemma 9.2.7(1)(a) and (1)(b). $\square$

9.3. Weak exchange condition and longest elements

We discuss general properties of Cartan graphs, Coxeter relations, a variant of the weak exchange condition, and the existence and uniqueness of longest elements in the Weyl groupoid.

LEMMA 9.3.1. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $X \in \mathcal{X}$, and $i, j \in I$ with $i \neq j$. The following are equivalent.*

(1) $a_{ij}^X = a_{ji}^X = 0$.
(2) $m_{ij}^X = 2$.
(3) $\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) = \{\alpha_i, \alpha_j\}$.

PROOF. Since $\alpha_i, \alpha_j \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$, (2) and (3) are equivalent. Moreover, (2) implies that $s_i^{r_i(X)}(\alpha_j) = \alpha_j$ and hence (1) holds. Assume (1). Then $a_{ij}^{r_i(X)} = 0 = a_{ji}^{r_i(X)}$ by (CG2) and since A^X is a Cartan matrix. Similarly, $a_{ij}^{r_j r_i(X)} = 0$. Hence $\kappa_{ij}^X = (i, j)$ and $m_{ij}^X = \overline{m}_{ij}^X = 2$ by Corollary 9.2.20. $\square$

REMARK 9.3.2. In any semi-Cartan graph $\mathcal{G}$, Lemma 9.3.1(2) implies (1), and (2) and (3) are equivalent.

LEMMA 9.3.3. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $i, j \in I$, and $X \in \mathcal{X}$ with $a_{ij}^X = 0$. Then $a_{jl}^X = a_{jl}^{r_i(X)}$ for all $l \in I$.*

PROOF. Since $a_{ij}^{r_i(X)} = a_{ij}^X = 0$, we observe that $a_{ji}^{r_i(X)} = 0$. Now $m_{ij}^X = 2$ by Lemma 9.3.1. Thus $s_i s_j^X(\alpha_l) = s_j s_i^X(\alpha_l)$ by Corollary 9.2.22, that is,

$$\alpha_l - a_{jl}^X \alpha_j - a_{il}^{r_j(X)} \alpha_i + a_{jl}^X a_{ij}^{r_j(X)} \alpha_i = \alpha_l - a_{il}^X \alpha_i - a_{jl}^{r_i(X)} \alpha_j + a_{il}^X a_{ji}^{r_i(X)} \alpha_j.$$

Then $a_{jl}^X = a_{jl}^{r_i(X)}$ by comparing the coefficients of α_j on both sides of the equation. $\square$

As an application of the Coxeter relations we now prove a **weak exchange condition** for Cartan graphs.

THEOREM 9.3.4. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $X \in \mathcal{X}$, $k \in \mathbb{N}$, and $i_1, \dots, i_k, i \in I$. Assume that $\mathrm{id}_X s_{i_1} \cdots s_{i_k}(\alpha_i) \in -\mathbb{N}_0^I$. Then there exist $j_1, \dots, j_k$ in I such that $j_k = i$ and*

$$\mathrm{id}_X s_{i_1} \cdots s_{i_k} = \mathrm{id}_X s_{j_1} \cdots s_{j_k}$$

in the Weyl groupoid of $\mathcal{G}$.

PROOF. Since $\mathcal{G}$ is a Cartan graph, it satisfies (CG3') and (CG4') by Corollary 9.2.20. Let $w = \mathrm{id}_X s_{i_1} \cdots s_{i_k}$ and $\kappa = (i_1, \dots, i_k)$. If κ is X-reduced, then the claim holds by Proposition 9.2.16.

Assume that κ is not X-reduced. By Lemma 9.2.7(1), there exists $1 \le l \le k$ such that $\beta_l^{X,\kappa} \in -\mathbb{N}_0^I$ and $\beta_n^{X,\kappa} \in \mathbb{N}_0^I$ for any $1 \le n < l$. By the same Lemma, $(i_1, \dots, i_{l-1})$ is X-reduced. Hence by Proposition 9.2.16 there exist $j_1, \dots, j_{l-1} \in I$ such that $\mathrm{id}_X s_{i_1} \cdots s_{i_{l-1}} = \mathrm{id}_X s_{j_1} \cdots s_{j_{l-1}}$ and $j_{l-1} = i_l$. Then

$$\begin{aligned} \mathrm{id}_X s_{i_1} \cdots s_{i_{l-1}} s_{i_l} \cdots s_{i_k} &= \mathrm{id}_X s_{j_1} \cdots s_{j_{l-1}} s_{i_l} \cdots s_{i_k} \\ &= \mathrm{id}_X s_{j_1} \cdots s_{j_{l-2}} s_{i_{l+1}} \cdots s_{i_k} s_i^2 \end{aligned}$$

which proves the theorem. □

The weak exchange condition in Theorem 9.3.4 is the main tool to understand the relation between reduced decompositions in the Weyl groupoid and X-reduced tuples of elements of I, and to prove the important equality $N(w) = l(w)$ for morphisms w in the Weyl groupoid.

THEOREM 9.3.5. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $i_1, \dots, i_l \in I$, and $X \in \mathcal{X}$. Let $w = \mathrm{id}_X s_{i_1} \dots s_{i_l} \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$, and*

$$\beta_k = \mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \textit{ for all } 1 \le k \le l.$$

(1) *The following are equivalent.*
   (a) *$(i_1, \dots, i_l)$ is a reduced decomposition of w.*
   (b) *$(i_1, \dots, i_l)$ is X-reduced.*
(2) *$N(w) = \ell(w)$, and if $(i_1, \dots, i_l)$ is a reduced decomposition of w, then $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \{\beta_1, \dots, \beta_l\} = \Lambda^X(i_1, \dots, i_l)$.*

PROOF. Let $\kappa = (i_1, \dots, i_l)$. Assume that κ is a reduced decomposition of w, and that κ is not X-reduced. By Lemma 9.2.7(1), there is an integer $2 \le k \le l$ such that $\beta_k \in -\mathbb{N}_0^I$. By Theorem 9.3.4, there are $j_1, \dots, j_{k-1} \in I$ such that $\mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}} = \mathrm{id}_X s_{j_1} \cdots s_{j_{k-1}}$, and $j_{k-1} = i_k$. Therefore,

$$\mathrm{id}_X s_{i_1} \cdots s_{i_k} = \mathrm{id}_X s_{j_1} \cdots s_{j_{k-2}} s_{i_k} s_{i_k} = \mathrm{id}_X s_{j_1} \cdots s_{j_{k-2}},$$

and then $\ell(w) < l$. Hence κ is X-reduced. Then Lemma 9.2.7(2) implies that $N(w) = l = \ell(w)$, and $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \{\beta_1, \dots, \beta_l\} = \Lambda^X(\kappa)$.

We have shown (2) and that (1)(a) implies (1)(b). To prove that (1)(b) implies (1)(a), assume that κ is X-reduced. Then $N(w) = \ell(w)$ by (2), and from Lemma 9.2.7 we obtain that $N(w) = |\boldsymbol{\Delta}^{X\,\mathrm{re}}(w)| = l$. Thus $\ell(w) = l$. □

COROLLARY 9.3.6. *Let $\mathcal{G}$ be a Cartan graph, $w \in \mathcal{W}(\mathcal{G})$ and i a label of $\mathcal{G}$.*

(1) *$w(\alpha_i) \in \mathbb{N}_0^I$ if and only if $\ell(ws_i) = \ell(w) + 1$.*
(2) *$w(\alpha_i) \in -\mathbb{N}_0^I$ if and only if $\ell(ws_i) = \ell(w) - 1$.*

PROOF. This holds by Lemma 9.1.21(2), since $N = \ell$ by Theorem 9.3.5. $\square$

COROLLARY 9.3.7. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $i \in I$, $X \in \mathcal{X}$, and $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$. If $\alpha_i \in \boldsymbol{\Delta}^{X\,\mathrm{re}}(w)$, then there exists a reduced decomposition $(i_1, \dots, i_l)$ of w with $i_1 = i$.*

PROOF. Assume that $\alpha_i \in \boldsymbol{\Delta}^{X\,\mathrm{re}}(w)$. Then $w^{-1}(\alpha_i) \in -\mathbb{N}_0^I$, and hence $\ell(w^{-1}s_i) = \ell(w^{-1}) - 1$ by Corollary 9.3.6. Let $w' = s_i w$. Then $w = s_i w'$, and $(i, i_2, \dots, i_l)$ is a reduced decomposition of w for any reduced decomposition $(i_2, \dots, i_l)$ of w'. $\square$

Theorem 9.3.5 is one of the main results in the general theory of Cartan graphs. In particular, it allows to prove for each point of a finite Cartan graph the existence of a unique longest element in the Weyl groupoid ending in this point. We also will see that the Weyl groupoid of a finite and connected Cartan graph has only finitely many objects and finitely many morphisms.

We begin with a criterion for equality of morphisms in the Weyl groupoid of a Cartan graph.

COROLLARY 9.3.8. *Let $\mathcal{G}$ be a Cartan graph, X a point of $\mathcal{G}$, and w, w' morphisms in $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$.*

(1) *Assume that $w(\alpha_i) \in \mathbb{N}_0^I$ for all labels $i \in I$ of $\mathcal{G}$. Then $w = \mathrm{id}_X$.*
(2) *Assume that $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \boldsymbol{\Delta}^{X\,\mathrm{re}}(w')$. Then $w = w'$.*
(3) *Assume that the linear functions $F(w)$ and $F(w')$ of w and w' coincide. Then $w = w'$.*

PROOF. (1) The assumption implies that $N(w^{-1}) = 0$. Hence $\ell(w) = 0$ by Theorem 9.3.5(2), and $w = \mathrm{id}_X$.

(2) Let $i \in I$. We show that $w^{-1}w'(\alpha_i) \in \mathbb{N}_0^I$. By (1), this proves the claim in (2).

If $w'(\alpha_i) \in \mathbb{N}_0^I$, then $w'(\alpha_i) \notin \boldsymbol{\Delta}^{X\,\mathrm{re}}(w') = \boldsymbol{\Delta}^{X\,\mathrm{re}}(w)$, and $w^{-1}w'(\alpha_i) \in \mathbb{N}_0^I$. On the other hand, if $w'(\alpha_i) \in -\mathbb{N}_0^I$, then $-w'(\alpha_i) \in \boldsymbol{\Delta}^{X\,\mathrm{re}}(w')$. This implies that $w^{-1}w'(\alpha_i) \in \mathbb{N}_0^I$, since $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w') = \boldsymbol{\Delta}^{X\,\mathrm{re}}(w)$.

(3) is a special case of (2). $\square$

PROPOSITION 9.3.9. *Let $\mathcal{G}$ be a finite Cartan graph, and X a point of $\mathcal{G}$. There is a unique morphism $w_0 \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ such that $\ell(w) \le \ell(w_0)$ for all morphisms $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$. Let $Y \in \mathcal{X}$ with $w_0 \in \mathrm{Hom}(Y, X)$. Then*

(1) $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w_0) = \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$,
(2) *$w_0 \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ is unique with the property that for all labels i of $\mathcal{G}$, $\ell(w_0 s_i) < \ell(w_0)$, and*
(3) *for all $\alpha \in \boldsymbol{\Delta}^{Y\,\mathrm{re}}$, α is simple if and only if $-w_0(\alpha)$ is simple.*

PROOF. Let I be the set of labels of $\mathcal{G}$. Since $\mathcal{G}$ is finite, by Theorem 9.1.22 with $R = \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ there exists a morphism $w' \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ with $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w') = \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$. Let $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ with $\ell(w) \ge \ell(w')$. Then Theorem 9.3.5 implies that $|\boldsymbol{\Delta}^{X\,\mathrm{re}}(w)| = \ell(w) \ge |\boldsymbol{\Delta}_+^{X\,\mathrm{re}}|$, and hence $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$. Therefore $w = w'$ by Corollary 9.3.8(2). This proves the first claim and (1) with $w_0 = w'$.

(2) We proved already that $\ell(w_0 s_i) \le \ell(w_0)$ (and hence $\ell(w_0 s_i) < \ell(w_0)$ by Corollary 9.3.6) for all $i \in I$. Conversely, let $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ such that for any $i \in I$, $\ell(w s_i) < \ell(w)$. Then $w(\alpha_i) \in -\mathbb{N}_0^I$ for any $i \in I$ by Corollary 9.3.6. Thus $\boldsymbol{\Delta}^{X\,\mathrm{re}}(w) = \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ and hence $w = w_0$ by the first paragraph of the proof.

(3) Since $w_0(\alpha_i) \in -\mathbb{N}_0^I$ by (1), the claim follows from Lemma 9.1.23. $\square$

DEFINITION 9.3.10. Let $\mathcal{G}$ be a finite Cartan graph, and let X be a point of $\mathcal{G}$. The element w_0 in Proposition 9.3.9 is called the **longest element** in $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$.

COROLLARY 9.3.11. *Let $\mathcal{G}$ be a finite Cartan graph, X, Z points of $\mathcal{G}$, and $w \in \mathrm{Hom}(Z, X)$. Then any reduced decomposition $(i_1, \dots, i_l)$ of w can be extended to a reduced decomposition $(i_1, \dots, i_l, \dots, i_m)$ of $w_0 \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$.*

PROOF. By Proposition 9.3.9, $\ell(w) < \ell(w_0)$ if $w \neq w_0$. Thus the claim follows from Proposition 9.3.9(2). □

COROLLARY 9.3.12. *Let $\mathcal{G}$ be a connected finite Cartan graph, and let $\mathcal{X}$ be the set of points of $\mathcal{G}$. Then $\mathcal{X}$ is finite and $\mathrm{Hom}(Y, X)$ is finite for all $X, Y \in \mathcal{X}$.*

PROOF. Let $X \in \mathcal{X}$. By Corollary 9.3.8(2), the map from $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ to the power set of $\mathbf{\Delta}_+^{X\,\mathrm{re}}$ sending w to $\mathbf{\Delta}^{X\,\mathrm{re}}(w)$ is injective. Since $\mathbf{\Delta}_+^{X\,\mathrm{re}}$ is finite, we conclude that $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ is finite. This proves the claim since $\mathcal{G}$ is connected. □

COROLLARY 9.3.13. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a finite Cartan graph, $X \in \mathcal{X}$, $n \in \mathbb{N}_0$, and $i_1, \dots, i_n \in I$ such that $w_0 = \mathrm{id}_X s_{i_1} \cdots s_{i_n}$ is the longest element in $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ and $\ell(w_0) = n$. Then $n = |\mathbf{\Delta}_+^{X\,\mathrm{re}}|$, and*

$$\mathbf{\Delta}_+^{X\,\mathrm{re}} = \{\mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \,|\, 1 \le k \le n\}.$$

PROOF. By Proposition 9.3.9(1), $\mathbf{\Delta}^{X\,\mathrm{re}}(w_0) = \mathbf{\Delta}_+^{X\,\mathrm{re}}$. Hence the claim follows from Theorem 9.3.5(2). □

Any reduced decomposition of a morphism w in the Weyl groupoid of a Cartan graph induces a total order on the set $\mathbf{\Delta}^{X\,\mathrm{re}}(w)$ in a natural way by Theorem 9.3.5. As in the case of Weyl groups, this order is convex in the strong sense of the next proposition.

PROPOSITION 9.3.14. *Let $\mathcal{G}$ be a Cartan graph, X a point of $\mathcal{G}$, w a morphism of the Weyl groupoid of $\mathcal{G}$, and $(i_1, \dots, i_l)$ with $l = \ell(w)$ a reduced decomposition of w. For any $1 \le k \le l$ let $\beta_k = \mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Then $\mathbf{\Delta}^{X\,\mathrm{re}}(w) = \{\beta_1, \dots, \beta_l\}$ is totally ordered by $\beta_p < \beta_q$ if and only if $p < q$. Let $k, k_1, \dots, k_r \in \{1, \dots, l\}$ with $k_1 \le k_2 \le \cdots \le k_r$. Assume that $\beta_k = \sum_{i=1}^r \beta_{k_i}$. Then either $r = 1$, $k = k_1$ or $k_1 < k < k_r$.*

PROOF. Let $v = \mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}}$. By Theorem 9.3.5, $\mathbf{\Delta}^{X\,\mathrm{re}}(w) = \{\beta_1, \dots, \beta_l\}$ and $\mathbf{\Delta}^{X\,\mathrm{re}}(v) = \{\beta_1, \dots, \beta_{k-1}\}$. Thus $v^{-1}(\beta_j) \in -\mathbb{N}_0^I$ for $1 \le j \le l$ if and only if $j < k$. Moreover, $\alpha_{i_k} = v^{-1}(\beta_k) = \sum_{i=1}^r v^{-1}(\beta_{k_i})$ by assumption. Hence either $r = 1$, $k = k_1$ or $r \ge 2$, $k_1 < k < k_r$. □

Finally we discuss a special property of standard Cartan graphs.

PROPOSITION 9.3.15. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a standard Cartan graph. Let X be a point of $\mathcal{G}$.*

(1) *Let $Y \in \mathcal{X}$ be any point, $k, l \in \mathbb{N}_0$, and $i_1, \dots, i_k, j_1, \dots, j_l \in I$ such that $\mathrm{id}_X s_{i_1} \cdots s_{i_k} = \mathrm{id}_X s_{j_1} \cdots s_{j_l}$. Then $\mathrm{id}_Y s_{i_1} \cdots s_{i_k} = \mathrm{id}_Y s_{j_1} \cdots s_{j_l}$.*
(2) *The set $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ is a group with*

$$(\mathrm{id}_X s_{i_1} \cdots s_{i_k})(\mathrm{id}_X s_{j_1} \cdots s_{j_l}) = \mathrm{id}_X s_{i_1} \cdots s_{i_k} s_{j_1} \cdots s_{j_l}$$

for all $k, l \in \mathbb{N}_0$ and all labels $i_1, \dots, i_k, j_1, \dots, j_l$ of $\mathcal{G}$.

(3) *The group* $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ *in (2) is isomorphic to the Weyl group of the Cartan matrix of* $\mathcal{G}$.

PROOF. (1) The assumptions imply that $F(\mathrm{id}_X s_{i_1} \cdots s_{i_k}) = F(\mathrm{id}_X s_{j_1} \cdots s_{j_l})$. Since $A^Z = A^X$ for all $Z \in \mathcal{X}$, it follows that $F(\mathrm{id}_Y s_{i_1} \cdots s_{i_k}) = F(\mathrm{id}_Y s_{j_1} \cdots s_{j_l})$. Hence $\mathrm{id}_Y s_{i_1} \cdots s_{i_k} = \mathrm{id}_Y s_{j_1} \cdots s_{j_l}$ by Corollary 9.3.8(3).

(2) The multiplication is well-defined by (1). The remaining group axioms follow directly from the definition of the multiplication.

(3) The group $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ is generated by the morphisms $\mathrm{id}_X s_i$ with $i \in I$, and fulfills the relations

$$((\mathrm{id}_X s_i)(\mathrm{id}_X s_j))^{m_{ij}^X} = \mathrm{id}_X$$

for all $i, j \in I$, since $\mathrm{id}_X (s_i s_j)^{m_{ij}^X} = \mathrm{id}_X$ by Corollary 9.2.22. Thus there is a unique surjective group homomorphism $W \to \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ sending s_i to $\mathrm{id}_X s_i$ for all $i \in I$, where W is the Weyl group of the Cartan matrix A of $\mathcal{G}$. If $i_1, \ldots, i_k \in I$ with $\mathrm{id}_X s_{i_1} \cdots s_{i_k} = \mathrm{id}_X$, then $s_{i_1} \cdots s_{i_k} = F(\mathrm{id}_X s_{i_1} \cdots s_{i_k}) = \mathrm{id}_{\mathbb{Z}^I}$. Thus the given group homomorphism is bijective. □

9.4. Coxeter groupoids

If the Coxeter relations hold for the generators in a group, then the exchange condition implies that the Coxeter relations are defining relations, that is, the group is a Coxeter group, see [**Bou68**, Ch. IV, 1.6]. We will extend this result to groupoids which are Weyl groupoids of a Cartan graph. In this case our proof of the weak exchange condition implies that the Coxeter relations of the Weyl groupoid are defining relations.

Coxeter groupoids are certain categories given by generators and relations. We recall the definition of such categories from [**ML98**, §II.7,8].

Let $\mathcal{X}$ be a set and let G be a directed graph with $\mathcal{X}$ as its set of vertices. One also says that G is an **$\mathcal{X}$-graph**. The **free category generated by** G is the category with $\mathcal{X}$ as the set of objects, where the morphisms are admissible finite compositions of arrows of G.

For a category $\mathcal{C}$ and any two objects X, Y of $\mathcal{C}$ let

$$R_{X,Y} \subseteq \mathrm{Hom}(X, Y) \times \mathrm{Hom}(X, Y)$$

be a relation, that is, a subset. Then there exists a category $\mathcal{C}/R$ and a functor $F_R : \mathcal{C} \to \mathcal{C}/R$ with the following properties.

(1) If $(f, f') \in R_{X,Y}$, then $F_R(f) = F_R(f')$.
(2) Let $\mathcal{D}$ be a category and $H : \mathcal{C} \to \mathcal{D}$ a functor. If $H(f) = H(f')$ for all $f, f' \in \mathrm{Hom}(X, Y)$, $X, Y \in \mathcal{C}$ with $(f, f') \in R_{X,Y}$, then there exists a unique functor $H' : \mathcal{C}/R \to \mathcal{D}$ such that $H'F_R = H$.

The second property of $\mathcal{C}/R$ is called the **universal property of** $\mathcal{C}/R$. The functor F_R is then necessarily a bijection between the objects of $\mathcal{C}$ and the objects of $\mathcal{C}/R$. If $\mathcal{C}$ is the free category generated by a graph G, then $\mathcal{C}/R$ is called the **category with generators** G **and relations** R.

DEFINITION 9.4.1. Let I be a non-empty finite set and let $\mathcal{X}$ be a non-empty set. Let G be a directed labeled graph with $\mathcal{X}$ as its set of objects, such that each object has for all $i \in I$ precisely one incoming and one outgoing arrow labeled by i. For all $X \in \mathcal{X}$ and $i \in I$ let $r_i(X)$ be the target of the i-arrow starting at X.

For all $X \in \mathcal{X}$ let $M^X = (m_{ij}^X)_{i,j\in I} \in (\mathbb{N} \cup \{\infty\})^{I\times I}$ be a symmetric matrix such that $m_{ii}^X = 1$ for all $i \in I$. Assume that $(r_i r_j)^{m_{ij}^X}(X) = X$ for all $X \in \mathcal{X}$ and $i, j \in I$ with $m_{ij}^X \neq \infty$. The **Coxeter groupoid** $\mathrm{Cox}(G, (M^X)_{X\in\mathcal{X}})$ is the category with generators G and relations

$$\mathrm{id}_X (s_i s_j)^{m_{ij}^X} = \mathrm{id}_X,$$

where $i, j \in I$, $X \in \mathcal{X}$ such that $m_{ij}^X \neq \infty$, and s_i^X (or simply s_i) is the morphism corresponding to the i-arrow of G starting at X.

The assumption $m_{ii}^X = 1$ for all objects X and labels i implies the equation $s_i^{r_i(X)} s_i^X = \mathrm{id}_X$. This equation will also be written as $\mathrm{id}_X s_i s_i = \mathrm{id}_X$.

EXAMPLE 9.4.2. Let $I = \{1, \dots, n\}$. Let G be a directed graph with one vertex and with one loop for each $i \in I$. Let $M = (m_{ij})_{i,j\in I} \in (\mathbb{N} \cup \{\infty\})^{n\times n}$ be a symmetric matrix with $m_{ii} = 1$ for all $i \in I$. Then $\mathrm{Cox}(G, M)$ is a Coxeter group viewed as a category.

DEFINITION 9.4.3. Let $\mathrm{Cox}(G, (M^X)_{X\in\mathcal{X}})$ be a Coxeter groupoid, where $\mathcal{X}$ is a set and G is an $\mathcal{X}$-graph. For all $X, Y \in \mathcal{X}$ and $w \in \mathrm{Hom}(Y, X)$ let $\ell(w)$ be the smallest integer $k \geq 0$ such that $w = \mathrm{id}_X s_{i_1} \cdots s_{i_k}$ for some $i_1, \dots, i_k \in I$. The family $(\ell : \mathrm{Hom}(X, Y) \to \mathbb{N}_0)_{X,Y\in\mathcal{X}}$ is called the **length function** and $\ell(w)$ is called the **length of** w.

Some properties of Coxeter groups immediately generalize to Coxeter groupoids. Recall the definition of the category $\mathcal{D}(\mathcal{X}, \{-1, 1\})$ from Definition 9.1.8, where $\{-1, 1\}$ is a monoid with respect to multiplication.

LEMMA 9.4.4. *Let $\mathrm{Cox}(G, (M^X)_{X\in\mathcal{X}})$ be a Coxeter groupoid, where $\mathcal{X}$ is a set and G is an $\mathcal{X}$-graph. There is a unique functor*

$$\det : \mathrm{Cox}(G, (M^X)_{X\in\mathcal{X}}) \to \mathcal{D}(\mathcal{X}, \{-1, 1\})$$

which is the identity on the objects $\mathcal{X}$ and sends any $s_i^X \in \mathrm{Hom}(X, r_i(X))$ to $(r_i(X), -1, X)$.

PROOF. This follows from the relations of the groupoid $\mathrm{Cox}(G, (M^X)_{X\in\mathcal{X}})$ and from its universal property as a quotient of a free category. □

LEMMA 9.4.5. *Let $\mathrm{Cox}(G, (M^X)_{X\in\mathcal{X}})$ be a Coxeter groupoid, where $\mathcal{X}$ is a set and G is an $\mathcal{X}$-graph. Let $X, Y, Z \in \mathcal{X}$, and let $w : X \to Y$, $w' : Y \to Z$ be morphisms in $\mathrm{Cox}(G, (M^X)_{X\in\mathcal{X}})$, $k \geq 0$, and $i_1, \dots, i_k \in I$. Then*

(1) $|\ell(w) - \ell(w')| \leq \ell(w'w) \leq \ell(w') + \ell(w)$, $\ell(w^{-1}) = \ell(w)$,
(2) $\ell(w'w) \equiv \ell(w') + \ell(w) \mod 2$,
(3) $\ell(s_i w), \ell(w s_i) \in \{\ell(w) + 1, \ell(w) - 1\}$ *for all* $i \in I$,
(4) $k - \ell(\mathrm{id}_X s_{i_1} \cdots s_{i_k})$ *is a non-negative even integer.*

PROOF. Follow the proof of Lemma 9.1.13 using Lemma 9.4.4. □

We will mainly be interested in Coxeter groupoids of Cartan graphs. In particular, we will show that the Weyl groupoid and the Coxeter groupoid of a Cartan graph are equivalent via a functor which is the identity on the points of the Cartan graph.

DEFINITION 9.4.6. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph. Let G be the $\mathcal{X}$-graph with arrows labeled by the elements of I, such that for any $i \in I$ and $X \in \mathcal{X}$ there is precisely one i-arrow starting at X, and the target of this arrow is $r_i(X)$. For all $X \in \mathcal{X}$ let $M^X = (m_{ij}^X)_{i,j \in I}$. We say that $\mathrm{Cox}(\mathcal{G}) = \mathrm{Cox}(G, (M^X)_{X \in \mathcal{X}})$ is the **Coxeter groupoid of** $\mathcal{G}$.

THEOREM 9.4.7. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $X \in \mathcal{X}$, $k \in \mathbb{N}$, and $i_1, \dots, i_k, i \in I$. Assume that $\mathrm{id}_X s_{i_1} \cdots s_{i_k}(\alpha_i) \in -\mathbb{N}_0^I$. Then there exist labels $j_1, \dots, j_k \in I$ such that $j_k = i$ and $\mathrm{id}_X s_{i_1} \cdots s_{i_k} = \mathrm{id}_X s_{j_1} \cdots s_{j_k}$ in $\mathrm{Cox}(\mathcal{G})$.*

PROOF. The claim is the Coxeter groupoid analogue of Theorem 9.3.4. The proof of Theorem 9.3.4 also works here without essential modifications. Let us recall the main steps.

(1) Assume that the sequence $(i_1, \dots, i_k)$ is X-reduced and $i \neq i_k$. Choose a pair $((j_1, \dots, j_k), p)$ in $I^k \times \mathbb{N}_0$ such that $\mathrm{id}_X s_{i_1} \cdots s_{i_k} = \mathrm{id}_X s_{j_1} \cdots s_{j_k}$ in $\mathrm{Cox}(\mathcal{G})$, $0 \le p < k$, and $j_n \in \{i, i_k\}$ for any $p < n \le k$. Assume that in all such pairs the second entry is at least p. Let $u = \mathrm{id}_X s_{j_1} \cdots s_{j_p}$ and $Y = r_{j_p} \cdots r_{j_1}(X)$. Then induction hypothesis implies that $u(\alpha_i), u(\alpha_{i_k}) \in \mathbb{N}_0^I$, $k - p = m_{i i_k}^Y$, and then $\mathrm{id}_Y s_{j_{p+1}} \cdots s_{j_k} = \mathrm{id}_Y s_{j_{p+2}} \cdots s_{j_{k-1}} s_{j_k} s_{j_{k-1}}$ in $\mathrm{Cox}(\mathcal{G})$. This implies the claim.

(2) Assume that $(i_1, \dots, i_k)$ is not X-reduced. Let $0 \le l < k$ such that $(i_1, \dots, i_l)$ is X-reduced and $\mathrm{id}_X s_{i_1} \cdots s_{i_l}(\alpha_{i_{l+1}}) \in -\mathbb{N}_0^I$. By (1), there exists a sequence $(j_1, \dots, j_l) \in I^l$ such that $j_l = i_{l+1}$ and $\mathrm{id}_X s_{i_1} \cdots s_{i_l} = \mathrm{id}_X s_{j_1} \cdots s_{j_l}$ in $\mathrm{Cox}(\mathcal{G})$. Then, by Lemma 9.4.5(4), $k - \ell(\mathrm{id}_X s_{i_1} \cdots s_{i_k})$ is a positive even integer from which one concludes the claim. $\square$

THEOREM 9.4.8. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph. Then the functor $W : \mathrm{Cox}(\mathcal{G}) \to \mathcal{W}(\mathcal{G})$ sending X to X, and s_i^X to s_i^X for all $X \in \mathcal{X}$ and $i \in I$, is an equivalence of categories.*

PROOF. By Corollary 9.2.22, $\mathrm{id}_X (s_i s_j)^{m_{ij}^X} = \mathrm{id}_X$ in $\mathcal{W}(\mathcal{G})$ for all $X \in \mathcal{X}$ and $i, j \in I$ with $m_{ij}^X < \infty$. Hence W is a well-defined functor.

Next we prove that

$$\ell(W(w)) = \ell(w) \text{ for any morphism } w \text{ in } \mathrm{Cox}(\mathcal{G}). \tag{9.4.1}$$

Let $X \in \mathcal{G}$, $l \ge 0$, $i_1, \dots, i_l \in I$, and $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$ in $\mathrm{Cox}(\mathcal{G})$. Assume that $\ell(w) = l$. Then $\ell(W(w)) \le l$. Moreover, for any $2 \le n \le l$ there is no $(j_1, \dots, j_{n-1}) \in I^{n-1}$ such that $\mathrm{id}_X s_{i_1} \cdots s_{i_{n-1}} = \mathrm{id}_X s_{j_1} \cdots s_{j_{n-1}}$ in $\mathrm{Cox}(\mathcal{G})$ and $j_{n-1} = i_n$. Therefore $\Lambda^X(i_1, \dots, i_l)$ consists of positive roots by Theorem 9.4.7, and hence $\ell(W(w)) = N(W(w)) = |\Lambda^X(i_1, \dots, i_l)| = l$ by Theorem 9.3.5(2) and Lemma 9.2.7.

By definition of the morphisms, W is surjective on the set of morphisms. To prove injectivity on the morphisms, let $X, Y \in \mathcal{G}$, and let $v, w : X \to Y$ be morphisms in $\mathrm{Cox}(\mathcal{G})$ with $W(v) = W(w)$. Then $W(w^{-1}v) = \mathrm{id}_X$. Hence $0 = \ell(\mathrm{id}_X) = \ell(W(w^{-1}v)) = \ell(w^{-1}v)$ by (9.4.1). Thus $w^{-1}v = \mathrm{id}_X$. $\square$

One application of Theorem 9.4.8 is the description of parabolic subgroupoids.

DEFINITION 9.4.9. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph and let $J \subseteq I$ be a non-empty subset. The quadruple

$$\mathcal{G}|J = \mathcal{G}(J, \mathcal{X}, r|(J \times \mathcal{X}), A|(J \times J \times \mathcal{X}))$$

is called the **restriction of** $\mathcal{G}$ **to** J.

LEMMA 9.4.10. *Any restriction of a semi-Cartan graph is a semi-Cartan graph. Any restriction of a Cartan graph is a Cartan graph.*

PROOF. The first claim is obvious. The second follows from Corollary 9.2.20, since for any point X, sequences of labels are X-reduced in the restriction if and only if they are X-reduced in the semi-Cartan graph. $\square$

REMARK 9.4.11. A restriction of a connected semi-Cartan graph is not necessarily connected. For example, $\mathcal{G}|\{1\}$ in Example 9.1.2 is connected, but $\mathcal{G}|\{2\}$ is not.

COROLLARY 9.4.12. *Let $\mathcal{G}$ be a Cartan graph with set I of labels, and let $J \subseteq I$ be a non-empty subset. Then there is a unique faithful functor*

$$\mathcal{W}(\mathcal{G}|J) \to \mathcal{W}(\mathcal{G})$$

which is the identity on the objects and sends each morphism s_j^X to s_j^X for all labels $j \in J$ and all points X of $\mathcal{G}$.

PROOF. Let X be a point of $\mathcal{G}$. Recall that $m_{ij}^X = \overline{m}_{ij}^X$ for any $i, j \in J$ by Corollary 9.2.20, and $\overline{m}_{ij}^X$ is the same in $\mathcal{G}$ and in $\mathcal{G}|J$. Thus, Theorem 9.4.8 implies that there is a unique functor $F_J : \mathcal{W}(\mathcal{G}|J) \to \mathcal{W}(\mathcal{G})$ which is the identity on the objects and sends any morphism s_i^Y, where $i \in J$ and Y is a point of $\mathcal{G}$, to s_i^Y.

Let X, Y be points of $\mathcal{G}$ and let $w, w' \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G}|J)}(X, Y)$. Assume that $F_J(w) = F_J(w')$. Then $F(w) = F(w')$ for all $j \in J$ and hence $w = w'$ in $\mathrm{Hom}_{\mathcal{W}(\mathcal{G}|J)}(X, Y)$ by Corollary 9.3.8(3). Thus F is faithful. $\square$

DEFINITION 9.4.13. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph, and let $J \subseteq I$ be a non-empty subset. Let $\mathcal{W}_J(\mathcal{G})$ be the subcategory of $\mathcal{W}(\mathcal{G})$ with objects the elements of $\mathcal{X}$ and with morphisms $s_{i_1} \cdots s_{i_k}^X \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(X, Y)$, where $k \in \mathbb{N}_0$, $i_1, \dots, i_k \in J$, and $X, Y \in \mathcal{X}$ such that $r_{i_1} \cdots r_{i_k}(X) = Y$. Then $\mathcal{W}_J(\mathcal{G})$ is a groupoid and is called a **parabolic subgroupoid of** $\mathcal{W}(\mathcal{G})$.

PROPOSITION 9.4.14. *Let $\mathcal{G}$ be a Cartan graph, and let $J \subseteq I$ be a non-empty subset of the set I of labels of $\mathcal{G}$. Then there is an equivalence of categories $\mathcal{W}(\mathcal{G}|J) \to \mathcal{W}_J(\mathcal{G})$ which is the identity on the points of $\mathcal{G}$ and sends s_j^X to s_j^X for all $j \in J$ and all points X of $\mathcal{G}$.*

PROOF. The functor in Corollary 9.4.12 is faithful and has its image in $\mathcal{W}_J(\mathcal{G})$. It is full by definition of $\mathcal{W}_J(\mathcal{G})$. This implies the claim. $\square$

The length function of a Cartan graph and on a parabolic subgroupoid coincide.

PROPOSITION 9.4.15. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph and $J \subseteq I$. Then any reduced decomposition of a morphism in $\mathcal{W}_J(\mathcal{G})$ is in $\mathcal{W}_J(\mathcal{G})$. In particular, any morphism $w \in \mathcal{W}_J(\mathcal{G})$ can be written as a product of $\ell(w)$ simple reflections s_j, $j \in J$.*

PROOF. Let $X \in \mathcal{X}$ and $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$ with $l \in \mathbb{N}_0$ and $i_1, \dots, i_l \in I$ be a reduced decomposition of a morphism $w \in \mathcal{W}_J(\mathcal{G})$. Assume to the contrary that $\{i_1, \dots, i_l\} \not\subseteq J$. Let $1 \le k \le l$ be minimal with $i_k \notin J$. Then

$$\alpha = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \boldsymbol{\Delta}^{X\,\mathrm{re}}(w)$$

by Theorem 9.3.5(2). Moreover, $\alpha \in \alpha_{i_k} + \sum_{j\in J} \mathbb{N}_0\alpha_j$ by the choice of k. Therefore

$$w^{-1}(\alpha) \in \alpha + \sum_{j\in J} \mathbb{Z}\alpha_j = \alpha_{i_k} + \sum_{j\in J} \mathbb{Z}\alpha_j$$

since w is a morphism in $\mathcal{W}_J(\mathcal{G})$. Thus $w^{-1}(\alpha) \in \mathbb{N}_0^I$, a contradiction. □

Corollary 9.4.16. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $J \subseteq I$, $X \in \mathcal{X}$, and $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$. If $w(\alpha_j) \in \mathbb{N}_0^I$ for all $j \in J$, then $\ell(wv) = \ell(w) + \ell(v)$ for all $v \in \mathcal{W}_J(\mathcal{G})$.*

Proof. Let $v = s_{i_1} \cdots s_{i_l}$ with $l = \ell(v)$. Then $i_1, \dots, i_l \in J$ by Proposition 9.4.15. Since $w(\alpha_j) \in \mathbb{N}_0^I$ for all $j \in J$, we conclude that

$$ws_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \mathbb{N}_0^I$$

for all $1 \le k \le l$ by Theorem 9.3.5(2). Thus $\ell(wv) = \ell(w) + l = \ell(w) + \ell(v)$ by Corollary 9.3.6. □

Corollary 9.4.17. *(Kostant's decomposition) Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $X \in \mathcal{X}$, $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$, and $J \subseteq I$. Then there exist uniquely determined $Y \in \mathcal{X}$, $u \in \mathrm{Hom}(Y, X)$, and $v \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), Y)$ such that $w = uv$, $\ell(w) = \ell(u) + \ell(v)$, $v \in \mathcal{W}_J(\mathcal{G})$, and $u(\alpha_j) \in \mathbb{N}_0^I$ for all $j \in J$. Moreover, $w = u'v'$ with $\ell(w) = \ell(u') + \ell(v')$ and $v' \in \mathcal{W}_J(\mathcal{G})$ implies that $\ell(u) \le \ell(u')$.*

Proof. We prove first the existence. Let M denote the set of all pairs (u', v') of morphisms in $\mathcal{W}(\mathcal{G})$ such that $w = u'v'$, $\ell(w) = \ell(u') + \ell(v')$, and $v' \in \mathcal{W}_J(\mathcal{G})$. Clearly, $(w, \mathrm{id}) \in M$. Let $(u, v) \in M$ be such that $\ell(u) \le \ell(u')$ for all $(u', v') \in M$. Then $u(\alpha_j) \in \mathbb{N}_0^I$ for all $j \in J$. Indeed, assume that $u(\alpha_j) \in -\mathbb{N}_0^I$ for some $j \in J$. Then $w = (us_j)(s_jv)$ and $\ell(us_j) = \ell(u) - 1$ by Corollary 9.3.6. Thus $(us_j, s_jv) \in M$, a contradiction to the choice of (u, v).

The last claim of the Corollary follows by definition of (u, v).

Let now $(u_1, v_1) \in M$ with $u_1(\alpha_j) \in \mathbb{N}_0^I$ for all $j \in J$. Then $\ell(u) \le \ell(u_1)$ and $\ell(u_1) \le \ell(u)$ by the last claim of the Corollary for (u, v) and (u_1, v_1), respectively. Hence $\ell(u) = \ell(u_1)$ and $u = wv^{-1} = u_1(v_1v^{-1})$. Since

$$\ell(u) = \ell(u_1(v_1v^{-1})) = \ell(u_1) + \ell(v_1v^{-1})$$

by Corollary 9.4.16, we conclude that $v_1v^{-1} = \mathrm{id}$ and hence $v_1 = v$, $u_1 = u$. □

The next Proposition is a result about real roots that are spanned by a subset of the simple roots.

Proposition 9.4.18. *Let $X \in \mathcal{X}$, $\emptyset \ne J \subseteq I$, and assume Axiom (CG3) in the connected component of X. If $\alpha \in \boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \sum_{j\in J} \mathbb{N}_0\alpha_j$, then there exist $k \in \mathbb{N}_0$, $i_1, \dots, i_k, l \in J$ such that $\alpha = \mathrm{id}_X s_{i_1} \cdots s_{i_k}(\alpha_l)$.*

Proof. It is enough to prove the following claim, where $\mathcal{Y}$ is the connected component of X.

(∗) Let $Y \in \mathcal{Y}$, $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), Y)$, $\alpha \in \boldsymbol{\Delta}^{Y\,\mathrm{re}}(w) \cap \sum_{j\in J} \mathbb{N}_0\alpha_j$. Then there exist $k \in \mathbb{N}_0$, $i_1, \dots, i_k, l \in J$ such that $\alpha = \mathrm{id}_Y s_{i_1} \cdots s_{i_k}(\alpha_l)$.

Indeed, if $\alpha \in \boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \sum_{j\in J} \mathbb{N}_0\alpha_j$, then there are $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ and $i \in I$ such that $w(\alpha_i) = \alpha$, hence $\alpha \in \boldsymbol{\Delta}^{X\,\mathrm{re}}(ws_i)$. Thus (∗) implies the claim of the Proposition.

Note that by the assumption in $(*)$, $w^{-1}(\alpha) \in -\mathbb{N}_0^I$, and $\alpha \in \sum_{j\in J}\mathbb{N}_0\alpha_j$. Hence there exists $n \in J$ such that $w^{-1}(\alpha_n) \in -\mathbb{N}_0^I$, and $\alpha_n \in \mathbf{\Delta}^{Y\,\mathrm{re}}(w)$. If $\alpha = \alpha_n$, then the claim in $(*)$ is obvious with $k = 0$ and $l = n$.

We prove $(*)$ by induction on $N(w) \geq 1$ which by Lemma 9.1.21(3) is a natural number. Assume that $N(w) = 1$. Then $\alpha = \alpha_n$, and we are done. Assume that $\alpha \neq \alpha_n$. Then we know from Lemma 9.1.19(1) that $s_n^Y(\alpha) \in \mathbf{\Delta}_+^{r_n(Y)\,\mathrm{re}}$. Since $(s_n w)^{-1}(s_n^Y(\alpha)) = w^{-1}(\alpha) \in -\mathbb{N}_0^I$, it follows that

$$s_n^Y(\alpha) \in \mathbf{\Delta}^{r_n(Y)\,\mathrm{re}}(s_n w) \cap \sum_{j\in J}\mathbb{N}_0\alpha_j.$$

On the other hand, $w^{-1}(\alpha_n) \in -\mathbb{N}_0^I$. Thus $N(s_n w) = N(w^{-1}s_n) = N(w) - 1$ by Lemma 9.1.21. By induction hypothesis there exist $k \geq 1$ and $i_2, \dots, i_k, l \in J$ such that $s_n^Y(\alpha) = \mathrm{id}_{r_n(Y)} s_{i_2}\cdots s_{i_k}(\alpha_l)$. Then $\alpha = \mathrm{id}_Y s_n s_{i_2}\cdots s_{i_k}(\alpha_l)$. ☐

Corollary 9.4.19. *Let $\mathcal{G}$ be a Cartan graph, let I be its set of labels, and let $J \subseteq I$ be a non-empty subset. Then for any point X of $\mathcal{G}$, the set $\mathbf{\Delta}^{X\,\mathrm{re}} \cap \sum_{j\in J}\mathbb{Z}\alpha_j$ is the set of real roots of the restriction $\mathcal{G}|J$ at X.*

Proof. Let X be a point of $\mathcal{G}$. A real root of $\mathcal{G}|J$ at X is a root of the form $\mathrm{id}_X s_{i_1}\cdots s_{i_k}(\alpha_j)$, where $k \in \mathbb{N}_0$ and $i_1, \dots, i_k, j \in J$. Since the entries of the Cartan matrices of the restriction come from the entries of the Cartan matrices of $\mathcal{G}$, these roots are indeed in $\mathbf{\Delta}^{X\,\mathrm{re}} \cap \sum_{j\in J}\mathbb{Z}\alpha_j$. Conversely, any root in the intersection $\mathbf{\Delta}^{X\,\mathrm{re}} \cap \sum_{j\in J}\mathbb{Z}\alpha_j$ is of the form $\mathrm{id}_X s_{i_1}\cdots s_{i_k}(\alpha_l)$, where $k \in \mathbb{N}_0$ and $i_1, \dots, i_k, l \in J$, by Proposition 9.4.18. ☐

9.5. Notes

Semi-Cartan graphs and attached sets of roots and Weyl groupoid appeared axiomatically first in [**HY08**]. In particular, variants of Theorem 9.3.5(2) and Proposition 9.3.9 have been proved there, and that the Coxeter relations hold in the Weyl groupoid of a Cartan graph, which is the essential part of Theorem 9.4.8.

A more structured approach was presented in [**CH09b**], where a semi-Cartan graph was called a Cartan scheme. There and in forthcoming papers, semi-Cartan graphs and Weyl groupoids were studied together with a root system, see the next Chapter for this notion.

The definition of X-reduced sequences and of a Cartan graph in the presented form, in particular, Axioms (CG3') and (CG4') as well as Theorem 9.2.18 and Corollary 9.2.20, are new.

The notion of a standard semi-Cartan graph originates from [**AHS10**], Definition 3.23 and was introduced in the combinatorial context in [**CH09b**], Definition 3.1.

Depending on emphasis, taste and intended applications and interpretations, (finite) Cartan graphs and their sets of real roots have several very different presentations in the literature. In one of these approaches, finite simply connected Cartan graphs are identified with crystallographic simplicial arrangements in [**Cun11**]. Rather differently, in [**Yam16**] and in [**BY18**], Section 5, a definition of a generalized root system is given using the notion of a base.

CHAPTER 10

The structure of Cartan graphs and root systems

Similarly to Coxeter groups, Cartan graphs have a very rich structure and can be studied from different perspectives. In this Chapter, we first point out some topological aspects in the theory of coverings and decompositions. Then we prove that finite Cartan graphs have a point with a Cartan matrix of finite type. We also classify finite Cartan graphs of rank two in terms of quiddity cycles, and study root systems of Cartan graphs emphasizing finite root systems.

10.1. Coverings and decompositions of Cartan graphs

Semi-Cartan graphs behave in some sense like a topological space with additional structure.

Definition 10.1.1. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ and $\mathcal{G}' = \mathcal{G}(I, \mathcal{Y}, t, B)$ be semi-Cartan graphs. Let $\pi : \mathcal{Y} \to \mathcal{X}$ be a map. We say that the triple $(\mathcal{G}', \mathcal{G}, \pi)$ is a **covering** of semi-Cartan graphs, $\mathcal{G}'$ **is a covering of** $\mathcal{G}$ and that π is a **covering map** if π is surjective and if $(\mathrm{id}, \pi) : \mathcal{G}' \to \mathcal{G}$ is a morphism, that is,

$$\pi(t_i(Y)) = r_i(\pi(Y)), \;\; b^Y_{ij} = a^{\pi(Y)}_{ij}$$

for all $i, j \in I$, $Y \in \mathcal{Y}$. We then also say that $\mathcal{G}$ is a **quotient semi-Cartan graph of** $\mathcal{G}'$.

Note that the surjectivity assumption in Definition 10.1.1 is superfluous if $\mathcal{G}$ is connected.

Remark 10.1.2. (1) Semi-Cartan graphs (as objects) and coverings (as morphisms) form a category.

(2) Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ and $\mathcal{G}' = \mathcal{G}(I, \mathcal{Y}, t, B)$ be semi-Cartan graphs and let $(\mathcal{G}', \mathcal{G}, \pi)$ be a covering. Then there is a unique covariant functor

$$F_\pi : \mathcal{W}(\mathcal{G}') \to \mathcal{W}(\mathcal{G})$$

sending any object $Y \in \mathcal{Y}$ to $\pi(Y)$ and any morphism s^Y_i to $s^{\pi(Y)}_i$, where $i \in I$. The assumption $B^Y = A^{\pi(Y)}$ for all $Y \in \mathcal{Y}$ implies that $w(\alpha_i) = F_\pi(w)(\alpha_i)$ for all $i \in I$, $w \in \mathrm{Hom}(Y, Z)$, and $Y, Z \in \mathcal{Y}$.

Proposition 10.1.3. *Let $\mathcal{G}' = \mathcal{G}(I, \mathcal{Y}, t, B)$ be a semi-Cartan graph. Let $\sim$ be an equivalence relation on $\mathcal{Y}$. Assume that $t_i(X) \sim t_i(Y)$ and that $B^X = B^Y$ for all $i \in I$ and $X, Y \in \mathcal{Y}$ with $X \sim Y$. Let $\mathcal{X}$ be the set of equivalence classes*

$$[X] = \{Y \in \mathcal{Y} \,|\, Y \sim X\}$$

and let $A^{[X]} = B^X$ for all $X \in \mathcal{Y}$. Let $r : I \times \mathcal{X} \to \mathcal{X}$, $r(i, [X]) = [t_i(X)]$, and $A : I \times I \times \mathcal{X} \to \mathbb{Z}$, $A(i, j, [X]) = B(i, j, X)$. Then $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is a semi-Cartan graph and $(\mathcal{G}', \mathcal{G}, \pi)$, where $\pi : \mathcal{Y} \to \mathcal{X}$, $X \mapsto [X]$, is a covering.

PROOF. It is clear that the map r and the matrices $A^{[X]}$ for all $X \in \mathcal{Y}$ are well-defined. Then the claim follows directly from the axioms of a semi-Cartan graph and a covering. □

LEMMA 10.1.4. *Let $(\mathcal{G}', \mathcal{G}, \pi)$ be a covering. Then $\mathbf{\Delta}^{X\,\mathrm{re}} = \mathbf{\Delta}^{\pi(X)\,\mathrm{re}}$ for all points X of $\mathcal{G}'$. In particular,*

(1) *$\mathcal{G}'$ is finite if and only if $\mathcal{G}$ is finite, and*
(2) *if $\mathcal{G}'$ is a Cartan graph, then $\mathcal{G}$ is a Cartan graph.*

PROOF. By Remark 10.1.2(2), $w(\alpha_i) = F_\pi(w)(\alpha_i)$ for all points X, Y of $\mathcal{G}'$, all labels i and all $w \in \mathrm{Hom}(Y, X)$. Hence $\mathbf{\Delta}^{\pi(X)\,\mathrm{re}} = \mathbf{\Delta}^{X\,\mathrm{re}}$ for all points X of $\mathcal{G}'$. Then (1) is clear and (2) follows from Axioms (CG3) and (CG4). □

Coverings of connected semi-Cartan graphs can be expressed in terms of automorphism groups of points.

PROPOSITION 10.1.5. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ and $\mathcal{G}' = \mathcal{G}(I, \mathcal{Y}, t, B)$ be connected semi-Cartan graphs and $\pi : \mathcal{Y} \to \mathcal{X}$ a map such that $(\mathcal{G}', \mathcal{G}, \pi)$ is a covering. Let $Y \in \mathcal{Y}$ and $X = \pi(Y)$.*

(1) *The map $F_\pi : \mathrm{Aut}(Y) \to \mathrm{Aut}(X)$ is injective.*
(2) *The map $F_\pi : \mathrm{Hom}(\mathcal{W}(\mathcal{G}'), Y) \to \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ is surjective.*
(3) *For any $Z \in \mathcal{Y}$ with $\pi(Z) = X$ the subgroups $F_\pi(\mathrm{Aut}(Y))$ and $F_\pi(\mathrm{Aut}(Z))$ of $\mathrm{Aut}(X)$ are conjugate.*
(4) *Let U be a subgroup of $\mathrm{Aut}(X)$ conjugate to $F_\pi(\mathrm{Aut}(Y))$. Then there exists $Z \in \mathcal{Y}$ such that $\pi(Z) = X$ and $U = F_\pi(\mathrm{Aut}(Z))$.*

PROOF. (1) Let $k \in \mathbb{N}$ and $i_1, \dots, i_k \in I$. Then $s_{i_1} \cdots s_{i_k}^Y(\alpha_i) = s_{i_1} \cdots s_{i_k}^X(\alpha_i)$ for all $i \in I$ by Remark 10.1.2(2). This implies the claim.

(2) Let $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$. Then $\mathrm{id}_X s_{i_1} \cdots s_{i_k} = F_\pi(\mathrm{id}_Y s_{i_1} \cdots s_{i_k})$ by Remark 10.1.2(2).

(3) Since $\mathcal{G}'$ is connected, there exists a morphism $w \in \mathrm{Hom}(Y, Z)$. Then $\mathrm{Aut}(Z) = w\mathrm{Aut}(Y)w^{-1}$, $F_\pi(w) \in \mathrm{Aut}(X)$, and

$$F_\pi(\mathrm{Aut}(Z)) = F_\pi(w)F_\pi(\mathrm{Aut}(Y))F_\pi(w^{-1}).$$

(4) Let $w \in \mathrm{Aut}(X)$ such that $wF_\pi(\mathrm{Aut}(Y))w^{-1} = U$. By (2) there exist $Z \in \mathcal{Y}$ and $w' \in \mathrm{Hom}(Y, Z)$ such that $F_\pi(w') = w$. Then

$$U = F_\pi(w'\mathrm{Aut}(Y)w'^{-1}) = F_\pi(\mathrm{Aut}(Z))$$

which proves the claim. □

In the next Proposition we discuss the construction of a covering of a connected semi-Cartan graph corresponding to a subgroup of the automorphism group of a point. For a special case of the construction we refer to Example 9.1.25.

PROPOSITION 10.1.6. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a connected semi-Cartan graph, $X \in \mathcal{X}$, and $U \subseteq \mathrm{Aut}(X)$ a subgroup.*

(1) *There exists a covering $(\mathcal{G}', \mathcal{G}, \pi)$ and a point Y of $\mathcal{G}'$ such that $\mathcal{G}'$ is connected, $\pi(Y) = X$, and*

$$F_\pi(\mathrm{Aut}(Y)) = U, \quad |\pi^{-1}(X)| = [\mathrm{Aut}(X) : U].$$

(2) *Assume that $\mathcal{G}$ is a Cartan graph. Then there exists a covering $(\mathcal{G}', \mathcal{G}, \pi)$ and a point Y of $\mathcal{G}'$ such that $\mathcal{G}'$ is a connected Cartan graph satisfying $\pi(Y) = X$, and $F_\pi(\mathrm{Aut}(Y)) = U$. Moreover, any two such coverings of $\mathcal{G}$ are isomorphic, and $|\pi^{-1}(X)| = [\mathrm{Aut}(X) : U]$.*

Proof. (1) We construct explicitly a covering of $\mathcal{G}$. Let

$$\mathcal{Y} = \{wU \mid w \in \mathrm{Hom}(X, X'), X' \in \mathcal{X}\}$$

be the set of left U-cosets. For all $i \in I$ let

$$t_i : \mathcal{Y} \to \mathcal{Y}, \quad wU \mapsto s_i wU$$

for all $w \in \mathrm{Hom}(X, X')$, $X' \in \mathcal{X}$. For all $wU \in \mathcal{Y}$, where $w \in \mathrm{Hom}(X, X')$, let $B^{wU} = A^{X'}$. Then $t_i^2 = \mathrm{id}_{\mathcal{Y}}$ since $s_i s_i^{X'} = \mathrm{id}_{X'}$ for all $X' \in \mathcal{X}$, and

$$B(i, j, wU) = A(i, j, X') = A(i, j, r_i(X')) = B(i, j, s_i wU)$$

for all $i, j \in I$, $w \in \mathrm{Hom}(X, X')$, $X' \in \mathcal{X}$. Thus $\mathcal{G}'$ is a connected semi-Cartan graph, since $\mathcal{G}$ is connected. The triple $(\mathcal{G}', \mathcal{G}, \pi)$ with $\pi : \mathcal{G}' \to \mathcal{G}$, $wU \mapsto X'$ for all $w \in \mathrm{Hom}(X, X')$, $X' \in \mathcal{X}$, is a covering. The automorphism group of $U \in \mathcal{Y}$ is isomorphic to U via F_π, and

$$|\pi^{-1}(X)| = |\{wU \mid w \in \mathrm{Aut}(X)\}| = [\mathrm{Aut}(X) : U].$$

(2) Let $\mathcal{G}'$ be the semi-Cartan graph constructed in (1). Let $i, j \in I$, $Z \in \mathcal{X}$, and $w \in \mathrm{Hom}(X, Z)$. Then $\boldsymbol{\Delta}^{Z\,\mathrm{re}} = \boldsymbol{\Delta}^{wU\,\mathrm{re}}$. Hence $m_{ij}^Z = m_{ij}^{wU}$ and $\boldsymbol{\Delta}^{wU\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$. Assume that m_{ij}^Z is finite. Then $\mathrm{id}_Z(s_i s_j)^{m_{ij}^Z} = \mathrm{id}_Z$ by Theorem 9.2.23 and by (CG4) for $\mathcal{G}$. Hence $(t_i t_j)^{m_{ij}^{wU}}(wU) = wU$. Therefore $\mathcal{G}'$ is a Cartan graph.

The uniqueness of $\mathcal{G}'$ follows from the fact that for any two coverings $(\mathcal{G}', \mathcal{G}, \pi')$ and $(\mathcal{G}'', \mathcal{G}, \pi'')$ with the required properties and any $Y \in \pi'^{-1}(X)$, $Z \in \pi''^{-1}(X)$ there is a unique isomorphism between $\mathcal{G}'$ and $\mathcal{G}''$ which is the identity on I and maps Y to Z. □

An important consequence of the proposition is the following.

Corollary 10.1.7. *Let $\mathcal{G}$ be a Cartan graph.*

(1) *There exists a covering $(\mathcal{G}', \mathcal{G}, \pi)$ such that $\mathcal{G}'$ is a simply connected Cartan graph.*
(2) *Let $(\mathcal{G}', \mathcal{G}, \pi)$ and $(\mathcal{G}'', \mathcal{G}, \pi'')$ be coverings such that $\mathcal{G}', \mathcal{G}''$ are Cartan graphs and $\mathcal{G}'$ is simply connected. Then there is a covering $(\mathcal{G}', \mathcal{G}'', \pi')$.*
(3) *Any two simply connected Cartan graph coverings of $\mathcal{G}$ are isomorphic.*

Proof. (1), (3) Apply Proposition 10.1.6 to all connected components of $\mathcal{G}$ by letting U be the trivial group.

(2) By (1), there is a covering $(\mathcal{G}''', \mathcal{G}'', \pi''')$ of $\mathcal{G}''$ such that $\mathcal{G}'''$ is simply connected. Then $(\mathcal{G}''', \mathcal{G}, \pi''\pi''')$ is a covering. By (3), $\mathcal{G}'''$ and $\mathcal{G}'$ are isomorphic. This implies the claim. □

We also identify another important class of semi-Cartan graphs.

Definition 10.1.8. A semi-Cartan graph $\mathcal{G}$ is called **incontractible**, if any covering $(\mathcal{G}, \mathcal{G}', \pi)$ is an isomorphism.

Lemma 10.1.9. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph and let $X \in \mathcal{X}$ and $i, j \in I$. If $i \neq j$ then*

$$-a_{ij}^X = \max\{m \in \mathbb{N}_0 \mid \alpha_j + m\alpha_i \in \boldsymbol{\Delta}^{X\,\mathrm{re}}\}.$$

PROOF. Assume that $i \neq j$. Then $a_{ij}^X = a_{ij}^{r_i(X)}$ and

$$\alpha_j - a_{ij}^X \alpha_i = s_i^{r_i(X)}(\alpha_j) \in \mathbf{\Delta}^{X\,\mathrm{re}}.$$

On the other hand, if $\alpha_j + m\alpha_i \in \mathbf{\Delta}^{X\,\mathrm{re}}$, then

$$s_i^X(\alpha_j + m\alpha_i) = \alpha_j + (-a_{ij}^X - m)\alpha_i \in \mathbf{\Delta}^{r_i(X)\,\mathrm{re}}.$$

Hence $m \le -a_{ij}^X$ by (CG3). $\square$

COROLLARY 10.1.10. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph. For all $X \in \mathcal{X}$ let $[X] = \{Y \in \mathcal{X} \mid \mathbf{\Delta}^{Y\,\mathrm{re}} = \mathbf{\Delta}^{X\,\mathrm{re}}\}$ and let $\mathcal{Y} = \{[X] \mid X \in \mathcal{X}\}$. Then $t : I \times \mathcal{Y} \to \mathcal{Y}$, $(i, [X]) \mapsto [r_i(X)]$, and $B : I \times I \times \mathcal{Y} \to \mathbb{Z}$, $(i, j, [X]) \mapsto a_{ij}^X$, are well-defined.*

(1) *The quadruple $\mathcal{G}' = \mathcal{G}(I, \mathcal{Y}, t, B)$ is an incontractible Cartan graph. Let $\pi : \mathcal{X} \to \mathcal{Y}$, $\pi(X) = [X]$. Then $(\mathcal{G}, \mathcal{G}', \pi)$ is a covering.*

(2) *Let $(\mathcal{G}, \mathcal{G}'', \pi'')$ be a covering. Then there is a covering $(\mathcal{G}'', \mathcal{G}', \pi')$.*

(3) *The Cartan graph $\mathcal{G}'$ is up to isomorphism the unique incontractible Cartan graph $\tilde{\mathcal{G}}$ which admits a covering $(\mathcal{G}, \tilde{\mathcal{G}}, \tilde{\pi})$.*

PROOF. The map t is well-defined if $\mathbf{\Delta}^{Y\,\mathrm{re}} = \mathbf{\Delta}^{X\,\mathrm{re}}$ for $X, Y \in \mathcal{X}$ implies that $\mathbf{\Delta}^{r_i(Y)\,\mathrm{re}} = \mathbf{\Delta}^{r_i(X)\,\mathrm{re}}$ for all $i \in I$. The latter holds since $A^Y = A^X$ by Lemma 10.1.9. By the same reason, B is well-defined.

(1) It is clear that $\mathcal{G}'$ is a Cartan graph and that π is a covering map. Let $(\mathcal{G}', \mathcal{G}'', \pi'')$ be a covering. Then for any $X \in \mathcal{X}$ the sets $\mathbf{\Delta}^{X\,\mathrm{re}} = \mathbf{\Delta}^{[X]\,\mathrm{re}}$ and $\mathbf{\Delta}^{\pi''[X]\,\mathrm{re}}$ coincide by Lemma 10.1.4, and hence π'' is injective. Thus π'' is an isomorphism and $\mathcal{G}'$ is incontractible.

(2) Let $\pi'(\pi''(X)) = \pi(X)$ for all $X \in \mathcal{X}$. This is well-defined, since by Lemma 10.1.4, $\mathbf{\Delta}^{\pi''(X)\,\mathrm{re}} = \mathbf{\Delta}^{X\,\mathrm{re}}$. The rest is clear.

(3) follows from (2). $\square$

EXAMPLE 10.1.11. A semi-Cartan graph is standard and incontractible if and only if it has precisely one point.

EXAMPLE 10.1.12. Let $\mathcal{G}$ be the Cartan graph in Example 9.1.15. Then $\mathcal{G}$ is incontractible, since $A^{X_1} \neq A^{X_2}$. The exchange graph of the unique simply connected covering of $\mathcal{G}$ is a cycle with 16 vertices. In general, the exchange graph of a finite connected simply connected Cartan graph of rank two is a cycle with as many vertices as the cardinality of the set of roots at a point. The latter is false for Cartan graphs of higher rank.

We turn our attention to products and decompositions of semi-Cartan graphs.

DEFINITION 10.1.13. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$, $\mathcal{G}' = \mathcal{G}(J, \mathcal{Y}, t, B)$ be semi-Cartan graphs. Assume that I and J are disjoint sets. The **product semi-Cartan graph** $\mathcal{G} \times \mathcal{G}'$ is the quadruple

$$\mathcal{G}(I \cup J, \mathcal{X} \times \mathcal{Y}, q = r \times t, C),$$

where

$$q_i(X, Y) = (r_i(X), Y),\ q_j(X, Y) = (X, t_j(Y))$$

for all $i \in I$, $j \in J$, $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and

$$c_{kl}^{(X,Y)} = \begin{cases} a_{kl}^X & \text{if } k, l \in I, \\ b_{kl}^Y & \text{if } k, l \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 10.1.14. *The product of two Cartan graphs is a Cartan graph.*

Proof. Let $\mathcal{G}_1 = \mathcal{G}(I_1, \mathcal{X}_1, r, A)$, $\mathcal{G}_2 = \mathcal{G}(I_2, \mathcal{X}_2, t, B)$ be Cartan graphs, where I_1 and I_2 are disjoint sets. We prove (CG3) and (CG4) for $\mathcal{G}_1 \times \mathcal{G}_2$. Let $I = I_1 \cup I_2$. By definition, $\boldsymbol{\Delta}^{(X,Y)\,\mathrm{re}} \subseteq \mathbb{Z}^I$ for all $X \in \mathcal{X}_1$, $Y \in \mathcal{X}_2$. Regard $\mathbb{Z}^{I_1}$ and $\mathbb{Z}^{I_2}$ as subgroups of $\mathbb{Z}^I$ via the identification of $\alpha_i \in \mathbb{Z}^{I_k}$ with $\alpha_i \in \mathbb{Z}^I$ for all $i \in I_k$ and $k \in \{1,2\}$. For all $X_1 \in \mathcal{X}_1$, $X_2 \in \mathcal{X}_2$, $j \in I_k$, $j' \in I \setminus I_k$, $\alpha \in \mathbb{Z}^{I_k}$, where $k \in \{1,2\}$, the definition of $\mathcal{G}_1 \times \mathcal{G}_2$ implies $s_j^{(X_1,X_2)}(\alpha) = s_j^{X_k}(\alpha), s_{j'}^{(X_1,X_2)}(\alpha) = \alpha$. Therefore $\boldsymbol{\Delta}^{(X_1,X_2)\,\mathrm{re}} = \boldsymbol{\Delta}^{X_1\,\mathrm{re}} \cup \boldsymbol{\Delta}^{X_2\,\mathrm{re}}$, and hence $\boldsymbol{\Delta}^{(X_1,X_2)\,\mathrm{re}} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$. Further,

$$((r \times t)_k (r \times t)_l)^{m_{kl}^{(X,Y)}}(X,Y) = ((r_k r_l)^{m_{kl}^X}(X), Y) = (X,Y)$$

for all $k,l \in I_1$, $X \in \mathcal{X}_1$, $Y \in \mathcal{X}_2$, and the analogous claim holds for all $k,l \in I_2$. If $(k,l) \in I_1 \times I_2$, then $m_{kl}^{(X,Y)} = 2$ for all $X \in \mathcal{X}_1$, $Y \in \mathcal{X}_2$, and $(r \times t)_k (r \times t)_l = r_k \times t_l$, and hence $((r \times t)_k (r \times t)_l)^2 = \mathrm{id}$. Thus $\mathcal{G}_1 \times \mathcal{G}_2$ is a Cartan graph. $\square$

Definition 10.1.15. A matrix $A = (a_{ij})_{i,j \in I} \in R^{I \times I}$, where R is any ring, is called **decomposable**, if there exist disjoint non-empty subsets $I_1, I_2 \subseteq I$ such that $I_1 \cup I_2 = I$ and $a_{ij} = a_{ji} = 0$ for all $i \in I_1$, $j \in I_2$. A semi-Cartan graph $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is said to be **decomposable** if A^X is decomposable for all $X \in \mathcal{X}$. Cartan matrices and semi-Cartan graphs, which are not decomposable, are called **indecomposable**.

Remark 10.1.16. Let $A = (a_{ij})_{i,j \in I} \in R^{I \times I}$, where R is a commutative integral domain. Assume that for all $i,j \in I$, $a_{ij} = 0$ implies that $a_{ji} = 0$. Then the following can be easily checked.

(1) For all $i,j \in I$ define $i \sim j$, if $i = j$ or there are $k > 0$ and $i_1, \dots, i_k \in I$ with $a_{ii_1} a_{i_1 i_2} \cdots a_{i_k j} \neq 0$. Then $\sim$ is an equivalence relation on I.
(2) Let $I = \cup_{1 \le l \le m} I_l$ be the decomposition of I into pairwise distinct equivalence classes by $\sim$, as defined in (1). Then the matrices $(a_{ij})_{i,j \in I_l}$, $1 \le l \le m$, are indecomposable, and $a_{ij} = 0 = a_{ji}$ for any $1 \le k < l \le m$, $i \in I_k$, and $j \in I_l$.

    Suppose that $I = \cup_{1 \le q \le r} J_q$ is the union of pairwise disjoint subsets $J_q \subseteq I$, $1 \le q \le r$, such that the matrices $(a_{ij})_{i,j \in J_q}$ with $1 \le q \le r$ are indecomposable, and that $i \in J_p$, $j \in J_q$ with $1 \le p < q \le r$ implies that $a_{ij} = 0 = a_{ji}$. Then $r = m$, and there is a permutation $w \in \mathbb{S}_m$ with $J_l = I_{w(l)}$ for all $1 \le l \le m$.

In particular, Cartan matrices can be uniquely decomposed into indecomposable Cartan matrices.

Proposition 10.1.17. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a connected Cartan graph of rank at least two, let $X \in \mathcal{X}$, and let $I = I_1 \cup I_2$ be a decomposition into disjoint non-empty subsets $I_1, I_2 \subseteq I$. Then the following are equivalent.*

(1) *For any $i \in I_1$ and $j \in I_2$, $a_{ij}^X = 0 = a_{ji}^X$.*
(2) *For any $Y \in \mathcal{X}$, $a_{ij}^Y = 0 = a_{ji}^Y$ for all $i \in I_1$ and $j \in I_2$.*
(3)
$$\boldsymbol{\Delta}^{X\,\mathrm{re}} = \Big(\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \sum_{i \in I_1} \mathbb{Z}\alpha_i\Big) \cup \Big(\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \sum_{i \in I_2} \mathbb{Z}\alpha_i\Big).$$

(4) *For all* $Y \in \mathcal{X}$,

$$\boldsymbol{\Delta}^{Y\,\mathrm{re}} = \Big(\boldsymbol{\Delta}^{Y\,\mathrm{re}} \cap \sum_{i\in I_1} \mathbb{Z}\alpha_i\Big) \cup \Big(\boldsymbol{\Delta}^{Y\,\mathrm{re}} \cap \sum_{i\in I_2} \mathbb{Z}\alpha_i\Big).$$

(5) *There is a covering* $\mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}$, *where* $\mathcal{G}_l$, $l = 1, 2$, *is the connected component of* $\mathcal{G}|I_l$ *containing* X.

Proof. Assume (1). We prove (2). By Lemma 9.3.3, for all $i \in I_1$, $j \in I_2$ the j-th rows of A^X and of $A^{r_i(X)}$ coincide. In particular, for all $i, l \in I_1$ and $j \in I_2$, $0 = a^X_{jl} = a^{r_i(X)}_{jl}$, and then $a^{r_i(X)}_{lj} = 0$, since $A^{r_i(X)}$ is a Cartan matrix. By symmetry, $a^{r_i(X)}_{jl} = 0$ for all $i, l \in I_2$ and $j \in I_1$. We proved that for all $l \in I_1$, $j \in I_2$ and for all $k \in I_1 \cup I_2 = I$, $a^{r_k(X)}_{lj} = 0 = a^{r_k(X)}_{jl}$. Now (2) follows since $\mathcal{G}$ is connected.

(2) implies (4) by the definition of $\boldsymbol{\Delta}^{Y\,\mathrm{re}}$. Moreover, (4) implies (3) trivially, and (3) implies (1) by Lemma 10.1.9.

(5) implies (2) because of Definition 10.1.13.

Finally, assume (2). We prove (5). Let

$$\mathcal{X}_l = \{r_{i_1} r_{i_2} \cdots r_{i_k}(X) \mid k \geq 0, i_1, i_2, \ldots, i_k \in I_l\},$$

$l = 1, 2$. Then, by definition, $\mathcal{G}_l = \mathcal{G}(I_l, \mathcal{X}_l, r|(I_l \times \mathcal{X}_l), A|(I_l \times I_l \times \mathcal{X}_l))$ for $l = 1, 2$. We define $\pi : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}$ by

$$\pi(a(X), b(X)) = ab(X),$$

where $a = r_{i_1} r_{i_2} \cdots r_{i_p}$, $b = r_{j_1} r_{j_2} \cdots r_{j_q}$, $i_1, i_2, \ldots, i_p \in I_1$, $j_1, j_2, \ldots, j_q \in I_2$, and $p, q \geq 0$.

To see that the map π is well-defined, we first note that $m^Y_{ij} = 2$ for all $i \in I_1$, $j \in I_2$ and $Y \in \mathcal{X}$ by (2) and Lemma 9.3.1, where we used that $\mathcal{G}$ is a Cartan graph. Therefore $(r_i r_j)^2(Y) = Y$ by (CG4) for $\mathcal{G}$. Thus $r_i r_j = r_j r_i$. We conclude that $\pi(a(X), b(X)) = ba(X)$ for all a, b, and hence π is well-defined. The map π is surjective, since $\mathcal{G}$ is connected and $\pi(\mathcal{X}_1 \times \mathcal{X}_2)$ is invariant under all r_i with $i \in I$.

We now prove that $(\mathrm{id}, \pi) : \mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}$ is a morphism. Let $a = r_{i_1} r_{i_2} \cdots r_{i_p}$ and $b = r_{j_1} r_{j_2} \cdots r_{j_q}$, where $p, q \geq 0$, $i_1, i_2, \ldots, i_p \in I_1$, $j_1, j_2, \ldots, j_q \in I_2$. Then $a(X) \in \mathcal{X}_1$, $b(X) \in \mathcal{X}_2$, and for all $i \in I_1$, $j \in I_2$,

$$\pi(r_i(a(Z), b(Z))) = \pi(r_i a(Z), b(Z)) = r_i ab(Z) = r_i(\pi(a(Z), b(Z))),$$
$$\pi(r_j(a(Z), b(Z))) = \pi(a(Z), r_j b(Z)) = a r_j b(Z) = r_j(\pi(a(Z), b(Z))),$$

since r_j commutes with a.

By definition, the entries of the Cartan matrix of $\mathcal{G}_1 \times \mathcal{G}_2$ at $(a(X), b(X))$ are

$$c^{(a(X),b(X))}_{kl} = \begin{cases} a^{a(X)}_{kl} & \text{if } k, l \in I_1, \\ a^{b(X)}_{kl} & \text{if } k, l \in I_2, \\ 0 & \text{otherwise.} \end{cases}$$

Since r_i and r_j commute for all $i \in I_1$, $j \in I_2$, it follows by repeatedly applying Lemma 9.3.3 that

$$c^{(a(X),b(X))}_{kl} = a^{ab(X)}_{kl} = a^{\pi(a(X),b(X))}_{kl}$$

for all $k, l \in I = I_1 \cup I_2$. Therefore $(\mathcal{G}_1 \times \mathcal{G}_2, \mathcal{G}, \pi)$ is a covering. □

As a corollary we now obtain the decomposition of a connected Cartan graph into indecomposable components.

COROLLARY 10.1.18. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a connected Cartan graph. Then there is a unique decomposition $I = \cup_{1\le l\le m} I_l$, where $m \ge 1$ and $I_k \cap I_l = \emptyset$ for all $1 \le k < l \le m$, such that the following hold.*

(1) *For all $X \in \mathcal{X}$, the matrices $(a^X_{ij})_{i,j\in I_l}$, $1 \le l \le m$, are indecomposable, and $a^X_{ij} = 0$ for all $i \in I_k$, $j \in I_l$, $1 \le k, l \le m$, $k \ne l$.*

(2) *For all $X \in \mathcal{X}$,*

$$\boldsymbol{\Delta}^{X\,\mathrm{re}} = \bigcup_{1\le l\le m} \boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \sum_{i\in I_l} \mathbb{Z}\alpha_i.$$

(3) *Let $X \in \mathcal{X}$. There is a covering $\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_m \to \mathcal{G}$, where $\mathcal{G}_l$, $1 \le l \le m$, is the connected component of $\mathcal{G}|I_l$ containing X, and $\mathcal{G}_l$ is an indecomposable Cartan graph.*

PROOF. This follows from Remark 10.1.16 and Proposition 10.1.17. In (3) we define $\pi : \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_m \to \mathcal{X}$ by

$$\pi(a_1(X), a_2(X), \dots, a_m(X)) = a_1 a_2 \cdots a_m(X),$$

where for all $1 \le l \le m$, $a_l = r_{i_1} r_{i_2} \cdots r_{i_{p_l}}$, $i_1, i_2, \dots, i_{p_l} \in I_l$, $p_l \ge 0$. $\square$

10.2. Types of Cartan matrices

We recall the classification of certain indecomposable matrices by Vinberg into three types: finite, affine, and indefinite. We use this classification to prove that any finite Cartan graph has a point with a Cartan matrix of finite type.

For any $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$ we write $x > 0$ ($x \ge 0$) if $x_i > 0$ ($x_i \ge 0$) for all $1 \le i \le n$.

In the theory of linear programming, there exist several variants of a so called Theorem of Alternatives. One of them is Gordan's Theorem.

THEOREM 10.2.1. *Let $m \in \mathbb{N}$, V a real vector space, and $\lambda_1, \dots, \lambda_m \in V^*$. Then either there exists $v \in V$ such that $\lambda_i(v) > 0$ for all $1 \le i \le m$, or there exists $y \in \mathbb{R}^m$ such that $\sum_{i=1}^m y_i \lambda_i = 0$, $y \ge 0$, $y \ne 0$.*

PROOF. The two cases are clearly mutually exclusive. We have to show that one of the two cases holds.

We proceed by induction on m. For $m = 1$ the claim is trivial.

Assume now that $m \ge 2$, and let

$$C = \{v \in V \mid \lambda_i(v) > 0 \text{ for all } 1 \le i < m\}.$$

If $C = \emptyset$, then by induction hypothesis there exists $z \in \mathbb{R}^{m-1}$, $z \ge 0$ and $z \ne 0$, such that $\sum_{i=1}^{m-1} z_i \lambda_i = 0$. Then $y = (z_1, \dots, z_{m-1}, 0)^t$ establishes the second case of the claim for m. Therefore we may assume that $C \ne \emptyset$, and hence $V \ne 0$. Further we may assume that $\lambda_m \ne 0$, since otherwise the second case holds in the claim with $y = (0, \dots, 0, 1)^t$.

If $\lambda_m(v) > 0$ for some $v \in C$, then the first case is established. If $\lambda_m(u) = 0$ for some $u \in C$, then choose $x \in V$, $\epsilon > 0$, such that $\lambda_m(x) = 1$ and $\lambda_i(u + \epsilon x) > 0$ for all $1 \le i < m$. Then the first case of the claim holds with $v = u + \epsilon x$. So we may assume that $\lambda_m(u) < 0$ for all $u \in C$.

Let $H = \ker(\lambda_m)$. Since $C \cap H = \emptyset$, induction hypothesis implies that there exists $z \in \mathbb{R}^{m-1}$ such that $\sum_{i=1}^{m-1} z_i \lambda_i|_H = 0$, and $z \ge 0$, $z \ne 0$. Then there exists $\mu \in \mathbb{R}$ such that $\sum_{i=1}^{m-1} z_i \lambda_i = \mu \lambda_m$. Evaluation of the latter at any $u \in C$ implies that $\mu < 0$. Then $y = (z_1, \dots, z_{m-1}, -\mu)^t$ establishes the second case of the claim. This finishes the proof of the theorem. □

COROLLARY 10.2.2. *(Gordan's Theorem, 1873) Let $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$. Then either there exists $x \in \mathbb{R}^n$ such that $Ax > 0$ or there exists $y \in \mathbb{R}^m$ such that $y^t A = 0$, $y \geq 0$, $y \neq 0$.*

PROOF. Apply Theorem 10.2.1 with $V = \mathbb{R}^n$, $\lambda_i(x) = \sum_{j=1}^n a_{ij}x_j$ for all $x \in V$, $1 \leq i \leq m$. □

COROLLARY 10.2.3. *Let $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$. Then either there exists $x \in \mathbb{R}^n$ such that $Ax < 0$, $x > 0$, or there exists $y \in \mathbb{R}^m$ such that $y^t A \geq 0$, $y \geq 0$, $y \neq 0$.*

PROOF. Let $B \in \mathbb{R}^{(m+n)\times n}$ such that the first m rows of B are the rows of $-A$ and the remaining rows form the identity in $\mathbb{R}^{n\times n}$. It follows that $Bx > 0$ if and only if $Ax < 0$ and $x > 0$. By Gordan's Theorem, the alternative of this case is the existence of $z \in \mathbb{R}^{m+n}$ such that $z^t B = 0$, $z \geq 0$, $z \neq 0$. The rows $m+1, \dots, m+n$ of B are linearly independent, and hence $z_i \neq 0$ for some $1 \leq i \leq m$. Let $y = (z_1, \dots, z_m)^t$. The assumptions on z are then equivalent to $y^t A \geq 0$, $y \geq 0$, $y \neq 0$. □

The classification of Vinberg applies to a special class of matrices which we introduce now.

DEFINITION 10.2.4. Let $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$. We say that A is a **Vinberg matrix** if

(1) A is indecomposable,
(2) $a_{ij} \leq 0$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, and
(3) $i, j \in \{1, \dots, n\}$, $a_{ij} = 0$ implies that $a_{ji} = 0$.

REMARK 10.2.5. A real square matrix satisfying the second condition in Definition 10.2.4 is usually called a **Z-matrix**.

LEMMA 10.2.6. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be a Vinberg matrix. If $x \in \mathbb{R}^n$ with $Ax \geq 0$ and $x \geq 0$, then $x > 0$ or $x = 0$.*

PROOF. Assume that $Ax \geq 0$, $x \geq 0$, and $x \neq 0$. By permuting the columns and the corresponding rows of A, we may assume that there exists $1 \leq s \leq n$ such that $x_i = 0$ for $1 \leq i < s$ and $x_j > 0$ for $s \leq j \leq n$. Since $a_{ij} \leq 0$ for all $i \neq j$, $Ax \geq 0$ implies that $a_{ij} = 0$ for all $1 \leq i < s \leq j \leq n$. Since A is Vinberg, we conclude that $s = 1$. Thus $x > 0$. □

THEOREM 10.2.7. *(Vinberg) Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be a Vinberg matrix. Then precisely one of the following cases appears.*

(1) $\det(A) \neq 0$*; there exists $u \in \mathbb{R}^n$ such that $Au > 0$, $u > 0$; $v \in \mathbb{R}^n$, $Av \geq 0$ implies that $v > 0$ or $v = 0$.*
(2) $\operatorname{rk} A = n - 1$*; there exists $u \in \mathbb{R}^n$ such that $Au = 0$, $u > 0$; $v \in \mathbb{R}^n$, $Av \geq 0$ implies that $Av = 0$.*
(3) *There exists $u \in \mathbb{R}^n$ such that $Au < 0$, $u > 0$; $v \in \mathbb{R}^n$, $Av \geq 0$, $v \geq 0$ implies that $v = 0$.*

In these cases A is called of **finite**, **affine**, *and* **indefinite type**, *respectively. Moreover, A^t is of the same type as A.*

Proof. It is clear that the three cases are mutually exclusive.

Let $C = \{u \in \mathbb{R}^n \mid u \geq 0\}$ and $K_A = \{u \in \mathbb{R}^n \mid Au \geq 0\}$. We distinguish three cases.

(1) $C \cap K_A \neq \{0\}$ and A is not invertible. Since A is not invertible, there exists $x \in \mathbb{R}^n$ such that $x \not\geq 0$, $Ax = 0$. Let $y \in C \cap K_A \setminus \{0\}$. By Lemma 10.2.6, $y > 0$ and the straight line containing x and y meets the boundary of C at 0. Hence $\operatorname{rk} A = n-1$ and there exists $u > 0$ such that $Au = 0$. Let $v \in \mathbb{R}^n \setminus \{u\}$ with $Av \geq 0$. The straight line $\{v + tu \mid t \in \mathbb{R}\}$ meets the boundary of C at 0. Hence $Av = 0$. Therefore A is of affine type.

(2) $C \cap K_A \neq \{0\}$ and A is invertible. Let $v \in \mathbb{R}^n$ with $Av \geq 0$ and let $y \in C \cap K_A \setminus \{0\}$. Then $y > 0$ by Lemma 10.2.6. If $v \in C$, then $v > 0$ or $v = 0$ by the same reason. Otherwise, the half line $\{v + ty \mid t > 0\}$ meets the boundary of C at 0, and hence $Av + tAy = 0$. Then $Av = Ay = 0$ since $v, y \in K_A$, a contradiction to the invertibility of A. Hence $v > 0$ or $v = 0$.

(3) $C \cap K_A = \{0\}$. Then $v \in \mathbb{R}^n$, $Av \geq 0$, $v \geq 0$ implies that $v = 0$.

The same arguments can be applied to A^t. We obtain the following cases.

(1) $C \cap K_{A^t} \neq \{0\}$. By cases (1) and (2) for A^t, $A^t x \geq 0$ implies that $A^t x = 0$ or $x > 0$ or $x = 0$. Thus Corollary 10.2.3 for A^t implies that there exists $y \in \mathbb{R}^n$ such that $Ay \geq 0$, $y \geq 0$, $y \neq 0$. Thus case (1) or case (2) holds for A. In particular, if A (and A^t) is not invertible, then A and A^t are of affine type. On the other hand, if A (and A^t) is invertible, then both A and A^t satisfy case (2). By Corollary 10.2.3 for $-A$ ($-A^t$, respectively) we conclude that A (A^t, respectively) is of finite type.

(2) $C \cap K_{A^t} = \{0\}$. Case (3) for A^t and Corollary 10.2.3 imply that there exists $u \in \mathbb{R}^n$ such that $Au < 0$, $u > 0$. In particular, cases (1) and (2) do not hold for A. Then A is of indefinite type.

This proves the theorem. □

We can say more about the three types of Vinberg matrices.

Corollary 10.2.8. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n \times n}$ be a Vinberg matrix. Then there exists $u \in \mathbb{R}^n$, $u > 0$, such that $Au > 0$ or $Au = 0$ or $Au < 0$. In these cases A is of finite, affine, and indefinite type, respectively.*

Proof. This follows directly from Theorem 10.2.7. □

Corollary 10.2.9. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n \times n}$ be a Vinberg matrix. Then there exists a unique $\lambda \in \mathbb{R}$ such that $A + \lambda \mathrm{id}$ is of affine type, $A + \mu \mathrm{id}$ is of finite type for all $\mu > \lambda$, and $A + \mu \mathrm{id}$ is of indefinite type for all $\mu < \lambda$.*

Proof. Suppose that A is of affine type. By Theorem 10.2.7, there exists $u \in \mathbb{R}^n$ such that $u > 0$ and $Au = 0$. Then $(A + \mu \mathrm{id})u > 0$ for all $\mu > 0$, and hence $A + \mu \mathrm{id}$ is of finite type. Similarly, $A + \mu \mathrm{id}$ is of indefinite type for all $\mu < 0$.

Assume now that A is of finite type. By Theorem 10.2.7, there exists $u > 0$ such that $Au > 0$. Then $(A + \mu \mathrm{id})u > 0$ for all $\mu \in \mathbb{R}$ in a small neighborhood of 0. Thus $A + \mu \mathrm{id}$ is of finite type for these μ. Further, $(A + \mu \mathrm{id})u < 0$ for some $\mu < 0$, and hence $A + \mu \mathrm{id}$ is of indefinite type for some $\mu < 0$.

Assume that A is of indefinite type. Let $u \in \mathbb{R}^n$ such that $u > 0$ and $Au < 0$. Similarly to the previous paragraph we conclude that $A + \mu \mathrm{id}$ is of indefinite type for all $\mu \in \mathbb{R}$ in some small neighborhood of 0, and that there exists $\mu_0 > 0$ such that $A + \mu\mathrm{id}$ is of finite type for all $\mu > \mu_0$. Let now λ be the supremum of all $\mu \in \mathbb{R}$ such that $A + \mu\mathrm{id}$ is of indefinite type. Then $A + \lambda\mathrm{id}$ is neither of indefinite nor of finite type. Hence $A + \lambda\mathrm{id}$ is of affine type, and the corollary follows from the first paragraph of the proof. □

LEMMA 10.2.10. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be a Vinberg matrix of finite or affine type. Let $J \subsetneq \{1,\dots,n\}$ be a proper subset such that $B = (a_{ij})_{i,j\in J}$ is indecomposable. Then B is a Vinberg matrix of finite type.*

PROOF. For any $v \in \mathbb{R}^n$ let $v_J = (v_j)_{j\in J}$. By assumption, there exists $u \in \mathbb{R}^n$ such that $u > 0$ and that either $Au > 0$ or $Au = 0$. Then $Bu_J \geq (Au)_J \geq 0$. Further, $Bu_J = 0$ if and only if $Au = 0$ and $a_{jk} = 0$ for all $j \in J$, $k \in I \setminus J$. Hence $Bu_J > 0$, since A is indecomposable and $J \subsetneq \{1,\dots,n\}$ is a proper subset. Thus B is of finite type. □

LEMMA 10.2.11. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be a Vinberg matrix. The following are equivalent.*

(1) *A is of finite type.*
(2) *All principal minors of A are positive.*
(3) $\det(A + \lambda\mathrm{id}) > 0$ *for all non-negative real numbers λ.*

REMARK 10.2.12. A real square matrix of which all principal minors are positive is also called a P-**matrix**.

PROOF. (2) implies (3) by the Leibniz formula for det.

Assume that (3) holds. Then $A + \lambda\mathrm{id}$ is not of affine type for all $\lambda \geq 0$, and hence A is of finite type by Corollary 10.2.9.

Assume now that (1) holds. In view of Lemma 10.2.10 it suffices to prove that $\det(A) > 0$. By Corollary 10.2.9, $A+\lambda\mathrm{id}$ is of finite type, and hence $\det(A+\lambda\mathrm{id}) \neq 0$, for all $\lambda \geq 0$. This implies that $\det(A + \lambda\mathrm{id}) > 0$ for all $\lambda \geq 0$. In particular, $\det(A) > 0$. Thus (2) holds. □

LEMMA 10.2.13. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be a Vinberg matrix. The following are equivalent.*

(1) *A is of affine type.*
(2) $\det(A) = 0$ *and all proper principal minors of A are positive.*
(3) $\det(A) = 0$ *and* $\det(A + \lambda\mathrm{id}) > 0$ *for all positive real numbers λ.*

PROOF. (1) implies (2) by Lemmas 10.2.10 and 10.2.11. The rest is similar to the proof of Lemma 10.2.11. □

Indecomposable Cartan matrices of finite and affine type, respectively, can be listed explicitly; for finite type, see Theorem 1.10.18. They are usually presented in terms of the associated Dynkin diagrams. We now prove that an indecomposable Cartan matrix is of finite type in the sense of Definition 1.10.17 if and only if it is of finite type in the sense of Theorem 10.2.7.

LEMMA 10.2.14. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be a Cartan matrix. If for all $i, j \in \{1,\dots,n\}$ with $i \neq j$ there is at most one sequence $i_1, i_2, \dots, i_k$, $k \geq 1$, $i_1 = i$, $i_2 = j$, of pairwise distinct elements of $\{1,\dots,n\}$ such that $\prod_{l=1}^{k-1} a_{i_l i_{l+1}} \neq 0$, then A is symmetrizable.*

PROOF. The claim follows easily by induction on n. $\square$

LEMMA 10.2.15. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be an indecomposable Cartan matrix of finite or affine type in the sense of Theorem* 10.2.7. *Then A is symmetrizable. Let $s \in \mathbb{N}$ with $s > 2$ and let $i_1, \dots, i_s \in \{1, \dots, n\}$ be pairwise distinct elements such that $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_s i_1} \neq 0$. Then $s = n$ and there is a permutation σ of $\{1, \dots, n\}$ such that*

$$a_{\sigma(i)\sigma(j)} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } (i,j) = (1,n) \text{ or } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{10.2.1}$$

In the latter case A is of affine type.

PROOF. If $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_s i_1} = 0$ for any $s > 2$ and any pairwise distinct elements $i_1, \dots, i_s \in \{1, \dots, n\}$, then A is symmetrizable by Lemma 10.2.14. So let us consider the opposite case.

By permuting the rows and columns of A we may assume that s is minimal and that $i_j = j$ for all $1 \le j \le s$. Then $a_{jk} \neq 0$ for $j, k \in \{1, \dots, s\}$ if and only if $|j - k| \le 1$ or $\{j, k\} = \{1, s\}$. Let $B = (a_{ij})_{1 \le j \le s}$. By Lemma 10.2.10, there exists $u \in \mathbb{R}^s$ such that $u > 0$ and that $Bu > 0$ or $Bu = 0$. Further, $Bu = 0$ if and only if A is of affine type and $s = n$. Let $v = (u_i^{-1})_{1 \le i \le s}$. Then $v^t Bu \ge 0$. But

$$\begin{aligned} 0 \le v^t Bu &= \sum_{i=1}^{s} 2u_i v_i + \sum_{1 \le i < j \le s} (a_{ij} u_i^{-1} u_j + a_{ji} u_j^{-1} u_i) \\ &\le 2s - \sum_{i=1}^{s-1} (u_i^{-1} u_{i+1} + u_i u_{i+1}^{-1}) - (u_1 u_s^{-1} + u_s^{-1} u_1) \le 0, \end{aligned}$$

and equality holds at all places if and only if $a_{ij} < 0$ implies $a_{ij} = -1$ and if $u_i = u_j$ for all $i, j \in \{1, \dots, s\}$. Then $Bu = 0$, and hence A is of affine type and $s = n$. This proves the lemma. $\square$

PROPOSITION 10.2.16. *Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be a Cartan matrix. Then A is of finite type in the sense of Definition* 1.10.17 *if and only if its indecomposable components are Vinberg matrices of finite type.*

PROOF. If A is of finite type, then all indecomposable components of A are Cartan matrices of finite type. Let B be an indecomposable component of A. Then all principal minors of B are positive and hence B is a Vinberg matrix of finite type by Lemma 10.2.11. For the converse, one concludes the symmetrizability of (the components of) A from Lemma 10.2.15. The rest follows again from Lemma 10.2.11. $\square$

We apply Vinberg's classification to finite semi-Cartan graphs.

LEMMA 10.2.17. *Let $\mathcal{G}$ be a finite semi-Cartan graph, X a point of $\mathcal{G}$, and $D \subseteq \boldsymbol{\Delta}^{X\,\mathrm{re}}$. Assume that $\gamma = 0$ or $\gamma \notin \sum_{i\in I} \mathbb{N}_0 \alpha_i$ for any $D' \subseteq \boldsymbol{\Delta}^{X\,\mathrm{re}}$ and $\gamma = \sum_{\beta \in D'} \beta - \sum_{\beta\in D} \beta$. Then $\alpha \in D$ and $-\alpha \notin D$ for any $\alpha \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$.*

PROOF. The claim follows by comparing D with $D' = D \cup \{\alpha\}$ and with $D' = D \setminus \{-\alpha\}$ for any $\alpha \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$. $\square$

THEOREM 10.2.18. *Let $\mathcal{G}$ be a finite semi-Cartan graph. Then there exists a point of $\mathcal{G}$ with a Cartan matrix of finite type.*

PROOF. Let X be a point of $\mathcal{G}$ and let $\mathcal{Y}$ be the set of points in the connected component of X. Let $Y \in \mathcal{Y}$, $D^Y \subseteq \mathbf{\Delta}^{Y\,\mathrm{re}}$, and $\delta^Y = \sum_{\beta\in D^Y}\beta$. Assume that $\delta^Y \in \sum_{i\in I}\mathbb{N}_0\alpha_i$ and that $Z \in \mathcal{Y}$, $D' \subseteq \mathbf{\Delta}^{Z\,\mathrm{re}}$, $\gamma = \sum_{\beta\in D'}\beta$ implies that $\gamma = \delta^Y$ or $\gamma - \delta^Y \notin \sum_{i\in I}\mathbb{N}_0\alpha_i$. (If $\mathcal{G}$ is a finite Cartan graph, then Lemma 10.2.17 implies that $D^Y = \mathbf{\Delta}^{Y\,\mathrm{re}}_+$.) Since $\mathcal{G}$ is finite, the set $\cup_{Y\in\mathcal{Y}}\mathbf{\Delta}^{Y\,\mathrm{re}}$ is finite by Lemma 9.1.18. Therefore Y and D^Y exist. In particular, $\alpha_i \in D^Y$ for all $i \in I$ by Lemma 10.2.17.

Let $x = (x_j)_{j\in I}$ such that $\delta^Y = \sum_{j\in I} x_j\alpha_j$. Then $x \geq 0$ by assumption. We show that $A^Y x > 0$. Let $i \in I$. By definition,

$$s_i^Y(\delta^Y - \alpha_i) = \alpha_i + \delta^Y - \sum_{j\in I} a_{ij}^Y x_j \alpha_i.$$

Since $s_i^Y(\delta^Y - \alpha_i)$ is a sum of roots of $r_i(Y) \in \mathcal{Y}$, the choice of Y and D^Y implies that $\sum_{j\in I} a_{ij}^Y x_j \geq 1$. Thus $A^Y x > 0$.

Now let B be an indecomposable component of A^Y and let x' be the corresponding component of x. Then $Bx' > 0$ and $x' \geq 0$. Hence the Vinberg matrix B is not of affine and not of indefinite type by Theorem 10.2.7. Therefore B is a Vinberg matrix of finite type. Thus A^Y is a Cartan matrix of finite type by Proposition 10.2.16. □

10.3. Classification of finite Cartan graphs of rank two

In this section we characterize finite connected Cartan graphs of rank two in terms of certain integer sequences. As a consequence, we obtain non-trivial local properties of such Cartan graphs. The structure discussed in this section appears in different forms at many places in mathematics, see also the Notes at the end of the chapter.

For all integers $1 < i \leq n$ let $V_i : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$,

$$V_i(c_1,\dots,c_n) = (c_1,\dots,c_{i-2},c_{i-1}+1,1,c_i+1,c_{i+1},\dots,c_n). \tag{10.3.1}$$

DEFINITION 10.3.1. Let $\mathcal{A}^+$ be the smallest subset of $\bigcup_{n\geq 2}\mathbb{N}_0^n$ such that

(1) $(0,0) \in \mathcal{A}^+$, and
(2) if $(c_1,\dots,c_n) \in \mathcal{A}^+$ and $1 < i \leq n$, then $V_i(c_1,\dots,c_n) \in \mathcal{A}^+$.

We say that two consecutive entries of a sequence in $\mathcal{A}^+$ are **neighbors** and that the first and the last entry are neighbors.

For all $n \geq 2$ let $\mathcal{A}^+(n)$ denote the set of all $(c_1,\dots,c_n) \in \mathcal{A}^+$.

The definition of $\mathcal{A}^+$ immediately implies the following.

LEMMA 10.3.2. *Let $n \geq 2$. Then $\sum_{i=1}^n c_i = 3n-6$ for all sequences $(c_1,\dots,c_n)$ in $\mathcal{A}^+(n)$.*

EXAMPLE 10.3.3. It follows directly from the definition that

$\mathcal{A}^+(2) = \{(0,0)\}$,

$\mathcal{A}^+(3) = \{(1,1,1)\}$,

$\mathcal{A}^+(4) = \{(1,2,1,2),(2,1,2,1)\}$,

$\mathcal{A}^+(5) = \{(1,2,2,1,3),(1,3,1,2,2),(2,1,3,1,2),(2,2,1,3,1),(3,1,2,2,1)\}$.

We also have other easy consequences of the definition.

LEMMA 10.3.4. *Let $c \in \mathcal{A}^+(n)$ with $n \geq 3$.*

(1) *For all $1 \leq i \leq n$, $c_i > 0$.*
(2) *There exist $1 \leq i < j \leq n$ with $(i,j) \neq (1,n)$ and $c_i = c_j = 1$.*
(3) *If $c_i = c_{i+1} = 1$ for some $1 \leq i < n$, then $n = 3$ and $c = (1,1,1)$.*

PROOF. The claim follows by induction on n, where for $n \leq 4$ it holds by Example 10.3.3. □

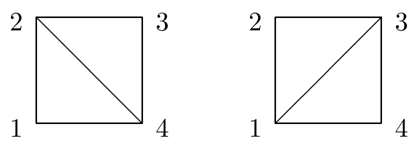
We relate sequences in $\mathcal{A}^+(n)$ to triangulations of labeled convex n-gons.

DEFINITION 10.3.5. Let $n \geq 2$ and let G be a convex n-gon. Enumerate the vertices of G from 1 to n such that consecutive integers correspond to neighboring vertices. We write $\mathcal{T}_n$ for the set of triangulations of G with non-intersecting diagonals. For any triangulation $T \in \mathcal{T}_n$ of G and any $i \in \{1, \dots, n\}$ let $c_i(T)$ be the number of triangles meeting at the i-th vertex.

EXAMPLE 10.3.6. (1) For $n = 2$, a convex n-gon G is just a line segment. A triangulation T of G is G itself. Then $c_1(T) = c_2(T) = 0$.

(2) For $n = 3$, a convex 3-gon G is a triangle. A triangulation T of G is G itself. Then $c_1(T) = c_2(T) = c_3(T) = 1$.

(3) Let $n = 4$. There are two triangulations T of a convex tetragon.



In the first case, $c_1(T) = 1$, $c_2(T) = 2$, $c_3(T) = 1$, $c_4(T) = 2$. In the second case, $c_1(T) = 2$, $c_2(T) = 1$, $c_3(T) = 2$, $c_4(T) = 1$.

PROPOSITION 10.3.7. *Let $n \geq 2$ and let G be a convex n-gon. Enumerate the vertices of G from 1 to n such that consecutive integers correspond to neighboring vertices. Then the map $\mathcal{T}_n \to \mathcal{A}^+(n)$, $T \mapsto (c_1(T), \dots, c_n(T))$, is a bijection.*

PROOF. We proceed by induction on n. For $n = 2$, the claim follows from Example 10.3.6(1). For $n \geq 3$, Axiom (2) for $\mathcal{A}^+$ corresponds to the rule to obtain a triangulation of a convex $n+1$-gon from a triangulation of a convex n-gon by adding a new triangle between two consecutive vertices, but not at the edge between the first and the last vertex. □

COROLLARY 10.3.8. *Let $n \geq 2$.*

(1) *Let $(c_1, \dots, c_n) \in \mathcal{A}^+(n)$. Then $\mathcal{A}^+(n)$ also contains $(c_n, c_{n-1}, \dots, c_1)$ and $(c_2, c_3, \dots, c_n, c_1)$. Thus the dihedral group $\mathbb{D}_n \subseteq \mathbb{S}_n$ of order $2n$ acts on $\mathcal{A}^+(n)$ by neighborhood preserving permutations of the entries:*

$$w(c_1, \dots, c_n) = (c_{w^{-1}(1)}, \dots, c_{w^{-1}(n)})$$

for all $w \in \mathbb{D}_n$, $(c_1, \dots, c_n) \in \mathcal{A}^+(n)$.

(2) *The dihedral group $\mathbb{D}_n$ acts on the set of triangulations of a convex n-gon by renumbering the vertices. The bijection in Proposition 10.3.7 commutes with the action of $\mathbb{D}_n$.*

PROOF. Both claims follow directly from the description of the bijection in Proposition 10.3.7. □

COROLLARY 10.3.9. *Let $n \geq 3$, $c = (c_1, \dots, c_n) \in \mathcal{A}^+(n)$, and let $1 < i < n$. If $c_i = 1$ then $c' \in \mathcal{A}^+(n-1)$, where*

$$c' = (c_1, \dots, c_{i-2}, c_{i-1} - 1, c_{i+1} - 1, c_{i+2}, \dots, c_n).$$

PROOF. The corresponding claim on triangulations of convex n-gons clearly holds. □

EXAMPLE 10.3.10. A short calculation shows that the sequences

$$(0,0),\ (1,1,1),\ (1,2,1,2),\ (1,2,2,1,3),$$
$$(1,2,2,2,1,4),\ (1,2,3,1,2,3),\ (1,3,1,3,1,3)$$

are representatives of the orbits of $\bigcup_{n=2}^{6} \mathcal{A}^+(n)$ under the action of the dihedral groups $\mathbb{D}_n$, $2 \leq n \leq 6$.

All sequences in $\mathcal{A}^+$ share a local property, which will enter prominently in the classification of Nichols algebras in Section 15.3. Under the **reversal** of a sequence $(c_1, c_2, \dots, c_k)$, $k \in \mathbb{N}$, we mean the sequence $(c_k, c_{k-1}, \dots, c_1)$.

PROPOSITION 10.3.11. *Let $n \geq 3$. Then any sequence $(c_1, \dots, c_n) \in \mathcal{A}^+$ contains a subsequence $(c_k)_{i \leq k \leq j}$, where $1 \leq i < j \leq n$, of the form*

$$(1,1),\ (1,2,a),\ (2,1,b),\ (1,3,1,b)$$

or their reversal, where $1 \leq a \leq 3$ and $3 \leq b \leq 5$.

PROOF. We give an indirect proof. Let $c = (c_1, \dots, c_n) \in \mathcal{A}^+$. Assume that the claim does not hold for this sequence. Then $n \geq 6$ by Example 10.3.3, since $(1,1)$, $(1,2,1)$, $(2,1,3)$ and $(3,1,2)$ are not subsequences. Since $n \geq 6$, Corollary 10.3.9 implies that c has no subsequence $(2,1,2)$. Let

$$\epsilon_{11} = (1),\ \epsilon_{12} = (2,1),\ \epsilon_{21} = (1,2),\ \epsilon_{22} = (1,3,1)$$

and let $E = \{\epsilon_{ij} \mid 1 \leq i, j \leq 2\}$. By assumption, both the left and the right neighbor of any subsequence ϵ_{ij} with $(i,j) \neq (1,1)$ is at least four. Thus there is a unique decomposition $d = (d_1, \dots, d_k)$, where $k \geq 2$, of c into disjoint subsequences of the form (a) and ϵ, where $a \geq 2$ and $\epsilon \in E$, such that

(a) $(\epsilon_{11}, 2)$, $(2, \epsilon_{11})$ and $(\epsilon_{11}, 3, \epsilon_{11})$ are not subsequences of d.

(For example, if $c = (1,4,6,1,3,1,7,2)$ then $d = (\epsilon_{11}, 4, 6, \epsilon_{22}, 7, 2)$.) Since c does not satisfy the claim of the proposition, we obtain the following information for d.

(b) No two consecutive entries of d belong to E.
(c) No entry $\epsilon \in E$ of d is preceded or followed by 2.
(d) If (ϵ_{21}, a) or (a, ϵ_{12}) is a subsequence of d, then $a \geq 4$.
(e) If (ϵ_{i2}, b) or (b, ϵ_{2i}) is a subsequence of d, where $i \in \{1,2\}$, then $b \geq 6$.

By iterated application of Corollary 10.3.9 we obtain further reductions of d.

$$(\dots, d_{m-1}, \epsilon_{ij}, d_{m+1}, \dots) \longrightarrow (\dots, d_{m-1} - i, d_{m+1} - j, \dots),$$
$$(\epsilon_{i2}, d_2, \dots) \longrightarrow (\epsilon_{i1}, d_2 - 1, \dots),$$
$$(\dots, \tilde{d}, \epsilon_{2i}) \longrightarrow (\dots, \tilde{d} - 1, \epsilon_{1i})$$

for all $i,j\in\{1,2\}$. Let us perform these reductions at all places $1\le m\le k$ in d, where an entry ϵ_{ij} with $1\le i,j\le 2$ appears. By (b) and (c), this results in a unique sequence d' without entries in E except maybe at the first or last place. Any intermediate entry of d' is at least 2 by the following.

(1) Any entry $d_m\notin E$ of d is decreased by at most 4. Hence if $d_m\ge 6$, then its value after the reductions is at least 2.
(2) If $4\le d_m<6$, then $d_{m-1}\ne\epsilon_{i2}$ and $d_{m+1}\ne\epsilon_{2i}$ for any $i\in\{1,2\}$ by (e). Hence d_m decreases by at most 2.
(3) If $d_m=3$ then (a),(d),(e) imply that at most one of d_{m-1}, d_{m+1} is in E, and this entry is ϵ_{11}. Hence d_m decreases by at most 1.
(4) If $d_m=2$ then d_m does not decrease by (c).

Thus by Corollary 10.3.9 there exists a sequence $d''=(d''_1,\dots,d''_l)\in\mathcal{A}^+$ with $l\ge 2$, where $d''_m\ge 2$ for all $1<m<l$ and $d''_1,d''_l\ge 1$. This is a contradiction to $d''\in\mathcal{A}^+$ and Lemma 10.3.4(2). $\square$

Sequences in $\mathcal{A}^+$ have an interesting number theoretic property, which we prove in Theorem 10.3.14 below. We start with general considerations.

Let

$$\eta:\mathbb{Z}\to\mathrm{SL}_2(\mathbb{Z}),\quad a\mapsto\begin{pmatrix}a&-1\\1&0\end{pmatrix}. \tag{10.3.2}$$

Recall that (α_1,α_2) is the standard basis of $\mathbb{Z}^2$.

LEMMA 10.3.12. *Let $n\in\mathbb{N}$ and $(c_k)_{1\le k\le n}\in\mathbb{Z}^n$. For all $1\le k\le n+1$ let $\beta_k=\eta(c_1)\cdots\eta(c_{k-1})(\alpha_1)$.*

(1) *Let $\beta_0=-\alpha_2$. Then $\beta_{k+1}=c_k\beta_k-\beta_{k-1}$ for all $1\le k\le n$.*
(2) *If $n\ge 3$ and $c_k=0$ for some $1\le k<n$, then $\beta_l\notin\mathbb{N}_0^2$ for some $1\le l\le n$.*
(3) *If $c_1\ge 1$ and $c_k\ge 2$ for all $1<k<n$, then $\beta_k\in\mathbb{N}_0^2$ for all $1\le k\le n$ and $\beta_k-\beta_{k-1}\in\mathbb{N}_0^2\setminus\{0\}$ for all $1<k\le n$. Further, if $n\ge 2$ then $\beta_{n+1}\in\mathbb{N}_0^2$ or $\beta_{n+1}+\beta_{n-1}\in-\mathbb{N}_0^2$.*

PROOF. (1) Since $\beta_1=\alpha_1$ and $\beta_2=\eta(c_1)(\alpha_1)=c_1\alpha_1+\alpha_2$, the claim holds for $k=1$. Let now $k\ge 2$. Then

$$\beta_{k+1}=\eta(c_1)\cdots\eta(c_k)(\alpha_1)=\eta(c_1)\cdots\eta(c_{k-1})(c_k\alpha_1+\alpha_2)=c_k\beta_k-\beta_{k-1}$$

since $\eta(c_{k-1})(\alpha_2)=-\alpha_1$.

(2) If $c_1=0$ then

$$\beta_3=\eta(c_1)(c_2\alpha_1+\alpha_2)=c_2\alpha_2-\alpha_1\notin\mathbb{N}_0^2.$$

If $1<k<n$ and $c_k=0$ then $\beta_{k+1}=-\beta_{k-1}$ by (1). Thus $\beta_{k-1}\notin\mathbb{N}_0^2$ or $\beta_{k+1}\notin\mathbb{N}_0^2$.

(3) Assume first that $c_k\ge 2$ for all $1\le k<n$. For all $0\le k\le n$ let $a_k,b_k\in\mathbb{Z}$ such that $\beta_k=a_k\alpha_1+b_k\alpha_2$, where $\beta_0=-\alpha_2$. We prove by induction on k for all $1\le k\le n$ that

$$a_k>b_k\ge 0,\ a_k>a_{k-1},\ b_k>b_{k-1},\ a_k-b_k-a_{k-1}+b_{k-1}\ge 0. \tag{10.3.3}$$

Since $\beta_1=\alpha_1$, (10.3.3) is valid for $k=1$. For $1\le k<n$ we get from (1) that

$$\begin{aligned}a_{k+1}-b_{k+1}=&(c_ka_k-a_{k-1})-(c_kb_k-b_{k-1})\\=&(c_k-1)(a_k-b_k)+(a_k-b_k-a_{k-1}+b_{k-1}).\end{aligned}$$

This is positive by induction hypothesis, since $c_k > 1$. Similarly,

$$a_{k+1} - a_k = c_k a_k - a_{k-1} - a_k = (c_k - 2)a_k + (a_k - a_{k-1}) > 0,$$
$$b_{k+1} - b_k = c_k b_k - b_{k-1} - b_k = (c_k - 2)b_k + (b_k - b_{k-1}) > 0.$$

In particular, $b_{k+1} > b_k \geq 0$. Finally,

$$\begin{aligned} a_{k+1} - b_{k+1} - a_k + b_k =& c_k(a_k - b_k) - a_{k-1} + b_{k-1} - a_k + b_k \\ =& (c_k - 2)(a_k - b_k) + (a_k - b_k - a_{k-1} + b_{k+1}) \geq 0, \end{aligned}$$

which completes the proof of (10.3.3) for all $1 \leq k \leq n$. Thus $\beta_k \in \mathbb{N}_0^2$ for all $1 \leq k \leq n$ and $\beta_k - \beta_{k-1} \in \mathbb{N}_0^2 \setminus \{0\}$ for all $1 < k \leq n$.

If $c_1 = 1$ and $c_k \geq 2$ for all $1 < k < n$, then $\beta_k \in \mathbb{N}_0^2$ for all $1 \leq k \leq n$ and $\beta_k - \beta_{k-1} \in \mathbb{N}_0^2 \setminus \{0\}$ for all $1 < k \leq n$ by a similar argument using the inequalities

$$b_k \geq a_k > 0,\ a_k \geq a_{k-1},\ b_k > b_{k-1},\ a_k - b_k - a_{k-1} + b_{k-1} < 0 \tag{10.3.4}$$

for all $1 < k \leq n$.

The last claim follows from (1). Indeed, if $c_n \geq 1$ then $\beta_{n+1} \in \mathbb{N}_0^2$, and if $c_n \leq 0$ then $\beta_{n+1} + \beta_{n-1} = c_n \beta_n \in -\mathbb{N}_0^2$. □

LEMMA 10.3.13. *Let $1 < i \leq n$, $c'_1, \dots, c'_n \in \mathbb{Z}$, and*

$$(c_1, \dots, c_{n+1}) = V_i(c'_1, \dots, c'_n) \in \mathbb{Z}^{n+1}.$$

Let $\beta'_k = \eta(c'_1)\cdots\eta(c'_{k-1})(\alpha_1)$ for all $1 \leq k \leq n$ and $\beta_k = \eta(c_1)\cdots\eta(c_{k-1})(\alpha_1)$ for all $1 \leq k \leq n+1$. Then

(1) *$\eta(c_1)\cdots\eta(c_{n+1}) = \eta(c'_1)\cdots\eta(c'_n)$, and*
(2) *$\beta_k = \beta'_k$ for all $1 \leq k < i$, $\beta_k = \beta'_{k-1}$ for all $i < k \leq n+1$, and $\beta_i = \beta'_{i-1} + \beta'_i = \beta_{i-1} + \beta_{i+1}$.*

PROOF. Direct calculation shows that

$$\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b+1) \quad \text{for all } a, b \in \mathbb{Z}. \tag{10.3.5}$$

This implies (1). Further, $\beta_k = \beta'_k$ for $1 \leq k < i$, and

$$\beta_{i+1} = \eta(c_1)\cdots\eta(c_i)(\alpha_1) = \eta(c'_1)\cdots\eta(c'_{i-2})\eta(c'_{i-1}+1)(\alpha_1 + \alpha_2) = \beta'_i,$$

since $\eta(c+1)(\alpha_1 + \alpha_2) = \eta(c)(\alpha_1)$ for all $c \in \mathbb{Z}$. Again by (10.3.5), $\beta_k = \beta'_{k-1}$ for all $i + 1 < k \leq n + 1$. Finally,

$$\begin{aligned} \beta_i =& \eta(c_1)\cdots\eta(c_{i-1})(\alpha_1) \\ =& \eta(c'_1)\cdots\eta(c'_{i-2})\eta(c'_{i-1}+1)(\alpha_1) \\ =& \eta(c'_1)\cdots\eta(c'_{i-2})(\eta(c'_{i-1})(\alpha_1) + \alpha_1) \\ =& \beta'_i + \beta'_{i-1} \end{aligned}$$

which implies (2). □

THEOREM 10.3.14. *Let $n \geq 2$ and let $(c_1, \dots, c_n) \in \mathbb{Z}^n$. The following are equivalent.*

(1) *$(c_1, \dots, c_n) \in \mathcal{A}^+$,*
(2) *$\eta(c_1)\cdots\eta(c_n) = -\mathrm{id}$, and the pairs $\beta_k = \eta(c_1)\cdots\eta(c_{k-1})(\alpha_1)$, $1 \leq k \leq n$, are in $\mathbb{N}_0^2$.*

PROOF. Assume (1). We prove (2) by induction on n. If $n = 2$, then trivially, $(c_1, c_2) = (0, 0)$, $\eta(0)^2 = -\mathrm{id}$, and $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$.

Assume now that $n \geq 3$. Then there exist $(c'_1, \dots, c'_{n-1}) \in \mathcal{A}^+(n-1)$ and $1 < i \leq n-1$ such that $(c_1, \dots, c_n) = V_i(c'_1, \dots, c'_{n-1})$. Hence (2) follows from Lemma 10.3.13.

Now assume (2). We prove (1) by induction on n. If $n = 2$, then

$$-\mathrm{id} = \eta(c_1)\eta(c_2) = \begin{pmatrix} c_1c_2 - 1 & -c_1 \\ c_2 & -1 \end{pmatrix}.$$

Thus $c_1 = c_2 = 0$, that is, $(c_1, c_2) \in \mathcal{A}^+$.

Assume that $n \geq 3$. For any $1 \leq k < n$, $c_k\beta_k = \beta_{k-1} + \beta_{k+1}$ because of Lemma 10.3.12(1), where $\beta_0 = -\alpha_2$. Since $\beta_l \in \mathbb{N}_0^2$ for any $1 \leq l \leq n$, we obtain that $c_k > 0$ for any $1 < k < n$ and $c_1 \geq 0$. Further, $c_1 > 0$ by Lemma 10.3.12(2). Since $\eta(c_1) \cdots \eta(c_n) = -\mathrm{id}$, we also get that $\beta_{n+1} = -\alpha_1$. Thus $c_i = 1$ for some $2 \leq i \leq n$ by the first claim in Lemma 10.3.12(3).

Assume first that $c_k \geq 2$ for any $1 < k < n$. Then, the first claim in Lemma 10.3.12(3) implies that $\beta_{n-1} \neq \alpha_1$. Moreover, $\beta_{n-1} - \alpha_1 \in -\mathbb{N}_0^2$ by the second claim in Lemma 10.3.12(3), which is a contradiction.

Let $i \in \{2, 3, \dots, n-1\}$ such that $c_i = 1$ and let $(c'_1, \dots, c'_{n-1}) \in \mathbb{Z}^{n-1}$ such that $(c_1, \dots, c_n) = V_i(c'_1, \dots, c'_{n-1})$. Then $\eta(c'_1) \cdots \eta(c'_{n-1}) = -\mathrm{id}$ and the pairs $\beta'_k = \eta(c'_1) \cdots \eta(c'_{k-1})(\alpha_1)$ for all $1 \leq k \leq n-1$ are in $\mathbb{N}_0^2$ because of Lemma 10.3.13. Hence $(c'_1, \dots, c'_{n-1}) \in \mathcal{A}^+$ by induction hypothesis, and then $(c_1, \dots, c_n) \in \mathcal{A}^+$. $\square$

We are going to characterize and classify finite connected Cartan graphs using their characteristic sequences.

DEFINITION 10.3.15. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph of rank two and let $X \in \mathcal{X}$ and $i \in I$. The **characteristic sequence of $\mathcal{G}$ with respect to X and i** is the infinite sequence $(c_k^{X,i})_{k \geq 1}$ of non-negative integers, where

$$\begin{aligned} c_{2k+1}^{X,i} &= -\, a_{ij}^{(r_jr_i)^k(X)} = -a_{ij}^{r_i(r_jr_i)^k(X)}, \\ c_{2k+2}^{X,i} &= -\, a_{ji}^{r_i(r_jr_i)^k(X)} = -a_{ji}^{(r_jr_i)^{k+1}(X)} \end{aligned}$$

for all $k \geq 0$ and $j \in I \setminus \{i\}$.

LEMMA 10.3.16. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph of rank two and let $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. Let $(c_k)_{k \geq 1}$ be the characteristic sequence of $\mathcal{G}$ with respect to X and i.*

(1) *The characteristic sequence of $\mathcal{G}$ with respect to $r_i(X)$ and j is $(c_{k+1})_{k \geq 2}$.*
(2) *Suppose that $(r_jr_i)^n(X) = X$ for some $n \geq 1$. Then $c_{2n+k} = c_k$ for all $k \geq 1$, and the characteristic sequence of $\mathcal{G}$ with respect to X and j is $(c_{2n+1-k})_{k \geq 1}$.*

PROOF. Both claims follows directly from the definition of a characteristic sequence. $\square$

For semi-Cartan graphs $\mathcal{G}$ of rank two we can use the map η to calculate $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ for any point X of $\mathcal{G}$.

Let $\tau \in \mathrm{Aut}(\mathbb{Z}^2)$, $(c_1, c_2) \mapsto (c_2, c_1)$.

DEFINITION 10.3.17. Let $I = \{1,2\}$, let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph of rank two, and let $X \in \mathcal{X}$ and $i \in I$. Let $(c_k)_{k\geq 1}$ be the characteristic sequence of $\mathcal{G}$ with respect to X and i. The **root sequence of $\mathcal{G}$ with respect to X and i** is the sequence $(\beta_k)_{k\geq 1}$ of elements of $\mathbb{Z}^2$, where

$$\beta_k = \eta(c_1)\cdots\eta(c_{k-1})(\alpha_1)$$

for all $k \geq 1$. In particular, $\beta_1 = \alpha_1$.

REMARK 10.3.18. Let $I = \{1,2\}$, let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph of rank two, and let $X \in \mathcal{X}$. Let $(\beta_k)_{k\geq 1}$ be the root sequence of $\mathcal{G}$ with respect to X and 1 and let $(\gamma_k)_{k\geq 1}$ be the root sequence of $\mathcal{G}$ with respect to X and 2. Then

$$\begin{aligned} \beta_{2k+1} &= \mathrm{id}_X(s_1s_2)^k(\alpha_1), & \beta_{2k+2} &= \mathrm{id}_X(s_1s_2)^k s_1(\alpha_2), \\ \tau\gamma_{2k+1} &= \mathrm{id}_X(s_2s_1)^k(\alpha_2), & \tau\gamma_{2k+2} &= \mathrm{id}_X(s_2s_1)^k s_2(\alpha_1) \end{aligned} \tag{10.3.6}$$

for all $k \geq 0$, since $s_1^Y = \eta(-a_{12}^Y)\tau$ and $s_2^Y = \tau\eta(-a_{21}^Y)$ for all $Y \in \mathcal{X}$. Thus

$$\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{\pm\beta_k, \pm\tau\gamma_k \mid k \geq 1\}. \tag{10.3.7}$$

For a finite sequence $(c_1, \ldots, c_n)$ of integers or vectors, where $n \geq 1$, let

$$(c_1, \ldots, c_n)^\infty = (d_k)_{k\geq 1}$$

be the sequence where $d_{mn+k} = c_k$ for all $1 \leq k \leq n$, $m \geq 0$.

EXAMPLE 10.3.19. Let $I = \{1,2\}$, let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a connected semi-Cartan graph of rank two, and let $X \in \mathcal{X}$. Assume that $a_{12}^X = 0$.

Since A^X is a Cartan matrix and $a_{12}^{r_1(X)} = a_{12}^X$, $a_{21}^{r_2(X)} = a_{21}^X$, we conclude that $a_{12}^X = a_{21}^X = 0$ and $a_{12}^{r_1(X)} = 0 = a_{21}^{r_2(X)}$. Since $\mathcal{G}$ is connected, the latter implies that $a_{12}^Y = a_{21}^Y = 0$ for all $Y \in \mathcal{X}$. One checks quickly that $\eta(0)^2 = -\mathrm{id}$, and hence the root sequence of $\mathcal{G}$ with respect of X and 1 is $(\alpha_1, \alpha_2, -\alpha_1, -\alpha_2)^\infty$. In particular, $m_{12}^X = 2$ by Remark 10.3.18. Therefore, $\mathcal{G}$ is a Cartan graph if and only if $(r_2r_1)^2(X) = X$. Up to isomorphism there exist precisely four such Cartan graphs: one with one object, two with two objects, and one with four objects. In fact, all of them are isomorphic to products of Cartan graphs of rank one (see Example 9.1.24) in the sense of Definition 10.1.13.

EXAMPLE 10.3.20. Let $I = \{1,2\}$ and let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a connected semi-Cartan graph of rank two. Assume that $a_{12}^Y, a_{21}^Y \leq -2$ for all $Y \in \mathcal{X}$.

Let $X \in \mathcal{X}$ and let $(c_k)_{k\geq 1}$ and $(\beta_k)_{k\geq 1}$ be the characteristic sequence and the root sequence of $\mathcal{G}$ with respect to X and 1, respectively. Then $c_k \geq 2$ for all $k \geq 1$ by assumption. Thus $\beta_k \in \mathbb{N}_0^2$ for any $k \geq 1$ and $\beta_k \neq \beta_l$ for any $1 \leq k < l$ by Lemma 10.3.12(3). By Remark 10.3.18, $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is infinite and is contained in $\mathbb{N}_0^2 \cup -\mathbb{N}_0^2$. Hence $\mathcal{G}$ is a Cartan graph in this case.

Now we characterize finite connected Cartan graphs of rank two.

THEOREM 10.3.21. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ with $I = \{i, j\}$ be a connected semi-Cartan graph of rank two such that $|\mathcal{X}|$ is finite. Let $X \in \mathcal{X}$ and let $n > 0$ be the integer with $(r_jr_i)^n(X) = X$, $(r_jr_i)^k(X) \neq X$ for any $1 \leq k < n$. Let $(c_k)_{k\geq 1}$ be the characteristic sequence of $\mathcal{G}$ with respect to X and i, and let $\varpi = 6n - \sum_{k=1}^{2n} c_k$. The following are equivalent.*

(1) *$\mathcal{G}$ is a finite Cartan graph.*

(2) $\varpi > 0$, $\varpi \mid 12$, $(c_1, \dots, c_{12n/\varpi}) \in \mathcal{A}^+$, *and*

$$(c_k)_{k\geq 1} = (c_1, \dots, c_{12n/\varpi})^\infty.$$

In this case $12n/\varpi = |\boldsymbol{\Delta}_+^{X\,\mathrm{re}}| = m_{ij}^X$.

Proof. Up to isomorphism we may assume that $I = \{1, 2\}$, $i = 1$, and $j = 2$. Let $(\beta_k)_{k\geq 1}$ be the root sequence of $\mathcal{G}$ with respect to X and 1.

Assume (1). We prove (2). Let $q = m_{ij}^X = \overline{m}_{ij}^X$, see Corollary 9.2.20. Then Remark 10.3.18 and Lemma 9.2.7 imply that $\beta_k \in \mathbb{N}_0^2$ for $1 \leq k \leq q$, and $\beta_{q+1} = -\beta_l$ for some $1 \leq l \leq q$, since $\overline{m}_{ij}^X = q$. Since $(c_k)_{k\geq 2}$ is the characteristic sequence of $\mathcal{G}$ with respect to $r_i(X)$ and j by Lemma 10.3.16(1), and since $\overline{m}_{ji}^{r_i(X)} = q$ by Proposition 9.2.14, it follows that $l = 1$, that is,

$$-\eta(c_1)\cdots\eta(c_q)(\alpha_1) = -\beta_{q+1} = \alpha_1.$$

Thus $-\eta(c_1)\cdots\eta(c_q) = \mathrm{id}$ by Lemma 9.2.19. Hence $(c_1, \dots, c_q) \in \mathcal{A}^+$ by Theorem 10.3.14. Therefore

$$\sum_{i=1}^{q} c_i = 3q - 6$$

by Lemma 10.3.2. Because of Lemma 10.3.16(1), the same reasoning for $r_i(X)$ and j shows that $(c_2, \dots, c_{q+1}) \in \mathcal{A}^+$ and that $\sum_{i=2}^{q+1} c_i = 3q - 6$. Hence $c_{q+1} = c_1$, and $(c_k)_{k\geq 1} = (c_1, \dots, c_q)^\infty$ by induction. In particular,

$$q\sum_{i=1}^{2n} c_i = \sum_{i=1}^{2qn} c_i = 2n\sum_{i=1}^{q} c_i = 2n(3q-6).$$

Therefore $\sum_{i=1}^{2n} c_i = (6nq - 12n)/q = 6n - 12n/q$. Hence $q \mid 12n$ and $12n/q = \varpi$. Moreover, $(r_j r_i)^q(X) = X$ by (CG4), and hence $n \mid q$. Therefore $\varpi \mid 12$, since $q = n \cdot 12/\varpi$. This proves (2).

Now assume that (2) holds. We prove (1). Let $q = 12n/\varpi$. Then $(c_1, \dots, c_q)$ is a sequence in $\mathcal{A}^+$, and hence $\beta_k \in \mathbb{N}_0^2$ for $1 \leq k \leq q$ and $\eta(c_1)\cdots\eta(c_q) = -\mathrm{id}$ by Theorem 10.3.14. Therefore, since $(c_k)_{k\geq 1} = (c_1, \dots, c_q)^\infty$, in the root sequence of $\mathcal{G}$ with respect to X and i only q elements of $\mathbb{N}_0^2$ and q elements of $-\mathbb{N}_0^2$ appear. Since $(c_q, \dots, c_1) \in \mathcal{A}^+$ by Corollary 10.3.8(1), Lemma 10.3.16 implies that the same holds for the root sequence of $\mathcal{G}$ with respect to X and j. Thus $\boldsymbol{\Delta}^{X\,\mathrm{re}} \subseteq \mathbb{N}_0^2 \cup -\mathbb{N}_0^2$ by Remark 10.3.18, and $\mathcal{G}$ is finite. Because of Lemma 10.3.16(1), the same arguments show that $\boldsymbol{\Delta}^{Y\,\mathrm{re}} \subseteq \mathbb{N}_0^2 \cup -\mathbb{N}_0^2$ for all $Y \in \mathcal{X}$. Therefore $|\boldsymbol{\Delta}_+^{X\,\mathrm{re}}| = m_{ij}^X = \overline{m}_{ij}^X = q$ by Corollary 9.2.20. Further, $n \mid q$ by assumption, and hence $(r_2 r_1)^q(X) = X$. Thus $\mathcal{G}$ is a Cartan graph. $\square$

Example 10.3.22. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a connected semi-Cartan graph of rank two. Let $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. Assume that $a_{ij}^X = a_{ji}^X = -1$. The only sequence in $\mathcal{A}^+$ with two consecutive entries 1 is $(1, 1, 1)$. Hence, by Theorem 10.3.21, $\mathcal{G}$ is a Cartan graph if and only if $a_{ij}^Y = a_{ji}^Y = -1$ for all $Y \in \mathcal{X}$ and if $(r_2 r_1)^3(X) = X$. Up to isomorphism there exist precisely four such Cartan graphs: one with one object, one with two objects (and $r_1 = r_2 \neq \mathrm{id}$), one with three objects, and one with six objects.

Theorem 10.3.21 and Lemma 10.3.13 provide us with a nice description of positive roots.

DEFINITION 10.3.23. For any $n \geq 1$, a set of non-zero vectors $v_1, v_2, \dots, v_k \in \mathbb{Z}^n$ with $k \geq 1$ is said to be **relatively prime**, if the elementary divisors of the matrix in $\mathbb{Z}^{n\times k}$ with columns $v_1, \dots, v_k$ are units.

COROLLARY 10.3.24. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ with $I = \{i, j\}$ be a finite Cartan graph of rank two. Let $X \in \mathcal{X}$, and let $(c_k)_{k\geq 1}$ be the characteristic sequence of $\mathcal{G}$ with respect to X and i. Let $q = m_{ij}^X$. Then $(c_1, \dots, c_q) \in \mathcal{A}^+$. For all $1 \leq k \leq q-2$ let $i_k \in \{2, 3, \dots, k+1\}$ such that*

$$(c_1, \dots, c_q) = V_{i_{q-2}} \cdots V_{i_2} V_{i_1}(0, 0).$$

Let $(\beta_1, \dots, \beta_q)$ be the sequence of elements of $\mathbb{Z}^2$ arising from (α_1, α_2) by inserting successively at the i_k-th place, where $1 \leq k \leq q-2$, the sum of the elements at place $i_k - 1$ and i_k.

(1) $\boldsymbol{\Delta}_+^{X\,\mathrm{re}} = \{\beta_1, \dots, \beta_q\}$.
(2) *For any $1 \leq k < q$, the matrix in $\mathbb{Z}^{2\times 2}$ with columns β_k, β_{k+1} has determinant 1. In particular, β_k and β_{k+1} are relatively prime.*
(3) *For any $1 < k < q$, β_k is a sum of two relatively prime positive real roots.*

PROOF. We may assume that $i = 1$ and $j = 2$. Since $\mathcal{G}$ is a finite Cartan graph, $r_j r_i$ has finite order. Then $(c_1, \dots, c_q) \in \mathcal{A}^+$ by Theorem 10.3.21. Let $(\beta_k)_{k\geq 1}$ be the root sequence of $\mathcal{G}$ with respect to X and i. Then

$$|\boldsymbol{\Delta}_+^{X\,\mathrm{re}}| = m_{ij}^X = \overline{m}_{ij}^X$$

(the length of κ_{ij}^X) by Corollary 9.2.20. Now Lemma 9.2.7(1) and Remark 10.3.18 imply that $\boldsymbol{\Delta}_+^{X\,\mathrm{re}} = \{\beta_1, \dots, \beta_q\}$. Thus the corollary follows from Lemma 10.3.13. Indeed, the property claimed in (2) holds for the sequence (α_1, α_2) and remains valid after each insertion of a new root. In particular, any inserted root (and hence each β_k with $1 < k < q$) is the sum of two relatively prime roots. $\square$

REMARK 10.3.25. If one identifies any positive real root $(a, b) \in \mathbb{Z}^2$ with the fraction $\frac{a}{b}$, then the construction of the set $\boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ in Corollary 10.3.24 parallels the iterated insertion of mediants of two neighboring rationals. The claim in Corollary 10.3.24(2) is a variant of a standard result in number theory in the context of Farey sequences.

EXAMPLE 10.3.26. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be the connected semi-Cartan graph from Example 9.1.2, see also Example 9.1.15. The characteristic sequence of $\mathcal{G}$ with respect to X_1 and 1 is $(c_k)_{k\geq 1} = (1, 4, 1, 3)^\infty$, and $n = 2$ is the smallest positive integer such that $(r_2 r_1)^n(X_1) = X_1$. We check the conditions in Theorem 10.3.21(2). We obtain that $\varpi = 6n - \sum_{i=1}^{2n} c_i = 3$, $\varpi \mid 12$, $12n/\varpi = 8$, and $(c_k)_{k\geq 1} = (c_1, \dots, c_8)^\infty$. Further,

$$\begin{aligned}(1, 4, 1, 3, 1, 4, 1, 3) &= V_3(1, 3, 2, 1, 4, 1, 3) = V_3V_4(1, 3, 1, 3, 1, 3)\\ &= V_3V_4V_3(1, 2, 2, 1, 3) = V_3V_4V_3V_4(1, 2, 1, 2)\\ &= V_3V_4V_3V_4V_3(1, 1, 1) = V_3V_4V_3V_4V_3V_2(0, 0)\end{aligned}$$

and hence $(c_1, \dots, c_8) \in \mathcal{A}^+$. We conclude from Theorem 10.3.21 that $\mathcal{G}$ is a Cartan graph and has $12n/\varpi = 8$ positive roots at each point. (We knew this already, but the proof in Example 9.1.15 is much more computational.) Further, by Corollary 10.3.24 we obtain easily the set of positive real roots at X_1. We again abbreviate $a\alpha_1 + b\alpha_2$ for $a, b \in \mathbb{N}_0$ by 1^a2^b.

$$\begin{aligned}\{1,2\} &\overset{V_3}{\to} \{1,12,2\} \overset{V_3}{\to} \{1,12,12^2,2\} \overset{V_4}{\to} \{1,12,12^2,12^3,2\}\\ &\overset{V_3}{\to} \{1,12,1^22^3,12^2,12^3,2\} \overset{V_4}{\to} \{1,12,1^22^3,1^32^5,12^2,12^3,2\}\\ &\overset{V_3}{\to} \{1,12,1^32^4,1^22^3,1^32^5,12^2,12^3,2\} = \boldsymbol{\Delta}_+^{X_1\,\mathrm{re}}.\end{aligned}$$

This coincides with the calculation in Example 9.1.15.

The next claim is part of Corollary 10.3.24.

Corollary 10.3.27. *Let $\mathcal{G}$ be a finite Cartan graph of rank two and let X be a point of $\mathcal{G}$. Then any non-simple positive real root at X is the sum of two relatively prime positive real roots.*

Proposition 10.3.11 and Theorem 10.3.21 imply another important fact about finite Cartan graphs.

Corollary 10.3.28. *Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a finite Cartan graph of rank two. Then there exist $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$ such that one of the following hold.*

(1) $a_{ij}^X = a_{ji}^X = 0$,
(2) $a_{ij}^X = a_{ji}^X = -1$,
(3) $a_{ji}^X = -1$, $a_{ij}^X = -2$, $-3 \le a_{ji}^{r_i(X)} \le -1$,
(4) $a_{ji}^X = -2$, $a_{ij}^X = -1$, $-5 \le a_{ji}^{r_i(X)} \le -3$,
(5) $a_{ji}^X = -1$, $a_{ij}^X = -3$, $a_{ji}^{r_i(X)} = -1$, $-5 \le a_{ij}^{r_j(X)} \le -3$.

Proof. We may assume that $\mathcal{G}$ is connected. If $m_{ij}^X = 2$ for all $X \in \mathcal{G}$ then $a_{ij}^X = a_{ji}^X = 0$ for all $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. Otherwise $m_{ij}^X \ge 3$ for all $X \in \mathcal{G}$ and $i, j \in I$ with $i \neq j$. Let $Y \in \mathcal{X}$ and $i \in I$. Let $(c_k)_{k\ge1}$ be the characteristic sequence of $\mathcal{G}$ with respect to Y and i. Then $(c_1, \dots, c_{m_{ij}^Y}) \in \mathcal{A}^+$ by Theorem 10.3.21. By Proposition 10.3.11 there exists a subsequence $(1,1)$, $(1,2,a)$, $(2,1,b)$ or $(1,3,1,b)$ with $1 \le a \le 3$, $3 \le b \le 5$ of $(c_1, \dots, c_{m_{ij}^Y})$ or its reversal. Thus the claim follows from Lemma 10.3.16. □

Theorem 10.3.21 also allows the classification of finite connected simply connected Cartan graphs of rank two.

Theorem 10.3.29. *The following hold.*

(1) *Let $n \ge 2$ be an integer, $(c_1, \dots, c_n) \in \mathcal{A}^+(n)$, $\mathcal{X} = \{1, 2, \dots, 2n\}$, and $I = \{1, 2\}$. Define $r_1, r_2 : \mathcal{X} \to \mathcal{X}$ by*

$$r_1 = (1\,2)(3\,4)\cdots(2n-1\,2n), \quad r_2 = (2\,3)(4\,5)\cdots(2n-2\,2n-1)(2n\,1).$$

Then there is a unique semi-Cartan graph $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$, such that the characteristic sequence of $\mathcal{G}$ with respect to $X = 1$ and 1 is $(c_1, \dots, c_n)^\infty$. This $\mathcal{G}$ is a connected simply connected finite Cartan graph, and for all $Y \in \mathcal{X}$, $m_{12}^Y = m_{21}^Y = n$.

(2) *Any finite connected simply connected Cartan graph of rank two is isomorphic to a Cartan graph in (1).*

(3) *Let $\mathcal{G}$ and $\mathcal{G}'$ be two semi-Cartan graphs as in (1) corresponding to a sequence $(c_1,\dots,c_q)\in\mathcal{A}^+$ and $(c'_1,\dots,c'_{q'})\in\mathcal{A}^+$, respectively. Then $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic if and only if $q=q'$ and $(c_1,\dots,c_q)$, $(c'_1,\dots,c'_q)$ are in the same orbit of $\mathcal{A}^+(q)$ under the action of $\mathbb{D}_q$.*

Proof. (1) The existence of $\mathcal{G}$ is easy to check. Clearly, $\mathcal{G}$ is connected. Moreover, $(r_2r_1)^k(X)=X$ if and only if $n\mid k$. Since $\sum_{i=1}^n c_i=3n-6$, Theorem 10.3.21 with $\varpi=12$ implies that $\mathcal{G}$ is a finite Cartan graph with $m_{ij}^X=n$. Finally, $\mathcal{G}$ is simply connected since for all $X\in\mathcal{X}$ the group $\mathrm{Hom}(X,X)$ is the cyclic group generated by $\mathrm{id}_X(s_1s_2)^n$, which is the identity by Corollary 9.2.22(1).

(2) Let $\mathcal{G}=\mathcal{G}(I,\mathcal{X},r,A)$ be a finite connected simply connected Cartan graph of rank two. We may assume that $I=\{1,2\}$. Let $X\in\mathcal{X}$, and let n be the smallest positive integer such that $(r_2r_1)^n(X)=X$. Since $\mathrm{id}_X(s_1s_2)^{m_{12}^X}=\mathrm{id}_X$ by Corollary 9.2.22, it follows that $n\le m_{12}^X$. Let $(c_k)_{k\ge1}$ be the characteristic sequence of $\mathcal{G}$ with respect to X and 1. By Theorem 9.2.23 and by (9.1.1), none of the morphisms $F(\mathrm{id}_X(s_1s_2)^k)$ with $0<k<m_{12}^X$ and $F(\mathrm{id}_X(s_1s_2)^ks_1)$ with $0\le k<m_{12}^X$ are the identity on $\mathbb{Z}^2$. Since $\mathcal{G}$ is simply connected, this implies that the $2m_{12}^X$ points $r_1(r_2r_1)^k(X)$ and $(r_2r_1)^k(X)$ with $0\le k<m_{12}^X$ are pairwise distinct. Thus $n=m_{12}^X$ and $|\mathcal{X}|=2m_{12}^X$, since $\mathcal{G}$ is connected and has rank two. Then $(c_1,\dots,c_n)\in\mathcal{A}^+$ and $(c_k)_{k\ge1}=(c_1,\dots,c_n)^\infty$ by Theorem 10.3.21. Thus (2) holds.

(3) Clearly, if $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic then they have the same number of points. Hence we may assume that $q=q'$. Then, by construction, $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic if and only if there exist $X\in\{1,\dots,2n\}$ and $i\in\{1,2\}$ such that the characteristic sequence of $\mathcal{G}$ with respect to X and i coincides with the characteristic sequence of $\mathcal{G}'$ with respect to $X'=1$ and $i=1$. Therefore (3) follows from Lemma 10.3.16. $\square$

The structure of root strings in root systems of finite Cartan graphs of rank two is more complicated than in usual root systems. We illustrate this in an example.

Example 10.3.30. Let $\mathcal{G}$ be the semi-Cartan graph of rank two with set of labels $I=\{1,2\}$, and with four points X_i, $i\in\mathbb{Z}_4=\{\bar0,\bar1,\bar2,\bar3\}$, such that

$$r_1(X_{\bar0})=X_{\bar1},\ r_1(X_{\bar2})=X_{\bar3},\quad r_2(X_{\bar0})=X_{\bar3},\ r_2(X_{\bar1})=X_{\bar2},$$

and the Cartan matrices of $\mathcal{G}$ are

$$A^{X_{\bar0}}=\begin{pmatrix}2&-3\\-3&2\end{pmatrix},\qquad A^{X_{\bar1}}=\begin{pmatrix}2&-3\\-2&2\end{pmatrix},$$
$$A^{X_{\bar3}}=\begin{pmatrix}2&-1\\-3&2\end{pmatrix},\qquad A^{X_{\bar2}}=\begin{pmatrix}2&-1\\-2&2\end{pmatrix}.$$

The characteristic sequence of $\mathcal{G}$ with respect to $X_{\bar0}$ and 1 is then $(3,2,1,3)^\infty$. The smallest positive integer n with $(r_2r_1)^n(X_{\bar0})=X_{\bar0}$ is $n=2$. Thus $\varpi=3$ in Theorem 10.3.21. Further, $\mathcal{G}$ is a finite Cartan graph by Theorem 10.3.21 with eight positive roots, since

$$(3,2,1,3,3,2,1,3)=V_3V_2V_2V_4V_3V_2(0,0)\in\mathcal{A}^+.$$

The set of positive roots at $X_{\bar0}$ is

$$\{1,1^32,1^52^2,1^22,12,12^2,12^3,2\}$$

by Corollary 10.3.24, where we abbreviate $a\alpha_1 + b\alpha_2$ by 1^a2^b for all $a, b \in \mathbb{N}_0$. We see that 1^52^2 and 12^2 are positive roots at $X_{\bar{0}}$, but 1^32^2 is not a positive root. However, $(4\alpha_1 + 2\alpha_2)/2$ and $(2\alpha_1 + 2\alpha_2)/2$ are positive roots.

We deduce a general claim supporting the observation in Example 10.3.30.

Proposition 10.3.31. *Let $\mathcal{G}$ be a finite Cartan graph of rank two. Let i, j be the labels of $\mathcal{G}$ and let X be a point of $\mathcal{G}$. Let $a, b \in \mathbb{N}_0$ with $b \geq 1$ such that $a\alpha_i + b\alpha_j \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$. Then $c\alpha_i + \alpha_j \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ for all $0 \leq c \leq a/b$.*

Proof. We proceed by induction on $a+b$. If $a+b = 1$ then $a = 0$ and the claim holds trivially. Assume now that $a+b \geq 2$. By Corollary 10.3.27, the root $a\alpha_i+b\alpha_j$ is the sum of two positive real roots $\beta_1 = a_1\alpha_i + b_1\alpha_j$ and $\beta_2 = a_2\alpha_i + b_2\alpha_j$, where $a_1, a_2, b_1, b_2 \in \mathbb{N}_0$. We distinguish two cases.

First assume that $b_1 = 0$. Then $\beta_1 = \alpha_i$. If $b = 1$ then $a\alpha_i + \alpha_j \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ by assumption and $c\alpha_i + \alpha_j \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ for all $0 \leq c \leq a - 1$ by induction hypothesis applied to $\beta_2 = (a - 1)\alpha_i + \alpha_j$.

If $b > 1$, then there is no integer c with $(a - 1)/b < c \leq a/b$. Hence the claim holds again by induction hypothesis applied to β_2.

Now assume that $b_1, b_2 > 0$. Then $a_1/b_1 \geq a/b$ or $a_2/b_2 \geq a/b$, since otherwise $a_1b < b_1a$, $a_2b < b_2a$, and hence

$$ab = (a_1 + a_2)b < (b_1 + b_2)a = ba,$$

which is absurd. Thus the Proposition holds by applying the induction hypothesis to β_1 and β_2. $\square$

10.4. Root systems

We are going to introduce root systems over Cartan graphs. We prove in Theorem 10.4.7 that finite Cartan graphs have a unique reduced root system, and that infinite Cartan graphs have no finite root system. We also show in Theorem 10.4.13 under some mild assumption that any positive real root is the sum of two positive real roots. Finally, we discuss the notion of irreducibility.

In the whole section let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph.

Definition 10.4.1. For all $X \in \mathcal{X}$ let R^X be a subset of $\mathbb{Z}^I$ with the following properties.

1. $0 \notin R^X$ and $\alpha_i \in R^X$ for all $X \in \mathcal{X}$ and $i \in I$.
2. $R^X \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ for all $X \in \mathcal{X}$.
3. For any $X \in \mathcal{X}$ and $i \in I$, $s_i^X(R^X) = R^{r_i(X)}$.

Then we say that the pair $(\mathcal{G}, (R^X)_{X\in\mathcal{X}})$ is a **root system over** $\mathcal{G}$. A root system over $\mathcal{G}$ is said to be **reduced** if for all $X \in \mathcal{X}$ and $\alpha \in R^X$ the roots α and $-\alpha$ are the only rational multiples of α in R^X. A root system over $\mathcal{G}$ is **finite** if R^X is a finite set for all $X \in \mathcal{X}$. The elements of $R_+^X = R^X \cap \mathbb{N}_0^I$ are called **positive roots** at X.

Remark 10.4.2. Our definition is very different from the usual definition of a root system, see for example [**Bou68**, Ch. VI, §1]. However it is known, that any finite reduced root system R in an n-dimensional Euclidean space has a basis $\alpha_1, \ldots, \alpha_n$. If one expresses the roots as linear combinations of the basis vectors, and lets s_i denote the simple reflection on α_i for all $1 \leq i \leq n$, then there is a Cartan graph with one point for which Axioms (1)–(3) in Definition 10.4.1 hold.

REMARK 10.4.3. Assume that $\mathcal{G}$ has precisely one point X. Then the root system in the sense of [**Kac90**, §1.3] associated to the Cartan matrix A^X satisfies the axioms of a root system over $\mathcal{G}$ in our sense.

There is always at least one reduced root system over $\mathcal{G}$, as the following example shows.

EXAMPLE 10.4.4. The pair $(\mathcal{G}, (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X\in\mathcal{X}})$ is a root system over $\mathcal{G}$. Indeed, Axioms (1) and (3) follow from the definition of $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ for all $X \in \mathcal{X}$, and (2) follows from (CG3). The root system $(\mathcal{G}, (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X\in\mathcal{X}})$ is reduced by Remark 9.1.16(2).

The root system in Example 10.4.4 is important for several reasons.

LEMMA 10.4.5. *Let $(\mathcal{G}, (R^X)_{X\in\mathcal{X}})$ be a root system over $\mathcal{G}$. Then for any $X \in \mathcal{X}$, $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is contained in R^X.*

PROOF. By Definition 10.4.1(1), $\alpha_i \in R^X$ for all $i \in I$ and $X \in \mathcal{X}$. Thus the claim follows from Definition 10.4.1(3) and the definition of $\boldsymbol{\Delta}^{X\,\mathrm{re}}$. ☐

Now we prove that there is at most one finite reduced root system over $\mathcal{G}$. If it exists, then it is of the form given in Example 10.4.4. For the proof we use an analogue of Lemma 9.1.19 for reduced root systems over $\mathcal{G}$.

LEMMA 10.4.6. *Let $(\mathcal{G}, (R^X)_{X\in\mathcal{X}})$ be a reduced root system over $\mathcal{G}$. Then for any $X \in \mathcal{X}$ and $i \in I$, s_i^X induces bijections*

$$s_i^X : R_+^X \setminus \{\alpha_i\} \to R_+^{r_i(X)} \setminus \{\alpha_i\}, \quad s_i^X : R_-^X \setminus \{-\alpha_i\} \to R_-^{r_i(X)} \setminus \{-\alpha_i\}.$$

PROOF. Follow the arguments in the proof of Lemma 9.1.19. ☐

THEOREM 10.4.7. *The following hold.*

(1) *Assume that $\mathcal{G}$ is finite. Then $(\mathcal{G}, (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X\in\mathcal{X}})$ is the only reduced root system over $\mathcal{G}$.*
(2) *Assume that $\mathcal{G}$ is not finite. Then there is no finite root system over $\mathcal{G}$.*

PROOF. (1) Since $\mathcal{G}$ is finite, $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is finite for all $X \in \mathcal{X}$. By Example 10.4.4, $(\mathcal{G}, (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X\in\mathcal{X}})$ is a finite reduced root system over $\mathcal{G}$. Let now $(\mathcal{G}, (R^X)_{X\in\mathcal{X}})$ be a reduced root system over $\mathcal{G}$. Let $X \in \mathcal{X}$ and let $\beta \in R_+^X$. By Proposition 9.3.9(1) there exist $Y \in \mathcal{X}$ and $w_0 \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y, X)$ such that $w_0^{-1}(\alpha) \in -\mathbb{N}_0^I$ for all $\alpha \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$. Thus $w_0^{-1}(\beta) \in -\mathbb{N}_0^I$, since β is a sum of positive real roots. Let $N = \ell(w_0)$ and $i_1, \dots, i_N \in I$ such that $w_0 = \mathrm{id}_X s_{i_1} \cdots s_{i_N}$. Then there exists $1 \le k \le N$ with $s_{i_k} \cdots s_{i_1}^X(\beta) \notin \mathbb{N}_0^I$. Let k be minimal. Then $s_{i_{k-1}} \cdots s_{i_1}^X(\beta) = \alpha_{i_k}$ by Lemma 10.4.6. Thus $\beta \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$.

(2) follows directly from Lemma 10.4.5. ☐

Next we develop some properties of finite reduced root systems over $\mathcal{G}$.

LEMMA 10.4.8. *Let $X \in \mathcal{X}$, $i, j \in I$, and $m \in \mathbb{Z}$. Assume that $\mathcal{G}$ is finite and $i \ne j$. Then $\alpha_j + m\alpha_i \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ if and only if $0 \le m \le -a_{ij}^X$.*

PROOF. Assume that $\alpha_j + m\alpha_i \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$. Then $0 \le m \le -a_{ij}^X$ by (CG3) and by Lemma 10.1.9. For the converse, by Corollary 9.4.19 it is enough to show that $\alpha_j + m\alpha_i$ for all $0 \le m \le -a_{ij}^X$ is a positive real root of $\mathcal{G}|\{i,j\}$ at X. The latter follows from Lemma 9.4.10 and Proposition 10.3.31, since $\alpha_j - a_{ij}^X\alpha_i \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ by Lemma 10.1.9. ☐

LEMMA 10.4.9. *Assume that $\mathcal{G}$ is finite. Let $X \in \mathcal{X}$, $1 \le k \le |I|$, and let $J \subseteq I$ with $|J| = k$, and let $\beta_1, \dots, \beta_k \in \boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \sum_{t\in J} \mathbb{Z}\alpha_t$ be linearly independent elements. Then there exist $j, i_1, \dots, i_l \in J$, where $l \ge 0$, and a point Y of $\mathcal{G}$, such that $w = s_{i_1} \cdots s_{i_l}^Y \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y, X)$, $\beta_1, \dots, \beta_{k-1} \in \sum_{t\in J\setminus\{j\}} \mathbb{Z}w(\alpha_t)$, and $\beta_k \in \sum_{t\in J} \mathbb{N}_0 w(\alpha_t)$.*

PROOF. Let $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^I, \mathbb{Z})$ such that $f(\beta_n) = 0$ for all $1 \le n < k$ and $f(\beta_k) > 0$. We proceed by induction on the cardinality of

$$N = \Big\{\alpha \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}} \cap \sum_{t\in J} \mathbb{N}_0\alpha_t \,\big|\, f(\alpha) < 0\Big\}.$$

If $|N| = 0$, then $f(\alpha_t) \ge 0$ for all $t \in J$, and $f(\alpha_j) > 0$ for some $j \in J$. Hence $\beta_k \in \sum_{t\in J} \mathbb{N}_0\alpha_t$ since $f(\beta_k) > 0$, and $\beta_n \in \sum_{t\in J\setminus\{j\}} \mathbb{Z}\alpha_t$ for all $1 \le n < k$ since $f(\beta_n) = 0$.

Assume that $|N| > 0$. Then $f(\alpha_i) < 0$ for some $i \in J$, and hence

$$\Big|\Big\{\alpha \in \boldsymbol{\Delta}_+^{r_i(X)\,\mathrm{re}} \cap \sum_{t\in J} \mathbb{N}_0\alpha_t \,\big|\, (f s_i^{r_i(X)})(\alpha) < 0\Big\}\Big| = |N| - 1$$

because of Lemma 9.1.19(1) and since $f s_i^{r_i(X)}(\alpha_i) > 0$. Moreover, the roots $s_i^X(\beta_n) \in \boldsymbol{\Delta}^{r_i(X)\,\mathrm{re}}$ with $1 \le n \le k$ are linearly independent in $\mathbb{Z}^I$ and

$$(f s_i^{r_i(X)})(s_i^X(\beta_k)) = f(\beta_k) > 0, \quad (f s_i^{r_i(X)})(s_i^X(\beta_n)) = f(\beta_n) = 0$$

for all $1 \le n < k$. Thus by induction hypothesis there exist $j \in J$, $Y \in \mathcal{X}$, $l \ge 1$ and $i_2, \dots, i_l \in J$ such that $w' = s_{i_2} \cdots s_{i_l}^Y \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y, r_i(X))$ and

$$s_i^X(\beta_1), \dots, s_i^X(\beta_{k-1}) \in \sum_{t\in J\setminus\{j\}} \mathbb{Z}w'(\alpha_t), \quad s_i^X(\beta_k) \in \sum_{t\in J} \mathbb{N}_0 w'(\alpha_t).$$

Therefore the claim holds with $w = s_i^{r_i(X)} w'$. □

PROPOSITION 10.4.10. *Assume that $\mathcal{G}$ is finite. Let $X \in \mathcal{X}$, $1 \le k \le |I|$, and let $\beta_1, \dots, \beta_k \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ be linearly independent elements in $\mathbb{Q}^I$. Then there exist $w \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y, X)$, where $Y \in \mathcal{X}$, and pairwise distinct elements $j_1, \dots, j_k \in I$, such that*

$$\beta_n \in \sum_{l=1}^{n} \mathbb{N}_0 w(\alpha_{j_l}) \tag{10.4.1}$$

for all $1 \le n \le k$.

PROOF. We may assume that $k = |I|$. We prove for all $0 \le m \le |I|$ by induction on $|I| - m$, that there exist $w_m \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y_m, X)$, where $Y_m \in \mathcal{X}$, and pairwise distinct elements $j_1, \dots, j_k \in I$, such that

$$\beta_n \in \sum_{l=1}^{m} \mathbb{Z} w_m(\alpha_{j_l}) \tag{10.4.2}$$

for all $1 \le n \le m$ and

$$\beta_n \in \sum_{l=1}^{n} \mathbb{N}_0 w_m(\alpha_{j_l}) \tag{10.4.3}$$

for all $m < n \le k$. For $m = |I|$ this claim holds trivially, and for $m = 0$ it is equivalent to the Proposition.

Let $0 \le m < |I|$ such that the claim in the previous paragraph holds for $m+1$. Then $w_{m+1}^{-1}(\beta_n) \in \sum_{l=1}^{m+1} \mathbb{Z}\alpha_{j_l}$ for all $1 \le n \le m+1$ by (10.4.2). Note that $j_1, \dots, j_{m+1}$ may be permuted without effect on the claim. By Lemma 10.4.9 we may choose the labels $j_1, \dots, j_{m+1}$ such that there exist a point $Y_m \in \mathcal{X}$, a morphism $u \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y_m, Y_{m+1})$, and $q \ge 0$, $i_1, \dots, i_q \in \{j_1, \dots, j_{m+1}\}$ such that $u = s_{i_1} \cdots s_{i_q}^{Y_m}$, $w_{m+1}^{-1}(\beta_n) \in \sum_{l=1}^{m} \mathbb{Z}u(\alpha_{j_l})$ for all $1 \le n \le m$, and $w_{m+1}^{-1}(\beta_{m+1}) \in \sum_{l=1}^{m+1} \mathbb{N}_0 u(\alpha_{j_l})$. Let $w_m = w_{m+1}u$. Then (10.4.2) holds for all $1 \le n \le m$ together with (10.4.3) for $n = m+1$. Since $i_1, \dots, i_q \in \{j_1, \dots, j_{m+1}\}$ and by induction hypothesis

$$w_{m+1}^{-1}(\beta_n) \in \sum_{l=1}^{n} \mathbb{N}_0\alpha_{j_l}, \quad w_{m+1}^{-1}(\beta_n) \notin \sum_{l=1}^{m+1} \mathbb{N}_0\alpha_{j_l}$$

for all $m+2 \le n \le k$, we conclude that $u^{-1}w_{m+1}^{-1}(\beta_n) \in \sum_{l=1}^{n} \mathbb{N}_0\alpha_{j_l}$ for all $m+2 \le n \le k$. This finishes the proof of (10.4.3) and the Proposition. $\square$

REMARK 10.4.11. In general, in the situation of Proposition 10.4.10 it is not true that $\sum_{l=1}^{k} \mathbb{Z}\beta_l = \sum_{l=1}^{k} \mathbb{Z}w(\alpha_{j_l})$. Indeed, assume that the rank of $\mathcal{G}$ is two, $k = 2$, and $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + 2\alpha_2$. Then $\mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 \ne \mathbb{Z}^2$.

COROLLARY 10.4.12. *Let $a \in \mathbb{N}$, $X \in \mathcal{X}$, and let $\alpha, \beta \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ be linearly independent elements. Assume that $a\alpha + \beta \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ and $\beta - m\alpha \notin \cup_{k\ge 2} k\mathbb{Z}^I$ for all $m \in \mathbb{N}_0$. Then $m\alpha + \beta \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ for all $0 \le m \le a$.*

PROOF. By Proposition 10.4.10 with $k = 2$ there exist $i, j \in I$, $a_1, a_2 \in \mathbb{N}_0$, $Y \in \mathcal{X}$, and $w \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y, X)$, such that $\alpha = w(\alpha_i)$, $\beta = w(a_1\alpha_i + a_2\alpha_j)$, and $a_2 > 0$. Since $\beta - a_1\alpha \notin k\mathbb{Z}^I$ for all $k \ge 2$, we conclude that $a_2 = 1$. Then $(a_1+a)\alpha_i + \alpha_j = w^{-1}(a\alpha + \beta) \in \boldsymbol{\Delta}^{Y\,\mathrm{re}}$ by assumption, and hence for all $0 \le m \le a$, $m\alpha + \beta = w((a_1+m)\alpha_i + \alpha_j) \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ by Lemma 10.4.8. $\square$

THEOREM 10.4.13. *Assume that m_{ij}^X is finite for all $X \in \mathcal{X}$ and $i, j \in I$. Let $X \in \mathcal{X}$. Then any positive real root at X is either simple or the sum of two relatively prime positive real roots.*

PROOF. Let $\alpha \in \boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ be a non-simple root. Among the pairs (u, j) in $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X) \times I$ with $u(\alpha_j) = \alpha$ pick (w, i) such that $\ell(w) \le \ell(u)$ for all u. Then $\ell(w) \ne 0$, since $\alpha \ne \alpha_i$. Let $N = \ell(w)$ and let $j \in I$ such that $\ell(ws_j) < \ell(w)$. Then $j \ne i$ by Corollary 9.3.6, since $w(\alpha_i) = \alpha \in \mathbb{N}_0^I$.

Let $Y, Z \in \mathcal{X}$, $u \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y, X)$, and $v \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Z, Y)$ such that $w = uv$, $\ell(w) = \ell(u) + \ell(v)$, and $v^{-1} = \mathrm{Prod}_{ji}^Z(\ell(v))$. Assume that $\ell(v)$ is maximal. Then

$$u(\alpha_i), u(\alpha_j) \in \mathbb{N}_0^I \tag{10.4.4}$$

by Theorem 9.3.4 and since $\ell(w) = \ell(u) + \ell(v)$. Further, $\ell(u) < \ell(w)$, and hence $\alpha \notin \{u(\alpha_i), u(\alpha_j)\}$ by the minimality of $\ell(w)$. The construction of v yields that $u^{-1}(\alpha) = v(\alpha_i) \in \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$. Since $\alpha \in \mathbb{N}_0^I$, we conclude from (10.4.4) that $u^{-1}(\alpha) \in \boldsymbol{\Delta}^{Y\,\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)$ is a non-simple positive root at Y of the restriction $\mathcal{G}|\{i,j\}$. Since $\mathcal{G}|\{i,j\}$ is a Cartan graph by Lemma 9.4.10, m_{ij}^Y is finite, and $\alpha \notin \{\alpha_i, \alpha_j\}$, we conclude from Corollary 10.3.27 that

$$u^{-1}(\alpha) = \gamma_1 + \gamma_2 \tag{10.4.5}$$

for two relatively prime positive real roots γ_1, γ_2 of $\mathcal{G}|\{i,j\}$ at Y. By Corollary 9.4.19, $\gamma_1, \gamma_2 \in \mathbf{\Delta}_+^{Y\,\mathrm{re}}$. Moreover, $u(\gamma_1), u(\gamma_2) \in \mathbf{\Delta}_+^{X\,\mathrm{re}}$ by (10.4.4). These two roots are relatively prime, since u is an isomorphism. Hence $\alpha = u(\gamma_1) + u(\gamma_2)$ is a sum of two relatively prime positive real roots by (10.4.5). □

A special case of the next proposition extends the characterization of connected indecomposable finite Cartan graphs in Proposition 10.1.17.

Proposition 10.4.14. *Let X be a point of $\mathcal{G}$, and let $J \subseteq I$ with $J \neq \emptyset$. Assume that $\mathcal{G}|J$ is finite. The following are equivalent.*

(1) *The Cartan matrix $(a_{ij}^X)_{i,j\in J}$ is indecomposable.*
(2) $\sum_{j\in J} \alpha_j \in \mathbf{\Delta}^{X\,\mathrm{re}}$.

Proof. Since $\mathbf{\Delta}^{X\,\mathrm{re}} \cap \sum_{j\in J} \mathbb{Z}\alpha_j$ is the set of real roots of $\mathcal{G}|J$ at X by Corollary 9.4.19, and since $\mathcal{G}|J$ is a Cartan graph by Lemma 9.4.10, we may assume that $I = J$ and that $\mathcal{G}$ is finite.

Assume (2). Then (1) follows from Proposition 10.1.17.

Assume now (1). We prove (2) by induction on $|I|$. If $|I| = 1$ then the claim is trivial. If $|I| = 2$ then Lemma 9.3.1 implies that $|\mathbf{\Delta}_+^{X\,\mathrm{re}}| > 2$. Thus $\sum_{i\in I} \alpha_i \in \mathbf{\Delta}_+^{X\,\mathrm{re}}$ by Corollary 10.3.24.

Assume now that $|I| = 3$. Let $i, j, k \in I$ be pairwise distinct elements. By (1) we may assume that $a_{ij}^X \neq 0$ and $a_{jk}^X \neq 0$. Then $\alpha_j + \alpha_k \in \mathbf{\Delta}_+^{X\,\mathrm{re}}$ by induction hypothesis applied to $\mathcal{G}|\{j,k\}$. We consider two cases. First, if $a_{ik}^X = 0$, then $a_{ik}^{r_i(X)} = 0$, and Proposition 10.1.17 implies that $a_{jk}^{r_i(X)} \neq 0$. Therefore $\alpha_j + \alpha_k$ is a real root in $\mathbf{\Delta}_+^{r_i(X)\,\mathrm{re}}$. Then

$$s_i^{r_i(X)}(\alpha_j + \alpha_k) = -a_{ij}^X \alpha_i + \alpha_j + \alpha_k \in \mathbf{\Delta}^{X\,\mathrm{re}}.$$

Since $a_{ij}^X < 0$, Corollary 10.4.12 with $\alpha = \alpha_i$, $\beta = \alpha_j + \alpha_k$ implies that $\alpha_i + \alpha_j + \alpha_k$ is a real root in $\mathbf{\Delta}^{X\,\mathrm{re}}$.

In the second of two cases, a_{ik}^X is non-zero. Then either

$$a_{jk}^{r_i(X)} = 0, \text{ and } \gamma = \alpha_i + \alpha_j + \alpha_k \in \mathbf{\Delta}^{r_i(X)\,\mathrm{re}}$$

by the previous paragraph, or

$$a_{jk}^{r_i(X)} \neq 0, \text{ and } \gamma = \alpha_j + \alpha_k \in \mathbf{\Delta}^{r_i(X)\,\mathrm{re}}$$

by induction hypothesis. In both cases,

$$s_i^{r_i(X)}(\gamma) = a\alpha_i + \alpha_j + \alpha_k \in \mathbf{\Delta}^{X\,\mathrm{re}}$$

for some $a \geq 1$. Hence $\alpha_i + \alpha_j + \alpha_k \in \mathbf{\Delta}^{X\,\mathrm{re}}$ by Corollary 10.4.12 with $\alpha = \alpha_i$, $\beta = \alpha_j + \alpha_k$.

Assume now that $|I| \geq 4$. Let $r = |I|$ and let $i_1, \dots, i_r \in I$ be pairwise distinct elements such that for any $2 \leq m \leq r$ there exists $1 \leq j < m$ with $a_{i_j i_m}^X \neq 0$. In particular, $a_{i_1 i_2}^X \neq 0$ and hence $\alpha_{i_1} + \alpha_{i_2} \in \mathbf{\Delta}^{X\,\mathrm{re}}$. Let

$$\beta_1 = \alpha_{i_1} + \alpha_{i_2}, \quad \beta_2 = \alpha_{i_3}, \quad \beta_3 = \alpha_{i_4}, \quad \dots, \quad \beta_{r-1} = \alpha_{i_r}.$$

By Proposition 10.4.10 there exist $Y \in \mathcal{X}$, a morphism $w \in \mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y, X)$ and labels $j_1, \dots, j_{r-1} \in I$ such that $\beta_m \in \sum_{l=1}^m \mathbb{N}_0 w(\alpha_{j_l})$ for all $1 \leq m \leq r-1$. For any $1 \leq m \leq r-1$, β_m is the only element in $\beta_m - \sum_{l=1}^{m-1} \mathbb{N}_0 \beta_l$ which is a multiple of a positive root. Thus $\beta_m = w(\alpha_{j_m})$ for all $1 \leq m \leq r-1$ by induction on m.

We prove that $A^Y|\{j_1,\dots,j_{r-1}\}$ is indecomposable. Then $\sum_{l=1}^{r-1}\alpha_{j_l} \in \boldsymbol{\Delta}^{Y\,\mathrm{re}}$ by induction hypothesis, and hence

$$\sum_{l=1}^{r-1} w(\alpha_{j_l}) = \sum_{l=1}^{r-1}\beta_l = \sum_{l=1}^{r}\alpha_{i_l} \in \boldsymbol{\Delta}^{X\,\mathrm{re}}.$$

Let $m \geq 2$. Since $a^X_{i_1 i_2} \neq 0$ and $a^X_{i_l i_{m+1}} \neq 0$ for some $1 \leq l \leq m$, induction hypothesis implies that $\beta_m + \beta_l \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ for some $1 \leq l < m$. Hence for this l we obtain that $\alpha_{j_m} + \alpha_{j_l} \in \boldsymbol{\Delta}^{Y\,\mathrm{re}}$. Therefore $A^Y|\{j_1,\dots,j_{r-1}\}$ is indecomposable and the proof is completed. $\square$

Definition 10.4.15. Let $(\mathcal{G},(R^X)_{X\in\mathcal{X}})$ be a root system over $\mathcal{G}$. We say that $(\mathcal{G},(R^X)_{X\in\mathcal{X}})$ is **reducible**, if there is a decomposition $I = I_1 \cup I_2$ into non-empty disjoint subsets such that

$$R^X = \Big(R^X \cap \sum_{i\in I_1}\mathbb{Z}\alpha_i\Big) \cup \Big(R^X \cap \sum_{i\in I_2}\mathbb{Z}\alpha_i\Big)$$

for all $X \in \mathcal{X}$. Root systems, which are not reducible, are called **irreducible**.

Corollary 10.4.16. *Assume that $\mathcal{G}$ is connected and finite. Let $X \in \mathcal{X}$. The following are equivalent.*

1. *$\mathcal{G}$ is indecomposable.*
2. *$(\mathcal{G},(\boldsymbol{\Delta}^{Y\,\mathrm{re}})_{Y\in\mathcal{X}})$ is irreducible.*
3. *$\sum_{i\in I}\alpha_i \in \boldsymbol{\Delta}^{Y\,\mathrm{re}}$ for all $Y \in \mathcal{X}$.*
4. *$\sum_{i\in I}\alpha_i \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$.*

Proof. (1) is equivalent to (2) by Proposition 10.1.17 without using the finiteness assumption. Further, (1) and Proposition 10.1.17 imply that A^Y is indecomposable for all $Y \in \mathcal{X}$. Thus (3) follows from (1) by Proposition 10.4.14. Finally, (4) follows from (3) trivially and (4) implies (1) by Proposition 10.4.14. $\square$

10.5. Notes

10.1. Coverings of semi-Cartan graphs have been introduced in [**CH09a**]. Decompositions of (semi-)Cartan graphs are discussed in [**CH09b**].

10.2. The classification of Cartan matrices into three types is due to Vinberg, see Section 4 in [**Vin71**]. Our presentation and nomenclature is based on [**Kac90**], Chapter 4.

10.3. The elements of $\mathcal{A}^+$ have been studied already in the work [**CC73**] of Conway and Coxeter, where they are called **quiddity cycles**. The relationship between finite Cartan graphs of rank two and sequences in $\mathcal{A}^+$, as well as many properties of the map η have been observed in [**CH09a**]. There also a classification of finite Cartan graphs of rank two is given. An interpretation of these results from the perspective of triangulations of convex n-gons was given in [**CH11**]. Proposition 10.3.11, Theorems 10.3.14 and 10.3.21, and Corollary 10.3.24 have been proven in [**HW15**], although the ideas behind Theorem 10.3.14 and Corollary 10.3.24 were available already in [**CH09a**] and [**CH11**]. A classification of finite Cartan graphs of rank three and higher was obtained in [**CH12**] and [**CH15**], respectively, using heavy computer calculations.

10.4. The concept of a root system over a Cartan graph appeared already in [**HY08**] and [**CH09b**]. Theorem 10.4.7 is a direct consequence of Propositions 2.9 and 2.12 in [**CH09b**]. Proposition 10.4.10 is [**CH12**], Theorem 2.4. Theorem 10.4.13 was proven in rank two in [**CH11**], Corollary 3.8, and in full generality in [**CH12**], Theorem 2.10.

CHAPTER 11

Cartan graphs of Lie superalgebras

Among the classical algebraic objects the regular Kac-Moody superalgebras, in particular the Kac-Moody algebras, admit a Cartan graph. We prove this in Theorem 11.2.10, and in Corollary 11.2.12 we discuss the finite-dimensional case. The Chapter starts with the basics of the theory of Lie superalgebras and then the structures needed for the construction of the Cartan graph are studied.

In this chapter, the ground field is the field of complex numbers.

11.1. Lie superalgebras

DEFINITION 11.1.1. A **Lie superalgebra** is a $\mathbb{Z}_2$-graded complex vector space $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (the **commutator**) satisfying the following axioms.

(1) $[x, y] \in \mathfrak{g}_{i+j}$ for all $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, $i, j \in \mathbb{Z}_2$,
(2) $[x, y] = -(-1)^{ij}[y, x]$ for all $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, $i, j \in \mathbb{Z}_2$,
(3) $[x, [y, z]] = [[x, y], z]+(-1)^{ij}[y, [x, z]]$ for all $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, $z \in \mathfrak{g}$, $i, j \in \mathbb{Z}_2$. (**Jacobi identity**)

The subspaces $\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{\bar 1}$ are called the **even** and **odd part of** $\mathfrak{g}$, respectively. The even part $\mathfrak{g}_{\bar 0}$ together with the restriction of $[\cdot,\cdot]$ to $\mathfrak{g}_{\bar 0} \times \mathfrak{g}_{\bar 0}$ is a Lie algebra. As usual, we let $\operatorname{ad} x(y) = [x, y]$ for all $x, y \in \mathfrak{g}$.

A graded linear map $f : \mathfrak{g} \to \mathfrak{h}$ between Lie superalgebras is a **homomorphism of Lie superalgebras** if $f([x, y]) = [f(x), f(y)]$ for all $x, y \in \mathfrak{g}$.

REMARK 11.1.2. The axioms of a Lie superalgebra $\mathfrak{g}$ imply that

$$[[x, y], z] = [x, [y, z]] + (-1)^{jk}[[x, z], y]$$

for all $x \in \mathfrak{g}$, $y \in \mathfrak{g}_j$, $z \in \mathfrak{g}_k$, $j, k \in \mathbb{Z}_2$.

EXAMPLE 11.1.3. Let $A = A_{\bar 0} \oplus A_{\bar 1}$ be a $\mathbb{Z}_2$-graded associative algebra (over the complex numbers) as defined in Section 5.1. Then A is a Lie superalgebra with commutator $[a, b] = ab - (-1)^{ij}ba$ for any $a \in A_i$, $b \in A_j$, $i, j \in \mathbb{Z}_2$.

LEMMA 11.1.4. *Let $\mathfrak{g}$ be a Lie superalgebra and let $x_1, \dots, x_k \in \mathfrak{g}$ be homogeneous elements with $k \geq 1$. Then any iterated bracket of $x_1, \dots, x_k$, in which x_1 appears at least once, is contained in the linear span of the elements $(\operatorname{ad} x_{i_1}) \cdots (\operatorname{ad} x_{i_m})(x_1)$ with $m \geq 0$ and $i_1, \dots, i_m \in \{1, \dots, k\}$.*

PROOF. By Axiom 11.1.1(2), any such iterated bracket is up to a sign equal to $(\operatorname{ad} y_1) \cdots (\operatorname{ad} y_l)(x_1)$, where $l \geq 0$ and $y_1, \dots, y_l$ are iterated brackets of $x_1, \dots, x_k$. The rest follows from the Jacobi identity. □

Among the Lie superalgebras there are the contragredient and the basic classical Lie superalgebras, which are related to Cartan graphs. We need some preparation before we introduce the definitions. Recall that $(\alpha_i)_{1\le i\le n}$ is the standard basis of $\mathbb{Z}^n$.

Definition 11.1.5. Let $n\in\mathbb{N}$,

$$B=(b_{ij})_{1\le i,j\le n}\in\mathbb{C}^{n\times n},\quad \tau=(\tau_i)_{1\le i\le n}\in\mathbb{Z}_2^n,$$

and let $\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}(B,\tau)$ be the Lie superalgebra given by generators e_i, f_i, and h_i with $1\le i\le n$, and relations

$$[h_i,h_j]=0,\quad [h_i,e_j]=b_{ij}e_j,\quad [h_i,f_j]=-b_{ij}f_j,\quad [e_i,f_j]=\delta_{ij}h_i,$$
$$h_i\in\tilde{\mathfrak{g}}_{\bar 0},\quad e_i,f_i\in\tilde{\mathfrak{g}}_{\tau_i}$$

for all $i,j\in\{1,\dots,n\}$. The Lie subsuperalgebras of $\tilde{\mathfrak{g}}$ generated by the sets

$$\{e_i\,|\,1\le i\le n\},\ \{f_i\,|\,1\le i\le n\},\ \text{and}\ \{h_i\,|\,1\le i\le n\}$$

are denoted by $\tilde{\mathfrak{n}}_+$, $\tilde{\mathfrak{n}}_-$, and $\mathfrak{h}$, respectively. For any $\alpha=\sum_{i=1}^n a_i\alpha_i\in\mathbb{Z}^n$ let

$$h_\alpha=\sum_{i=1}^n a_ih_i\in\mathfrak{h},\quad \tau_\alpha=\sum_{i=1}^n a_i\tau_i\in\mathbb{Z}_2.$$

We write $\langle\cdot,\cdot\rangle_B$ for the bilinear form on $\mathbb{C}^n$ with $\langle\alpha,\beta\rangle_B=\alpha^tB\beta$ for all $\alpha,\beta\in\mathbb{C}^n$.

Remark 11.1.6. It is more common to take for the definition of the Lie superalgebras $\tilde{\mathfrak{g}}(B,\tau)$ a larger Cartan subalgebra $\mathfrak{h}$ in order to implement the $\mathbb{Z}^n$-grading in Lemma 11.1.9 below using inner superderivations. For our purposes, the given less technical definition of $\tilde{\mathfrak{g}}(B,\tau)$ together with the grading will be sufficient.

Remark 11.1.7. In the setting of Definition 11.1.5, let $1\le i\le n$. If $\tau_i=\bar 0$, then $[e_i,e_i]=[f_i,f_i]=0$ in $\tilde{\mathfrak{g}}$ because of Definition 11.1.1(2). Similarly, if $\tau_i=\bar 1$, then the axioms of a Lie superalgebra imply that

$$[[e_i,e_i],e_i]=[f_i,[f_i,f_i]]=0.$$

(The proof uses that the characteristic of $\mathbb{C}$ is not 3.)

Remark 11.1.8. We construct a non-trivial homomorphism of Lie superalgebras from $\tilde{\mathfrak{g}}$ in Definition 11.1.5 to a $\mathbb{Z}_2$-graded associative algebra, and prove that the elements $e_1,\dots,e_n,h_1,\dots,h_n,f_1,\dots,f_n$ of $\tilde{\mathfrak{g}}$ are linearly independent.

Let $n\in\mathbb{N}$, let $B\in\mathbb{C}^{n\times n}$, and let V be an n-dimensional $\mathbb{Z}_2$-graded vector space. Let $x_1,\dots,x_n$ be a basis of V consisting of homogeneous elements, and for each $1\le i\le n$ let $\tau_i\in\mathbb{Z}_2$ be the degree of x_i. The polynomial ring $H=\mathbb{C}[h_1,\dots,h_n]$ in n indeterminates $h_1,\dots,h_n$ is a Hopf algebra, where $h_1,\dots,h_n$ are primitive elements. The free algebra $T(V)$ has a unique H-module algebra structure with action $\triangleright$ of H on $T(V)$ satisfying $h_i\triangleright x_j=b_{ij}x_j$ for all $1\le i,j\le n$.

Let $A=T(V)\#H$, see Definition 2.6.8. Then A is a $\mathbb{Z}_2$-graded algebra such that for any $1\le i\le n$, x_i has degree τ_i and h_i is even. It can be presented by generators $x_1,\dots,x_n$ and $h_1,\dots,h_n$ and relations

$$h_ix_j=x_jh_i+b_{ij}x_j,\quad h_ih_j=h_jh_i$$

for all $1\le i,j\le n$. Moreover, since $\mathbb{Z}_2$ is finite, $\mathrm{End}(A)$ becomes a $\mathbb{Z}_2$-graded algebra with

$$\mathrm{End}(A)_p=\{f\in\mathrm{End}(A)\mid\forall x\in A_{p'},p'\in\mathbb{Z}_2: f(x)\in A_{p+p'}\}$$

for all $p\in\mathbb{Z}_2$.

For any $1 \le i \le n$, left multiplication by x_i (denoted by e'_i) and left multiplication by h_i (denoted by h'_i) are graded endomorphisms of A of degree τ_i and $\bar{0}$, respectively. Moreover, for any $1 \le i \le n$, there is a unique algebra automorphism σ_i of A of degree $\bar{0}$ with

$$\sigma_i(x_j) = (-1)^{\tau_i \tau_j} x_j, \quad \sigma_i(h_j) = h_j + b_{ji} 1,$$

and a unique (σ_i, id)-derivation f'_i of degree τ_i with

$$f'_i(x_j) = -(-1)^{\tau_i} \delta_{ij} h_i, \quad f'_i(h_j) = 0$$

for all $1 \le j \le n$. One then checks that $\rho : \tilde{\mathfrak{g}}(B, \tau) \to \mathrm{End}(A)$ with

$$e_i \mapsto e'_i, \quad h_i \mapsto h'_i, \quad f_i \mapsto f'_i$$

for any $1 \le i \le n$, is a homomorphism of Lie superalgebras, and the endomorphisms $e'_1, \dots, e'_n, h'_1, \dots, h'_n, f'_1, \dots, f'_n$ of A are linearly independent. Thus the elements $e_1, \dots, e_n, h_1, \dots, h_n, f_1, \dots, f_n$ of $\tilde{\mathfrak{g}}(B, \tau)$ are linearly independent.

LEMMA 11.1.9. *Let $n \in \mathbb{N}$, $B \in \mathbb{C}^{n \times n}$, and let $\tau \in \mathbb{Z}_2^n$.*

(1) *The Lie superalgebra $\tilde{\mathfrak{g}}(B, \tau)$ has a unique $\mathbb{Z}^n$-grading $\tilde{\mathfrak{g}} = \oplus_{\alpha \in \mathbb{Z}^n} \tilde{\mathfrak{g}}_\alpha$ with $\deg(e_i) = -\deg(f_i) = \alpha_i$ and $\deg(h_i) = 0$ for all $1 \le i \le n$.*
(2) *$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$ and $\dim \mathfrak{h} = n$.*

PROOF. (1) The Lie superalgebra $\tilde{\mathfrak{g}}$ is $\mathbb{Z}^n$-graded, since the defining relations are homogeneous.

(2) Let

$$\begin{aligned} G &= \{e_i, h_i, f_i \mid 1 \le i \le n\}, \\ G^+ &= \{e_i \mid 1 \le i \le n\}, \\ G^- &= \{f_i \mid 1 \le i \le n\}. \end{aligned}$$

For any $k \ge 0$ let $F_k(\tilde{\mathfrak{g}})$, $F_k^+(\tilde{\mathfrak{g}})$, and $F_k^-(\tilde{\mathfrak{g}})$ denote the linear span of all elements $(\mathrm{ad}\, x_1)(\mathrm{ad}\, x_2) \cdots (\mathrm{ad}\, x_l)(x_{l+1}) \in \tilde{\mathfrak{g}}$, such that $0 \le l \le k$ and $x_1, x_2, \dots, x_{l+1} \in G$, $x_1, x_2, \dots, x_{l+1} \in G^+$, and $x_1, x_2, \dots, x_{l+1} \in G^-$, respectively. Using the Jacobi identity and the defining relations of $\tilde{\mathfrak{g}}$, one proves the following.

(a) $\tilde{\mathfrak{g}} = \bigcup_{k \ge 0} F_k(\tilde{\mathfrak{g}})$.
(b) $\mathrm{ad}\, e_i(\mathfrak{h}) \subseteq F_0^+(\tilde{\mathfrak{g}})$, $\mathrm{ad}\, h_i(\mathfrak{h}) = 0$, $\mathrm{ad}\, f_i(\mathfrak{h}) \subseteq F_0^-(\tilde{\mathfrak{g}})$ for any $1 \le i \le n$.
(c) For any $k \ge 0$, $\mathrm{ad}\, G^+(F_k^+(\tilde{\mathfrak{g}})) \subseteq F_{k+1}^+(\tilde{\mathfrak{g}})$, $\mathrm{ad}\, \mathfrak{h}(F_k^+(\tilde{\mathfrak{g}})) \subseteq F_k^+(\tilde{\mathfrak{g}})$, and $\mathrm{ad}\, G^-(F_k^+(\tilde{\mathfrak{g}})) \subseteq F_k(\tilde{\mathfrak{g}})$.
(d) For any $k \ge 0$ the relations $\mathrm{ad}\, G^+(F_k^-(\tilde{\mathfrak{g}})) \subseteq F_k(\tilde{\mathfrak{g}})$, $\mathrm{ad}\, \mathfrak{h}(F_k^-(\tilde{\mathfrak{g}})) \subseteq F_k^-(\tilde{\mathfrak{g}})$, and $\mathrm{ad}\, G^-(F_k^-(\tilde{\mathfrak{g}})) \subseteq F_{k+1}^-(\tilde{\mathfrak{g}})$ hold.

By induction on k then it follows that $F_k(\tilde{\mathfrak{g}}) = F_k^+(\tilde{\mathfrak{g}}) + \mathfrak{h} + F_k^-(\tilde{\mathfrak{g}})$ for any $k \ge 0$. Then Lemma 11.1.4 implies that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ + \mathfrak{h} + \tilde{\mathfrak{n}}_-$. The last sum is direct by (1). Moreover, $\dim \mathfrak{h} = n$ by Remark 11.1.8. $\square$

We continue with the study of the structure of the Lie superalgebras in Definition 11.1.5.

LEMMA 11.1.10. *Let $n \in \mathbb{N}$, $B \in \mathbb{C}^{n \times n}$, $\tau \in \mathbb{Z}_2^n$, and let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(B, \tau)$. The set $\mathfrak{I}$ of $\mathbb{Z}^n$-graded ideals of $\tilde{\mathfrak{g}}$ contained in $\tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_-$ contains a unique element $\mathfrak{r}$ such that $\mathfrak{m} \subseteq \mathfrak{r}$ for all $\mathfrak{m} \in \mathfrak{I}$. The quotient Lie superalgebra $\mathfrak{g}(B, \tau) = \tilde{\mathfrak{g}}/\mathfrak{r}$ is $\mathbb{Z}^n$-graded and is called a* **contragredient Lie superalgebra***.*

PROOF. The ideal $\mathfrak{r}$ is unique, since $\tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_-$ is a subspace of $\tilde{\mathfrak{g}}$ and the sum of $\mathbb{Z}^n$-graded ideals of $\tilde{\mathfrak{g}}$ is a $\mathbb{Z}^n$-graded ideal of $\tilde{\mathfrak{g}}$. Existence of $\mathfrak{r}$ is clear. Since $\mathfrak{r}$ is $\mathbb{Z}^n$-graded, the quotient $\mathfrak{g}(B,\tau)$ is $\mathbb{Z}^n$-graded as well. □

The following lemma is elementary but of relevance in view of the Weyl groupoid of a contragredient Lie superalgebra.

LEMMA 11.1.11. *Let $n \in \mathbb{N}$, $B, C \in \mathbb{C}^{n\times n}$, and $\tau \in \mathbb{Z}_2$. Then the following are equivalent.*

(1) *There exists a surjective homomorphism $\varphi : \tilde{\mathfrak{g}}(B,\tau) \to \mathfrak{g}(C,\tau)$ of Lie superalgebras with $\varphi(\tilde{\mathfrak{g}}(B,\tau)_\alpha) = \mathfrak{g}(C,\tau)_\alpha$ for all $\alpha \in \mathbb{Z}^n$.*
(2) *There exists an invertible diagonal matrix D with $B = DC$.*

PROOF. By the definitions of $\tilde{\mathfrak{g}}(B,\tau)$ and $\mathfrak{g}(C,\tau)$, (1) is equivalent to the existence of non-zero numbers λ_i, μ_i with $1 \le i \le n$ such that there is a homomorphism of Lie superalgebras $\varphi : \tilde{\mathfrak{g}}(B,\tau) \to \mathfrak{g}(C,\tau)$ with $\varphi(e_i) = \lambda_i e_i$, $\varphi(f_i) = \mu_i f_i$ for all $1 \le i \le n$. Any such homomorphism satisfies

$$\varphi(h_i) = \varphi([e_i, f_i]) = [\lambda_i e_i, \mu_i f_i] = \lambda_i \mu_i h_i$$

and

$$b_{ij}\lambda_j e_j = \varphi([h_i, e_j]) = [\lambda_i\mu_i h_i, \lambda_j e_j] = \lambda_i\mu_i c_{ij}\lambda_j e_j$$

for all $1 \le i, j \le n$. Thus, in view of Remark 11.1.8, (1) implies (2) with $d_{ii} = \lambda_i\mu_i$ for all $1 \le i \le n$.

Assume now (2). Then there is a unique homomorphism of Lie superalgebras $\varphi : \tilde{\mathfrak{g}}(B,\tau) \to \tilde{\mathfrak{g}}(C,\tau)$ with $\varphi(e_i) = d_{ii}e_i$, $\varphi(f_i) = f_i$ for all $1 \le i \le n$. This implies (1). □

REMARK 11.1.12. Let $n \in \mathbb{N}$ and $B, C \in \mathbb{C}^{n\times n}$ be symmetric matrices. Assume that $B = DC$ for some invertible diagonal matrix D and that C is not decomposable in the sense of Definition 10.1.15. Then $B = dC$ for some non-zero $d \in \mathbb{C}$. Indeed, assume that D is not a multiple of the identity. Let

$$I_1 = \{1 \le i \le n \mid d_{ii} = d_{11}\}, \quad I_2 = \{1, \dots, n\} \setminus I_1.$$

Let $i \in I_1$ and $j \in I_2$ such that $c_{ij} \ne 0$. Then

$$d_{ii}c_{ij} = b_{ij} = b_{ji} = d_{jj}c_{ji} = d_{jj}c_{ij}.$$

Hence $d_{ii} = d_{jj}$, a contradiction to the definition of I_1 and I_2.

LEMMA 11.1.13. *Let $n \in \mathbb{N}$, $B \in \mathbb{C}^{n\times n}$, $\tau \in \mathbb{Z}_2^n$, and let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(B,\tau)$. Let $1 \le i \le n$, $\beta, \beta' \in \mathbb{Z}^n \setminus \mathbb{Z}\alpha_i$, $x \in \tilde{\mathfrak{g}}_\beta$, $y \in \tilde{\mathfrak{g}}_{\beta'}$, and $\lambda \in \mathbb{C}$.*

(1) *If $\beta + \beta' = -\alpha_i$, $[\tilde{\mathfrak{g}}_\gamma, \tilde{\mathfrak{g}}_{-\gamma}] \subseteq \mathbb{C}h_\gamma$ for all $\gamma \in \{\beta, \beta+\alpha_i\}$, and $[x, y] = \lambda f_i$, then $[[e_i, x], y] = \lambda h_{\beta+\alpha_i}$, $[x, [e_i, y]] = -(-1)^{\tau_i\tau_\beta}\lambda h_\beta$.*
(2) *If $\beta + \beta' = \alpha_i$, $[\tilde{\mathfrak{g}}_\gamma, \tilde{\mathfrak{g}}_{-\gamma}] \subseteq \mathbb{C}h_\gamma$ for all $\gamma \in \{\beta, \beta-\alpha_i\}$, and $[x, y] = \lambda e_i$, then $[x, [y, f_i]] = \lambda h_\beta$ and $[[x, f_i], y] = -(-1)^{\tau_i\tau_{\beta'}}\lambda h_{\beta-\alpha_i}$.*

PROOF. We prove (1). The proof of (2) is similar.

By assumption and by Jacobi identity,

$$\lambda h_i = \lambda[e_i, f_i] = [e_i, [x, y]] = [[e_i, x], y] + (-1)^{\tau_i\tau_\beta}[x, [e_i, y]]. \tag{11.1.1}$$

Since $[\tilde{\mathfrak{g}}_\gamma, \tilde{\mathfrak{g}}_{-\gamma}] \subseteq \mathbb{C}h_\gamma$ for $\gamma \in \{\beta, \beta+\alpha_i\}$, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$[[e_i, x], y] = \lambda_1 h_{\beta+\alpha_i}, \quad [x, [e_i, y]] = \lambda_2 h_\beta. \tag{11.1.2}$$

Since $\beta \notin \mathbb{C}\alpha_i$, h_β and h_i are linearly independent by Lemma 11.1.9(2). Thus $\lambda_1+(-1)^{\tau_i\tau_\beta}\lambda_2 = 0$ and $\lambda_1 = \lambda$ by (11.1.1) and (11.1.2). This implies the claim. $\square$

LEMMA 11.1.14. *Let $n \in \mathbb{N}$, $B \in \mathbb{C}^{n\times n}$, $\tau \in \mathbb{Z}_2^n$, and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(B,\tau)$. Assume that B is symmetric. Then $[\tilde{\mathfrak{g}}_\beta, \tilde{\mathfrak{g}}_{-\beta}] \subseteq \mathbb{C}h_\beta$ for all $\beta \in \mathbb{Z}^n$.*

PROOF. By Definition 11.1.1(2) and by Lemma 11.1.9 it suffices to prove the claim for $\beta \in \sum_{i=1}^n \mathbb{N}_0\alpha_i$. We proceed by induction on the sum $|\beta|$ of the coefficients of β. For $\beta = 0$ the claim is trivial. If $\beta \in \mathbb{N}\alpha_i$ with $1 \le i \le n$, then $[\tilde{\mathfrak{g}}_\beta, \tilde{\mathfrak{g}}_{-\beta}]$ is contained both in $\mathfrak{h}$ and in the Lie subsuperalgebra of $\tilde{\mathfrak{g}}$ generated by e_i and f_i, and hence is contained in $\mathbb{C}h_i$.

Let $\beta = \sum_{k=1}^n b_k\alpha_k$ with $\sum_{k=1}^n b_k \ge 2$. Assume that $\beta \notin \mathbb{N}_0\alpha_i$ for any $1 \le i \le n$ and that $[\tilde{\mathfrak{g}}_\gamma, \tilde{\mathfrak{g}}_{-\gamma}] \subseteq \mathbb{C}h_\gamma$ for all $\gamma \in \mathbb{Z}^n$ with $|\gamma| < |\beta|$. By Lemma 11.1.4 it suffices to show that $[[x,e_i],[f_j,y]] \in \mathbb{C}h_\beta$ for all $1 \le i,j \le n$ with $b_i, b_j > 0$, $x \in \tilde{\mathfrak{g}}_{\beta-\alpha_i}$, and $y \in \tilde{\mathfrak{g}}_{-(\beta-\alpha_j)}$. So let $i,j \in \{1,\dots,n\}$ with $b_i,b_j > 0$ and let $x \in \tilde{\mathfrak{g}}_{\beta-\alpha_i}$, $y \in \tilde{\mathfrak{g}}_{-(\beta-\alpha_j)}$. By Lemma 11.1.4, if $\beta = m\alpha_k + \alpha_l$ for some $m \ge 1$ and $1 \le k,l \le n$ with $k \ne l$, then we may also assume that $i = j = k$.

By induction hypothesis and by Lemma 11.1.9, there exist complex numbers μ_1,μ_2,μ_3,μ_4 such that

$$[x,y] = \delta_{ij}\mu_1 h_{\beta-\alpha_i}, \qquad [[x,f_j],y] = -(-1)^{\tau_i\tau_{\beta-\alpha_i-\alpha_j}}\mu_3 f_i, \tag{11.1.3}$$

$$[[x,f_j],[e_i,y]] = \mu_2 h_{\beta-\alpha_i-\alpha_j}, \qquad [x,[e_i,y]] = -(-1)^{\tau_j\tau_{\beta-\alpha_i-\alpha_j}}\mu_4 e_j. \tag{11.1.4}$$

Induction hypothesis and Lemma 11.1.13 imply that $\mu_3 = \mu_2$ and $\mu_4 = \mu_2$. Then

$$\begin{aligned}
[[x,e_i],[f_j,y]] &= [x,[e_i,[f_j,y]]] - (-1)^{\tau_{\beta-\alpha_i}\tau_i}[e_i,[x,[f_j,y]]]\\
&= [x,[\delta_{ij}h_i,y]] + (-1)^{\tau_i\tau_j}[x,[f_j,[e_i,y]]]\\
&\quad - (-1)^{\tau_{\beta-\alpha_i}\tau_i}[e_i,[[x,f_j],y]] - (-1)^{\tau_{\beta-\alpha_i}(\tau_i+\tau_j)}[e_i,[f_j,[x,y]]]\\
&= -\delta_{ij}\langle\alpha_i,\beta-\alpha_i\rangle_B\mu_1 h_{\beta-\alpha_i} + (-1)^{\tau_i\tau_j}[[x,f_j],[e_i,y]]\\
&\quad + (-1)^{\tau_\beta\tau_j}[f_j,[x,[e_i,y]]] + (-1)^{\tau_i\tau_j}\mu_3 h_i - \delta_{ij}\langle\beta-\alpha_i,\alpha_i\rangle_B\mu_1 h_i\\
&= -\delta_{ij}\langle\alpha_i,\beta-\alpha_i\rangle_B\mu_1 h_\beta + (-1)^{\tau_i\tau_j}\mu_2 h_{\beta-\alpha_i-\alpha_j}\\
&\quad + (-1)^{\tau_i\tau_j}\mu_4 h_j + (-1)^{\tau_i\tau_j}\mu_3 h_i,
\end{aligned}$$

where the first two equations follow from the Jacobi identity, the third from (11.1.3) and the Jacobi identity, and the last one from the symmetry of B and from (11.1.4). Thus $[[x,e_i],[f_j,y]] \in \mathbb{C}h_\beta$ since $\mu_2 = \mu_3 = \mu_4$. $\square$

We now start the discussion of contragredient and basic classical Lie superalgebras.

DEFINITION 11.1.15. Let $\mathfrak{g}$ be a Lie superalgebra. A complex valued bilinear form f on $\mathfrak{g}$ is called

(1) **invariant** if $f([x,y],z) = f(x,[y,z])$ for all $x,y,z \in \mathfrak{g}$, and
(2) **supersymmetric** if $f(x,y) = (-1)^{ij}f(y,x)$ for all $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, and $i,j \in \mathbb{Z}_2$.

The Lie superalgebra $\mathfrak{g}$ is **basic classical** if $\mathfrak{g}$ is finite-dimensional, simple (that is, it has no proper ideals), its even part is a reductive Lie algebra (the direct sum of abelian and of simple ideals), and $\mathfrak{g}$ admits a non-degenerate invariant bilinear form.

REMARK 11.1.16. Let $\mathfrak{g}$ be a Lie superalgebra and let $X \subseteq \mathfrak{g}$ be a homogeneous subset. Suppose that $\mathfrak{g}$ is generated by X, that is, spanned by iterated brackets of elements of X. Then a bilinear form $f : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is invariant if and only if $f([x, y], z) = f(x, [y, z])$ for all $x, z \in \mathfrak{g}$ and $y \in X$. Indeed, let

$$\mathfrak{g}^f = \{y \in \mathfrak{g} \mid \forall x, z \in \mathfrak{g} : f([x, y], z) = f(x, [y, z])\}.$$

Clearly, $\mathfrak{g}^f$ is a subspace of $\mathfrak{g}$. Moreover, for any $y_1 \in \mathfrak{g}^f_{i_1}$, $y_2 \in \mathfrak{g}^f_{i_2}$, and any $x, z \in \mathfrak{g}$, where $i_1, i_2 \in \mathbb{Z}_2$,

$$\begin{aligned} f([x, [y_1, y_2]], z) &= f([[x, y_1], y_2], z) - (-1)^{i_1 i_2} f([[x, y_2], y_1], z) \\ &= f([x, y_1], [y_2, z]) - (-1)^{i_1 i_2} f([x, y_2], [y_1, z]) \\ &= f(x, [y_1, [y_2, z]]) - (-1)^{i_1 i_2} f(x, [y_2, [y_1, z]]) \\ &= f(x, [[y_1, y_2], z]) \end{aligned}$$

by Jacobi identity and the definition of $\mathfrak{g}^f$. This implies the claim.

PROPOSITION 11.1.17. *Let $n \in \mathbb{N}$, $B \in \mathbb{C}^{n\times n}$, $\tau \in \mathbb{Z}_2^n$, and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(B, \tau)$. Assume that B is symmetric. There is a unique complex valued invariant bilinear form $(\cdot\,|\,\cdot)$ on $\tilde{\mathfrak{g}}$ with*

$$(e_i \mid f_i) = (-1)^{\tau_i}(f_i \mid e_i) = 1, \quad (h_i \mid h_j) = b_{ij}, \quad (\tilde{\mathfrak{g}}_\alpha \mid \tilde{\mathfrak{g}}_\beta) = 0$$

for all $1 \le i, j \le n$ and $\alpha, \beta \in \mathbb{Z}^n$ with $\beta \ne -\alpha$. This form is $\mathbb{Z}^n$-graded, supersymmetric, and $[x, y] = (x \mid y)h_\alpha$ for all $x \in \tilde{\mathfrak{g}}_\alpha$, $y \in \tilde{\mathfrak{g}}_{-\alpha}$, $\alpha \in \mathbb{Z}^n$.

PROOF. Any invariant bilinear form $(\cdot\,|\,\cdot)$ on $\tilde{\mathfrak{g}}$ is uniquely determined by its values $(x\,|\,y)$ for $x \in \{e_i, f_i, h_i \mid 1 \le i \le n\}$ and $y \in \tilde{\mathfrak{g}}$. Thus the uniqueness of $(\cdot\,|\,\cdot)$ in the Proposition follows from Lemma 11.1.9.

By Lemma 11.1.14 and Lemma 11.1.9 there exists a unique $\mathbb{Z}^n$-graded complex valued bilinear form $(\cdot\,|\,\cdot)$ on $\tilde{\mathfrak{g}}$ such that

$$(h_i \mid h_j) = b_{ij}, \quad [x, y] = (x \mid y)h_\alpha$$

for all $1 \le i, j \le n$ and $x \in \tilde{\mathfrak{g}}_\alpha$, $y \in \tilde{\mathfrak{g}}_{-\alpha}$ with $\alpha \in \mathbb{Z}^n \setminus \{0\}$. The required properties of the form, except invariance, are clearly satisfied. By Remark 11.1.16, it suffices to prove for any $y \in \{h_i, e_i, f_i \mid 1 \le i \le n\}$ that $([x, y] \mid z) = (x \mid [y, z])$. The latter is clear for $y = h_i$, $1 \le i \le n$, by construction.

Let $1 \le i \le n$, $x \in \tilde{\mathfrak{g}}_\alpha$, and $z \in \tilde{\mathfrak{g}}_\beta$ with $\alpha, \beta \in \mathbb{Z}^n$. Then

$$([x, e_i] \mid z) = (x \mid [e_i, z]) \tag{11.1.5}$$

whenever $\alpha+\alpha_i+\beta \ne 0$. The same equation also holds if $\alpha \notin \mathbb{Z}\alpha_i$ and $\alpha+\alpha_i+\beta = 0$ because of Lemma 11.1.13. Moreover, if $\alpha, \beta \in \mathbb{Z}\alpha_i$ and $\alpha + \alpha_i + \beta = 0$, then the claim follows by easy calculations using Lemma 11.1.4 and Remark 11.1.7. Finally, the analog of (11.1.5) with f_i instead of e_i holds by similar reasons. □

REMARK 11.1.18. It is known, that the odd part of a basic classical Lie superalgebra $\mathfrak{g}$ is an irreducible module over the even part. Basic classical Lie superalgebras are classified by Kac, see [**Kac77**]. They can be presented as follows [**Kac77**, 2.5.1, Th. 3].

Let $n \in \mathbb{N}$, $\tau \in \mathbb{Z}_2^n$, and let $B = (b_{ij})_{1\le i,j\le n} \in \mathbb{C}^{n\times n}$ be a symmetric matrix. The center C of the contragredient Lie superalgebra $\mathfrak{g}(B, \tau)$ is contained in $\mathfrak{h}$ and is of dimension at most one. By [**Kac77**, Prop. 2.5.2], the quotient $\mathfrak{g}(B, \tau)/C$

is simple if and only if for all $i, j \in \{1, \dots, n\}$ there exist $t \geq 2$ and a family $(i_k)_{1 \leq k \leq t} \in \{1, \dots, n\}^t$ such that $i = i_1$, $j = i_t$, and

$$b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_{t-1} i_t} \neq 0. \tag{11.1.6}$$

(Equivalently, $n = 1$, $b_{11} \neq 0$ or $n \geq 2$ and B is indecomposable.) In this case, $\mathfrak{g}(B, \tau)/C$ is a basic classical Lie superalgebra if and only if it is finite-dimensional.

The Lie superalgebra $\mathfrak{g}(B, \tau)/C$ is isomorphic as a Lie superalgebra to the quotient of $\tilde{\mathfrak{g}}$ by the radical of the invariant form in Proposition 11.1.17.

11.2. Cartan graphs of regular Kac-Moody superalgebras

In this section we construct the Cartan graph of a regular Kac-Moody superalgebra attached to a symmetric indecomposable matrix. The construction can be easily adapted to basic classical Lie superalgebras as well.

LEMMA 11.2.1. *Let $n \in \mathbb{N}$, $B \in \mathbb{C}^{n \times n}$, $\tau \in \mathbb{Z}_2^n$, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(B, \tau)$, and $1 \leq i, j \leq n$.*

(1) *If $i \neq j$, then for any $m \geq 1$,*

$$(\operatorname{ad} e_j)(\operatorname{ad} f_i)^m(f_j) = (-1)^{m\tau_i\tau_j} b_{ji} (\operatorname{ad} f_i)^{m-1}(f_i),$$
$$(\operatorname{ad} f_j)(\operatorname{ad} e_i)^m(e_j) = (-1)^{m\tau_i\tau_j}(-1)^{\tau_j} b_{ji} (\operatorname{ad} e_i)^{m-1}(e_i).$$

(2) *If $\tau_i = \bar{0}$ and $i \neq j$ then for any $m \geq 0$,*

$$(\operatorname{ad} e_i)(\operatorname{ad} f_i)^m(f_j) = -m\Big(\frac{m-1}{2} b_{ii} + b_{ij}\Big)(\operatorname{ad} f_i)^{m-1}(f_j),$$
$$(\operatorname{ad} f_i)(\operatorname{ad} e_i)^m(e_j) = -m\Big(\frac{m-1}{2} b_{ii} + b_{ij}\Big)(\operatorname{ad} e_i)^{m-1}(e_j).$$

(3) *If $\tau_i = \bar{1}$ then for any $m \geq 0$,*

$$(\operatorname{ad} e_i)(\operatorname{ad} f_i)^m(f_j) = \begin{cases} -\frac{m}{2} b_{ii} (\operatorname{ad} f_i)^{m-1}(f_j) & \text{if } m \text{ is even,} \\ -\big(\frac{m-1}{2} b_{ii} + b_{ij}\big)(\operatorname{ad} f_i)^{m-1}(f_j) & \text{if } m \text{ is odd,} \end{cases}$$

$$(\operatorname{ad} f_i)(\operatorname{ad} e_i)^m(e_j) = \begin{cases} \frac{m}{2} b_{ii} (\operatorname{ad} e_i)^{m-1}(e_j) & \text{if } m \text{ is even,} \\ \big(\frac{m-1}{2} b_{ii} + b_{ij}\big)(\operatorname{ad} e_i)^{m-1}(e_j) & \text{if } m \text{ is odd.} \end{cases}$$

PROOF. The claim follows by induction on m from the defining relations of $\tilde{\mathfrak{g}}$ and the axioms of a Lie superalgebra. (Define $x_m = (\operatorname{ad} f_i)^m(f_j)$ for $m \geq 0$. Regarding (2) and (3), prove that $[e_i, x_m] = \lambda_m x_{m-1}$ for any $m \geq 1$, where $\lambda_m \in \mathbb{C}$, and that $\lambda_1 = -b_{ij}$ and $\lambda_m = (1-m)b_{ii} - b_{ij} + (-1)^{\tau_i}\lambda_{m-1}$ for any $m \geq 2$.) $\square$

Recall from Lemma 11.1.10 the definition of a contragredient Lie superalgebra.

LEMMA 11.2.2. *Let $\mathfrak{g}$ be a contragredient Lie superalgebra of rank $n \geq 1$ and let $\alpha \in \mathbb{N}_0^n$ with $\alpha \neq 0$.*

(1) *Let $x \in \mathfrak{g}_\alpha$. If $[f_i, x] = 0$ for all $1 \leq i \leq n$ then $x = 0$.*
(2) *Let $x \in \mathfrak{g}_{-\alpha}$. If $[e_i, x] = 0$ for all $1 \leq i \leq n$ then $x = 0$.*

PROOF. (1) Let $\mathfrak{k}$ be the ideal of $\mathfrak{n}_+ = (\tilde{\mathfrak{n}}_+ + \mathfrak{r})/\mathfrak{r}$ generated by x. Since $x \in \mathfrak{g}_\alpha$, $\mathfrak{k}$ becomes an ideal of $(\tilde{\mathfrak{n}}_+ + \mathfrak{h} + \mathfrak{r})/\mathfrak{r}$. Since $(\operatorname{ad} f_i)(\tilde{\mathfrak{n}}_+) \subseteq \tilde{\mathfrak{n}}_+ + \mathfrak{h}$ for all $1 \leq i \leq n$, the assumption and Jacobi identity imply that $(\operatorname{ad} f_i)(\mathfrak{k}) \subseteq \mathfrak{k}$ for all $1 \leq i \leq n$. Hence $\mathfrak{k} = 0$ by the definition of $\mathfrak{g}$.

(2) is proven similarly to (1). $\square$

REMARK 11.2.3. Lemma 11.2.2 implies that for any $1 \le i \le n$, $[e_i, e_i] = 0$ in $\mathfrak{g}$ if and only if $\tau_i = \bar{0}$ or $b_{ii} = 0$. Indeed,

$$[f_i, [e_i, e_i]] = [[f_i, e_i], e_i] + (-1)^{\tau_i}[e_i, [f_i, e_i]] = (1 - (-1)^{\tau_i})b_{ii}e_i$$

and $[f_j, [e_i, e_i]] = 0$ for any $j \neq i$.

DEFINITION 11.2.4. Let $n \in \mathbb{N}$, $B \in \mathbb{C}^{n\times n}$, and $\tau \in \mathbb{Z}_2^n$. Assume that for any $1 \le i, j \le n$, $b_{ij} = 0$ implies that $b_{ji} = 0$. For any $1 \le i, j \le n$ let

$$a_{ij}^{B,\tau} = a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j,\ b_{ij} = 0, \\ -m & \text{if } i \neq j,\ b_{ij} \neq 0,\ \tau_i = \bar{0},\ b_{ij} = -\frac{m}{2}b_{ii}, \\ -1 & \text{if } i \neq j,\ b_{ij} \neq 0,\ \tau_i = \bar{1},\ b_{ii} = 0, \\ -m & \text{if } i \neq j,\ b_{ij} \neq 0,\ \tau_i = \bar{1},\ b_{ij} = -\frac{m}{2}b_{ii},\ m \ge 2 \text{ is even}, \\ -\infty & \text{otherwise.} \end{cases}$$

The matrix $A^{B,\tau} = (a_{ij})_{1\le i,j\le n}$ is called the **Cartan matrix of** the pair (B, τ) (and of $\mathfrak{g}(B, \tau)$).

LEMMA 11.2.5. *Let $n \in \mathbb{N}$, $B \in \mathbb{C}^{n\times n}$, and $\tau \in \mathbb{Z}_2^n$ such that $b_{ij} = 0$ implies $b_{ji} = 0$ for any $1 \le i, j \le n$. Let $\mathfrak{g} = \mathfrak{g}(B, \tau)$, $1 \le i, j \le n$, and $a = a_{ij}^{B,\tau}$. Assume that $i \neq j$.*

(1) *If $a = -\infty$ then $(\operatorname{ad} e_i)^m(e_j) \neq 0$ and $(\operatorname{ad} f_i)^m(f_j) \neq 0$ in $\mathfrak{g}$ for all $m \ge 1$.*
(2) *Suppose that $a \in \mathbb{Z}$. Then $(\operatorname{ad} e_i)^{1-a}(e_j) = 0$, $(\operatorname{ad} f_i)^{1-a}(f_j) = 0$, and $[f_i, (\operatorname{ad} e_i)^k(e_j)] \neq 0$, $[e_i, (\operatorname{ad} f_i)^k(e_j)] \neq 0$ in $\mathfrak{g}$ for all $1 \le k \le -a$.*

PROOF. By Remark 11.1.8, the elements e_i and f_i with $1 \le i \le n$ are nonzero in $\mathfrak{g}$. Now combine Lemmas 11.2.2 and 11.2.1 as well as Remarks 11.1.7 and 11.2.3. $\square$

DEFINITION 11.2.6. Let $n \ge 1$, $B \in \mathbb{C}^{n\times n}$ a symmetric matrix, and $\tau \in \mathbb{Z}_2^n$. Let $\mathcal{X}$ be the smallest set of pairs (C, σ) containing (B, τ) such that

($*$) for any $(C, \sigma) \in \mathcal{X}$ and any $1 \le i \le n$, such that $a_{ij} = a_{ij}^{C,\sigma} \in \mathbb{Z}$ for all $1 \le j \le n$, the pair $r_i(C, \sigma) = (C', \sigma') \in \mathbb{C}^{n\times n} \times \mathbb{Z}_2^n$ is contained in $\mathcal{X}$, where $c'_{jk} = \langle \alpha_j - a_{ij}\alpha_i, \alpha_k - a_{ik}\alpha_i \rangle_C$ and $\sigma'_j = \sigma_j - a_{ij}\sigma_i \in \mathbb{Z}_2$ for all $1 \le j, k \le n$.

Assume that $A^{C,\sigma} \in \mathbb{Z}^{n\times n}$ for all $(C, \sigma) \in \mathcal{X}$. Then $\mathfrak{g}(B, \tau)$ is called a **regular Kac-Moody superalgebra**, and the quadruple $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$, where $I = \{1, \dots, n\}$, $r : I \times \mathcal{X} \to \mathcal{X}$ with $r(i, (C, \sigma)) = r_i(C, \sigma)$, and $A = (A^{C,\sigma})_{(C,\sigma)\in\mathcal{X}}$, is called the **Cartan graph of** (B, τ) and of $\mathfrak{g}(B, \tau)$.

The terminology for $\mathcal{G}$ will be justified in Theorem 11.2.10.

LEMMA 11.2.7. *Let $n \ge 1$, $C \in \mathbb{C}^{n\times n}$ a symmetric matrix, $\sigma \in \mathbb{Z}_2^n$, and $1 \le i \le n$. Assume that $a_{ij}^{C,\sigma} \in \mathbb{Z}$ for all $1 \le j \le n$. Let $(C', \sigma') = r_i(C, \sigma)$.*

(1) *If $c_{ii} \neq 0$ then $C' = C$ and $\sigma' = \sigma$.*

(2) *Assume that* $c_{ii} = 0$. *Then* C' *is symmetric and*

$$c'_{jk} = \begin{cases} -c_{jk} & \text{if } i = j \text{ or } i = k, \\ c_{jk} & \text{if } i \neq j,\ i \neq k, \text{ and } c_{ij}c_{ik} = 0, \\ c_{jk} + c_{ik} + c_{ji} & \text{if } i \neq j,\ i \neq k, \text{ and } c_{ij}c_{ik} \neq 0 \end{cases}$$

for any $1 \leq j, k \leq n$.

PROOF. The claim follows from Definition 11.2.4 and the definition of (C', σ'). If $c_{ii} \neq 0$, then $a_{ij} = 2c_{ij}/c_{ii}$ for any $1 \leq j \leq n$, which implies that $C' = C$. Moreover, if $\sigma_i \neq \bar{0}$ then a_{ij} is even for any $1 \leq j \leq n$, and hence $\sigma' = \sigma$. If $c_{ii} = 0$, then the claim follows by direct calculations. □

REMARK 11.2.8. Assume that in the setting of Lemma 11.2.7 the matrix C is decomposable in the sense of Definition 10.1.15. Let I_1, I_2 be non-empty subsets of $\{1, \dots, n\}$ such that $I_1 \cup I_2 = \{1, \dots, n\}$, $I_1 \cap I_2 = \emptyset$, $i \in I_1$, and $c_{jk} = 0$ for any $j \in I_1$, $k \in I_2$. Then the lemma implies that C' is decomposable and that $c_{jk} = c'_{jk}$ whenever $j \in I_2$ or $k \in I_2$, since $c_{ij}c_{ik} = 0$ for these pairs (j, k). Similarly, $\sigma'_j = \sigma_j$ for any $j \in I_2$.

LEMMA 11.2.9. *Let* $n \geq 1$, $C \in \mathbb{C}^{n \times n}$ *a symmetric matrix,* $\sigma \in \mathbb{Z}_2^n$, *and* $1 \leq i \leq n$. *Assume that* $a_{ij} = a_{ij}^{C,\sigma} \in \mathbb{Z}$ *for all* $1 \leq j \leq n$. *Let* $(C', \sigma') = r_i(C, \sigma)$ *and let*

$$e'_i = f_i,\ f'_i = (-1)^{\sigma_i} e_i,\ e'_j = (\operatorname{ad} e_i)^{-a_{ij}}(e_j),\ f''_j = (\operatorname{ad} f_i)^{-a_{ij}}(f_j) \in \mathfrak{g}(C, \sigma)$$

for all $1 \leq j \leq n$ *with* $j \neq i$. *Then there is a unique isomorphism of Lie superalgebras* $R_i : \mathfrak{g}(C', \sigma') \to \mathfrak{g}(C, \sigma)$ *with*

$$R_i(e_j) = e'_j, \quad R_i(f_j) = f'_j \quad \text{for all } 1 \leq j \leq n,$$

where $f'_j = (e'_j \mid f''_j)^{-1} f''_j$ *for all* $1 \leq j \leq n$ *with* $j \neq i$. *Moreover,*

$$R_i(\mathfrak{g}(C', \sigma')_\alpha) = \mathfrak{g}(C, \sigma)_{s_i(\alpha)}$$

for any $\alpha \in \mathbb{Z}^n$, *where* $s_i \in \operatorname{Aut}(\mathbb{Z}^n)$ *is defined by* $s_i(\alpha_j) = \alpha_j - a_{ij}^{C,\sigma}\alpha_i$ *for all* $1 \leq j \leq n$.

PROOF. By construction, e'_j, f''_j (where $f''_i = f'_i$) have $\mathbb{Z}_2$-degree σ'_j for all $1 \leq j \leq n$. For all $1 \leq j \leq n$ let $h'_j = h_j - a_{ij}h_i \in \mathfrak{h}$. Then

$$[h'_j, h'_k] = 0, \qquad [h'_j, e'_k] = c'_{jk}e'_k, \qquad [h'_j, f''_k] = -c'_{jk}f''_k$$

for all $1 \leq j, k \leq n$ by Definition 11.2.6. Moreover,

$$\begin{aligned} [e'_i, f'_i] &= (-1)^{\sigma_i}[f_i, e_i] = -[e_i, f_i] = h'_i, \\ [e'_i, f''_j] &= [f_i, (\operatorname{ad} f_i)^{-a_{ij}}(f_j)] = 0, \\ [f'_i, e'_j] &= (-1)^{\sigma_i}[e_i, (\operatorname{ad} e_i)^{-a_{ij}}(e_j)] = 0 \end{aligned}$$

for all $1 \leq j \leq n$ with $j \neq i$ by the definitions and by Lemma 11.2.5(2). For any $1 \leq j \leq n$ with $j \neq i$ we obtain from Proposition 11.1.17 that

$$[e'_j, f''_j] = (e'_j \mid f''_j)h'_j.$$

Moreover, the invariance of $(\cdot \mid \cdot)$ and Lemma 11.2.5(2) imply that in this setting $(e'_j \mid f''_j) \neq 0$ and hence f'_j is well-defined and $[e'_j, f'_j] = h'_j$. Also, for $1 \leq j, k \leq n$ with $j, k \neq i$ and $j \neq k$ we know that $[e'_j, f'_k] = 0$ by degree reasons. Thus there is a unique homomorphism $\tilde{R}_i : \tilde{\mathfrak{g}}(C',\sigma') \to \mathfrak{g}(C,\sigma)$ of Lie superalgebras sending e_j to e'_j and f_j to f'_j for all $1 \le j \le n$. Since $e'_i = f_i$ and $f'_i = (-1)^{\sigma_i} e_i$, Lemma 11.2.5(2) implies that $e_j, f_j \in \tilde{R}_i(\tilde{\mathfrak{g}}(C',\sigma'))$ for all $1 \le j \le n$, and hence $\tilde{R}_i$ is surjective. Moreover,

$$\tilde{R}_i(\tilde{\mathfrak{g}}_\alpha) = \mathfrak{g}_{s_i(\alpha)} \tag{11.2.1}$$

for all $\alpha \in \mathbb{Z}^n$, where $s_i \in (\mathrm{Aut}(\mathbb{Z}^n))$ with $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all $1 \le j \le n$. We conclude that $\tilde{R}_i(\mathfrak{r})$ is an ideal of $\mathfrak{g}(C,\sigma)$ contained in $\mathfrak{n}_+ + \mathfrak{n}_-$ and hence 0. Thus $\tilde{R}_i$ induces a surjective map $R_i : \mathfrak{g}(C',\sigma') \to \mathfrak{g}(C,\sigma)$ of Lie superalgebras. Clearly, R_i restricted to $\mathfrak{h}$ is injective, and hence $\ker(R_i)$ is an ideal of $\mathfrak{g}(C',\sigma')$ contained in $\mathfrak{n}_+ + \mathfrak{n}_-$. But 0 is the only such ideal of $\mathfrak{g}(C',\sigma')$, and hence R_i is bijective. The last claim follows from (11.2.1). □

THEOREM 11.2.10. *The Cartan graph of a regular Kac-Moody superalgebra is a connected Cartan graph in the sense of Definition* 9.1.14.

PROOF. Let $n \ge 1$, $B \in \mathbb{C}^{n\times n}$ a symmetric matrix, and $\tau \in \mathbb{Z}_2^n$. Assume that $\mathfrak{g}(B,\tau)$ is a regular Kac-Moody superalgebra. Let $\mathcal{G} = \mathcal{G}(I,\mathcal{X},r,A)$ be the Cartan graph of (B,τ). Thus, $A^{C,\sigma} \in \mathbb{Z}^{n\times n}$ for all $(C,\sigma) \in \mathcal{X}$. By assumption, $r : I \times \mathcal{X} \to \mathcal{X}$ and $A : I \times I \times \mathcal{X} \to \mathbb{Z}$ are well-defined maps. We have to prove axioms (CG1)–(CG4) of a Cartan graph. Connectedness follows from the definition of $\mathcal{X}$.

Axioms (CG1) and (CG2) follow easily from Lemma 11.2.7 and Definition 11.2.4. Hence $\mathcal{G}$ is a semi-Cartan graph. In order to verify Axioms (CG3) and (CG4), for any $(C,\sigma) \in \mathcal{X}$ let

$$\boldsymbol{\Delta}^{(C,\sigma)} = \{\alpha \in \mathbb{Z}^n \setminus \{0\} \mid \mathfrak{g}(C,\sigma)_\alpha \ne 0\}. \tag{11.2.2}$$

Then $\boldsymbol{\Delta}^{(C,\sigma)} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ by Lemma 11.1.9. Lemma 11.2.9 implies that for all $(C,\sigma) \in \mathcal{X}$, $\boldsymbol{\Delta}^{r_i(C,\sigma)} = s_i^{(C,\sigma)}(\boldsymbol{\Delta}^{(C,\sigma)})$. Hence

$$\boldsymbol{\Delta}^{(C,\sigma)\,\mathrm{re}} \subseteq \boldsymbol{\Delta}^{(C,\sigma)} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$$

for all $(C,\sigma) \in \mathcal{X}$, that is, Axiom (CG3) is fulfilled.

Let $i,j \in I$ and $X = (C,\sigma) \in \mathcal{X}$. Assume that $i \ne j$ and $m_{ij}^X < \infty$. Then $F(\mathrm{id}_X(s_is_j)^{m_{ij}^X}) = \mathrm{id}_{\mathbb{Z}^n}$ by Theorem 9.2.23. Let $(C',\sigma') = (r_jr_i)^{m_{ij}^X}(X)$. In order to prove (CG4), we have to show that $(C',\sigma') = (C,\sigma)$.

Assume first that C is indecomposable. Lemma 11.2.9 implies that there is an isomorphism $\varphi : \mathfrak{g}(C,\sigma) \to \mathfrak{g}(C',\sigma')$ of Lie superalgebras such that for all $\alpha \in \mathbb{Z}^n$, $\varphi(\mathfrak{g}(C,\sigma)_\alpha) = \mathfrak{g}(C',\sigma')_\alpha$. Thus $(C',\sigma') = (C,\sigma)$ by Lemma 11.1.11 and Remark 11.1.12, since C is indecomposable. Hence Axiom (CG4) is fulfilled in this case.

Finally, assume that C is decomposable. Let I_1, I_2 be non-empty subsets of $\{1,\dots,n\}$ such that $I_1 \cup I_2 = \{1,\dots,n\}$, $I_1 \cap I_2 = \emptyset$, and $c_{kl} = 0$ whenever $k \in I_1$, $l \in I_2$. If $i \in I_1$ and $j \in I_2$, then $m_{ij}^X = 2$ and Remark 11.2.8 implies that

$$r_jr_i(C,\sigma) = r_ir_j(C,\sigma).$$

Hence $(r_jr_i)^{m_{ij}^X}(C,\sigma) = (C,\sigma)$ by (CG1). On the other hand, if $i,j \in I_1$, then Remark 11.2.8 implies that r_i, r_j don't change the block decomposition of C and the entries of C and σ in the components away from i,j. Moreover, Corollary 9.2.20 implies that $m_{ij}^X = \overline{m}_{ij}^X$ does not change by passing from $\mathcal{G}$ to its restriction to I_1.

Thus $(r_j r_i)^{m_{ij}^X}(C,\sigma)=(C,\sigma)$ by the previous paragraph applied to the restriction of $\mathcal{G}$ to I_1. This completes the proof. □

REMARK 11.2.11. It is known that the Cartan graph of a regular Kac-Moody superalgebra is not standard in general, and some points may have Cartan matrices which are not of finite type. Further, a regular Kac-Moody superalgebra may have roots which are the double of another root, which is due to the fact that $[e_i,e_i]$ may be non-zero for some $1\le i\le n$.

COROLLARY 11.2.12. *Let $n\ge 1$, $B\in\mathbb{C}^{n\times n}$ a symmetric matrix, and $\tau\in\mathbb{Z}_2^n$. Assume that $\mathfrak{g}(B,\tau)$ is finite-dimensional. Then $\mathcal{G}=\mathcal{G}(I,\mathcal{X},r,A)$ is a finite Cartan graph.*

PROOF. As argued in the proof of Theorem 11.2.10, $\boldsymbol{\Delta}^{(C,\sigma)\,\mathrm{re}}\subseteq\boldsymbol{\Delta}^{(C,\sigma)}$ for any $(C,\sigma)\in\mathcal{X}$. Since $\mathfrak{g}(B,\tau)$ is finite-dimensional, the set $\boldsymbol{\Delta}^{(B,\tau)}$ is finite. Since $\mathcal{G}$ is connected, it is a finite Cartan graph. □

EXAMPLE 11.2.13. Here we construct explicitly the Cartan graph of the regular Kac-Moody superalgebra $\mathfrak{g}=\mathfrak{g}(B,\tau)$ with

$$B=\begin{pmatrix}0&-1&0\\-1&2&-2\\0&-2&4\end{pmatrix},\qquad \tau=(\bar 1,\bar 0,\bar 0).$$

This Lie superalgebra is usually denoted by $C(3)$ or $osp(2|4)$. By Definition 11.2.4, the Cartan matrix of $\mathfrak{g}$ is

$$A=\begin{pmatrix}2&-1&0\\-1&2&-2\\0&-1&2\end{pmatrix},$$

and $r_2(B,\tau)=r_3(B,\tau)=(B,\tau)$ by Lemma 11.2.7. Let $(B',\tau')=r_1(B,\tau)$. Then

$$B'=\begin{pmatrix}0&1&0\\1&0&-2\\0&-2&4\end{pmatrix},\qquad \tau'=(\bar 1,\bar 1,\bar 0).$$

by Definition 11.2.6 and by Lemma 11.2.7. The Cartan matrix of (B',τ') is

$$A'=\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}.$$

Then $r_1(B',\tau')=r_3(B',\tau')=(B,\tau)$. Let $(B'',\tau'')=r_2(B',\tau')$. Then

$$B''=\begin{pmatrix}2&-1&-1\\-1&0&2\\-1&2&0\end{pmatrix},\qquad \tau''=(\bar 0,\bar 1,\bar 1),$$

and the Cartan matrix of (B'',τ'') is

$$A''=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix},$$

which is not of finite type.

Continuing this way we obtain that $r_1(B'',\tau'')=(B'',\tau'')$, $r_2(B'',\tau'')=(B',\tau')$ and $r_3(B'',\tau'')$ is the pair (B',τ') up to permutation of the indices 2 and 3 in

$\{1,2,3\}$. The exchange graph of $\mathcal{G}$ is displayed in Figure 11.2.1. Instead of a pair (C,σ) for a point of $\mathcal{G}$ we write C in top of σ.

$$
\begin{tikzcd}[ampersand replacement=\&]
\begin{array}{c}\begin{pmatrix}0 \& -1 \& 0\\ -1 \& 2 \& -2\\ 0 \& -2 \& 4\end{pmatrix}\\ (\bar 1,\bar 0,\bar 0)\end{array} \arrow[r, dash, "1"] \& \begin{array}{c}\begin{pmatrix}0 \& 1 \& 0\\ 1 \& 0 \& -2\\ 0 \& -2 \& 4\end{pmatrix}\\ (\bar 1,\bar 1,\bar 0)\end{array} \arrow[dr, dash, "2"] \& \\
\& \& \begin{array}{c}\begin{pmatrix}2 \& -1 \& -1\\ -1 \& 0 \& 2\\ -1 \& 2 \& 0\end{pmatrix}\\ (\bar 0,\bar 1,\bar 1)\end{array} \\
\begin{array}{c}\begin{pmatrix}0 \& 0 \& -1\\ 0 \& 4 \& -2\\ -1 \& -2 \& 2\end{pmatrix}\\ (\bar 1,\bar 0,\bar 0)\end{array} \arrow[r, dash, "1"] \& \begin{array}{c}\begin{pmatrix}0 \& 0 \& 1\\ 0 \& 4 \& -2\\ 1 \& -2 \& 0\end{pmatrix}\\ (\bar 1,\bar 0,\bar 1)\end{array} \arrow[ur, dash, "3"'] \&
\end{tikzcd}
$$

FIGURE 11.2.1. Exchange graph of the Lie superalgebra $osp(2|4)$

11.3. Notes

11.1. For a much more detailed exposition of the theory of Lie superalgebras and historical remarks we refer to [**Mus12**] and [**BM$^+$92**].

11.2. Our definition of a regular Kac-Moody superalgebra follows [**HS07**]. In [**Ser11**], the Weyl groupoid of a contragredient Lie superalgebra is defined. This Weyl groupoid of a regular Kac-Moody superalgebra and our definition of the Weyl groupoid of the Cartan graph of a regular Kac-Moody superalgebra are different, but closely related. Note that the former has more objects and more (iso)morphisms.

Part 3

Weyl groupoids and root systems of Nichols algebras

CHAPTER 12

A braided monoidal isomorphism of Yetter-Drinfeld modules

Let H be a Hopf algebra with bijective antipode, and $\mathcal{C} = {}_H^H\mathcal{YD}$. We discuss dual pairs (A, B) of graded Hopf algebras in $\mathcal{C}$ and rational modules over graded algebras. In this context, there is a monoidal isomorphism between categories of comodules and of rational modules. In Theorem 12.3.2 we construct a braided monoidal isomorphism

$$(\Omega, \omega) : {}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}_A^A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$$

between categories of rational Yetter-Drinfeld modules. In the applications in Chapter 13, the dual pair (A, B) is the pair $(\mathcal{B}(V^*), \mathcal{B}(V))$ of Corollary 7.2.8, where V is a finite-dimensional object in ${}_H^H\mathcal{YD}$.

In Theorem 12.3.3, we construct a Hopf algebra isomorphism T which relates $K\#B$ and $\Omega(K)\#A$, where K is a Hopf algebra in ${}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$. In the applications later, $V = M_i$ is irreducible, $K\#\mathcal{B}(M_i)$ is the Hopf algebra of a Nichols system, and $\Omega(K)\#\mathcal{B}(M_i^*)$ is the Hopf algebra of the i-th reflection of the Nichols system.

The Hopf algebra isomorphism T is then used in Section 12.4 to compare one-sided coideal subalgebras of $K\#B$ and of $\Omega(K)\#A$. In our theory, T plays the role of the Lusztig isomorphisms of quantum groups to construct right coideal subalgebras and PBW-bases of Nichols systems and Nichols algebras.

Most of this Chapter depends on the general theory of braided strict monoidal categories in Chapter 3.

12.1. Dual pairs of Yetter-Drinfeld Hopf algebras

Recall the notion of a Hopf pairing in a braided strict monoidal category from Definition 3.3.7.

Definition 12.1.1. Let $A = \bigoplus_{n\geq 0} A(n)$ and $B = \bigoplus_{n\geq 0} B(n)$ be locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C} = {}_H^H\mathcal{YD}$, and

$$\langle\ ,\ \rangle : A \otimes B \to \Bbbk, \quad a \otimes b \mapsto \langle a, b\rangle,$$

a Hopf pairing in $\mathcal{C}$. Then $(A, B, \langle\ ,\ \rangle)$ is called a **dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in** $\mathcal{C}$ if $\langle\ ,\ \rangle$ is non-degenerate, and if

$$\langle A(m), B(n)\rangle = 0 \text{ for all } m \neq n. \tag{12.1.1}$$

Remark 12.1.2. Let $A = \bigoplus_{n\geq 0} A(n)$ and $B = \bigoplus_{n\geq 0} B(n)$ be locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C} = {}_H^H\mathcal{YD}$, and $\langle\ ,\ \rangle : A \otimes B \to \Bbbk$ a bilinear non-degenerate form satisfying (12.1.1). Then $(A, B, \langle\ ,\ \rangle)$ is a dual pair of locally finite

$\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$ if and only if the following axioms hold.

$$\langle h \cdot a, b\rangle = \langle a, \mathcal{S}(h)\cdot b\rangle, \tag{12.1.2}$$

$$a_{(-1)}\langle a_{(0)}, b\rangle = \mathcal{S}^{-1}(b_{(-1)})\langle a, b_{(0)}\rangle, \tag{12.1.3}$$

$$\langle a, bb'\rangle = \langle a^{(1)}, b'\rangle\langle a^{(2)}, b\rangle, \qquad \langle 1, b\rangle = \varepsilon(b), \tag{12.1.4}$$

$$\langle aa', b\rangle = \langle a, b^{(2)}\rangle\langle a', b^{(1)}\rangle, \qquad \langle a, 1\rangle = \varepsilon(a), \tag{12.1.5}$$

for all $a, a' \in A$, $b, b' \in B$ and $h \in H$.

The bilinear form $\langle\ ,\ \rangle : A \otimes B \to \Bbbk$ is a morphism in $\mathcal{C}$ if and only if (12.1.2) and (12.1.3) are satisfied, and it is a Hopf pairing in $\mathcal{C}$ if and only if (12.1.2)–(12.1.5) are satisfied.

In (12.1.4) and (12.1.5), the equations

$$\langle a, bb'\rangle = \langle a^{(1)}, b\rangle\langle a^{(2)}, b'\rangle,$$
$$\langle aa', b\rangle = \langle a, b^{(1)}\rangle\langle a', b^{(2)}\rangle$$

seem to look more natural. But for braided monoidal categories the natural definition of a Hopf pairing is given in Definition 3.3.7.

In view of (12.1.1), non-degeneracy of the pairing means that for all $n \in \mathbb{N}_0$, the maps

$$A(n) \to (B(n))^*, \quad a \mapsto (b \mapsto \langle a, b\rangle),$$
$$B(n) \to (A(n))^*, \quad b \mapsto (a \mapsto \langle a, b\rangle),$$

are isomorphisms.

If we extend the $\mathbb{N}_0$-gradings to $\mathbb{Z}$-gradings by $A(n) = 0$, $B(n) = 0$ for all $n < 0$, and if we define a new $\mathbb{Z}$-grading on A by $\deg(A(n)) = -n$ for all $n \in \mathbb{Z}$ (as we will do later for $A = \mathcal{B}(M^*)$), then (12.1.1) just says that $\langle\ ,\ \rangle : A \otimes B \to \Bbbk$ is $\mathbb{Z}$-graded. Here, the grading of $\Bbbk$ is given by $\Bbbk(n) = 0$, if $n \neq 0$, and $\Bbbk(0) = \Bbbk$.

The main example we have in mind comes from the theory of Nichols algebras. If $V \in \mathcal{C}$ is a finite-dimensional object, then by Corollary 7.2.8, there is a bilinear form

$$\langle\ ,\ \rangle : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to \Bbbk$$

such that $(\mathcal{B}(V^*), \mathcal{B}(V), \langle\ ,\ \rangle)$ is a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$.

We note that finite-dimensional (non-graded) Hopf algebras in $\mathcal{C}$ and their opcop-duals are another example.

PROPOSITION 12.1.3. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}^{\mathrm{fd}}$, and R^* its left dual. Then $(R^{*\mathrm{op\,cop}}, R, \langle\ ,\ \rangle)$ is a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras, where $\langle\ ,\ \rangle$ is the evaluation map*

$$R^* \otimes R \to \Bbbk, \quad f \otimes x \mapsto f(x),$$

and where $R(n) = 0$, $R^*(n) = 0$ for all $n \neq 0$, and $R(0) = R$, $R^*(0) = R^*$.

PROOF. By Corollary 4.2.6, R^* is a Hopf algebra in ${}^H_H\mathcal{YD}^{\mathrm{fd}}$ with multiplication and comultiplication defined for all $f, g \in R^*$ and $x, y \in H$ by

$$(fg)(x) = f((x^{(1)})_{(0)})g((x^{(1)})_{(-1)} \cdot x^{(2)}), \tag{12.1.6}$$

$$f(xy) = f^{(1)}(x_{(0)})f^{(2)}(x_{(-1)} \cdot y), \tag{12.1.7}$$

where $\Delta_R(x) = x^{(1)} \otimes x^{(2)}$, $\mu_R(x \otimes y) = xy$. $R^{*\mathrm{opcop}} = ((R^*)^{\mathrm{op}})^{\mathrm{cop}}$ is the Hopf algebra

$$(R^*, \mu_{R^*}\overline{c}_{R^*,R^*}, \eta_{R^*}, c_{R^*,R^*}\Delta_{R^*}, \varepsilon_{R^*}, \mathcal{S}_{R^*}).$$

By Theorem 4.4.11(1), the antipode of R is bijective. Hence $R^{*\mathrm{opcop}}$ is a Hopf algebra in ${}^H_H\mathcal{YD}^{\mathrm{fd}}$ by Corollary 3.2.16(2). For all $f, g \in R^*$ let

$$f \circ g = g_{(0)}(\mathcal{S}^{-1}(g_{(-1)}) \cdot f), \quad f^{[1]} \otimes f^{[2]} = (f^{(1)})_{(-1)} \cdot f^{(2)} \otimes (f^{(1)})_{(0)}.$$

Then $f \circ g$ is the multiplication of $f \otimes g$ and $f^{[1]} \otimes f^{[2]}$ is the comultiplication of f with respect to $R^{*\mathrm{opcop}}$. For all $x, y \in R$,

$$\begin{aligned} f^{[2]}(x) f^{[1]}(y) &= (f^{(1)})_{(0)}(x)\big((f^{(1)})_{(-1)} \cdot f^{(2)}\big)(y) \\ &= \big((f^{(1)})_{(-1)}(f^{(1)})_{(0)}(x) \cdot f^{(2)}\big)(y) \\ &= f(xy), \end{aligned}$$

where the last equality follows from Lemma 4.2.2(1) and (12.1.7), and

$$\begin{aligned} (f \circ g)(x) &= \big(g_{(0)}(\mathcal{S}^{-1}(g_{(-1)}) \cdot f)\big)(x) \\ &= g_{(0)}((x^{(1)})_{(0)})(\mathcal{S}^{-1}(g_{(-1)}) \cdot f)((x^{(1)})_{(-1)} \cdot x^{(2)}) \\ &= g(x^{(1)}) f(x^{(2)}), \end{aligned}$$

where the second equation follows from (12.1.6) and the last from Lemma 4.2.2(1). We have shown (12.1.4) and (12.1.5) for $(R^{*\mathrm{op\ cop}}, R, \langle\ ,\ \rangle)$, since the claims for unit and counit are obvious. □

Proposition 12.1.4. *Let $(A, B, \langle\ ,\ \rangle)$ be a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$.*

(1) *The antipodes of A and B and of their bosonizations are bijective.*
(2) *For all $a \in A$, $b \in B$,*

$$\langle \mathcal{S}_A(a), b\rangle = \langle a, \mathcal{S}_B(b)\rangle. \tag{12.1.8}$$

(3) *Define $\langle\ ,\ \rangle^+ = \langle\ ,\ \rangle c(\mathcal{S}_B \otimes \mathcal{S}_A) : B \otimes A \to \Bbbk$. Then*

$$\langle b, a\rangle^+ = \langle a, \mathcal{S}^2(b)\rangle \tag{12.1.9}$$

for all $b \in B$, $a \in A$, where $\mathcal{S}$ is the antipode of $B\#H$, and $(B, A, \langle\ ,\ \rangle^+)$ is a dual pair of locally finite Hopf algebras in $\mathcal{C}$.

Proof. (1) By Theorem 4.4.11(1), the antipode of a finite-dimensional Hopf algebra in $\mathcal{C}$ is bijective. Hence (1) follows from Proposition 6.4.2 and Corollary 3.8.11. (2) follows from Proposition 3.3.8(1).

(3) For all $b \in B$ and $a \in A$,

$$\begin{aligned} \langle b, a\rangle^+ &= \langle \mathcal{S}_A(b_{(-1)} \cdot a), \mathcal{S}_B(b_{(0)})\rangle \\ &= \langle b_{(-1)} \cdot a, \mathcal{S}_B^2(b_{(0)})\rangle && \text{(by (12.1.8))} \\ &= \langle a, \mathcal{S}_B^2(\mathcal{S}(b_{(-1)}) \cdot b_{(0)})\rangle && \text{(by (12.1.2))} \\ &= \langle a, \mathcal{S}^2(b)\rangle. && \text{(by Corollary 4.3.5(2)(a))} \end{aligned}$$

By Proposition 3.3.8(2), $\langle\ ,\ \rangle^+$ is a Hopf pairing in $\mathcal{C}$. Hence (3) follows, since $\mathcal{S}^2 : B \to B$ is graded by Corollary 5.1.3 and bijective by (1). □

12.2. Rational modules

Let A and B be objects in $\mathcal{C} = {}^H_H\mathcal{YD}$, and $\langle\ ,\ \rangle$ a **pairing** of A, B in $\mathcal{C}$, that is, a morphism $\langle\ ,\ \rangle : A \otimes B \to \Bbbk$, $a \otimes b \mapsto \langle a, b\rangle$, in $\mathcal{C}$. For subsets $X \subseteq A$ and $Y \subseteq B$, we define

$$X^\perp = \{b \in B \mid \langle x, b\rangle = 0 \text{ for all } x \in X\},$$
$$Y^\perp = \{a \in A \mid \langle a, y\rangle = 0 \text{ for all } y \in Y\}.$$

The pairing $\langle\ ,\ \rangle$ is non-degenerate, if $A^\perp = 0$ and $B^\perp = 0$.

REMARK 12.2.1. Let $A, B \in \mathcal{C}$ and $\langle\ ,\ \rangle$ a pairing in $\mathcal{C}$.

(1) Let $E \subseteq A$ and $F \subseteq B$ be subobjects in $\mathcal{C}$. Then $E^\perp \subseteq B$ and $F^\perp \subseteq A$ are subobjects in $\mathcal{C}$.

(2) Assume that the pairing $\langle\ ,\ \rangle$ is non-degenerate and B is finite-dimensional. Then the map

$$A \to B^*,\ a \mapsto (b \mapsto \langle a, b\rangle), \tag{12.2.1}$$

is an isomorphism in $\mathcal{C}$, where B^* is the left dual of B of Definition 4.2.3. Let $(b_i)_{1\le i\le n}$ be a basis of B with dual basis $(f_i)_{1\le i\le n}$ of B^*. For all i, let a_i be the inverse image of f_i under the isomorphism (12.2.1). It follows from Lemma 4.2.2 that $(A, \mathrm{ev}_B, \mathrm{coev}_B)$ is a left dual of B, where

$$\mathrm{ev}_B = \langle\ ,\ \rangle, \quad \mathrm{coev}_B : \Bbbk \to B \otimes A,\ 1 \mapsto \sum_{i=1}^n b_i \otimes a_i.$$

(3) Let $F \subseteq B$ in $\mathcal{C}$ be a finite-dimensional subobject in $\mathcal{C}$. Then

$$A/F^\perp \otimes F \to \Bbbk,\ \overline{a} \otimes b \mapsto \langle a, b\rangle, \tag{12.2.2}$$

is a non-degenerate pairing in $\mathcal{C}$, if $A^\perp = 0$.

Let $V, W \in \mathcal{C}$, and $\langle\ ,\ \rangle$ a pairing of A, B in $\mathcal{C}$. We denote by $\mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A\otimes V, W)$ the set of all g in $\mathrm{Hom}_{\mathcal{C}}(A\otimes V, W)$ such that for all $v \in V$ there is a finite-dimensional subobject $F \subseteq B$ in $\mathcal{C}$ with $g(F^\perp \otimes v) = 0$.

PROPOSITION 12.2.2. *Let $A, B \in \mathcal{C}$, $\langle\ ,\ \rangle$ a non-degenerate pairing of A, B in $\mathcal{C}$, and $W \in \mathcal{C}$. Assume that for any $b \in B$ there is a finite-dimensional subobject $F \subseteq B$ in $\mathcal{C}$ containing b. For all $V \in \mathcal{C}$, the map*

$$D_V : \mathrm{Hom}_{\mathcal{C}}(V, B \otimes W) \to \mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A \otimes V, W),$$
$$f \mapsto (A \otimes V \xrightarrow{\mathrm{id}_A \otimes f} A \otimes B \otimes W \xrightarrow{\langle\ ,\ \rangle \otimes \mathrm{id}_W} W),$$

is bijective.

PROOF. (1) Let $f \in \mathrm{Hom}_{\mathcal{C}}(V, B \otimes W)$, and $g = D_V(f)$. For each $v \in V$ there is a finite-dimensional subobject $F \subseteq B$ in $\mathcal{C}$ with $f(v) \in F \otimes W$. This follows from the assumption on B. Hence $g(F^\perp \otimes v) = 0$. Thus D_V is well-defined. Note that D_V is injective.

(2) Assume that B is finite-dimensional. Since $(A, \langle\ ,\ \rangle, \mathrm{coev}_B)$ is a left dual of B by Remark 12.2.1(2), D_V is bijective by (3.5.3).

(3) Let $V' \subseteq V$ be a finite-dimensional H-subcomodule. Then $HV' \subseteq V$ is a subobject in $\mathcal{C}$. Assume that $HV' = V$. We prove that then D_V is surjective.

Let $g \in \mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A \otimes V, W)$. Since V' is finite-dimensional, there is a finite-dimensional subobject $F \subseteq B$ in $\mathcal{C}$ with $g(F^\perp \otimes V') = 0$. By Remark 12.2.1(1), $F^\perp$ is an H-submodule of A. Thus for all $h \in H$,

$$g(F^\perp \otimes hV') = h_{(2)} g(\mathcal{S}^{-1}(h_{(1)})F^\perp \otimes V') = 0.$$

Hence $g(F^\perp \otimes V) = 0$. The pairing

$$A/F^\perp \otimes F \to \Bbbk, \quad \overline{a} \otimes b \mapsto \langle a, b \rangle,$$

is non-degenerate. By (2), the map

$$D_V : \mathrm{Hom}_{\mathcal{C}}(V, F \otimes W) \to \mathrm{Hom}_{\mathcal{C}}(A/F^\perp \otimes V, W)$$

for this pairing is bijective. Let $f \in \mathrm{Hom}_{\mathcal{C}}(V, F \otimes W)$ be the inverse of $\overline{g}$ in $\mathrm{Hom}_{\mathcal{C}}(A/F^\perp \otimes V, W)$, where $\overline{g}$ is induced by g. Then f composed with the inclusion $F \otimes W \to B \otimes W$ is the preimage of g under D.

(4) The family $(D_V)_{V \in \mathcal{C}}$ is a natural transformation. Let

$$\mathcal{U} = \{HV' \mid V' \subseteq V \text{ finite-dimensional } H\text{-subcomodule}\}.$$

Note that for all $U_1, U_2 \in \mathcal{U}$ there is an element $U \in \mathcal{U}$ with $U_1 \subseteq U$ and $U_2 \subseteq U$, since $\mathcal{U}$ is closed under sums. By Theorem 2.1.3, V is the union of all $U \in \mathcal{U}$. Let $g \in \mathrm{Hom}_{\mathrm{rat}}(A \otimes V, W)$. For any $U \in \mathcal{U}$, let g_U be the restriction of g to $A \otimes U$. It follows from (3) that for any U there is a morphism $f_U : U \to B \otimes W$ in $\mathcal{C}$ with $D_U(f_U) = g_U$. For all $U_1, U_2 \in \mathcal{U}$ with $U_1 \subseteq U_2$, $f_{U_2}|U_1 = f_{U_1}$, since D_{U_1} is injective. Hence the maps f_U define a linear map $f : V \to B \otimes W$ by $f(v) = f_U(v)$, where U is an element in $\mathcal{U}$ containing v. Then $D(f) = g$. □

Definition 12.2.3. Let $R = \bigoplus_{n \geq 0} R(n)$ be an $\mathbb{N}_0$-graded algebra (in $\mathcal{M}_\Bbbk$). A left or right R-module X is called **rational** if for any element $x \in X$ there is a natural number n_0 such that $R(n)x = 0$ and $xR(n) = 0$ for all $n \geq n_0$, respectively.

Let R be a left H-module algebra, and $R\#H$ the corresponding smash product algebra. A left or right $R\#H$-module V is called **rational over** R if V is a rational R-module by restriction. We denote the categories of left and of right $R\#H$-modules which are rational over R by ${}_{R\#H}\mathcal{M}_{\mathrm{rat}}$ and ${}_{\mathrm{rat}}\mathcal{M}_{R\#H}$, respectively.

Let R be an $\mathbb{N}_0$-graded algebra in $\mathcal{C}$. The subcategories of rational left and rational right R-modules in $\mathcal{C}$ are denoted by ${}_R\mathcal{C}_{\mathrm{rat}}$ and ${}_{\mathrm{rat}}\mathcal{C}_R$, respectively.

Lemma 12.2.4. *Let $R = \bigoplus_{n \geq 0} R(n)$ be an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$ with bosonization $R\#H$.*

1. *${}_{R\#H}\mathcal{M}_{\mathrm{rat}}$ is a monoidal subcategory of ${}_{R\#H}\mathcal{M}$ which is closed under arbitrary direct sums, subobjects and quotient objects.*
2. *${}_R\mathcal{C}_{\mathrm{rat}}$ is a monoidal subcategory of ${}_R\mathcal{C}$ which is closed under arbitrary direct sums, subobjects and quotient objects in ${}_R\mathcal{C}$.*
3. *The tensor algebra of any $V \in {}_{R\#H}\mathcal{M}_{\mathrm{rat}}$ is an object in ${}_{R\#H}\mathcal{M}_{\mathrm{rat}}$. The Nichols algebra $\mathcal{B}(V)$ of any $V \in {}^{R\#H}_{R\#H}\mathcal{YD}$ is rational over R if V is.*
4. *Let A be a left $R\#H$-module algebra, and $V \subseteq A$ an $R\#H$-submodule which is rational over R. Assume that A is generated as an algebra by V. Then A is rational over R.*

Proof. (1) Let $V, W \in {}_{R\#H}\mathcal{M}_{\mathrm{rat}}$, $v \in V$, $w \in W$. Then there is a natural number n_0 such that such that $(R(n)\#1)v = 0$, $(R(n)\#1)w = 0$ for all $n \geq n_0$.

Note that $(R(n)\#H)v = 0$ for all $n \geq n_0$, since

$$(1\#h_{(2)})(\mathcal{S}^{-1}(h_{(1)}) \cdot r\#1)v = (r\#h)v$$

for all $h \in H$, $r \in R$.

Let $n \geq 2n_0$ and $r \in R(n)$. Then $(r\#1)(v \otimes w) = (r^{(1)}\#r^{(2)}_{(-1)})v \otimes (r^{(2)}_{(0)}\#1)w = 0$, since $\Delta_R(r) = r^{(1)} \otimes r^{(2)} \in \bigoplus_{i+j=n} R(i) \otimes R(j)$.

The remaining claims in (1) are obvious.

(2) follows from (1), since the functor $F_1 : {}_R({}^H_H\mathcal{YD}) \to {}_R({}_H\mathcal{M}) \cong {}_{R\#H}\mathcal{M}$ of Definition 3.8.3 is strict monoidal by Proposition 3.8.4(3).

(3) follows from (1).

(4) Since A is a left $R\#H$-module algebra, and the algebra A is generated by V, A is an $R\#H$-module quotient of $T(V)$. Hence A is rational as an R-module by (3). □

We want to restrict the Yetter-Drinfeld criterion in Proposition 3.4.5(2) to rational left modules.

LEMMA 12.2.5. *Let R be a Hopf algebra in $\mathcal{C}$ and $(V, \lambda) \in {}_R\mathcal{C}$, $(V, \delta) \in {}^R\mathcal{C}$ with $V \in \mathcal{C}$. Let $X \in {}_R\mathcal{C}$, and assume that there is an index set I, a family X_i, $i \in I$, of objects in ${}_R\mathcal{C}$, and morphisms $f_i : X \to X_i$ in ${}_R\mathcal{C}$ for all $i \in I$ with $\bigcap_{i\in I} \ker(f_i) = 0$. If $c^{\mathcal{YD}}_{V,X_i}$ is a morphism in ${}_R\mathcal{C}$ for all $i \in I$, then $c^{\mathcal{YD}}_{V,X}$ is a morphism in ${}_R\mathcal{C}$.*

PROOF. Let $c = c^{\mathcal{YD}}_{V,X}$ and $c_i = c^{\mathcal{YD}}_{V,X_i}$, $i \in I$. For all $i \in I$, the diagrams

$$\begin{tikzcd} V \otimes X \arrow[r, "c"] \arrow[d, "\mathrm{id}\otimes f_i"'] & X \otimes V \arrow[d, "f_i\otimes\mathrm{id}"] \\ V \otimes X_i \arrow[r, "c_i"] & X_i \otimes V \end{tikzcd}$$

commute, since the f_i are left R-linear. Hence for all $r \in R$, $v \in V$ and $x \in X$,

$$\begin{aligned}(f_i \otimes \mathrm{id})(c(r(v \otimes x))) &= c_i(\mathrm{id} \otimes f_i)(r(v \otimes x)) = rc_i(\mathrm{id} \otimes f_i)(v \otimes x) \\ &= (f_i \otimes \mathrm{id})(rc(v \otimes x)),\end{aligned}$$

and $c(r(v \otimes x)) - rc(v \otimes x) \in \bigcap_{i\in I} \ker(f_i \otimes \mathrm{id}) = 0$. Thus c is an R-linear map. □

PROPOSITION 12.2.6. *Let R be an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$, V an object in $\mathcal{C}$, $(V, \lambda) \in {}_R\mathcal{C}$, and $(V, \delta) \in {}^R\mathcal{C}$. The following are equivalent.*

(1) *For all $(X, \lambda_X) \in {}_R\mathcal{C}_{\mathrm{rat}}$,*

$$c^{\mathcal{YD}}_{V,X} = \left(V \otimes X \xrightarrow{\delta\otimes\mathrm{id}} R \otimes V \otimes X \xrightarrow{\mathrm{id}\otimes c_{V,X}} R \otimes X \otimes V \xrightarrow{\lambda_X\otimes\mathrm{id}} X \otimes V\right)$$

is a morphism in ${}_R\mathcal{C}$.

(2) $V \in {}^R_R\mathcal{YD}(\mathcal{C})$.

PROOF. (1) ⇒ (2). By Proposition 3.4.5, it is enough to prove that

$$c^{\mathcal{YD}}_{V,R} : V \otimes R \to R \otimes V$$

is left R-linear, where R is a left R-module by multiplication in R. For all $n \geq 0$, let $X_n = R/\bigoplus_{i\geq n} R(i)$ with left (and right) R-linear quotient map $\pi_n : R \to X_n$. Then $\bigcap_{n\geq 0} \ker(\pi_n) = 0$, $R(m)X_n = 0$ for all $m \geq n$, and X_n is a rational R-module quotient of R. Hence by Lemma 12.2.5, $c^{\mathcal{YD}}_{V,R}$ is left R-linear.

(2) ⇒ (1) is clear from Proposition 3.4.5. □

PROPOSITION 12.2.7. *Let $(A, B, \langle\ ,\ \rangle)$ be a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$. The functor*

$$\overline{D}^l : {}^B\mathcal{C} \to {}_{A^{\mathrm{cop}}}\overline{\mathcal{C}}_{\mathrm{rat}}, \quad (V,\delta) \mapsto (V,\lambda),$$

where $\lambda = \big(A \otimes V \xrightarrow{\mathrm{id}\otimes\delta} A \otimes B \otimes V \xrightarrow{\langle\ ,\ \rangle\otimes\mathrm{id}} V\big)$, *and where morphisms f are mapped onto f, is a strict monoidal isomorphism of categories.*

PROOF. (1) Let V be an object in $\mathcal{C} = {}^H_H\mathcal{YD}$. By Proposition 12.2.2,

$$\begin{aligned} D_V : \mathrm{Hom}_{\mathcal{C}}(V, B \otimes V) &\to \mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A \otimes V, V), \\ \delta &\mapsto \lambda = (\langle\ ,\ \rangle \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \delta), \end{aligned}$$

is bijective. Note that $\mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A \otimes V, V)$ is the set of all λ in $\mathrm{Hom}_{\mathcal{C}}(A \otimes V, V)$ such that for all $v \in V$ there is a natural number n_0 with $\lambda(A(n) \otimes v) = 0$ for all $n \geq n_0$.

We claim that the map

$$\{\delta \mid (V,\delta) \in {}^B\mathcal{C}\} \to \{\lambda \mid (V,\lambda) \in {}_A\mathcal{C}_{\mathrm{rat}}\},\ \delta \mapsto D_V(\delta),$$

is bijective. Let $\delta \in \mathrm{Hom}_{\mathcal{C}}(V, B \otimes V)$, and $\lambda = D_V(\delta) \in \mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A \otimes V, V)$, We have to show that $(V,\delta) \in {}^B\mathcal{C}$ if and only if $(V,\lambda) \in {}_A\mathcal{C}$.

Let $v \in V$. We introduce the notation

$$v_{[-1]} \otimes v_{[0]} = \delta(v), \quad av = \lambda(a \otimes v), \text{ for all } a \in A.$$

The following are equivalent, since the pairing is non-degenerate.

(a) $v_{[-1]} \otimes (v_{[0]})_{[-1]} \otimes (v_{[0]})_{[0]} = \Delta_B(v_{[-1]}) \otimes v_{[0]}$.
(b) For all $a, a' \in A$,
$\langle a, v_{[-1]}\rangle\langle a', (v_{[0]})_{[-1]}\rangle(v_{[0]})_{[0]} = \langle a, (v_{[-1]})^{(1)}\rangle\langle a', (v_{[-1]})^{(2)}\rangle v_{[0]}$.

This proves our claim, since (a) is equivalent to $(\mathrm{id}_B \otimes \delta)\delta = (\Delta_B \otimes \mathrm{id}_V)\delta$, and (b) to the equality $a'(av) = (a'a)v$ for all $a, a' \in A$ by (12.1.5). Note that by (12.1.4), $1v = \varepsilon(v_{[-1]})v_{[0]}$.

(2) Let $(V,\delta), (V',\delta') \in {}^B\mathcal{C}$, and $(V,\lambda), (V',\lambda')$ the corresponding modules in ${}_A\mathcal{C}_{\mathrm{rat}}$, where $\lambda = D_V(\delta)$, $\lambda' = D_{V'}(\delta')$. It is easy to see that a map $f \in \mathrm{Hom}_{\mathcal{C}}(V, V')$ is B-colinear if and only if is A-linear. Since ${}_A\mathcal{C}_{\mathrm{rat}}$ is a monoidal subcategory of ${}_A\mathcal{C}$ by Lemma 12.2.4(1), the Proposition follows from (1) and Proposition 3.3.9. □

DEFINITION 12.2.8. Let R be an $\mathbb{N}_0$-graded algebra and C an $\mathbb{N}_0$-graded coalgebra in $\mathcal{C}$. For all $X \in {}_R\mathcal{C}$, $(Y, \delta_Y) \in {}^C\mathcal{C}$, and $n \geq 0$ let

$$\mathcal{F}_n X = \{x \in X \mid R(i)x = 0 \text{ for all } i > n\}, \tag{12.2.3}$$

$$\mathcal{F}^n Y = \{y \in Y \mid \delta_Y(y) \in \bigoplus_{i=0}^{n} C(i) \otimes Y\}. \tag{12.2.4}$$

LEMMA 12.2.9. *Let R be an $\mathbb{N}_0$-graded algebra and C an $\mathbb{N}_0$-graded coalgebra in $\mathcal{C}$, $(X, \lambda_X) \in {}_R\mathcal{C}$, and $(Y, \delta_Y) \in {}^C\mathcal{C}$.*

(1) *$(\mathcal{F}_n X)_{n\geq 0}$ is an $\mathbb{N}_0$-filtration in $\mathcal{C}$ of the largest rational R-submodule of X.*
(2) *$(\mathcal{F}^n Y)_{n\geq 0}$ is an $\mathbb{N}_0$-filtration in $\mathcal{C}$ of Y.*

PROOF. (1) Let $x \in X$ and $i \in \mathbb{N}_0$, and assume that $R(i)x = 0$. Then for all $r \in R(i)$,

(a) $0 = h_{(2)}((\mathcal{S}^{-1}(h_{(1)}) \cdot r)x) = r(h \cdot x)$, since λ_X is H-linear, and $R(i) \subseteq R$ is an H-submodule.

(b) $0 = r_{(-1)} \otimes \delta(r_{(0)}x) = r_{(-2)} \otimes r_{(-1)}x_{(-1)} \otimes r_{(0)}x_{(0)}$, since λ_X is H-colinear, and $R(i)$ is an H-subcomodule. Hence $0 = x_{(-1)} \otimes rx_{(0)}$.

By (a) and (b), $\mathcal{F}_n X \subseteq X$ is a subobject in $\mathcal{C}$ for all $n \in \mathbb{N}_0$. By the definition of rational R-modules, $\bigcup_{n\geq 0} \mathcal{F}_n X$ is the largest rational R-submodule of X.

(2) Let $n \in \mathbb{N}_0$. Since δ_Y is a map in $\mathcal{C}$, and $\bigoplus_{i=0}^n C(i) \otimes Y \subseteq C \otimes Y$ is a subobject in $\mathcal{C}$, (2) follows. $\square$

LEMMA 12.2.10. *Let $(A, B, \langle\ ,\ \rangle)$ be a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$, and $V \in {}^B\mathcal{C}$.*

1. *$\mathcal{F}_n \overline{D}^l(V) = \mathcal{F}^n V$ for all $n \geq 0$.*
2. *We view A and B as $\mathbb{Z}$-graded Hopf algebras in $\mathcal{C}$, where for all $n < 0$, $A(n) = 0$ and $B(n) = 0$. If V is a $\mathbb{Z}$-graded B-comodule, then $\overline{D}^l(V)$ is a $\mathbb{Z}$-graded A-module with $\overline{D}^l(V)(n) = V(-n)$ for all $n \in \mathbb{Z}$.*

PROOF. (1) For all $i \geq 0$, the kernel of the induced map

$$B \otimes V \to \mathrm{Hom}(A(i), V), \quad b \otimes v \mapsto (a \mapsto \langle a, b\rangle v)$$

is $\bigoplus_{j\neq i} B(j) \otimes V$ by (12.1.1) and non-degeneracy of the form. This implies (1).

(2) Let $m, n \in \mathbb{Z}$. Then

$$A(m)\overline{D}^l(V)(n) = A(m)V(-n) \subseteq V(-m-n) = \overline{D}^l(V)(m+n),$$

since $\delta(V(-n)) \subseteq \bigoplus_{i+j=-n} B(i) \otimes V(j)$. $\square$

LEMMA 12.2.11. *Let R be an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$, $V = (V, \delta) \in {}^R\mathcal{C}$, and $m \in \mathbb{Z}$. Define $\delta^{(m)} = \big(V \xrightarrow{\delta} R \otimes V \xrightarrow{\mathcal{S}_R^{2m} \otimes \mathrm{id}} R \otimes V \xrightarrow{c_{R,V}^{2m}} R \otimes V\big)$.*

1. *$V^{(m)} = (V, \delta^{(m)})$ is an object in ${}^R\mathcal{C}$.*
2. *For all $n \geq 0$, $\mathcal{F}^n V = \mathcal{F}^n V^{(m)}$.*
3. *If V is a $\mathbb{Z}$-graded object in ${}^R\mathcal{C}$, then $V^{(m)}$ with the grading of V is a $\mathbb{Z}$-graded object in ${}^R\mathcal{C}$.*

PROOF. (1) follows from Corollary 3.3.6, since

$$(F_+^{rl} F_+^{lr})^m(V) = V^{(m)}, \quad (F_-^{rl} F_-^{lr})^m(V) = V^{(-m)}$$

for all $m \geq 0$.

(2) and (3) are obvious, since $c_{R,V}^{2m}(\mathcal{S}_R^{2m} \otimes \mathrm{id})(R(n) \otimes V') = R(n) \otimes V'$ for all $n \geq 0$ and all subobjects $V' \subseteq V$ in $\mathcal{C}$. $\square$

12.3. The braided monoidal isomorphism (Ω, ω)

For an $\mathbb{N}_0$-graded Hopf algebra A in $\mathcal{C} = {}^H_H\mathcal{YD}$ we denote by ${}^A_A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ and ${}_{\mathrm{rat}}\mathcal{YD}(\mathcal{C})^A_A$ the full subcategories of the left and the right Yetter-Drinfeld modules over A which are rational A-modules, respectively.

In this section we assume that $(A, B, \langle\ ,\ \rangle)$ is a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$.

The main results are Theorem 12.3.2, which says that there is an isomorphism $(\Omega, \omega) : {}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \xrightarrow{\cong} {}^A_A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ of braided monoidal categories, and Theorem 12.3.3, which describes for any Hopf algebra K in ${}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ the bosonization $\Omega(K)\#A$ in terms of K.

We begin with a result on categories of Yetter-Drinfeld modules which is very similar to Theorem 3.4.16. There we assumed strict monoidal isomorphisms between module categories and between comodule categories to derive a braided monoidal isomorphism of Yetter-Drinfeld modules. Here we do the same assuming isomorphisms between module and comodule categories.

Recall from Proposition 3.3.8 that

$$\langle\ ,\ \rangle^+ = \langle\ ,\ \rangle c_{B,A}(\mathcal{S}_B \otimes \mathcal{S}_A) : B \otimes A \to \Bbbk,$$
$$\langle\ ,\ \rangle^{+\mathrm{cop}} = \langle\ ,\ \rangle^+(\mathrm{id}_B \otimes \mathcal{S}_A^{-1}) : B^{\mathrm{cop}} \otimes A^{\mathrm{cop}} \to \Bbbk$$

give rise to dual pairs of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$ and $\overline{\mathcal{C}}$, respectively, and that

$$\langle\ ,\ \rangle^{+\mathrm{cop}} = \langle\ ,\ \rangle c_{B,A}(\mathrm{id}_B \otimes \mathcal{S}_A) = \langle\ ,\ \rangle c_{B,A}(\mathcal{S}_B \otimes \mathrm{id}_A). \tag{12.3.1}$$

Let

$$D_1 : {}^B\mathcal{C} \to {}_{A^{\mathrm{cop}}}\overline{\mathcal{C}}_{\mathrm{rat}}, \qquad D_2 : {}^{A^{\mathrm{cop}}}\overline{\mathcal{C}} \to {}_B\mathcal{C}_{\mathrm{rat}}$$

be the strict monoidal isomorphisms $D_1 = \overline{D}^l$ for the pairing $\langle\ ,\ \rangle$, and $D_2 = \overline{D}^l$ for the pairing $\langle\ ,\ \rangle^{+\mathrm{cop}}$ in Proposition 12.2.7. By definition,

$$D_1(V, \delta) = (V, \overline{\lambda}), \quad \overline{\lambda} = \big(A \otimes V \xrightarrow{\mathrm{id}\otimes\delta} A \otimes B \otimes V \xrightarrow{\langle\ ,\ \rangle\otimes\mathrm{id}} V\big), \tag{12.3.2}$$

$$D_2(V, \overline{\delta}) = (V, \lambda), \quad \lambda = \big(B \otimes V \xrightarrow{\mathrm{id}\otimes\overline{\delta}} B \otimes A \otimes V \xrightarrow{\langle\ ,\ \rangle^{+\mathrm{cop}}\otimes\mathrm{id}} V\big), \tag{12.3.3}$$

for all $(V, \delta) \in {}^B\mathcal{C}$, $(V, \overline{\delta}) \in {}^{A^{\mathrm{cop}}}\overline{\mathcal{C}}$.

THEOREM 12.3.1. *The functor*

$$D : \overline{{}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}} \to {}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})_{\mathrm{rat}}, \quad (V, \lambda, \delta) \mapsto (V, \overline{\lambda}, \overline{\delta}),$$

where $\overline{\lambda}$ and $\overline{\delta}$ are defined by (12.3.2) *and* (12.3.3). *and where morphisms f are mapped onto f, is a braided strict monoidal isomorphism.*

PROOF. (1) Let $(X, \overline{\delta}_X) \in {}^{A^{\mathrm{cop}}}\overline{\mathcal{C}}$, $(V, \delta) \in {}^B\mathcal{C}$, and define

$$(X, \lambda_X) = D_2(X, \overline{\delta}_X), \quad (V, \overline{\lambda}) = D_1(V, \delta).$$

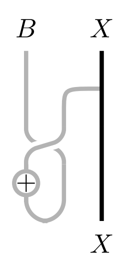
We first prove the equality

$$\overline{c}^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)} = c^{\mathcal{YD}}_{(X,\overline{\delta}_X),(V,\overline{\lambda})}. \tag{12.3.4}$$

Let $\delta =$ 

, $\overline{\delta}_X =$ 

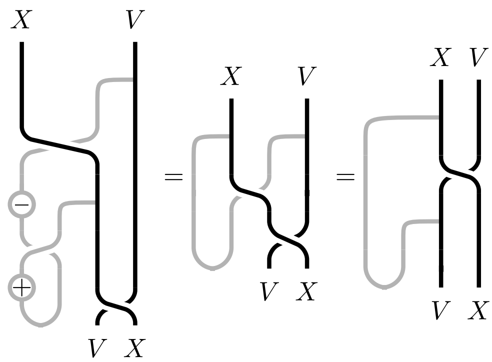
. Then $\overline{\lambda} =$ 

, $\lambda_X =$ 

.

Hence

$$\overline{c}^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)} =$$



$$= c^{\mathcal{YD}}_{(X,\overline{\delta}_X),(V,\overline{\lambda})},$$

where the second equality follows from (3.2.13) with $h = \overline{\delta}_X$ and (3.2.9), and since $\mathcal{S}_B^{-1}$ and $\mathcal{S}_A$ cancel by Proposition 3.3.8(1), and the third from (3.2.12) with $h = \delta$.

(2) Let $V \in \mathcal{C}$, and

$$\mathcal{P}^l(V) = \{(\lambda,\delta) \mid (V,\lambda) \in {}_B\mathcal{C}_{\mathrm{rat}}, (V,\delta) \in {}^B\mathcal{C}\},$$
$$\mathcal{P}^r(V) = \{(\overline{\lambda},\overline{\delta}) \mid (V,\overline{\lambda}) \in {}_{A^{\mathrm{cop}}}\overline{\mathcal{C}}_{\mathrm{rat}}, (V,\overline{\delta}) \in {}^{A^{\mathrm{cop}}}\overline{\mathcal{C}}\}.$$

By Proposition 12.2.7, the map $\Phi : \mathcal{P}^l(V) \to \mathcal{P}^r(V),\ (\lambda,\delta) \mapsto (\overline{\lambda},\overline{\delta})$, defined by

$$D_1(V,\delta) = (V,\overline{\lambda}), \qquad D_2(V,\overline{\delta}) = (V,\lambda),$$

is bijective.

Let $(\lambda,\delta) \in \mathcal{P}^l(\mathrm{V})$, and $(\overline{\lambda},\overline{\delta}) = \Phi(\lambda,\delta)$. We claim that the following are equivalent.

(a) $(V,\lambda,\delta) \in {}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$.
(b) $(V,\overline{\lambda},\overline{\delta}) \in {}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})_{\mathrm{rat}}$.
(c) For all $(X,\lambda_X) \in {}_B\mathcal{C}_{\mathrm{rat}}$, the morphism

$$\overline{c}^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)} : (X,\lambda_X) \otimes (V,\lambda) \to (V,\lambda) \otimes (X,\lambda_X) \text{ is in } {}_B\mathcal{C}_{\mathrm{rat}}.$$

(d) For all $(X,\overline{\delta}_X) \in {}^{A^{\mathrm{cop}}}\overline{\mathcal{C}}$, the morphism

$$c^{\mathcal{YD}}_{(X,\overline{\delta}_X),(V,\overline{\lambda})} : (X,\overline{\delta}_X) \otimes (V,\overline{\delta}) \to (V,\overline{\delta}) \otimes (X,\overline{\delta}_X) \text{ is in } {}^{A^{\mathrm{cop}}}\overline{\mathcal{C}}.$$

By Proposition 12.2.6 and Proposition 3.4.8, (a) is equivalent to (c). By Proposition 3.4.5, (b) is equivalent to (d). The equivalence of (c) and (d) follows from (12.3.4), since D_2 is a strict monoidal isomorphism.

(3) Since D_1 and D_2 are strict monoidal isomorphisms, it follows from (1) and (2), that D is a well-defined strict monoidal isomorphism.

To show that D is braided, let $X = (X,\lambda_X,\delta_X), V = (V,\lambda,\delta) \in {}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$. Define $D(X) = (X,\overline{\lambda}_X,\overline{\delta}_X)$, $D(V) = (V,\overline{\lambda},\overline{\delta})$. Then $\overline{c}^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)}$ is the inverse braiding of X,V in ${}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, and $c^{\mathcal{YD}}_{(X,\overline{\delta}_X),(V,\overline{\lambda})}$ is the braiding of $D(X), D(V)$ in ${}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$. Hence D is braided by (12.3.4). $\square$

If $(G,\psi) : \mathcal{A} \to \mathcal{B}$ is a braided monoidal functor, then (G,ψ) is also braided monoidal with respect to the inverse braidings of $\mathcal{A}$ and $\mathcal{B}$. We denote this functor again by $(G,\psi) : \overline{\mathcal{A}} \to \overline{\mathcal{B}}$.

It follows from Proposition 3.3.4 and Lemma 12.2.9(1) that the functors

$$
\begin{aligned}
&{}_A\mathcal{C}_{\mathrm{rat}} \to {}_{\mathrm{rat}}\mathcal{C}_A,\ (V,\lambda) \mapsto (V,\lambda_+),\ \text{where } \lambda_+ = \lambda c_{V,A}(\mathrm{id}_V \otimes \mathcal{S}_A),\\
&{}_{\mathrm{rat}}\mathcal{C}_A \to {}_A\mathcal{C}_{\mathrm{rat}},\ (V,\lambda) \mapsto (V,\lambda_-),\ \text{where } \lambda_- = \lambda \overline{c}_{A,V}(\mathcal{S}_A^{-1} \otimes \mathrm{id}_V),
\end{aligned}
$$

are well-defined inverse isomorphisms. Hence the braided monoidal isomorphisms in Theorems 3.4.15, 3.4.16 and Corollary 3.4.17 for A restrict to braided monoidal isomorphisms again denoted by

$$
\begin{aligned}
&(F_{rl}^{\mathcal{YD}},\rho) : {}_{\mathrm{rat}}\mathcal{YD}(\mathcal{C})_A^A \to {}_A^A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}},\ (V,\lambda,\delta) \mapsto (V,\lambda_-,(\mathcal{S}_A \otimes \mathrm{id}_V)c_{V,A}\delta),\\
&\overline{F}_{lr}^{\mathcal{YD}} : {}_{A^{\mathrm{cop}}}^{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})_{\mathrm{rat}} \to \overline{{}_{\mathrm{rat}}\mathcal{YD}(\mathcal{C})_A^A},\ (V,\lambda,\delta) \mapsto (V,\lambda_+,c_{A,V}\delta),\\
&(F,\varphi) : \overline{{}_A^A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}} \to {}_{A^{\mathrm{cop}}}^{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})_{\mathrm{rat}},\ (V,\lambda,\delta) \mapsto (V,\lambda,(\mathcal{S}_A^{-1} \otimes \mathrm{id}_V)\overline{c}^2_{A,V}\delta).
\end{aligned}
$$

THEOREM 12.3.2. *The functor*

$$\Omega : {}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}_A^A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}, \quad (V,\lambda,\delta) \mapsto (V,\lambda_1,\delta_1),\ \textit{with}$$

$$\lambda_1 = \big(A \otimes V \xrightarrow{\mathrm{id}\otimes\delta} A \otimes B \otimes V \xrightarrow{\langle\ ,\ \rangle\otimes\mathrm{id}} V\big),\ \textit{and}$$

$$\delta_1 = \big(V \xrightarrow{\delta_2} A \otimes V \xrightarrow{\mathcal{S}_A^2\otimes\mathrm{id}} A \otimes V \xrightarrow{c^2_{A,V}} A \otimes V\big),\ \textit{where } \delta_2 \textit{ is defined by}$$

$$\lambda = \big(B \otimes V \xrightarrow{\mathrm{id}\otimes\delta_2} B \otimes A \otimes V \xrightarrow{\langle\ ,\ \rangle^+\otimes\mathrm{id}} V\big),$$

and where morphisms f are mapped onto f, is an isomorphism of categories, and

$$(\Omega,\omega) : {}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}_A^A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}},\ \textit{where } \omega_{X,Y} = c^{{}_B^B\mathcal{YD}(\mathcal{C})}_{Y,X}\overline{c}_{X,Y}$$

for all $X,Y \in {}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, is a braided monoidal isomorphism.

The diagram

$$
\begin{tikzcd}
\overline{{}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}} \arrow[rr, "(\Omega,\omega)"] \arrow[dr, "D"'] & & \overline{{}_A^A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}} \arrow[dl, "(F,\varphi)"] \\
& {}_{A^{\mathrm{cop}}}^{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})_{\mathrm{rat}} &
\end{tikzcd}
$$

of braided monoidal isomorphisms commutes.

PROOF. By Corollary 3.4.17, (F,φ) is an isomorphism. Since the inverse of (F,φ) is $(F_{rl}^{\mathcal{YD}},\rho)\overline{F}_{lr}^{\mathcal{YD}}$, we define (Ω,ω) as the composition

$$\overline{{}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}} \xrightarrow{D} {}_{A^{\mathrm{cop}}}^{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})_{\mathrm{rat}} \xrightarrow{\overline{F}_{lr}^{\mathcal{YD}}} \overline{{}_{\mathrm{rat}}\mathcal{YD}(\mathcal{C})_A^A} \xrightarrow{(F_{rl}^{\mathcal{YD}},\rho)} \overline{{}_A^A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}}.$$

We compute the functor Ω. Let $(V,\lambda,\delta) \in {}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$. Then

$$\Omega(V,\lambda,\delta) = F_{rl}^{\mathcal{YD}}\overline{F}_{lr}^{\mathcal{YD}}D(V,\lambda,\delta) = (V,\overline{\lambda}_{+-},(\mathcal{S}_A \otimes \mathrm{id}_V)c^2_{A,V}\overline{\delta}),$$

where $\overline{\lambda}$ and $\overline{\delta}$ are defined by

$$\overline{\lambda} = \big(A \otimes V \xrightarrow{\mathrm{id}\otimes\delta} A \otimes B \otimes V \xrightarrow{\langle\ ,\ \rangle\otimes\mathrm{id}} V\big), \tag{12.3.5}$$

$$\lambda = \big(B \otimes V \xrightarrow{\mathrm{id}\otimes\overline{\delta}} B \otimes A \otimes V \xrightarrow{\langle\ ,\ \rangle^{+\mathrm{cop}}\otimes\mathrm{id}} V\big). \tag{12.3.6}$$

Note that $\overline{\lambda}_{+-} = \lambda_1$. We have to prove that $(\mathcal{S}_A \otimes \mathrm{id}_V)c^2_{A,V}\overline{\delta} = \delta_1$, that is,

$$\delta_2 = (\mathcal{S}_A^{-2} \otimes \mathrm{id}_V)\overline{c}^2_{A,V}(\mathcal{S}_A \otimes \mathrm{id}_V)c^2_{A,V}\overline{\delta} = (\mathcal{S}_A^{-1} \otimes \mathrm{id}_V)\overline{\delta}$$

satisfies the equation

$$\lambda = \big(B \otimes V \xrightarrow{\mathrm{id}\otimes\delta_2} B \otimes A \otimes V \xrightarrow{\langle\, ,\, \rangle^{+}\otimes\mathrm{id}} V\big).$$

This follows from (12.3.6), since $\langle\, ,\, \rangle^{+\mathrm{cop}} = \langle\, ,\, \rangle^{+}(\mathrm{id}_B \otimes \mathcal{S}_A^{-1})$.

To compute the monoidal structure ω, let $G = \overline{F}_{lr}^{\mathcal{YD}} D$. Then G is a braided strict monoidal functor, $\Omega = F_{rl}^{\mathcal{YD}} G$, and for all $X, Y \in {}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$,

$$\omega_{X,Y} = \rho_{G(X),G(Y)} : F_{rl}^{\mathcal{YD}} G(X) \otimes F_{rl}^{\mathcal{YD}} G(Y) \to F_{rl}^{\mathcal{YD}} G(X \otimes Y),$$

where $\rho_{G(X),G(Y)} = c^{\mathcal{YD}(\mathcal{C})_A^A}_{G(Y),G(X)} \overline{c}_{G(X),G(Y)}$.

The functor $G : {}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}_{\mathrm{rat}}\mathcal{YD}(\mathcal{C})_A^A$ with the Yetter-Drinfeld braidings of ${}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ and of ${}_{\mathrm{rat}}\mathcal{YD}(\mathcal{C})_A^A$ is braided and strict monoidal. Hence for all $X, Y \in {}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$,

$$\omega_{X,Y} = c^{\mathcal{YD}(\mathcal{C})_A^A}_{G(Y),G(X)} \overline{c}_{G(X),G(Y)} = c^{{}_B^B\mathcal{YD}(\mathcal{C})}_{Y,X} \overline{c}_{X,Y}.$$

□

Let K be a Hopf algebra in ${}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ with bijective antipode. We denote by $\Omega(K)$ the Hopf algebra in ${}_A^A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ given by the braided isomorphism (Ω, ω). The bosonization $(\widetilde{S}, \widetilde{\pi}, \widetilde{\gamma})$ of $\Omega(K)$ is a Hopf algebra triple over A in $\mathcal{C}$, and $(\widetilde{S}^{\mathrm{cop}}, \widetilde{\pi}, \widetilde{\gamma})$ is a Hopf algebra triple over A^{cop} in $\overline{\mathcal{C}}$ with commutative diagrams

$$\begin{tikzcd} & A \arrow[dl, "\widetilde{\gamma}"'] \arrow[d, "="] \\ \widetilde{S} = \Omega(K)\#A \arrow[r, "\widetilde{\pi}"] & A \end{tikzcd} \qquad \begin{tikzcd} & A^{\mathrm{cop}} \arrow[dl, "\widetilde{\gamma}"'] \arrow[d, "="] \\ (\Omega(K)\#A)^{\mathrm{cop}} \arrow[r, "\widetilde{\pi}"] & A^{\mathrm{cop}} \end{tikzcd}$$

Since $\mathcal{S}_K = \mathcal{S}_{\Omega(K)}$ by Remark 3.1.8, the antipodes of $\Omega(K)$ and $\widetilde{S}$ are bijective by Corollary 3.8.11.

Then $\Omega(K) = \widetilde{S}^{\mathrm{co}\,A}$ is the set of right coinvariant elements of the projection $\widetilde{\pi}$ (where we identify $x\#1$ with x for all $x \in \Omega(K)$). Let $\widetilde{L} \subseteq \widetilde{S}^{\mathrm{cop}}$ be the braided Hopf algebra of right coinvariant elements of the braided Hopf algebra projection $\widetilde{S}^{\mathrm{cop}} \xrightarrow{\widetilde{\pi}} A^{\mathrm{cop}}$ in $\overline{\mathcal{C}}$. Thus $\widetilde{L}$ is a Hopf algebra in ${}_{A^{\mathrm{cop}}}^{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$, and by Theorem 3.10.4 on Hopf algebra triples in $\overline{\mathcal{C}}$, the multiplication map

$$\widetilde{L}\#A^{\mathrm{cop}} \cong \widetilde{S}^{\mathrm{cop}}$$

is an isomorphism of Hopf algebras in $\overline{\mathcal{C}}$. By Theorem 12.3.1, we may view the Hopf algebra K^{cop} in $\overline{{}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}}$ as a Hopf algebra in ${}_{A^{\mathrm{cop}}}^{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$. This Hopf algebra turns out to be isomorphic to $\widetilde{L}$.

THEOREM 12.3.3. *Let K be a Hopf algebra in ${}_B^B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ with bijective antipode. Let $(\widetilde{S}, \widetilde{\pi}, \widetilde{\gamma})$ be the bosonizations of $\Omega(K)$, and $\widetilde{L}$ the set of right coinvariant elements of the projection $\widetilde{S}^{\mathrm{cop}} \xrightarrow{\widetilde{\pi}} A^{\mathrm{cop}}$.*

Then the morphism $T : \widetilde{L} \to K$, $x \mapsto \mathcal{S}_K^{-1}\mathcal{S}_{\widetilde{S}}(x)$, in $\mathcal{C}$ is an isomorphism

$$T : \widetilde{L} \to D(K^{\mathrm{cop}})$$

of Hopf algebras in ${}_{A^{\mathrm{cop}}}^{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$.

PROOF. We denote by $F(\Omega(K)^{\mathrm{cop}})$ the image of the Hopf algebra $\Omega(K)^{\mathrm{cop}}$ of the braided strict monoidal isomorphism $(F, \varphi) : {}^A_A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}}))_{\mathrm{rat}}$. Let $T : \widetilde{L} \to F(\Omega(K)^{\mathrm{cop}})$ be the isomorphism of Hopf algebras in ${}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$ of Theorem 3.10.6 for the Hopf algebra triple $(\widetilde{S}, \widetilde{\pi}, \widetilde{\gamma})$.

Note that $\Omega(K)^{\mathrm{cop}} = \Omega(K^{\mathrm{cop}})$, since (Ω, ω) is a braided monoidal functor. Hence

$$F(\Omega(K)^{\mathrm{cop}}) = F(\Omega(K^{\mathrm{cop}})) = D(K^{\mathrm{cop}})$$

by the commutative diagram in Theorem 12.3.2.

By Theorem 3.10.6, $\iota_{\widetilde{L}}T^{-1} = \mathcal{S}_{\widetilde{S}}^{-1}\iota\mathcal{S}_{\Omega(K)}$. Since $\mathcal{S}_{\Omega(K)} = \mathcal{S}_K$, it follows that $T^{-1}(y) = \mathcal{S}_{\widetilde{S}}^{-1}\mathcal{S}_K(y)$ for all $y \in K$. Hence $\mathcal{S}_{\widetilde{S}}(\widetilde{L}) = K$, and $T(x) = \mathcal{S}_K^{-1}\mathcal{S}_{\widetilde{S}}(x)$ for all $x \in \widetilde{L}$. □

REMARK 12.3.4. Since D is strict monoidal, the Hopf algebra $D(K^{\mathrm{cop}})$ is described as follows. Let μ_K, Δ_K, λ and δ be multiplication, comultiplication, B-action and B-coaction of K. Then multiplication, comultiplication, A^{cop}-action and A^{cop}-coaction of the Hopf algebra $D(K^{\mathrm{cop}})$ are μ_K, $\overline{c}^{{}^B_B\mathcal{YD}(\mathcal{C})}_{K,K}\Delta_K$, and $\overline{\lambda}$, $\overline{\delta}$ defined in (12.3.2) and (12.3.3).

We close this section with an immediate corollary of Theorem 12.3.3. Let c be a braiding of the monoidal category ${}^H_H\mathcal{YD}$. Let P be a Hopf algebra in $({}^H_H\mathcal{YD}, c)$, and X a Hopf algebra in ${}^P_P\mathcal{YD}({}^H_H\mathcal{YD}, c)$. A Hopf ideal I of X is a subobject $I \subseteq X$ in ${}^P_P\mathcal{YD}({}^H_H\mathcal{YD}, c)$ which is an ideal and a coideal of X with $\mathcal{S}_X(I) \subseteq I$. Hopf ideals $I \subseteq X$ are the subobjects in ${}^P_P\mathcal{YD}({}^H_H\mathcal{YD}, c)$ such that the quotient map $P \to P/I$ is a morphism of Hopf algebras in ${}^P_P\mathcal{YD}({}^H_H\mathcal{YD}, c)$. We denote by $\mathfrak{I}(P)$ the set of all Hopf ideals of P.

COROLLARY 12.3.5. *Under the assumptions of Theorem* 12.3.3 *the map*

$$\mathfrak{I}(\widetilde{L}) \to \mathfrak{I}(K),\ I \mapsto T(I),$$

is bijective, where $\mathfrak{I}(\widetilde{L})$ *and* $\mathfrak{I}(K)$ *are the set of Hopf ideals of the Hopf algebra* $\widetilde{L}$ *in* ${}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$ *and of the Hopf algebra* K *in* ${}^B_B\mathcal{YD}(\mathcal{C})$.

PROOF. By Theorem 12.3.3, the map

$$\mathfrak{I}(\widetilde{L}) \to \mathfrak{I}(D(K^{\mathrm{cop}})),\ I \mapsto T(I),$$

is bijective. By Theorem 12.3.1, $\mathfrak{I}(D(K^{\mathrm{cop}})) = \mathfrak{I}(K^{\mathrm{cop}})$. Since Hopf ideals of K are Yetter-Drinfeld subobjects, it is clear that $\mathfrak{I}(K^{\mathrm{cop}}) = \mathfrak{I}(K)$. □

COROLLARY 12.3.6. *Let* V *be an object in* ${}^B_B\mathcal{YD}_{\mathrm{rat}}$.

(1) $\mathcal{F}_n\Omega(V) = \mathcal{F}^nV$, $\mathcal{F}^n\Omega(V) = \mathcal{F}_nV$ *for all* $n \geq 0$.
(2) *If* V *is a* $\mathbb{Z}$*-graded object in* ${}^B_B\mathcal{YD}_{\mathrm{rat}}$*, then* $\Omega(V)$ *is a* $\mathbb{Z}$*-graded object in* ${}^A_A\mathcal{YD}_{\mathrm{rat}}$*, where* $\Omega(V)(n) = V(-n)$ *for all* $n \in \mathbb{Z}$.

PROOF. (1) By Lemma 12.2.10(1), $\mathcal{F}_n\Omega(V) = \mathcal{F}^nV$ and $\mathcal{F}^n(V, \delta_2) = \mathcal{F}_nV$. By Lemma 12.2.11, $\mathcal{F}^n(V, \delta_2) = \mathcal{F}^n(V, \delta_1) = \mathcal{F}^n\Omega(V)$.

(2) follows from Lemma 12.2.10(2) and Lemma 12.2.11(2). □

We introduce a notation for the special case of Theorem 12.3.2 we need later-on.

DEFINITION 12.3.7. Let $V \in \mathcal{C} = {}^H_H\mathcal{YD}$ be finite-dimensional. We denote by

$$(\Omega_V, \omega_V) : {}^{\mathcal{B}(V)}_{\mathcal{B}(V)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^{\mathcal{B}(V^*)}_{\mathcal{B}(V^*)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$$

the braided monoidal isomorphism of Theorem 12.3.2 for $(\mathcal{B}(V^*), \mathcal{B}(V), \langle\ ,\ \rangle)$ with Hopf pairing $\langle\ ,\ \rangle : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to \Bbbk$ of Corollary 7.2.5.

REMARK 12.3.8. If R is an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$ with bijective antipode, ${}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$ denotes the full subcategory of Yetter-Drinfeld modules in ${}^{R\#H}_{R\#H}\mathcal{YD}$ which are rational as modules over R. In the situation of Theorem 12.3.2, we use the braided, strict monoidal isomorphism of Theorem 3.8.7 to obtain a braided monoidal isomorphism

$${}^{B\#H}_{B\#H}\mathcal{YD}_{\mathrm{rat}} \cong {}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^A_A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \cong {}^{A\#H}_{A\#H}\mathcal{YD}_{\mathrm{rat}}$$

which we again denote by (Ω, ω).

For $Q \in {}^{B\#H}_{B\#H}\mathcal{YD}_{\mathrm{rat}}$, the Nichols algebra $\mathcal{B}(Q)$ (defined with respect to the braiding of ${}^{B\#H}_{B\#H}\mathcal{YD}$) is an $\mathbb{N}_0$-graded Hopf algebra in ${}^{B\#H}_{B\#H}\mathcal{YD}$. By Lemma 12.2.4, $\mathcal{B}(Q)$ is again an object in ${}^{B\#H}_{B\#H}\mathcal{YD}_{\mathrm{rat}}$.

COROLLARY 12.3.9. *Let Q be an object in ${}^{B\#H}_{B\#H}\mathcal{YD}_{\mathrm{rat}}$, and $\mathcal{B}(Q)$ its Nichols algebra. Then*

$$(\Omega, \omega)(\mathcal{B}(Q)) \cong \mathcal{B}((\Omega, \omega)(Q))$$

as $\mathbb{N}_0$-graded Hopf algebras in ${}^{A\#H}_{A\#H}\mathcal{YD}_{\mathrm{rat}}$.

PROOF. It is easy to see that $(\Omega, \omega)(\mathcal{B}(Q))$ is a connected $\mathbb{N}_0$-graded Hopf algebra which is generated as an algebra by $\Omega(Q)$. Moreover, any homogeneous primitive element of degree ≥ 2 is zero. Hence $(\Omega, \omega)(\mathcal{B}(Q))$ is a Nichols algebra of $\Omega(Q)$, and the claim follows from Theorem 7.1.14. $\square$

12.4. One-sided coideal subalgebras of braided Hopf algebras

The isomorphism $T : \widetilde{L} \to K$ of Theorem 12.3.3 shows that the Hopf algebras S and $\widetilde{S}$ are closely related. In this section we use T to study one-sided coideal subalgebras in S and in $\widetilde{S}$.

We begin with some general remarks about one-sided coideal subalgebras in Hopf algebra triples.

Let H be a Hopf algebra with bijective antipode, and let $\mathcal{C} = ({}^H_H\mathcal{YD}, c)$ be a braided monoidal category with underlying monoidal category ${}^H_H\mathcal{YD}$ and some braiding c. In particular, c could be the Yetter-Drinfeld braiding $c^{{}^H_H\mathcal{YD}}$ or its inverse $\overline{c}^{{}^H_H\mathcal{YD}}$.

Let X be a bialgebra in $\mathcal{C}$. A **left (right) coideal subalgebra** of X in $\mathcal{C}$ is a subobject $E \subseteq X$ and an algebra in $\mathcal{C}$ such that the inclusion map $E \subseteq X$ is an algebra morphism in $\mathcal{C}$ and $\Delta_X(E) \subseteq X \otimes E$ ($\Delta_X(E) \subseteq E \otimes X$).

Let P be a Hopf algebra in $\mathcal{C}$, $Q \subseteq P$ a Hopf subalgebra, and $\pi : P \to Q$ a Hopf algebra morphism in $\mathcal{C}$ with $\pi|Q = \mathrm{id}_Q$. Let $R = P^{\mathrm{co}\, Q}$ be the space of right coinvariant elements of P with respect to π. Thus we are in the situation of

Theorem 3.10.4 of a Hopf algebra triple (P, π, γ), where γ is the inclusion map.

$$\begin{tikzcd}
 & & Q \arrow[dl, "\subseteq"'] \arrow[d, "="] \\
R \arrow[r, "\subseteq"] \arrow[d, "="'] & P \arrow[r, "\pi"] \arrow[dl, "\vartheta"] & Q \\
R & &
\end{tikzcd}$$

Let $\Delta_P : P \to P \otimes P$, $x \mapsto \Delta_P(x) = x^{(1)} \otimes x^{(2)}$, denote the comultiplication of P. Recall that for all $x \in P$,

$$\vartheta(x) = x^{(1)} \mathcal{S}_Q(\pi(x^{(2)})), \tag{12.4.1}$$

$R \subseteq P$ is a subalgebra in $\mathcal{C}$ with $\Delta_P(R) \subseteq P \otimes R$, and R is a Hopf algebra in ${}^Q_Q\mathcal{YD}(\mathcal{C})$ with Q-action $\mathrm{ad} : Q \otimes R \to R$ and Q-coaction $\delta : R \to Q \otimes R$. The multiplication map

$$R \otimes Q \to P, \quad x \otimes q \mapsto xq, \text{ is bijective} \tag{12.4.2}$$

with inverse $P \to R \otimes Q$, $x \mapsto \vartheta(x^{(1)}) \otimes \pi(x^{(2)})$. For all $x \in R$, $y \in P$, $q \in Q$,

$$\Delta_R(x) = \vartheta(x^{(1)}) \otimes x^{(2)}, \tag{12.4.3}$$

$$\delta(x) = \pi(x^{(1)}) \otimes x^{(2)}, \tag{12.4.4}$$

$$\Delta_P(x) = \vartheta(x^{(1)})\pi(x^{(2)}) \otimes x^{(3)}, \tag{12.4.5}$$

$$\vartheta(qx) = (\mathrm{ad}\, q)(x), \tag{12.4.6}$$

$$\vartheta(yq) = \vartheta(y)\varepsilon(q). \tag{12.4.7}$$

In the next two lemmas we relate right coideal subalgebras of P containing Q and left coideal subalgebras of P contained in R to the Hopf algebra structure of R in ${}^Q_Q\mathcal{YD}(\mathcal{C})$.

Definition 12.4.1. Let

$$\mathcal{E}_r^+(P) = \{E \mid E \subseteq P \text{ right coideal subalgebra in } \mathcal{C},\ Q \subseteq E\},$$
$$\mathcal{E}_r(P, X) = \{E \mid E \subseteq P \text{ right coideal subalgebra in } \mathcal{C},\ E \subseteq X\},$$

where $X \subseteq P$ is a subobject in ${}^H_H\mathcal{YD}$, and

$$\mathcal{F}_r(P) = \{F \mid F \subseteq R \text{ subalgebra in } \mathcal{C},$$
$$\Delta_R(F) \subseteq F \otimes R,\ F \subseteq R\ Q\text{-submodule}\}.$$

For left coideal subalgebras we define

Definition 12.4.2. Let

$$\mathcal{E}_l^+(P) = \{E \mid E \subseteq P \text{ left coideal subalgebra in } \mathcal{C},\ Q \subseteq E\},$$
$$\mathcal{E}_l(P, X) = \{E \mid E \subseteq P \text{ left coideal subalgebra in } \mathcal{C},\ E \subseteq X\},$$

where $X \subseteq P$ is a subobject in ${}^H_H\mathcal{YD}$, and

$$\mathcal{F}_l(P) = \{F \mid F \subseteq R \text{ subalgebra in } \mathcal{C},$$
$$\Delta_R(F) \subseteq R \otimes F,\ F \subseteq R\ Q\text{-subcomodule}\}.$$

Note that the sets in the previous definitions depend on the Hopf algebra triple (P, π, γ).

LEMMA 12.4.3. (1) *For all $E \in \mathcal{E}_r^+(P)$, the multiplication map*

$$(E \cap R) \otimes Q \to E$$

is an isomorphism in ${}^H_H\mathcal{YD}$.

(2) *The map $\mathcal{E}_r^+(P) \to \mathcal{F}_r(P)$, $E \mapsto E \cap R$, is bijective with inverse given by $F \mapsto FQ$.*

PROOF. (1) The map

$$E \to (E \cap R) \otimes Q, \quad x \mapsto x^{(1)}\mathcal{S}_Q(\pi(x^{(2)})) \otimes \pi(x^{(3)}) = \vartheta(x^{(1)}) \otimes \pi(x^{(2)}),$$

is inverse to the multiplication map. This can be checked using (12.4.7).

(2) We first show that both maps are well-defined. Let $E \in \mathcal{E}_r^+(P)$. Then $E \cap R \subseteq R$ is a subalgebra in $\mathcal{C}$. For all $x \in E \cap R$,

$$\Delta_R(x) = x^{(1)}\mathcal{S}_Q(\pi(x^{(2)})) \otimes x^{(3)} \in (E \otimes P) \cap (R \otimes R) = (E \cap R) \otimes R$$

by (12.4.1) and (12.4.3). Moreover,

$$\Delta_P(qx) = (q^{(1)} \otimes q^{(2)})(x^{(1)} \otimes x^{(2)}) \in QE \otimes QP \subseteq E \otimes P$$

for all $x \in E$, $q \in Q$, since $c(Q \otimes E) = E \otimes Q$. Hence $E \cap R$ is a Q-submodule of R, since

$$(\operatorname{ad} q)(x) = \vartheta(qx) = (qx)^{(1)}\mathcal{S}_Q\pi((qx)^{(2)}) \in E \cap R$$

by (12.4.1) and (12.4.6).

Let $F \in \mathcal{F}_r(P)$. By (12.4.3) and (12.4.5), $\Delta_P(F) \subseteq FQ \otimes R$. Hence for all $x \in F$, $q \in Q$, $\Delta_P(xq) = \Delta_P(x)\Delta_P(q) \in FQ \otimes P$.

To see that $FQ \subseteq P$ is a subalgebra, we have to prove that $QF \subseteq FQ$. For all $q \in Q$, $x \in F$,

$$\Delta_P(qx) = \Delta_P(q)\Delta_P(x) \in QFQ \otimes P,$$

since $\Delta_P(F) \subseteq FQ \otimes R$. Hence

$$qx = \vartheta((qx)^{(1)})\pi((qx)^{(2)}) \in \vartheta(QFQ)Q = (\operatorname{ad} Q)(F)Q \subseteq FQ$$

by (12.4.6), (12.4.7), and since $F \subseteq R$ is a Q-submodule.

Finally it follows that the two maps are inverse bijections. If $E \in \mathcal{E}_r^+(P)$, then $E = (E \cap R)Q$ by (1). If $F \in \mathcal{F}_r(P)$, then $(FQ \cap R)Q = FQ$. By (12.4.2), the multiplication maps $(FQ \cap R) \otimes Q \to (FQ \cap R)Q = FQ$ and $F \otimes Q \to FQ$ are bijective. Hence $F = FQ \cap R$. $\square$

LEMMA 12.4.4. $\mathcal{E}_l(P, R) = \mathcal{F}_l(P)$.

PROOF. The inclusion $\mathcal{E}_l(P, R) \subseteq \mathcal{F}_l(P)$ follows from (12.4.3) and (12.4.4), and $\supseteq$ follows from (12.4.5). $\square$

Now we assume the situation of Theorem 12.3.3. Thus $(A, B, \langle\ ,\ \rangle)$ is a dual pair of locally finite Hopf algebras in $\mathcal{C} = {}^H_H\mathcal{YD}$ (with the Yetter-Drinfeld braiding),

$$(\Omega, \omega) : {}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^A_A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$$

is the braided monoidal isomorphism of Theorem 12.3.2, K is a Hopf algebra in ${}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, and (S,π,γ) and $(\widetilde{S},\widetilde{\pi},\widetilde{\gamma})$ are the bosonizations of K and of $\Omega(K)$ with commutative diagrams

$$\begin{tikzcd} & B \arrow[dl, "\gamma"'] \arrow[d, "="] \\ S = K\#B \arrow[r, "\pi"] & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\widetilde{\gamma}"'] \arrow[d, "="] \\ \widetilde{S} = \Omega(K)\#A \arrow[r, "\widetilde{\pi}"] & A \end{tikzcd}$$

The triple $(\widetilde{S}^{\mathrm{cop}},\widetilde{\pi},\widetilde{\gamma})$ is a Hopf algebra triple over A^{cop} in $\overline{\mathcal{C}}$, and $\widetilde{L}$ denotes the set of right coinvariant elements of the projection $\widetilde{S}^{\mathrm{cop}} \xrightarrow{\widetilde{\pi}} A^{\mathrm{cop}}$.

The bijections in the following theorems are induced by the Hopf algebra isomorphism $T : \widetilde{L} \to K$ of Theorem 12.3.3.

THEOREM 12.4.5. *Under the assumptions of Theorem* 12.3.3 *the map*

$$\mathcal{E}_r(\widetilde{S},\widetilde{L}) \to \mathcal{E}_r^+(S),\ E \mapsto T(E)B,$$

is an inclusion-preserving bijection with inverse given by $E \mapsto T^{-1}(E \cap K)$. *For all* $E \in \mathcal{E}_r(\widetilde{S},\widetilde{L})$, *the multiplication map*

$$T(E) \otimes B \to T(E)B$$

is bijective.

PROOF. By Theorem 12.3.3, T is an A^{cop}-colinear isomorphism of algebras and coalgebras in $\mathcal{C}$. The A^{cop}-comodule structure of K is $(\mathcal{S}_A \otimes \mathrm{id})\delta_2$, where the category isomorphism $D^l_+ : {}^A\mathcal{C} \to {}_B\mathcal{C}_{\mathrm{rat}}$ maps (K,δ_2) onto (K,λ). Note that a subobject $F \subseteq K$ in $\mathcal{C}$ is a B-submodule if and only if F is an A^{cop}-subcomodule. This follows from the category isomorphism D^l_+, and since $\mathcal{S}_A : A \to A^{\mathrm{cop}}$ is a coalgebra isomorphism. Hence T induces a bijection between

$$\begin{aligned}\mathcal{F}_l(\widetilde{S}^{\mathrm{cop}}) = \{F \mid F \subseteq \widetilde{L} \text{ subalgebra in } {}^H_H\mathcal{YD},\ \Delta_{\widetilde{L}}(F) \subseteq \widetilde{L} \otimes F,\\ F \subseteq \widetilde{L}\ A^{\mathrm{cop}}\text{-subcomodule}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{F} = \{F \mid F \subseteq K \text{ subalgebra in } {}^H_H\mathcal{YD},\ \overline{c}^{{}^B_B\mathcal{YD}(\mathcal{C})}_{K,K}\Delta_K(F) \subseteq K \otimes F,\\ F \subseteq K\ B\text{-submodule}\}.\end{aligned}$$

If $F \subseteq K$ is a subobject in $\mathcal{C}$ and a left B-submodule, then

$$\overline{c}^{{}^B_B\mathcal{YD}(\mathcal{C})}_{K,K}\Delta_K(F) \subseteq K \otimes F \iff \Delta_K(F) \subseteq F \otimes K,$$

since by Proposition 3.4.5, the braiding of $K \otimes K$ in ${}^B_B\mathcal{YD}(\mathcal{C})$ defines an isomorphism $K \otimes F \cong F \otimes K$ for B-stable subobjects $F \subseteq K$. Hence $\mathcal{F} = \mathcal{F}_r(S)$.

Lemma 12.4.4 for the projection $\widetilde{S}^{\mathrm{cop}} \xrightarrow{\widetilde{\pi}} A^{\mathrm{cop}}$ gives the equality

$$\mathcal{E}_r(\widetilde{S},\widetilde{L}) = \mathcal{E}_l(\widetilde{S}^{\mathrm{cop}},\widetilde{L}) = \mathcal{F}_l(\widetilde{S}^{\mathrm{cop}}).$$

The first claim of the theorem follows by composing the bijection $\mathcal{E}_r(\widetilde{S},\widetilde{L}) \to \mathcal{F}_r(S)$ induced by T and the bijection $\mathcal{F}_r(S) \to \mathcal{E}_r^+(S)$ in Lemma 12.4.3(2). The second claim then holds by Lemma 12.4.3(1). □

THEOREM 12.4.6. *Under the assumptions of Theorem* 12.3.3 *the map*

$$\mathcal{E}_l^+(\widetilde{S}) \to \mathcal{E}_l(S,K), \quad E \mapsto T(E \cap \widetilde{L}),$$

is an inclusion-preserving bijection with inverse given by $E \mapsto T^{-1}(E)A$. *For all* $E \in \mathcal{E}_l(S,K)$, *the multiplication map*

$$T^{-1}(E) \otimes A \to T^{-1}(E)A$$

is bijective.

PROOF. By Theorem 12.3.3, $T : \widetilde{L} \to K$ is an A^{cop}-linear, that is, A-linear isomorphism of algebras and coalgebras in the monoidal category $\mathcal{C}$. Recall that K is an A-module with module structure λ_1, where the category isomorphism $D^l : {}^B\mathcal{C} \to {}_A\mathcal{C}_{\mathrm{rat}}$ maps (K,δ) onto (K,λ_1). Thus a subobject $F \subseteq K$ in $\mathcal{C}$ is an A-submodule if and only if F is a B-subcomodule. Hence T induces a bijection between

$$\mathcal{F}_r(\widetilde{S}^{\mathrm{cop}}) = \{F \mid F \subseteq \widetilde{L} \text{ subalgebra in } {}^H_H\mathcal{YD},\ \Delta_{\widetilde{L}}(F) \subseteq F \otimes \widetilde{L},\\ F \subseteq \widetilde{L}\ A\text{-submodule}\}$$

and

$$\mathcal{F} = \{F \mid F \subseteq K \text{ subalgebra in } {}^H_H\mathcal{YD},\ \overline{c}^{\,{}^B_B\mathcal{YD}(\mathcal{C})}_{K,K}\Delta_K(F) \subseteq F \otimes K,\\ F \subseteq K\ B\text{-subcomodule}\}.$$

If $F \subseteq K$ is a subobject in $\mathcal{C}$ and a left B-subcomodule, then

$$\overline{c}^{\,{}^B_B\mathcal{YD}(\mathcal{C})}_{K,K}\Delta_K(F) \subseteq F \otimes K \iff \Delta_K(F) \subseteq K \otimes F,$$

since by Proposition 3.4.5, the braiding of $K \otimes K$ in ${}^B_B\mathcal{YD}(\mathcal{C})$ defines an isomorphism $F \otimes K \cong K \otimes F$ for B-costable subobjects $F \subseteq K$. Hence $\mathcal{F} = \mathcal{F}_l(S)$.

Note that $\mathcal{F}_l(S) = \mathcal{E}_l(S,K)$ by Lemma 12.4.4 for the projection $S \xrightarrow{\pi} B$. Since $\mathcal{E}_l^+(\widetilde{S}) = \mathcal{E}_r^+(\widetilde{S}^{\mathrm{cop}})$, the first part of the theorem follows by composing the bijection $\mathcal{E}_r^+(\widetilde{S}^{\mathrm{cop}}) \to \mathcal{F}_r(\widetilde{S}^{\mathrm{cop}})$ in Lemma 12.4.3(2) with $\mathcal{C} = \overline{{}^H_H\mathcal{YD}}$, and the bijection $\mathcal{F}_r(\widetilde{S}^{\mathrm{cop}}) \to \mathcal{F}_l(S)$ induced by T. The second claim holds by Lemma 12.4.3(1). □

12.5. Notes

12.3. The braided monoidal isomorphism (Ω,ω) first appeared in [**HS13b**] in the form ${}^{B\#H}_{B\#H}\mathcal{YD}_{\mathrm{rat}} \cong {}^{A\#H}_{A\#H}\mathcal{YD}_{\mathrm{rat}}$, however, without the factorization in Theorem 12.3.2. Then in [**BLS15**] a proof of the category isomorphism was given for finite-dimensional pairs A, B of braided Hopf algebras and replacing ${}^{A\#H}_{A\#H}\mathcal{YD}$ by ${}^A_A\mathcal{YD}(\mathcal{C})$, $\mathcal{C} = {}^H_H\mathcal{YD}$. In fact, in [**BLS15**], $\mathcal{C}$ was just a braided monoidal category, but the braided Hopf algebras A, B were related by a non-degenerate pairing $A \otimes B \to I$ together with an inverse copairing $I \to B \otimes A$; in particular, A was a left dual of B.

Our proof of Theorem 12.3.2 is inspired from [**BLS15**]. To cover the case of the dual pair $\mathcal{B}(V^*), \mathcal{B}(V)$, $V \in {}^H_H\mathcal{YD}$ finite-dimensional (where the existence of a copairing is not assumed), we have to introduce in Section 12.2 Yetter-Drinfeld modules which are rational as modules. Working with Yetter-Drinfeld modules over smash products $A\#H$, as we did in our first proof, easily gets technically very complicated. For the main results in this chapter we need Yetter-Drinfeld modules

${}^{A}_{A}\mathcal{YD}(\mathcal{C})$, where $\mathcal{C} = {}^{H}_{H}\mathcal{YD}$ or $\mathcal{C} = \overline{{}^{H}_{H}\mathcal{YD}}$. Hence we presented the general theory of braided monoidal categories in the long Chapter 3.

The factorization of Ω in Theorem 12.3.2, and Theorem 12.3.3 are published here for the first time.

12.4. The Hopf algebra isomorphism T of Theorem 12.3.3 is the main tool to compare right or left coideal subalgebras of $K\#B$ and of $\Omega(K)\#A$, K a Hopf algebra in ${}^{B}_{B}\mathcal{YD}(\mathcal{C})$, in Theorem 12.4.5 and 12.4.6. Special cases of these results for one-sided coideal subalgebras of Nichols algebras have been obtained in [**HS13a**] with another method of proof.

CHAPTER 13

Nichols systems, and semi-Cartan graph of Nichols algebras

Let H be a Hopf algebra with bijective antipode. After a discussion of some subtle technicalities on graded objects and bosonization, we introduce and study reflections of tuples of Yetter-Drinfeld modules over H in Section 13.4. Using the functor Ω from the previous Chapter, we show how these can be extended to reflections of Nichols systems if all entries of the tuples are simple. In the ideal case, as shown in Section 13.6, the reflections give rise to a semi-Cartan graph.

13.1. $\mathbb{Z}$-graded Yetter-Drinfeld modules

The functor $\mathbb{N}_0\text{-Gr}\,\mathcal{M}_{\Bbbk} \to \mathbb{Z}\text{-Gr}\,\mathcal{M}_{\Bbbk}$ which extends the $\mathbb{N}_0$-grading of an object V in $\mathbb{N}_0\text{-Gr}\,\mathcal{M}_{\Bbbk}$ to a $\mathbb{Z}$-grading by setting $V(n) = 0$ for all $n < 0$, is strict monoidal. Hence an $\mathbb{N}_0$-graded algebra R is naturally a $\mathbb{Z}$-graded algebra by setting $R(n) = 0$ for all $n < 0$. In the same way we view $\mathbb{N}_0$-graded coalgebras and Hopf algebras as $\mathbb{Z}$-graded coalgebras and Hopf algebras, respectively.

Let H be a $\mathbb{Z}$-graded Hopf algebra. A $\mathbb{Z}$-graded Yetter-Drinfeld module V over H is by definition an object V in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-Gr}\,\mathcal{M}_{\Bbbk})$ (see Section 5.5). In other words, V is an object in ${}^H_H\mathcal{YD}$ and a $\mathbb{Z}$-graded vector space such that the module and comodule structure maps $H \otimes V \to V$ and $V \to H \otimes V$ are graded.

We next characterize irreducible $\mathbb{Z}$-graded Yetter-Drinfeld modules over an $\mathbb{N}_0$-graded Hopf algebra.

Let R be an $\mathbb{N}_0$-graded algebra, C an $\mathbb{N}_0$-graded coalgebra, $X \in {}_R\mathcal{M}$, and $Y \in {}^C\mathcal{M}$ with comodule structure $\delta : Y \to C \otimes Y$. Recall that

$$\mathcal{F}_0 X = \{x \in X \mid R(i)x = 0 \text{ for all } i > 0\},$$
$$\mathcal{F}^0 Y = \{y \in Y \mid \delta(y) \in C(0) \otimes Y\}.$$

LEMMA 13.1.1. *Let R be an $\mathbb{N}_0$-graded algebra, C an $\mathbb{N}_0$-graded coalgebra, and H an $\mathbb{N}_0$-graded Hopf algebra.*

1. *Let X be a left R-submodule. Then $\mathcal{F}_0 X \subseteq X$ is an R-submodule. If X is a $\mathbb{Z}$-graded R-module, then $\mathcal{F}_0 X$ is a $\mathbb{Z}$-graded submodule.*
2. *Let $Y \neq 0$ be a left C-comodule with coaction $\delta : Y \to C \otimes Y$. Then $\mathcal{F}^0 Y \subseteq Y$ is a C-subcomodule with $\delta(\mathcal{F}^0 Y) \subseteq C(0) \otimes \mathcal{F}^0 Y$, and $\mathcal{F}^0 Y \neq 0$. If Y is a $\mathbb{Z}$-graded C-comodule, then $\mathcal{F}^0 Y$ is a $\mathbb{Z}$-graded subcomodule.*
3. *Let V be a $\mathbb{Z}$-graded Yetter-Drinfeld module over H. Then the homogeneous components $V(n)$, $n \in \mathbb{Z}$, are objects in ${}^{H(0)}_{H(0)}\mathcal{YD}$, where the $H(0)$-action is given by restriction with respect to the Hopf algebra inclusion $H(0) \subseteq H$, and the $H(0)$-coaction is defined by the Hopf algebra projection $H \to H(0)$.*

PROOF. (1) and (3) are easy to check.

(2) By Remark 2.2.10(3) and Corollary 5.2.6, $\delta(\mathcal{F}^0 Y) \subseteq C(0) \otimes \mathcal{F}^0 Y$, and $\mathcal{F}^0 Y \neq 0$. If Y is a $\mathbb{Z}$-graded C-comodule, then $\mathcal{F}^0 Y = \delta^{-1}(H(0) \otimes Y)$ is a $\mathbb{Z}$-graded subcomodule of Y, since $H(0) \otimes Y \subseteq H \otimes Y$ is a graded subspace. □

PROPOSITION 13.1.2. *Let H be an $\mathbb{N}_0$-graded Hopf algebra with bijective antipode, and V a $\mathbb{Z}$-graded Yetter-Drinfeld module over H. The following are equivalent.*

(1) *V is an irreducible object in ${}^H_H\mathcal{YD}$.*
(2) *V is an irreducible $\mathbb{Z}$-graded Yetter-Drinfeld module over H.*
(3) *There is an integer n_0 such that*
  (a) *$V(n_0)$ is irreducible in ${}^{H(0)}_{H(0)}\mathcal{YD}$,*
  (b) *$V(n_0) = \mathcal{F}^0 V$,*
  (c) *$V(n_0)$ generates V as an H-module, that is,*

$$V(n) = \begin{cases} H(n-n_0)V(n_0) & \text{for all } n \geq n_0, \\ 0 & \text{for all } n < n_0. \end{cases}$$

PROOF. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Let $\delta : V \to H \otimes V$ denote the left coaction of H on V. By Lemma 13.1.1(2), $\mathcal{F}^0 V$ is a non-zero $\mathbb{Z}$-graded H-subcomodule of V satisfying $\delta(\mathcal{F}^0 V) \subseteq H(0) \otimes \mathcal{F}^0 V$. Let n_0 be an integer such that the homogeneous component $\mathcal{F}^0 V \cap V(n_0)$ of degree n_0 of $\mathcal{F}^0 V$ is non-zero. Then

$$\delta(\mathcal{F}^0 V \cap V(n_0)) \subseteq H(0) \otimes (\mathcal{F}^0 V \cap V(n_0)),$$

and $\mathcal{F}^0 V \cap V(n_0) \subseteq V$ is an H-subcomodule. Let $X \subseteq \mathcal{F}^0 V \cap V(n_0)$ be a non-zero H-subcomodule. By Lemma 5.5.1(2), $HX \subseteq V$ is a $\mathbb{Z}$-graded subobject in ${}^H_H\mathcal{YD}$. Hence $HX = V$ by (2). Thus $H(n)X = V(n+n_0)$ for all $n \geq 0$. In particular, $V(n) = 0$ for all $n < n_0$, and $H(0)X = V(n_0)$. Then $\mathcal{F}^0 V = \mathcal{F}^0 V \cap V(n_0)$. Since $\delta(V(n_0)) \in H(0) \otimes V(n_0)$, it follows that $\mathcal{F}^0 V = V(n_0)$. We proved (3)(b) and (3)(c).

To prove (3)(a), let $X \subseteq V(n_0)$ be a non-zero Yetter-Drinfeld submodule over $H(0)$. Then $X = H(0)X = V(n_0)$.

(3) $\Rightarrow$ (1). Let $X \subseteq V$ be a non-zero subobject in ${}^H_H\mathcal{YD}$. Then $\mathcal{F}^0 X$ is a subobject of $\mathcal{F}^0 V$ in ${}^{H(0)}_{H(0)}\mathcal{YD}$. By Lemma 13.1.1(2), $\mathcal{F}^0 X$ is non-zero, hence $\mathcal{F}^0 X = \mathcal{F}^0 V = V(n_0) \subseteq X$ by (3)(a) and (3)(b). Thus $X = V$ by (3)(c). □

PROPOSITION 13.1.3. *Let H be an $\mathbb{N}_0$-graded Hopf algebra with bijective antipode, and V a $\mathbb{Z}$-graded Yetter-Drinfeld module over H. Assume that there are integers $n_0 \leq n_1$ such that*

$$V = V(n_0) \oplus V(n_0+1) \oplus \cdots \oplus V(n_1),\ V(n_0) \neq 0,\ V(n_1) \neq 0,$$

is the decomposition of V into $\mathbb{Z}$-homogeneous components. The following are equivalent.

(1) *V is an irreducible object in ${}^H_H\mathcal{YD}$.*
(2) (a) *$V(n_0)$ is irreducible in ${}^{H(0)}_{H(0)}\mathcal{YD}$,*
  (b) *$V(n_0) = \mathcal{F}^0 V$,*
  (c) *$V(n_0)$ generates V as an H-module.*
(3) (a) *$V(n_1)$ is irreducible in ${}^{H(0)}_{H(0)}\mathcal{YD}$,*

(b) $V(n_1) = \mathcal{F}_0 V$,
(c) $V(n_1)$ *generates* V *as an* H*-comodule.*

PROOF. (1) $\Leftrightarrow$ (2) follows from Proposition 13.1.2.

(1) $\Rightarrow$ (3). By Lemma 13.1.1(1), $\mathcal{F}_0 V \subseteq V$ is a $\mathbb{Z}$-graded H-submodule. Note that $V(n_1) \subseteq \mathcal{F}_0 V$, hence $\mathcal{F}_0 V \neq 0$. Let l be an integer such that $\mathcal{F}_0 V \cap V(l) \neq 0$. Then $\mathcal{F}_0 V \cap V(l) \subseteq V$ is an H-submodule, since $V(l) \subseteq H$ is an $H(0)$-submodule. Let $X \subseteq \mathcal{F}_0 V \cap V(l)$ be a non-zero H-submodule. By Lemma 5.5.1(3) and (1), $XH^* = V$. Since $XH^* \subseteq \bigoplus_{j \leq l} V(j)$ it follows that $l = n_1$. Thus $\mathcal{F}_0 V = V(n_1)$. We have shown (3)(b) and (3)(c).

To prove (3)(a), let $0 \neq X \subseteq V(n_1)$ be a Yetter-Drinfeld submodule over $H(0)$, where $V(n_1)$ is a Yetter-Drinfeld module over $H(0)$ as defined in Lemma 13.1.1(3). Then $V = XH^* \subseteq \bigoplus_{j<n_1} V(n_j) \oplus X$, and $X = V(n_1)$.

(3) $\Rightarrow$ (1). By Proposition 13.1.2 it is enough to show that V is an irreducible $\mathbb{Z}$-graded Yetter-Drinfeld module over H. Let $X \subseteq V$ be a non-zero $\mathbb{Z}$-graded Yetter-Drinfeld module over H. Thus

$$X = X \cap V(n_0) \oplus X \cap V(n_0+1) \oplus \cdots \oplus X \cap V(n_1).$$

Since $\mathcal{F}_0 X$ contains the non-zero homogeneous component of X of maximal degree, it follows that $0 \neq \mathcal{F}_0 X \subseteq \mathcal{F}_0 V = V(n_1)$ by (b). Therefore $V(n_1) = \mathcal{F}_0 X \subseteq X$ by (a). By (c), $V = V(n_1)H^* \subseteq XH^* = X$. $\square$

13.2. Projections of Nichols algebras

For any Yetter-Drinfeld module $V \in {}^H_H\mathcal{YD}$, the bosonization $\mathcal{B}(V)\#H$ is an $\mathbb{N}_0$-graded Hopf algebra with $\deg(V) = 1$ and $\deg(H) = 0$ by Corollaries 4.3.5 and 4.3.6. We call this grading the standard grading of the bosonization $\mathcal{A}(V) = \mathcal{B}(V)\#H$. Let $\pi_H = \varepsilon \otimes \mathrm{id}_H : \mathcal{A}(V) \to H$ be the Hopf algebra projection onto H. Thus the diagram

$$\begin{tikzcd} & & H \arrow[dl, "\subseteq"'] \arrow[d, "="] \\ \mathcal{B}(V) \subseteq & \mathcal{A}(V) \arrow[r, "\pi_H"] & H \end{tikzcd}$$

commutes, and $\mathcal{B}(V) = \mathcal{A}(V)^{\mathrm{co}\,H}$. We use the notation

$$\Delta_{\mathcal{A}(V)}(a) = a_{(1)} \otimes a_{(2)}, \quad \Delta_{\mathcal{B}(V)}(b) = b^{(1)} \otimes b^{(2)}$$

for all $a \in \mathcal{A}(V)$, $b \in \mathcal{B}(V)$. Let $\vartheta = \mathrm{id}_{\mathcal{B}(V)} \otimes \varepsilon : \mathcal{A}(V) \to \mathcal{B}(V)$ be the coalgebra projection onto $\mathcal{B}(V)$. Then $\Delta_{\mathcal{B}(V)}(a) = (\vartheta \otimes \mathrm{id})\Delta_{\mathcal{A}(V)}(a)$ for any $a \in \mathcal{B}(V)$. In particular, for all $x \in V$,

$$\Delta_{\mathcal{B}(V)}(x) = x \otimes 1 + 1 \otimes x, \quad \Delta_{\mathcal{A}(V)}(x) = x \otimes 1 + x_{(-1)} \otimes x_{(0)}.$$

See Theorem 3.8.7 and Corollary 4.3.3 for the theory of bosonization.

In this subsection we fix a Yetter-Drinfeld module $V \in {}^H_H\mathcal{YD}$ with subobjects U and W in ${}^H_H\mathcal{YD}$ such that $V = U \oplus W$. Thus

$$\begin{tikzcd} & & U \arrow[dl, "\subseteq"'] \arrow[d, "="] \\ W \subseteq & V \arrow[r, "\pi"] & U \end{tikzcd}$$

is a commutative diagram in ${}^H_H\mathcal{YD}$, where $\pi : V = W \oplus U \to U$ is the projection with kernel W.

LEMMA 13.2.1. *There is a unique Hopf algebra map $\pi : \mathcal{A}(V) \to \mathcal{A}(U)$ which is the identity on U and H and vanishes on W. The map π is $\mathbb{N}_0$-graded with respect to the gradings given by*

$$\deg(H) = 0, \quad \deg(U) = 0, \quad \deg(W) = 1,$$

and also with respect to the standard gradings.

PROOF. The algebra $\mathcal{A}(V)$ is generated by V and H. This implies the uniqueness of π. On the other hand, $V \in {}^H_H\mathcal{YD}$ is $\mathbb{N}_0$-graded with $V(0) = U$, $V(1) = W$, and $V(n) = 0$ for all $n \geq 2$. Then $\mathcal{B}(V)$ is an $\mathbb{N}_0$-graded bialgebra by Corollary 7.1.15(1), and $\mathcal{A}(U)$ is the degree zero part of $\mathcal{A}(V)$. Let $\pi : \mathcal{A}(V) \to \mathcal{A}(U)$ be the graded projection. Then π is a Hopf algebra map vanishing on W, and it is graded in the standard gradation. $\square$

Let $K = \{x \in \mathcal{A}(V) \mid (\mathrm{id} \otimes \pi)\Delta_{\mathcal{A}(V)}(x) = x \otimes 1\}$. Hence

$$\begin{tikzcd} & \mathcal{A}(U) \arrow[dl, "\subseteq"'] \arrow[d, "="] \\ K \subseteq \mathcal{A}(V) \arrow[r, "\pi"] & \mathcal{A}(U) \end{tikzcd}$$

commutes, and $K = \mathcal{A}(V)^{\mathrm{co}\,\mathcal{A}(U)}$.

We first view π as an $\mathbb{N}_0$-graded map with respect to the standard gradings of $\mathcal{A}(V)$ and $\mathcal{A}(U)$. By Theorem 5.5.6, K is an $\mathbb{N}_0$-graded Hopf algebra in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ with grading $K(n) = \mathcal{A}(V)(n) \cap K$ for all $n \geq 0$, and with action, coaction and comultiplication

$$\begin{aligned} \mathrm{ad}\, &: \mathcal{A}(U) \otimes K \to K, \; a \otimes x \mapsto \mathrm{ad}\, a(x), \\ \delta_K &: K \to \mathcal{A}(U) \otimes K, \; x \mapsto (\pi \otimes \mathrm{id})\Delta_{\mathcal{A}(U)}(x), \\ \Delta_K &: K \to K \otimes K, \; x \mapsto \vartheta_K(x_{(1)}) \otimes x_{(2)}, \end{aligned}$$

with $\vartheta_K : \mathcal{A}(V) \to K$, $a \mapsto a_{(1)}\pi\mathcal{S}(a_{(2)})$.

The multiplication map

$$K\#\mathcal{A}(U) \xrightarrow{\cong} \mathcal{A}(V)$$

is an $\mathbb{N}_0$-graded Hopf algebra isomorphism.

We denote the primitive elements of K by

$$P(K) = \{x \in K \mid \Delta_K(x) = x \otimes 1 + 1 \otimes x\}.$$

LEMMA 13.2.2. (1) *$K = \{x \in \mathcal{B}(V) \mid (\mathrm{id} \otimes \pi)\Delta_{\mathcal{B}(V)}(x) = x \otimes 1\}$.*
(2) *$P(K) \subseteq K$ is an $\mathbb{N}_0$-graded subobject in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$.*

PROOF. (1) Let $\pi_H = \varepsilon \otimes \mathrm{id} : \mathcal{A}(U) \to H$ be the projection onto H, and $\vartheta = \mathrm{id} \otimes \varepsilon : \mathcal{A}(V) \to \mathcal{B}(V)$. If $x \in K$, then $(\mathrm{id} \otimes \pi_H\pi)\Delta_{\mathcal{A}(V)}(x) = x \otimes 1$, hence $x \in \mathcal{A}(V)^{\mathrm{co}\,H} = \mathcal{B}(V)$, and $x \otimes 1 = x^{(1)}x^{(2)}{}_{(-1)} \otimes \pi(x^{(2)}{}_{(0)})$. Hence

$$x \otimes 1 = \vartheta(x^{(1)}x^{(2)}{}_{(-1)}) \otimes \pi(x^{(2)}{}_{(0)}) = x^{(1)} \otimes \pi(x^{(2)}).$$

Conversely, let $x \in \mathcal{B}(V)$ with $x^{(1)} \otimes \pi(x^{(2)}) = x \otimes 1$. Then $x \in K$, since the projection $\pi : \mathcal{B}(V) \to \mathcal{B}(U)$ is left H-colinear.

(2) follows from Lemma 5.5.2. □

For the proof of the important Proposition 13.2.4 below we need a general result on the existence of specific elements in a subcomodule of a graded comodule.

Let C be an $\mathbb{N}_0$-graded coalgebra, and X an $\mathbb{N}_0$-graded left C-comodule with structure map $\delta : X \to C \otimes X$. We define the components of δ as we did before for Δ. For all $i, j \geq 0$ let $\delta_{ij} : X(i+j) \to C(i) \otimes X(j)$ be the composition

$$X(i+j) \subseteq X \xrightarrow{\delta} C \otimes X \xrightarrow{\pi_i \otimes \pi_j} C(i) \otimes X(j).$$

In the next proposition we consider graded comodules with injective components $\delta_{n-1,1}$ for all $n \geq 1$.

Proposition 13.2.3. *Let C be an $\mathbb{N}_0$-graded coalgebra, X an $\mathbb{N}_0$-graded left C-comodule, and Y a C-subcomodule of X. Let $k \geq 0$ be an integer. Assume that $\delta_{n-k,k} : X(n) \to C(n-k) \otimes X(k)$ is injective for all $n \geq k$, and that Y is not contained in $\bigoplus_{i=0}^{k-1} X(i)$. Then $Y \cap \bigoplus_{i=0}^{k} X(i) \neq 0$.*

Proof. By assumption there is an element $0 \neq y = \sum_{i=0}^{n} x(i) \in Y$, $n \geq k$, with homogeneous components $x(i) \in X(i)$ for all $0 \leq i \leq n$, and $x(n) \neq 0$. Let $x = x(n)$, $z = y - x$. Since $\delta_{n-k,k}$ is injective,

$$0 \neq (\pi_{n-k} \otimes \pi_k)(\delta(x)) \in C(n-k) \otimes X(k).$$

Hence there exists $f \in C^*$ with $0 \neq f(x_{(-1)})x_{(0)} \in X(k)$ and $f(C(i)) = 0$ for all $i \neq n-k$. Note that $f(z_{(-1)})z_{(0)} \in \bigoplus_{i=0}^{k-1} X(i)$. Thus

$$f(y_{(-1)})y_{(0)} = f(x_{(-1)})x_{(0)} + f(z_{(-1)})z_{(0)} \in \bigoplus_{i=0}^{k} X(i)$$

is a non-zero element in $Y \cap \bigoplus_{i=0}^{k} X(i)$. □

Proposition 13.2.4. *Let Z be a nonzero subobject of $W \subseteq \mathcal{A}(W)$ in ${}^H_H\mathcal{YD}$, and let $Q = \operatorname{ad} \mathcal{A}(U)(Z)$.*

(1) *$Q \subseteq P(K)$ is an $\mathbb{N}_0$-graded subobject in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$, and*

$$Q(0) = 0, \quad Q(1) = Z, \quad Q(n) = (\operatorname{ad} U)^{n-1}(Z) \text{ for all } n \geq 2.$$

(2) *For all $x \in Q$,*
   (a) *$\Delta_{\mathcal{A}(V)}(x) \in x \otimes 1 + \mathcal{A}(U) \otimes Q$,*
   (b) *$\Delta_{\mathcal{B}(V)}(x) \in x \otimes 1 + \mathcal{B}(U) \otimes Q$.*

(3) *$Z \subseteq Q$ is a large left $\mathcal{A}(U)$-subcomodule, that is, if $Q' \subseteq Q$ is a non-zero $\mathcal{A}(U)$-subcomodule, then $Q' \cap Z \neq 0$.*

Proof. (1) Let $x \in W$. Then $\Delta_{\mathcal{A}(V)}(x) = x \otimes 1 + x_{(-1)} \otimes x_{(0)}$, hence $x \in K$, since $\pi(W) = 0$. By definition of Δ_K,

$$\Delta_K(x) = \vartheta_K(x) \otimes 1 + \vartheta_K(x_{(-1)}) \otimes x_{(0)} = x \otimes 1 + 1 \otimes x.$$

Hence $W \subseteq P(K)$. Let $a \in \mathcal{B}(U)$ and $h \in H$. Then

$$\operatorname{ad}(a\#h)(Z) = \operatorname{ad} a(h \cdot Z) \subseteq \operatorname{ad} a(Z).$$

Hence $Q = \operatorname{ad} \mathcal{A}(U)(Z) = \operatorname{ad} \mathcal{B}(U)(Z) = \bigoplus_{n \geq 0} (\operatorname{ad} U)^n(Z)$, where for all $n \geq 0$, $\deg((\operatorname{ad} U)^n(Z)) = n+1$.

For all $x \in Z$, $\delta_K(z) = \pi(x) \otimes 1 + \pi(x_{(-1)}) \otimes x_{(2)} = x_{(-1)} \otimes x_{(0)}$. Hence $Z \subseteq P(K)$ is an $\mathcal{A}(U)$-subcomodule, and by Lemma 5.5.1(2), $Q \subseteq P(K)$ is a graded subobject in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$.

(2)(a) Let $a \in \mathcal{A}(U)$, $z \in Z$. Then $\Delta_{\mathcal{A}(V)}(z) = z \otimes 1 + z_{(-1)} \otimes z_{(0)}$. Hence

$$\begin{aligned}\Delta_{\mathcal{A}(V)}(\operatorname{ad} a(z)) &= \Delta_{\mathcal{A}(V)}(a_{(1)} z \mathcal{S}(a_{(2)})) \\ &= a_{(1)} z_{(1)} \mathcal{S}(a_{(4)}) \otimes a_{(2)} z_{(2)} \mathcal{S}(a_{(3)}) \\ &= a_{(1)} z \mathcal{S}(a_{(4)}) \otimes a_{(2)} \mathcal{S}(a_{(3)}) \\ &\quad + a_{(1)} z_{(-1)} \mathcal{S}(a_{(4)}) \otimes a_{(2)} z_{(0)} \mathcal{S}(a_{(3)}) \\ &\in \operatorname{ad} a(z) \otimes 1 + \mathcal{A}(U) \otimes Q.\end{aligned}$$

(2)(b) Let $x \in Q$. Then by (2)(a),

$$\Delta_{\mathcal{B}(V)}(x) = \vartheta(x_{(1)}) \otimes x_{(2)} \in x \otimes 1 + \mathcal{B}(U) \otimes Q.$$

(3) $\mathcal{B}(V)$ is a left $\mathcal{B}(U)$-comodule in ${}^H_H\mathcal{YD}$ with comodule structure

$$\delta : \mathcal{B}(V) \xrightarrow{\Delta_{\mathcal{B}(V)}} \mathcal{B}(V) \otimes \mathcal{B}(V) \xrightarrow{\pi \otimes \mathrm{id}} \mathcal{B}(U) \otimes \mathcal{B}(V).$$

Let $x \in Q$. Then $\delta(x) \in \mathcal{B}(U) \otimes Q$ by (2)(b), since $\pi(x) = 0$. Thus Q is an $\mathbb{N}_0$-graded left $\mathcal{B}(U)$-comodule via $\delta : Q \to \mathcal{B}(U) \otimes Q$, and for all $n \geq 1$ and $x \in Q(n)$, $\delta_{n-1,1}(x) = \Delta_{\mathcal{B}(V)_{n-1,1}}(x)$ by (2)(b), since $\pi_1(1) = 0$. It follows that $\delta_{n-1,1} : Q(n) \to \mathcal{B}(U)(n-1) \otimes Q(1)$ is injective, since the Nichols algebra $\mathcal{B}(V)$ is strictly graded. Hence (3) follows from Proposition 13.2.3 with $k = 1$, since $Q(0) = 0$ by (1). □

Corollary 13.2.5. (1) *Let Z_i with $i \in I$ be subobjects of W in ${}^H_H\mathcal{YD}$, and let $Z \in {}^H_H\mathcal{YD}$ such that $Z = \sum_{i \in I} Z_i = \bigoplus_{i \in I} Z_i$. For all $i \in I$ let $Q_i = \operatorname{ad} \mathcal{A}(U)(Z_i)$, and let $Q = \operatorname{ad} \mathcal{A}(U)(Z)$. Then $Q = \bigoplus_{i \in I} Q_i$.*

(2) *Let $Z \subseteq W$ be an irreducible subobject in ${}^H_H\mathcal{YD}$. Then $\operatorname{ad} \mathcal{A}(U)(Z)$ is an irreducible object in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$.*

Proof. (1) If the sum of the Q_i is not direct, there is an index $k \in I$ such that $0 \neq Q_k \cap \sum_{i \neq k} Q_i$. Hence $0 \neq Z_k \cap \sum_{i \neq k} Q_i$, since Z_k is a large $\mathcal{A}(U)$-subcomodule of Q_k by Proposition 13.2.4(3). Since Z_k is of degree one, we obtain the contradiction $0 \neq Z_k \cap \sum_{i \neq k} Z_i$.

(2) Let $0 \neq Q' \subseteq \operatorname{ad} \mathcal{A}(U)(Z)$ be a subobject in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$. Then $0 \neq Q' \cap Z$ by Proposition 13.2.4(3). Since $Q' \cap Z$ is an H-submodule and an H-subcomodule of Z, and since Z is irreducible in ${}^H_H\mathcal{YD}$, it follows that $Q' \cap Z = Z \subseteq Q'$, hence $\operatorname{ad} \mathcal{A}(U)(Z) = Q'$. □

Definition 13.2.6. Let Q be a non-zero $\mathbb{N}_0$-graded object in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$. Assume that Q has only finitely many non-zero homogeneous components. Let

$$Q^{\max} = Q(n), \text{ where } n \geq 0,\ Q(n) \neq 0,\ Q(m) = 0 \text{ for all } m > n.$$

Note that the homogeneous components of an $\mathbb{N}_0$-graded object in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ are objects in ${}^H_H\mathcal{YD}$, where the H-action is induced from the inclusion $H \subseteq \mathcal{A}(U)$, and the H-coaction from the projection $\pi_H : \mathcal{A}(U) \to H$.

THEOREM 13.2.7. *Let $Z \subseteq W$ be an irreducible subobject in ${}^H_H\mathcal{YD}$, and assume that $Q = \operatorname{ad}\mathcal{A}(U)(Z)$ has only finitely many non-zero homogeneous components. Then $Q^{\max}$ is irreducible in ${}^H_H\mathcal{YD}$, and Q is generated as an $\mathcal{A}(U)$-comodule by $Q^{\max}$.*

PROOF. This follows from Corollary 13.2.5(2) and Proposition 13.1.3(3). $\square$

In the next theorem, $\mathcal{B}(\operatorname{ad}\mathcal{A}(U)(W))$ denotes the Nichols algebra of the Yetter-Drinfeld module $\operatorname{ad}\mathcal{A}(U)(W)$ in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$.

THEOREM 13.2.8. *There is a unique isomorphism*

$$K \cong \mathcal{B}(\operatorname{ad}\mathcal{A}(U)(W))$$

of braided Hopf algebras in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ which is the identity on $\operatorname{ad}\mathcal{A}(U)(W)$. In particular, $P(K) = \operatorname{ad}\mathcal{A}(U)(W)$.

PROOF. Let $Q = \operatorname{ad}\mathcal{A}(U)(W)$. By Lemma 2.6.25, K is generated as an algebra by Q.

We go back to the non-standard gradings in Lemma 13.2.1, where

$$\deg(H) = 0,\ \deg(U) = 0,\ \deg(W) = 1.$$

The map π is $\mathbb{N}_0$-graded, where now $\mathcal{A}(U)$ is trivially graded. By Theorem 5.5.6, K is an $\mathbb{N}_0$-graded Hopf algebra in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ with homogeneous components in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$, where by (1), $K(0) = \Bbbk$ and $K(n) = Q^n$ for all $n \geq 1$. By Lemma 5.5.2, $P(K)$ is an $\mathbb{N}_0$-graded object in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ with homogeneous components $P(K)(n) = P(K) \cap Q^n$ in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$. In particular, $P(K)(n)$, $n \geq 1$, is a subcomodule of the left $\mathcal{A}(U)$-comodule K with comodule structure $\delta_K : K \to \mathcal{A}(U) \otimes K$, $x \mapsto \pi(x_{(1)}) \otimes x_{(2)}$. It remains to show that $P(K)(n) = 0$ for all $n \geq 2$.

Assume that $P(K)(n)$ is non-zero for some $n \geq 2$. The Hopf algebra $\mathcal{A}(U)$ is $\mathbb{N}_0$-graded with $\mathcal{A}(U)(n) = \mathcal{B}^n(U)\#H$ for all $n \geq 0$. In particular $\mathcal{A}(U)$ is an $\mathbb{N}_0$-filtered coalgebra with $\mathcal{F}_0(\mathcal{A}(U)) = H$. By Corollary 5.2.6, there is a non-zero element $x \in P(K)(n)$ with $\delta_K(x) \in H \otimes P(K)(n)$, hence $\delta_K(x) = \pi_H(x_{(1)}) \otimes x_{(2)}$. Thus

$$\Delta_{\mathcal{A}(V)}(x) = x \otimes 1 + \pi_H(x_{(1)}) \otimes x_{(2)},$$

since x is primitive in K, and $K\#\mathcal{A}(U) \cong \mathcal{A}(V)$. Hence

$$\Delta_{\mathcal{B}(V)}(x) = x \otimes 1 + \vartheta\pi_H(x_{(1)}) \otimes x_{(2)} = x \otimes 1 + 1 \otimes x.$$

We have found in Q^n a non-zero primitive element x of the braided Hopf algebra $\mathcal{B}(V)$. Since $Q = \bigoplus_{m\geq 1}(\operatorname{ad} U)^{m-1}(W)$ by Proposition 13.2.4(1), in the standard gradation x is a sum of homogeneous elements of degree ≥ 2. This is impossible since primitive elements in $\mathcal{B}(V)$ have degree one. $\square$

Starting with a direct sum decomposition of V, by Theorem 13.2.8 we obtain a smash product decomposition of braided Hopf algebras

$$\mathcal{B}(Q)\#\mathcal{B}(U) \cong \mathcal{B}(V), \quad Q = \operatorname{ad}\mathcal{A}(U)(W).$$

We need to prove a kind of converse. First we prove a converse of Corollary 13.2.5.

LEMMA 13.2.9. *Let $U \in {}^H_H\mathcal{YD}$, $Q \in {}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$, and $\mathcal{B}(Q)$ the Nichols algebra of Q with bosonization $\mathcal{B}(Q)\#\mathcal{A}(U)$. Assume that $Q = \mathcal{A}(U) \cdot \mathcal{F}^0 Q$. Then the algebra $\mathcal{B}(Q)\#\mathcal{B}(U) = (\mathcal{B}(Q)\#\mathcal{A}(U))^{\operatorname{co} H}$ is generated by $\mathcal{F}^0 Q$ and U.*

PROOF. By definition, the algebras $\mathcal{B}(U)$ and $\mathcal{B}(Q)$ are generated by U and $\mathcal{A}(U)\cdot\mathcal{F}^0Q=\mathcal{B}(U)\cdot\mathcal{F}^0Q$, respectively. To see that $\mathcal{B}(U)\cdot\mathcal{F}^0Q$ is contained in the subalgebra generated by $\mathcal{B}(U)$ and $\mathcal{F}^0Q$, let $b\in\mathcal{B}(U)$ and $w\in\mathcal{F}^0Q$. Then in the smash product algebra $\mathcal{B}(Q)\#\mathcal{A}(U)$, $bw=(b_{(1)}\cdot w)b_{(2)}$, hence

$$\begin{aligned} b\cdot w &= b_{(1)}w\mathcal{S}_{\mathcal{A}(U)}(b_{(2)})\\ &= b^{(1)}b^{(2)}{}_{(-1)}w\mathcal{S}_{\mathcal{A}(U)}(b^{(2)}{}_{(0)})\\ &= b^{(1)}b^{(2)}{}_{(-2)}w\mathcal{S}_H(b^{(2)}{}_{(-1)})\mathcal{S}_{\mathcal{B}(U)}(b^{(2)}{}_{(0)})\\ &= b^{(1)}(b^{(2)}{}_{(-1)}\cdot w)\mathcal{S}_{\mathcal{B}(U)}(b^{(2)}{}_{(0)}). \end{aligned}$$

This implies the claim since $\mathcal{F}^0Q$ is an H-submodule of Q. □

THEOREM 13.2.10. *Let $U\in{}^H_H\mathcal{YD}$, and let Q be a semisimple object in the category of $\mathbb{Z}$-graded objects in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$, where $\mathcal{A}(U)$ is $\mathbb{N}_0$-graded by the standard grading. Let $\mathcal{B}(Q)$ be the Nichols algebra of $Q\in{}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$, and $W=\mathcal{F}^0Q$. Then there is a unique isomorphism*

$$\mathcal{B}(Q)\#\mathcal{B}(U)\cong\mathcal{B}(W\oplus U)$$

of braided Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on $W\oplus U$.

PROOF. Note that $\mathcal{F}^0$ commutes with direct sums of comodules. Let

$$Q=\bigoplus_{i\in I}Q_i\ \text{ and } W=\bigoplus_{i\in I}W_i$$

be the decomposition of Q into irreducible $\mathbb{Z}$-graded objects Q_i in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ and of W into irreducible objects $W_i=\mathcal{F}^0Q_i$ in ${}^H_H\mathcal{YD}$ with $Q_i=\mathcal{A}(U)\cdot W_i$ by Proposition 13.1.2.

We change the $\mathbb{Z}$-grading of Q by shifting the degree in each Q_i. By Proposition 13.1.2, for all $i\in I$, $Q_i(n_i)=W_i$, where n_i is the smallest degree of a non-zero homogeneous component of Q_i. Let Q_i' be the Yetter-Drinfeld module Q_i with grading $Q_i'(n)=Q_i(n+n_i-1)$ for all $n\in\mathbb{Z}$. Since degree shifting preserves graded modules and graded comodules, Q_i' is again a graded Yetter-Drinfeld module.

Let $Q'=\bigoplus_{i\in I}Q_i'$. Then $Q'=Q$ as objects in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$, $Q'(n)=0$ for all $n\le 0$, and Q' is an $\mathbb{N}_0$-graded object in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ with

$$Q'(1)=\bigoplus_{i\in I}W_i=W.$$

By Corollary 7.1.15, the Nichols algebra $\mathcal{B}(Q')$ is an $\mathbb{N}_0$-graded Hopf algebra quotient of $T(Q')$ in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$, and

$$\mathcal{B}(Q')(0)=\Bbbk,\ \mathcal{B}(Q')(1)=W,$$

since $Q'(0)=0$. By Theorem 5.5.6(1), the bosonization $\mathcal{B}(Q')\#\mathcal{A}(U)$ is an $\mathbb{N}_0$-graded Hopf algebra with $(\mathcal{B}(Q')\#\mathcal{A}(U))(0)=H$, $(\mathcal{B}(Q')\#\mathcal{A}(U))(1)=(W\oplus U)H$. By Theorem 5.5.6(2), $\mathcal{B}(Q')\#\mathcal{B}(U)=(\mathcal{B}(Q')\#\mathcal{A}(U))^{\operatorname{co}H}$ is an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$, where the H-coinvariant elements are defined with respect to the projection onto degree 0, and

$$(\mathcal{B}(Q')\#\mathcal{B}(U))(0)=\Bbbk,\quad(\mathcal{B}(Q')\#\mathcal{B}(U))(1)=W\oplus U.$$

By Lemma 13.2.9, the algebra $\mathcal{B}(Q')\#\mathcal{B}(U)$ is generated by $W \oplus U$. Therefore $\mathcal{B}(Q')\#\mathcal{B}(U)$ is a pre-Nichols algebra of $W \oplus U$, and there is an $\mathbb{N}_0$-graded surjective morphism $\varphi : \mathcal{B}(Q')\#\mathcal{B}(U) \to \mathcal{B}(W \oplus U)$ of Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on $W \oplus U$.

Let $\Phi = \varphi\#\mathrm{id}_H$. Then the following diagram of $\mathbb{N}_0$-graded Hopf algebras is commutative, where $\pi' = \varepsilon \otimes \mathrm{id}_{\mathcal{A}(U)}$ and π is the projection map from the beginning of this section.

$$\begin{tikzcd}
\mathcal{B}(Q')\#\mathcal{A}(U) \arrow[r, "\Phi"] \arrow[d, "\pi'"] & \mathcal{A}(W \oplus U) \arrow[d, "\pi"] \\
\mathcal{A}(U) \arrow[r, "\mathrm{id}"'] & \mathcal{A}(U)
\end{tikzcd}$$

Let $K = \mathcal{A}(W \oplus U)^{\mathrm{co}\,\mathcal{A}(U)}$. By Theorem 13.2.8, $K = \mathcal{B}(\mathrm{ad}\,\mathcal{B}(U)(W))$. Hence Φ induces a map

$$\Phi_K : \mathcal{B}(Q') \to \mathcal{B}(\mathrm{ad}\,\mathcal{A}(U)(W))$$

in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ between the right coinvariant elements of π' and of π, respectively, where $\Phi_K|W = \mathrm{id}$. Recall from Corollary 4.3.3(1) that the given Hopf algebra structure of $\mathcal{B}(Q')$ in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$ coincides with the structure on the coinvariant elements of π'. By Corollary 13.2.5, $\mathrm{ad}\,\mathcal{A}(U)(W) = \bigoplus_{i\in I} \mathrm{ad}\,\mathcal{A}(U)(W_i)$ is a decomposition into irreducible objects $\mathrm{ad}\,\mathcal{A}(U)(W_i)$ in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$. The map Φ induces surjective $\mathbb{N}_0$-graded maps

$$\Phi_i : Q'_i = \mathcal{A}(U) \cdot W_i \to \mathrm{ad}\,\mathcal{A}(U)(W_i),\ i \in I,$$

in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$, since $\Phi|W_i = \mathrm{id}$. For all $i \in I$, Φ_i is bijective, since Q'_i is irreducible as an $\mathbb{N}_0$-graded object in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$. Hence Φ induces an isomorphism $Q' \to \mathrm{ad}\,\mathcal{A}(U)(W)$ in ${}^{\mathcal{A}(U)}_{\mathcal{A}(U)}\mathcal{YD}$. Thus Φ_K is bijective, and it follows from Corollary 4.3.3 that Φ is an isomorphism. □

13.3. The adjoint action in Nichols algebras

Let $U, W \in {}^H_H\mathcal{YD}$. Then $\mathcal{B}(U) \subseteq \mathcal{A}(U \oplus W)$ acts on $\mathcal{B}(U \oplus W)$ via the left adjoint action. In Theorem 13.3.1 we give a description of the $\mathcal{B}(U)$-submodule of $\mathcal{B}(U \oplus W)$ generated by W which does not dependend on the explicit structure of the Nichols algebra $\mathcal{B}(U \oplus W)$. This description can be used to compute reflections of Yetter-Drinfeld modules defined in Section 13.4.

Recall the definitions of $T_n, \varphi_n \in \mathbb{Z}\mathbb{B}_{n+1}$ for all $n \geq 1$ from Corollary 1.8.14. We also write T_n and φ_n for the image of T_n and φ_n, respectively, in $\mathrm{End}(U^{\otimes n} \otimes W)$ under the representation $\mathbb{Z}\mathbb{B}_{n+1} \to \mathrm{End}((U \oplus W)^{\otimes n+1})$ introduced in Section 1.7. Let $X_0^{U,W} = W$ and for all $n \geq 1$ let

$$X_n^{U,W} = (S_n \otimes \mathrm{id}_W)T_n(U^{\otimes n} \otimes W) \subseteq U^{\otimes n} \otimes W.$$

THEOREM 13.3.1. *Let $U, W \in {}^H_H\mathcal{YD}$ and let $X_n = X_n^{U,W}$ for all $n \in \mathbb{N}_0$.*

(1) *$X_n \subseteq U \otimes X_{n-1}$ and $X_n = \varphi_n(U \otimes X_{n-1})$ for all $n \geq 1$.*
(2) *For all $n \in \mathbb{N}_0$ there is an isomorphism $X_n \to (\mathrm{ad}\,U)^n(W)$ in ${}^H_H\mathcal{YD}$, where $(\mathrm{ad}\,U)^n(W) \subseteq \mathcal{B}(U \oplus W)$.*

Proof. (1) We proceed by induction on n. First, $X_1 = \varphi_1(U \otimes X_0)$ since $T_1 = \varphi_1$ by definition. Moreover,

$$X_1 = T_1(U \otimes W) = (\mathrm{id}_{U\otimes W} - c_{W,U}c_{U,W})(U \otimes W) \subseteq U \otimes W = U \otimes X_0.$$

Assume now that $n \geq 2$. Then

$$S_nT_n(U^{\otimes n} \otimes W) = \varphi_n(\mathrm{id}_U \otimes S_{n-1}T_{n-1})(U^{\otimes n} \otimes W) = \varphi_n(U \otimes X_{n-1})$$

by Corollary 1.8.14(4) and by definition of X_{n-1}. Moreover,

$$X_n = \varphi_n(U \otimes X_{n-1}) \subseteq \varphi_n(U \otimes U \otimes X_{n-2})$$

by induction hypothesis. Hence $X_n \subseteq U \otimes X_{n-1}$ by Corollary 1.8.14(3) and induction hypothesis.

(2) For any $u \in U$ and $x \in \mathcal{B}(U\oplus W)$, $\mathrm{ad}\, u(x) = ux-(u_{(-1)}\cdot x)u_{(0)}$ by definition. Hence for any $n \in \mathbb{N}_0$, $u_1, \dots, u_n \in U$ and $w \in W$, $\mathrm{ad}\, u_1 \cdots \mathrm{ad}\, u_n(w) \in \mathcal{B}(U \oplus W)$ is the multiplication of $\mathcal{B}(U \oplus W)$ composed with

$$(\mathrm{id} - c_n \cdots c_2c_1) \cdots (\mathrm{id} - c_nc_{n-1})(\mathrm{id} - c_n)(u_1 \otimes \cdots \otimes u_n \otimes w).$$

Since $S_{n+1} : \mathcal{B}(U\oplus W)(n+1) \to (U\oplus W)^{\otimes n+1}$ is injective, $(\mathrm{ad}\, U)^n(W)$ is isomorphic via S_{n+1} to

$$S_{n+1}(\mathrm{id} - c_n \cdots c_2c_1) \cdots (\mathrm{id} - c_nc_{n-1})(\mathrm{id} - c_n)(U^{\otimes n} \otimes W).$$

The latter equals $S_nT_n(U^{\otimes n} \otimes W) = X_n$ by Corollary 1.8.14(2). □

13.4. Reflections of Yetter-Drinfeld modules

We are going to define, under some assumptions, the reflection of a tuple of finite-dimensional Yetter-Drinfeld modules. In Theorem 13.4.9 we relate the Nichols algebra of a tuple to the one of its reflection.

Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, \dots, \theta\}$. Let H be a Hopf algebra with bijective antipode, and let $\mathcal{F}_\theta^H$ denote the category of families $M = (M_i)_{i\in\mathbb{I}}$, where $M_1, \dots, M_\theta \in {}^H_H\mathcal{YD}$ are finite-dimensional. A morphism $f : M \to N$ is a family $f = (f_i)_{i\in\mathbb{I}}$, where $f_i : M_i \to N_i$ is a morphism in ${}^H_H\mathcal{YD}$ for all $i \in \mathbb{I}$. The identity of M is $(\mathrm{id}_{M_i})_{i\in\mathbb{I}}$. The isomorphism class of any $M \in \mathcal{F}_\theta^H$ is denoted by $[M]$.

For any two $M, N \in \mathcal{F}_\theta^H$ which are isomorphic we write $M \cong N$.

As in Section 9.1, let $(\alpha_i)_{i\in\mathbb{I}}$ be the standard basis of $\mathbb{Z}^\mathbb{I}$. Then the Yetter-Drinfeld module $M_1 \oplus \cdots \oplus M_\theta \in {}^H_H\mathcal{YD}$ is $\mathbb{Z}^\theta$-graded with homogeneous component M_i of degree α_i for all $i \in \mathbb{I}$.

For all $M \in \mathcal{F}_\theta^H$ let $\mathcal{B}(M)$ denote the Nichols algebra $\mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$.

Corollary 13.4.1. *Let $M, N \in \mathcal{F}_\theta^H$. If $M \cong N$ then $\mathcal{B}(M)$ and $\mathcal{B}(N)$ are isomorphic as $\mathbb{Z}^\theta$-graded algebras and coalgebras in ${}^H_H\mathcal{YD}$.*

Proof. Apply Corollary 7.1.15(2) with the trivial $\mathbb{Z}^\theta$-grading of H. □

Definition 13.4.2. Let $M \in \mathcal{F}_\theta^H$ and $i \in \mathbb{I}$. We say that M is ***i*-finite** if for all $j \in \mathbb{I} \setminus \{i\}$ there exists $m \in \mathbb{N}$ such that $(\mathrm{ad}\, M_i)^m(M_j) = 0$ in $\mathcal{B}(M)$.

Assume that M is i-finite. For all $j \in \mathbb{I} \setminus \{i\}$ let

$$a_{ij}^M = -\max\{m \in \mathbb{N}_0 \mid (\mathrm{ad}\, M_i)^m(M_j) \neq 0\},$$

and let $a_{ii}^M = 2$. These so called **Cartan integers** allow us to define the reflection

$$s_i^M \in \mathrm{Aut}(\mathbb{Z}^\theta), \quad s_i^M(\alpha_j) = \alpha_j - a_{ij}^M\alpha_i$$

for all $j \in \mathbb{I}$. The family $R_i(M) = (R_i(M)_j)_{j\in\mathbb{I}} \in \mathcal{F}_\theta^H$, where

$$R_i(M)_j = \begin{cases} M_i^* & \text{if } j = i, \\ (\operatorname{ad} M_i)^{-a_{ij}^M}(M_j) & \text{if } j \neq i, \end{cases}$$

is called the ***i*-th reflection of** M.

COROLLARY 13.4.3. *Let $M \in \mathcal{F}_\theta^H$ and $i \in \mathbb{I}$. Assume that M is i-finite and that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}\setminus\{i\}$. Then the components $R_i(M)_j$ with $j \in \mathbb{I}\setminus\{i\}$ are irreducible objects in ${}^H_H\mathcal{YD}$.*

PROOF. Let $j \in \mathbb{I}\setminus\{i\}$ and let $a_{ij} = a_{ij}^M$. By assumption,

$$\operatorname{ad}\mathcal{B}(M_i)(M_j) = M_j \oplus (\operatorname{ad} M_i)(M_j) \oplus \cdots \oplus (\operatorname{ad} M_i)^{-a_{ij}}(M_j),$$

and $(\operatorname{ad} M_i)^{-a_{ij}}(M_j) \neq 0$. Hence $(\operatorname{ad}\mathcal{B}(M_i)(M_j))^{\max} = (\operatorname{ad} M_i)^{-a_{ij}}(M_j)$ is irreducible in ${}^H_H\mathcal{YD}$ by Theorem 13.2.7. $\square$

LEMMA 13.4.4. *Let $M \in \mathcal{F}_\theta^H$. Assume that M is i-finite for all $i \in \mathbb{I}$. Then $A^M = (a_{ij}^M)_{i,j\in\mathbb{I}}$ is a Cartan matrix.*

PROOF. Let $i, j \in \mathbb{I}$ with $i \neq j$ and let $a_{ij} = a_{ij}^M$. By Theorem 13.3.1(2) with $n = 1$, $a_{ij} = 0$ if and only if

$$0 = X_1^{M_i,M_j} = T_1(M_i \otimes M_j) = (\mathrm{id} - c^2)|M_i \otimes M_j.$$

Thus $a_{ij} = 0$ if and only if $c_{M_i,M_j} = (c_{M_j,M_i})^{-1}$, which in turn is equivalent to $a_{ji}^M = 0$. The remaining properties of A^M are clearly fulfilled. $\square$

Reflections and Cartan matrices of objects in $\mathcal{F}_\theta^H$ are compatible with isomorphisms.

LEMMA 13.4.5. *Let $M, N \in \mathcal{F}_\theta^H$ such that $M \cong N$. Let $i \in \mathbb{I}$. If M is i-finite, then N is i-finite, $R_i(M) \cong R_i(N)$, and $a_{ij}^N = a_{ij}^M$ for all $j \in \mathbb{I}$.*

PROOF. The claim follows from Corollary 7.1.15(2). $\square$

DEFINITION 13.4.6. Let $M \in \mathcal{F}_\theta^H$ and $i \in \mathbb{I}$. Let $\pi_{M_i} : \mathcal{B}(M) \to \mathcal{B}(M_i)$ be the Hopf algebra projection in ${}^H_H\mathcal{YD}$ induced by the i-th projection of the direct sum $\bigoplus_{j\in\mathbb{I}} M_j$. Let

$$K_i^{\mathcal{B}(M)} = \mathcal{B}(M)^{\operatorname{co}\mathcal{B}(M_i)}, \quad L_i^{\mathcal{B}(M)} = {}^{\operatorname{co}\mathcal{B}(M_i)}\mathcal{B}(M)$$

be the set of right and left coinvariant elements of $\mathcal{B}(M)$ with respect to π_{M_i}, respectively.

REMARK 13.4.7. In Definition 13.4.6, $\mathcal{B}(M)$ and $\mathcal{B}(M_i)$ are Hopf algebras in ${}^H_H\mathcal{YD}$. Moreover, $(\mathcal{B}(M), \pi_{M_i}, \iota_{M_i})$ is a Hopf algebra triple over $\mathcal{B}(M_i)$ in ${}^H_H\mathcal{YD}$, where $\iota_{M_i} : \mathcal{B}(M_i) \to \mathcal{B}(M)$ is the Hopf algebra map induced by the canonical embedding $M_i \to \bigoplus_{j\in\mathbb{I}} M_j$. Thus $K_i^{\mathcal{B}(M)}$ is a Hopf algebra in ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}({}^H_H\mathcal{YD})$ by Theorem 3.10.4, and $K_i^{\mathcal{B}(M)} = F(K_i^{\mathcal{B}(M)})$ is a Hopf algebra in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$ by Theorem 3.8.7.

On the other hand, $\mathcal{B}(M)\#H$ and $\mathcal{B}(M_i)\#H$ are Hopf algebras and

$$(\mathcal{B}(M)\#H, \pi_{M_i}\#\mathrm{id}_H, \iota_{M_i}\#\mathrm{id}_H)$$

is a Hopf algebra triple over $\mathcal{B}(M_i)\#H$. Let $K_i^{\mathcal{B}(M)\#H} = (\mathcal{B}(M)\#H)^{\mathrm{co}\,\mathcal{B}(M_i)\#H}$. Then $K_i^{\mathcal{B}(M)\#H} \in {}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$ by Theorem 3.10.4. Moreover, by Proposition 4.3.9, the embedding $\iota_{\mathcal{B}(M)} = \mathrm{id}_{\mathcal{B}(M)} \otimes \eta : \mathcal{B}(M) \to \mathcal{B}(M)\#H$ induces an isomorphism $K_i^{\mathcal{B}(M)} \to K_i^{\mathcal{B}(M)\#H}$ of Hopf algebras in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$.

Recall the notation of rational Yetter-Drinfeld modules from Definition 12.2.3 and Section 12.3.

LEMMA 13.4.8. *Let $M \in \mathcal{F}_\theta^H$ and $i \in \mathbb{I}$. Then the following are equivalent.*

(1) *M is i-finite.*
(2) $K_i^{\mathcal{B}(M)} \in {}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$.

PROOF. Assume (1). By Remark 13.4.7, $K_i^{\mathcal{B}(M)}$ and $K_i^{\mathcal{B}(M)\#H}$ are isomorphic as algebras in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$. Thus $K_i^{\mathcal{B}(M)}$ is generated as an algebra by $\sum_{j\neq i} \mathrm{ad}\,(\mathcal{B}(M_i)\#H)(M_j)$ because of Theorem 2.6.23 and Lemma 2.6.25 applied to the right $\mathcal{B}(M_i)\#H$-comodule algebra $\mathcal{B}(M)\#H$, as the latter is generated as an algebra by $\bigoplus_{j\neq i} M_j$ and $\mathcal{B}(M_i)\#H$. Then (2) follows from Lemma 12.2.4(4).

Conversely, (2) implies (1) since $M_j \in K_i^{\mathcal{B}(M)}$ for all $j \in \mathbb{I} \setminus \{i\}$. □

The next theorem gives a natural explanation of reflections of tuples of Yetter-Drinfeld modules. All the deeper results on $R_i(M)$ depend on this description. Recall the notation (Ω_V, ω_V) in Definition 12.3.7 for finite-dimensional $V \in {}^H_H\mathcal{YD}$.

THEOREM 13.4.9. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and $(\Omega_{M_i}, \omega_{M_i}) = (\Omega, \omega)$. Assume that M is i-finite and that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Then there is an isomorphism*

$$\Theta : \mathcal{B}(R_i(M)) \xrightarrow{\cong} \Omega(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*)$$

of Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on the components of $R_i(M)$.

PROOF. Let $W = \bigoplus_{j\neq i} M_j$, and $Q = \mathrm{ad}\,\mathcal{B}(M_i)(W)$. Lemma 13.4.8 implies that $Q \in {}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$. For all $j \in \mathbb{I}$, $j \neq i$, let

$$Q_j = \mathrm{ad}\,\mathcal{B}(M_i)(M_j) = M_j \oplus \mathrm{ad}\,M_i(M_j) \oplus \cdots \oplus (\mathrm{ad}\,M_i)^{-a_{ij}^M}(M_j).$$

Then $Q = \bigoplus_{j\neq i} Q_j$, and for all $j \neq i$, Q_j is irreducible in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$ by Corollary 13.2.5. By Theorem 13.2.8, $K_i^{\mathcal{B}(M)} \cong \mathcal{B}(Q)$, and by Corollary 12.3.9, $\Omega(K_i^{\mathcal{B}(M)}) \cong \mathcal{B}(\Omega(Q))$, hence

$$\Omega(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*) \cong \mathcal{B}(\Omega(Q))\#\mathcal{B}(M_i^*).$$

Since Q is a $\mathbb{Z}$-graded semisimple object in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$ with $Q(n) = 0$ for all $n < 0$, it follows from Theorem 12.3.2 and Corollary 12.3.6(2) that $\Omega(Q)$ is a $\mathbb{Z}$-graded semisimple object in ${}^{\mathcal{B}(M_i^*)\#H}_{\mathcal{B}(M_i^*)\#H}\mathcal{YD}_{\mathrm{rat}}$. Thus Theorem 13.2.10 applies, and

$$\mathcal{B}(\Omega(Q))\#\mathcal{B}(M_i^*) \cong \mathcal{B}(\mathcal{F}^0\Omega(Q) \oplus M_i^*) = \mathcal{B}(R_i(M)),$$

since

$$\mathcal{F}^0\Omega(Q) = \mathcal{F}_0 Q = \bigoplus_{j\neq i} (\mathrm{ad}\,M_i)^{-a_{ij}^M}(M_j),$$

where the first equality follows from Corollary 12.3.6(1), and the second from Proposition 13.1.3(3b). ☐

COROLLARY 13.4.10. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and assume that M is i-finite, and M_j is irreducible in $\mathcal{C} = {}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Let $K_i = K_i^{\mathcal{B}(M)}$, and $D(K_i^{\mathrm{cop}})$ the Hopf algebra in ${}^{\mathcal{B}(M_i^*)^{\mathrm{cop}}}_{\mathcal{B}(M_i^*)^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$, where D is defined in Theorem 12.3.1. Let $D(K_i^{\mathrm{cop}})\#\mathcal{B}(M_i^*)^{\mathrm{cop}}$ be the bosonization which is a Hopf algebra in $\overline{\mathcal{C}}$. Then there is an isomorphism*

$$\widetilde{\Theta} : \mathcal{B}(R_i(M)) \xrightarrow{\cong} \big(D(K_i^{\mathrm{cop}})\#\mathcal{B}(M_i^*)^{\mathrm{cop}}\big)^{\mathrm{cop}}$$

of Hopf algebras in ${}^H_H\mathcal{YD}$. Let $K_i\#\mathcal{B}(M_i^*)$ be the vector space $K_i \otimes \mathcal{B}(M_i^*)$ with algebra structure given by

$$(x\#f)(y\#g) = x\langle f^{(2)}, \pi_{M_i}(y^{(1)})\rangle(y^{(2)})_{(0)}\#\big(\mathcal{S}^{-1}((y^{(2)})_{(-1)}) \cdot f^{(1)}\big)g$$

for all $x, y \in K_i$, $f, g \in \mathcal{B}(M_i^*)$. Then $\widetilde{\Theta}$ is the algebra map

$$\widetilde{\Theta} : \mathcal{B}(R_i(M)) \xrightarrow{\cong} K_i\#\mathcal{B}(M_i^*)$$

which is the identity on the components of $R_i(M)$.

(Here, xy and fg denote the product of x, y in K_i, and of f, g in $\mathcal{B}(M_i^*)$, respectively, and $\Delta_{\mathcal{B}(M)}(x) = x^{(1)} \otimes x^{(2)}$, $\Delta_{\mathcal{B}(M_i^*)}(f) = f^{(1)} \otimes f^{(2)}$.)

PROOF. Let $\widetilde{L_i}$ be the space of right coinvariant elments of the projection $(\Omega(K_i)\#\mathcal{B}(M_i^*))^{\mathrm{cop}} \to \mathcal{B}(M_i^*)^{\mathrm{cop}}$, and $T_i : \widetilde{L_i} \to D(K_i^{\mathrm{cop}})$, $x \mapsto \mathcal{S}_{K_i}^{-1}\mathcal{S}_{\widetilde{S}}$, the Hopf algebra isomorphism in Theorem 12.3.3, where $\widetilde{S} = \Omega(K_i)\#\mathcal{B}(M_i^*)$. Let $\Phi : \widetilde{L_i}\#\mathcal{B}(M_i^*)^{\mathrm{cop}} \to \big(\Omega(K_i)\#\mathcal{B}(M_i^*)\big)^{\mathrm{cop}}$ be the Hopf algebra isomorphism of Corollary 4.3.1. We define $\widetilde{\Theta}$ as the composition

$$\begin{aligned}\mathcal{B}(R_i(M))^{\mathrm{cop}} &\xrightarrow{\Theta} \big(\Omega(K_i)\#\mathcal{B}(M_i^*)\big)^{\mathrm{cop}} \xrightarrow{\Phi^{-1}} \widetilde{L_i}\#\mathcal{B}(M_i^*)^{\mathrm{cop}}\\ &= \widetilde{L_i}\#\mathcal{B}(M_i^*)^{\mathrm{cop}} \xrightarrow{T_i \otimes \mathrm{id}} D(K_i^{\mathrm{cop}})\#\mathcal{B}(M_i^*)^{\mathrm{cop}}.\end{aligned}$$

Then $\widetilde{\Theta}$ is an isomorphism of Hopf algebras in $\overline{{}^H_H\mathcal{YD}}$.

Let $j \in \mathbb{I} \setminus \{i\}$, and $x \in M_j' = (\mathrm{ad}_{\mathcal{B}(M)} M_i)^{1-a_{ij}^M}(M_j)$. Then $x \otimes 1$ is a primitive element in the Hopf algebra $\widetilde{S} = \Omega(K_i)\#\mathcal{B}(M_i^*)$, since Θ is a Hopf algebra isomorphism by Theorem 13.4.9. Hence $-x = \mathcal{S}_{\widetilde{S}}^{-1}(x) \in \widetilde{L_i}$, and

$$T_i(x) = \mathcal{S}_{K_i}^{-1}\mathcal{S}_{\widetilde{S}}(x) = -\mathcal{S}_{K_i}^{-1}(x) = x,$$

since x is a primitive element of K_i. We have shown that $\widetilde{\Theta}(x \otimes 1) = x \otimes 1$. This proves that $\widetilde{\Theta}$ is the identity on the components of $R_i(M)$, since by definition, $\widetilde{\Theta}(1 \otimes f) = 1 \otimes f$ for all $f \in \mathcal{B}(M_i^*)$.

We now describe the algebra structure of $D(K_i^{\mathrm{cop}})\#\mathcal{B}(M_i^*)^{\mathrm{cop}}$ with underlying vector space $K_i \otimes \mathcal{B}(M_i^*)$. We write $\pi = \pi_{M_i} : \mathcal{B}(M) \to \mathcal{B}(M_i)$. By Remark 12.3.4 (applied to the canonical form $\mathcal{B}(M_i^*) \otimes \mathcal{B}(M_i) \xrightarrow{\langle\,,\,\rangle} \Bbbk$), $D(K_i^{\mathrm{cop}})$ is an algebra in ${}_{\mathcal{B}(M_i^*)^{\mathrm{cop}}}\overline{\mathcal{C}}$, $\mathcal{C} = {}^H_H\mathcal{YD}$, where $D(K_i^{\mathrm{cop}}) = K_i$ as an algebra with action

$$\mathcal{B}(M_i^*)^{\mathrm{cop}} \otimes K_i \to K_i, \quad f \otimes x \mapsto \langle f, \pi(x^{(1)})\rangle x^{(2)}.$$

Let $f \in \mathcal{B}(M_i^*)$, $x \in K_i$. We compute the commutation law for $(1 \otimes f)(x \otimes 1)$ in the smash product algebra $D(K_i^{\mathrm{cop}}) \# \mathcal{B}(M_i^*)^{\mathrm{cop}}$. By definition,

$$\Delta_{\mathcal{B}(M_i^*)^{\mathrm{cop}}}(f) = (f^{(2)})_{(0)} \otimes \mathcal{S}^{-1}((f^{(2)})_{(-1)}) \cdot f^{(1)}.$$

Hence in $K_i \# \mathcal{B}(M_i^*)^{\mathrm{cop}}$,

$$\begin{aligned}
&(1 \otimes f)(x \otimes 1) = \\
&= \langle (f^{(2)})_{(0)}, \pi((x_{(0)})^{(1)}) \rangle (x_{(0)})^{(2)} \otimes \big(\mathcal{S}^{-1}(x_{(-1)}) \mathcal{S}^{-1}((f^{(2)})_{(-1)})\big) \cdot f^{(1)} \\
&= (x^{(2)})_{(0)} \otimes \big(\mathcal{S}^{-1}((x^{(2)})_{(-1)}) \mathcal{S}^{-1}(\pi(x^{(1)})_{(-1)}) \langle (f^{(2)})_{(0)}, \pi(x^{(1)})_{(0)} \rangle \\
&\quad \mathcal{S}^{-1}((f^{(2)})_{(-1)})\big) \cdot f^{(1)} \\
&= \langle f^{(2)}, \pi(x^{(1)}) \rangle (x^{(2)})_{(0)} \otimes \mathcal{S}^{-1}((x^{(2)})_{(-1)}) \cdot f^{(1)},
\end{aligned}$$

where the third equality follows from

$$x_{(-1)} \otimes \pi((x_{(0)})^{(1)}) \otimes (x_{(0)})^{(2)} = \pi(x^{(1)})_{(-1)} (x^{(2)})_{(-1)} \otimes \pi(x^{(1)})_{(0)} \otimes (x^{(2)})_{(0)},$$

and the last equality from the rule (12.1.3). □

13.5. Nichols systems and their reflections

As in the previous section, let $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, \dots, \theta\}$.

We are going to introduce and to discuss pre-Nichols systems of M and Nichols systems of (M, i), where $M \in \mathcal{F}_\theta^H$ and $i \in \mathbb{I}$. These will be used to develop criteria to decide whether a given pre-Nichols algebra is in fact a Nichols algebra. As an application, we will prove in Chapter 16 that some (small) quantum groups are Nichols algebras.

Definition 13.5.1. Let S be a Hopf algebra in ${}^H_H\mathcal{YD}$, $N_1, \dots, N_\theta$ be finite-dimensional subobjects of S in ${}^H_H\mathcal{YD}$, and $N = (N_1, \dots, N_\theta)$. Let

$$f = (f_j)_{j \in \mathbb{I}} : N \to M$$

be an isomorphism of tuples in $\mathcal{F}_\theta^H$ for some $M \in \mathcal{F}_\theta^H$. The triple $\mathcal{N} = \mathcal{N}(S, N, f)$ is called a **pre-Nichols system of** M if

- (Sys1) S is generated as an algebra by $N_1, \dots, N_\theta$, and
- (Sys2) S is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ with $\deg(N_j) = \alpha_j$ for all integers $1 \le j \le \theta$.

Remark 13.5.2. For $\theta = 1$ a pre-Nichols system of M is nothing but a pre-Nichols algebra of M_1, see Definition 7.1.6.

Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of a tuple $M \in \mathcal{F}_\theta^H$. Note that $S(0) = \Bbbk 1$ and $\sum_{j=1}^\theta N_j = \bigoplus_{j=1}^\theta N_j$ by (Sys1) and (Sys2). Hence the antipode of S is bijective by Proposition 6.4.2. We will use the notation

$$\mathcal{N}_j = N_j, \quad 1 \le j \le \theta.$$

Let

$$p^{\mathcal{N}} : S \to \mathcal{B}(M)$$

be the surjective map of $\mathbb{N}_0^\theta$-graded Hopf algebras in ${}^H_H\mathcal{YD}$ which is defined by $f_j : N_j \xrightarrow{\cong} M_j \subseteq \mathcal{B}(M)$ on N_j, $j \in \mathbb{I}$. It is called the **canonical map of** $\mathcal{N}$.

We note that the Hopf algebra map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ exists for a pre-Nichols system of M by the definition of the Nichols algebra $\mathcal{B}(M)$.

A pre-Nichols system gives rise to many other pre-Nichols systems by changing the grading.

EXAMPLE 13.5.3. Let $M \in \mathcal{F}_\theta^H$ and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $\theta' \in \mathbb{N}$, $\mathbb{I}' = \{1, \dots, \theta'\}$, and $h : \mathbb{I} \to \mathbb{I}'$ be any map. Let $h_0 : \mathbb{Z}^\theta \to \mathbb{Z}^{\theta'}$ be the homomorphism with $h_0(\alpha_i) = \alpha_{h(i)}$ for all $i \in \mathbb{I}$. For any $P \in \mathcal{F}_\theta^H$ let $h_1(P) = (P'_1, \dots, P'_{\theta'}) \in \mathcal{F}_{\theta'}^H$, where

$$P'_j = \bigoplus_{i \in \mathbb{I}, h(i) = j} P_i$$

for any $j \in \mathbb{I}'$. Then

$$h_*(\mathcal{N}) = \mathcal{N}(S, h_1(N), h_2(f)),$$

where

$$h_2(f)_j = \bigoplus_{i \in \mathbb{I}, h(i) = j} f_i : h_1(N)_j \to h_1(M)_j$$

for all $j \in \mathbb{I}'$, is a pre-Nichols system of $h_1(M)$. Indeed, the Yetter-Drinfeld module $\sum_{j=1}^{\theta'} h_1(N)_j = \sum_{i=1}^{\theta} N_i$ generates S. Moreover, S is $\mathbb{N}_0^{\theta'}$-graded Hopf algebra in ${}_H^H\mathcal{YD}$, where for any $\beta \in \mathbb{N}_0^{\theta'}$ the homogeneous component of S of degree β is

$$\bigoplus_{\alpha \in \mathbb{N}_0^\theta, h_0(\alpha) = \beta} S(\alpha).$$

In the special case, where $\theta' = 1$, this construction results in the pre-Nichols algebra S of $\bigoplus_{i=1}^\theta M_i$.

Now we define Nichols systems of (M, i).

DEFINITION 13.5.4. Let $M \in \mathcal{F}_\theta^H$, and $\mathcal{N} = \mathcal{N}(S, N, f)$ a pre-Nichols system of M. Let $i \in \mathbb{I}$. Then $\mathcal{N}$ is called a **Nichols system of** (M, i), if $p^{\mathcal{N}}$ defines bijective maps

(Sys3) $\Bbbk[N_i] \cong \mathcal{B}(M_i)$, and
(Sys4) $(\mathrm{ad}_S N_i)^n(N_j) \cong (\mathrm{ad}_{\mathcal{B}(M)} M_i)^n(M_j)$ for all $j \in \mathbb{I} \setminus \{i\}$ and $n \geq 0$.

(Here, ad_S and $\mathrm{ad}_{\mathcal{B}(M)}$ denote the adjoint actions of S and $\mathcal{B}(M)$, respectively.)

Note that $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$ is a Nichols system of (M, i) with canonical map $p^{\mathcal{N}_0} = \mathrm{id}_{\mathcal{B}(M)}$.

In the following three lemmas we discuss properties of pre-Nichols systems related to Axiom (Sys4).

LEMMA 13.5.5. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and $\mathcal{N} = \mathcal{N}(S, N, f)$ a pre-Nichols system of M. Assume that $p^{\mathcal{N}} | \Bbbk[N_i] : \Bbbk[N_i] \to \mathcal{B}(M_i)$ is bijective. Then for any $j \in \mathbb{I} \setminus \{i\}$ the following are equivalent.*

(1) *The restriction of $p^{\mathcal{N}}$ to $\oplus_{n \geq 1} (\mathrm{ad}_S N_i)^n(N_j)$ is bijective.*
(2) *There is no non-zero primitive element of S in $\oplus_{n \geq 1} (\mathrm{ad}_S N_i)^n(N_j)$.*

Moreover, if M_j is irreducible, then these properties are equivalent to

(3) *The Yetter-Drinfeld module $\mathrm{ad}_S \Bbbk[N_i](N_j) \in {}_{\Bbbk[N_i]\#H}^{\Bbbk[N_i]\#H}\mathcal{YD}$ is irreducible.*

PROOF. Assume that $\theta \geq 2$ and let $j \in \mathbb{I}$ with $j \neq i$.

Clearly, (1) implies (2) by the definition of $p^{\mathcal{N}}$. Now we prove that (2) implies (1). Let $m \in \mathbb{N}$ and $x \in (\mathrm{ad}_S N_i)^m(N_j)$. Assume that $p^{\mathcal{N}}(x) = 0$, and the restriction of $p^{\mathcal{N}}$ to $\oplus_{n=1}^{m-1}(\mathrm{ad}_S N_i)^n(N_j)$ is bijective. Then

$$\Delta(x) - x \otimes 1 - 1 \otimes x \in \ker(p^{\mathcal{N}} \otimes p^{\mathcal{N}}) \cap \big(\Bbbk[N_i] \otimes \oplus_{n=0}^{m-1}(\mathrm{ad}_S N_i)^n(N_j)\big).$$

Since $p^{\mathcal{N}}|\Bbbk[N_i]$ and $p^{\mathcal{N}}|N_j$ are bijective, we obtain that x is primitive. Then $x = 0$ by (2).

Assume now that M_j is irreducible. Then $\mathrm{ad}_{\mathcal{B}(M)}(\mathcal{B}(M_i)\#H)(M_j)$ is irreducible by Corollary 13.2.5(2). In particular, (1) implies (3). Finally, the kernel of the restriction of $p^{\mathcal{N}}$ to $\oplus_{n\geq 1}(\mathrm{ad}_S N_i)^n(N_j)$ is a Yetter-Drinfeld submodule of $\mathrm{ad}_S\Bbbk[N_i](N_j)$ and hence (3) implies (1). □

Lemma 13.5.6. *Let $\mathcal{N}(S,N,f)$ be a pre-Nichols system of M, $m \in \mathbb{N}_0$, and $i,j \in \mathbb{I}$. Assume that $i \neq j$ and $\dim N_i = \dim N_j = 1$. Let $x_i \in N_i$ and $x_j \in N_j$ be non-zero elements. Then $(\mathrm{ad}_S x_i)^m(x_j) = 0$ if and only if $\dim S(\alpha_j + m\alpha_i) < m+1$.*

Proof. The assumptions imply that $S(\alpha_j + m\alpha_i)$ is the linear span of the $m+1$ monomials $x_i^k x_j x_i^{m-k}$ with $0 \leq k \leq m$. If $x_i^m = 0$ then $\dim S(\alpha_j + m\alpha_i) < m+1$. Moreover, $(\mathrm{ad}_S x_i)^m(x_j) = (\mathrm{ad}_S x_i^m)(x_j) = 0$ by Lemma 4.3.11. Therefore we may suppose that $x_i^m \neq 0$.

If $(\mathrm{ad}_S x_i)^m(x_j) \neq 0$, then $0 \neq (\mathrm{ad}_S x_i)^k(x_j) \in K_i^{\mathcal{N}}$ for any $0 \leq k \leq m$. In this case the isomorphism $K_i^{\mathcal{N}}\#\Bbbk[x_i] \cong S$ in Theorem 3.9.2(6) implies that the elements

$$(\mathrm{ad}_S x_i)^k(x_j)x_i^{m-k}, \quad 0 \leq k \leq m, \tag{13.5.1}$$

are linearly independent in S. Therefore $\dim S(\alpha_j + m\alpha_i) = m+1$.

Conversely, if $(\mathrm{ad}_S x_i)^m(x_j) = 0$, then $\dim S(\alpha_j + m\alpha_i) < m+1$ since the monomials $x_i^k x_j x_i^{m-k}$ with $0 \leq k \leq m$ are linearly dependent. □

Lemma 13.5.7. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and $\mathcal{N} = \mathcal{N}(S,N,f)$ a pre-Nichols system of M satisfying (Sys4). Let $x \in \Bbbk[N_i] \cap S(n)$ be a primitive element of degree $n \geq 2$. Then*

$$\mathrm{ad}_S x(y) = 0, \quad (\mathrm{id} - c^2)(x \otimes y) = 0$$

for any $y \in N_j$ with $j \neq i$.

Proof. If $y \in N_j$ with $j \in \mathbb{I}$, $j \neq i$, then $\mathrm{ad}_S x(y) \in \mathrm{ad}_S\Bbbk[N_i](N_j) \cap S(n+1)$ and $\pi(x) = 0$, where $\pi : S \to S(1)$ is the homogeneous projection. Then

$$\Delta(\mathrm{ad}_S x(y)) = \mathrm{ad}_S x(y) \otimes 1 + 1 \otimes \mathrm{ad}_S x(y) + (\mathrm{id} - c^2)(x \otimes y) \tag{13.5.2}$$

by Proposition 6.2.17(2), and hence $(\pi \otimes \mathrm{id})\Delta(\mathrm{ad}_S x(y)) = 0$. Since $\mathcal{B}(M)$ is strictly graded, it follows that $p^{\mathcal{N}}(\mathrm{ad}_S x(y)) = 0$. Thus (Sys4) implies that $\mathrm{ad}_S x(y) = 0$. Then the claim follows from Equation (13.5.2). □

Lemma 13.5.8. *Let $M \in \mathcal{F}_\theta^H$, $V = M_1 \oplus \cdots \oplus M_\theta$, and let $R = \bigoplus_{n\geq 0} R(n)$ be an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Assume that R is a pre-Nichols algebra of V with surjective map $\pi : R \to \mathcal{B}(V)$ of $\mathbb{N}_0$-graded Hopf algebras. Let $\mathrm{gr}\, R$ be the $\mathbb{N}_0^\theta$-graded Hopf algebra defined in Proposition 5.2.21. Then $\mathcal{N} = \mathcal{N}(\mathrm{gr}\, R, N, f)$ is a pre-Nichols system of M, where for all $i \in \mathbb{I}$, $N_i = \mathrm{gr}\,(R)(\alpha_i)$ is isomorphic to M_i via f_i. Assume that $p^{\mathcal{N}} : \mathrm{gr}\, R \to \mathcal{B}(M)$ is an isomorphism. Then $\pi : R \to \mathcal{B}(M)$ is an isomorphism.*

PROOF. Since R is generated by V, it is clear that $\mathcal{N}$ is a pre-Nichols system of M. We prove the last claim of the lemma.

(1) For all $\gamma \in \mathbb{N}_0^\theta$ let

$$R_\gamma = \sum_{\substack{1\le i_1,\dots,i_n\le\theta \\ \alpha_{i_1}+\cdots+\alpha_{i_n}=\gamma}} M_{i_1}\cdots M_{i_n} \subseteq R.$$

Then by definition of $\operatorname{gr} R$ in Proposition 5.2.21, for all $0 \neq \alpha \in \mathbb{N}_0^\theta$,

$$\begin{aligned} F_\alpha(R) &= \sum_{\gamma\le\alpha} R_\gamma, \quad F_{<\alpha}(R) = F_\beta, \quad \text{where } \beta = \max\{\gamma \mid \gamma < \alpha\}, \\ \operatorname{gr}(R)(\alpha) &= F_\alpha(R)/F_{<\alpha}(R). \end{aligned}$$

For any $x \in R \setminus \{0\}$ we can write

$$x = x_{\beta_1} + \cdots + x_{\beta_t}, \quad x_{\beta_l} \in R_{\beta_l} \setminus \{0\}, \quad \beta_l \in \mathbb{N}_0^\theta, \quad 1 \le l \le t,$$

such that $\beta_k < \beta_l$ whenever $k < l$. Then $\pi(x_{\beta_l}) \in \mathcal{B}(M)(\beta_l)$ for all l. Moreover, $x \in F_{\beta_t}(R)$ and $p^{\mathcal{N}}(x + F_{<\beta_t}) = \pi(x_{\beta_t})$.

(2) Let $x \in R$ with $\pi(x) = 0$. Assume that $x \neq 0$. Then there exists a minimal $\alpha \in \mathbb{N}_0^\theta$ with respect to $<$ such that $x \in F_\alpha(R)$. Note that $\alpha \neq 0$. Since $p^{\mathcal{N}} : \operatorname{gr} R \to \mathcal{B}(M)$ is an $\mathbb{N}_0^\theta$-graded isomorphism, the residue class of x in $\operatorname{gr}(R)(\alpha) = F_\alpha(R)/F_{<\alpha}(R)$ is zero. Thus $x \in F_\beta(R)$, where $\beta = \max\{\gamma \mid \gamma < \alpha\}$. This is a contradiction to the minimality of α. □

DEFINITION 13.5.9. Let $i \in \mathbb{I}$, $M \in \mathcal{F}_\theta^H$, and let $\pi_i : \mathcal{B}(M) \to \mathcal{B}(M_i)$ be the Hopf algebra projection defined by the projection $\bigoplus_{j=1}^\theta M_j \to M_i$ in ${}^H_H\mathcal{YD}$.

Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. We write

$$\tilde{\pi}_i^{\mathcal{N}} : S \to \Bbbk[N_i], \quad \tilde{\gamma}_i^{\mathcal{N}} : \Bbbk[N_i] \to S$$

for the canonical $\mathbb{N}_0^\theta$-graded maps which are the identity on N_i. Moreover, let

$$K_i^{\mathcal{N}} = S^{\operatorname{co}\Bbbk[N_i]}, \quad L_i^{\mathcal{N}} = {}^{\operatorname{co}\Bbbk[N_i]}S,$$

where the left and right coinvariant elements are defined with respect to $\tilde{\pi}_i^{\mathcal{N}}$.

If $p^{\mathcal{N}}$ induces an isomorphism $p^{\mathcal{N}}|\Bbbk[N_i] : \Bbbk[N_i] \to \mathcal{B}(M_i)$, we also define the maps

$$\pi_i^{\mathcal{N}} = p^{\mathcal{N}}\tilde{\pi}_i^{\mathcal{N}} : S \to \mathcal{B}(M_i), \quad \gamma_i^{\mathcal{N}} = \tilde{\gamma}_i^{\mathcal{N}}\big(p^{\mathcal{N}}|\Bbbk[N_i]\big)^{-1} : \mathcal{B}(M_i) \to S.$$

REMARK 13.5.10. Let $i \in \mathbb{I}$ and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Then $(S, \tilde{\pi}_i^{\mathcal{N}}, \tilde{\gamma}_i^{\mathcal{N}})$ is a braided Hopf algebra triple over $\Bbbk[N_i]$.

Assume that $p^{\mathcal{N}}$ induces an isomorphism $p^{\mathcal{N}}|\Bbbk[N_i] : \Bbbk[N_i] \to \mathcal{B}(M_i)$. Then $\pi_i p^{\mathcal{N}} = \pi_i^{\mathcal{N}}$ and

$$K_i^{\mathcal{N}} = S^{\operatorname{co}\mathcal{B}(M_i)}, \quad L_i^{\mathcal{N}} = {}^{\operatorname{co}\mathcal{B}(M_i)}S,$$

where the left and right coinvariant elements are defined with respect to $\pi_i^{\mathcal{N}}$.

Note that $K_i^{\mathcal{B}(M)} = K_i^{\mathcal{N}_0}$ and $L_i^{\mathcal{B}(M)} = L_i^{\mathcal{N}_0}$, where $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$. The following diagram commutes.

$$\begin{tikzcd}
 & & \mathcal{B}(M_i) \arrow[dl, "\gamma_i^{\mathcal{N}}"'] \arrow[d, "="'] \\
K_i^{\mathcal{N}} \arrow[r, "\subseteq"] \arrow[d] & S \arrow[r, "\pi_i^{\mathcal{N}}"] \arrow[d, "p^{\mathcal{N}}"'] & \mathcal{B}(M_i) \arrow[d, "="'] \\
K_i^{\mathcal{B}(M)} \arrow[r, "\subseteq"] & \mathcal{B}(M) \arrow[r, "\pi_i"] & \mathcal{B}(M_i)
\end{tikzcd}$$

LEMMA 13.5.11. *Let $i \in \mathbb{I}$, $M \in \mathcal{F}_\theta^H$, and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system of (M, i).*

1. $L_i^{\mathcal{N}} = \mathcal{S}_S^{-1}(K_i^{\mathcal{N}})$.
2. *The algebras $K_i^{\mathcal{N}}$ and $L_i^{\mathcal{N}}$ are generated by the subspaces $(\mathrm{ad}_S N_i)^n(N_j)$ and $\mathcal{S}_S^{-1}((\mathrm{ad}_S N_i)^n(N_j))$, respectively, with $j \neq i$ and $n \geq 0$.*
3. *$K_i^{\mathcal{N}}$ and $L_i^{\mathcal{N}}$ are $\mathbb{N}_0^\theta$-graded subalgebras of S.*
4. *Assume that M is i-finite. Then $K_i^{\mathcal{N}}$ is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$, and $p^{\mathcal{N}}$ induces a surjective map $K_i^{\mathcal{N}} \to K_i^{\mathcal{B}(M)}$ of Hopf algebras in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$.*

PROOF. (1) Since S is a braided Hopf algebra and since $\pi_i^{\mathcal{N}}$ is a Hopf algebra map, Proposition 3.2.12 implies that

$$\begin{aligned}
(\mathrm{id} \otimes \pi_i^{\mathcal{N}})\Delta(\mathcal{S}_S(x)) =&(\mathrm{id} \otimes \pi_i^{\mathcal{N}})(\mathcal{S}_S \otimes \mathcal{S}_S)c_{S,S}\Delta(x)\\
=&c_{\mathcal{B}(M_i),S}(\pi_i^{\mathcal{N}} \otimes \mathrm{id})(\mathcal{S}_S \otimes \mathcal{S}_S)\Delta(x)\\
=&c_{\mathcal{B}(M_i),S}(\mathcal{S}_{\mathcal{B}(M_i)} \otimes \mathcal{S}_S)(\pi_i^{\mathcal{N}} \otimes \mathrm{id})\Delta(x)
\end{aligned}$$

for any $x \in S$. Hence $\mathcal{S}_S(x) \in K_i^{\mathcal{N}}$ if and only if $x \in L_i^{\mathcal{N}}$.

The claim on $K_i^{\mathcal{N}}$ in (2) follows from Theorem 2.6.23 and Lemma 2.6.25 with $R = K_i^{\mathcal{N}}$, $A = S\#H$ and $W = \sum_{j \neq i} N_j$. The claim on $L_i^{\mathcal{N}}$ then follows from (1).

(3) holds since S, $\mathcal{B}(M_i)$, $\pi_i^{\mathcal{N}}$ and Δ_S are $\mathbb{N}_0^\theta$-graded.

(4) Since M is i-finite, the vector space $\sum_{n \geq 0}(\mathrm{ad}_S N_i)^n(N_j)$ is finite-dimensional for all $j \neq i$ by (Sys4), and (2) and (3) imply that $K_i^{\mathcal{N}} \in {}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$, see the proof of Lemma 13.4.8. By Theorem 5.5.6(2), $K_i^{\mathcal{N}}$ is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$. The rest holds since $p^{\mathcal{N}}$ is a surjective Hopf algebra map. $\square$

From the next theorem we will derive a construction which is fundamental for our analysis of Nichols algebras. Under reasonable assumptions we obtain from a Nichols system of (M, i) a new Nichols system of $(R_i(M), i)$ (see Proposition 13.5.14).

THEOREM 13.5.12. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M is i-finite and M_j is irreducible for all $j \in \mathbb{I}$ with $j \neq i$. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system of (M, i), $\widetilde{N}_i = M_i^*$, and $\widetilde{N}_j = (\mathrm{ad}_S N_i)^{-a_{ij}^M}(N_j)$ for all $j \in \mathbb{I}$ with $j \neq i$. Let $(\Omega, \omega) = (\Omega_{M_i}, \omega_{M_i})$ and let $\cdot_\Omega$ denote the $\mathcal{B}(M_i^*)$-action on $\Omega(K_i^{\mathcal{N}})$.*

(1) *For all* $j \neq i$ *and* $n \geq 0$,
$$(M_i^*)^n \cdot_\Omega \widetilde{N}_j = \begin{cases} (\mathrm{ad}_S N_i)^{-a_{ij}^M - n}(N_j) & \text{if } 0 \leq n \leq -a_{ij}^M, \\ 0 & \text{if } n > -a_{ij}^M. \end{cases}$$

(2) *The Yetter-Drinfeld modules* $\widetilde{N}_j \in {}^H_H\mathcal{YD}$, *where* $j \neq i$, *are irreducible.*

(3) *The algebra* $\Omega(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*)$ *is generated by* $\bigcup_{j\in\mathbb{I}} \widetilde{N}_j$.

(4) $\Omega(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*)$ *is an* $\mathbb{N}_0^\theta$*-graded Hopf algebra in* ${}^H_H\mathcal{YD}$ *with*
$$\deg(x \otimes y) = s_i^M(\deg^S(x) + \deg(y))$$
for all homogeneous elements $x \in K_i^{\mathcal{N}}$ *and* $y \in \mathcal{B}(M_i^*)$, *where* $\mathcal{B}(M_i^*)$ *is a* $\mathbb{Z}^\theta$*-graded algebra with* $\deg(M_i^*) = -\alpha_i$, *and* $\deg^S$ *is the degree of the graded algebra* S. *In particular,* $\deg(\widetilde{N}_j) = \alpha_j$ *for all* $j \in \mathbb{I}$. *(Here,* $\widetilde{N}_j$, $j \neq i$, *is identified with* $\widetilde{N}_j \otimes 1$, *and* $\widetilde{N}_i$ *with* $1 \otimes M_i^*$.*)*

PROOF. (1) Assume that $\theta \geq 2$. Let $j \in \mathbb{I} \setminus \{i\}$ and $Q_j = \mathrm{ad}_S \Bbbk[N_i](N_j)$. Since M is i-finite,
$$\mathrm{ad}_{\mathcal{B}(M)}\mathcal{B}(M_i)(M_j) = \bigoplus_{n=0}^{-a_{ij}^M} (\mathrm{ad}_{\mathcal{B}(M)} M_i)^n(M_j).$$
Since $\mathcal{N}$ is a Nichols system of (M, i), it follows that
$$Q_j = N_j \oplus \mathrm{ad}_S N_i(N_j) \oplus \cdots \oplus (\mathrm{ad}_S N_i)^{-a_{ij}^M}(N_j),$$
and $p^{\mathcal{N}}$ induces an isomorphism $Q_j \cong \mathrm{ad}_{\mathcal{B}(M)}\mathcal{B}(M_i)(M_j)$ of objects in the category ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$. Since N_j is irreducible, also Q_j is irreducible in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$ by Corollary 13.2.5(2). Moreover, Q_j is a $\mathbb{Z}$-graded object in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$ with
$$Q_j(n) = \begin{cases} (\mathrm{ad}_S N_i)^{n-1}(N_j) & \text{if } 1 \leq n \leq 1 - a_{ij}^M, \\ 0 & \text{if } n \leq 0 \text{ or } n > 1 - a_{ij}^M. \end{cases}$$
Therefore $\Omega(Q_j)$ is irreducible and $\mathbb{Z}$-graded in ${}^{\mathcal{B}(M_i^*)\#H}_{\mathcal{B}(M_i^*)\#H}\mathcal{YD}$. The non-zero homogeneous component of $\Omega(Q_j)$ of smallest degree is $\widetilde{N}_j$ of degree $n_0 = -1 + a_{ij}^M$, since for all integers n, $\Omega(Q_j)(n) = Q_j(-n)$. Hence we obtain from Proposition 13.1.2(3c) that for all $n \geq 0$,
$$\begin{aligned}(M_i^*)^n \cdot_\Omega \widetilde{N}_j &= \Omega(Q_j)(n + n_0) \\ &= Q_j(-n + 1 - a_{ij}^M) = \begin{cases} (\mathrm{ad}_S N_i)^{-n-a_{ij}^M}(N_j) & \text{if } n \leq -a_{ij}^M, \\ 0 & \text{if } n > -a_{ij}^M. \end{cases}\end{aligned}$$

(2) For any $j \in \mathbb{I} \setminus \{i\}$, $Q_j = \mathrm{ad}_S \Bbbk[N_i](N_j)$ is irreducible by the proof of (1) and has only finitely many non-zero homogeneous components. Hence $\widetilde{N}_j$ is irreducible by Theorem 13.2.7.

(3) Let $\widetilde{S}'$ be the subalgebra of $\Omega(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*)$ generated by $\widetilde{N}_1, \ldots, \widetilde{N}_\theta$. For all $\xi \in M_i^*$ and $x \in \Omega(K_i^{\mathcal{N}})$, the product of $1 \otimes \xi$ and $x \otimes 1$ in $\Omega(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*)$ is
$$(1 \otimes \xi)(x \otimes 1) = \xi \cdot_\Omega x \otimes 1 + \xi_{(-1)} \cdot x \otimes \xi_{(0)},$$
since ξ is primitive in $\mathcal{B}(M_i^*)$. Hence for any H-stable subspace X of $\Omega(K_i^{\mathcal{N}})$ with $X \otimes 1 \subseteq \widetilde{S}'$ it follows that $\xi \cdot_\Omega X \otimes 1$ is contained in $\widetilde{S}'$. In particular, we see

by induction on n that $(M_i^*)^n \cdot_\Omega \widetilde{N}_j \otimes 1$ is contained in $\widetilde{S}'$ for all $n \geq 0$. Then $\mathrm{ad}_S \Bbbk[N_i](N_j) \otimes 1 \subseteq \widetilde{S}'$ by (1). The subspaces $\mathrm{ad}_S \Bbbk[N_i](N_j) \subseteq K_i^{\mathcal{N}}$ with $j \in \mathbb{I} \setminus \{i\}$ generate the algebra $K_i^{\mathcal{N}}$ by Lemma 13.5.11(2). They are objects in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$. Since for all subobjects $X, Y \subseteq K_i^{\mathcal{N}}$ in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$, $\Omega(X)\Omega(Y) = \Omega(XY)$, we conclude that $\Omega(K_i^{\mathcal{N}})$ is generated by the subspaces $\mathrm{ad}_S \Bbbk[N_i](N_j)$, $j \in \mathbb{I} \setminus \{i\}$. Hence $\Omega(K_i^{\mathcal{N}}) \otimes 1 \subseteq \widetilde{S}'$. This implies (3).

(4) Recall that the Hopf algebra $\mathcal{B}(M_i)\#H$ is $\mathbb{Z}^\theta$-graded with $\deg(M_i) = \alpha_i$ and $\deg(H) = 0$. By Lemma 13.5.11(3), $K_i^{\mathcal{N}}$ is an $\mathbb{N}_0^\theta$-graded Hopf algebra. We extend the grading of $K_i^{\mathcal{N}}$ to a $\mathbb{Z}^\theta$-grading by $K_i^{\mathcal{N}}(\alpha) = 0$ for all $\alpha \notin \mathbb{N}_0^\theta$. Then Lemma 13.5.11(4) implies that $K_i^{\mathcal{N}}$ is a $\mathbb{Z}^\theta$-graded Hopf algebra in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}_{\mathrm{rat}}$. Then $\Omega(K_i^{\mathcal{N}})$ is a $\mathbb{Z}^\theta$-graded Hopf algebra in ${}^{\mathcal{B}(M_i^*)\#H}_{\mathcal{B}(M_i^*)\#H}\mathcal{YD}_{\mathrm{rat}}$ with the same grading

$$\Omega(K_i^{\mathcal{N}})(\alpha) = K_i^{\mathcal{N}}(\alpha) \text{ for all } \alpha \in \mathbb{Z}^\theta,$$

where $\mathcal{B}(M_i^*)\#H$ is a $\mathbb{Z}^\theta$-graded Hopf algebra with $\deg(M_i^*) = -\alpha_i$, $\deg(H) = 0$. This follows from the definition of Ω and ω, since $\langle\ ,\ \rangle : \mathcal{B}(M_i^*) \otimes \mathcal{B}(M_i) \to \Bbbk$ is $\mathbb{Z}^\theta$-graded, where $\Bbbk(0) = \Bbbk$ and $\Bbbk(\alpha) = 0$ for all $\alpha \neq 0$. By Theorem 5.5.6(1), $\Omega(K_i^{\mathcal{N}})\#(\mathcal{B}(M_i^*)\#H)$ is a $\mathbb{Z}^\theta$-graded Hopf algebra. Hence $\Omega(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*)$ is a $\mathbb{Z}^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Shifting the degree by s_i^M defines the grading of $\Omega(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*)$. Hence $\deg(\widetilde{N}_i) = s_i^M(-\alpha_i) = \alpha_i$, and for all $j \neq i$,

$$\deg(\widetilde{N}_j) = s_i^M(-a_{ij}^M \alpha_i + \alpha_j) = \alpha_j.$$

The proof of the theorem is completed. □

Definition 13.5.13. Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system of (M, i). Assume that M_j is irreducible for all $j \neq i$ and that M is i-finite. Let

$$\widetilde{N} = (\widetilde{N}_1, \dots, \widetilde{N}_\theta), \quad \widetilde{f} = (\widetilde{f}_1, \dots, \widetilde{f}_\theta), \quad R_i(\mathcal{N}) = \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}),$$

where $\widetilde{S} = \Omega_{M_i}(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*)$ is the Hopf algebra in Theorem 13.5.12 with generators $\widetilde{N}_1, \dots, \widetilde{N}_\theta$, $\widetilde{f}_i$ is the identity on M_i^*, and $\widetilde{f}_j : \widetilde{N}_j \to R_i(M)_j$ for any $j \in \mathbb{I} \setminus \{i\}$ is the isomorphism induced by $p^{\mathcal{N}}$. The triple $R_i(\mathcal{N})$ is called the **i-th reflection of $\mathcal{N}$**.

Proposition 13.5.14. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Assume that M is i-finite. Let $\mathcal{N}$ be a Nichols system of (M, i). Then $R_i(\mathcal{N})$ is a Nichols system of $(R_i(M), i)$.*

Proof. Let $\mathcal{N}(S, N, f) = \mathcal{N}$ and $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_i(\mathcal{N})$. For all $j \in \mathbb{I} \setminus \{i\}$, the map

$$\widetilde{f}_j : \widetilde{N}_j = (\mathrm{ad}_S N_i)^{-a_{ij}^M}(N_j) \to (\mathrm{ad}_{\mathcal{B}(M)} M_i)^{-a_{ij}^M}(M_j) = R_i(M)_j$$

induced by $p^{\mathcal{N}}$ is an isomorphism, since $\mathcal{N}$ is a Nichols system of (M, i). Hence $R_i(\mathcal{N})$ is a pre-Nichols system of $R_i(M)$ by Theorem 13.5.12(3),(4).

The canonical map $p^{R_i(\mathcal{N})}$ of $R_i(\mathcal{N})$ sends $\widetilde{N}_j$ to $R_i(M)_j$ for all $j \in \mathbb{I}$. Moreover, $\Bbbk[\widetilde{N}_i] = \mathcal{B}(M_i^*) \subseteq \widetilde{S}$. Hence (Sys3) holds for $R_i(\mathcal{N})$ with respect to $i \in \mathbb{I}$.

Let now $j \in \mathbb{I} \setminus \{i\}$ and $Q_j = \oplus_{n=0}^{-a_{ij}^M} (\mathrm{ad}_S N_i)^n(N_j)$. The left action of $\mathcal{B}(M_i^*)$ on $\Omega_{M_i}(K_i^{\mathcal{N}})$ coincides with the restriction of the adjoint action of $\widetilde{S}$, and hence

$$\mathrm{ad}_{\widetilde{S}} \mathcal{B}(M_i^*)(\widetilde{N}_j) = \Omega(Q_j) \tag{13.5.3}$$

by Theorem 13.5.12(1). Now (Sys4) for $\mathcal{N}$ implies that $Q_j \in {}^{\Bbbk[N_i]\#H}_{\Bbbk[N_i]\#H}\mathcal{YD}$ is irreducible, see Lemma 13.5.5. Then $\Omega(Q_j) \in {}^{\Bbbk[\widetilde{N}_i]\#H}_{\Bbbk[\widetilde{N}_i]\#H}\mathcal{YD}$ is irreducible and hence (Sys4) holds for $R_i(\mathcal{N})$ because of Lemma 13.5.5. $\square$

REMARK 13.5.15. In the proof of Proposition 13.5.14 we also observed that $\Omega_{M_i}(\mathrm{ad}_S \Bbbk[N_i](N_j))\#1$ is invariant under the adjoint action of $\mathcal{B}(M_i^*)$ and that

$$\begin{aligned}(\mathrm{ad}\, M_i^*)^{1-a_{ij}^M}(\widetilde{N}_j) &= 0\\ (\mathrm{ad}\, M_i^*)^n(\widetilde{N}_j) &= (\mathrm{ad}_S N_i)^{-a_{ij}^M-n}(N_j)\end{aligned}$$

for any $j \in \mathbb{I} \setminus \{i\}$, $0 \le n \le -a_{ij}^M$.

If the canonical map of a Nichols system of (M, i) is an isomorphism, more detailed information can be obtained.

LEMMA 13.5.16. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Assume that M is i-finite. Let $\mathcal{N}$ be a pre-Nichols system of M such that the canonical map of $\mathcal{N}$ is an isomorphism. Then $\mathcal{N}$ and $R_i(\mathcal{N})$ are Nichols systems of (M, i) and $(R_i(M), i)$, respectively, and the canonical map of $R_i(\mathcal{N})$ is an isomorphism.*

PROOF. It is clear from the definition that $\mathcal{N}$ is a Nichols system of (M, i), and that $p^{\mathcal{N}}$ induces an isomorphism $p^{\mathcal{N}} : K_i^{\mathcal{N}} \to K_i^{\mathcal{B}(M)}$. Hence the canonical map of $R_i(\mathcal{N})$ is the composition

$$\Omega_{M_i}(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*) \xrightarrow{\Omega_{M_i}(p^{\mathcal{N}})\otimes \mathrm{id}} \Omega_{M_i}(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*) \xrightarrow{\Theta^{-1}} \mathcal{B}(R_i(M)),$$

where Θ is the isomorphism of Theorem 13.4.9. Thus the Lemma follows from Proposition 13.5.14. $\square$

Recall from Theorem 3.5.8 the isomorphism ψ_V between any $V \in {}^H_H\mathcal{YD}$ and its double dual.

DEFINITION 13.5.17. Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Assume that M is i-finite. Let Θ be the isomorphism in Theorem 13.4.9. For all $j \in \mathbb{I}$ let

$$f_j^M : M_j \to (R_i^2(M))_j,\ x \mapsto \begin{cases} \Theta^{-1}(x \otimes 1) & \text{if } j \ne i,\\ \psi_{M_i}(x) & \text{if } j = i.\end{cases}$$

Let $f^M = (f_j^M)_{1 \le j \le \theta} : M \to R_i^2(M)$.

REMARK 13.5.18. The tuple f^M in Definition 13.5.17 is well-defined by Remark 13.5.15 applied to the Nichols system $\mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$.

PROPOSITION 13.5.19. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Assume that M is i-finite. Then*

(1) *$R_i(M)_j$ is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$ and $R_i(M)$ is i-finite,*

(2) $a_{ij}^{R_i(M)} = a_{ij}^M$ *for all* $j \in \mathbb{I}$, *and*
(3) $f^M : M \to R_i^2(M)$ *is an isomorphism in* $\mathcal{F}_\theta^H$.

Proof. The claims of the Proposition follow from Remark 13.5.15 applied to the Nichols system $\mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$. □

Definition 13.5.20. Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Assume that M is i-finite. For any Nichols system $\mathcal{N}$ of (M, i) let

$$T_i^{\mathcal{N}} : L_i^{R_i(\mathcal{N})} = {}^{\mathrm{co}\,\mathcal{B}(M_i^*)}(\Omega_{M_i}(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*)) \xrightarrow{\cong} D((K_i^{\mathcal{N}})^{\mathrm{cop}}) = K_i^{\mathcal{N}}$$

be the isomorphism T in Theorem 12.3.3 with $B = \mathcal{B}(M_i)$ and $\mathcal{C} = {}_H^H\mathcal{YD}$.

Corollary 13.5.21. *Let* $i \in \mathbb{I}$ *and* $M \in \mathcal{F}_\theta^H$ *such that* M_j *is irreducible in* ${}_H^H\mathcal{YD}$ *for all* $j \in \mathbb{I} \setminus \{i\}$. *Assume that* M *is* i*-finite. Let* $\mathcal{N}$ *be a Nichols system of* (M, i) *and let* $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_i(\mathcal{N})$. *Then* $T_i^{\mathcal{N}} : L_i^{R_i(\mathcal{N})} \to K_i^{\mathcal{N}}$ *is an algebra isomorphism in* ${}_H^H\mathcal{YD}$ *such that*

(1) *For all* $j \in \mathbb{I} \setminus \{i\}$, $0 \le n \le -a_{ij}^M$, *and* $y \in (\mathrm{ad}_{\widetilde{S}} M_i^*)^n(\widetilde{N}_j)$,

$$T_i^{\mathcal{N}}(\mathcal{S}_{\widetilde{S}}^{-1}(y)) = -y,$$
$$T_i^{\mathcal{N}}\big(\mathcal{S}_{\widetilde{S}}^{-1}((\mathrm{ad}_{\widetilde{S}} M_i^*)^n(\widetilde{N}_j))\big) = (\mathrm{ad}_S N_i)^{-a_{ij}^M - n}(N_j).$$

(2) *Let* $\alpha \in \mathbb{N}_0^\theta$, *and let* $x \in L_i^{R_i(\mathcal{N})}(\alpha)$ *be a non-zero homogeneous element. Then* $\deg(T_i^{\mathcal{N}}(x)) = s_i^{R_i(M)}(\alpha)$. *In particular,* $s_i^{R_i(M)}(\alpha) \in \mathbb{N}_0^\theta$. *Here,* $L_i^{R_i(\mathcal{N})}$ *and* $K_i^{\mathcal{N}}$ *are* $\mathbb{N}_0^\theta$*-graded subalgebras of* $\widetilde{S}$ *and* S*, respectively.*

Proof. If $\theta = 1$ then the claim is trivial. Assume that $\theta \ge 2$. Let $j \in \mathbb{I} \setminus \{i\}$, $0 \le n \le -a_{ij}^M$, $y \in (\mathrm{ad}_{\widetilde{S}} M_i^*)^n(\widetilde{N}_j)$, and $x = \mathcal{S}_{\widetilde{S}}^{-1}(y)$. Remark 13.5.15 implies that the elements in $(\mathrm{ad}_{\widetilde{S}} M_i^*)^n(\widetilde{N}_j)$ are primitive in $K_i^{\mathcal{N}}$. Hence $T_i^{\mathcal{N}}(x) = \mathcal{S}_{K_i^{\mathcal{N}}}^{-1}(y) = -y$. This proves (1). Moreover, $\deg(x) = \deg(y) = n\alpha_i + \alpha_j$, since $\mathcal{S}_{\widetilde{S}}^{-1}$ is graded by Corollary 5.1.3. On the other hand,

$$\deg(T_i^{\mathcal{N}}(x)) = (-a_{ij}^M - n)\alpha_i + \alpha_j = s_i^M(\deg(x)),$$

since $T_i^{\mathcal{N}}(x) \in (\mathrm{ad}_S N_i)^{-a_{ij}^M - n}(N_j)$. Now, as $s_i^M = s_i^{R_i(M)}$ as a consequence of Proposition 13.5.19(2), (2) follows from Lemma 13.5.11(2). □

Finally we introduce morphisms of pre-Nichols systems. The results in the remaining part of this section will be needed in Section 16.3 for the study of small quantum groups.

Definition 13.5.22. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ and $\mathcal{N}' = \mathcal{N}(S', N', f')$ be pre-Nichols systems of M for some $M \in \mathcal{F}_\theta^H$. A **morphism** $p : \mathcal{N} \to \mathcal{N}'$ **of pre-Nichols systems of** M is a Hopf algebra morphism $p : S \to S'$ such that for any $j \in \mathbb{I}$, p induces an isomorphism $p_j = p|N_j : N_j \to N_j'$ satisfying $f_j = f_j' p_j$.

An example of a morphism of pre-Nichols systems of M is the canonical map $p^{\mathcal{N}} : \mathcal{N} \to \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$.

Remark 13.5.23. Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$. The class of Nichols systems of (M, i) forms a category $\mathfrak{N}_i^M$ with morphisms as in Definition 13.5.22. We list some properties of the category which follow from the definitions.

(1) For any two pre-Nichols systems $\mathcal{N},\mathcal{N}'$ of M, there is at most one morphism $p:\mathcal{N}\to\mathcal{N}'$. In particular, $\mathfrak{N}_i^M$ is a **thin category**.
(2) Let $p:\mathcal{N}\to\mathcal{N}'$ be a morphism of pre-Nichols systems of M. Then p is a surjective morphism of $\mathbb{N}_0^\theta$-graded Hopf algebras in ${}^H_H\mathcal{YD}$. For all $j\in\mathbb{I}$, p_j is an isomorphism, in particular, $\ker(p)\cap\mathcal{N}_j=0$.
(3) Let $p:\mathcal{N}\to\mathcal{N}'$ be a morphism of pre-Nichols systems of M. If $\mathcal{N}$ is a Nichols system of (M,i), then $\mathcal{N}'$ is a Nichols system of (M,i).
(4) A morphism p of pre-Nichols systems of M is an isomorphism if and only if p is bijective.
(5) Let $p':\mathcal{N}\to\mathcal{N}'$ and $p'':\mathcal{N}\to\mathcal{N}''$ be morphisms of pre-Nichols systems of M with $\ker(p')\subseteq\ker(p'')$. Then there is a morphism $p:\mathcal{N}'\to\mathcal{N}''$ satisfying $pp'=p''$. If $\ker(p')=\ker(p'')$, then p is an isomorphism.

The following proposition states the existence of terminal and initial objects in $\mathfrak{N}_i^M$ for any $i\in\mathbb{I}$ and $M\in\mathcal{F}_\theta^H$. Note that such objects are unique up to isomorphism.

PROPOSITION 13.5.24. *Let $i\in\mathbb{I}$ and $M\in\mathcal{F}_\theta^H$.*

(1) *For any Nichols system $\mathcal{N}$ of (M,i), the canonical map is the unique morphism from $\mathcal{N}$ to $\mathcal{N}(\mathcal{B}(M),M,\mathrm{id}_M)$.*
(2) *There is a Nichols system $\mathcal{N}_{\mathrm{ini}}=\mathcal{N}(\widehat{S},\widehat{N},\hat{f})$ of (M,i), such that for any Nichols system $\mathcal{N}=\mathcal{N}(S,N,f)$ of (M,i) there is a unique morphism $q:\mathcal{N}_{\mathrm{ini}}\to\mathcal{N}$.*

PROOF. (1) follows directly from Remark 13.5.23(1). Now we prove (2). Let $V=\oplus_{i=1}^\theta M_i$ and let $I_i=I(M_i)$ and $I(V)$ be the defining ideals of $\mathcal{B}(M_i)$ and $\mathcal{B}(V)$, respectively, see Definition 7.1.1. For all $j\in\mathbb{I}\setminus\{i\}$ let

$$I_j=I(V)\cap\oplus_{n=0}^\infty(\mathrm{ad}_{T(V)}M_i)^n(M_j).$$

Let $\widehat{S}$ be the quotient of $T(V)$ by the ideal generated by $I_1,\dots,I_\theta$. Since $I(V)$ is an $\mathbb{N}_0^\theta$-graded Hopf ideal, we conclude that $\widehat{S}$ is an $\mathbb{N}_0^\theta$-graded Hopf algebra generated by $M_1,\dots,M_\theta$. Moreover, $\mathcal{N}_{\mathrm{ini}}=(\widehat{S},M,\mathrm{id})$ is a Nichols system of (M,i) by construction.

Let $\mathcal{N}=\mathcal{N}(S,N,f)$ be a Nichols system of (M,i). Then $f^{-1}:M\to N$ induces a Hopf algebra map $q:T(V)\to S$. Moreover, $I_1,\dots,I_\theta\subseteq\ker q$ because of (Sys3) and (Sys4) for S. Hence q factors to a Hopf algebra map $q:\widehat{S}\to S$. Thus q is a morphism from $\mathcal{N}_{\mathrm{ini}}$ to $\mathcal{N}$ since $f_jq|_{M_j}=\mathrm{id}_{M_j}$ for all $j\in\mathbb{I}$. The uniqueness of q follows from Remark 13.5.23(1). □

PROPOSITION 13.5.25. *Let $i\in\mathbb{I}$ and $M\in\mathcal{F}_\theta^H$ such that M is i-finite and M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j\in\mathbb{I}$. Let $\mathcal{N}=\mathcal{N}(S,N,f)$ and $\mathcal{N}'=\mathcal{N}(S',N',f')$ be Nichols systems of (M,i) and $(R_i^2(M),i)$, respectively, with $\dim S(\alpha)=\dim S'(\alpha)$ for any $\alpha\in\mathbb{N}_0^\theta$. Then any morphism $p:\mathcal{N}'\to R_i^2(\mathcal{N})$ of Nichols systems of $(R_i^2(M),i)$ is an isomorphism.*

PROOF. As any morphism of Nichols systems, p is surjective and graded. Let $R_i^2(\mathcal{N})=\mathcal{N}(\widetilde{S},\widetilde{N},\widetilde{f})$. By Definition 13.5.13 and Proposition 13.5.14,

$$\widetilde{S}=\Omega_{M_i^*}(\Omega_{M_i}(K_i^{\mathcal{N}}))\#\mathcal{B}(M_i^{**}).$$

By Theorem 13.5.12(4), $\dim \widetilde{S}\big(s_i^{R_i(M)} s_i^M(\alpha)\big) = \dim S(\alpha)$ for all $\alpha \in \mathbb{N}_0^\theta$. Moreover, $s_i^{R_i(M)} s_i^M = \mathrm{id}_{\mathbb{Z}^\theta}$ because of Proposition 13.5.19(2). It follows that

$$\dim \widetilde{S}(\alpha) = \dim S(\alpha) = \dim S'(\alpha)$$

for all $\alpha \in \mathbb{N}_0^\theta$. Thus p is injective. □

Let $i \in \mathbb{I}$, $M \in \mathcal{F}_\theta^H$, and let $\mathcal{N} = \mathcal{N}(S, N, f)$, $\mathcal{N}' = \mathcal{N}(S', N', f')$ be Nichols systems of (M, i). We note that a morphism $p : \mathcal{N} \to \mathcal{N}'$ satisfies

$$p^{\mathcal{N}'} p = p^{\mathcal{N}}, \quad \pi_i^{\mathcal{N}'} p = \pi_i^{\mathcal{N}} \text{ for all } 1 \le i \le \theta. \tag{13.5.4}$$

We denote the induced morphism of Hopf algebras in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$ by

$$p_K : K_i^{\mathcal{N}} \to K_i^{\mathcal{N}'}.$$

Definition 13.5.26. Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M is i-finite. For any morphism $p : \mathcal{N} \to \mathcal{N}'$ of Nichols systems of (M, i), we define

$$R_i(p) = \Omega_{M_i}(p_K)\#\mathrm{id} : \Omega_{M_i}(K_i^{\mathcal{N}}) \otimes \mathcal{B}(M_i^*) \to \Omega_{M_i}(K_i^{\mathcal{N}'}) \otimes \mathcal{B}(M_i^*).$$

Lemma 13.5.27. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M is i-finite and M_j is irreducible for all $j \in \mathbb{I} \setminus \{i\}$.*

(1) *$R_i : \mathfrak{N}_i^M \to \mathfrak{N}_i^{R_i(M)}$, where $\mathcal{N} \in \mathfrak{N}_i^M$ is mapped to $R_i(\mathcal{N})$ and a morphism $p : \mathcal{N} \to \mathcal{N}'$ to $R_i(p)$, is a functor.*

(2) *Let $p : \mathcal{N} \to \mathcal{N}'$ be a morphism of Nichols systems of (M, i). The diagram*

$$\begin{tikzcd} L_i^{R_i(\mathcal{N})} \arrow[r, "T_i^{\mathcal{N}}"] \arrow[d, "R_i(p)_L"] & K_i^{\mathcal{N}} \arrow[d, "p_K"] \\ L_i^{R_i(\mathcal{N}')} \arrow[r, "T_i^{\mathcal{N}'}"] & K_i^{\mathcal{N}'} \end{tikzcd}$$

commutes, where $R_i(p)_L$ denotes the restriction of $R_i(p)$ to the left coinvariant elements.

Proof. (1) By Proposition 13.5.14, for any $\mathcal{N} \in \mathfrak{N}_i^M$, $R_i(\mathcal{N})$ is a Nichols system of $(R_i(M), i)$. It remains to show that $R_i(p)$ is a morphism for any morphism p. This follows from Corollary 4.3.3 and from (Sys4).

(2) It is enough to check commutativity of the diagram on generators of the form $\mathcal{S}_{\widetilde{S}}^{-1}(y)$, where $y \in (\mathrm{ad}_{\widetilde{S}} M_i^*)^n(\widetilde{N}_j)$, $n \ge 0$, and $j \ne i$. (We use the notation above the lemma.) Now (1) implies that $R_i(p)\mathcal{S}_{\widetilde{S}}^{-1} = \mathcal{S}_{\widetilde{S}'}^{-1} R_i(p)$. Moreover, $\Omega(p_K) = p_K$ by definition. Hence (2) follows from Corollary 13.5.21(1). □

Now we discuss the compatibility of morphisms of Nichols systems of (M, i) and quotient constructions.

Proposition 13.5.28. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system of (M, i) and let J be an $\mathbb{N}_0^\theta$-graded Hopf ideal of S in ${}^H_H\mathcal{YD}$.*

(1) *Assume that $N_i \cap J = 0$. Then*

$$J = (J \cap K_i^{\mathcal{N}})\Bbbk[N_i] = \Bbbk[N_i](J \cap K_i^{\mathcal{N}}) = (J \cap L_i^{\mathcal{N}})\Bbbk[N_i] = \Bbbk[N_i](J \cap L_i^{\mathcal{N}}).$$

(2) *Assume that* $N_j \cap J = 0$ *for any* $j \in \mathbb{I}$. *Then* $\overline{\mathcal{N}} = \mathcal{N}(S/J, N, f)$ *is a Nichols system of* (M, i), *and the canonical map* $p : S \to S/J$ *is a morphism* $p : \mathcal{N} \to \overline{\mathcal{N}}$ *of pre-Nichols systems.*

PROOF. (1) Let $\pi = \tilde{\pi}_i^{\mathcal{N}} : S \to \Bbbk[N_i]$ be the graded projection from Definition 13.5.9. Then $\pi(J) = \bigoplus_{n\geq 0} S(n\alpha_i) \cap J = \Bbbk[N_i] \cap J$. Since $N_i \cap J = 0$, (Sys3) and Corollary 1.3.11(1) imply that $\pi(J) = \Bbbk[N_i] \cap J = 0$. Hence π induces a Hopf algebra morphism $\overline{\pi} : S/J \to \Bbbk[N_i]$ in ${}^H_H\mathcal{YD}$, and $\overline{\pi}\overline{\gamma} = \mathrm{id}_{\Bbbk[N_i]}$, where $\overline{\gamma}$ is the composition of the inclusion map $\Bbbk[N_i] \to S$ with the canonical map $S \to S/J$. Let $\overline{K} = (S/J)^{\mathrm{co}\,\Bbbk[N_i]}$. By Theorem 3.9.2(6) the diagram

$$\begin{tikzcd} K_i^{\mathcal{N}} \otimes \Bbbk[N_i] \arrow[r, "\cong"] \arrow[d] & S \arrow[d] \\ \overline{K} \otimes \Bbbk[N_i] \arrow[r, "\cong"] & S/J \end{tikzcd} \tag{13.5.5}$$

commutes, where the horizontal maps are multiplication and the vertical maps are the canonical maps. We conclude from the diagram that $J = (J \cap K_i^{\mathcal{N}})\Bbbk[N_i]$. Corollary 3.9.3 allows us to interchange the tensor factors in (13.5.5), and the equality $J = \Bbbk[N_i](J \cap K_i^{\mathcal{N}})$ follows by the same argument. By applying the antipode of S we obtain the remaining equations.

(2) Since J is an $\mathbb{N}_0^\theta$-graded Hopf ideal of S and $N_j \cap J = 0$ for all $j \in J$, $\overline{\mathcal{N}} = \mathcal{N}(S/J, N, f)$ is a pre-Nichols system of M. In the proof of (1) we saw that $\pi(J) = 0$. Hence $\pi_i^{\mathcal{N}}(J) = 0$ and (Sys3) holds for $\overline{\mathcal{N}}$ and i. Finally, for any $j \in \mathbb{I}\setminus\{i\}$ the canonical map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ is injective on $\mathrm{ad}_S\Bbbk[N_i](N_j)$ by assumption and it factorizes via S/J. Hence $p^{\overline{\mathcal{N}}} : S \to \mathcal{B}(M)$ is injective on $\mathrm{ad}_{S/J}\Bbbk[N_i](N_j)$. Thus $\overline{\mathcal{N}}$ is a Nichols system of (M, i). Finally, p is a morphism, since for all $j \in \mathbb{I}$, $p_j = \mathrm{id}_{N_j}$. □

PROPOSITION 13.5.29. *Let* $i \in \mathbb{I}$ *and* $M \in \mathcal{F}_\theta^H$ *such that* M *is* i*-finite and* M_j *is irreducible for all* $j \in \mathbb{I} \setminus \{i\}$.

(1) *Let* $p : \mathcal{N} \to \mathcal{N}'$ *be a morphism in* $\mathfrak{N}_i^M$. *Then*

$$\ker(R_i(p)) = (T_i^{\mathcal{N}})^{-1}\big(\ker(p) \cap K_i^{\mathcal{N}}\big)\mathcal{B}(M_i^*),$$
$$\ker(p) = T_i^{\mathcal{N}}\big(\ker(R_i(p)) \cap L_i^{R_i(\mathcal{N})}\big)\Bbbk[\mathcal{N}_i].$$

(2) *Let* $\mathcal{N} \in \mathfrak{N}_i^M$, *and let* $q : R_i(\mathcal{N}) \to \mathcal{N}''$ *be a morphism in* $\mathfrak{N}_i^{R_i(M)}$ *for some* $\mathcal{N}'' \in \mathfrak{N}_i^{R_i(M)}$. *Then there exist a morphism* $p : \mathcal{N} \to \mathcal{N}'$ *in* $\mathfrak{N}_i^M$ *and an isomorphism* $r : R_i(\mathcal{N}') \to \mathcal{N}''$ *in* $\mathfrak{N}_i^{R_i(M)}$ *such that* $\ker(R_i(p)) = \ker(q)$ *and* $rR_i(p) = q$.

PROOF. (1) By Proposition 13.5.28 for $R_i(\mathcal{N})$,

$$\ker(R_i(p)) = \big(\ker(R_i(p)) \cap L_i^{R_i(\mathcal{N})}\big)\mathcal{B}(M_i^*) = \ker(R_i(p)_L)\mathcal{B}(M_i^*).$$

Since $T_i^{\mathcal{N}}$ and $T_i^{\mathcal{N}'}$ are isomorphisms, from Lemma 13.5.27(2) it follows that

$$\ker(R_i(p)_L) = (T_i^{\mathcal{N}})^{-1}\ker(p_K) = (T_i^{\mathcal{N}})^{-1}\big(\ker(p) \cap K_i^{\mathcal{N}}\big).$$

The claim on $\ker(p)$ is obtained similarly.

(2) We construct a morphism p using (1) and Proposition 13.5.28(2).

Let $\mathcal{C} = {}^H_H\mathcal{YD}$, $\mathcal{N}(S,N,f) = \mathcal{N}$, $\mathcal{N}(\widetilde{S},\widetilde{N},\widetilde{f}) = R_i(\mathcal{N})$, and $\widetilde{J} = \ker(q)$. Then $\widetilde{J}$ is a graded Hopf ideal of the $\mathbb{N}_0^\theta$-graded Hopf algebra $\widetilde{S}$ in $\mathcal{C}$ and for all $j \in \mathbb{I}$, $\widetilde{N}_j \cap \widetilde{J} = 0$ by Remark 13.5.23(2). Hence $\widetilde{S}^{\mathrm{cop}}/\widetilde{J} = (\widetilde{S}/\widetilde{J})^{\mathrm{cop}}$ is a braided Hopf algebra in $\overline{\mathcal{C}}$ with projection to A^{cop}. Let $\widetilde{L} = L_i^{R_i(\mathcal{N})}$. Then $\widetilde{J} \cap \widetilde{L}$ is a graded Hopf ideal of the $\mathbb{N}_0^\theta$-graded Hopf algebra $\widetilde{L}$ in ${}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$. Now by Theorem 12.3.3(2), $T_i^{\mathcal{N}}(\widetilde{J} \cap \widetilde{L}) \subseteq K_i^{\mathcal{N}}$ is a graded Hopf ideal of the $\mathbb{N}_0^\theta$-graded Hopf algebra $K_i^{\mathcal{N}}$ in ${}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})$. By Corollary 12.3.5 and since $T_i^{\mathcal{N}}$ is graded, $T_i^{\mathcal{N}}(\widetilde{J} \cap \widetilde{L})$ is a graded Hopf ideal of the $\mathbb{N}_0^\theta$-graded Hopf algebra $K_i^{\mathcal{N}}$ in ${}^B_B\mathcal{YD}(\mathcal{C})$. Hence $J = T_i^{\mathcal{N}}(\widetilde{J} \cap \widetilde{L})\Bbbk[N_i]$ is a graded Hopf ideal of S. By definition, $N_i \cap J = 0$. Further, it follows from (Sys4) and Lemma 13.5.5 that $N_j \cap J = 0$ for any $j \in \mathbb{I} \setminus \{i\}$. Let $p : \mathcal{N} \to \mathcal{N}(S/J, N, f)$ be the morphism from Proposition 13.5.28(2). Then $\ker(R_i(p)) = \ker(q)$ by (1). The existence and claimed properties of r follow from Remark 13.5.23(5) applied to the morphisms $R_i(p)$ and q. □

13.6. The semi-Cartan graph of a Nichols algebra

As before, let H be a Hopf algebra with bijective antipode, let $\theta \geq 1$ be a natural number, and $\mathbb{I} = \{1, \dots, \theta\}$.

Definition 13.6.1. Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Let $l \in \mathbb{N}_0$ and $i_1, \dots, i_l \in \mathbb{I}$. Let $\mathcal{N}$ be a pre-Nichols system of M.

1. We say that M **admits the reflection sequence** $(i_1, \dots, i_l)$ if $l = 0$ or if M is i_1-finite and $R_{i_1}(M)$ admits the reflection sequence $(i_2, \dots, i_l)$.
2. We say that $\mathcal{N}$ **admits the reflection sequence** $(i_1, \dots, i_l)$ if $l = 0$ or if $\mathcal{N}$ is a Nichols system of (M, i_1), M is i_1-finite, and $R_{i_1}(\mathcal{N})$ admits the reflection sequence $(i_2, \dots, i_l)$.
3. We say that M **admits all reflections** if M admits all reflection sequences $(j_1, \dots, j_k)$ with $k \in \mathbb{N}_0$ and $j_1, \dots, j_k \in \mathbb{I}$.
4. We say that $\mathcal{N}$ **admits all reflections** if $\mathcal{N}$ admits all reflection sequences $(j_1, \dots, j_k)$ with $k \in \mathbb{N}_0$ and $j_1, \dots, j_k \in \mathbb{I}$.
5. Assume that M admits all reflections. Let
$$\mathcal{F}_\theta^H(M) = \{R_{j_1}(\cdots R_{j_k}(M)) \mid k \in \mathbb{N}_0, j_1, \dots, j_k \in \mathbb{I}\}.$$

Let $i \in \mathbb{I}$. According to Lemma 13.4.5, if M is i-finite, then the isomorphism class $r_i([M]) = [R_i(M)]$ and the Cartan integers a_{ij}^M with $j \in \mathbb{I}$ do not depend on the choice of the representative of the isomorphism class of M.

Theorem 13.6.2. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $\mathcal{X} = \{[P] \mid P \in \mathcal{F}_\theta^H(M)\}$, and let $r : \mathbb{I} \times \mathcal{X} \to \mathcal{X}$, $(i, [P]) \mapsto [R_i(P)]$. Then*
$$\mathcal{G}(M) = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, (A^X)_{X \in \mathcal{X}}),$$
where $A^{[P]} = (a_{ij}^P)_{i,j \in \mathbb{I}}$ for all $[P] \in \mathcal{X}$, is a semi-Cartan graph.

Proof. Lemma 13.4.5 implies that r and the family $(A^X)_{X \in \mathcal{X}}$ are well-defined. For any $X \in \mathcal{X}$, A^X is a Cartan matrix by Lemma 13.4.4. According to Definition 9.1.1, it remains to show that $\mathcal{G}(M)$ fulfills Axioms (CG1) and (CG2). This in turn follows from Proposition 13.5.19. □

DEFINITION 13.6.3. Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Assume that M admits all reflections. We call $\mathcal{G}(M)$ the **semi-Cartan graph of** M, and $\mathcal{W}(M) = \mathcal{W}(\mathcal{G}(M))$ the **Weyl groupoid of** M. Often it will be more convenient to say that $\mathcal{G}(M)$ is the Cartan graph of $\mathcal{B}(M)$ and $\mathcal{W}(M)$ is the Weyl groupoid of $\mathcal{B}(M)$.

PROPOSITION 13.6.4. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Assume that $\mathcal{B}(M)$ is a finite-dimensional vector space over $\Bbbk$. Then M admits all reflections, and $\dim \mathcal{B}(P) = \dim \mathcal{B}(M)$ for each $P \in \mathcal{F}_\theta^H(M)$.*

PROOF. Since $\mathcal{B}(M)$ is finite-dimensional, M is i-finite for any $i \in \mathbb{I}$ by degree reasons. Moreover, $R_i(M)_j$ is irreducible for all $j \in \mathbb{I}$ by Corollary 13.4.3 and since $R_i(M)_i = M_i^*$. Hence $\mathcal{B}(R_i(M))$ is finite-dimensional for all $i \in \mathbb{I}$ by Theorem 13.4.9. By induction on l it follows that for any $\kappa = (i_1, \dots, i_l) \in \mathbb{I}^l$ with $l \geq 0$, the tuple M admits the reflection sequence κ and that $\mathcal{B}(R_{i_l} \cdots R_{i_1}(M))$ and $\mathcal{B}(M)$ have the same dimension. This implies the claim. $\square$

An important fact relating reflections of tuples in $\mathcal{F}_\theta^H$ to reflections of Nichols systems is the following.

PROPOSITION 13.6.5. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Let $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id}_M)$.*

(1) *Let $l \in \mathbb{N}_0$ and $i_1, \dots, i_l \in \mathbb{I}$. Then M admits the reflection sequence $(i_1, \dots, i_l)$ if and only if $\mathcal{N}_0$ does.*
(2) *M admits all reflections if and only if $\mathcal{N}_0$ does.*

PROOF. (1) Let $\mathcal{N}$ be a pre-Nichols system of M such that $p^{\mathcal{N}}$ is an isomorphism. We prove by induction on l that M admits the reflection sequence $(i_1, \dots, i_l)$ if and only if $\mathcal{N}$ does. Since $p^{\mathcal{N}_0} = \mathrm{id}$ is an isomorphism, this proves the proposition.

For $l = 0$ the claim is trivial. Assume now that $l \geq 1$. By Lemma 13.5.16, $\mathcal{N}$ is a Nichols system of (M, i_1), $R_i(\mathcal{N})$ is a pre-Nichols system of $R_i(M)$, and $p^{R_i(\mathcal{N})}$ is an isomorphism. Hence the claim follows from the definitions and the induction hypothesis.

(2) follows from (1). $\square$

13.7. Notes

13.2. In the discussion of projections of Nichols algebras we follow [**AHS10**, Section 3], where Theorem 13.2.7 is shown. Theorem 13.2.8 is a result from [**HS13b**].

13.3. The computation of the adjoint action in Theorem 13.3.1 first appeared in [**HS10b**, Proposition 6.5].

13.4. Theorem 13.4.9 was shown in [**HS13b**, Theorem 8.9]. The existence of the algebra isomorphism $\widetilde{\Theta} : \mathcal{B}(R_i(M)) \xrightarrow{\cong} K_i^{\mathcal{B}(M)} \# \mathcal{B}(M_i^*)$ in Corollary 13.4.10 (without the Hopf algebra structure on the right-hand side) was one of the main results in [**AHS10**]. The algebra structure of the smash product was defined by quantum differential operators or as a subalgebra of a Heisenberg double. The somewhat lengthy proof of the isomorphism in [**AHS10**, Theorem 3.12] used families of braided derivations. Our categorical proof of the existence of the Hopf algebra isomorphism $\widetilde{\Theta}$ is completely different.

13.5. The notion of Nichols systems and their reflections is new.

13.6. Theorem 13.6.2 was first shown in [**AHS10**].

CHAPTER 14

Right coideal subalgebras of Nichols systems, and Cartan graph of Nichols algebras

We use the theory of reflections from the previous Chapter to study graded right coideal subalgebras of Nichols systems in the category of Yetter-Drinfeld modules over Hopf algebras with bijective antipode. In the basic Theorem 14.1.9 we construct right coideal subalgebras of pre-Nichols systems stepwise starting from an $[M]$-reduced representation of a morphism in the semi-Cartan graph $\mathcal{G}(M)$. Having introduced the correct notions, at this point the proof follows easily by induction. In Section 14.2 we introduce exact factorizations of bialgebras and of Nichols systems. As applications, among others we prove that a semi-Cartan graph of a Nichols system is a Cartan graph, and provide a structural result on commutation relations and a criterion for the finiteness of the Nichols algebra of a semi-simple Yetter-Drinfeld module. In the finite case a PBW type decomposition is given.

Throughout, let H be a Hopf algebra with bijective antipode.

14.1. Right coideal subalgebras of Nichols systems

We specialize the bijective correspondence of Theorem 12.4.5 to graded right coideal subalgebras of Nichols systems.

LEMMA 14.1.1. *Let $M \in {}^H_H\mathcal{YD}$ be an irreducible object. Then $\Bbbk 1$ and $\mathcal{B}(M)$ are the only right or left coideal subalgebras of the Nichols algebra $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$.*

PROOF. Recall from Theorem 7.1.2 that $\mathcal{B}(M)$ is a strictly graded coalgebra. Let $\Bbbk 1 \neq E \subseteq \mathcal{B}(M)$ be a right or left coideal subalgebra in ${}^H_H\mathcal{YD}$. Then $0 \neq E \cap M$ by Corollary 1.3.11(3). Since M is an irreducible object in ${}^H_H\mathcal{YD}$, it follows that $M \subseteq E$, hence $\mathcal{B}(M) = E$. □

Let $\theta \geq 1$ and $\mathbb{I} = \{1, \dots, \theta\}$. In what follows we will heavily use the notation introduced in Definitions 13.5.9 and 13.5.20.

LEMMA 14.1.2. *Let $M \in \mathcal{F}^H_\theta$, $i \in \mathbb{I}$, and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system of (M, i). Assume that M_i is irreducible. Let $E \subseteq S$ be a right coideal subalgebra in ${}^H_H\mathcal{YD}$. Consider the following conditions.*

(1) *$E \subseteq L^{\mathcal{N}}_i$.*
(2) *$N_i \not\subseteq E$.*

Then (1) implies (2). If E is an $\mathbb{N}^\theta_0$-graded subspace of S, then (2) implies (1).

PROOF. Assume first that (1) holds and that $N_i \subseteq E$. Then $N_i \subseteq L^{\mathcal{N}}_i$ by (1). Since $\pi^{\mathcal{N}}_i | L^{\mathcal{N}}_i = \varepsilon | L^{\mathcal{N}}_i$, we obtain that $N_i = 0$, which is excluded, since $N_i \cong M_i$ is irreducible.

Assume now that (2) holds. Then $N_i \cap E = 0$ by the irreducibility of M_i. We prove (1). Since E is a graded subspace of the $\mathbb{N}^\theta_0$-graded Hopf algebra S, and since the projection $\pi_i^{\mathcal{N}} : S \to \mathcal{B}(M_i)$ is graded, (2) implies that the homogeneous part of $\pi_i^{\mathcal{N}}(E)$ of degree α_i is zero. Hence $M_i \not\subseteq \pi_i^{\mathcal{N}}(E)$. Since $\pi_i^{\mathcal{N}}(E)$ is a right coideal subalgebra of $\mathcal{B}(M_i)$, $\pi_i^{\mathcal{N}}(E) = \Bbbk 1$ by Lemma 14.1.1. Thus $E \subseteq L_i^{\mathcal{N}}$ by Lemma 2.5.6(2). $\square$

Definition 14.1.3. For any $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and for any Nichols system $\mathcal{N} = \mathcal{N}(S, N, f)$ of (M, i) we define

$$\begin{aligned}
\mathcal{K}(\mathcal{N}) &= \{E \mid E \subseteq S \ \mathbb{N}_0^\theta\text{-graded right coideal subalgebra in } {}^H_H\mathcal{YD}\},\\
\mathcal{K}_i^+(\mathcal{N}) &= \{E \mid E \in \mathcal{K}(\mathcal{N}),\, \mathcal{N}_i \subseteq E\},\\
\mathcal{K}_i^-(\mathcal{N}) &= \{E \mid E \in \mathcal{K}(\mathcal{N}),\, \mathcal{N}_i \not\subseteq E\}.
\end{aligned}$$

Theorem 14.1.4. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N}$ be a Nichols system of (M, i). Assume that M is i-finite, and that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$.*

(1) *The map*
$$t_i^{\mathcal{N}} : \mathcal{K}_i^-(R_i(\mathcal{N})) \to \mathcal{K}_i^+(\mathcal{N}), \quad E \mapsto T_i^{\mathcal{N}}(E)\Bbbk[\mathcal{N}_i],$$
is bijective with inverse given by $E \mapsto (T_i^{\mathcal{N}})^{-1}(E \cap K_i^{\mathcal{N}})$.

(2) *The multiplication map $T_i^{\mathcal{N}}(E) \otimes \Bbbk[\mathcal{N}_i] \to T_i^{\mathcal{N}}(E)\Bbbk[\mathcal{N}_i]$ is bijective for all $E \in \mathcal{K}_i^-(R_i(\mathcal{N}))$.*

Proof. Let $\mathcal{N} = \mathcal{N}(S, N, f)$, $R_i(\mathcal{N}) = \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f})$, and $K = K_i^{\mathcal{N}}$. In order to apply Theorem 12.4.5, let $\langle\ ,\ \rangle$ be the canonical pairing with $A = \mathcal{B}(M_i^*)$, $B = \mathcal{B}(M_i)$. Then

$$K\#B \cong S$$

by multiplication and the isomorphism $\mathcal{B}(M_i) \cong \Bbbk[N_i]$, and

$$\Omega_{M_i}(K)\#A = \widetilde{S}.$$

By Theorem 12.4.5, the map $\mathcal{E}_r(\widetilde{S}, L_i^{R_i(\mathcal{N})}) \to \mathcal{E}_r^+(S)$, $E \mapsto T_i^{\mathcal{N}}(E)\Bbbk[N_i]$, is bijective with inverse $E \mapsto (T_i^{\mathcal{N}})^{-1}(E \cap K_i^{\mathcal{N}})$. By Corollary 13.5.21(2), this bijection can be restricted to the $\mathbb{N}_0^\theta$-graded subalgebras. Hence the theorem follows from Lemma 14.1.2. (2) holds by Theorem 12.4.5. $\square$

Lemma 14.1.5. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $p : \mathcal{N} \to \mathcal{N}'$ be a morphism of Nichols systems of (M, i). Assume that M is i-finite, and that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Let $E \in \mathcal{K}_i^-(R_i(\mathcal{N}))$ and $E' \in \mathcal{K}_i^-(R_i(\mathcal{N}'))$, and assume that $R_i(p)$ induces an isomorphism $E \to E'$. Then p induces an isomorphism*

$$t_i^{\mathcal{N}}(E) \to t_i^{\mathcal{N}'}(E').$$

Proof. By Lemma 14.1.2, $E \subseteq L_i^{R_i(\mathcal{N})}$ and $E' \subseteq L_i^{R_i(\mathcal{N}')}$. Hence the commutative diagram in Lemma 13.5.27(2) induces a commutative diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & T_i^{\mathcal{N}}(E) \arrow[d] \\
E' \arrow[r] & T_i^{\mathcal{N}'}(E')
\end{tikzcd},$$

where the horizontal maps are isomorphisms induced by $T_i^{\mathcal{N}}$ and $T_i^{\mathcal{N}'}$, and the vertical maps are induced by $R_i(p)$ and p. Since the left vertical map is an isomorphism by assumption, p induces an isomorphism

$$T_i^{\mathcal{N}}(E) \to T_i^{\mathcal{N}'}(E').$$

Moreover, p induces an isomorphism $\Bbbk[\mathcal{N}_i] \to \Bbbk[\mathcal{N}_i']$ by the choice of p. Hence the claim follows from Theorem 14.1.4(2). $\square$

REMARK 14.1.6. Let $M \in \mathcal{F}_\theta^H$ such that M_i is irreducible in ${}_H^H\mathcal{YD}$ for all $i \in \mathbb{I}$, and let $\mathcal{N}$ be a Nichols system of M. Let $i_1, \dots, i_l \in \mathbb{I}$, $l \geq 1$. Assume that $\mathcal{N}$ admits the reflection sequence $(i_1, \dots, i_l)$. Let $w = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_l}$,

$$[R_{i_l} \cdots R_{i_1}(M)] \xrightarrow{s_{i_l}} [R_{i_{l-1}} \cdots R_{i_1}(M)] \cdots \xrightarrow{s_{i_2}} [R_{i_1}(M)] \xrightarrow{s_{i_1}} [M]$$

be a morphism in the Weyl groupoid of M.

By abuse of notation, for all $1 \leq k \leq l$ let $T_{i_k} = T_{i_k}^{R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})}$. Recall the definition of

$$L_{i_k}^{R_{i_k} \cdots R_{i_1}(\mathcal{N})}, \quad K_{i_k}^{R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})}$$

from Definition 13.5.9. We denote the isomorphism

$$L_{i_k}^{R_{i_k} \cdots R_{i_1}(\mathcal{N})} \xrightarrow{T_{i_k}} K_{i_k}^{R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})} \qquad \text{by}$$

$$R_{i_k} \cdots R_{i_1}(\mathcal{N}) \overset{T_{i_k}}{\rightsquigarrow} R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N}).$$

Let $t_{i_k} = t_{i_k}^{R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})}$. By definition

$$\mathcal{K}_{i_k}^-(R_{i_k} \cdots R_{i_1}(\mathcal{N})) \subseteq \mathcal{K}(R_{i_k} \cdots R_{i_1}(\mathcal{N})),$$
$$K_{i_k}^+(R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})) \subseteq \mathcal{K}(R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N}))$$

are subsets. We denote the bijective map

$$\mathcal{K}_{i_k}^-(R_{i_k} \cdots R_{i_1}(\mathcal{N})) \xrightarrow{t_{i_k}} \mathcal{K}_{i_k}^+(R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})) \qquad \text{by}$$

$$\mathcal{K}(R_{i_k} \cdots R_{i_1}(\mathcal{N})) \overset{t_{i_k}}{\rightsquigarrow} \mathcal{K}(R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})).$$

Thus $\overset{T_{i_k}}{\rightsquigarrow}$ and $\overset{t_{i_k}}{\rightsquigarrow}$ are "partially defined maps", and we can look at their composition (where it is defined).

$$R_{i_l} \cdots R_{i_1}(\mathcal{N}) \overset{T_{i_l}}{\rightsquigarrow} R_{i_{l-1}} \cdots R_{i_1}(\mathcal{N}) \cdots \overset{T_{i_2}}{\rightsquigarrow} R_{i_1}(\mathcal{N}) \overset{T_{i_1}}{\rightsquigarrow} \mathcal{N}$$

$$\mathcal{K}(R_{i_l} \cdots R_{i_1}(\mathcal{N})) \overset{t_{i_l}}{\rightsquigarrow} \mathcal{K}(R_{i_{l-1}} \cdots R_{i_1}(\mathcal{N})) \cdots \overset{t_{i_2}}{\rightsquigarrow} \mathcal{K}(R_{i_1}(\mathcal{N})) \overset{t_{i_1}}{\rightsquigarrow} \mathcal{K}(\mathcal{N})$$

Let $\beta_k = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Recall from Section 13.5 that $R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})_{i_k}$ is the direct summand of $R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})$ of degree α_{i_k}. Under suitable assumptions we will show in the next theorem that

$$\mathcal{N}_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(R_{i_{k-1}} \cdots R_{i_1}(\mathcal{N})_{i_k}) \quad \text{and}$$
$$E^{\mathcal{N}}_{(i_1,\dots,i_l)} = t_{i_1} \cdots t_{i_l}(\mathbb{k})$$

are well-defined. Here, $\mathbb{k}$ is the trivial object in $\mathcal{K}(R_{i_l} \cdots R_{i_1}(\mathcal{N}))$.

The irreducible Yetter-Drinfeld modules $\mathcal{N}_{\beta_k}$ correspond to the higher root vectors in quantum groups, and the right coideal subalgebra $E^{\mathcal{N}}_{(i_1,\dots,i_l)}$ of the pre-Nichols algebra (that is, the first entry) of the Nichols system $\mathcal{N}$ is decomposed into the tensor product of the Nichols algebras of the $\mathcal{N}_{\beta_k}$.

In the next definition we describe this construction in a more formal way.

Definition 14.1.7. Let $M \in \mathcal{F}^H_\theta$ such that M_i is irreducible in ${}^H_H\mathcal{YD}$ for all $i \in \mathbb{I}$. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $l \in \mathbb{N}_0$ and let $i_1, \dots, i_l \in \mathbb{I}$. Assume that $\mathcal{N}$ admits the reflection sequence $(i_1, \dots, i_l)$. Let

$$R_{()}(\mathcal{N}) = \mathcal{N}, \quad L^{\mathcal{N}}_{()} = S, \quad T^{\mathcal{N}}_{()} = \mathrm{id}_S, \quad \mathcal{K}^-_{()}(\mathcal{N}) = \mathcal{K}(\mathcal{N}), \quad t^{\mathcal{N}}_{()} = \mathrm{id}_{\mathcal{K}(\mathcal{N})},$$

and for any $1 \le k \le l$ define inductively

$$\begin{aligned} R_{(i_1,\dots,i_k)}(\mathcal{N}) &= R_{i_k}(\cdots R_{i_1}(\mathcal{N})), \\ L^{\mathcal{N}}_{(i_1,\dots,i_k)} &= \left(T^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})}_{i_k}\right)^{-1}\left(K^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})}_{i_k} \cap L^{\mathcal{N}}_{(i_1,\dots,i_{k-1})}\right), \\ T^{\mathcal{N}}_{(i_1,\dots,i_k)} &= T^{\mathcal{N}}_{i_1} T^{R_{i_1}(\mathcal{N})}_{i_2} \cdots T^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})}_{i_k} : L^{\mathcal{N}}_{(i_1,\dots,i_k)} \to S \end{aligned}$$

and

$$\begin{aligned} &\mathcal{K}^-_{(i_1,\dots,i_k)}(R_{(i_1,\dots,i_k)}(\mathcal{N})) = \\ &\left(t^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})}_{i_k}\right)^{-1}\left(\mathcal{K}^+_{i_k}(R_{(i_1,\dots,i_{k-1})}(\mathcal{N})) \cap \mathcal{K}^-_{(i_1,\dots,i_{k-1})}(R_{(i_1,\dots,i_{k-1})}(\mathcal{N}))\right), \\ &t^{\mathcal{N}}_{(i_1,\dots,i_k)} = t^{\mathcal{N}}_{i_1} \cdots t^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})}_{i_k} : \mathcal{K}^-_{(i_1,\dots,i_k)}(R_{(i_1,\dots,i_k)}(\mathcal{N})) \to \mathcal{K}(\mathcal{N}). \end{aligned}$$

Remark 14.1.8. In Definition 14.1.7, both the objects $L^{\mathcal{N}}_{(i_1,\dots,i_k)}$ and the sets $\mathcal{K}^-_{(i_1,\dots,i_k)}(R_{(i_1,\dots,i_k)}(\mathcal{N}))$ are largest with respect to inclusion such that $T^{\mathcal{N}}_{(i_1,\dots,i_k)}$ and $t^{\mathcal{N}}_{(i_1,\dots,i_k)}$, respectively, are well-defined maps. Moreover, for $k = 1 \le l$ the definitions yield that

$$R_{(i_1)}(\mathcal{N}) = R_{i_1}(\mathcal{N}), \quad L^{\mathcal{N}}_{(i_1)} = L^{\mathcal{N}}_{i_1}, \quad T^{\mathcal{N}}_{(i_1)} = T^{\mathcal{N}}_{i_1},$$

and that

$$\mathcal{K}^-_{(i_1)}(R_{(i_1)}(\mathcal{N})) = \mathcal{K}^-_{i_1}(R_{i_1}(\mathcal{N})), \quad t^{\mathcal{N}}_{(i_1)} = t^{\mathcal{N}}_{i_1}.$$

Recall the definitions of an $[M]$-reduced sequence from Definition 9.2.1 and of the semi-Cartan graph of M from Definition 13.6.3.

Theorem 14.1.9. *Let $M \in \mathcal{F}^H_\theta$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $l \ge 1$ and $i_1, \dots, i_l \in \mathbb{I}$. Assume that $(i_1, \dots, i_l)$ is $[M]$-reduced in the semi-Cartan graph $\mathcal{G}(M)$ and that $\mathcal{N}$ admits the reflection sequence $(i_1, \dots, i_l)$. For any $1 \le k \le l$, let $\beta_k = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$.*

(1) *$\beta_1, \dots, \beta_l$ are pairwise distinct non-zero elements of $\mathbb{N}_0^\theta$.*

(2) *For any* $1 \le k \le l$, $R_{(i_1,\dots,i_{k-1})}(\mathcal{N})_{i_k} \subseteq L^{\mathcal{N}}_{(i_1,\dots,i_{k-1})}$. *Let*

$$N_{\beta_k} = N_k^{\mathcal{N}}(i_1,\dots,i_l) = T^{\mathcal{N}}_{(i_1,\dots,i_{k-1})}(R_{(i_1,\dots,i_{k-1})}(\mathcal{N})_{i_k}).$$

(3) $\Bbbk 1 \in \mathcal{K}^-_{(i_1,\dots,i_l)}(R_{(i_1,\dots,i_l)}(\mathcal{N}))$. *Let* $E^{\mathcal{N}}(i_1,\dots,i_l) = t^{\mathcal{N}}_{(i_1,\dots,i_l)}(\Bbbk 1)$.

(4) *For any* $1 \le k \le l$, $N_{\beta_k} \subseteq E^{\mathcal{N}}(i_1,\dots,i_l)$ *is a finite-dimensional irreducible subobject in* ${}^H_H\mathcal{YD}$ *of degree* β_k.

(5) *For any* $1 \le k \le l$, *the identity on* N_{β_k} *induces a graded isomorphism* $\mathcal{B}(N_{\beta_k}) \cong \Bbbk[N_{\beta_k}] \subseteq S$ *of* $\mathbb{N}_0^\theta$*-graded algebras in* ${}^H_H\mathcal{YD}$.

(6) *The multiplication map* $\Bbbk[N_{\beta_l}] \otimes \cdots \otimes \Bbbk[N_{\beta_1}] \to E^{\mathcal{N}}(i_1,\dots,i_l)$ *is an isomorphism of* $\mathbb{N}_0^\theta$*-graded objects in* ${}^H_H\mathcal{YD}$.

(7) *Let* $E^{\mathcal{B}(M)}(i_1,\dots,i_l) = E^{\mathcal{N}_0}(i_1,\dots,i_l)$ *with* $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$. *The canonical map* $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ *in* ${}^H_H\mathcal{YD}$ *induces an isomorphism*

$$E^{\mathcal{N}}(i_1,\dots,i_l) \to E^{\mathcal{B}(M)}(i_1,\dots,i_l).$$

PROOF. By Definition 9.2.1, $\Lambda^{[M]}(i_1,\dots,i_l) = \{\beta_1,\dots,\beta_l\}$.
We proceed by induction on l, where (7) is replaced by

(7') Let $p : \mathcal{N} \to \mathcal{N}' = \mathcal{N}(S', N', f')$ be a morphism of pre-Nichols systems of M. Then $\mathcal{N}'$ admits the reflection sequence $(i_1,\dots,i_l)$, and p induces an isomorphism $E^{\mathcal{N}}(i_1,\dots,i_l) \to E^{\mathcal{N}'}(i_1,\dots,i_l)$ in ${}^H_H\mathcal{YD}$.

Since $\mathcal{N}$ admits the reflection sequence $(i_1,\dots,i_l)$, Remark 13.5.23(3) implies that $\mathcal{N}'$ admits the reflection sequence $(i_1,\dots,i_l)$. Hence (7') is equivalent to (7) by (13.5.4).

Let $l = 1$. Then $\beta_1 = \alpha_{i_1}$, $N_{\beta_1} = N_{i_1}$, and $E^{\mathcal{N}}(i_1) = \Bbbk[N_{i_1}]$. Hence (1)–(6) and (7') are obvious.

Let $l > 1$, and assume that $(i_1,\dots,i_l)$ is $[M]$-reduced. Then $(i_2,\dots,i_l)$ is $[R_{i_1}(M)]$-reduced in $\mathcal{G}(R_{i_1}(M))$ by Lemma 9.2.2. To prove the theorem for the reflection sequence $(i_1,\dots,i_l)$, we may assume by induction that the theorem holds for the pre-Nichols system $R_{i_1}(\mathcal{N}) = \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f})$ of $R_{i_1}(M)$, for the reflection sequence $(i_2,\dots,i_l)$, and (regarding the proof of (7')) for the morphism

$$R_{i_1}(p) : R_{i_1}(\mathcal{N}) \to R_{i_1}(\mathcal{N}') = \mathcal{N}(\widetilde{S'}, \widetilde{N'}, \widetilde{f'})$$

of pre-Nichols systems of $R_{i_1}(M)$. Explicitly, for any $2 \le k \le l$, we define the roots $\gamma_k = \mathrm{id}_{[R_{i_1}(M)]} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Then the following are assumed.

(a) $\gamma_2,\dots,\gamma_l$ are pairwise distinct non-zero elements of $\mathbb{N}_0^\theta$.

(b) For any $2 \le k \le l$, $R_{(i_1,\dots,i_{k-1})}(\mathcal{N})_{i_k} \subseteq L^{R_{i_1}(\mathcal{N})}_{(i_2,\dots,i_{k-1})}$. Let

$$\widetilde{N}_{\gamma_k} = T^{R_{i_1}(\mathcal{N})}_{(i_2,\dots,i_{k-1})}(R_{(i_1,\dots,i_{k-1})}(\mathcal{N})_{i_k}).$$

(c) $\Bbbk 1 \in \mathcal{K}^-_{(i_2,\dots,i_l)}(R_{(i_1,\dots,i_l)}(\mathcal{N}))$. Let $E^{R_{i_1}(\mathcal{N})}(i_2,\dots,i_l) = t^{R_{i_1}(\mathcal{N})}_{(i_2,\dots,i_l)}(\Bbbk 1)$.

(d) For any $2 \le k \le l$, $\widetilde{N}_{\gamma_k} \subseteq E^{R_{i_1}(\mathcal{N})}(i_2,\dots,i_l)$ is a finite-dimensional irreducible subobject in ${}^H_H\mathcal{YD}$ of degree γ_k.

(e) For any $2 \le k \le l$, the identity on $\widetilde{N}_{\gamma_k}$ induces a graded isomorphism $\mathcal{B}(\widetilde{N}_{\gamma_k}) \cong \Bbbk[\widetilde{N}_{\gamma_k}] \subseteq \widetilde{S}$ of $\mathbb{N}_0^\theta$-graded algebras in ${}^H_H\mathcal{YD}$.

(f) The multiplication map $\Bbbk[\widetilde{N}_{\gamma_l}] \otimes \cdots \otimes \Bbbk[\widetilde{N}_{\gamma_2}] \to E^{R_{i_1}(\mathcal{N})}(i_2,\dots,i_l)$ is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$.

(g) The morphism $R_{i_1}(p) : R_{i_1}(\mathcal{N}) \to R_{i_1}(\mathcal{N}')$ in ${}_H^H\mathcal{YD}$ induces an isomorphism $E^{R_{i_1}(\mathcal{N})}(i_2, \dots, i_l) \to E^{R_{i_1}(\mathcal{N}')}(i_2, \dots, i_l)$.

Since $(i_1, \dots, i_l)$ is $[M]$-reduced, $\alpha_{i_1} \neq \gamma_k$ for any $2 \le k \le l$ by Definition 9.2.1. By (b) and by Corollary 13.5.21(2), $\widetilde{N}_{\gamma_k}$ has degree γ_k for all $2 \le k \le l$. Thus

$$\widetilde{N}_{i_1} \not\subseteq E^{R_{i_1}(\mathcal{N})}(i_2, \dots, i_l) \tag{14.1.1}$$

by (a) and (f), and hence

$$E^{R_{i_1}(\mathcal{N})}(i_2, \dots, i_l) \in \mathcal{K}_{i_1}^-(R_{i_1}(\mathcal{N})) \tag{14.1.2}$$

by (c). This and Remark 14.1.8 imply (3). Moreover,

$$E^{R_{i_1}(\mathcal{N})}(i_2, \dots, i_l) \subseteq L_{i_1}^{R_{i_1}(\mathcal{N})} \tag{14.1.3}$$

by (14.1.1) and by Lemma 14.1.2. Hence, by (d),

$$\widetilde{N}_{\gamma_k} \subseteq E^{R_{i_1}(\mathcal{N})}(i_2, \dots, i_l) \subseteq L_{i_1}^{R_{i_1}(\mathcal{N})}$$

for any $2 \le k \le l$. This proves (2) by Remark 14.1.8 and that $T_{i_1}^{\mathcal{N}}(\widetilde{N}_{\gamma_k}) \subseteq K_{i_1}^{\mathcal{N}}$ for any $2 \le k \le l$. Therefore $\beta_k = s_{i_1}^{R_{i_1}(M)}(\gamma_k) \in \mathbb{N}_0^\theta$ for any $2 \le k \le l$, and (1) follows. Further, we obtain from Theorem 14.1.4 that the multiplication map

$$T_{i_1}^{\mathcal{N}}\big(E^{R_{i_1}(\mathcal{N})}(i_2, \dots, i_l)\big) \otimes \Bbbk[N_{i_1}] \to E^{\mathcal{N}}(i_1, \dots, i_l) \tag{14.1.4}$$

is bijective. Since $T_{i_1}^{\mathcal{N}} : L_{i_1}^{R_{i_1}(\mathcal{N})} \to K_{i_1}^{\mathcal{N}}$ is an algebra isomorphism, we obtain from (f) that the multiplication map

$$\Bbbk[T_{i_1}^{\mathcal{N}}(\widetilde{N}_{\gamma_l})] \otimes \cdots \otimes \Bbbk[T_{i_1}^{\mathcal{N}}(\widetilde{N}_{\gamma_2})] \to T_{i_1}^{\mathcal{N}}\big(E^{R_{i_1}(\mathcal{N})}(i_2, \dots, i_l)\big) \tag{14.1.5}$$

is bijective.

Since $T_{i_1}^{\mathcal{N}}$ is an isomorphism in ${}_H^H\mathcal{YD}$, $N_{\beta_k} = T_{i_1}^{\mathcal{N}}(\widetilde{N}_{\gamma_k})$ is irreducible in ${}_H^H\mathcal{YD}$ by (d) for any $2 \le k \le l$. We saw already that the degree of $N_{\beta_k} = T_{i_1}^{\mathcal{N}}(\widetilde{N}_{\gamma_k})$ is β_k. This proves (4).

Since $\mathcal{N}$ is a Nichols system of (M, i_1), the identity on N_{i_1} induces an isomorphism $\mathcal{B}(N_{i_1}) \cong \Bbbk[N_{i_1}]$. Since $T_{i_1}^{\mathcal{N}}$ is an algebra isomorphism in ${}_H^H\mathcal{YD}$ mapping $\widetilde{N}_{\gamma_k}$ with any $2 \le k \le l$ onto N_{β_k}, the following chain of algebra isomorphisms in ${}_H^H\mathcal{YD}$ proves (5).

$$\mathcal{B}(N_{\beta_k}) \cong \mathcal{B}(\widetilde{N}_{\gamma_k}) \cong \Bbbk[\widetilde{N}_{\gamma_k}] \cong \Bbbk[N_{\beta_k}].$$

Here, the second isomorphism is given in (e), and the first and third isomorphism are induced by the isomorphism $N_{\beta_k} = T_{i_1}^{\mathcal{N}}(\widetilde{N}_{\gamma_k}) \cong \widetilde{N}_{\gamma_k}$ of objects in ${}_H^H\mathcal{YD}$.

Claim (6) follows from the bijectivity of the maps in (14.1.4) and (14.1.5). Finally, (7') follows from (g) and Lemma 14.1.5 in view of (14.1.2). $\square$

In Theorem 14.1.9, the notation $N_{\beta_k} = N_k^{\mathcal{N}}(i_1, \dots, i_l)$ is somewhat misleading, since N_{β_k} does depend on the $[M]$-reduced sequence $(i_1, \dots, i_l)$. We follow here the convention for the higher root vectors in quantum groups.

REMARK 14.1.10. Under the assumptions of Theorem 14.1.9, the following inductive properties are clear from Theorem 14.1.9 and its proof.

$$E^{\mathcal{N}}(i_1, \dots, i_k) = \Bbbk[N_{\beta_k}] \cdots \Bbbk[N_{\beta_1}] \subseteq E^{\mathcal{N}}(i_1, \dots, i_l) \tag{14.1.6}$$

for any $1 \le k \le l$,

$$E^{\mathcal{N}}(i_1,\dots,i_l) = T_{i_1}^{\mathcal{N}}\big(E^{R_{i_1}(\mathcal{N})}(i_2,\dots,i_l)\big)\Bbbk[N_{\beta_1}], \tag{14.1.7}$$

$$\Bbbk[N_{\beta_l}]\cdots\Bbbk[N_{\beta_2}] = E^{\mathcal{N}}(i_1,\dots,i_l) \cap K_{i_1}^{\mathcal{N}}. \tag{14.1.8}$$

Recall the definition of $\mathcal{F}_\theta^H(M)$ from Definition 13.6.1(5).

Corollary 14.1.11. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Then for any $P \in \mathcal{F}_\theta^H(M)$ and any $[P]$-reduced sequence κ, $\Lambda^{[P]}(\kappa) \subseteq \mathbb{N}_0^{\mathbb{I}}$.*

Proof. By Theorem 13.6.2, $\mathcal{G}(M) = \mathcal{G}(\mathbb{I},\mathcal{X},r,A)$ is a semi-Cartan graph. Let $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$. Then $\mathcal{N}_0$ admits all reflections by Proposition 13.6.5. Let $X \in \mathcal{X}$. Then, by definition, there exist $k \ge 0$ and $j_1,\dots,j_k \in \mathbb{I}$ such that $X = [P]$, where $P = R_{j_k}\cdots R_{j_1}(M)$. Clearly, P admits all reflections. Moreover, the pre-Nichols system $\mathcal{N}_{[P]} = R_{j_k}\cdots R_{j_1}(\mathcal{N}_0)$ of P is isomorphic to $\mathcal{N}(\mathcal{B}(P), P, \mathrm{id}_P)$ via the canonical map because of Lemma 13.5.16.

Let now κ be a $[P]$-reduced sequence. Then $\Lambda^{[P]}(\kappa) \subseteq \mathbb{N}_0^{\mathbb{I}}$ by Theorem 14.1.9(1) applied to $\mathcal{N}_{[P]}$. □

Theorem 14.1.12. *Under the assumptions of Theorem 14.1.9, the following commutation rules hold. For any $1 \le p < q \le l$, $x \in N_{\beta_p}$, $y \in N_{\beta_q}$,*

$$xy - (x_{(-1)}\cdot y)x_{(0)} \in \Bbbk[N_{\beta_{q-1}}]\Bbbk[N_{\beta_{q-2}}]\cdots\Bbbk[N_{\beta_{p+1}}].$$

Proof. Let $\mathcal{N}(\widetilde{S},\widetilde{N},\widetilde{f}) = R_{i_1}(\mathcal{N})$. Then $N_{\beta_k} = T_{i_1}^{\mathcal{N}}(\widetilde{N}_{\gamma_k})$ in (14.1.8) for any $2 \le k \le l$, where $\gamma_k = \mathrm{id}_{[R_{i_1}(N)]}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k})$. Since $T_{i_1}^{\mathcal{N}}$ is an algebra isomorphism in ${}_H^H\mathcal{YD}$, using (14.1.7) and induction on l we may assume that $p = 1$. By (14.1.6), it is enough to consider the case $q = l$. Then by (14.1.8),

$$\begin{aligned} xy - (x_{(-1)}\cdot y)x_{(0)} = (\mathrm{ad}_S x)(y) &\in E^{\mathcal{N}}(i_1,\dots,i_l)\cap K_{i_1}^{\mathcal{N}} \\ &= \Bbbk[N_{\beta_l}]\Bbbk[N_{\beta_{l-1}}]\cdots\Bbbk[N_{\beta_2}], \end{aligned}$$

since $y \in K_{i_1}^{\mathcal{N}}$ and $x \in N_{i_1}$. Hence there are integers $a_j \in \mathbb{N}_0$ for all $2 \le j \le l$ such that $\deg((\mathrm{ad}_S x)(y)) = \beta_1 + \beta_l = \sum_{j=2}^l a_j\beta_j$. Assume that $a_l \ge 1$. Then $\alpha_{i_1} = \beta_1 \in \sum_{j=2}^l \mathbb{N}_0\beta_j$, which is not possible, since $\mathbb{N}_0^\theta \ni \beta_j \ne \alpha_{i_1}$ for all $2 \le j \le l$. Hence $a_l = 0$, which proves the claim. □

Corollary 14.1.13. *Under the assumptions of Theorem 14.1.9, for any permutation σ of $\{1,\dots,l\}$, the multiplication map*

$$\Bbbk[N_{\beta_{\sigma(l)}}]\otimes\cdots\otimes\Bbbk[N_{\beta_{\sigma(1)}}] \to E^{\mathcal{N}}(i_1,\dots,i_l)$$

is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$.

Proof. Let $h : \mathbb{N}_0^\theta \to \mathbb{N}_0$ be an additive map such that $h(\beta) > 0$ for any $\beta \ne 0$. Let

$$\bar{h} : \mathbb{N}_0^l \to \mathbb{N}_0, \quad \bar{h}(k_1,\dots,k_l) = \sum_{i=1}^l k_i h(\beta_i).$$

Let $\Gamma = \mathbb{N}_0^l$ together with the weighted lexicographic ordering $<$:

$$(k_1,\dots,k_l) < (m_1,\dots,m_l) \Leftrightarrow \bar{h}(k_1,\dots,k_l) < \bar{h}(m_1,\dots,m_l) \text{ or}$$
$$\bar{h}(k_1,\dots,k_l) = \bar{h}(m_1,\dots,m_l),\ k_1 = m_1,\dots,$$
$$k_{i-1} = m_{i-1},\ k_i < m_i \text{ for some } 1 \le i \le l.$$

Then Γ is a totally ordered abelian monoid satisfying axioms (M1) and (M2) in Section 5.2.

We introduce a filtration $\mathcal{F}$ of $E^{\mathcal{N}}(i_1,\dots,i_l)$ by Γ. For any $\alpha \in \Gamma$, let us define $F_\alpha(E^{\mathcal{N}}(i_1,\dots,i_l))$ to be the sum of all subspaces $N_{j_1}\cdots N_{j_m}$ with $m \in \mathbb{N}_0$ and $j_1,\dots,j_m \in \{1,\dots,l\}$, such that $(n_1,\dots,n_m) \le \alpha$, where for any $1 \le k \le l$ the number n_k counts the appearances of k in $(j_1,\dots,j_m)$.

Theorem 14.1.12 implies that in the graded algebra associated to the filtration $\mathcal{F}(E^{\mathcal{N}}(i_1,\dots,i_l))$ the relations

$$xy = (x_{(-1)} \cdot y)x_{(0)} \tag{14.1.9}$$

hold for any $1 \le i < j \le l$ and any $x \in N_{\beta_i}$, $y \in N_{\beta_j}$. Then the surjectivity of the multiplication map in the Corollary follows from (14.1.9), the invertibility of the braidings $c_{N_{\beta_i},N_{\beta_j}}$ for all $1 \le i < j \le l$, and from the surjectivity of the map in Theorem 14.1.9(6). Finally, the injectivity follows from surjectivity and from the bijectivity of the map in Theorem 14.1.9(6), since for any $\beta \in \mathbb{N}_0^\theta$, the dimension of the homogeneous component of $\Bbbk[N_{\beta_{\sigma(l)}}] \otimes \cdots \otimes \Bbbk[N_{\beta_{\sigma(1)}}]$ of degree β does not depend on σ. □

COROLLARY 14.1.14. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Let $l \ge 1$ and $i_1,\dots,i_l \in \mathbb{I}$. Assume that M admits all reflections and that $\kappa = (i_1,\dots,i_l)$ is $[M]$-reduced in the semi-Cartan graph $\mathcal{G}(M)$. If $\alpha_i \in \Lambda^{[M]}(\kappa)$ for all $i \in \mathbb{I}$, then the following hold.*

(1) *$E^{\mathcal{B}(M)}(i_1,\dots,i_l) = \mathcal{B}(M)$.*
(2) *For any pre-Nichols system $\mathcal{N} = \mathcal{N}(S,N,f)$ of M admitting the reflection sequence κ, the map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ is bijective.*

PROOF. (1) Since $E^{\mathcal{B}(M)}(i_1,\dots,i_l) \subseteq \mathcal{B}(M)$ is a subalgebra, it is enough to prove that $M_1,\dots,M_\theta \subseteq E^{\mathcal{B}(M)}(i_1,\dots,i_l)$. For any $i \in \mathbb{I}$ there exists $1 \le k_i \le l$ with $\alpha_i = \beta_{k_i}^{[M],\kappa}$. Hence $M_i = M_{\alpha_i} \subseteq E^{\mathcal{B}(M)}(i_1,\dots,i_l)$ by degree reasons.

(2) As in (1) it is clear that $E^{\mathcal{N}}(i_1,\dots,i_l) = S$. Hence (2) follows from Theorem 14.1.9(7) and from (1). □

14.2. Exact factorizations of Nichols systems

Given a group G, an exact factorization of G is a pair of subgroups (G_1, G_2), such that the multiplication map $G_1 \times G_2 \to G$ is bijective. We discuss here a related notion for braided bialgebras and for pre-Nichols systems. As an application we deduce Theorem 14.2.12, which tells that the semi-Cartan graph of a tuple $M \in \mathcal{F}_\theta^H$ of irreducible objects, such that M admits all reflections, is a Cartan graph.

DEFINITION 14.2.1. Let B be a bialgebra in the category ${}^H_H\mathcal{YD}$. Let F and E be a left and a right coideal subalgebra of B, respectively. The pair (F,E) is called an **exact factorization of** B, if the multiplication map $F \otimes E \to B$ is an isomorphism in ${}^H_H\mathcal{YD}$.

EXAMPLE 14.2.2. Let G be a finite group and let $B = \Bbbk G$ be the group algebra of G. A left (right) coideal subalgebra of B is nothing but a subalgebra of B spanned by the elements of a subgroup of G. A pair (G_1, G_2) of subgroups of G is an exact factorization of G if and only if the pair $(\Bbbk G_1, \Bbbk G_2)$ is an exact factorization of B. This correspondence provides a bijection between exact factorizations of G and exact factorizations of B.

Let $\theta \in \mathbb{N}$ and let $\mathbb{I} = \{1, 2, \dots, \theta\}$.

DEFINITION 14.2.3. Let $M \in \mathcal{F}_\theta^H$ and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let F and E be $\mathbb{N}_0^\theta$-graded left and right coideal subalgebras of S, respectively. The pair (F, E) is called an **exact factorization of** $\mathcal{N}$, if the multiplication map $F \otimes E \to S$ is an isomorphism in ${}^H_H\mathcal{YD}$.

EXAMPLE 14.2.4. Let P be a Hopf algebra in ${}^H_H\mathcal{YD}$ and let $\pi : P \to Q$ be a projection to a Hopf subalgebra Q of P in ${}^H_H\mathcal{YD}$. Let $R = P^{\mathrm{co}\,Q}$ be the space of right coinvariant elements of P with respect to π. Then R is a left coideal subalgebra of P in ${}^H_H\mathcal{YD}$, and (R, Q) is an exact factorization of P because of (12.4.2).

Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $\pi : S \to \Bbbk[N_i]$ be the unique Hopf algebra map such that $\pi(N_j) = 0$ for any $j \neq i$ and $\pi|N_i = \mathrm{id}_{N_i}$. The subspaces $\Bbbk[N_i]$ and $S^{\mathrm{co}\,\Bbbk[N_i]}$ of S are $\mathbb{N}_0^\theta$-graded, since π is $\mathbb{N}_0^\theta$-graded. Hence, by the previous paragraph, $(S^{\mathrm{co}\,\Bbbk[N_i]}, \Bbbk[N_i])$ is an exact factorization of $\mathcal{N}$.

Reflections of Nichols systems provide non-trivial exact factorizations. To deal with them, we will need a variant of the maps $t_i^\mathcal{N}$ in Theorem 14.1.4 for left coideal subalgebras.

DEFINITION 14.2.5. For any $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and for any Nichols system $\mathcal{N} = \mathcal{N}(S, N, f)$ of (M, i) we define

$$\begin{aligned} \mathcal{L}(\mathcal{N}) &= \{F \mid F \subseteq S \ \mathbb{N}_0^\theta\text{-graded left coideal subalgebra in } {}^H_H\mathcal{YD}\}, \\ \mathcal{L}_i^+(\mathcal{N}) &= \{F \mid F \in \mathcal{L}(\mathcal{N}), \mathcal{N}_i \subseteq F\}, \\ \mathcal{L}_i^-(\mathcal{N}) &= \{F \mid F \in \mathcal{L}(\mathcal{N}), \mathcal{N}_i \not\subseteq F\}. \end{aligned}$$

LEMMA 14.2.6. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system of (M, i). Assume that M_i is irreducible. Let $F \subseteq S$ be a left coideal subalgebra in ${}^H_H\mathcal{YD}$. Consider the following conditions.*

(1) $F \subseteq K_i^\mathcal{N}$.
(2) $N_i \not\subseteq F$.

Then (1) *implies* (2). *If F is a graded subspace of S, then* (2) *implies* (1).

PROOF. Adapt the proof of Lemma 14.1.2 accordingly. □

For any pre-Nichols system $\mathcal{N}(S, N, f)$ of some $M \in \mathcal{F}_\theta^H$, let $\mathrm{Sub}(\mathcal{N})$ denote the set of all Yetter-Drinfeld submodules of S.

THEOREM 14.2.7. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N}$ be a Nichols system of (M, i). Assume that M is i-finite, and that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$.*

(1) *The map*

$$\bar{t}_i^\mathcal{N} : \mathrm{Sub}(R_i(\mathcal{N})) \to \mathrm{Sub}(\mathcal{N}), \quad F \mapsto T_i^\mathcal{N}(F \cap L_i^{R_i(\mathcal{N})}),$$

induces a bijection $\bar{t}_i^{\mathcal{N}} : \mathcal{L}_i^+(R_i(\mathcal{N})) \to \mathcal{L}_i^-(\mathcal{N})$, *with inverse given by* $F \mapsto (T_i^{\mathcal{N}})^{-1}(F)\Bbbk[\mathcal{N}_i^*]$.

(2) *The multiplication map* $(T_i^{\mathcal{N}})^{-1}(F) \otimes \Bbbk[\mathcal{N}_i^*] \to (T_i^{\mathcal{N}})^{-1}(F)\Bbbk[\mathcal{N}_i^*]$ *is bijective for all* $F \in \mathcal{L}_i^-(\mathcal{N})$.

PROOF. The claim follows from Theorem 12.4.6 following the arguments in the proof of Theorem 14.1.4. In the last part of the proof one needs Lemma 14.2.6. □

Analogously to Definition 14.1.7, we introduce a notation for compositions of maps $\bar{t}_i^{\mathcal{N}}$.

DEFINITION 14.2.8. Let $M \in \mathcal{F}_\theta^H$ such that M_i is irreducible in ${}^H_H\mathcal{YD}$ for all $i \in \mathbb{I}$. Let $l \in \mathbb{N}_0$ and let $i_1, \dots, i_l \in \mathbb{I}$. For any pre-Nichols system $\mathcal{N} = \mathcal{N}(S, N, f)$ of M admitting the reflection sequence $(i_1, \dots, i_l)$ define

$$\bar{t}^{\mathcal{N}}_{(i_1,\dots,i_l)} = \bar{t}^{\mathcal{N}}_{i_1}\bar{t}^{R_{i_1}(\mathcal{N})}_{i_2}\cdots\bar{t}^{R_{(i_1,\dots,i_{l-1})}(\mathcal{N})}_{i_l} : \mathrm{Sub}(R_{(i_1,\dots,i_l)}(\mathcal{N})) \to \mathrm{Sub}(\mathcal{N}).$$

THEOREM 14.2.9. *Let* $M \in \mathcal{F}_\theta^H$ *such that* M_j *is irreducible in* ${}^H_H\mathcal{YD}$ *for all* $j \in \mathbb{I}$. *Assume that* M *admits all reflections. Let* $\kappa = (i_1, \dots, i_l) \in \mathbb{I}^l$ *be an* $[M]$*-reduced sequence in the semi-Cartan graph* $\mathcal{G}(M)$, *where* $l \geq 1$. *Let* $\mathcal{N}$ *be a pre-Nichols system of* M, *and assume that* $\mathcal{N}$ *admits the reflection sequence* κ. *Let* $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_\kappa(\mathcal{N})$. *and* $F = \bar{t}_\kappa^{\mathcal{N}}(\widetilde{S})$.

(1) $F \in \mathcal{L}_{i_1}^-(\mathcal{N})$.

(2) $(F, E^{\mathcal{N}}(\kappa))$ *is an exact factorization of* $\mathcal{N}$.

PROOF. We prove (1) and (2) by induction on l. For $l = 1$, by definition we have $E^{\mathcal{N}}(i_1) = \Bbbk[\mathcal{N}_{i_1}]$ (see Theorem 14.1.4(3)) and

$$F = \bar{t}_{i_1}^{\mathcal{N}}(\widetilde{S}) = T_{i_1}^{\mathcal{N}}(L_{i_1}^{R_{i_1}(\mathcal{N})}) = K_{i_1}^{\mathcal{N}} \in \mathcal{L}_{i_1}^-(\mathcal{N}).$$

Hence $(F, E^{\mathcal{N}}(i_1))$ is an exact factorization of $\mathcal{N}$ by Example 14.2.4.

Assume that $l \geq 2$. Let $\mathcal{N}' = R_{i_1}(\mathcal{N}) = \mathcal{N}(S', N', f')$ and let $\kappa' = (i_2, \dots, i_l)$. By assumption, $\mathcal{N}'$ admits the reflection sequence κ'. Thus $F' = \bar{t}_{\kappa'}^{\mathcal{N}'}(\widetilde{S}) \in \mathcal{L}(\mathcal{N}')$, and $(F', E^{\mathcal{N}'}(\kappa'))$ is an exact factorization of $\mathcal{N}'$ by induction hypothesis. Theorem 14.1.9 implies that the homogeneous component of $E^{\mathcal{N}'}(\kappa')$ of degree α_{i_1} is zero. Hence $N'_{i_1} \subseteq F'$, since the multiplication map $F' \otimes E^{\mathcal{N}'}(\kappa') \to S'$ is surjective. Thus $F' \in \mathcal{L}_{i_1}^+(\mathcal{N}')$ and $F = \bar{t}_{i_1}^{\mathcal{N}}(F') \in \mathcal{L}_{i_1}^-(\mathcal{N})$ by Theorem 14.2.7.

It remains to prove that the multiplication map $F \otimes E^{\mathcal{N}}(\kappa) \to S$ is bijective, where $\mathcal{N} = \mathcal{N}(S, N, f)$. By Theorem 14.2.7(2), the multiplication map

$$(T_{i_1}^{\mathcal{N}})^{-1}(F) \otimes \Bbbk[N'_{i_1}] \to F'$$

is bijective. Hence the multiplication map

$$(T_{i_1}^{\mathcal{N}})^{-1}(F) \otimes \Bbbk[N'_{i_1}] \otimes E^{\mathcal{N}'}(\kappa') \to S' \tag{14.2.1}$$

is bijective. Moreover, $(T_{i_1}^{\mathcal{N}})^{-1}(F), E^{\mathcal{N}'}(\kappa') \subseteq L_{i_1}^{\mathcal{N}'}$ by definition of $T_{i_1}^{\mathcal{N}}$ and by Theorem 14.1.9(3), respectively. Thus the multiplication map

$$(T_{i_1}^{\mathcal{N}})^{-1}(F) \otimes E^{\mathcal{N}'}(\kappa') \to L_{i_1}^{\mathcal{N}'} \tag{14.2.2}$$

is injective. Moreover, for any $\alpha \in \mathbb{N}_0^{\mathbb{I}}$,

$$\dim S'(\alpha) = \sum_{k=0}^{\infty} \dim \big((T_{i_1}^{\mathcal{N}})^{-1}(F) \otimes E^{\mathcal{N}'}(\kappa')\big)(\alpha - k\alpha_{i_1}) \dim \Bbbk[N'_{i_1}](k\alpha_{i_1})$$

by the bijectivity of the map in (14.2.1), and

$$\begin{aligned}\dim S'(\alpha) &= \sum_{k=0}^{\infty} \dim K_{i_1}^{\mathcal{N}'}(\alpha - k\alpha_{i_1}) \dim \Bbbk[N'_{i_1}](k\alpha_{i_1}) \\ &= \sum_{k=0}^{\infty} \dim L_{i_1}^{\mathcal{N}'}(\alpha - k\alpha_{i_1}) \dim \Bbbk[N'_{i_1}](k\alpha_{i_1})\end{aligned}$$

by (12.4.2) and by Lemma 13.5.11(1),(3). This implies that

$$\dim \big((T_{i_1}^{\mathcal{N}})^{-1}(F) \otimes E^{\mathcal{N}'}(\kappa')\big)(\alpha) = \dim L_{i_1}^{\mathcal{N}'}(\alpha)$$

for any $\alpha \in \mathbb{N}_0^{\mathbb{I}}$. Thus the map in (14.2.2) is bijective. Therefore, since $T_{i_1}^{\mathcal{N}}$ is an algebra map, also the multiplication map $F \otimes T_{i_1}^{\mathcal{N}}\big(E^{\mathcal{N}'}(\kappa')\big) \to K_{i_1}^{\mathcal{N}}$ is bijective. Then the multiplication map

$$F \otimes T_{i_1}^{\mathcal{N}}\big(E^{\mathcal{N}'}(\kappa')\big) \otimes \Bbbk[N_{i_1}] \to S$$

is bijective by (12.4.2). Thus the multiplication map $F \otimes E^{\mathcal{N}}(\kappa) \to S$ is bijective because of Theorem 14.1.4(2). $\square$

In Proposition 14.2.11 below we will identify the exact factorization in Theorem 14.2.9 with a familiar one for a special reduced sequence. To do so, we will use a variant of Lemma 14.2.6.

LEMMA 14.2.10. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M such that the canonical map $p^{\mathcal{N}}$ induces an isomorphism $\Bbbk[N_i] \cong \mathcal{B}(M_i)$. Let $F \subseteq S$ be an $\mathbb{N}_0^\theta$-graded left coideal in ${}^H_H\mathcal{YD}$ with $N_i \cap F = 0$. Then $F \subseteq K_i^{\mathcal{N}}$.*

PROOF. Since F is a graded subspace of the $\mathbb{N}_0^\theta$-graded Hopf algebra S, and since the projection $\pi_i^{\mathcal{N}} : S \to \mathcal{B}(M_i)$ is graded, the homogeneous component of $\pi_i^{\mathcal{N}}(F)$ of degree α_i is $\pi_i^{\mathcal{N}}(N_i \cap F)$ and hence zero. Therefore $M_i \cap \pi_i^{\mathcal{N}}(F) = 0$. Moreover, $\pi_i^{\mathcal{N}}(F)$ is a left coideal of $\mathcal{B}(M_i)$. Since $\mathcal{B}(M_i)$ is a strictly graded coalgebra, Corollary 1.3.11(3) implies that $\pi_i^{\mathcal{N}}(F) = 0$ or $\pi_i^{\mathcal{N}}(F) = \Bbbk 1$. In particular,

$$\pi_i^{\mathcal{N}}(f) = \varepsilon(\pi_i^{\mathcal{N}}(f))1 = \varepsilon(f)1$$

for any $f \in F$. Therefore $F \subseteq S^{\operatorname{co} \pi_i^{\mathcal{N}}(S)}$ by Lemma 2.5.6(1). Since

$$\pi_i^{\mathcal{N}}(S) = \mathcal{B}(M_i) \cong \Bbbk[N_i],$$

we conclude that $F \subseteq K_i^{\mathcal{N}}$. $\square$

Recall the definitions of $\overline{m}_{ij}^X$ and κ_{ij}^X from Remark 9.2.9.

PROPOSITION 14.2.11. *Assume that $\theta \geq 2$. Let $M \in \mathcal{F}_\theta^H$ such that M_n is irreducible in ${}^H_H\mathcal{YD}$ for all $n \in \mathbb{I}$, and let $i, j \in \mathbb{I}$ with $i \neq j$. Assume that M admits all reflections and $\overline{m}_{ij}^{[M]} < \infty$. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M admitting the reflection sequence $\kappa = \kappa_{ij}^{[M]}$. Let $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_\kappa(\mathcal{N})$. Then*

$$\bar{t}_\kappa^{\mathcal{N}}(\widetilde{S}) = S^{\operatorname{co} \Bbbk[N_i + N_j]}, \quad E^{\mathcal{N}}(\kappa) = \Bbbk[N_i + N_j].$$

Proof. Let $\mathcal{G}(M) = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ be the semi-Cartan graph of M. Then $\mathcal{G}(M)$ satisfies (CG3') by Corollary 14.1.11. Let $m = \overline{m}_{ij}^{[M]}$. Then $\beta_1^{[M],\kappa} = \alpha_i$ and $\beta_m^{[M],\kappa} = \alpha_j$ by Lemma 9.2.15. In particular, $N_i + N_j \subseteq E^{\mathcal{N}}(\kappa)$. Conversely,

$$E^{\mathcal{N}}(\kappa) \subseteq \bigoplus_{k_1,k_2 \in \mathbb{N}_0} S(k_1\alpha_i + k_2\alpha_j) = \Bbbk[N_i + N_j]$$

by Theorem 14.1.9(4),(6) and since S is $\mathbb{N}_0^\theta$-graded. Thus $E^{\mathcal{N}}(\kappa) = \Bbbk[N_i + N_j]$.

By Theorem 14.2.9, $F = \bar{t}_\kappa^{\mathcal{N}}(\widetilde{S})$ is a left coideal of S in ${}^H_H\mathcal{YD}$ and $(F, E^{\mathcal{N}}(\kappa))$ is an exact factorization of S. Then $(N_i + N_j) \cap F = 0$. We prepare to apply Lemma 14.2.10. Let $h : \{1, \dots, \theta\} \to \{1, \dots, \theta - 1\}$ be a surjective map with $h(i) = h(j)$. Then $h_*(\mathcal{N})$ is a pre-Nichols system of $h_1(M)$ in the terminology of Example 13.5.3. Moreover,

$$h_1(N)_{h(i)} = N_i + N_j$$

by construction. The canonical map $p^{h_*(\mathcal{N})}$ induces an isomorphism

$$\Bbbk[N_i + N_j] = E^{\mathcal{N}}(\kappa) \xrightarrow{\cong} \mathcal{B}(M_i + M_j)$$

by Theorem 14.1.9(7). Therefore Lemma 14.2.10 applies, and $F \subseteq S^{\mathrm{co}\,\Bbbk[N_i+N_j]}$. Hence, and since $(F, \Bbbk[N_i + N_j])$, $(S^{\mathrm{co}\,\Bbbk[N_i+N_j]}, \Bbbk[N_i + N_j])$ are exact factorizations of S, it follows that $F = S^{\mathrm{co}\,\Bbbk[N_i+N_j]}$. $\square$

Theorem 14.2.12. *Let $M \in \mathcal{F}_\theta^H$. Assume that M_j is irreducible for all $j \in \mathbb{I}$ and that M admits all reflections. Then $\mathcal{G}(M)$ is a Cartan graph.*

Proof. By Theorem 13.6.2 and Corollary 14.1.11, $\mathcal{G}(M) = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ is a semi-Cartan graph satisfying (CG3'). Because of Corollary 9.2.20 it suffices to prove that $\mathcal{G}(M)$ satisfies (CG4').

Let $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$. Then, by Proposition 13.6.5, $\mathcal{N}_0$ admits all reflections. Let $X \in \mathcal{X}$. By definition, there exist $k \geq 0$ and $j_1, \dots, j_k \in \mathbb{I}$ such that $X = [P]$, where $P = R_{j_k} \cdots R_{j_1}(M)$. Clearly, P admits all reflections. Moreover, the pre-Nichols system

$$\mathcal{N}_{[P]} = \mathcal{N}(S, N, f) = R_{j_k} \cdots R_{j_1}(\mathcal{N}_0)$$

of P is isomorphic to $\mathcal{N}(\mathcal{B}(P), P, \mathrm{id}_P)$ via the canonical map by Lemma 13.5.16.

Let $i, j \in \mathbb{I}$. Assume that $i \neq j$ and that $m = \overline{m}_{ij}^{[P]} < \infty$. Let

$$\kappa' = (i_1, \dots, i_m, k) \in \mathbb{I}^{m+1},$$

where $(i_1, \dots, i_m) = \kappa_{ij}^{[P]}$. Assume that $k \neq i$ and $k \neq j$. Then

$$\mathrm{id}_{[P]} s_{i_1} \cdots s_{i_m}(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$$

and $\beta_n^{[P],\kappa'} \in \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$ for any $1 \leq n \leq m$. Hence κ' is $[P]$-reduced by Lemma 9.2.5 and since $\kappa_{ij}^{[P]}$ is $[P]$-reduced. Let

$$\mathcal{N}' = \mathcal{N}(S', N', f') = R_{(i_1,\dots,i_m)}(\mathcal{N}_{[P]}).$$

Then $N'_k \subseteq L_{(i_1,\dots,i_m)}^{\mathcal{N}_{[P]}}$ by Theorem 14.1.9(2), and

$$T_{(i_1,\dots,i_m)}^{\mathcal{N}_{[P]}}(N'_k) \subseteq \bar{t}_{(i_1,\dots,i_m)}^{\mathcal{N}}(S') = S^{\mathrm{co}\,\Bbbk[N_i+N_j]}$$

by Proposition 14.2.11.

By Proposition 9.2.14, $m = \overline{m}_{ji}^{[P]}$. Hence $\kappa_{ji}^{[P]} = (i_2, \dots, i_m, i_{m+1})$, where $i_{m+1} = i_{m-1}$. Let $k \in \mathbb{I}$ and $\kappa'' = (i_2, \dots, i_m, i_{m+1}, k) \in \mathbb{I}^{m+1}$. Assume that $k \neq i$ and $k \neq j$. By interchanging i and j in the previous paragraph we obtain that κ'' is $[P]$-reduced. Let

$$\mathcal{N}'' = \mathcal{N}(S'', N'', f'') = R_{(i_2,\dots,i_m,i_{m+1})}(\mathcal{N}_{[P]}).$$

Then $N''_k \subseteq L^{\mathcal{N}_{[P]}}_{(i_1,\dots,i_m,i_{m+1})}$ by Theorem 14.1.9(2).

Since S'' is $\mathbb{N}_0^\theta$-graded, the map $\big(T^{\mathcal{N}_{[P]}}_{(i_2,\dots,i_m,i_{m+1})}\big)^{-1} T^{\mathcal{N}_{[P]}}_{(i_1,\dots,i_m)}$ sends N'_k to an irreducible Yetter-Drinfeld submodule of S'' of degree

$$\begin{aligned} \mathrm{id}_{[N'']} s_{i_{m+1}} s_{i_m} \cdots s_{i_2} s_{i_1} \cdots s_{i_m}(\alpha_k) &= \mathrm{id}_{[N'']}(s_{i_{m+1}} s_{i_m})^m(\alpha_k) \\ &\in \alpha_k + \mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j. \end{aligned}$$

By Lemma 9.2.15 and Proposition 9.2.14,

$$\mathrm{id}_{[N'']}(s_{i_{m+1}} s_{i_m})^m(\alpha_i) = \alpha_i, \quad \mathrm{id}_{[N'']}(s_{i_{m+1}} s_{i_m})^m(\alpha_j) = \alpha_j.$$

Similarly, by looking at the degree of $\big(T^{\mathcal{N}_{[P]}}_{(i_1,\dots,i_m)}\big)^{-1} T^{\mathcal{N}_{[P]}}_{(i_2,\dots,i_m,i_{m+1})}(N''_k)$ we obtain that

$$(\mathrm{id}_{[N'']}(s_{i_{m+1}} s_{i_m})^m)^{-1}(\alpha_k) \in \alpha_k + \mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j.$$

The previous results hold for any $k \in \mathbb{I} \setminus \{i,j\}$. Hence

$$\mathrm{id}_{[N'']}(s_{i_{m+1}} s_{i_m})^m(\alpha_k) = \alpha_k$$

for any $k \in \mathbb{I}$ by (9.1.1) and Lemma 9.2.19. Thus, for any $k \in \mathbb{I} \setminus \{i,j\}$, the map $\big(T^{\mathcal{N}_{[P]}}_{(i_2,\dots,i_m,i_{m+1})}\big)^{-1} T^{\mathcal{N}_{[P]}}_{(i_1,\dots,i_m)}$ provides an isomorphism of the Yetter-Drinfeld modules N'_k and N''_k. Moreover,

$$N'_{i_m} = R_{(i_1,\dots,i_m)}(\mathcal{N}_{[P]})_{i_m} = R_{(i_1,\dots,i_{m-1})}(\mathcal{N}_{[P]})^*_{i_m}.$$

Lemma 9.2.13 implies that $T^{\mathcal{N}}_{(i_1,\dots,i_{m-1})}(R_{(i_1,\dots,i_{m-1})}(\mathcal{N}_{[P]})_{i_m})$ is a Yetter-Drinfeld submodule of S of degree $\mathrm{id}_{[P]} s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m}) = \alpha_j$. Hence $N'_{i_m} \cong P^*_j$. On the other hand,

$$N'_{i_{m+1}} = R_{(i_1,\dots,i_m)}(\mathcal{N}_{[P]})_{i_{m+1}} = R_{(i_2,\dots,i_m)}(R_{i_1}(\mathcal{N}_{[P]}))_{i_{m+1}}.$$

Again, Lemma 9.2.13 implies that $N'_{i_{m+1}}$ is isomorphic to $R_{i_1}(P)_{i_1} = P^*_i$. Using similar arguments one shows that $N''_{i_m} \cong P^*_j$ and $N''_{i_{m+1}} \cong P^*_i$. Indeed,

$$\begin{aligned} N''_{i_m} &= R_{(i_2,\dots,i_m,i_{m+1})}(\mathcal{N}_{[P]})_{i_m} \\ &= R_{(i_1,\dots,i_{m-1})}(R_{i_2}(\mathcal{N}_{[P]}))_{i_m} \cong R_j(\mathcal{N}_{[P]})_j = P^*_j, \\ N''_{i_{m+1}} &= R_{(i_2,\dots,i_m,i_{m+1})}(\mathcal{N}_{[P]})_{i_{m+1}} \\ &= R_{(i_2,\dots,i_m)}(\mathcal{N}_{[P]})^*_{i_{m+1}} \cong P^*_i. \end{aligned}$$

Thus $[N'] = [N'']$ and $\mathrm{id}_{[N']}(s_i s_j)^m = \mathrm{id}_{[N']}$. Since $[N'] = r_{i_m} \cdots r_{i_1}([P])$, this implies that $\mathrm{id}_{[P]}(s_i s_j)^m = \mathrm{id}_{[P]}$. Therefore (CG4') holds. $\square$

14.3. Hilbert series of right coideal subalgebras of Nichols algebras

In this section we specialize results from Section 14.1 to Nichols algebras and analyze the setting further in view of the notion of Hilbert series.

Let $\theta \in \mathbb{N}$ and let $\mathbb{I} = \{1, 2, \dots, \theta\}$.

In the next definition, the t_i should not be confused with the maps $t_i^{\mathcal{N}}$ in Theorem 14.1.4.

DEFINITION 14.3.1. Let X be an $\mathbb{N}_0^\theta$-graded vector space, and assume that $X(\alpha)$ is finite-dimensional for all $\alpha \in \mathbb{N}_0^\theta$. For any $\alpha = \sum_{i=1}^\theta n_i\alpha_i \in \mathbb{N}_0^\theta$ let $t^\alpha = t_1^{n_1} \cdots t_\theta^{n_\theta}$ in the polynomial algebra $\Bbbk[t_1, \dots, t_\theta]$. The (multivariate) **Hilbert series** of X is the formal power series

$$\mathcal{H}_X(t) = \sum_{\alpha \in \mathbb{N}_0^\theta} (\dim X(\alpha)) t^\alpha \in \mathbb{N}_0[\![t_1, \dots, t_\theta]\!].$$

We denote the support of X by

$$\operatorname{supp}(X) = \{\alpha \in \mathbb{N}_0^\theta \mid X(\alpha) \neq 0\}.$$

Let $s : \operatorname{supp}(X) \to \mathbb{N}_0^\theta$ be a mapping. Then we define

$$s(\mathcal{H}_X(t)) = \sum_{\alpha \in \mathbb{N}_0^\theta} (\dim X(\alpha)) t^{s(\alpha)}.$$

PROPOSITION 14.3.2. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N}$ be a Nichols system of (M, i). Assume that M is i-finite, and that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Let $E \in \mathcal{K}_i^+(\mathcal{N})$. Then*

$$\mathcal{H}_E(t) = s_i^{R_i(M)}(\mathcal{H}_{E'}(t))\mathcal{H}_{\mathcal{B}(M_i)}(t)$$

with $E' = (t_i^{\mathcal{N}})^{-1}(E)$.

PROOF. Theorem 14.1.4 implies that $E' \in \mathcal{K}_i^-(R_i(\mathcal{N}))$ and that

$$\mathcal{H}_E(t) = \mathcal{H}_{T_i^{\mathcal{N}}(E')}(t)\mathcal{H}_{\Bbbk[\mathcal{N}_i]}(t).$$

Since $E' \subseteq L_i^{\mathcal{B}(R_i(M))}$, the claim of the Proposition follows from Corollary 13.5.21(2) and from (Sys3). $\square$

DEFINITION 14.3.3. Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$. Assume that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$ and that M is i-finite. Let

$$T_i^{\mathcal{B}(M)} : L_i^{\mathcal{B}(R_i(M))} \xrightarrow{\cong} {}^{\operatorname{co}\mathcal{B}(M_i^*)}(\Omega_{M_i}(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*)) \xrightarrow{\cong} K_i^{\mathcal{B}(M)}$$

be the composition of two isomorphisms in ${}^H_H\mathcal{YD}$, where the first one is defined by restriction of the isomorphism Θ in Theorem 13.4.9, and the second one is $T_i^{\mathcal{N}_0}$, where $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$.

COROLLARY 14.3.4. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$. Assume that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$ and that M is i-finite. Then $T_i^{\mathcal{B}(M)}$ is an algebra isomorphim in ${}^H_H\mathcal{YD}$ and the following hold.*

(1) *For all $j \in \mathbb{I} \setminus \{i\}$ and $0 \le n \le -a_{ij}^M$,*

$$T_i^{\mathcal{B}(M)}\Big(\mathcal{S}^{-1}_{\mathcal{B}(R_i(M))}\big((\operatorname{ad} M_i^*)^n(R_i(M)_j)\big)\Big) = (\operatorname{ad} M_i)^{-a_{ij}^M - n}(M_j).$$

(2) *Let* $\alpha \in \mathbb{N}_0^\theta$, *and let* $x \in L_i^{\mathcal{B}(R_i(M))}$ *be a non-zero homogeneous element with* $\deg(x) = \alpha$. *Then* $\deg(T_i^{\mathcal{B}(M)}(x)) = s_i^{R_i(M)}(\alpha)$. *In particular,* $s_i^{R_i(M)}(\alpha) \in \mathbb{N}_0^\theta$.

Here, $K_i^{\mathcal{B}(M)} \subseteq \mathcal{B}(M)$ *and* $L_i^{\mathcal{B}(R_i(M))} \subseteq \mathcal{B}(R_i(M))$ *are* $\mathbb{N}_0^\theta$*-graded subalgebras with respect to the standard grading of the Nichols algebras* $\mathcal{B}(M)$ *and* $\mathcal{B}(R_i(M))$*, respectively.*

PROOF. The isomorphism $\Theta : \mathcal{B}(R_i(M)) \to \Omega(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*)$ discussed in Theorem 13.4.9 commutes with the projection $\pi_i : \mathcal{B}(R_i(M)) \to \mathcal{B}(M_i^*)$ and with the projection of the smash product $\widetilde{\pi}_i : \Omega(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*) \to \mathcal{B}(M_i^*)$, since $\widetilde{\pi}_i\Theta(x) = \pi_i(x)$ for all algebra generators $x \in R_i(M)_j \subseteq \mathcal{B}(R_i(M))$ with $j \in \mathbb{I}$. Hence Θ defines by restriction an isomorphism between $L_i^{\mathcal{B}(R_i(M))}$ and ${}^{\mathrm{co}\,\mathcal{B}(M_i^*)}(\Omega(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*))$, and the corollary follows from the case of Nichols algebras of Corollary 13.5.21. $\square$

In analogy to Definition 14.1.3, we introduce a notation for certain sets of right coideal subalgebras of Nichols algebras, that is, of the pre-Nichols system $\mathcal{N}_0$ of M.

DEFINITION 14.3.5. For any $M \in \mathcal{F}_\theta^H$ let $\mathcal{K}(\mathcal{B}(M))$ denote the set of all $\mathbb{N}_0^\theta$-graded right coideal subalgebras of $\mathcal{B}(M)$ in ${}_H^H\mathcal{YD}$, and

$$\mathcal{K}_i^+(\mathcal{B}(M)) = \{E \mid E \in \mathcal{K}(\mathcal{B}(M)), M_i \subseteq E\},$$
$$\mathcal{K}_i^-(\mathcal{B}(M)) = \{E \mid E \in \mathcal{K}(\mathcal{B}(M))), M_i \not\subseteq E\}.$$

COROLLARY 14.3.6. *Let* $i \in \mathbb{I}$ *and* $M \in \mathcal{F}_\theta^H$. *Assume that* M_j *is irreducible in* ${}_H^H\mathcal{YD}$ *for all* $j \in \mathbb{I}$ *and that* M *is* i*-finite.*

(1) *The map*
$$t_i^{\mathcal{B}(M)} : \mathcal{K}_i^-(\mathcal{B}(R_i(M))) \to \mathcal{K}_i^+(\mathcal{B}(M)), \quad E \mapsto T_i^{\mathcal{B}(M)}(E)\Bbbk[M_i],$$
is bijective with inverse given by $E \mapsto (T_i^{\mathcal{B}(M)})^{-1}(E \cap K_i^{\mathcal{B}(M)})$.

(2) *The multiplication map* $T_i^{\mathcal{B}(M)}(E) \otimes \Bbbk[M_i] \to T_i^{\mathcal{B}(M)}(E)\Bbbk[M_i]$ *is bijective for all* $E \in \mathcal{K}_i^-(\mathcal{B}(R_i(M)))$.

PROOF. The corollary follows from Theorem 14.1.4 applied to $\mathcal{N}_0$. $\square$

COROLLARY 14.3.7. *Let* $i \in \mathbb{I}$ *and* $M \in \mathcal{F}_\theta^H$. *Assume that* M_j *is irreducible in* ${}_H^H\mathcal{YD}$ *for all* $j \in \mathbb{I}$ *and that* M *is* i*-finite. Let* $E \in \mathcal{K}_i^+(\mathcal{B}(M))$. *Then*

$$\mathcal{H}_E(t) = s_i^{R_i(M)}(\mathcal{H}_{E'}(t))\mathcal{H}_{\mathcal{B}(M_i)}(t)$$

with $E' = (t_i^{\mathcal{B}(M)})^{-1}(E)$.

PROOF. Similarly to the proof of Proposition 14.3.2, the claim follows from Corollaries 14.3.6 and 14.3.4(2). $\square$

We now define a variant of the right coideal subalgebras $E^{\mathcal{B}(M)}(i_1, \dots, i_l)$ in Theorem 14.1.9(7).

DEFINITION 14.3.8. Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. For any $[M]$-reduced sequence $(i_1, \dots, i_l)$ in the semi-Cartan graph $\mathcal{G}(M)$, where $l \in \mathbb{N}_0$ and $i_1, \dots, i_l \in \mathbb{I}$, let

$$\widehat{E}^{\mathcal{B}(M)}(i_1, \dots, i_l) = t_{i_1}^{\mathcal{B}(M)} t_{i_2}^{\mathcal{B}(R_{i_1}(M))} \cdots t_{i_l}^{\mathcal{B}(R_{(i_1,\dots,i_{l-1})}(M))}(\Bbbk 1).$$

We note an important uniqueness property of this construction.

COROLLARY 14.3.9. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $(i_1, \dots, i_l)$ be an $[M]$-reduced sequence in the semi-Cartan graph $\mathcal{G}(M)$, where $l \in \mathbb{N}_0$ and $i_1, \dots, i_l \in \mathbb{I}$. Then*

$$\widehat{E}^{\mathcal{B}(M)}(i_1, \dots, i_l) = E^{\mathcal{B}(M)}(i_1, \dots, i_l)$$

is the only element in $\mathcal{K}(\mathcal{B}(M))$ with the same Hilbert series as of $E^{\mathcal{B}(M)}(i_1, \dots, i_l)$.

PROOF. We proceed by induction on l. For $l = 0$ the claim is trivial.
Assume that $l \geq 1$. Then

$$\widehat{E}^{\mathcal{B}(R_{i_1}(M))}(i_2, \dots, i_l) = E^{\mathcal{B}(R_{i_1}(M))}(i_2, \dots, i_l)$$

by induction hypothesis, and $E^{\mathcal{B}(R_{i_1}(M))}(i_2, \dots, i_l)$ is the only element in the set $\mathcal{K}(\mathcal{B}(R_{i_1}(M)))$ which has the same Hilbert series as $E^{\mathcal{B}(R_{i_1}(M))}(i_2, \dots, i_l)$. Let $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id}_M)$. Then Proposition 14.3.2 and Corollary 14.3.7 imply that $\widehat{E}^{\mathcal{B}(R_{i_1}(M))}(i_2, \dots, i_l)$ and $E^{R_{i_1}(\mathcal{N}_0)}(i_2, \dots, i_l)$ have the same Hilbert series. Thus

$$\widehat{E}^{\mathcal{B}(R_{i_1}(M))}(i_2, \dots, i_l) \in \mathcal{K}_{i_1}^-(\mathcal{B}(R_{i_1}(M))),$$

since $E^{R_{i_1}(\mathcal{N}_0)}(i_2, \dots, i_l) \in \mathcal{K}_{i_1}^-(R_{i_1}(\mathcal{N}_0))$. Again by Proposition 14.3.2 and Corollary 14.3.7, $\widehat{E}^{\mathcal{B}(M)}(i_1, \dots, i_l)$ and $E^{\mathcal{B}(M)}(i_1, \dots, i_l)$ have the same Hilbert series.

Finally, let $E \in \mathcal{K}(\mathcal{B}(M))$ and assume that E has the same Hilbert series as $\widehat{E}^{\mathcal{B}(M)}(i_1, \dots, i_l)$. Then $(t_{i_1}^{\mathcal{B}(M)})^{-1}(E)$ and $\widehat{E}^{\mathcal{B}(R_{i_1}(M))}(i_2, \dots, i_l)$ have the same Hilbert series by Corollary 14.3.7. Thus they coincide by induction hypothesis and hence $E = \widehat{E}^{\mathcal{B}(M)}(i_1, \dots, i_l)$. □

14.4. Tensor decomposable Nichols algebras

Assume that the Cartan graph of a given tuple of irreducible Yetter-Drinfeld modules is finite. We provide in this section an algebraic interpretation of the (real) roots of this Cartan graph. We also give a characterization of the finiteness of the Cartan graph.

For any $\alpha \in \mathbb{N}_0^\theta$, $\alpha = \sum_{i=1}^\theta n_i \alpha_i$ with $n_i \geq 0$ for all $i \in \mathbb{I}$, we will write $|\alpha| = \sum_{i=1}^\theta n_i$. For any $\mathbb{N}_0^\theta$-graded object V in ${}_H^H\mathcal{YD}$ and $n \geq 0$ we define

$$V(n) = \bigoplus_{\substack{\alpha \in \mathbb{N}_0^\theta \\ |\alpha| = n}} V(\alpha). \tag{14.4.1}$$

Then $V = \bigoplus_{n \geq 0} V(n)$ is a decomposition into $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$.

For the n-fold tensor product $V_1 \otimes \cdots \otimes V_n$ of objects $V_1, \dots, V_n$ in the monoidal category ${}_H^H\mathcal{YD}$, where $n \in \mathbb{N}_0$, we will also write $\bigotimes_{l=1}^n V_l$. Note that

$$\bigotimes_{l=1}^n V_l \cong \bigotimes_{l=1}^n V_{\sigma(l)}$$

for any permutation $\sigma \in \mathbb{S}_n$, since ${}_H^H\mathcal{YD}$ is braided.

DEFINITION 14.4.1. Let V be an $\mathbb{N}_0^\theta$-graded object in ${}_H^H\mathcal{YD}$, and $M \in \mathcal{F}_\theta^H$.

(1) We say that V is **tensor decomposable** if there are an integer $n \geq 0$, irreducible Yetter-Drinfeld modules $Q_1, \dots, Q_n \in {}^H_H\mathcal{YD}$ of finite dimension and pairwise distinct elements $\beta_1, \dots, \beta_n$ in $\mathbb{N}_0^\theta \setminus \{0\}$ such that

$$V \cong \bigotimes_{l=1}^{n} \mathcal{B}(Q_l)$$

as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$, where the gradings of $\mathcal{B}(Q_l)$ are given by $\deg(Q_l) = \beta_l$, $1 \leq l \leq n$ (and where H is trivially $\mathbb{N}_0^\theta$-graded).

By convention, tensor decomposability with $n = 0$ means that $V \cong \Bbbk 1$.

(2) The Nichols algebra $\mathcal{B}(M)$ is called **tensor decomposable** if $\mathcal{B}(M)$ is tensor decomposable as an $\mathbb{N}_0^\theta$-graded object in ${}^H_H\mathcal{YD}$ with the standard grading, where $\deg(M_i) = \alpha_i$ for all $i \in \mathbb{I}$.

REMARK 14.4.2. The notion in Definition 14.4.1 has very strong consequences. We will not discuss weaker concepts here, because the examples we have in mind in this book are covered by the definition.

EXAMPLE 14.4.3. We show that the right coideal subalgebras $E^{\mathcal{N}}(i_1, \dots, i_l)$ in Theorem 14.1.9 are tensor decomposable.

Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $l \geq 1$ and $i_1, \dots, i_l \in \mathbb{I}$. Assume that $(i_1, \dots, i_l)$ is $[M]$-reduced in the semi-Cartan graph $\mathcal{G}(M)$ and that $\mathcal{N}$ admits the reflection sequence $(i_1, \dots, i_l)$. Let $E^{\mathcal{N}}(i_1, \dots, i_l)$ be as in Theorem 14.1.9.

For any $1 \leq k \leq l$, let $\beta_k = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Then $\beta_1, \dots, \beta_l$ are pairwise distinct non-zero elements of $\mathbb{N}_0^\theta$ by Theorem 14.1.9(1). For each $1 \leq k \leq l$, $N_{\beta_k} = N_k^{\mathcal{N}}(i_1, \dots, i_l)$ is a finite-dimensional irreducible object in ${}^H_H\mathcal{YD}$ of degree β_k by Theorem 14.1.9(4). Moreover, by Theorem 14.1.9(6), the multiplication map $\Bbbk[N_{\beta_l}] \otimes \cdots \otimes \Bbbk[N_{\beta_1}] \to E^{\mathcal{N}}(i_1, \dots, i_l)$ is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$. Finally, for each $1 \leq k \leq l$, $\mathcal{B}(N_{\beta_k}) \cong \Bbbk[N_{\beta_k}]$ by Theorem 14.1.9(5).

We prove some properties of the tensor decompositions in Definition 14.4.1 such as uniqueness and cancellation which essentially follow from the theorem of Krull-Schmidt.

LEMMA 14.4.4. *Let U, V and W be $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$ with finite-dimensional homogeneous components. Assume that $W(0) \cong \Bbbk$ in ${}^H_H\mathcal{YD}$. If*

$$U \otimes W \cong V \otimes W \quad \text{or} \quad W \otimes U \cong W \otimes V$$

as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$, then $U \cong V$ as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$.

PROOF. Let $n \geq 0$. Since $W(0)$ is the trivial object,

$$(U \otimes W)(n) \cong U(n) \oplus \bigoplus_{i=0}^{n-1} U(i) \otimes W(n-i),$$
$$(V \otimes W)(n) \cong V(n) \oplus \bigoplus_{i=0}^{n-1} V(i) \otimes W(n-i)$$

as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$. Since the homogenous components of U, V and W are finite-dimensional, the claim follows by induction from Krull-Schmidt. □

LEMMA 14.4.5. *Let $n,m \geq 1$ be integers and $Q_1,\dots,Q_n$, $P_1,\dots,P_m$ finite-dimensional irreducible objects in ${}^H_H\mathcal{YD}$ with $\deg(Q_l),\deg(P_k) \in \mathbb{N}_0^\theta \setminus \{0\}$ for all $1 \leq l \leq n$, $1 \leq k \leq m$. Assume that $\bigotimes_{l=1}^n \mathcal{B}(Q_l) \cong \bigotimes_{k=1}^m \mathcal{B}(P_k)$ as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$. Then $n = m$, and there is a permutation $\sigma \in \mathbb{S}_n$ such that $P_l \cong Q_{\sigma(l)}$ as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$ for all $1 \leq l \leq n$.*

PROOF. Let $r = \min\{|\deg(Q_l)| \,\big|\, 1 \leq l \leq n\}$, and let L be the set of all l such that $1 \leq l \leq n$ and $|\deg(Q_l)| = r$. Then $\bigoplus_{l\in L} Q_l$ is the $\mathbb{N}_0$-homogeneous component of $\bigotimes_{l=1}^n \mathcal{B}(Q_l)$ of minimal positive degree. Hence

$$r = \min\{|\deg(P_k)| \mid 1 \leq k \leq m\}$$

and $\bigoplus_{l\in L} Q_l \cong \bigoplus_{k\in K} P_k$, where $K = \{1 \leq k \leq m \mid |\deg(P_k)| = r\}$. By Krull-Schmidt there are indices $l \in L$ and $k \in K$ such that $Q_l \cong P_k$. The claim follows now by induction and Lemma 14.4.4 using that ${}^H_H\mathcal{YD}$ is braided. $\square$

Lemma 14.4.4 is of particular relevance when we decompose graded right coideal subalgebras of graded Hopf algebras.

PROPOSITION 14.4.6. *Let $\mathcal{N} = \mathcal{N}(S,N,f)$ be a pre-Nichols system of some $M \in \mathcal{F}_\theta^H$, let E be a tensor decomposable $\mathbb{N}_0^\theta$-graded right coideal subalgebra of S and let $1 \leq i \leq \theta$. Assume that $E(\alpha_i) = N_i$ is irreducible in ${}^H_H\mathcal{YD}$ and that the canonical map $\Bbbk[N_i] \to \mathcal{B}(N_i)$ is bijective. Then $E \cap S^{\mathrm{co}\,\Bbbk[N_i]}$ is tensor decomposable and there exist tensor decompositions*

$$E \cap S^{\mathrm{co}\,\Bbbk[N_i]} \cong \bigotimes_{l=1}^n \mathcal{B}(Q_l), \quad E \cong \mathcal{B}(N_i) \otimes \bigotimes_{l=1}^n \mathcal{B}(Q_l),$$

where $n \geq 0$ and $\deg(Q_l) \notin \mathbb{N}_0\alpha_i$ for all $1 \leq l \leq n$.

PROOF. By assumption, S admits an $\mathbb{N}_0^\theta$-graded projection to its Hopf subalgebra

$$\bigoplus_{k\geq 0} S(k\alpha_i) = \Bbbk[N_i] \cong \mathcal{B}(N_i). \tag{14.4.2}$$

By Lemma 12.4.3, the multiplication map $(E \cap S^{\mathrm{co}\,\Bbbk[N_i]}) \otimes \Bbbk[N_i] \to E$ is bijective. Clearly, this map is an $\mathbb{N}_0^\theta$-graded morphism in ${}^H_H\mathcal{YD}$. By assumption, there exists a tensor decomposition

$$E \cong \bigotimes_{l=0}^n \mathcal{B}(Q_l)$$

of E. Since $N_i = E(\alpha_i)$ is irreducible, we may assume that $Q_0 = N_i$. Moreover, the homogeneous components of E are finite-dimensional, and hence Lemma 14.4.4 with $W = \Bbbk[N_i]$ implies that $E \cap S^{\mathrm{co}\,\Bbbk[N_i]} \cong \bigotimes_{l=1}^n \mathcal{B}(Q_l)$. Finally, from (14.4.2) it follows that $\deg(Q_l) \notin \mathbb{N}_0\alpha_i$ for all $1 \leq l \leq n$. $\square$

DEFINITION 14.4.7. Let $s \in \mathrm{Aut}(\mathbb{Z}^\theta)$, $\alpha \in \mathbb{Z}^\theta$, and $Q \in {}^H_H\mathcal{YD}$. Let V be an $\mathbb{N}_0^\theta$-graded object in ${}^H_H\mathcal{YD}$, and assume that V is tensor decomposable. We define

$$\begin{aligned} s(([Q],\alpha)) &= ([Q], s(\alpha)), \\ \Phi_+^V &= \{([Q_l],\beta_l) \mid 1 \leq l \leq n\}, \\ \Phi_-^V &= \{([Q_l^*],-\beta_l) \mid 1 \leq l \leq n\}, \\ \Phi^V &= \Phi_+^V \cup \Phi_-^V, \end{aligned}$$

where Q_l and β_l, $1 \le l \le n$, are the irreducible Yetter-Drinfeld modules and their degrees, respectively, in the tensor decomposition of V in Definition 14.4.1(1), and $[\ ]$ means isomorphism class.

In Definition 14.4.7, $\Phi_+^V \cap \Phi_-^V = \emptyset$ and the set Φ^V has precisely $2n$ elements by Definition 14.4.1. By Lemma 14.4.5, Φ^V is well-defined, and if V and W are tensor decomposable $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$, then $\Phi^V = \Phi^W$ if and only if $V \cong W$.

LEMMA 14.4.8. *Let $M \in \mathcal{F}_\theta^H$ and $i \in \mathbb{I}$ be such that M_i is irreducible in ${}_H^H\mathcal{YD}$. Assume that $\mathcal{B}(M)$ is tensor decomposable. Then $([M_i], \alpha_i) \in \Phi_+^{\mathcal{B}(M)}$.*

PROOF. Take a tensor decomposition of $\mathcal{B}(M)$. Then for any $1 \le l \le n$, $\mathcal{B}(Q_l) = \bigoplus_{r \ge 0} \mathcal{B}(Q_l)(r\beta_l)$. Hence

$$M_i \cong \mathcal{B}(M)(\alpha_i) \cong \bigoplus_{\substack{r_1,\dots,r_n \ge 0 \\ \sum_{1\le j\le n} r_j\beta_j = \alpha_i}} \bigotimes_{l=1}^n \mathcal{B}(Q_l)(r_l\beta_l).$$

Since $\beta_l \in \mathbb{N}_0^\theta \setminus \{0\}$ for all l, it follows that $M_i \cong \bigoplus_{1\le j\le n, \beta_j = \alpha_i} Q_j$. This proves the lemma, since M_i is irreducible. $\square$

PROPOSITION 14.4.9. *Let $M \in \mathcal{F}_\theta^H$ and $i \in \mathbb{I}$. Assume that M_j is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I}$ and that $\mathcal{B}(M)$ is tensor decomposable. Then M is i-finite, $\mathcal{B}(R_i(M))$ is tensor decomposable, and $\Phi^{\mathcal{B}(R_i(M))} = s_i^M(\Phi^{\mathcal{B}(M)})$.*

PROOF. Note that $\mathcal{B}(M) \cong K_i^{\mathcal{B}(M)} \otimes \mathcal{B}(M_i)$ and that $K_i^{\mathcal{B}(M)} = \mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}$ is tensor decomposable by Proposition 14.4.6 with $S = E = \mathcal{B}(M)$ and $N = M$. By Lemma 13.5.11(2),(3), $K_i^{\mathcal{B}(M)}$ is an $\mathbb{N}_0^\theta$-graded subalgebra of $\mathcal{B}(M)$ generated by all $(\mathrm{ad}_{\mathcal{B}(M)} M_i)^n(M_j)$ with $j \in \mathbb{I} \setminus \{i\}$ and $n \ge 0$. Let

$$\phi = \{n\alpha_i + \alpha_j \mid n \ge 0, j \in \mathbb{I} \setminus \{i\}, (\mathrm{ad}_{\mathcal{B}(M)} M_i)^n(M_j) \ne 0\} \subseteq \mathbb{N}_0^\theta.$$

None of the elements of ϕ is a sum of the others. Hence for any $\alpha \in \phi$, the set $\Phi_+^{K_i^{\mathcal{B}(M)}}$ has to contain a pair $([Q_\alpha], \alpha)$ with $Q_\alpha \in {}_H^H\mathcal{YD}$. Thus M is i-finite since $\Phi_+^{K_i^{\mathcal{B}(M)}}$ is finite.

By Corollary 14.3.4(2), $T_i^{\mathcal{B}(M)}$ defines an isomorphism

$$L_i^{\mathcal{B}(R_i(M))} \cong \big(K_i^{\mathcal{B}(M)}\big)'$$

of $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$, where $\big(K_i^{\mathcal{B}(M)}\big)' = K_i^{\mathcal{B}(M)}$ as Yetter-Drinfeld modules, and $\big(K_i^{\mathcal{B}(M)}\big)'(\alpha) = K_i^{\mathcal{B}(M)}(s_i^{R_i(M)}(\alpha))$ for all $\alpha \in \mathbb{N}_0^\theta$. By the first sentence of the proof and by Lemma 14.4.11 below, $\big(K_i^{\mathcal{B}(M)}\big)'$ is tensor decomposable. Hence $L_i^{\mathcal{B}(R_i(M))}$ is tensor decomposable. The multiplication map

$$L_i^{\mathcal{B}(R_i(M))} \otimes \mathcal{B}(M_i^*) \xrightarrow{\cong} \mathcal{B}(R_i(M)) \tag{14.4.3}$$

is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$, and hence $\mathcal{B}(R_i(M))$ is tensor decomposable. Moreover,

$$\Phi_+^{L_i^{\mathcal{B}(R_i(M))}} = s_i^M\Big(\Phi_+^{K_i^{\mathcal{B}(M)}}\Big), \tag{14.4.4}$$

since $s_i^M = (s_i^{R_i(M)})^{-1}$. Thus

$$\Phi^{\mathcal{B}(R_i(M))} = \Phi^{L_i^{\mathcal{B}(R_i(M))}} \cup \{([M_i], \alpha_i), ([M_i^*], -\alpha_i)\} = s_i^M(\Phi^{\mathcal{B}(M)})$$

which completes the proof of the proposition. □

Recall the functor $F : \mathcal{W}(M) \to \mathbb{Z}^\theta$ from Section 9.1.

COROLLARY 14.4.10. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections and that $\mathcal{B}(P)$ is tensor decomposable for some $P \in \mathcal{F}_\theta^H(M)$.*

1. *For each $P' \in \mathcal{F}_\theta^H(M)$, $\mathcal{B}(P')$ is tensor decomposable.*
2. *For any $Q', Q'' \in \mathcal{F}_\theta^H(M)$ and any morphism $w : [Q'] \to [Q'']$ in $\mathcal{W}(M)$, $F(w)(\Phi^{\mathcal{B}(Q')}) = \Phi^{\mathcal{B}(Q'')}$.*
3. *For any $\alpha \in \boldsymbol{\Delta}^{\mathcal{B}(M)\,\mathrm{re}}$ there exists $Q \in {}^H_H\mathcal{YD}$ with $([Q, \alpha]) \in \Phi^{\mathcal{B}(M)}$.*

PROOF. By Proposition 13.5.19 and the definition of $\mathcal{F}_\theta^H(M)$, for any tuple $P' \in \mathcal{F}_\theta^H(M)$ there exist $m \geq 0$, $i_1, \dots, i_m \in \mathbb{I}$ with $P' \cong R_{i_m} \dots R_{i_1}(P)$. Thus (1) and (2) follow from Proposition 14.4.9 and the definition of a morphism.

(3) Let $\alpha \in \boldsymbol{\Delta}^{\mathcal{B}(M)\,\mathrm{re}}$. Then $\alpha = w(\alpha_i)$ for some $w \in \mathrm{Hom}([P'], [M])$, $i \in \mathbb{I}$ with $P' \in \mathcal{F}_\theta^H(M)$. By (1) and Lemma 14.4.8, $([P_i'], \alpha_i) \in \Phi^{\mathcal{B}(P')}$. Therefore $([P_i'], \alpha) \in \Phi^{\mathcal{B}(M)}$ by (2). □

LEMMA 14.4.11. *Let $S \subseteq \mathbb{N}_0^\theta$ be a submonoid, and $s : S \to \mathbb{N}_0^\theta$ be an injective monoid morphism. For all $\mathbb{N}_0^\theta$-graded objects V in ${}^H_H\mathcal{YD}$ with $\mathrm{supp}(V) \subseteq S$, define $V' = V$ as Yetter-Drinfeld module with $\mathbb{N}_0^\theta$-grading*

$$V'(\alpha) = \begin{cases} V(s^{-1}(\alpha)), & \text{if } \alpha \in s(\mathrm{supp}(V)), \\ 0, & \text{otherwise,} \end{cases}$$

for all $\alpha \in \mathbb{N}_0^\theta$. Let X be a tensor decomposable $\mathbb{N}_0^\theta$-graded object in ${}^H_H\mathcal{YD}$ with $\mathrm{supp}(X) \subseteq S$. Then X' is tensor decomposable, and $\Phi_+^{X'} = s(\Phi_+^X)$.

PROOF. Since X is tensor decomposable, there is an isomorphism

$$X \cong \bigotimes_{l=1}^n \mathcal{B}(Q_l)$$

as in Definition 14.4.1. Let $1 \leq l \leq n$ and $\beta_l = \deg(Q_l)$. Then $\mathrm{supp}(Q_l) \subseteq \mathrm{supp}(X)$. Moreover, $\mathcal{B}(Q_l) = \bigoplus_{k \geq 0} Q_l^k$, where Q_l^k is the subspace of $\mathcal{B}(Q_l)$ spanned by the k-fold products of elements of Q_l. Hence

$$\mathcal{B}(Q_l)'(s(k\beta_l)) = \mathcal{B}(Q_l)(k\beta_l) = Q_l^k = (Q_l')^k = \mathcal{B}(Q_l')(ks(\beta_l))$$

in ${}^H_H\mathcal{YD}$ for all $k \geq 0$. Thus $\mathcal{B}(Q_l)' = \mathcal{B}(Q_l')$, where $\deg(Q_l') = s(\beta_l)$. It follows that

$$X' \cong \Big(\bigotimes_{l=1}^n \mathcal{B}(Q_l)\Big)' \cong \bigotimes_{l=1}^n \mathcal{B}(Q_l').$$

Hence $\Phi_+^{X'} = \{([Q_l'], s(\beta_l)) \mid 1 \leq l \leq n\}$. The injectivity of s ensures that $s(\beta_1), \dots, s(\beta_n)$ are non-zero and pairwise distinct. This implies the lemma. □

LEMMA 14.4.12. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $m \geq 0$. Assume that for each $P \in \mathcal{F}_\theta^H(M)$, any $[P]$-reduced sequence has length at most m. Then $\mathcal{B}(P)$ is tensor decomposable for some $P \in \mathcal{F}_\theta^H(M)$.*

PROOF. Let $n \geq 0$, $P \in \mathcal{F}_\theta^H(M)$, and $\kappa = (i_1, \dots, i_n)$ be a $[P]$-reduced sequence. Assume that for each $Q \in \mathcal{F}_\theta^H(M)$, the length of any $[Q]$-reduced sequence is at most n. Then for any $i \in \mathbb{I}$, $(i, i_1, \dots, i_n)$ is not an $r_i([P])$-reduced sequence. Hence $\alpha_i \in \Lambda^{[P]}(\kappa)$ for all $i \in \mathbb{I}$ by Lemma 9.2.2(2). It follows that $\mathcal{B}(P) = E^{\mathcal{B}(P)}(\kappa)$ by Corollary 14.1.14, and $E^{\mathcal{B}(P)}(\kappa)$ is tensor decomposable by Theorem 14.1.9(5),(6),(7). $\square$

PROPOSITION 14.4.13. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. The following are equivalent.*

(1) *The tuple M admits all reflections and $\mathcal{G}(M)$ is finite.*
(2) *The tuple M admits all reflections and $\mathcal{B}(P)$ is tensor decomposable for some $P \in \mathcal{F}_\theta^H(M)$.*
(3) *The Nichols algebra $\mathcal{B}(M)$ is tensor decomposable.*

PROOF. Assume that M admits all reflections and that $\mathcal{G}(M)$ is finite. Let $m = |\boldsymbol{\Delta}_+^{[M]\,\mathrm{re}}|$. Theorem 14.1.9(1) implies that for each $P \in \mathcal{F}_\theta^H(M)$, any $[P]$-reduced sequence has length at most m. Then $\mathcal{B}(P)$ is tensor decomposable for some $P \in \mathcal{F}_\theta^H(M)$ by Lemma 14.4.12. Thus (1) implies (2).

If $M \in \mathcal{F}_\theta^H$ admits all reflections and $\mathcal{B}(P)$ is tensor decomposable for some $P \in \mathcal{F}_\theta^H(M)$, then $\mathcal{B}(M)$ is tensor decomposable by Corollary 14.4.10(1). Therefore (2) implies (3).

Assume (3). Then M is i-finite and $\mathcal{B}(R_i(M))$ is tensor decomposable by Proposition 14.4.9. By induction on k it follows that M admits the reflection sequence $(i_1, \dots, i_k)$ and $\mathcal{B}(R_{i_k} \cdots R_{i_1}(M))$ is tensor decomposable for any $k \geq 0$ and any $i_1, \dots, i_k \in \mathbb{I}$. Thus M admits all reflections. Moreover, $\Phi^{\mathcal{B}(M)}$ is finite since $\mathcal{B}(M)$ is tensor decomposable. Thus, by Corollary 14.4.10(3), $\boldsymbol{\Delta}^{[M]\,\mathrm{re}}$ is finite. Hence (3) implies (1). $\square$

The following Theorem is already known due to Theorem 14.2.12. Nevertheless, the previous considerations allow us to provide a different proof based on the axioms (CG3) and (CG4). Note that compared with Theorem 14.2.12, we additionally assume that the semi-Cartan graph is finite.

THEOREM 14.4.14. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections and that the semi-Cartan graph $\mathcal{G}(M)$ is finite. Then $\mathcal{G}(M)$ is a Cartan graph.*

PROOF. By Theorem 13.6.2, $\mathcal{G}(M)$ is a semi-Cartan graph. Moreover, the finiteness of $\mathcal{G}(M)$ implies that $\mathcal{B}(P)$ is tensor decomposable for any $P \in \mathcal{F}_\theta^H(M)$ by Proposition 14.4.13 and by Corollary 14.4.10(1).

Let α be a real root of the semi-Cartan graph $\mathcal{G}(M)$ at a point X. Then $\alpha \in \mathbb{N}_0^\theta \cup -\mathbb{N}_0^\theta$ by Corollary 14.4.10(3). This proves (CG3).

To prove (CG4), let $Q' \in \mathcal{F}_\theta^H(M)$, $X = [Q']$, and $i, j \in \mathbb{I}$ with $i \neq j$. (We don't need to consider the case $i = j$ by Remark 9.1.16(4).) Then m_{ij}^X is finite, since $\mathcal{G}(M)$ is finite. Let $Q'' = (R_i R_j)^{m_{ij}^X}(Q')$ and $Y = [Q'']$. Then $Y = (r_i r_j)^{m_{ij}^X}(X)$. We have to show that $Y = X$.

Let $w = \mathrm{id}_Y(s_is_j)^{m_{ij}^X} : X \to Y$. Then $F(w) = \mathrm{id}_{\mathbb{Z}^\theta}$ by Theorem 9.2.23 and (CG3). By Corollary 14.4.10(2),

$$\Phi^{\mathcal{B}(Q')} = F(w)(\Phi^{\mathcal{B}(Q')}) = \Phi^{\mathcal{B}(Q'')},$$

and therefore $Y = X$. $\square$

14.5. Nichols algebras with finite Cartan graph

We prove that semi-simple Yetter-Drinfeld modules with a finite-dimensional Nichols algebra have a finite Cartan graph. Based on the previous Sections 14.1–14.4, we establish structural results on Nichols algebras with finite Cartan graph. A criterion for the finiteness of a Cartan graph in terms of reduced sequences was given in Proposition 9.2.25. Another one for the Cartan graph of a Nichols algebra was formulated in Proposition 14.4.13.

COROLLARY 14.5.1. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections and that $\mathcal{G}(M)$ is finite. Let $P \in \mathcal{F}_\theta^H(M)$, and let $\kappa = (i_1, \dots, i_m) \in \mathbb{I}^m$ with $m \in \mathbb{N}_0$ be a reduced decomposition of the longest element w_0 in* $\mathrm{Hom}(\mathcal{W}(M), [P])$*. Then κ is $[P]$-reduced. For all $1 \le k \le m$ let $\beta_k = \beta_k^{[P],\kappa}$ and $P_{\beta_k} \subseteq \mathcal{B}(P)$ in ${}^H_H\mathcal{YD}$ be as in Theorem* 14.1.9*(2).*

(1) *The multiplication map $\Bbbk[P_{\beta_m}] \otimes \cdots \otimes \Bbbk[P_{\beta_1}] \to \mathcal{B}(P)$ is bijective.*
(2) $\boldsymbol{\Delta}_+^{[P]\,\mathrm{re}} = \{\beta_1, \dots, \beta_m\}$.
(3) *Let $Q \in \mathcal{F}_\theta^H(M)$, $1 \le k \le m$, $i \in \mathbb{I}$, and let $w : [Q] \to [P]$ be a morphism in $\mathcal{W}(M)$. Assume that $\beta_k = w(\alpha_i)$. Then $P_{\beta_k} \cong Q_i$ in ${}^H_H\mathcal{YD}$.*
(4) *Let $Q \in \mathcal{F}_\theta^H(M)$ and $i \in \mathbb{I}$. Then $Q_i \cong P_{\beta_k}$ or $Q_i \cong P_{\beta_k}^*$ in ${}^H_H\mathcal{YD}$ for some $1 \le k \le m$.*
(5) *Let $i, j \in \mathbb{I}$, $i \ne j$, and $0 \le t \le -a_{ij}^{[P]}$. Then there exists $1 \le k \le m$ such that $\alpha_j + t\alpha_i = \beta_k$ and $(\mathrm{ad}\, P_i)^t(P_j) \cong P_{\beta_k}$ in ${}^H_H\mathcal{YD}$. In particular, $(\mathrm{ad}\, P_i)^t(P_j)$ is irreducible in ${}^H_H\mathcal{YD}$.*

PROOF. By Theorem 14.2.12 or by Theorem 14.4.14, $\mathcal{G}(M)$ is a finite Cartan graph. Thus (2) holds by Corollary 9.3.13. By Theorem 9.3.5(1), κ is $[P]$-reduced. Hence (1) follows from Theorem 14.1.9 and Corollary 14.1.14(1), since for all $i \in \mathbb{I}$, $\alpha_i \in \boldsymbol{\Delta}_+^{[P]\,\mathrm{re}}$. In particular, $\mathcal{B}(P)$ is tensor decomposable.

(3) By Corollary 14.4.10(2), $F(w)(\Phi^{[Q]}) = \Phi^{[P]}$. Since $([Q_i], \alpha_i) \in \Phi^{[Q]}$ by Lemma 14.4.8, it follows that

$$F(w)([Q_i], \alpha_i) = ([Q_i], w(\alpha_i)) = ([Q_i], \beta_k) \in \Phi^{[P]}.$$

Hence $Q_i \cong P_{\beta_k}$, since the elements $\beta_1, \dots, \beta_m$ are pairwise distinct.

(4) Since $\mathcal{G}(M)$ is connected, there is a morphism $w : [Q] \to [P]$, and by Corollary 14.4.10(2), $F(w)(\Phi^{[Q]}) = \Phi^{[P]}$. By Lemma 14.4.8, $([Q_i], \alpha_i) \in \Phi^{[Q]}$. Hence $([Q_i], w(\alpha_i)) \in \Phi^{[P]}$.

If $w(\alpha_i) \in \mathbb{N}_0^\theta$, then $Q_i \cong P_{\beta_k}$ for some k. If $w(\alpha_i) \in -\mathbb{N}_0^\theta$, then by Definition 14.4.7, $([Q_i^*], -w(\alpha_i)) \in \Phi_+^{\mathcal{B}(P)}$, hence $Q_i^* \cong P_{\beta_k}$ for some k.

(5) Let $i \in \mathbb{I}$. By Lemma 14.4.8, there is an index $1 \le h \le m$ such that $\beta_h = \alpha_i$ and $P_{\beta_h} \cong P_i$. Since $K_i^{\mathcal{B}(P)} \otimes \mathcal{B}(P_i) \cong \mathcal{B}(P)$, it follows from the remark above

Definition 14.4.1 and from Lemma 14.4.4, that

$$K_i^{\mathcal{B}(P)} \cong \bigotimes_{\substack{1\le k\le m\\ k\neq h}} \mathcal{B}(P_{\beta_k}). \tag{14.5.1}$$

We know from Theorem 13.2.8 that the algebra $K_i^{\mathcal{B}(P)}$ is generated by the homogeneous subspaces $(\operatorname{ad} P_i)^t(P_j)$ of degree $\alpha_j + t\alpha_i$, $j \neq i$, $j \in \mathbb{I}$, and $0 \le t \le -a_{ij}^{[P]}$. Since these subspaces have pairwise distinct degrees, we see that for all $j \neq i$, and $0 \le t \le -a_{ij}^{[P]}$,

$$K_i^{\mathcal{B}(P)}(\alpha_j + t\alpha_i) = (\operatorname{ad} P_i)^t(P_j)$$

as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$. On the other hand, the homogeneous part of degree $\alpha_j + t\alpha_i$ of the right hand side of (14.5.1) is the direct sum of all tensor products

$$\bigotimes_{\substack{1\le k\le m\\ k\neq h}} (P_{\beta_k})^{n_k},\ n_k \ge 0 \text{ for all } k,$$

where $\sum_{\substack{1\le k\le m\\ k\neq h}} n_k\beta_k = \alpha_j + t\alpha_i$. For all $k \neq h$, $\beta_k \notin \mathbb{N}_0\alpha_i$ since β_k and α_i are real roots of $[P]$. Thus the sum can have only one non-zero summand $n_k\beta_k$, and $n_k = 1$, for some k. Hence $(\operatorname{ad} P_i)^t(P_j) \cong P_{\beta_k}$ in ${}^H_H\mathcal{YD}$ for some $1 \le k \le m$. □

COROLLARY 14.5.2. *Under the assumptions of Corollary* 14.5.1, *let* $x \in P_{\beta_k}$, *where* $1 \le k \le m$. *Then*

$$\Delta_{\mathcal{B}(P)}(x) \in x \otimes 1 + 1 \otimes x + \Bbbk[P_{\beta_{k-1}}]\Bbbk[P_{\beta_{k-2}}]\cdots\Bbbk[P_{\beta_1}] \otimes \mathcal{B}(P).$$

PROOF. Let $n = |\deg(\beta_k)|$. Then, by Lemma 1.3.6,

$$\Delta_{\mathcal{B}(P)}(x) \in x \otimes 1 + 1 \otimes x + \bigoplus_{i=1}^{n-1} \mathcal{B}^i(P) \otimes \mathcal{B}^{n-i}(P).$$

By (14.1.6) we may assume that $k = m$. Since $E^{\mathcal{B}(P)}(i_1, \dots, i_k)$, as defined in Theorem 14.1.9(7), is an $\mathbb{N}_0^\theta$-graded right coideal subalgebra of $\mathcal{B}(P)$, and since $\Delta_{\mathcal{B}(P)}$ is graded, the claim follows by degree reasons from the tensor decomposition of $E^{\mathcal{B}(P)}(i_1, \dots, i_m)$ in Theorem 14.1.9(6). □

COROLLARY 14.5.3. *Let* $M \in \mathcal{F}_\theta^H$ *such that* M_j *is irreducible in* ${}^H_H\mathcal{YD}$ *for all* $j \in \mathbb{I}$. *The following are equivalent.*

(1) $\mathcal{B}(M)$ *is finite-dimensional.*
(2) M *admits all reflections, and*
  (a) $\mathcal{G}(M)$ *is finite.*
  (b) $\mathcal{B}(P_i)$ *is finite-dimensional for all* $P \in \mathcal{F}_\theta^H(M)$ *and* $i \in \mathbb{I}$.

In particular, if $\mathcal{B}(M)$ *is finite-dimensional, then* $\mathcal{G}(M)$ *is a finite Cartan graph.*

PROOF. Assume that $\mathcal{B}(M)$ is finite-dimensional. Then M admits all reflections and $\dim \mathcal{B}(P) = \dim \mathcal{B}(M)$ for any $P \in \mathcal{F}_\theta^H(M)$ by Proposition 13.6.4. This implies (2)(b). By Theorem 14.1.9(1) and (4), for each $P \in \mathcal{F}_\theta^H(M)$ the length of any $[P]$-reduced sequence is at most $\dim \mathcal{B}(M)$. Then $\mathcal{G}(M)$ is finite by Lemma 14.4.12 and Proposition 14.4.13.

(2) implies (1) by Corollary 14.5.1(1) and (3). □

THEOREM 14.5.4. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Assume that $\mathcal{N}$ admits all reflections and $\mathcal{G}(M)$ is finite. Then the canonical map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ is bijective.*

PROOF. By Theorem 14.2.12 or 14.4.14, $\mathcal{G}(M)$ is a Cartan graph. Moreover, $\mathcal{G}(M)$ is finite by assumption. By Proposition 9.3.9 there exist $l \geq 0$ and $i_1, \dots, i_l \in \mathbb{I}$ such that $w_0 = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_l}$ is a longest element of $\mathrm{Hom}(\mathcal{W}(M), [M])$ and $\ell(w_0) = l$. Then $(i_1, \dots, i_l)$ is $[M]$-reduced and $\boldsymbol{\Delta}_+^{[M]\,\mathrm{re}} = \Lambda^{[M]}(i_1, \dots, i_l)$ by Corollary 14.5.1. Hence $\alpha_i \in \Lambda^{[M]}(i_1, \dots, i_l)$ for all $i \in \mathbb{I}$. Thus the claim follows from Corollary 14.1.14(2). $\square$

14.6. Tensor decomposable right coideal subalgebras

We are going to determine all tensor decomposable $\mathbb{N}_0^\theta$-graded right coideal subalgebras of Nichols algebras $\mathcal{B}(M)$, where $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$, and M admits all reflections. In Theorem 14.6.6 we relate the poset structure of the set of these right coideal subalgebras to the right Duflo order on the Weyl groupoid. In Corollary 14.6.8 we provide a variant of this correspondence for those M with finite Cartan graph. We also prove freeness of right coideal subalgebras over each other.

LEMMA 14.6.1. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Let $E \subseteq \mathcal{B}(M)$ be an $\mathbb{N}_0^\theta$-graded right coideal in ${}^H_H\mathcal{YD}$. If $E \neq \Bbbk 1$, then $M_i \subseteq E$ for some $i \in \mathbb{I}$.*

PROOF. Let $E = \bigoplus_{n \geq 0} E(n)$ be the natural $\mathbb{N}_0$-grading of (14.4.1). Since $E \neq \Bbbk 1$, $E(n) \neq 0$ for some $n \geq 1$. The map

$$\mathcal{B}^n(M) \subseteq \mathcal{B}(M) \xrightarrow{\Delta_{\mathcal{B}(M)}} \mathcal{B}(M) \otimes \mathcal{B}(M) \xrightarrow{\pi_1 \otimes \mathrm{id}} (M_1 \oplus \cdots \oplus M_\theta) \otimes \mathcal{B}(M)$$

is injective, since $\mathcal{B}(M)$ is strictly graded. Hence

$$(\pi_1 \otimes \mathrm{id}) \Delta_{\mathcal{B}(M)}(E(n)) \subseteq \pi_1(E) \otimes \mathcal{B}(M) \neq 0.$$

Thus $E \cap (M_1 \oplus \cdots \oplus M_\theta) = \pi_1(E)$ is non-zero. Then $E \cap M_i \neq 0$ for some i, since E is $\mathbb{N}_0^\theta$-graded. Since M_i is irreducible, $M_i \subseteq E$. $\square$

LEMMA 14.6.2. *Let $M \in \mathcal{F}_\theta^H$ be such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $w \in \mathrm{Hom}(\mathcal{W}(M), [M])$, and let E be an $\mathbb{N}_0^\theta$-graded right coideal subalgebra of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$ with a tensor decomposition $E \cong \bigotimes_{l=1}^n \mathcal{B}(Q_l)$ such that*

$$\{\deg(Q_l) \mid 1 \leq l \leq n\} = \boldsymbol{\Delta}^{[M]\,\mathrm{re}}(w).$$

Then $E = \widehat{E}^{\mathcal{B}(M)}(\kappa)$ for any reduced decomposition κ of w.

PROOF. By Theorem 14.2.12, $\mathcal{G}(M)$ is a Cartan graph.

We proceed by induction on $\ell(w)$. If $\ell(w) = 0$, then $E = \Bbbk 1 = \widehat{E}^{\mathcal{B}(M)}()$.

Assume that $\ell(w) \geq 1$, and let $\kappa = (i_1, \dots, i_{\ell(w)})$ be a reduced decomposition of w. Then $\alpha_{i_1} \in \boldsymbol{\Delta}^{[M]\,\mathrm{re}}(w)$ by Theorem 9.3.5(2), and hence $E(\alpha_{i_1}) \neq 0$. Since

$$E(\alpha_{i_1}) \subseteq \mathcal{B}(M)(\alpha_{i_1}) = M_{i_1}$$

and M_{i_1} is irreducible, it follows that $E(\alpha_{i_1}) = M_{i_1}$. Let $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id}_M)$. By Proposition 14.4.6 for $\mathcal{N}_0$, without loss of generality $Q_1 \cong M_{i_1}$, $\deg(Q_1) = \alpha_{i_1}$, and $E \cap K_{i_1}^{\mathcal{B}(M)} \cong \mathcal{B}(Q_2) \otimes \cdots \otimes \mathcal{B}(Q_n)$. Then, by Corollaries 14.3.6(1) and 14.3.4(2),

$$E' := (T_{i_1}^{\mathcal{B}(M)})^{-1}(E \cap K_{i_1}^{\mathcal{B}(M)}) \in \mathcal{K}_{i_1}^-(\mathcal{B}(R_{i_1}(M)))$$

and

$$E' \cong \mathcal{B}(Q_2') \otimes \cdots \otimes \mathcal{B}(Q_l')$$

with $Q_l' = Q_l$ in ${}^H_H\mathcal{YD}$ and $\deg(Q_l') = s_{i_1}^M(\deg(Q_l))$ for all $2 \le l \le n$. It follows from Theorem 9.3.5(2) that

$$\{\deg(Q_l') \mid 2 \le l \le n\} = \boldsymbol{\Delta}^{[R_{i_1}(M)]\,\mathrm{re}}(\mathrm{id}_{[R_{i_1}(M)]} s_{i_2} \cdots s_{i_{\ell(w)}}).$$

Therefore $E' = \widehat{E}^{\mathcal{B}(R_{i_1}(M))}(i_2, \dots, i_n)$ by induction hypothesis, and we may conclude directly that $E = t_{i_1}^{\mathcal{B}(M)}(E') = \widehat{E}^{\mathcal{B}(M)}(\kappa)$. $\square$

PROPOSITION 14.6.3. *Let $M \in \mathcal{F}_\theta^H$ be such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let E be an $\mathbb{N}_0^\theta$-graded right coideal subalgebra of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$.*

(1) *The following are equivalent.*
   (a) *There exists $w \in \mathrm{Hom}(\mathcal{W}(M), [M])$, such that for any reduced decomposition κ of w, $E = E^{\mathcal{B}(M)}(\kappa) = \widehat{E}^{\mathcal{B}(M)}(\kappa)$.*
   (b) *There exists an $[M]$-reduced sequence κ such that $E = \widehat{E}^{\mathcal{B}(M)}(\kappa)$.*
   (c) *E is tensor decomposable.*

(2) *If $\mathcal{B}(Q_1) \otimes \cdots \otimes \mathcal{B}(Q_n)$ is a tensor decomposition of E then in* (1) *one has $\boldsymbol{\Delta}^{[M]\,\mathrm{re}}(w) = \{\deg(Q_l) : 1 \le l \le n\}$.*

PROOF. By Theorem 14.2.12, $\mathcal{G}(M)$ is a Cartan graph.

(1) (a) implies (b) since any reduced decomposition of w is $[M]$-reduced because of Theorem 9.3.5. (b) implies (c) by Example 14.4.3 and by Corollary 14.3.9.

Assume that E is tensor decomposable. We prove (a) and (2). Let $n \ge 0$ and let $Q_1, \dots, Q_n \in {}^H_H\mathcal{YD}$ be irreducible objects such that

$$E \cong \mathcal{B}(Q_1) \otimes \cdots \otimes \mathcal{B}(Q_n)$$

as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$. For each $1 \le l \le n$ let $\beta_l = \deg(Q_l)$. We prove by induction on n that there exists an element $w \in \mathrm{Hom}(\mathcal{W}(M), [M])$ such that $\boldsymbol{\Delta}^{[M]\,\mathrm{re}}(w) = \{\beta_1, \dots, \beta_n\}$ and that $E = \widehat{E}^{\mathcal{B}(M)}(\kappa)$ for any reduced decomposition κ of w.

If $n = 0$ then $E = \Bbbk 1$ and (a) and (2) hold for $w = \mathrm{id}_{[M]}$.

Assume that $n \ge 1$. Then by Lemma 14.6.1 there exists $1 \le i \le \theta$ with $M_i \subseteq E$. Let $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id}_M)$. By Proposition 14.4.6 for $\mathcal{N}_0$, without loss of generality $Q_1 \cong M_i$, $\beta_1 = \alpha_i$, and $E \cap K_i^{\mathcal{B}(M)} \cong \mathcal{B}(Q_2) \otimes \cdots \otimes \mathcal{B}(Q_n)$. Then, by Corollaries 14.3.6(1) and 14.3.4(2),

$$E' := (T_i^{\mathcal{B}(M)})^{-1}(E \cap K_i^{\mathcal{B}(M)}) \in \mathcal{K}_i^-(\mathcal{B}(R_i(M)))$$

and

$$E' \cong \mathcal{B}(Q_2') \otimes \cdots \otimes \mathcal{B}(Q_l')$$

with $Q_l' = Q_l$ in ${}^H_H\mathcal{YD}$ and $\deg(Q_l') = s_i^M(\deg(Q_l))$ for all $2 \le l \le n$. Thus, by induction hypothesis, there exists $w' \in \mathrm{Hom}(\mathcal{W}(M), [R_i(M)])$ with

$$\{s_i^{[M]}(\beta_l) : 2 \le l \le n\} = \boldsymbol{\Delta}^{[R_i(M)]\,\mathrm{re}}(w').$$

Let

$$w = s_i^{[R_i(M)]} w' \in \mathrm{Hom}(\mathcal{W}(M), [M]).$$

By definition, $w'^{-1}(\alpha_i) \in \mathbb{N}_0^\theta$ since $\alpha_i \notin \boldsymbol{\Delta}^{[R_i(M)]\,\mathrm{re}}(w')$. Hence $\ell(w) = \ell(w') + 1$ by Lemma 9.1.21 and Theorem 9.3.5(2). Thus (2) holds with $w \in \mathrm{Hom}(\mathcal{W}(M), [M])$, and then (a) is true because of Lemma 14.6.2 and Corollary 14.3.9. $\square$

DEFINITION 14.6.4. Let $\mathcal{G}$ be a semi-Cartan graph, and let X be a point of $\mathcal{G}$. For all $w_1, w_2 \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$, we define $w_1 \le_D w_2$ if and only if any reduced decomposition $(i_1, \dots, i_k)$ of w_1, where $k = \ell(w_1)$, can be extended to a reduced decomposition $(i_1, \dots, i_k, \dots, i_l)$ of w_2, where $l = \ell(w_2)$. The partial order $\le_D$ on $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ is called the (right) **Duflo order** or the **weak Bruhat order**.

PROPOSITION 14.6.5. *Let $\mathcal{G}$ be a semi-Cartan graph, X a point of $\mathcal{G}$, and $w_1, w_2 \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$. Let $k = \ell(w_1)$ and $l = \ell(w_2)$. The following are equivalent.*

1. *$w_1 \le_D w_2$.*
2. *$k \le l$, and there is a reduced decomposition $(i_1, \dots, i_l)$ of w_2, such that $(i_1, \dots, i_k)$ is a reduced decomposition of w_1.*
3. *$\ell(w_2) = \ell(w_1) + \ell(w_1^{-1} w_2)$.*

PROOF. (1) implies (2) trivially.

Assume (2). Let $Y = r_{i_k} \cdots r_{i_1}(X)$. Then

$$w_1^{-1} w_2 = \mathrm{id}_Y s_{i_k} \cdots s_{i_1} s_{i_1} \cdots s_{i_l} = \mathrm{id}_Y s_{i_{k+1}} \cdots s_{i_l},$$

and hence $\ell(w_1^{-1} w_2) \le l - k$. Assume that $m = \ell(w_1^{-1} w_2) < l - k$, and let $(j_1, \dots, j_m)$ be a reduced decomposition of $w_1^{-1} w_2$. Then

$$w_2 = w_1 w_1^{-1} w_2 = \mathrm{id}_X s_{i_1} \cdots s_{i_k} s_{j_1} \cdots s_{j_m},$$

a contradiction to $\ell(w_2) = l$. Thus (2) implies (3).

Assume (3). Then $l - k = \ell(w_1^{-1} w_2) \ge 0$. We prove that $w_1 \le_D w_2$. Let $(i_1, \dots, i_k)$ be a reduced decomposition of w_1, and let $(j_1, \dots, j_{l-k})$ be a reduced decomposition of $w_1^{-1} w_2$. Then

$$w_2 = w_1 w_1^{-1} w_2 = s_{i_1} \cdots s_{i_k} s_{j_1} \cdots s_{j_{l-k}}.$$

Since $\ell(w_2) = l$, $(i_1, \dots, i_k, j_1, \dots, j_{l-k})$ is a reduced decomposition of w_2. Hence $w_1 \le_D w_2$. $\square$

For any $M \in \mathcal{F}_\theta^H$ let

$$\mathcal{K}^{\mathrm{td}}(\mathcal{B}(M)) = \{E \in \mathcal{K}(\mathcal{B}(M)) \mid E \text{ is tensor decomposable}\}.$$

THEOREM 14.6.6. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Then for all $P \in \mathcal{F}_\theta^H(M)$ the map*

$$E^{\mathcal{B}(P)} : \mathrm{Hom}(\mathcal{W}(M), [P]) \to \mathcal{K}^{\mathrm{td}}(\mathcal{B}(P)), \quad w \mapsto E^{\mathcal{B}(P)}(w),$$

is bijective, order preserving and order reflecting, where $E^{\mathcal{B}(P)}(w)$ is defined in Proposition 14.6.3*(1), and* $\mathrm{Hom}(\mathcal{W}(M), [P])$ *and* $\mathcal{K}^{\mathrm{td}}(\mathcal{B}(P))$ *are ordered by the Duflo order and by inclusion, respectively.*

PROOF. By Theorem 14.2.12, $\mathcal{G}(M)$ is a Cartan graph, and the general theory in Section 9.3 applies. The map $E^{\mathcal{B}(P)}$ is well-defined by Proposition 14.6.3. Let $w_1, w_2 \in \mathrm{Hom}(\mathcal{W}(M), [P])$ with $w_1 \leq_D w_2$, and let $k = \ell(w_1)$, $l = \ell(w_2)$. Then $k \leq l$, and there are labels $i_1, \dots, i_l$ of $\mathcal{G}(M)$ such that $(i_1, \dots, i_k)$ and $(i_1, \dots, i_l)$ are reduced decompositions of w_1 and w_2, respectively. Hence

$$E^{\mathcal{B}(P)}(w_1) = E^{\mathcal{B}(P)}(i_1, \dots, i_k) \subseteq E^{\mathcal{B}(P)}(i_1, \dots, i_l) = E^{\mathcal{B}(P)}(w_2).$$

To prove injectivity of $E^{\mathcal{B}(P)}$, let $w_1, w_2 \in \mathrm{Hom}(\mathcal{W}(M), [P])$, and assume that $E^{\mathcal{B}(P)}(w_1) = E^{\mathcal{B}(P)}(w_2)$. Let κ_1 and κ_2 be a reduced decomposition of w_1 and w_2, respectively. Then $\widehat{E}^{\mathcal{B}(P)}(\kappa_1) = \widehat{E}^{\mathcal{B}(P)}(\kappa_2)$ by Corollary 14.3.9. Hence, by Proposition 14.6.3, $\boldsymbol{\Delta}^{[P]\,\mathrm{re}}(w_1) = \boldsymbol{\Delta}^{[P]\,\mathrm{re}}(w_2)$. Now Corollary 9.3.8(2) implies that $w_1 = w_2$.

The map $E^{\mathcal{B}(P)}$ is surjective by Proposition 14.6.3(1).

Finally, let $w_1, w_2 \in \mathrm{Hom}(\mathcal{W}(M), [P])$ with $E^{\mathcal{B}(P)}(w_1) \subseteq E^{\mathcal{B}(P)}(w_2)$. We have to prove that $w_1 \leq_D w_2$. Let $(i_1, \dots, i_k)$ with $k \geq 0$ be a reduced decomposition of w_1. By assumption,

$$E^{\mathcal{B}(P)}(i_1, \dots, i_k) = E^{\mathcal{B}(P)}(w_1) \subseteq E^{\mathcal{B}(P)}(w_2).$$

We proceed by induction on k. If $k = 0$ then $w_1 = \mathrm{id}_{[P]}$ and we are done. Assume that $k > 0$. Then

$$M_{i_1} \subseteq E^{\mathcal{B}(P)}(i_1, \dots, i_k) \subseteq E^{\mathcal{B}(P)}(w_2).$$

Now Proposition 14.4.6 implies that there is a tensor decomposition of $E^{\mathcal{B}(P)}(w_2)$ with tensor factor $\mathcal{B}(M_{i_1})$ such that $\deg(M_{i_1}) = \alpha_{i_1}$. Therefore, using that $E^{\mathcal{B}(P)}$ is injective, Proposition 14.6.3(2) implies that $\alpha_{i_1} \in \boldsymbol{\Delta}^{[P]\,\mathrm{re}}(w_2)$. Then, by Corollary 9.3.7, there exists a reduced decomposition $(j_1, j_2, \dots, j_l)$ of w_2 with $i_1 = j_1$. Using this, Corollary 14.3.6(1) implies that

$$\begin{aligned} E^{\mathcal{B}(R_{i_1}(P))}(i_2, \dots, i_k) &= (t_{i_1}^{\mathcal{B}(P)})^{-1}(E^{\mathcal{B}(P)}(i_1, \dots, i_k)) \\ &\subseteq (t_{i_1}^{\mathcal{B}(P)})^{-1}(E^{\mathcal{B}(P)}(j_1, \dots, j_l)) = E^{\mathcal{B}(R_{i_1}(P))}(j_2, \dots, j_l). \end{aligned}$$

Thus $s_{i_1} w_1 \leq_D s_{i_1} w_2$ by induction hypothesis, and hence $w_1 \leq_D w_2$. □

For tuples M with finite Cartan graph, Theorem 14.6.6 has a slightly simpler variant which we will state in Corollary 14.6.8 below.

LEMMA 14.6.7. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $E, E' \in \mathcal{K}(\mathcal{B}(M))$ such that $E \subseteq E'$ and E' is tensor decomposable. Then E is tensor decomposable.*

PROOF. By Theorem 14.6.6, there exists a morphism $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), [M])$ with $E' = E^{\mathcal{B}(M)}(w)$. We proceed by induction on $\ell(w)$.

If $E = \Bbbk 1$, (which holds in particular for $\ell(w) = 0$,) then E is tensor decomposable. Assume that $E \neq \Bbbk 1$. Then $M_i \subseteq E$ for some $i \in \mathbb{I}$ by Lemma 14.6.1. Thus

$$E^{\mathcal{B}(M)}(s_i^{[R_i(M)]}) \subseteq E \subseteq E'.$$

It follows that

$$(t_i^{\mathcal{B}(M)})^{-1}(E) \subseteq (t_i^{\mathcal{B}(M)})^{-1}(E') = E^{\mathcal{B}(R_i(M))}(s_i w)$$

and $\ell(s_i w) = \ell(w) - 1$ as in the last paragraph of the proof of Theorem 14.6.6. Thus $(t_i^{\mathcal{B}(M)})^{-1}(E)$ is tensor decomposable by induction hypothesis. By Theorem 14.6.6,

$(t_i^{\mathcal{B}(M)})^{-1}(E) = E^{\mathcal{B}(R_i(M))}(v)$ for some $v \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), [R_i(M)])$, and therefore $E = E^{\mathcal{B}(M)}(s_i v)$ is tensor decomposable. □

COROLLARY 14.6.8. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections, and that $\mathcal{G}(M)$ is finite. For all $P \in \mathcal{F}_\theta^H(M)$, the map*

$$E^{\mathcal{B}(P)} : \mathrm{Hom}(\mathcal{W}(M), [P]) \to \mathcal{K}(\mathcal{B}(P)), \quad w \mapsto E^{\mathcal{B}(P)}(w),$$

is bijective, order preserving and order reflecting, where $E^{\mathcal{B}(P)}(w)$ is defined in Proposition 14.6.3(1), and $\mathrm{Hom}(\mathcal{W}(M), [P])$ and $\mathcal{K}(\mathcal{B}(P))$ are ordered by the Duflo order and by inclusion, respectively.

PROOF. By Proposition 14.4.13, $\mathcal{B}(M)$ is tensor decomposable since $\mathcal{G}(M)$ is finite. Thus

$$\mathcal{K}^{\mathrm{td}}(\mathcal{B}(M)) = \mathcal{K}(\mathcal{B}(M))$$

by Lemma 14.6.7. Hence the claim follows from Theorem 14.6.6. □

COROLLARY 14.6.9. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections, and that $\mathcal{G}(M)$ is finite. Let*

$$E_1, E_2 \in \mathcal{K}(\mathcal{B}(P))$$

with $P \in \mathcal{F}_\theta^H(M)$ and $E_1 \subseteq E_2 \subseteq \mathcal{B}(P)$. Then there are integers $0 \le l \le m$ and a $[P]$-reduced sequence $(i_1, \dots, i_m) \in \mathbb{I}^m$ with finite-dimensional irreducible subobjects P_{β_k} of $\mathcal{B}(P)$ in ${}^H_H\mathcal{YD}$ for all $1 \le k \le m$, as defined in Theorem 14.1.9, such that $\Bbbk[P_{\beta_k}] \cong \mathcal{B}(P_{\beta_k})$ for all $1 \le k \le m$, and the multiplication maps

$$\Bbbk[P_{\beta_l}] \otimes \cdots \otimes \Bbbk[P_{\beta_1}] \to E_1,$$
$$\Bbbk[P_{\beta_m}] \otimes \cdots \otimes \Bbbk[P_{\beta_1}] \to E_2$$

are bijective. In particular, E_2 is a free right module over E_1.

PROOF. By Corollary 14.6.8, there are $w_1, w_2 \in \mathrm{Hom}(\mathcal{W}(M), [P])$ such that $E^{\mathcal{B}(P)}(w_1) = E_1$ and $E^{\mathcal{B}(P)}(w_2) = E_2$. Moreover, by the definition of the Duflo order, there is a reduced decomposition $(i_1, \dots, i_m)$ of w_2, such that $(i_1, \dots, i_l)$ is a reduced decomposition of w_1, where $0 \le l \le m$. Then $E_2 = E^{\mathcal{B}(P)}(i_1, \dots, i_m)$ and $E_1 = E^{\mathcal{B}(P)}(i_1, \dots, i_l)$ by Proposition 14.6.3(1).

The bijectivity of the multiplication maps for w_1 and w_2 follows from Theorem 14.1.9. Since the multiplication map $\Bbbk[P_{\beta_m}] \cdots \Bbbk[P_{\beta_{l+1}}] \otimes E_1 \to E_2$ is bijective, E_2 is free over E_1. □

14.7. Notes

The content of Chapter 14 is mostly new.

14.2. In [**Dru11**], Section 4.3, it was noted that decompositions of the longest element of a finite Weyl group into the product of two elements can be realized algebraically as a tensor product decomposition of a left and a right coideal subalgebra of the positive part of the associated quantized enveloping algebra. Section 14.2 is partially motivated by this observation.

14.3. A variant of Corollary 14.3.6 was proven in [**HS13a**], Theorem 5.6.

14.6. A variant of Theorem 14.6.6 was proven in [**HS13a**], Theorem 6.12.

Part 4

Applications

CHAPTER 15

Nichols algebras of diagonal type

We are going to discuss the general reflection theory of pre-Nichols systems for pre-Nichols algebras of diagonal type. We study root vector sequences in analogy to tensor decompositions of graded right coideal subalgebras of Nichols algebras. In Section 15.3 we classify rank two Nichols algebras of diagonal type with finite Cartan graph and those of finite dimension. Partial results are provided in rank three in Section 15.4. In Section 15.5 we prove that finite-dimensional pre-Nichols algebras of diagonal type are Nichols and that finite-dimensional pointed Hopf algebras with abelian coradical are generated as algebras by group-like and skew-primitive elements (over algebraically closed fields of characteristic 0). The proofs are based on the reflection theory in Chapter 14 and the previous sections in this Chapter.

15.1. Reflections of Nichols algebras of diagonal type

In this section we study Nichols algebras of braided vector spaces of diagonal type in more detail. Among such Nichols algebras, the tensor decomposable ones in the sense of Definition 14.4.1 are best understood. Decomposability was characterized in Proposition 14.4.13 in terms of reflections.

Let H be a Hopf algebra with bijective antipode, $\theta \in \mathbb{N}$, $\mathbb{I} = \{1, \dots, \theta\}$, and let $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta^H$ be a tuple of one-dimensional Yetter-Drinfeld modules. A characterization of one-dimensional Yetter-Drinfeld modules was given in Example 1.4.3 if H is a group algebra, and in Example 3.4.3 in general. In Proposition 15.1.10 we will define the small Cartan graph $\mathcal{G}_{\mathrm{s}}(M)$ of M whenever M admits all reflections. In Theorem 15.1.14 we show that if M is of Cartan type, then $\mathcal{G}_{\mathrm{s}}(M)$ has only one point, and that $\mathcal{G}_{\mathrm{s}}(M)$ is finite if and only if the Cartan matrix of M is of finite type.

We start the section with a Lemma of Rosso, which is fundamental to deal with reflections of tuples in $\mathcal{F}_\theta^H$. Recall the definition of $\varphi_n \in \mathbb{Z}\mathbb{B}_{n+1}$ for $n \geq 1$ from Corollary 1.8.14 and let $\varphi_0 = 0$. For any braided vector space V and for all $n \geq 0$, let φ_n also denote the image of φ_n under the representation of $\mathbb{Z}\mathbb{B}_{n+1}$ on $V^{\otimes n+1}$ introduced in Section 1.7.

LEMMA 15.1.1. *(Rosso's Lemma) Let V be a braided vector space of dimension at least two and let $(q_{ij})_{1\leq i,j\leq 2} \in (\Bbbk^\times)^{2\times 2}$. Choose linearly independent elements x_1, x_2 of V. Assume that $c_{V,V}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $1 \leq i, j \leq 2$. Let $n \in \mathbb{N}_0$. Then the following hold.*

(1) *For each $i \in \{1, 2\}$, $x_i^n = 0$ in $\mathcal{B}(V)$ if and only if $(n)^!_{q_{ii}} = 0$.*
(2) $\varphi_n(x_1^{\otimes n} \otimes x_2) = (n)_{q_{11}}(1 - q_{11}^{n-1} q_{12} q_{21}) x_1^{\otimes n} \otimes x_2$.
(3) *The following are equivalent.*
    (a) $(\operatorname{ad} x_1)^n(x_2) \neq 0$ *and* $(\operatorname{ad} x_1)^{n+1}(x_2) = 0$ *in* $\mathcal{B}(V)$.

(b) $(n+1)_{q_{11}}(q_{11}^n q_{12}q_{21}-1)=0$ *and* $(k+1)_{q_{11}}(q_{11}^k q_{12}q_{21}-1)\neq 0$ *for any* $0\leq k<n$.

PROOF. (1) Let $i\in\{1,2\}$. By Theorem 7.1.2(3), $x_i^n=0$ if and only if $S_n(x_i^{\otimes n})=0$. By (1.9.3), this is equivalent to $(n)^!_{q_{ii}}x_i^{\otimes n}=0$.

(2) We proceed by induction on n. For $n=0$ the claim is obviously true. For $n\geq 1$ use Corollary 1.8.14(3) to conclude that

$$\begin{aligned}\varphi_n(x_1^{\otimes n}\otimes x_2)&=(1-c_1c_2\cdots c_{n-1}c_n^2c_{n-1}\cdots c_1+\varphi_{n-1}{}^{\uparrow 1}c_1)(x_1^{\otimes n}\otimes x_2)\\&=(1-q_{11}^{2n-2}q_{12}q_{21}+(n-1)_{q_{11}}(1-q_{11}^{n-2}q_{12}q_{21})q_{11})x_1^{\otimes n}\otimes x_2\\&=(n)_{q_{11}}(1-q_{11}^{n-1}q_{12}q_{21})x_1^{\otimes n}\otimes x_2.\end{aligned}$$

(3) Let $V_i=\Bbbk x_i$ for $i\in\{1,2\}$. Then $\mathcal{B}(V_1\oplus V_2)\subseteq\mathcal{B}(V)$ by Corollary 7.1.15(2). Let $m\in\mathbb{N}_0$. Then $(\operatorname{ad}x_1)^m(x_2)=0$ if and only if $X_m^{V_1,V_2}=0$ by Theorem 13.3.1(2). Now recall that $X_m^{V_1,V_2}=\varphi_m\varphi_{m-1}{}^{\uparrow 1}\cdots\varphi_1{}^{\uparrow m}$ by the definition of $X_m^{V_1,V_2}$ and by Corollary 1.8.14(4). Hence (3) follows from (2). □

For all $j\in\mathbb{I}$ let x_j be a basis of M_j, and let $g_j\in H$, $\chi_j\in\operatorname{Alg}(H,\Bbbk)$ such that

$$\delta_{M_j}(x_j)=g_j\otimes x_j,\quad h\cdot x_j=\chi_j(h)x_j \tag{15.1.1}$$

for all $h\in H$. Then g_j is an invertible group-like element for all $j\in\mathbb{I}$. For all $j,k\in\mathbb{I}$ let $q_{jk}\in\Bbbk^\times$ such that

$$c_{M_j,M_k}(x_j\otimes x_k)=q_{jk}x_k\otimes x_j.$$

REMARK 15.1.2. By Example 3.4.3, $g_ig_j=g_jg_i$ and $\chi_i\chi_j=\chi_j\chi_i$ for all $i,j\in\mathbb{I}$. For any $\alpha=\sum_{i\in\mathbb{I}}a_i\alpha_i$ in $\mathbb{Z}^\theta$ let

$$g_\alpha=\prod_{i\in\mathbb{I}}g_i^{a_i}\in H,\quad \chi_\alpha=\prod_{i\in\mathbb{I}}\chi_i^{a_i}\in\operatorname{Alg}(H,\Bbbk). \tag{15.1.2}$$

Let $k\geq 0$, $i_1,\dots,i_k\in\mathbb{I}$, $V=M_{i_1}\otimes\cdots\otimes M_{i_k}$, and $v\in V$. Then, by definition,

$$\delta_V(x)=g_\alpha\otimes x,\quad hv=\chi_\alpha(h)v$$

for any $h\in H$, where $\alpha=\sum_{n=1}^k\alpha_{i_n}$.

The elements $(\operatorname{ad}x_i)^m(x_j)$ in $T(M)$, where $i,j\in\mathbb{I}$ with $i\neq j$ and $m\geq 0$, will play a crucial role in the sequel. We give an explicit form of them in the following Lemma.

LEMMA 15.1.3. *Assume that* $\theta\geq 2$. *Let* $i,j\in\mathbb{I}$ *with* $i\neq j$ *and let* $m\geq 0$. *Then*

$$(\operatorname{ad}x_i)^m(x_j)=\sum_{k=0}^m(-1)^kq_{ii}^{k(k-1)/2}q_{ij}^k\binom{m}{k}_{q_{ii}}x_i^{m-k}x_jx_i^k.$$

PROOF. By Remark 15.1.2, $g_ig_j=g_jg_i$. Thus the claim holds by Proposition 4.3.12(1). □

We discuss the structure of $R_i(M)$, where $i\in\mathbb{I}$, see Definition 13.4.2.

LEMMA 15.1.4. *Let* $i\in\mathbb{I}$. *Then* M *is* i*-finite if and only if for all integers* $j\in\mathbb{I}\setminus\{i\}$ *there exists* $m\in\mathbb{N}_0$ *such that* $(m+1)_{q_{ii}}(q_{ii}^mq_{ij}q_{ji}-1)=0$.

PROOF. This follows from Lemma 15.1.1(3). □

LEMMA 15.1.5. *Let $i, j \in \mathbb{I}$. Assume that $i \neq j$ and that M is i-finite.*

(1) $a_{ij}^M = -\min\{m \in \mathbb{N}_0 \mid (m+1)_{q_{ii}}(q_{ii}^m q_{ij} q_{ji} - 1) = 0\}$.

(2) $R_i(M)_j = \Bbbk(\operatorname{ad} x_i)^{-a_{ij}^M}(x_j)$, *and* $\dim R_i(M)_j = 1$.

(3) *Let $m \in \mathbb{N}_0$. Then*

$$\delta_{T(M)}((\operatorname{ad} x_i)^m(x_j)) = g_j g_i^m \otimes (\operatorname{ad} x_i)^m(x_j),$$
$$h \cdot (\operatorname{ad} x_i)^m(x_j) = \chi_j \chi_i^m(h)(\operatorname{ad} x_i)^m(x_j)$$

in $T(M)$ for all $h \in H$.

(4) $\delta_{R_i(M)_k}(y_k) = g_k g_i^{-a_{ik}^M} \otimes y_k$ *and* $h \cdot y_k = \chi_k \chi_i^{-a_{ik}^M}(h) y_k$ *for all* $k \in \mathbb{I}$, $y_k \in R_i(M)_k$, *and* $h \in H$, *where* $\chi_i^{-1} = \chi_i \circ \mathcal{S}$.

PROOF. (1) follows from Lemma 15.1.1(3), and (2) holds by the definitions of $R_i(M)_j$ and M, see Definition 13.4.2.

(3) follows from Remark 15.1.2, since $(\operatorname{ad} x_i)^m(x_j)$ is a linear combination of the monomials $x_i^{m-k} x_j x_i^k$ with $0 \leq k \leq m$.

(4) Let $y_i \in M_i^*$ with $\langle y_i, x_i \rangle = 1$. Then

$$\delta_{M_i^*}(y_i) = g_i^{-1} \otimes y_i, \quad h \cdot y_i = \chi_i^{-1}(h) y_i$$

for all $h \in H$ by Lemma 4.2.2. Thus (4) holds for $k = i$. For $k \neq i$ the claim follows from (3). $\square$

The following lemma is an immediate consequence of Lemma 15.1.5(1).

LEMMA 15.1.6. *Let $i, j \in \mathbb{I}$. Assume that $i \neq j$ and that M is i-finite. Let $m \in \mathbb{N}_0$. Then $a_{ij}^M = -m$ if and only if one of the following holds.*

(1) $m = 0$, $q_{ij} q_{ji} = 1$.

(2) $m \geq 1$, $q_{ij} q_{ji} = q_{ii}^{-m}$, *and* $q_{ii}^k \neq 1$ *for all* $1 \leq k \leq m$.

(3) $m \geq 1$, q_{ii} *is a primitive* $m+1$*-st root of unity, and* $(q_{ij} q_{ji})^{m+1} \neq 1$.

(4) $m \geq 1$, $\operatorname{char}(\Bbbk) = m + 1$, $q_{ii} = 1$, *and* $q_{ij} q_{ji} \neq 1$.

Moreover, no two of the conditions in (1)–(4) can hold simultaneously.

A graph (I, E) is a pair, where I is a set, called the set of vertices, and E, the set of edges, is a subset of the set of subsets of I of two elements. A labeled graph with labels in $\Bbbk$ is a quadruple (I, E, f_I, f_E), where (I, E) is a graph, and $f_I : I \to \Bbbk$, $f_E : E \to \Bbbk$ are functions. If $i \in I$ is a vertex, and $\{i, j\} \in E$ is an edge, then $f_I(i)$ and $f_E(\{i, j\})$ are called the labels of i and $\{i, j\}$, respectively.

An isomorphism between labeled graphs (I, E, f_I, f_E) and (J, F, f_J, f_F) is a bijective map $\sigma : I \to J$ which induces a bijection

$$\widetilde{\sigma} : E \to F, \ \{i, j\} \mapsto \{\sigma(i), \sigma(j)\} \text{ with } f_I = f_J \sigma, f_E = f_F \widetilde{\sigma}.$$

DEFINITION 15.1.7. Let V be a θ-dimensional braided vector space of diagonal type. Let $(x_i)_{i \in \mathbb{I}}$ be a basis of V and let $\boldsymbol{q} = (q_{ij})_{i,j \in \mathbb{I}}$ be a matrix of non-zero scalars in $\Bbbk$ with $c_{V,V}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $i, j \in \mathbb{I}$, see Remark 1.5.4. The **Dynkin diagram of** V with respect to the basis $(x_i)_{i \in \mathbb{I}}$ is a labeled graph $\mathcal{D}$ with θ vertices. The vertices of $\mathcal{D}$ correspond to the integers $i \in \mathbb{I}$ and are labeled by q_{ii}. For any $1 \leq i < j \leq \theta$, there is an edge between vertex i and vertex j if and only if $q_{ij} q_{ji} \neq 1$. In this case, $q_{ij} q_{ji}$ is the label of this edge.

The Dynkin diagram of $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta^H$ with $\dim M_j = 1$ for all $j \in \mathbb{I}$ is the Dynkin diagram of the braided vector space $M_1 \oplus \cdots \oplus M_\theta$ with respect to the basis $(x_i)_{i \in \mathbb{I}}$, where $0 \neq x_i \in M_i$ for all i.

Proposition 4.5.9 implies that up to isomorphism of labeled graphs, the Dynkin diagram of a braided vector space V of diagonal type does not depend on the choice of the braiding matrix of V.

LEMMA 15.1.8. *Let $i \in \mathbb{I}$. Assume that M is i-finite. Let $a_{ij} = a_{ij}^M$ for all $j \in \mathbb{I}$ and let $W = \bigoplus_{j\in\mathbb{I}} R_i(M)_j$.*

(1) *The braiding matrix of W is $(q'_{jk})_{j,k\in\mathbb{I}}$, where*

$$q'_{jk} = q_{jk} q_{ik}^{-a_{ij}} q_{ji}^{-a_{ik}} q_{ii}^{a_{ij}a_{ik}}. \tag{15.1.3}$$

(2) *The labels of the Dynkin diagram of W are*

$$q'_{jj} = \begin{cases} q_{ii} & \text{if } j = i, \\ q_{jj} & \text{if } j \neq i,\ q_{ii}^{-a_{ij}} q_{ij} q_{ji} = 1, \\ q_{jj}(q_{ij}q_{ji})^{-a_{ij}} q_{ii} & \text{if } j \neq i,\ (1-a_{ij})_{q_{ii}} = 0, \end{cases}$$

$$q'_{jk}q'_{kj} = \begin{cases} q_{ik}q_{ki} & \text{if } j = i,\ k \neq i, \\ q_{jk}q_{kj} & \text{if } j,k \neq i,\ q_{ii}^{-a_{ij}} q_{ij}q_{ji} = 1, \end{cases}$$

if $q_{ii}^{-a_{ik}} q_{ik}q_{ki} = 1$, and

$$q'_{jk}q'_{kj} = \begin{cases} q_{ii}^2 (q_{ik}q_{ki})^{-1} & \text{if } j = i,\ k \neq i, \\ q_{jk}q_{kj}(q_{ik}q_{ki}q_{ii}^{-1})^{-a_{ij}} & \text{if } j,k \neq i,\ q_{ii}^{-a_{ij}} q_{ij}q_{ji} = 1, \\ q_{jk}q_{kj}(q_{ij}q_{ji})^{-a_{ik}}(q_{ik}q_{ki})^{-a_{ij}} q_{ii} & \text{if } j,k \neq i,\ (1-a_{ij})_{q_{ii}} = 0, \end{cases}$$

if $(1-a_{ik})_{q_{ii}} = 0$.

PROOF. By Lemma 15.1.5(2), $y_j = (\operatorname{ad} x_i)^{-a_{ij}}(x_j)$ is a basis of $R_i(M)_j$ for $j \in \mathbb{I} \setminus \{i\}$, and $y_i \in M_i^*$ with $\langle y_i, x_i\rangle = 1$ is a basis of $R_i(M)_i = M_i^*$. Since ad is a morphism in ${}^H_H\mathcal{YD}$, $\Bbbk y_j \in {}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Lemma 15.1.5(4) implies that the braiding is given by

$$c_{R_i(M)_j, R_i(M)_k}(y_j \otimes y_k) = \chi_k \chi_i^{-a_{ik}}(g_j g_i^{-a_{ij}}) y_k \otimes y_j$$

for all $j,k \in \mathbb{I}$. This implies (1). Since $(1-a_{ij})_{q_{ii}}(q_{ii}^{-a_{ij}} q_{ij}q_{ji} - 1) = 0$ by Lemma 15.1.5(1), we obtain (2) directly from (1). $\square$

Assume that M admits all reflections. Then, by Theorem 14.2.12, the semi-Cartan graph $\mathcal{G}(M) = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ as defined in Theorem 13.6.2 is a Cartan graph. We use Lemmas 15.1.5 and 15.1.8 to define a quotient Cartan graph of it in Proposition 15.1.10 below. We need some preparation.

Let $\mathcal{Y}$ be the set of equivalence classes

$$[N]_{\mathrm{s}} = \{P \in \mathcal{F}_\theta^H(M) \mid P \sim_{\mathrm{s}} N\}$$

with respect to the equivalence relation $\sim_{\mathrm{s}}$ on $\mathcal{F}_\theta^H$, where

$$N' \sim_{\mathrm{s}} N'' \quad \Leftrightarrow \quad \text{for all } j,k \in \mathbb{I},\ q'_{jj} = q''_{jj} \text{ and } q'_{jk}q'_{kj} = q''_{jk}q''_{kj} \tag{15.1.4}$$

for any $N', N'' \in \mathcal{F}_\theta^H$ with braiding matrix $(q'_{jk})_{j,k\in\mathbb{I}}$ and $(q''_{jk})_{j,k\in\mathbb{I}}$, respectively. Moreover, let $r : \mathbb{I} \times \mathcal{Y} \to \mathcal{Y}$, $(j, [N]_{\mathrm{s}}) \mapsto [R_j(N)]_{\mathrm{s}}$.

REMARK 15.1.9. By definition, the Dynkin diagrams of all points of $\mathcal{G}(M)$ in an equivalence class $[N]_{\mathrm{s}}$ coincide. More generally, if $\tau : \mathbb{I} \to \mathbb{I}$ is a bijection and $N', N'' \in \mathcal{F}_\theta^H$ with $N' \sim_{\mathrm{s}} N''$, then N' and $(N''_{\tau(j)})_{j\in\mathbb{I}}$ have the same Dynkin diagram.

PROPOSITION 15.1.10. *Assume that M admits all reflections. Then the map $r : \mathbb{I} \times \mathcal{Y} \to \mathcal{Y}$, $(j, [N]_{\mathrm{s}}) \mapsto [R_j(N)]_{\mathrm{s}}$, is well-defined. The tuple*

$$\mathcal{G}_{\mathrm{s}}(M) = \mathcal{G}(\mathbb{I}, \mathcal{Y}, r, A_{\mathrm{s}})$$

with $A_{\mathrm{s}} : \mathbb{I}\times\mathbb{I}\times\mathcal{Y}$, $(j, k, [N]_{\mathrm{s}}) \mapsto a^N_{jk}$ is a Cartan graph. The triple $(\mathcal{G}(M), \mathcal{G}_{\mathrm{s}}(M), \pi)$ with $\pi : \mathcal{X} \to \mathcal{Y}$, $N \mapsto [N]_{\mathrm{s}}$ is a covering.

The Cartan graph $\mathcal{G}_{\mathrm{s}}(M)$ is called the **small Cartan graph of M**.

PROOF. Let $N, P \in \mathcal{F}^H_\theta$ with $N \sim_{\mathrm{s}} P$. Then $A^N = A^P$ by Lemma 15.1.5(1). Thus, for any $Y \in \mathcal{Y}$, A^Y_{s} is well-defined. Further, Lemma 15.1.8 implies that $R_j(N) \sim_{\mathrm{s}} R_j(P)$ for any $j \in \mathbb{I}$, and hence r is well-defined. Thus the proposition holds by Proposition 10.1.3 and by Lemma 10.1.4. □

The following definition uses various notions from Definition 8.2.1.

DEFINITION 15.1.11. Let (V, c) be a finite-dimensional braided vector space of diagonal type and let $\boldsymbol{q} = (q_{ij})_{i,j\in\mathbb{I}}$ be a braiding matrix of V. We say that (V, c) is **generic**, **quasi-generic**, and of **(finite) Cartan type**, respectively, if $\boldsymbol{q}$ is.

Recall from Definition 8.2.1 that $\boldsymbol{q}$ is of (finite) Cartan type if there exists a Cartan matrix $A = (a_{ij})_{i,j\in\mathbb{I}}$ (of finite type) such that for all $i, j \in \mathbb{I}$,

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \quad \text{where } 0 \le -a_{ij} < \mathrm{ord}(q_{ii}) \text{ if } i \ne j. \tag{15.1.5}$$

Proposition 4.5.9 implies that the definitions of a braided vector space of Cartan type and of a generic braided vector space do not depend on the choice of the braiding matrix.

Recall that $M_1 \oplus \cdots \oplus M_\theta \in {}^H_H\mathcal{YD}$ is a braided vector space of diagonal type. We say that M is generic, quasi-generic, and of (finite) Cartan type, respectively, if the braided vector space $M_1 \oplus \cdots \oplus M_\theta$ is.

LEMMA 15.1.12. *Assume that M is of Cartan type with Cartan matrix A. Then M is i-finite for all $i \in \mathbb{I}$ and $a^M_{ij} = a_{ij}$ for all $i, j \in \mathbb{I}$.*

PROOF. Let $i \in \mathbb{I}$. The i-finiteness of M follows from Lemma 15.1.4. Condition (15.1.5) implies that if $q_{ii}^{m+1} = 1$ for some $m \in \mathbb{N}_0$ then $(q_{ij}q_{ji})^{m+1} = 1$ for any $j \in \mathbb{I} \setminus \{i\}$. Thus in Lemma 15.1.6 only the first two cases occur and $a^M_{ij} = a_{ij}$ by the assumptions on A in Definition 8.2.2 and by Lemma 15.1.6. □

LEMMA 15.1.13. *Assume that M is of Cartan type with Cartan matrix A. Let $i \in \mathbb{I}$ and let $W = \bigoplus_{j\in\mathbb{I}} R_i(M)_j$. Then the following hold.*

(1) *The braiding matrix of W is $(q'_{jk})_{j,k\in\mathbb{I}}$, where*

$$q'_{jk} = q_{jk}q_{ik}^{-a_{ij}}q_{ij}^{a_{ik}} \quad \text{for all } j, k \in \mathbb{I}.$$

(2) *The labels of the Dynkin diagram of W and of $R_i(M)$ are*

$$q'_{jj} = q_{jj}, \qquad q'_{jk}q'_{kj} = q_{jk}q_{kj}$$

for all $j, k \in \mathbb{I}$.

(3) *The tuple $R_i(M)$ is of Cartan type with Cartan matrix A.*

PROOF. Since M is i-finite for all $i \in \mathbb{I}$ by Lemma 15.1.12, the tuple $R_i(M)$ is well-defined. The claims in (1) and (2) on the braiding matrix and the Dynkin diagram follow directly from Lemma 15.1.8. (3) follows from (2). □

THEOREM 15.1.14. *Assume that M is of Cartan type. Then the following hold.*

(1) *M admits all reflections.*
(2) *Let $M' \in \mathcal{F}_\theta^H(M)$. Then M' is of Cartan type.*
(3) *The Cartan graph of M is standard.*
(4) *The small Cartan graph of M has only one point.*
(5) *The Cartan graph of M is finite if and only if A^M is of finite type.*
(6) *The Nichols algebra $\mathcal{B}(M)$ is finite-dimensional if and only if A^M is of finite type and if for all $i \in \mathbb{I}$ there exists $m \in \mathbb{N}_0$ such that $(m+1)_{q_{ii}} = 0$.*

Note that by Lemma 10.1.4, the Cartan graph of M is finite if and only if the small Cartan graph of M is finite.

PROOF. Let A be the Cartan matrix of M. Then, by Lemma 15.1.12 and Lemma 15.1.13(3), M is i-finite for all $i \in \mathbb{I}$ and all tuples $R_i(M)$ with $i \in \mathbb{I}$ are of Cartan type with Cartan matrix A. Thus any $N \in \mathcal{F}_\theta^H(M)$ is i-finite and of Cartan type with Cartan matrix A. This implies (1), (2), and (3). (4) follows from (15.1.4) and Lemma 15.1.13(2). By (4) and by Example 9.1.17, $\mathcal{G}_\mathrm{s}(M)$ is finite if and only if A^M is of finite type. Thus (5) holds because of Lemma 10.1.4. Finally, (6) follows from Corollary 14.5.3 because of (1),(4),(5), and Example 1.10.1. □

COROLLARY 15.1.15. *Let $q \in \Bbbk^\times$. Assume that $\theta \geq 2$ and $q_{ij} = q$ for all $i, j \in \mathbb{I}$. The following are equivalent.*

(1) *$\mathcal{B}(M)$ is finite-dimensional,*
(2) *$q = 1$, $\mathrm{char}(\Bbbk) \neq 0$ or $q = -1$ or $\theta = 2$, $\mathrm{ord}(q) = 3$.*

PROOF. If q is not a root of 1, then M is not i-finite by Lemma 15.1.4, and hence $\mathcal{B}(M)$ is infinite-dimensional.

Assume that q is a root of 1 of order $N \geq 1$. Then M is i-finite for all $i \in \mathbb{I}$ by Lemma 15.1.4. Let A be the Cartan matrix of M. Then, by Lemma 15.1.6, M is of Cartan type with $a_{ij} = 0$ if $q = 1$, and $a_{ij} = N - 2$ otherwise. Hence the Cartan matrix A is of finite type if and only if $q^2 = 1$ or $\theta = 2$, $N = 3$. Thus the claim follows from Theorem 15.1.14(6) and Example 1.10.1. □

COROLLARY 15.1.16. *Assume that $\Bbbk$ is algebraically closed of characteristic 0. Let G be a finite group of odd order. Then there exist only finitely many isomorphism classes of Yetter-Drinfeld modules over G with finite-dimensional Nichols algebra.*

PROOF. By Maschke's theorem, the group algebra of any subgroup of G is semisimple, and has only finitely many isomorphism classes of simple modules. Thus, by Corollary 1.4.18 there exist only finitely many isomorphism classes of simple Yetter-Drinfeld modules over G. Since any finite-dimensional Yetter-Drinfeld module over G is semisimple by Proposition 1.4.20, it suffices to prove that for any simple $V \in {}^G_G\mathcal{YD}$, $\mathcal{B}(V \oplus V \oplus V)$ is infinite-dimensional.

So let $V \in {}^G_G\mathcal{YD}$ be a simple object. Since $\Bbbk$ is algebraically closed, by Proposition 1.4.21 there exists $q \in \Bbbk^\times$ and $g \in G$ with $V_g \neq 0$ and $g \cdot v = qv$ for all $v \in V_g$. Note that $q \neq -1$ since the order of g is odd and $\mathrm{char}(\Bbbk) = 0$. Thus $W = V_g \oplus V_g \oplus V_g$ is a braided subspace of V of dimension at least 3, and

$$c_{W,W}(w \otimes w') = g \cdot w' \otimes w = qw' \otimes w$$

for all $w, w' \in W$. Then the claim follows from Corollary 15.1.15 for W. □

We note that by Example 1.10.15 there are finite-dimensional Yetter-Drinfeld modules over the group $\mathbb{Z}/(2)$ with Nichols algebra of dimension 2^n, $n \geq 1$.

Another consequence of Theorem 15.1.14(6) is a general result on Nichols algebras over symmetric groups. Recall the definition of the Yetter-Drinfeld modules $M(g, V)$ over groups from Definition 1.4.15.

COROLLARY 15.1.17. *Assume that* $\Bbbk$ *is an algebraically closed field of characteristic* 0. *Let* $n \geq 1$, $g \in \mathbb{S}_n$, *and let* $V \neq 0$ *be an* $\mathbb{S}_n^g$*-module. Assume that the Nichols algebra of* $M(g, V) \in {}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$ *is finite-dimensional. Then* g *has even order and* $g \cdot v = -v$ *for all* $v \in V$.

PROOF. Since $\mathcal{B}(M(g, V))$ is finite-dimensional, also V is finite-dimensional. Thus the action of g on V is diagonalizable by the assumptions of $\Bbbk$.

First we prove that there is no $0 \neq v \in V$ with $g \cdot v = v$. In particular, $g \neq 1$. Indeed, otherwise $\Bbbk v$ is a braided subspace of $M(g, V)$ for such a v, and $\mathcal{B}(\Bbbk v)$ is infinite-dimensional by Example 1.10.1, a contradiction.

Assume now that the order of g is two. Then $(g+1)(g-1)v = 0$ for all $v \in V$. By the above it follows that $g \cdot v = -v$ for all $v \in V$, and hence the claim is proven in this case.

Finally, assume that the order of g is at least three. Then g is a product of pairwise disjoint cycles, and at least one of the cycles has order at least three. Hence $g^{-1} \neq g$ and there exists $h \in \mathbb{S}_n$ with $hgh^{-1} = g^{-1}$. (Indeed, g and g^{-1} have the same cycle type, and any two permutations of the same cycle type are conjugate in $\mathbb{S}_n$.) Choose now $0 \neq v \in V$ and $q \in \Bbbk^\times$ with $g \cdot v = qv$. By the second paragraph we know that $q \neq 1$. Then $W = \Bbbk 1 \otimes v + \Bbbk h \otimes v$ is a braided subspace of $M(g, V)$ of diagonal type with braiding matrix

$$\boldsymbol{q} = \begin{pmatrix} q & q^{-1} \\ q^{-1} & q \end{pmatrix}$$

since $g(h \otimes v) = hg^{-1} \otimes v = q^{-1} h \otimes v$. If $q \neq -1$, then $\boldsymbol{q}$ and A are of Cartan type with $a_{12} = a_{21} = -2$. In this case A is not of finite type, $\mathcal{B}(W)$ is infinite-dimensional by Theorem 15.1.14(6), which is a contradiction. Thus $g \cdot v' = -v'$ for all $v' \in V$. Consequently, g has even order N, because $v' = g^N \cdot v' = (-1)^N v'$ for all $v' \in V$. □

We also formulate an important general finiteness condition for $\mathcal{B}(M)$ based on the Cartan graph of M and Example 1.10.1.

PROPOSITION 15.1.18. *Assume that* $\mathcal{B}(M)$ *is finite-dimensional. Then for all* $\alpha \in \boldsymbol{\Delta}_+^{[M]\,\mathrm{re}}$ *there exists* $n \geq 1$ *such that* $(n+1)_q = 0$, *where* $q = \chi_\alpha(g_\alpha)$. *In particular, if the Dynkin diagram* $\mathcal{D}$ *of* M *is connected, and* q *is the product of all labels of* $\mathcal{D}$, *then* $(n+1)_q = 0$ *for some* $n \geq 1$.

PROOF. By Corollary 14.5.3, $\mathcal{G}(M)$ is finite and $\mathcal{B}(N)$ is finite-dimensional for all $N \in \mathcal{F}_\theta^H(M)$. Let $\alpha \in \boldsymbol{\Delta}_+^{[M]\,\mathrm{re}}$. Then Corollary 14.5.1 implies that there exist an irreducible Yetter-Drinfeld submodule M_α of $\mathcal{B}(M)$ of $\mathbb{N}_0^\theta$-degree α and $N \in \mathcal{F}_\theta^H(M)$, $i \in \mathbb{I}$ such that $M_\alpha \cong N_i$ in ${}^H_H\mathcal{YD}$. Let v be a basis of N_i. Since M_α has degree α, it follows from Remark 15.1.2 that

$$\delta_{N_i}(v) = g_\alpha \otimes v, \quad hv = \chi_\alpha(h)v$$

for any $h \in H$. Since $\mathcal{B}(N_i)$ is finite-dimensional, we conclude from Example 1.10.1 that $(n+1)_q = 0$ for some $n \geq 1$.

Assume that $\mathcal{D}$ is connected. Then Lemma 15.1.6 implies that A^M is indecomposable. By Proposition 10.4.14, $\alpha = \sum_{i=1}^{\theta} \alpha_i \in \boldsymbol{\Delta}_+^{M\,\mathrm{re}}$. Since $q = \chi_\alpha(g_\alpha)$, the second claim follows from the first one. $\square$

An immediate consequence of Proposition 15.1.18 is the following.

COROLLARY 15.1.19. *Assume that* $\mathrm{char}(\Bbbk) = 0$. *If the Dynkin diagram* $\mathcal{D}$ *of* M *is connected, and the product of all labels of* $\mathcal{D}$ *is* 1 *or not a root of* 1, *then* $\mathcal{B}(M)$ *is infinite-dimensional.*

15.2. Root vector sequences

Let H be a Hopf algebra with bijective antipode, $\theta \in \mathbb{N}$, $\mathbb{I} = \{1, \dots, \theta\}$, and let $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta^H$ be a tuple of one-dimensional Yetter-Drinfeld modules admitting all reflections. Then $\mathcal{G}(M)$ is a Cartan graph by Theorem 14.2.12. Let $\boldsymbol{q} = (q_{ij})_{i,j \in \mathbb{I}} \in \Bbbk^{\times\,\theta \times \theta}$ be the braiding matrix of M. We introduce the notion of root vector sequences for pre-Nichols systems of M, which is based on reduced sequences and right coideal subalgebras. Note that reduced sequences correspond to reduced decompositions of morphisms in the Weyl groupoid of $\mathcal{G}(M)$ by Theorem 9.3.5. An important application of root vector sequences is to construct PBW bases. A general result on Nichols algebras in this direction is Theorem 15.2.7 below. We will also prove a similar result on quantum groups in Sections 16.2 and 16.3.

DEFINITION 15.2.1. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $\kappa = (i_1, \dots, i_t) \in \mathbb{I}^t$ with $t \geq 0$ be an $[M]$-reduced sequence. A sequence $x_1, \dots, x_t$ of elements of S is called a **root vector sequence for** κ, if

(1) $x_j \in S(\beta_j^{[M],\kappa}) \setminus \{0\}$ for any $1 \leq j \leq t$, and
(2) for any $1 \leq j \leq t$, the products $x_j^{n_j} \cdots x_2^{n_2} x_1^{n_1}$ with $n_1, \dots, n_j \in \mathbb{N}_0$, span a right coideal subalgebra of S in ${}_H^H\mathcal{YD}$.

Existence and uniqueness of root vector sequences will be discussed under additional assumptions in Proposition 15.2.6.

For an example of a root vector sequence we refer to Remark 16.2.6.

REMARK 15.2.2. (1) In the setting of Definition 15.2.1, for any root vector sequence $x_1, \dots, x_t$ for κ in S and for any $\lambda_1, \dots, \lambda_t \in \Bbbk^\times$, the sequence $\lambda_1 x_1, \dots, \lambda_t x_t$ is a root vector sequence for κ in S.

(2) Let $p : \mathcal{N} \to \mathcal{N}'$ with $\mathcal{N} = \mathcal{N}(S, N, f)$ and $\mathcal{N}' = \mathcal{N}(S', N', f')$ be a morphism of pre-Nichols systems of M, and let $\kappa = (i_1, \dots, i_t) \in \mathbb{I}^t$ with $t \geq 0$ be an $[M]$-reduced sequence. Then for any root vector sequence $x_1, \dots, x_t$ for κ in S, $p(x_1), \dots, p(x_t)$ is a root vector sequence for κ in S' if and only if $p(x_j) \neq 0$ for any $1 \leq j \leq t$. Indeed, p is a graded Hopf algebra map in ${}_H^H\mathcal{YD}$ and $p(C)$ is a right coideal subalgebra of S' for any right coideal subalgebra C of S.

The combination of the two properties in Definition 15.2.1 has strong consequences. Recall the notation $K_i^{\mathcal{N}}$ from Definition 13.5.9.

LEMMA 15.2.3. *Let* $\mathcal{N} = \mathcal{N}(S, N, f)$ *be a pre-Nichols system of* M *and let* $\kappa = (i_1, \dots, i_t) \in \mathbb{I}^t$ *with* $t \geq 0$ *be an* $[M]$*-reduced sequence. Let* $x_1, \dots, x_t$ *be a root vector sequence for* κ.

(1) *For any* $2 \leq j \leq t$, $x_j \in K_{i_1}^{\mathcal{N}}$.

(2) *For any* $2 \le j \le t$, *the products* $x_j^{n_j} \cdots x_2^{n_2}$ *with* $n_2, \dots, n_j \in \mathbb{N}_0$, *span* $C_j \cap K_{i_1}^{\mathcal{N}}$, *where* C_j *is the subalgebra of* S *generated by* $x_1, \dots, x_j$.

Proof. Assume that $t \ge 2$ and let $2 \le j \le t$.

(1) Let C be the subalgebra of S generated by $x_1, \dots, x_{j-1}$, and for all $1 \le l \le j$ let $\beta_l = \beta_l^{[M],\kappa}$. By assumption,

$$\Delta_S(x_j) - x_j \otimes 1 \in C \otimes S.$$

Let $\pi_{i_1} : S \to \Bbbk[N_{i_1}]$ be the homogeneous Hopf algebra projection with kernel $\bigoplus_{\beta \notin \mathbb{N}_0 \alpha_{i_1}} S(\beta)$. Then $x_j \in K_{i_1}^{\mathcal{N}}$ if and only if the homogeneous summand of $\Delta_S(x_j)$ in $C(\beta_j - n\alpha_{i_1}) \otimes S(n\alpha_{i_1})$ is zero for any $n \ge 1$. Since $x_l \in S(\beta_l)$ for any $1 \le l \le j-1$, the latter property follows from Proposition 9.3.14 and the definition of C. Indeed, otherwise there exist $n_1, \dots, n_{j-1}, n \in \mathbb{N}_0$ such that $\sum_{l=1}^{j-1} n_l \beta_l + n\beta_1 = \beta_j$, which is a contradiction.

(2) Note first that $\Bbbk[x_1] = \Bbbk[N_{i_1}]$, since $x_1 \in S(\alpha_{i_1}) \setminus \{0\}$ and N_{i_1} is one-dimensional. Hence $K_{i_1}^{\mathcal{N}} \# \Bbbk[x_1] \cong S$ via canonical embedding and multiplication by Theorem 3.9.2(6). By (1), $x_j^{n_j} \cdots x_2^{n_2} \in C_j \cap K_{i_1}^{\mathcal{N}}$ for any $n_2, \dots, n_j \in \mathbb{N}_0$. Thus the second part of the definition of a root vector sequence implies the claim. $\square$

Recall the maps $T_i = T_i^{\mathcal{N}}$ from Theorem 12.3.3 and Corollary 13.5.21. They allow transformations of root vector sequences for Nichols systems.

Proposition 15.2.4. *Let* $i \in \mathbb{I}$, $\mathcal{N} = \mathcal{N}(S, N, f)$ *be a Nichols system of* (M, i), *and* $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_i(\mathcal{N})$. *Let* $t \ge 1$, $\kappa = (i, i_2, \dots, i_t) \in \mathbb{I}^t$ *be an* $[M]$*-reduced sequence, and* $x_1, \dots, x_t$ *be a root vector sequence for* κ *in* S. *Then*

$$T_i^{-1}(x_2), \dots, T_i^{-1}(x_t)$$

is a root vector sequence for $(i_2, \dots, i_t)$ *in* $\widetilde{S}$.

Proof. The elements $T_i^{-1}(x_l) \in \widetilde{S}$ with $2 \le l \le t$ are well-defined since $x_l \in K_i^{\mathcal{N}}$ for any $2 \le l \le t$ by Lemma 15.2.3(1). Moreover,

$$\deg(T_i^{-1}(x_l)) = s_i^M(\deg x_l) = \mathrm{id}_{R_i(M)} s_{i_2} \cdots s_{i_{l-1}}(\alpha_{i_l})$$

for any $2 \le l \le t$ by Corollary 13.5.21(2).

If $t = 1$ then the Lemma is trivial. Assume now that $t \ge 2$ and let $2 \le j \le t$. Let C_j be the subalgebra of S generated by $x_1, \dots, x_j$. Then C_j is a right coideal subalgebra of S in ${}^H_H\mathcal{YD}$ containing $\Bbbk[N_i]$ by assumption. Hence $\widetilde{C} = T_i^{-1}(C_j \cap K_i^{\mathcal{N}})$ is a right coideal subalgebra of $\widetilde{S}$ in ${}^H_H\mathcal{YD}$ by Theorem 12.4.5. Since T_i is an algebra isomorphism, Lemma 15.2.3(2) implies that $\widetilde{C}$ is spanned by the products $T_i^{-1}(x_j)^{n_j} \cdots T_i^{-1}(x_2)^{n_2}$ with $n_2, \dots, n_j \in \mathbb{N}_0$. This implies the claim. $\square$

Proposition 15.2.5. *Let* $i \in \mathbb{I}$, $\mathcal{N} = \mathcal{N}(S, N, f)$ *be a Nichols system of* (M, i), *and* $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_i(\mathcal{N})$. *Let* $t \ge 1$, $\kappa = (i, i_2, \dots, i_t) \in \mathbb{I}^t$ *be an* $[M]$*-reduced sequence, and* $x_1 \in N_i \setminus \{0\}$. *For any root vector sequence* $x_2, \dots, x_t$ *in* $\widetilde{S}$ *for* $(i_2, \dots, i_t)$,

$$x_1, T_{i_1}(x_2), \dots, T_{i_1}(x_t)$$

is a root vector sequence in S *for* κ.

PROOF. For $t=1$ the Proposition is trivial.

Assume that $t \geq 2$ and let $2 \leq j \leq t$. Then $x_2, \dots, x_j$ generate a right coideal subalgebra $\widetilde{C}$ of $\widetilde{S}$ in ${}_H^H\mathcal{YD}$ by assumption. Moreover, $\widetilde{N}_i \not\subseteq \widetilde{C}$. Indeed, $\alpha_i \neq \deg(x_l) \in \mathbb{N}_0^\theta$ for each $2 \leq l \leq j$, since κ is $[M]$-reduced. Hence $\widetilde{C} \subseteq L_i^{R_i(\mathcal{N})}$ by Lemma 14.1.2. Let $C = T_i(\widetilde{C})\Bbbk[N_i] \subseteq S$. Then C is a right coideal subalgebra of S in ${}_H^H\mathcal{YD}$ by Theorem 12.4.5. Since T_i is an algebra map by Theorem 12.3.3, it follows that C is spanned by the monomials $T_i(x_j)^{n_j} \cdots T_i(x_2)^{n_2} x_1^{n_1}$ with $n_1, \dots, n_j \in \mathbb{N}_0$. By choice of x_1, $\deg(x_1) = \alpha_i$. Moreover, $\deg\big(T_i(x_l)\big) = \beta_l^{[M],\kappa}$ for any $2 \leq l \leq j$ by assumption on the degrees of the elements $x_2, \dots, x_t$ and by Corollary 13.5.21(2). Finally, $T_i(x_l) \neq 0$ for all $2 \leq l \leq t$, since T is injective. This implies the claim. $\square$

PROPOSITION 15.2.6. *Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $\kappa = (i_1, \dots, i_t) \in \mathbb{I}^t$ with $t \geq 0$ be an $[M]$-reduced sequence. Assume that $\mathcal{N}$ admits the reflection sequence κ.*

(1) *There exists a root vector sequence for κ in S.*
(2) *Let $x_1, \dots, x_t$ and $y_1, \dots, y_t$ be root vector sequences for κ in S. Then there are $\lambda_1, \dots, \lambda_t \in \Bbbk^\times$ such that $y_l = \lambda_l x_l$ for all $1 \leq l \leq t$.*

PROOF. (1) Since $\dim M_j = 1$ for all $j \in \mathbb{I}$, the vector spaces $N_j^{\mathcal{N}}(\kappa)$ in Theorem 14.1.9(2) are one-dimensional. For all $1 \leq j \leq t$ choose a non-zero vector $x_j \in N_j^{\mathcal{N}}(\kappa)$. By Theorem 14.1.9(4), the elements x_j have the correct degree. By Theorem 14.1.9(6), for any $1 \leq j \leq t$ the monomials $x_j^{n_j} \cdots x_1^{n_1}$ with $n_1, \dots, n_j \geq 0$ span $E^{\mathcal{N}}(i_1, \dots, i_j)$. By Theorem 14.1.9(3) and Theorem 14.1.4(1), $E^{\mathcal{N}}(i_1, \dots, i_j)$ is a right coideal subalgebra of S for all $1 \leq j \leq t$. Thus $x_1, \dots, x_t$ is a root vector sequence for κ in S.

(2) We proceed by induction on t. For $t=0$ the claim is trivial. Assume that $t \geq 1$ and that the claim holds for all reduced sequences of length at most $t-1$. Let $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_{i_1}(\mathcal{N})$. Then $T_{i_1}^{-1}(x_2), \dots, T_{i_1}^{-1}(x_t)$ and $T_{i_1}^{-1}(y_2), \dots, T_{i_1}^{-1}(y_t)$ are root vector sequences for $(i_2, \dots, i_t)$ in $\widetilde{S}$ by Proposition 15.2.4. By induction hypothesis there exist scalars $\lambda_2, \dots, \lambda_t \in \Bbbk^\times$ such that $T_i^{-1}(y_l) = \lambda_l T_i^{-1}(x_l)$ for all $2 \leq l \leq t$. Moreover, $y_1 = \lambda_1 x_1$ for some $\lambda_1 \in \Bbbk^\times$ since $\dim N_{i_1} = 1$. This implies the claim. $\square$

For any $\alpha = \sum_{i=1}^\theta a_i \alpha_i \in \mathbb{Z}^\theta$ let $g_\alpha \in H$ and $\chi_\alpha \in \mathrm{Alg}(H, \Bbbk)$ be as in Equation (15.1.2), and let $q_{\alpha\alpha} = \chi_\alpha(g_\alpha)$ and $N(q_{\alpha\alpha})$ be as in Example 1.10.1.

THEOREM 15.2.7. *Let $M \in \mathcal{F}_\theta^H$ such that $\dim M_i = 1$ for all $i \in \mathbb{I}$. Assume that M admits all reflections. Let $\kappa = (i_1, \dots, i_t)$ with $t \geq 0$ be an $[M]$-reduced sequence, and for all $1 \leq k \leq t$ let $\beta_k = \beta_k^{[M],\kappa}$. Let $x_1, \dots, x_t$ be a root vector sequence for κ in $\mathcal{B}(M)$.*

(1) *For any $0 \leq k \leq t$ let $(q_{ij}^{(k)})_{i,j \in \mathbb{I}}$ be the braiding matrix of the tuple $R_{i_k} \cdots R_{i_1}(M) \in \mathcal{F}_\theta^H$. Then $q_{\beta_k\beta_k} = q_{i_k i_k}^{(k-1)}$ for any $1 \leq k \leq \theta$.*
(2) *The elements $x_t^{n_t} \cdots x_1^{n_1}$ with $0 \leq n_k < N(q_{\beta_k\beta_k})$ for all $1 \leq k \leq t$ form a basis of the right coideal subalgebra $E^{\mathcal{B}(M)}(i_1, \dots, i_t)$ of the Nichols algebra $\mathcal{B}(M)$.*
(3) *Assume that for all $i \in \mathbb{I}$, $\alpha_i \in \Lambda^{[M]}(\kappa)$. Then $\mathcal{B}(M) = E^{\mathcal{B}(M)}(\kappa)$ and the elements $x_t^{n_t} \cdots x_1^{n_1}$ such that $0 \leq n_k < N(q_{\beta_k\beta_k})$ for all $1 \leq k \leq t$ form a basis of $\mathcal{B}(M)$.*

PROOF. By Proposition 15.2.6(2) and the proof of Proposition 15.2.6(1) for the pre-Nichols system $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id}_M)$, x_k is a basis of $N_k^{\mathcal{N}_0}(\kappa)$ (in the notation of Theorem 14.1.9(2)). Since $x_k \in \mathcal{B}(M)(\beta_k)$, Remark 15.1.2 implies that

$$c_{\mathcal{B}(M),\mathcal{B}(M)}(x_k \otimes x_k) = \chi_{\beta_k}(g_{\beta_k})x_k \otimes x_k = q_{\beta_k\beta_k} x_k \otimes x_k. \tag{15.2.1}$$

(1) follows from Theorem 14.1.9(2) and from (15.2.1), since $T^{\mathcal{N}_0}_{(i_1,\dots,i_{k-1})}$ is an isomorphism of Yetter-Drinfeld modules.

(2) Example 1.10.1 and (15.2.1) imply that for any $1 \le k \le t$, the Hopf algebras $\mathcal{B}(\Bbbk x_k)$ and $\Bbbk[x]/(x^N)$ in ${}^H_H\mathcal{YD}$, where $N = N(q_{\beta_k\beta_k})$, are isomorphic. Thus the claim holds by Theorem 14.1.9(5),(6).

(3) is a consequence of (2) and Corollary 14.1.14(1). □

Existence and uniqueness of root vector sequences in a more general context is less clear. With Proposition 15.2.9 we provide a tool which will be used in Section 16.3.

Motivated by the notation from Section 12.4, for any $i \in \mathbb{I}$ and any pre-Nichols system $\mathcal{N}(S, N, f)$ of M we define

$$\begin{aligned}
\mathcal{E}_r^{+i}(S) &= \{C \mid C \subseteq S \text{ right coideal subalgebra in } {}^H_H\mathcal{YD},\ N_i \subseteq C\},\\
\mathcal{F}_r^{i}(S) &= \{C \mid C \subseteq K_i^{\mathcal{N}} \text{ subalgebra in } {}^H_H\mathcal{YD},\\
&\qquad\qquad \Delta_{K_i^{\mathcal{N}}}(C) \subseteq C \otimes K_i^{\mathcal{N}}, C \text{ is } \operatorname{ad}\Bbbk[N_i]\text{-invariant}\}.
\end{aligned}$$

LEMMA 15.2.8. *Let $\gamma : \overline{S} \to S$ and $\pi : S \to \overline{S}$ be Hopf algebra maps in ${}^H_H\mathcal{YD}$ with $\pi \circ \gamma = \mathrm{id}_{\overline{S}}$. Let R be the algebra of right coinvariants*

$$R = \{x \in S \mid (\mathrm{id} \otimes \pi)\Delta_S(x) = x \otimes 1\}.$$

Let $\overline{J}$ be a Hopf ideal of $\overline{S}$ such that $(\operatorname{ad}_S\gamma(x))(y) = 0$ for any $x \in \overline{J}$, $y \in R$, and let J be the ideal of S generated by $\gamma(\overline{J})$.

(1) *$J \cap \gamma(\overline{S}) = \gamma(\overline{J})$ and $J \cap R = 0$.*

(2) *The canonical map $p : S \to S/J$ induces by restriction an isomorphism $p_0 : R \to (S/J)^{\mathrm{co}\,\gamma(\overline{S}/\overline{J})}$ of algebras, coalgebras and left $\operatorname{ad}\gamma(\overline{S})$-modules.*

PROOF. The multiplication map $R \otimes \gamma(\overline{S}) \to S$ is bijective by Theorem 3.9.2(6). Moreover, $\overline{J}$ is a Hopf ideal of $\overline{S}$ such that $(\operatorname{ad}_S\gamma(\overline{J}))(R) = 0$, and hence

$$\gamma(\overline{J})R \subseteq (\operatorname{ad}_S\gamma(\overline{J}))(R)\gamma(\overline{S}) + (\operatorname{ad}_S\gamma(\overline{S}))(R)\gamma(\overline{J}) \subseteq R\gamma(\overline{J})$$

by the restriction of the formula in Proposition 3.7.2(1)(a) for $V = S$ and $H = \gamma(\overline{S})$ to $\gamma(\overline{J}) \otimes R$. We conclude that $J = R\gamma(\overline{J})$ is a Hopf ideal of S, $J \cap \gamma(\overline{S}) = \gamma(\overline{J})$, and $J \cap R = 0$. Thus p induces a linear isomorphism $p_0 : R \to (S/J)^{\mathrm{co}\,\gamma(\overline{S}/\overline{J})}$. The rest follows from the fact that J is a Hopf ideal of S. □

PROPOSITION 15.2.9. *Let $i \in \mathbb{I}$ and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M satisfying (Sys4) for i. Let $x \in N_i \setminus \{0\}$, $n = \operatorname{ord}(q_{ii})$, and let J be the ideal of S generated by x^n. Let*

$$\psi : \mathcal{E}_r^{+i}(S) \to \mathcal{E}_r^{+i}(S/J), \quad \psi(C) = p(C),$$

where $p : S \to S/J$ is the canonical morphism. Let $\bar{J} = J \cap \Bbbk[N_i]$.

(1) *The map ψ is bijective. For any $E \in \mathcal{E}_r^{+i}(S/J)$,*

$$\psi^{-1}(E) = \Big(p^{-1}\big(E \cap (S/J)^{\mathrm{co}\,\Bbbk[N_i]/\bar{J}}\big) \cap K_i^{\mathcal{N}}\Big)\Bbbk[N_i].$$

(2) *For any family $(y_j)_{j\in I}$ of generators of a right coideal subalgebra E in $\mathcal{E}_r^{+i}(S/J)$ with $y_j \in (S/J)^{\mathrm{co}\,\Bbbk[N_i]/\bar{J}} \cup N_i$ for all $j \in I$, there is a unique family $(x_j)_{j\in I}$ of generators of $\psi^{-1}(E) \in \mathcal{E}_r^{+i}(S)$ such that $p(x_j) = y_j$ and $x_j \in K_i^{\mathcal{N}} \cup N_i$ for all $j \in I$.*

Proof. By Proposition 2.4.2(5), x^n is primitive in S. Moreover, x^n is homogeneous with respect to the $\mathbb{N}_0^\theta$-grading, and $\Bbbk x^n \in {}^H_H\mathcal{YD}$. We conclude that J is a Hopf ideal of S in ${}^H_H\mathcal{YD}$ and a graded subspace of S.

(1) Let $\bar{S} = \Bbbk[N_i]$ and let $\bar{J}$ be the (Hopf) ideal of $\bar{S}$ generated by x^n. By Lemma 12.4.3, the maps

$$\mathcal{E}_r^{+i}(S) \to \mathcal{F}_r^i(S), \quad C \mapsto C \cap S^{\mathrm{co}\,\bar{S}},$$

and

$$\mathcal{E}_r^{+i}(S/J) \to \mathcal{F}_r^i(S/J), \quad C \mapsto C \cap (S/J)^{\mathrm{co}\,\bar{S}/\bar{J}},$$

are bijective. Moreover, $(\mathrm{ad}\,x^n)(y) = 0$ for any $y \in N_j$, $j \in \mathbb{I}\setminus\{i\}$ by Lemma 13.5.7. Since $(\mathrm{ad}\,x^n)(x) = 0$, it follows that $\mathrm{ad}\,x^n = 0$ in $\mathrm{End}(S)$. Now Lemma 15.2.8 with $\bar{S} = \Bbbk[N_i]$ applies. In particular, $\bar{J} = J \cap \Bbbk[N_i]$, and the canonical map $S \to S/J$ induces an isomorphism $S(\alpha_i) \to (S/J)(\alpha_i)$ and a bijection between $\mathcal{F}_r^i(S)$ and $\mathcal{F}_r^i(S/J)$. This implies the bijectivity of ψ and the description of $\psi^{-1}(E)$, $E \in \mathcal{E}_r^{+i}(S/J)$.

(2) Since p restricted to $K_i^{\mathcal{N}} \cup N_i$ is injective, the uniqueness in the claim clearly holds. The existence follows from the description of $\psi^{-1}(E)$ in (1). $\square$

Remark 15.2.10. In the setting of Proposition 15.2.9, the direct analogue of the map ψ between the sets of right coideal subalgebras in ${}^H_H\mathcal{YD}$ contained in ${}^{\mathrm{co}\,\Bbbk[N_i]}S$ and ${}^{\mathrm{co}\,\Bbbk[N_i]}(S/J)$, respectively, fails to be a bijection. Indeed, assume that $\theta = 2$, $i = 1$, and that $q_{11} = q_{12}q_{21} = q_{22} = -1 \neq 1$. Let $E_1 \in M_1$, $E_2 \in M_2$ be non-zero elements. Then $\mathcal{N}(S, M, \mathrm{id}_M)$ is a pre-Nichols system of M, where $S = T(M)/((\mathrm{ad}\,E_1)^2(E_2), (\mathrm{ad}\,E_2)^2(E_1))$. Let $E_{12} = E_1E_2 - q_{12}E_2E_1$. Then

$$\begin{aligned}\Delta(E_{12}) &= E_{12} \otimes 1 + 2E_1 \otimes E_2 + 1 \otimes E_{12},\\ \Delta(E_{12}^2) &= E_{12}^2 \otimes 1 + 4q_{21}E_1^2 \otimes E_2^2 + 1 \otimes E_{12}^2.\end{aligned}$$

Therefore $\Bbbk[E_{12}^2]$ is a right coideal subalgebra of $S/(E_1^2)$ and is left coinvariant with respect to $\Bbbk[E_1]$, but $\Bbbk[E_{12}^2]$ is a not a right coideal subalgebra in S.

15.3. Rank two Nichols algebras of diagonal type

The Nichols algebra of a one-dimensional braided vector space was studied in Example 1.10.1. Here we classify two-dimensional braided vector spaces of diagonal type. By Remark 1.5.4, these can be realized as Yetter-Drinfeld modules over the group algebra H of $\mathbb{Z}^2$. We assume that $\mathrm{char}(\Bbbk) = 0$.

Let $\mathbb{I} = \{1, 2\}$. Let V be a two-dimensional braided vector space of diagonal type and let $q = (q_{ij})_{i,j\in\mathbb{I}}$ be its braiding matrix with respect to a basis x_1, x_2 of V. Then $\Bbbk x_1$ and $\Bbbk x_2$ are one-dimensional Yetter-Drinfeld modules over H. We determine whether $\mathcal{B}(V)$ is finite-dimensional in terms of the Dynkin diagram of V. Our proof uses the existence of a finite Cartan graph of a pair $(\Bbbk x_1, \Bbbk x_2)$ with finite-dimensional Nichols algebra and the classification of all $M \in \mathcal{F}_2^H$ such that $\mathcal{G}(M)$ is finite.

For all $n \in \mathbb{N}$ let P_n denote the set of primitive n-th roots of unity in $\Bbbk$.

THEOREM 15.3.1. *Assume that* $\operatorname{char}(\Bbbk) = 0$. *Let* V *be a two-dimensional braided vector space of diagonal type. Let* x_1, x_2 *be a basis of* V *such that*

$$c(x_i \otimes x_j) \in \Bbbk x_j \otimes x_i$$

for all $i, j \in \{1, 2\}$. *Then the following are equivalent.*

(1) *The pair* $M = (\Bbbk x_1, \Bbbk x_2)$ *admits all reflections, and* $\mathcal{G}(M)$ *is finite.*

(2) *The Dynkin diagram* $\mathcal{D}$ *of* V *appears in Table* 15.1 *(up to isomorphism).*

In this case, the Dynkin diagrams of the points of $\mathcal{G}_{\mathrm{s}}(M)$ *appear in the row of* $\mathcal{D}$, *and the same row of Table* 15.2 *contains the exchange graph of* $\mathcal{G}_{\mathrm{s}}(M)$.

REMARK 15.3.2. We describe the labeled graphs with labels in $\Bbbk$ and set of vertices $\{1, 2\}$ as follows.

Here, q_i is the label of $i \in \{1, 2\}$, and q is the label of the edge between 1 and 2, if the set of edges it not empty. Isomorphic labeled graphs are obtained by interchanging the labels q_1 and q_2.

In order to be able to display the exchange graphs of the Cartan graphs appearing in Theorem 15.3.1, we introduce the following notation. In row n of Table 15.1, where $1 \leq n \leq 18$, let $\mathcal{D}_{n,k}$ be the k-th Dynkin diagram for all $k \geq 1$ (if it exists). For the presentation of the exchange graph of $\mathcal{G}_{\mathrm{s}}(M)$ it is important to distinguish between vertex 1 (on the left) and vertex 2 (on the right) of $\mathcal{D}_{n,k}$. Therefore we write $\tau\mathcal{D}_{n,k}$ for the graph $\mathcal{D}_{n,k}$, if vertex 1 is on the right and vertex 2 is on the left. As a further simplification, we just write k for $\mathcal{D}_{n,k}$ in Table 15.2.

PROOF. First we prove that (2) implies (1). Let $M_1 = \Bbbk x_1$, $M_2 = \Bbbk x_2$ as Yetter-Drinfeld modules over $H = \Bbbk\mathbb{Z}^2$ and let $M = (M_1, M_2)$. Then, by construction, $M \in \mathcal{F}_2^H$. Assume that the Dynkin diagram $\mathcal{D}$ of M appears in Table 15.1. Then, by Lemma 15.1.4, M is i-finite for all $i \in \{1, 2\}$. Moreover, using Lemma 15.1.5(1) and Lemma 15.1.8(2) one checks that the Dynkin diagram of $R_i(M)$ for all $1 \leq i \leq 2$ appears in the same row of Table 15.1 as $\mathcal{D}$. Doing the same for all diagrams in the row of $\mathcal{D}$ implies that M admits all reflections. Moreover, we obtain that the objects of the small Cartan graph $\mathcal{G}_{\mathrm{s}}(M)$, defined in Proposition 15.1.10, correspond to the Dynkin diagrams in the row of $\mathcal{D}$. We will apply Theorem 10.3.21 in order to show that $\mathcal{G}_{\mathrm{s}}(M)$ is finite. Then $\mathcal{G}(M)$ is finite by Lemma 10.1.4. Our strategy is the following.

The above calculations allow us to check that the exchange graph of $\mathcal{G}_{\mathrm{s}}(M)$ is the one in Table 15.2. In that table, $\mathcal{D}_{m,k}$ is just abbreviated by k. Then we calculate the minimal number n such that $(r_2 r_1)^n(\mathcal{D}) = \mathcal{D}$. We compute the characteristic sequence $(c_k)_{k\geq 1}$ of $\mathcal{C}_{\mathrm{s}}(M)$ with respect to the first object in the row of $\mathcal{D}$ and the label $i = 1$. This is just the infinite power of the sequence in the last column of Table 15.2 in the row of $\mathcal{D}$. Then we calculate $\kappa = 6n - \sum_{k=1}^{2n} c_k$, and we check that $(c_1, \dots, c_{12n/\kappa})$ is the sequence in the last column of Table 15.2 in the row of $\mathcal{D}$. Now, using Corollary 10.3.9, one verifies that $(c_1, \dots, c_{12n/\kappa}) \in \mathcal{A}^+$. Then Theorem 10.3.21 implies that $\mathcal{G}_{\mathrm{s}}(M)$ is finite.

Now we prove that (1) implies (2). To do so, we use Corollary 10.3.28.

By Proposition 15.1.10, the assumptions in (1) imply that $\mathcal{G}_{\mathrm{s}}(M)$ is a finite Cartan graph. It suffices to show that the Dynkin diagram of one point of $\mathcal{G}_{\mathrm{s}}(M)$ is contained in Table 15.1. Indeed, by the first part of the proof of the theorem, then all points of $\mathcal{G}_{\mathrm{s}}(M)$ have such a Dynkin diagram. In fact, Corollary 10.3.28 claims the existence of a point with particular properties. We assume that $X = [M]_{\mathrm{s}}$ is such a point with $i = 1$ and $j = 2$, and we prove that the Dynkin diagram of this point appears in Table 15.1. We proceed case by case and use Lemma 15.1.6.

Step 1. $a_{12}^X = a_{21}^X = 0$. Then $q_{12}q_{21} = 1$, and hence $\mathcal{D} = \mathcal{D}_{1,1}$.

Step 2. $a_{12}^X = a_{21}^X = -1$. Then

$$q_{12}q_{21} = q_{11}^{-1},\ q_{11} \neq 1, \quad \text{or} \quad q_{11} = -1,\ (q_{12}q_{21})^2 \neq 1,$$

and

$$q_{12}q_{21} = q_{22}^{-1},\ q_{22} \neq 1, \quad \text{or} \quad q_{22} = -1,\ (q_{12}q_{21})^2 \neq 1$$

by Lemma 15.1.6. If $q_{12}q_{21} = q_{11}^{-1} = q_{22}^{-1}$, then $\mathcal{D} = \mathcal{D}_{2,1}$. Otherwise we obtain that $q_{12}q_{21} \notin \{1, -1\}$, and one of the following hold.

(1) $q_{11} = q_{22} = -1$,
(2) $q_{22} = -1$, $q_{12}q_{21} = q_{11}^{-1}$,
(3) $q_{11} = -1$, $q_{12}q_{21} = q_{22}^{-1}$.

Then $\mathcal{D} = \mathcal{D}_{3,2}$, $\mathcal{D} = \mathcal{D}_{3,1}$, and $\mathcal{D} = \tau\mathcal{D}_{3,1}$, respectively.

Step 3. $a_{12}^X = -2$, $a_{21}^X = -1$, $a_{21}^{r_1(X)} \in \{-1, -2, -3\}$. Then

$$q_{12}q_{21} = q_{11}^{-2},\ q_{11} \notin \{1, -1\}, \quad \text{or} \quad q_{11} \in P_3,\ (q_{12}q_{21})^3 \neq 1,$$

and

$$q_{12}q_{21} = q_{22}^{-1},\ q_{22} \neq 1, \quad \text{or} \quad q_{22} = -1,\ (q_{12}q_{21})^2 \neq 1$$

by Lemma 15.1.6. If $q_{12}q_{21} = q_{11}^{-2}$, $q_{11}^2 \neq 1$, $q_{12}q_{21} = q_{22}^{-1}$, then $\mathcal{D} = \mathcal{D}_{4,1}$. If $q_{12}q_{21} = q_{11}^{-2}$, $q_{22} = -1$, $(q_{12}q_{21})^2 \neq 1$, then $\mathcal{D} = \mathcal{D}_{5,1}$. If $q_{11} \in P_3$, $(q_{12}q_{21})^3 \neq 1$, and $q_{12}q_{21} = q_{22}^{-1}$, then $\mathcal{D} = \mathcal{D}_{6,1}$ for $q_{22} = -q_{11}^{-1}$ and $\mathcal{D} = \mathcal{D}_{7,1}$ for $q_{22} \neq -q_{11}^{-1}$.

Assume now that $q_{11} \in P_3$, $(q_{12}q_{21})^3 \neq 1$, $q_{22} = -1$, and $(q_{12}q_{21})^2 \neq 1$. Let $W = R_1(M)_1 \oplus R_1(M)_2$ and let $p = (p_{ij})_{1 \leq i,j \leq 2}$ be the braiding matrix of W. Then

$$p_{11} = q_{11}, \quad p_{12}p_{21} = q_{11}^2(q_{12}q_{21})^{-1}, \quad p_{22} = -q_{11}(q_{12}q_{21})^2$$

by Lemma 15.1.8(2).

(a) $a_{21}^{r_1(X)} = -1$. Then

$$p_{12}p_{21} = p_{22}^{-1},\ p_{22} \neq 1, \quad \text{or} \quad p_{22} = -1,\ (p_{12}p_{21})^2 \neq 1$$

by Lemma 15.1.6. In the first case, $q_{11}^2(q_{12}q_{21})^{-1} = -q_{11}^{-1}(q_{12}q_{21})^{-2}$. This is a contradiction to $q_{11} \in P_3$, $(q_{12}q_{21})^2 \neq 1$. In the second case, $q_{11}(q_{12}q_{21})^2 = 1$. Since $q_{11} \in P_3$ and $(q_{12}q_{21})^3 \neq 1$, we conclude that $q_{12}q_{21} = -q_{11}$. Therefore $\mathcal{D} = \mathcal{D}_{8,1}$.

(b) $a_{21}^{r_1(X)} = -2$. Then

$$p_{12}p_{21} = p_{22}^{-2},\ p_{22}^2 \neq 1, \quad \text{or} \quad p_{22} \in P_3,\ (p_{12}p_{21})^3 \neq 1$$

by Lemma 15.1.6. In the first case, $q_{11}^2(q_{12}q_{21})^{-1} = q_{11}^{-2}(q_{12}q_{21})^{-4}$, and hence $q_{12}q_{21} \in P_9$ and $q_{11} = (q_{12}q_{21})^{-3}$. Then $\mathcal{D} = \mathcal{D}_{11,2}$. In the second case we have that $-q_{11}(q_{12}q_{21})^2 \in P_3$. Since $P_3 = \{q_{11}, q_{11}^{-1}\}$, we conclude that $(q_{12}q_{21})^2 = -1$ or $-(q_{12}q_{21})^2 = q_{11}$. If $(q_{12}q_{21})^2 = -1$, then with $\zeta = (q_{11}q_{12}q_{21})^{-1}$ we obtain that $\zeta \in P_{12}$, $q_{11} = -\zeta^2$, $q_{12}q_{21} = \zeta^3$, and $\mathcal{D} = \mathcal{D}_{10,2}$. On the other hand, if $q_{11} = -(q_{12}q_{21})^2$, then $q_{12}q_{21} \in P_{12}$ and $\mathcal{D} = \mathcal{D}_{9,2}$ with $\zeta = (q_{12}q_{21})^{-1}$.

(c) $a_{21}^{r_1(X)} = -3$. Then

$$p_{12}p_{21} = p_{22}^{-3},\ p_{22}^2, p_{22}^3 \neq 1, \quad \text{or} \quad p_{22} \in P_4,\ (p_{12}p_{21})^4 \neq 1$$

by Lemma 15.1.6.

(c1) Assume that $p_{12}p_{21} = p_{22}^{-3}$, that is, $q_{11}^2(q_{12}q_{21})^{-1} = -q_{11}^{-3}(q_{12}q_{21})^{-6}$. Then $(-q_{12}q_{21})^5 = q_{11}$, and hence $-q_{12}q_{21} = q_{11}^{-1}$ or $-q_{12}q_{21} \in P_{15}$. If $q_{11} = -(q_{12}q_{21})^{-1}$, then the braiding matrix $(\tilde{p}_{ij})_{1\le i,j\le 2}$ of $R_2(M)$ satisfies $\tilde{p}_{11} = q_{11}q_{12}q_{21}q_{22} = 1$ by Lemma 15.1.8(2), which is a contradiction to $a_{12}^{R_2(M)} < 0$. If $-q_{12}q_{21} \in P_{15}$, then $\mathcal{D} = \mathcal{D}_{17,2}$ with $\zeta = (-q_{12}q_{21})^7$.

(c2) Assume that $p_{22} \in P_4$. Then $q_{11}^2(q_{12}q_{21})^4 = -1$, and thus $q_{11} = -(q_{12}q_{21})^4$. We conclude that $(q_{12}q_{21})^{12} = -1$, and hence $q_{12}q_{21} \in P_{24}$ since $q_{11} \neq 1$. Then $\mathcal{D} = \mathcal{D}_{14,2}$ with $\zeta = (q_{12}q_{21})^5$.

Step 4. $a_{12}^X = -1$, $a_{21}^X = -2$, $a_{21}^{r_1(X)} \in \{-3,-4,-5\}$. Then

$$q_{12}q_{21} = q_{11}^{-1},\ q_{11} \neq 1, \quad \text{or} \quad q_{11} = -1,\ (q_{12}q_{21})^2 \neq 1,$$

and

$$q_{12}q_{21} = q_{22}^{-2},\ q_{22} \notin \{1,-1\}, \quad \text{or} \quad q_{22} \in P_3,\ (q_{12}q_{21})^3 \neq 1$$

by Lemma 15.1.6. As in Step 3, we distinguish four different cases. In three of these cases we identified $\mathcal{D}$ (more precisely, $\tau\mathcal{D}$) already in Step 3.

Assume now that $q_{11} = -1$, $q_{22} \in P_3$, and $(q_{12}q_{21})^2, (q_{12}q_{21})^3 \neq 1$. Let us define $p = (p_{ij})_{1\le i,j\le 2}$ to be the braiding matrix of the first reflection of M. Then

$$p_{11} = -1, \quad p_{12}p_{21} = (q_{12}q_{21})^{-1}, \quad p_{22} = -q_{12}q_{21}q_{22}$$

by Lemma 15.1.8(2).

(a) $a_{21}^{r_1(X)} = -3$. Then

$$p_{12}p_{21} = p_{22}^{-3},\ p_{22}^2, p_{22}^3 \neq 1, \quad \text{or} \quad p_{22} \in P_4,\ (p_{12}p_{21})^4 \neq 1$$

by Lemma 15.1.6. In the first case, $(q_{12}q_{21})^{-1} = -(q_{12}q_{21}q_{22})^{-3}$, and hence $(q_{12}q_{21})^2 = -1$. Let $\zeta = (q_{12}q_{21}q_{22})^{-1}$. Then $\zeta \in P_{12}$, $q_{12}q_{21} = \zeta^3$, and $q_{22} = -\zeta^2$, and $\mathcal{D} = \tau\mathcal{D}_{10,2}$. In the second case, $(q_{12}q_{21}q_{22})^2 = -1$. Then $q_{22} = -(q_{12}q_{21})^2$, and hence $q_{12}q_{21} \in P_{12}$. Then $\mathcal{D} = \tau\mathcal{D}_{9,2}$.

(b) $a_{21}^{r_1(X)} = -4$. Then

$$p_{12}p_{21} = p_{22}^{-4},\ p_{22}^3, p_{22}^4 \neq 1, \quad \text{or} \quad p_{22} \in P_5,\ (p_{12}p_{21})^5 \neq 1.$$

In the first case, $(q_{12}q_{21})^{-1} = (q_{12}q_{21}q_{22})^{-4}$, and hence $q_{22} = (q_{12}q_{21})^{-3}$. Then $q_{12}q_{21} \in P_9$ and $\mathcal{D} = \tau\mathcal{D}_{11,2}$. In the second case, $-q_{12}q_{21}q_{22} \in P_5$. It follows that $-q_{12}q_{21} \in P_{15}$ and $-(q_{12}q_{21})^5 = q_{22}$. Then $\mathcal{D} = \tau\mathcal{D}_{17,2}$.

(c) $a_{21}^{r_1(X)} = -5$. Then

$$p_{12}p_{21} = p_{22}^{-5},\ p_{22}^3, p_{22}^4, p_{22}^5 \neq 1, \quad \text{or} \quad -p_{22} \in P_3,\ (p_{12}p_{21})^6 \neq 1.$$

In the first case, $(q_{12}q_{21})^{-1} = -(q_{12}q_{21}q_{22})^{-5}$, and hence $q_{22} = -(q_{12}q_{21})^4$. Then $q_{12}q_{21} \in P_{24}$, and $\mathcal{D} = \tau\mathcal{D}_{14,2}$. In the second case, $q_{12}q_{21}q_{22} \in P_3$ and $(q_{12}q_{21})^6 \neq 1$. But this is impossible.

Step 5. $a_{21}^X = -1$, $a_{12}^X = -3$, $a_{21}^{r_1(X)} = -1$, $a_{12}^{r_2(X)} \in \{-3,-4,-5\}$. Then

$$q_{12}q_{21} = q_{11}^{-3},\ q_{11}^2, q_{11}^3 \neq 1, \quad \text{or} \quad q_{11} \in P_4,\ (q_{12}q_{21})^4 \neq 1,$$

and

$$q_{12}q_{21} = q_{22}^{-1},\ q_{22} \neq 1, \quad \text{or} \quad q_{22} = -1,\ (q_{12}q_{21})^2 \neq 1$$

by Lemma 15.1.6.

(a) Assume that $q_{12}q_{21} = q_{11}^{-3}$, $q_{11}^2, q_{11}^3 \neq 1$, and $q_{12}q_{21} = q_{22}^{-1}$. Then $\mathcal{D} = \mathcal{D}_{12,1}$.

(b) Assume that $q_{12}q_{21} = q_{11}^{-3}$, $q_{11}^2, q_{11}^3 \neq 1$, $q_{22} = -1$ and $(q_{12}q_{21})^2 \neq 1$. Let $q = q_{11}$. Then $r_2(X)$ has Dynkin diagram

$$\overset{-q^{-2}}{\circ}\!\!\overset{q^3}{\text{———}}\!\!\overset{-1}{\circ}$$

by Lemma 15.1.8(2). Now we are going to analyze the consequences of the assumption $a_{12}^{r_2(X)} \in \{-3, -4, -5\}$.

(b1) Assume that $a_{12}^{r_2(X)} = -3$. Then $q^3 = -q^6$, $q^4 \neq 1$, or $-q^{-2} \in P_4$. In the first case, $-q \in P_3$ and $\mathcal{D} = \mathcal{D}_{12,1}$. In the second case, $q \in P_8$ and $\mathcal{D} = \mathcal{D}_{13,3}$.

(b2) Assume that $a_{12}^{r_2(X)} = -4$. Then $q^3 = q^8$ or $-q^{-2} \in P_5$. In the first case, $q \in P_5$ since $q \neq 1$, and hence $\mathcal{D} = \mathcal{D}_{15,1}$. In the second case, $q \in P_{20}$ and $\mathcal{D} = \mathcal{D}_{16,1}$.

(b3) Assume that $a_{12}^{r_2(X)} = -5$. Then $q^3 = -q^{10}$ or $-q^{-2} \in P_6$, $q^{18} \neq 1$. In the first case, $-q \in P_7$ since $q^2 \neq 1$, and hence $\mathcal{D} = \mathcal{D}_{18,1}$. In the second case, $q^2 \in P_3$, which is a contradiction to $q^{18} \neq 1$.

(c) Assume that $q_{11} \in P_4$, $(q_{12}q_{21})^4 \neq 1$, and $q_{12}q_{21} = q_{22}^{-1}$. Let $\xi = q_{11}$ and $q = q_{22}$. The first reflection of M has Dynkin diagram

$$\overset{\xi}{\circ}\!\!\overset{-q}{\text{———}}\!\!\overset{\xi q^{-2}}{\circ}$$

by Lemma 15.1.8(2). Since $a_{21}^{r_1(X)} = -1$, this implies that $-\xi q^{-1} = 1$ or $\xi q^{-2} = -1$. In the first case $q = -\xi$, which contradicts to $\xi \in P_4$, $q^4 \neq 1$. In the second case $\xi = -q^2$, and hence $q \in P_8$. Then $\mathcal{D} = \mathcal{D}_{13,1}$.

(d) Assume now that $q_{11} \in P_4$, $(q_{12}q_{21})^4 \neq 1$, and $q_{22} = -1$. Let $\xi = q_{11}$ and $q = q_{12}q_{21}$. The first and second reflections of M have Dynkin diagrams

$$\overset{\xi}{\circ}\!\!\overset{-q^{-1}}{\text{———}}\!\!\overset{-\xi q^3}{\circ} \quad \text{and} \quad \overset{-\xi q}{\circ}\!\!\overset{q^{-1}}{\text{———}}\!\!\overset{-1}{\circ}$$

respectively, by Lemma 15.1.8(2). Since $a_{21}^{r_1(X)} = -1$, this implies that $\xi q^2 = 1$ or $\xi q^3 = 1$. In the first case $q \in P_8$, $\xi = q^{-2}$, and hence $\mathcal{D} = \mathcal{D}_{13,2}$. In the second case $q^6 = -1$. Then $(-\xi q)^3 = (-q^{-2})^3 = 1$, a contradiction to $a_{12}^{r_2(X)} \leq -3$.

Now all cases in Corollary 10.3.28 are checked, and the proof of the theorem is completed. $\square$

THEOREM 15.3.3. *Assume that* $\mathrm{char}(\Bbbk) = 0$. *Let V be a two-dimensional braided vector space of diagonal type. Let $\mathcal{D}$ be the Dynkin diagram of V. Then $\mathcal{B}(V)$ is finite-dimensional if and only if the following hold.*

1. *The graph $\mathcal{D}$ appears in Table* 15.1.
2. *The labels of all vertices of the Dynkin diagrams in the row of $\mathcal{D}$ are non-trivial roots of* 1.

	Dynkin diagrams	fixed parameters
1	$\overset{q}{\circ}\qquad\overset{r}{\circ}$	$q, r \in \Bbbk^\times$
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$	$q \in \Bbbk^\times \setminus \{1\}$
3	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}\quad\overset{-1}{\circ}\overset{q}{\text{———}}\overset{-1}{\circ}$	$q \in \Bbbk^\times$, $q^2 \neq 1$
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$	$q \in \Bbbk^\times$, $q^2 \neq 1$
5	$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{-1}{\circ}\quad\overset{-q^{-1}}{\circ}\overset{q^2}{\text{———}}\overset{-1}{\circ}$	$q \in \Bbbk^\times$, $q^4 \neq 1$
6	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-\zeta^{-1}}{\circ}$	$\zeta \in P_3$
7	$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}\quad\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{———}}\overset{\zeta q^{-1}}{\circ}$	$\zeta \in P_3$, $q \in \Bbbk^\times$, $q^3 \neq 1$, $q \neq -\zeta^{-1}$
8	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in P_3$
9	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-\zeta^2}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^3}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^3}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^2}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$	$\zeta \in P_{12}$
10	$\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^2}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-\zeta^2}{\circ}$	$\zeta \in P_{12}$
11	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{-\zeta}{\circ}$	$\zeta \in P_9$
12	$\overset{q}{\circ}\overset{q^{-3}}{\text{———}}\overset{q^3}{\circ}$	$q \in \Bbbk^\times$, $q^2, q^3 \neq 1$
13	$\overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}\quad\overset{\zeta^2}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$	$\zeta \in P_8$
14	$\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-4}}{\circ}\overset{\zeta^5}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-4}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{\zeta^6}{\circ}\quad\overset{\zeta^{-1}}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^6}{\circ}$	$\zeta \in P_{24}$
15	$\overset{\zeta}{\circ}\overset{\zeta^2}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in P_5$
16	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in P_{20}$
17	$\overset{\zeta^5}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-\zeta}{\circ}\quad\overset{\zeta^5}{\circ}\overset{-\zeta^{-2}}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{-\zeta^2}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{-\zeta^4}{\text{———}}\overset{-\zeta^{-4}}{\circ}$	$\zeta \in P_{15}$
18	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}$	$\zeta \in P_7$

TABLE 15.1. Dynkin diagrams of 2-dimensional braided vector spaces of diagonal type with finite Cartan graph

	exchange graphs	n	κ	sequence in $\mathcal{A}^+$
1	1	1	6	$(0,0)$
2	1	1	4	$(1,1,1)$
3	$1 \overset{2}{\text{——}} 2 \overset{1}{\text{——}} \tau 1$	3	12	$(1,1,1)$
4	1	1	3	$(2,1,2,1)$
5	$1 \overset{2}{\text{——}} 2$	2	6	$(2,1,2,1)$
6	1	1	3	$(2,1,2,1)$
7	$1 \overset{1}{\text{——}} 2$	2	6	$(2,1,2,1)$
8	$1 \overset{2}{\text{——}} 2$	2	6	$(2,1,2,1)$
9	$\begin{array}{ccccccccc} 1 & \overset{1}{\text{——}} & 2 & \overset{2}{\text{——}} & 3 & \overset{1}{\text{——}} & 4 & \overset{2}{\text{——}} & 5 \\ 2 \vert & & & & & & & & \vert 1 \\ \tau 5 & \overset{1}{\text{——}} & \tau 4 & \overset{2}{\text{——}} & \tau 3 & \overset{1}{\text{——}} & \tau 2 & \overset{2}{\text{——}} & \tau 1 \end{array}$	5	12	$(2,1,3,1,2)$
10	$1 \overset{2}{\text{——}} 2 \overset{1}{\text{——}} 3 \overset{2}{\text{——}} \tau 2 \overset{1}{\text{——}} \tau 1$	5	12	$(3,1,2,2,1)$
11	$1 \overset{2}{\text{——}} 2 \overset{1}{\text{——}} 3$	3	6	$(4,1,2,2,2,1)$
12	1	1	2	$(3,1,3,1,3,1)$
13	$1 \overset{1}{\text{——}} 2 \overset{2}{\text{——}} 3$	3	6	$(3,1,3,1,3,1)$
14	$1 \overset{2}{\text{——}} 2 \overset{1}{\text{——}} 3 \overset{2}{\text{——}} 4$	4	6	$(5,1,2,3,1,3,2,1)$
15	$1 \overset{2}{\text{——}} 2$	2	3	$(3,1,4,1,3,1,4,1)$
16	$1 \overset{2}{\text{——}} 2 \overset{1}{\text{——}} 3 \overset{2}{\text{——}} 4$	4	6	$(3,1,4,1,3,1,4,1)$
17	$1 \overset{1}{\text{——}} 2 \overset{2}{\text{——}} 3 \overset{1}{\text{——}} 4$	4	6	$(2,1,4,1,4,1,2,3)$
18	$1 \overset{2}{\text{——}} 2$	2	2	$(3,1,5,1,3,1,5,1,3,1,5,1)$

TABLE 15.2. The exchange graphs of the small Cartan graphs in Theorem 15.3.1

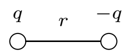
PROOF. Let $(V, (x_i, g_i, \chi_i)_{1\le i\le 2})$ be a realization of the braiding matrix of V over $G = \mathbb{Z}^2$. Let $M_1 = \Bbbk x_1$, $M_2 = \Bbbk x_2$ as Yetter-Drinfeld modules over $\Bbbk G$, and let $M = (M_1, M_2)$. Then $\mathcal{B}(M) = \mathcal{B}(V)$ as $\mathbb{N}_0$-graded algebras and coalgebras. By Corollary 14.5.3, the Nichols algebra $\mathcal{B}(M)$ is finite-dimensional if and only if M admits all reflections, $\mathcal{G}(M)$ is finite, and for all $N = (N_1, N_2) \in \mathcal{F}_\theta^H(M)$ the Nichols algebras $\mathcal{B}(N_1)$ and $\mathcal{B}(N_2)$ are finite-dimensional. By Example 1.10.1, $\mathcal{B}(N_1)$ and $\mathcal{B}(N_2)$ are finite-dimensional if and only if the diagonal entries of their braiding matrices are non-trivial roots of 1. By Remark 15.1.9, the set of Dynkin diagrams of the points of $\mathcal{G}(M)$ is the same as the set of Dynkin diagrams of the points of $\mathcal{G}_{\mathrm{s}}(M)$. Thus the claim follows from Theorem 15.3.1. $\square$

LEMMA 15.3.4. *Let V be a two-dimensional braided vector space of diagonal type, and let $(q_{ij})_{1\le i,j\le 2}$ be the braiding matrix of V. If the Dynkin diagram of $\mathcal{D}$ appears in Table* 15.1, *then one of the following hold.*

(1) $q_{12}q_{21} \in \{1, q_{11}^{-1}, q_{22}^{-1}\}$,
(2) $q_{11} = -1$ *or* $q_{22} = -1$,
(3) $q_{11}(q_{12}q_{21})^2 q_{22} = -1$, $\mathcal{D} \in \{\mathcal{D}_{9,1}, \mathcal{D}_{10,3}, \mathcal{D}_{11,3}, \mathcal{D}_{14,3}, \mathcal{D}_{17,1}\}$, *and* $q_{11} \in P_3$ *or* $q_{22} \in P_3$.

Proof. Check the diagrams in Table 15.1 case by case. ☐

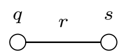
Lemma 15.3.5. *Let* $q, r \in \Bbbk^\times$. *The Dynkin diagram*



appears in Table 15.1 *if and only if* $r = 1$ *or* $q \in P_3$, $r = -q^{-1}$ *or* $-q \in P_3$, $r = q^{-1}$, *or* $q \in P_4$, $r \in P_4$.

Proof. Check the diagrams in Table 15.1 case by case. ☐

Lemma 15.3.6. *Let* $q, r, s \in \Bbbk^\times$ *such that* $r \neq 1$. *If the Dynkin diagram*



appears in Table 15.1 *then* $q^k r = 1$ *for some* $1 \le k \le 5$ *or* $q \in P_k$, $r^k \neq 1$ *for some* $2 \le k \le 5$.

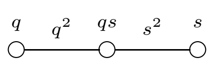
Proof. Check the diagrams in Table 15.1 case by case. ☐

15.4. Application to Nichols algebras of rank three

We now detect some three-dimensional vector spaces of diagonal type which have infinite dimensional Nichols algebras. These will be used to prove in Section 15.5 that any finite-dimensional pre-Nichols algebra over $\Bbbk$ in the category ${}^G_G\mathcal{YD}$, where G is a finite abelian group and $\mathrm{char}(\Bbbk) = 0$, is a Nichols algebra.

In the following, we will often apply claims on one- or two-dimensional braided vector spaces, such as the classification in Theorem 15.3.3 of two-dimensional braided vector spaces of diagonal type with finite-dimensional Nichols algebra, to a braided subspace of a larger braided vector space. For a braided vector space of diagonal type with given Dynkin diagram we will say that we apply a claim to a subset of vertices, if we mean the braided subspace generated by the basis vectors corresponding to the given subset of vertices.

Lemma 15.4.1. *Let V be a three-dimensional braided vector space of diagonal type. Let* $q, s \in \Bbbk^\times$. *Assume that the Dynkin diagram of* $\mathcal{D}$ *is*



Then $\mathcal{B}(V)$ *is infinite-dimensional.*

Proof. We prove the Lemma indirectly by assuming that $\mathcal{B}(V)$ is finite-dimensional.

Apply Lemma 15.3.4 to the first two and the last two vertices of $\mathcal{D}$, respectively. We obtain that

$$(q^2 - 1)(q^3 - 1)(q^3 s - 1)(qs + 1)(q^6 s + 1) = 0, \tag{15.4.1}$$

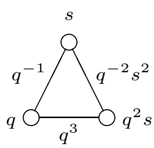
$$(s^2 - 1)(s^3 - 1)(qs^3 - 1)(qs + 1)(qs^6 + 1) = 0. \tag{15.4.2}$$

If $q=1$ or q is not a root of 1, then $\mathcal{B}(V)$ is infinite-dimensional by Example 1.10.1. If $q^6s=-1$ and $(q^2-1)(q^3-1)(q^3s-1)(qs+1)\neq 0$, then Lemma 15.3.4 and the shape of the Dynkin diagrams $\mathcal{D}_{9,1}$, $\mathcal{D}_{10,3}$, $\mathcal{D}_{11,3}$, $\mathcal{D}_{14,3}$ and $\mathcal{D}_{17,1}$ yield a contradiction. It follows that one of the first four factors in (15.4.1) are 0, and similarly one of the first four factors in (15.4.2) have to be 0.

Assume that $q=-1$. Then Lemma 15.3.5 implies that the Dynkin diagram with the last two vertices appears in Table 15.1 if and only if $s^2=1$. In this case, $\mathcal{B}(V)$ is infinite-dimensional by Example 1.10.1. Otherwise it is infinite dimensional by Theorem 15.3.3. We argue similarly if $s=-1$.

Assume that $q^2\neq 1$ and $s^2\neq 1$. The products of the labels of $\mathcal{D}$ is $(qs)^4$. Hence we may assume that $qs\neq -1$ by Corollary 15.1.19. Then $q\in P_3$ or $q^3s=1$. Similarly, $s\in P_3$ or $qs^3=1$. If $q^3s=qs^3=1$ or $q=s^{-1}\in P_3$, then $q^4s^4=1$, and $\mathcal{B}(V)$ is again infinite dimensional. Otherwise $q=s\in P_3$ or $s\in P_9$, $q=s^{-3}$, or $q\in P_9$, $s=q^{-3}$. In all cases V is of infinite Cartan type, and hence $\mathcal{B}(V)$ is infinite-dimensional by Theorem 15.1.14. $\square$

LEMMA 15.4.2. *Let V be a three-dimensional braided vector space of diagonal type. Let $q,s\in\Bbbk^\times$ such that $q\neq -1$. Assume that the Dynkin diagram $\mathcal{D}$ of V is*



Then $\mathcal{B}(V)$ is infinite-dimensional.

PROOF. Assume that $\mathcal{B}(V)$ is finite-dimensional. By Theorem 15.3.3, any subdiagram of $\mathcal{D}$ with two vertices has to appear in Table 15.1.

By Example 1.10.1 applied to the vertices of $\mathcal{D}$ we obtain that q, s, and q^2s are non-trivial roots of 1.

Apply Lemma 15.3.4 to the two vertices at the bottom of $\mathcal{D}$. We obtain that

$$(q^3-1)(q^4-1)(q^5s-1)(q^2s+1)(q^9s+1)=0.$$

If $q^3=1$, then we obtain a contradiction to the finite-dimensionality of $\mathcal{B}(V)$ from Lemma 15.4.1.

Assume that $q^9s=-1$ and $q^3\neq 1$. Then the labels of the Dynkin diagrams in Lemma 15.3.4(3) imply that $-q\in P_9\cup P_{15}$ and $q^2s\in P_3$. Using again that $q^9s=-1$, we conclude that $(-q)^7=(q^2s)^{-1}\in P_3$, a contradiction.

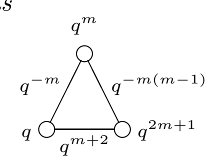
If $q^4=1$, then $q\in P_4$, since $q^2\neq 1$ by assumption. Apply Lemma 15.3.5 to the two vertices on the right. Since $s\neq 1$ and $-s=q^2s\neq 1$, we conclude that $s\in P_3\cup P_4\cup P_6$. If $s\in P_3$, then the Dynkin diagram with the lower two vertices does not appear in Table 15.1. If $s\in P_6$, then the same is true for the Dynkin diagram with the two vertices on the left. Finally, if $s\in P_4$, then $s=q$ or $s=-q$. Then $\mathcal{D}$ is of infinite Cartan type, and hence $\mathcal{B}(V)$ is infinite dimensional by Theorem 15.1.14.

Assume that $q^2s=-1$. Then $q^{-2}s^2=q^{-6}$. By Lemma 15.3.6 applied to the two vertices at the bottom of $\mathcal{D}$ we conclude that $q^k=1$ for some $k\leq 8$. By the above, we may assume that $k\geq 5$. The product of the labels of $\mathcal{D}$ is $q^3s^4=q^{-5}$. Hence, if $q^5=1$, then we obtain a contradiction to Corollary 15.1.19. If $q\in P_6$, then $\mathcal{D}$ is of infinite Cartan type, which is a contradiction to Theorem 15.1.14. If $q\in P_7$, then $s=-q^{-2}\in P_{14}$ and $s^{10}q^{-1}=1$. Hence the subdiagram of $\mathcal{D}$ corresponding to the two vertices on the left does not appear in Table 15.1 by Lemma 15.3.6, a contradiction. Finally, if $q \in P_8$, then $q^4 = -1$, and hence Corollary 15.1.19 applied to the subdiagram of the two vertices at the bottom of $\mathcal{D}$ yields a contradiction.

Assume that $q^5 s = 1$. Then $s = q^{-5}$. Since the subdiagram of $\mathcal{D}$ containing the two vertices at the bottom is of Cartan type, Theorem 15.1.14 implies that $q^{k+3} = 1$ for some $0 \le k \le 3$. Further, $q^k \neq 1$ for $1 \le k \le 4$ by the above considerations, and $q^5 \neq 1$ since $s \neq 1$. Finally, if $q \in P_6$ then $\mathcal{D}$ is of infinite Cartan type, a contradiction to Theorem 15.1.14.

Now all cases are considered and the Lemma is proven. $\square$

LEMMA 15.4.3. *Let V be a three-dimensional braided vector space of diagonal type. Let $m \ge 2$ and let $q \in \Bbbk^\times$ such that $q^k \neq 1$ for all $1 \le k \le m+1$. Assume that the Dynkin diagram $\mathcal{D}$ of V is*



Then $\mathcal{B}(V)$ is infinite-dimensional.

PROOF. Assume that $\mathcal{B}(V)$ is finite-dimensional. By Theorem 15.3.3, any subdiagram of $\mathcal{D}$ with two vertices has to appear in Table 15.1.

By Example 1.10.1 applied to the vertices of $\mathcal{D}$ we obtain that q is a non-trivial root of 1, and that $q^{2m+1} \neq 1$.

By Theorem 15.1.14 applied to the two vertices on the left of $\mathcal{D}$ we obtain that $m \in \{2, 3\}$.

Assume that $m = 2$. Then $q^2, q^3, q^5 \neq 1$. By Lemma 15.3.4 applied to the two vertices at the bottom of $\mathcal{D}$ we conclude that

$$(q^4 - 1)(q^5 + 1)(q^9 - 1) = 0$$

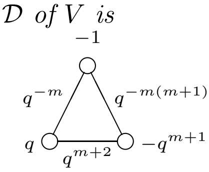
or $q^{14} = -1$, $q^5 \in P_3$. The last case is impossible since $q \neq -1$. Hence $q \in P_4 \cup P_9$ or $-q \in P_5$. If $q \in P_4 \cup P_9$ then $\mathcal{D}$ is of infinite Cartan type, which is a contradiction to Theorem 15.1.14. If $-q \in P_5$ then $q^k q^{m+2} \neq 1$ for $0 \le k \le 5$, and hence Lemma 15.3.6 applied to the two vertices at the bottom of $\mathcal{D}$ yields a contradiction.

Assume now that $m = 3$. Then $q^2, q^3, q^4, q^7 \neq 1$. By Lemma 15.3.6 applied to the two vertices at the bottom of $\mathcal{D}$ we conclude that $q \in P_k$, where $5 \le k \le 10$ and $k \neq 7$. On the other hand, by Lemma 15.3.4 applied to the same two vertices we obtain that

$$(q^5 - 1)(q^6 - 1)(q^{12} - 1)(q^7 + 1) = 0$$

or $q^{18} = -1$, $q^7 \in P_3$. The last two relations have no solution for q. Hence $q \in P_5 \cup P_6$ by our restriction on the order of q. In both cases, $\mathcal{D}$ is of infinite Cartan type, which is a contradiction to Theorem 15.1.14. This proves the Lemma. $\square$

LEMMA 15.4.4. *Let V be a three-dimensional braided vector space of diagonal type. Let $m \ge 2$ and let $q \in \Bbbk^\times$ such that $q^k \neq 1$ for all $1 \le k \le m+1$ and $q^{2m} \neq 1$. Assume that the Dynkin diagram $\mathcal{D}$ of V is*



Then $\mathcal{B}(V)$ is infinite-dimensional.

Proof. Assume that $\mathcal{B}(V)$ is finite-dimensional. By Theorem 15.3.3, any subdiagram of $\mathcal{D}$ with two vertices has to appear in Table 15.1.

By Example 1.10.1 applied to the vertices of $\mathcal{D}$ we obtain that q is a non-trivial root of 1, and that $q^{m+1} \neq -1$.

By Lemma 15.3.6 applied to the two vertices on the left of $\mathcal{D}$ we obtain that $m \in \{2,3,4,5\}$.

Since $q \notin P_2 \cup P_3$ and $q^{m+1} \neq 1$, Lemma 15.3.4 applied to the two vertices at the bottom of $\mathcal{D}$ implies that

$$(q^{m+2}-1)(q^{m+3}-1)(q^{2m+3}+1) = 0$$

or $q^{3m+6} = 1$, $-q^{m+1} \in P_3$.

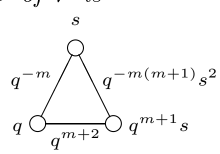
If $q^{m+2} = 1$, then $q \in P_{m+2}$, $q^{-m} = q^2$, $-q^{m+1} = -q^{-1}$, $q^{-m(m+1)} = q^{-2}$, and hence we obtain a contradiction to Lemma 15.4.1.

If $q^{m+3} = 1$, then $q \in P_{m+3}$, $q^{-m} = q^3$, $-q^{m+1} = -q^{-2}$, $q^{-m(m+1)} = q^{-6}$, and hence we obtain a contradiction to Lemma 15.4.2 with $s = -q^{-2}$.

If $q^{3m+6} = 1$, $-q^{m+1} \in P_3$, and $q^k \neq 1$ for all $1 \leq k \leq m+3$, then $q^{3m+3} = -1$, $-q^3 = 1$, and hence $q \in P_6$, $m = 2$. This is a contradiction to $q^{m+1} \neq -1$.

Finally, if $q^{2m+3} = -1$ and $q^{m+2}, q^{m+3} \neq 1$, then $-q^{m+1} = (q^{m+2})^{-1}$, and hence Theorem 15.1.14 applied to the two vertices at the bottom of $\mathcal{D}$ implies that $q^{m+2+k} = 1$ for some $k \in \{2,3\}$. If $k = 2$, then $q \in P_{m+4}$, $q^{m-1} = -1$, and hence $q \in P_{10}$, a contradiction to $m \leq 5$. If $k = 3$, then $q \in P_{m+5}$, $q^{m-2} = -1$, and hence $q \in P_{14}$, a contradiction to $m \leq 5$. This completes the proof of the lemma. $\square$

Proposition 15.4.5. *Let V be a three-dimensional braided vector space of diagonal type. Let $m \in \mathbb{N}_0$ and let $q, s \in \Bbbk^\times$ such that $q^k \neq 1$ for all $1 \leq k \leq m+1$. Assume that the Dynkin diagram $\mathcal{D}$ of V is*



Then $\mathcal{B}(V)$ is infinite-dimensional.

Proof. Assume that $\mathcal{B}(V)$ is finite-dimensional. By Theorem 15.3.3, any subdiagram of $\mathcal{D}$ with two vertices has to appear in Table 15.1.

We assumed that $q^k \neq 1$ for $1 \leq k \leq m+1$. In particular, if $q^{-m} = 1$ then $m = 0$, and if $q^{-m} = q^{-1}$, then $m = 1$. By Lemma 15.3.4 applied to the two vertices on the left of $\mathcal{D}$ we obtain that $m = 0$, $m = 1$, $s = q^m$, $q = -1$, $s = -1$, or $q^{1-2m}s = -1$.

If $m = 0$, then we obtain a contradiction to Lemma 15.4.1. If $m = 1$, then Lemma 15.4.2 yields a contradiction. If $m =\geq 2$ and $s = q^m$, then we obtain a contradiction to Lemma 15.4.3. If $q = -1$, then $q^2 = 1$ and hence $m = 0$. If $s = -1$, $s \neq q^m$, and $m \geq 2$, then a contradiction is obtained by Lemma 15.4.4.

Assume now that $m \geq 2$, $s \neq q^m$, and $s \neq -1$. Then $q^{1-2m}s = -1$, and since $q^3 \neq 1$, Lemma 15.3.4 further implies that $q^{m+1}s \in P_3$. By analyzing the Dynkin diagrams $\mathcal{D}_{9,1}$, $\mathcal{D}_{10,3}$, $\mathcal{D}_{11,3}$, $\mathcal{D}_{14,3}$, and $\mathcal{D}_{17,1}$, and using that q^{-m} is a power of q, we also obtain that $m = 2$, $-q \in P_9$, or $m = 3$, $-q \in P_{15}$. If $m = 2$ and $q \in P_{18}$, then $q^{14}q^{m+2} = 1$. If $m = 3$ and $q \in P_{30}$, then $q^{25}q^{m+2} = 1$. In both cases, Lemma 15.3.6 applied to the two vertices at the bottom of $\mathcal{D}$ gives a contradiction. This completes the proof of the proposition. $\square$

15.5. Primitively generated braided Hopf algebras

Let $\theta \geq 1$ and $\mathbb{I} = \{1, \dots, \theta\}$. The main result in this section is the following.

THEOREM 15.5.1. *Assume that* $\mathrm{char}(\Bbbk) = 0$. *Let* $M \in \mathcal{F}_\theta^H$*, and assume that the Yetter-Drinfeld modules* M_i *with* $i \in \mathbb{I}$ *are one-dimensional. Let* R *be a finite-dimensional pre-Nichols algebra of* $M_1 \oplus \cdots \oplus M_\theta$*. Then the canonical Hopf algebra map* $R \to \mathcal{B}(M)$ *is bijective.*

REMARK 15.5.2. Assume that $p = \mathrm{char}(\Bbbk) > 0$. Let V be the one-dimensional braided vector space with trivial braiding. Then the polynomial ring $\Bbbk[x]$ is a commutative cocommutative Hopf algebra, where x is primitive. It is the coordinate ring of the additive group. Further, $\mathcal{B}(V) = \Bbbk[x]/(x^p)$ by Example 1.10.1. The Hopf algebra $\Bbbk[x]/(x^{p^r})$ for any $r \geq 1$ is a finite-dimensional pre-Nichols algebra, which is also known as the coordinate ring of the r-th Frobenius kernel of the additive group. Thus finite-dimensional pre-Nichols algebras over fields of positive characteristic are not necessarily Nichols algebras.

Before we prove Theorem 15.5.1, we need some preparations. Assume for the rest of the section that $\mathrm{char}(\Bbbk) = 0$. Let $M \in \mathcal{F}_\theta^H$ and assume that $\dim M_i = 1$ for all $i \in \mathbb{I}$. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M with finite-dimensional algebra S. Thus, S is a Hopf algebra in ${}^H_H\mathcal{YD}$, $N_1, \dots, N_\theta$ are one-dimensional subobjects of S in ${}^H_H\mathcal{YD}$, $N = (N_1, \dots, N_\theta)$, and $f = (f_j)_{j \in \mathbb{I}} : N \to M$ is an isomorphism of tuples in $\mathcal{F}_\theta^H$ such that

(1) S is generated as an algebra by $N_1, \dots, N_\theta$, and
(2) S is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ with $\deg(N_j) = \alpha_j$ for all $j \in \mathbb{I}$.

For any $i \in \mathbb{I}$, let x_i and y_i be bases of N_i and M_i, respectively, such that $f_i(x_i) = y_i$. According to Example 3.4.3, there exist $g_1, \dots, g_\theta \in G(H)$ and characters $\chi_1, \dots, \chi_\theta \in \mathrm{Alg}(H, \Bbbk)$ such that for any $i \in \mathbb{I}$ the Yetter-Drinfeld structures of N_i and M_i are given by

$$\delta_{N_i}(x_i) = g_i \otimes x_i, \qquad \delta_{M_i}(y_i) = g_i \otimes y_i,$$
$$h \cdot x_i = \chi_i(h) x_i, \qquad h \cdot y_i = \chi_i(h) y_i$$

for all $h \in H$, respectively. Hence the braiding matrix of $\bigoplus_{i \in \mathbb{I}} N_i$ and $\bigoplus_{i \in \mathbb{I}} M_i$ with respect to the bases $(x_i)_{i \in \mathbb{I}}$ and $(y_i)_{i \in \mathbb{I}}$ is $(q_{ij})_{i,j \in \mathbb{I}}$, where $q_{ij} = \chi_j(g_i)$ for all $i, j \in \mathbb{I}$. Let

$$p = p^{\mathcal{N}} : S \to \mathcal{B}(M), \; p(x_i) = y_i \text{ for all } i \in \mathbb{I}$$

be the canonical map of $\mathbb{N}_0^\theta$-graded Hopf algebras in ${}^H_H\mathcal{YD}$.

LEMMA 15.5.3. *Let* $i \in \mathbb{I}$ *and let* $t \geq 2$ *such that* $x_i^{t-1} \neq 0$ *and* $x_i^t = 0$ *in* S*. Then* $\mathrm{ord}(q_{ii}) = t$, $y_i^{t-1} \neq 0$ *and* $y_i^t = 0$ *in* $\mathcal{B}(M)$.

PROOF. Since S is finite-dimensional and $\mathbb{N}_0^\theta$-graded, there exists $t \geq 1$ such that $x_i^t = 0$. Then $y_i^t = 0$. By Corollary 7.1.15(2), $\mathcal{B}(\Bbbk y_i)$ is a subalgebra of $\mathcal{B}(M)$. Let $n = \mathrm{ord}(q_{ii})$. By Example 1.10.1, $n < \infty$ and $y_i^{n-1} \neq 0$, $y_i^n = 0$. Hence $x_i^{n-1} \neq 0$. It suffices to prove that $x_i^n = 0$.

Assume that $x_i^n \neq 0$. Proposition 2.4.2(5) implies that x_i^n is primitive in S. Further,

$$c_{S,S}(x_i^n \otimes x_i^n) = q_{ii}^{n^2} x_i^n \otimes x_i^n = x_i^n \otimes x_i^n.$$

Hence $1 \otimes x_i^n$ and $x_i^n \otimes 1$ commute in the algebra $S \otimes S$. Since $x_i^n \neq 0$ and $x_i^t = 0$, there exists $k \geq 2$ such that $x_i^{(k-1)n} \neq 0$, $x_i^{kn} = 0$. For this k we obtain that

$$0 = \Delta(x_i^{kn}) = \Delta(x_i^n)^k = (x_i^n \otimes 1 + 1 \otimes x_i^n)^k = \sum_{l=0}^{k} \binom{k}{l} x_i^{nl} \otimes x_i^{n(k-l)},$$

a contradiction since $\mathrm{char}(\Bbbk) = 0$ and S is graded. Thus $x_i^n = 0$. $\square$

Since S is finite-dimensional and $\mathbb{N}_0^\theta$-graded, for any $i, j \in \mathbb{I}$ with $i \neq j$ there exists $m \geq 1$ with $(\operatorname{ad} x_i)^{m+1}(x_j) = 0$. Then $(\operatorname{ad} y_i)^{m+1}(y_j) = 0$ and hence M is i-finite for all $i \in \mathbb{I}$.

LEMMA 15.5.4. *For any $i, j \in \mathbb{I}$ with $i \neq j$, $(\operatorname{ad} x_i)^{m+1}(x_j)$ is primitive in S for $m = -a_{ij}^M$.*

PROOF. Let $m = -a_{ij}^M$. It follows from Lemma 15.1.6 that one of the following conditions is satisfied.

(a) $m \geq 0$ and $q_{ij}q_{ji} = q_{ii}^{-m}$,
(b) $m \geq 1$ and $\mathrm{ord}(q_{ii}) = m + 1$.

We have shown in Proposition 4.3.12 that

$$\begin{aligned} \Delta((\operatorname{ad} x_i)^{m+1}(x_j)) = &\ (\operatorname{ad} x_i)^{m+1}(x_j) \otimes 1 + 1 \otimes (\operatorname{ad} x_i)^{m+1}(x_j) \\ &+ \sum_{k=1}^{m+1} \binom{m+1}{k}_{q_{ii}} \prod_{l=m+1-k}^{m} (1 - q_{ii}^l q_{ij} q_{ji}) x_i^k \otimes (\operatorname{ad} x_i)^{m+1-k}(x_j). \end{aligned}$$

Thus we have to prove that

$$\sum_{k=1}^{m+1} \binom{m+1}{k}_{q_{ii}} \prod_{l=m+1-k}^{m} (1 - q_{ii}^l q_{ij} q_{ji}) x_i^k \otimes (\operatorname{ad} x_i)^{m+1-k}(x_j) = 0. \tag{15.5.1}$$

This is clear in case (a). Assume (b). The summand with $k = m + 1$ in (15.5.1) vanishes by Lemma 15.5.3, since $\mathrm{ord}(q_{ii}) = m + 1 \geq 2$. The other summands in (15.5.1) are zero since $\binom{m+1}{k}_{q_{ii}} = 0$ for any $1 \leq k \leq m$ by Lemma 1.9.4. This proves the lemma. $\square$

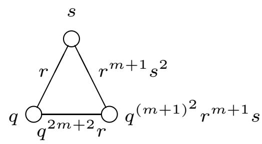
PROPOSITION 15.5.5. *Assume that $\theta \geq 2$. Let $i, j \in \mathbb{I}$ with $i \neq j$, and let $m = -a_{ij}^M$. Then $(\operatorname{ad} x_i)^m(x_j) \neq 0$ and $(\operatorname{ad} x_i)^{m+1}(x_j) = 0$ in S.*

PROOF. By definition of a_{ij}^M, $(\operatorname{ad} y_i)^m(y_j) \neq 0$ and $(\operatorname{ad} y_i)^{m+1}(y_j) = 0$ in $\mathcal{B}(M)$. Hence $(\operatorname{ad} x_i)^m(x_j) \neq 0$ in S.

Let $x = (\operatorname{ad} x_i)^{m+1}(x_j)$. Assume that $x \neq 0$. By Lemma 15.5.4, x is primitive, and $x_i^{m+1} \neq 0$, since $(\operatorname{ad} x_i)^{m+1}(x_j) = (\operatorname{ad} x_i^{m+1})(x_j)$. Since x_i is nilpotent, Lemma 15.5.3 implies that $\mathrm{ord}(q_{ii}) \geq m + 2$.

Let $\operatorname{gr} S$ denote the $\mathbb{N}_0$-graded braided Hopf algebra corresponding to the coradical filtration $S_0 = \Bbbk 1 \subseteq S_1 \subseteq \cdots$ of S. Then $S_1 = \Bbbk 1 \oplus S_1^+$. Note that $p(x) = 0$, since $p(x) = (\operatorname{ad} y_i)^{m+1}(y_j)$ is a primitive element of degree $m + 2$ in the Nichols algebra $\mathcal{B}(M)$. Moreover, $y_i = p(x_i)$, $y_j = p(x_j)$ are linearly independent in $\mathcal{B}(M)$. Hence the elements $x_i, x_j, (\operatorname{ad} x_i)^{m+1}(x_j)$ are linearly independent in $\operatorname{gr} S$. Let $\widehat{S}$ be the Hopf subalgebra of $\operatorname{gr} S$ generated by x_i, x_j, and x. Then $\widehat{S}$ is a pre-Nichols algebra of $\widehat{S}(1)$, and $\widehat{S}$ is finite-dimensional since $\dim \widehat{S} \leq \dim S$. Hence Theorem 7.1.7 implies that $\mathcal{B}(\widehat{S}(1))$ is finite-dimensional.

Let $q = q_{ii}$, $s = q_{jj}$, and $r = q_{ij}q_{ji}$. Lemma 15.1.5(3) implies that the Dynkin diagram of $\widehat{S}(1)$ with respect to the basis x_i, x_j, x is



Since $a_{ij}^M = -m$ and $\operatorname{ord}(q) \geq m+2$, by Lemma 15.1.6 it follows that $q^m r = 1$. Then $\mathcal{B}(\widehat{S}(1))$ is infinite-dimensional by Proposition 15.4.5, which is a contradiction. Hence $x = 0$. $\square$

COROLLARY 15.5.6. *Assume that* $\operatorname{char}(\Bbbk) = 0$. *Let* $M \in \mathcal{F}_\theta^H$ *and assume that* $\dim M_i = 1$ *for all* $i \in \mathbb{I}$. *Let* $\mathcal{N} = \mathcal{N}(S, N, f)$ *be a pre-Nichols system of* M *with finite-dimensional algebra* S. *Then the canonical map* $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ *is bijective.*

PROOF. For our proof we are going to apply Theorem 14.5.4. We prove first by induction on k the following claim:

Let $k \geq 0$ *and* $i_1, \dots, i_k \in \mathbb{I}$. *Then* $\mathcal{N}$ *admits the reflection sequence* $(i_1, \dots, i_k)$.

Since $\mathcal{B}(M)$ is finite-dimensional, M admits all reflections by Proposition 13.6.4. Thus it suffices to show that for any $P \in \mathcal{F}_\theta^H(M)$, any $i \in \mathbb{I}$, and any pre-Nichols system $\widetilde{\mathcal{N}} = \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f})$ of P with $\dim \widetilde{S} < \infty$, $\widetilde{\mathcal{N}}$ is a Nichols system of (P, i).

Let $P \in \mathcal{F}_\theta^H(M)$, $\widetilde{\mathcal{N}} = \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f})$ a pre-Nichols system of P with $\dim \widetilde{S} < \infty$, and $i \in \mathbb{I}$. Then $\dim P_l = \dim \widetilde{N}_l = 1$ for any $l \in \mathbb{I}$, since $\dim M_l = 1$ for any $l \in \mathbb{I}$. By Lemma 15.5.3, the canonical map $p^{\widetilde{\mathcal{N}}}$ induces an isomorphism

$$\Bbbk[\widetilde{N}_i] \xrightarrow{\cong} \mathcal{B}(P_i).$$

By Proposition 15.5.5, for any $j \in \mathbb{I}$ with $j \neq i$ and any $n \geq 0$,

$$(\operatorname{ad}_{\widetilde{S}} \widetilde{N}_i)^n(\widetilde{N}_j) \neq 0 \text{ if and only if } (\operatorname{ad}_{\mathcal{B}(P)} P_i)^n(P_j) \neq 0.$$

Thus $\widetilde{\mathcal{N}}$ is a Nichols system of (P, i).

Now the above claim implies that $\mathcal{N}$ admits all reflections. By Corollary 14.5.3, $\mathcal{G}(M)$ is finite. Then Theorem 14.5.4 says that the canonical map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ is bijective. $\square$

Finally, we prove Theorem 15.5.1.

PROOF. In Proposition 5.2.21 and Lemma 13.5.8, starting with R we constructed a pre-Nichols system $\mathcal{N} = \mathcal{N}(\operatorname{gr} R, N, f)$ of M. Thus $\operatorname{gr} R$ is finite-dimensional since R is. The canonical map $p^{\mathcal{N}} : \operatorname{gr} R \to \mathcal{B}(M)$ is bijective by Corollary 15.5.6. Therefore the canonical map $R \to \mathcal{B}(M)$ is bijective by Lemma 13.5.8. $\square$

COROLLARY 15.5.7. *Assume that* $\Bbbk$ *is algebraically closed, and* $\operatorname{char}(\Bbbk) = 0$. *Let A be a finite-dimensional pointed Hopf algebra with abelian coradical. Then A is generated as an algebra by group-like and skew-primitive elements.*

Proof. Let $(A_n)_{n\geq 0}$ be the coradical filtration of A. Then $A_0 = \Bbbk G$ is the group algebra of the group $G = G(A)$, and G is abelian by assumption. Let $R = (\operatorname{gr} A)^{\operatorname{co} A_0}$ be the $\mathbb{N}_0$-graded strictly graded Hopf algebra in ${}^G_G\mathcal{YD}$ of Corollary 5.3.16. By Theorem 5.4.7, A is generated by group-like and skew-primitive elements if and only if A is generated by A_1. Hence by Corollary 5.3.16, it remains to be shown that R is generated by $R(1)$.

Let $S = R^{*\mathrm{gr}} \cong R^*$ be the $\mathbb{N}_0$-graded Hopf algebra defined by the braided duality in Corollary 4.2.9. By Corollary 4.2.10, S is generated by $S(1)$ since R is strictly graded. By the assumptions on $\Bbbk$ and G, $S(1)$ is a direct sum of one-dimensional Yetter-Drinfeld modules over $\Bbbk G$. Thus S is a finite-dimensional pre-Nichols algebra, hence a Nichols algebra by Theorem 15.5.1. Then R is generated by $R(1)$ by Corollary 4.2.10. □

Corollary 5.4.9 of the weak Theorem of Taft-Wilson allows to describe the skew-primitive generators of the previous corollary more precisely.

Corollary 15.5.8. *Assume that $\Bbbk$ is algebraically closed, and* $\operatorname{char}(\Bbbk) = 0$. *Let A be a finite-dimensional pointed Hopf algebra with abelian group $G = G(A)$, and coradical filtration $(A_n)_{n\geq 0}$. Let $R = A^{\operatorname{co}\Bbbk G}$, and $V = R(1) \in {}^G_G\mathcal{YD}$. Choose a decomposition of the Yetter-Drinfeld module $V \in {}^G_G\mathcal{YD}$,*

$$V = \bigoplus_{i=1}^{\theta} \Bbbk x_i, \quad 0 \neq x_i \in V_{g_i}^{\chi_i},\, g_i \in G,\, \chi_i \in \widehat{G} \text{ for all } 1 \leq i \leq \theta,$$

and preimages a_i of x_i, $1 \leq i \leq \theta$, under the canonical map $A_1 \to A_1/A_0$.

Then A is generated as an algebra by $\{a_1, \ldots, a_\theta\} \cup G$, the elements $1, a_1, \ldots, a_\theta$ are a basis of A_1 as a right $\Bbbk G$-module by restriction and

$$\Delta(a_i) = g_i \otimes a_i + a_i \otimes 1, \quad g a_i g^{-1} = \chi_i(g) a_i, \quad 1 \leq i \leq \theta,\, g \in G.$$

Proof. The multiplication map $V \# \Bbbk G \to A_1/A_0$ is an isomorphism, and for all $g \in G$, $x_i g \in P_{g_i g, g}^{\chi_i}(\operatorname{gr} A)$. Hence $A_1/A_0 = \bigoplus_{1\leq i\leq\theta, g\in G} \Bbbk x_i g$, and for all i, g, $\Bbbk x_i g \subseteq P_{g_i g, g}^{\chi_i}(\operatorname{gr} A)$. Note that possibly there are indices $i \neq j$ with $g_i = g_j$, $\chi_i = \chi_j$. The corollary follows from Corollary 5.4.9 and Corollary 15.5.7. □

Sometimes the information in the last corollary about the generators of A is sufficient to find defining relations for A. A very easy example is the following.

Proposition 15.5.9. *Assume that $\Bbbk$ is algebraically closed, and* $\operatorname{char}(\Bbbk) = 0$. *Let A be a finite-dimensional pointed Hopf algebra with group $G(A) = G = \{1, g\}$ of order two. Let χ be the non-trivial character of G with $\chi(g) = -1$. Then*

$$A \cong \mathcal{B}(V) \# \Bbbk G,\ \dim A = 2^{n+1},$$

where $V = V_g^\chi \in {}^G_G\mathcal{YD}$, $\dim V = n$, and $\mathcal{B}(V) \cong \Lambda(V)$.

Proof. By Example 1.10.15, $V = V_g^\chi \in {}^G_G\mathcal{YD}$, and $\mathcal{B}(V) \cong \Lambda(V)$. Let $x_1, \ldots, x_n$ be a basis of V, and choose elements $a_i \in P_{g,1}^\chi(A)$ as in Corollary 15.5.8. Then for all i,

$$\Delta_A(a_i^2) = (g \otimes a_i + a_i \otimes 1)^2 = 1 \otimes a_i^2 + a_i^2 \otimes 1.$$

Assume that $a_i^2 \neq 0$ for some i. Then it follows from the binomial formula that the elements $(a_i^{2n})_{n\geq 0}$ are linearly independent. This contradicts our assumption on the dimension of A. Hence for all i, $a_i^2 = 0$, and for all $i \neq j$, $(a_i + a_j)^2 = 0$, and $a_i a_j + a_j a_i = 0$. By Example 1.10.15,

$$\Phi : \mathcal{B}(V)\#\Bbbk G \to A, \quad x_i \mapsto a_i,\, 1 \leq i \leq n,\, g \mapsto g,$$

is a well-defined Hopf algebra map. By Corollary 15.5.8, Φ is surjective. The first term of the coradical filtration of $\mathcal{B}(V)\#\Bbbk G$ is $\Bbbk G \oplus (V\#\Bbbk G)$, since $\mathcal{B}(V)\#\Bbbk G$ is coradically graded by Proposition 5.3.18. Hence Φ is injective by Theorem 5.4.5. □

15.6. Notes

15.1. Lemma 15.1.1(1) and (3) is [**Ros98**, Lemma 14].

Theorem 15.1.14 describes the basic properties of diagonal braidings of Cartan type. In [**AS00a**, Theorem 1.1], it was shown (under some restrictions for small primes) that a finite-dimensional Nichols algebra of Cartan type must be of finite Cartan type. The first success of the idea of the root system of a Nichols algebra, where the roots were defined as the degrees of Kharchenko's PBW-basis of a Nichols algebra of diagonal type, was [**Hec06**, Theorem 1], which says that these restrictions can be removed.

Corollaries 15.1.15 and 15.1.16 are taken from [**Gn00b**]. Corollary 15.1.17 was proven originally in [**AZ07**].

15.2. The definition and the theory of root vector sequences is new. Note that for the definition of a root vector sequence the maps T_i from Theorem 12.3.3 and Corollary 13.5.21 are not needed.

15.3. Theorem 15.3.1 was proved first in [**Hec08**]. That proof also used Kharchenko's theory of Lyndon words. The proof in the book is based on [**HW15**].

The classification of finite-dimensional rank two Nichols algebras of diagonal type in Theorem 15.3.3 was obtained first in [**Hec07**] and in the unpublished paper [**Hec04**] based on Kharchenko's theory. A closer look at the dimensions of the obtained Nichols algebras resulted in the observation that there should exist an equivalence relation preserving the dimension but not necessarily the Hilbert series of the Nichols algebras. This lead to the discovery of the Weyl groupoid in [**Hec06**] and the explicit description of the equivalence relation in [**Hec05**] as well as to a new classification in [**Hec08**].

15.5. Corollary 15.5.7 was shown in [**AS10**, Theorem 5.5], under additional assumptions on the braiding.

An equivalent version of Theorem 15.5.1 was proven in [**Ang13**]. Our proof uses Theorem 14.5.4. Thus we have to show in Corollary 15.5.6 that certain pre-Nichols systems are Nichols systems. This follows mainly from the equality $(\mathrm{ad}^S x_i)^{m+1}(x_j) = 0$ in Proposition 15.5.5. This equality is the first Proposition in Angiono's proof, [**Ang13**, Proposition 4.1]; it was shown by similar methods in [**AS10**, Lemma 5.4], under additional assumptions on the braiding. In the remaining part of his proof Angiono needs his description of Nichols algebras by generators and relations in [**Ang13**, Theorem 3.1].

Proposition 15.5.9 is a very early classification result in [**Nic78**, Theorem 4.2.1]. A rather large class of finite-dimensional pointed Hopf algebras A was classified in [**AS10**] starting from the lifted generators a_i of the basis elements x_i of the braided vector space $R(1)$ of diagonal type in Corollary 15.5.8.

CHAPTER 16

Nichols algebras of Cartan type

Let G be an abelian group, $K_1, \dots, K_\theta \in G$, and $\chi_1, \dots, \chi_\theta \in \widehat{G}$, and for all $1 \le i \le \theta$ let $M_i \in {}^G_G\mathcal{YD}$ be one-dimensional with basis $E_i \in (M_i)^{\chi_i}_{K_i}$. Assume that the braiding matrix $\boldsymbol{q} = (q_{ij})_{1 \le i,j \le \theta}$, $q_{ij} = \chi_j(K_i)$ for all i, j, is of finite Cartan type. We are going to give presentations of the Nichols algebra of the tuple $M = (M_1, \dots, M_\theta)$ by generators and relations, and determine PBW bases attached to reduced decompositions of the longest element of the Weyl group of the Cartan matrix. In particular, our results apply to the positive parts U_q^+ and u_q^+ of quantum groups in the generic case and of small quantum groups. In Section 16.2 we assume that the braiding matrix is quasi-generic. In Section 16.3 we consider the case when all q_{ii} are roots of 1. In this case a technical assumption is added in order to ensure that all defining relations are quantum Serre or root vector relations. To be able to apply reflection theory, we develop first a theory of Yetter-Drinfeld modules over bosonizations of Nichols algebras of one-dimensional Yetter-Drinfeld modules, which in fact is a variation of the well-studied representation theory of $U_q(\mathfrak{sl}_2)$. In the last two sections of the Chapter we characterize Nichols algebras of diagonal type which are domains of finite Gelfand-Kirillov dimension, and pointed Hopf algebras of finite Gelfand-Kirillov dimension with abelian coradical and generic braiding.

16.1. Yetter-Drinfeld modules over a Hopf algebra of polynomials

Let G be an abelian group, let $\chi \in \widehat{G}$ be a character of G and let $g \in G$. We write $\Bbbk[x; \chi, g]$ for the Hopf algebra $\Bbbk[x]\#\Bbbk G$, where $\Bbbk x$ is a one-dimensional Yetter-Drinfeld module over G such that

$$h \cdot x = \chi(h)x, \quad \delta(x) = g \otimes x$$

for all $h \in G$. By Example 2.6.13, the elements $x^k h$ with $h \in G$, $k \in \mathbb{N}_0$, form a $\Bbbk$-basis of $\Bbbk[x; \chi, g]$.

Lemma 16.1.1. *For all $k \in \mathbb{N}_0$ let $\Bbbk[x; \chi, g](k)$ be the $\Bbbk$-span of all $x^k h$ with $h \in G$. Then*

$$\Bbbk[x; \chi, g] = \bigoplus_{k \in \mathbb{N}_0} \Bbbk[x; \chi, g](k)$$

is an $\mathbb{N}_0$-graded Hopf algebra with coradical $\Bbbk[x; \chi, g](0) = \Bbbk 1 \# \Bbbk G$. In particular, $\Bbbk[x; \chi, g]$ is pointed and has a bijective antipode.

Proof. Clearly, $\Bbbk[x; \chi, g]$ is an $\mathbb{N}_0$-graded bialgebra, where x has degree 1 and the elements of G have degree 0. In particular, the vector space filtration

$\mathcal{F}(\Bbbk[x;\chi,g]) = (F_k(\Bbbk[x;\chi,g]))_{k\geq 0}$, where

$$F_k(\Bbbk[x;\chi,g]) = \bigoplus_{i=0}^{k} \Bbbk[x;\chi,g](i)$$

for all $k \in \mathbb{N}_0$, is a coalgebra filtration of $\Bbbk[x;\chi,g]$. Therefore $\Bbbk[x;\chi,g]$ is pointed with coradical $\Bbbk 1\#\Bbbk G$ by Proposition 5.4.2(1). Then $\Bbbk[x;\chi,g]$ is a Hopf algebra with bijective antipode by Corollary 5.2.11(2). $\square$

PROPOSITION 16.1.2. *Assume that* $\mathrm{char}(\Bbbk) = 0$.

(1) *Assume that* $\chi(g) = 1$ *or* $\chi(g)$ *is not a root of* 1. *Then* $(\Bbbk[x;\chi,g]_j)_{j\geq 0}$, *where* $\Bbbk[x;\chi,g]_j = \bigoplus_{i=0}^{j} \Bbbk[x;\chi,g](i)$ *for all* $j \geq 0$, *is the coradical filtration of* $\Bbbk[x;\chi,g]$.

(2) *Let* $n > 1$ *and assume that* $\chi(g)$ *is a primitive* n*-th root of* 1. *Then* $(\Bbbk[x;\chi,g]'_j)_{j\geq 0}$, *where*

$$\Bbbk[x;\chi,g]'_j = \sum_{i=0}^{j}\sum_{k=0}^{j-i} \Bbbk[x;\chi,g](i+nk)$$

for all $j \geq 0$, *is the coradical filtration of* $\Bbbk[x;\chi,g]$.

PROOF. Let $j \in \mathbb{N}_0$. Since $\Delta(x^j) = \Delta(x)^j$ and since $\Delta(x) = x\otimes 1 + g\otimes x$ and $(g\otimes x)(x\otimes 1) = \chi(g)(x\otimes 1)(g\otimes x)$, Proposition 1.9.5 implies that

$$\Delta(x^j) = \sum_{i=0}^{j} \binom{j}{i}_{\chi(g)} x^{j-i}g^i \otimes x^i. \tag{16.1.1}$$

(1) For any $j \geq 2$, the map

$$\Delta_{1,j-1} : \Bbbk[x;\chi,g](j) \to \Bbbk[x;\chi,g](1)\otimes \Bbbk[x;\chi,g](j-1)$$

is injective if and only if $(j)_{\chi(g)} \neq 0$. Therefore, if $\chi(g) = 1$ or $\chi(g)$ is not a root of 1, then $\Bbbk[x;\chi,g]$ is coradically graded by Proposition 5.3.13.

(2) For any $j \in \mathbb{N}_0$ let $X'(j) = \bigoplus_{k=0}^{\min\{j,n-1\}} \Bbbk[x;\chi,g](k+n(j-k))$. Since $\chi(g)$ is a primitive n-th root of 1, $x^n \in P_{g^n,1}(\Bbbk[x;\chi,g])$ by Proposition 2.4.2(5). Moreover, $g^n\otimes x^n$ and $x^n \otimes 1$ commute in $\Bbbk[x;\chi,g]\otimes\Bbbk[x;\chi,g]$. Therefore

$$\begin{aligned}\Delta(x^{k+n(j-k)}) &= \Delta(x)^k\Delta(x^n)^{j-k}\\ &= \sum_{i=0}^{k}\binom{k}{i}_{\chi(g)} x^{k-i}g^i\otimes x^i \cdot \sum_{m=0}^{j-k}\binom{j-k}{m} x^{(j-k-m)n}g^{mn}\otimes x^{mn}\\ &= \sum_{i=0}^{k}\sum_{m=0}^{j-k}\binom{k}{i}_{\chi(g)}\binom{j-k}{m} x^{(j-k-m)n+k-i}g^{mn+i}\otimes x^{mn+i}\end{aligned}$$

for any $0 \leq k \leq \min\{j, n-1\}$ and any $j \in \mathbb{N}_0$. We conclude that

$$\Delta(X'(j)) \subseteq \bigoplus_{i=0}^{j} X'(j-i)\otimes X'(i),$$

and hence $\Bbbk[x;\chi,g] = \bigoplus_{j=0}^{\infty} X'(j)$ is an $\mathbb{N}_0$-graded coalgebra. Since

$$\begin{aligned}\Delta_{1,j-1}(x^{k+n(j-k)}) =& (k)_{\chi(g)} x g^{(j-k)n+k-1}\otimes x^{(j-k)n+k-1}\\ &+ (j-k)x^n g^{(j-k-1)n+k}\otimes x^{(j-k-1)n+k}\end{aligned}$$

for any $j \geq 2$, $1 \leq k \leq \min\{n-1, j\}$, Proposition 5.3.13 implies that $\Bbbk[x;\chi,g]$ is a coradically graded coalgebra. Then the claim in (2) follows from the equation $\Bbbk[x;\chi,g]'_j = \bigoplus_{m=0}^{j} X'(m)$. $\square$

In the remaining part of this section we study Yetter-Drinfeld modules over $\Bbbk[x;\chi,g]$. We are particularly interested in weight modules.

DEFINITION 16.1.3. A Yetter-Drinfeld module $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ is said to be a **weight module** if the action of g on V is diagonalizable.

EXAMPLE 16.1.4. Let V be a $\Bbbk G$-module. Then V becomes a $\Bbbk[x;\chi,g]$-module via $xv = 0$ for all $v \in V$. Define a trivial $\Bbbk[x;\chi,g]$-comodule structure on V via $\delta_V(v) = 1 \otimes v$ for all $v \in V$. If these module and comodule structures define a Yetter-Drinfeld module structure, then

$$\begin{aligned}0 =&\delta(xv)\\ =&x_{(1)}\mathcal{S}(x_{(3)}) \otimes x_{(2)}v\\ =&x \otimes v + g \otimes xv + g(-g^{-1}x) \otimes gv\\ =&x \otimes (v - gv)\end{aligned}$$

for all $v \in V$, and hence $gv = v$ for all $v \in V$. In particular, V is a weight module. Conversely, if $gv = v$ and $xv = 0$ for all $v \in V$, then δ_V as above defines a Yetter-Drinfeld module structure on V over $\Bbbk[x;\chi,g]$.

Let $\pi : \Bbbk[x;\chi,g] \to \Bbbk G = \Bbbk 1 \# \Bbbk G$ be the homogeneous projection.

LEMMA 16.1.5. *Let $V \in {}^{\Bbbk[x;\chi,g]}\mathcal{M}$ and let $v \in V$ and $h \in G$. Assume that $(\pi \otimes \mathrm{id})\delta_V(v) = h \otimes v$. Then*

$$\delta_V(v) = h \otimes v + \sum_{n>0} x^n g^{-n} h \otimes v_n$$

for some $v_n \in V$, $n > 0$, where $v_n = 0$ for all but finitely many n.

PROOF. Since $V \in {}^{\Bbbk[x;\chi,g]}\mathcal{M}$ and $v \in V$, for any $n \in \mathbb{N}_0$ and $f \in G$ there exists $v_{n,f} \in V$ such that $\delta_V(v) = \sum_{n,f} x^n f \otimes v_{n,f}$. Since $(\varepsilon \otimes \mathrm{id})\delta_V(v) = v$, we conclude that $v = \sum_{f \in G} v_{0,f}$. Moreover,

$$(\pi \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\delta_V(v) = \sum_{n,f} g^n f \otimes x^n f \otimes v_{n,f} \tag{16.1.2}$$

since $(\pi \otimes \mathrm{id})\Delta$ is an algebra map and since

$$(\pi \otimes \mathrm{id})\Delta(x) = g \otimes x, \quad (\pi \otimes \mathrm{id})\Delta(f) = f \otimes f$$

for all $f \in G$. On the other hand, the expression in (16.1.2) is equal to

$$(\pi \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \delta_V)\delta_V(v) = (\mathrm{id} \otimes \delta_V)(\pi \otimes \mathrm{id})\delta_V(v) = h \otimes \sum_{n,f} x^n f \otimes v_{n,f}.$$

In particular, $v_{n,f} = 0$ whenever $g^n f \neq h$. This implies the claim. $\square$

PROPOSITION 16.1.6. *Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$. For any $h \in G$ let*

$$V_h = \{v \in V \mid (\pi \otimes \mathrm{id})\delta_V(v) = h \otimes v\}.$$

Then $V = \bigoplus_{h\in G} V_h$, $GV_f = V_f$, $xV_f \subseteq V_{gf}$, *and*

$$\delta_V(v) \in \sum_{n=0}^{\infty} \Bbbk[x;\chi,g](n) \otimes V_{fg^{-n}}$$

for any $f \in G$, $v \in V_f$.

Proof. For any $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$, the map $\delta'_V = (\pi \otimes \mathrm{id})\delta_V : V \to \Bbbk G \otimes V$ defines a left $\Bbbk G$-comodule structure on V. By Proposition 1.1.17 we obtain that $V = \bigoplus_{h\in G} V_h$. The Yetter-Drinfeld condition implies that

$$\delta_V(hv) = hv_{(-1)}h^{-1} \otimes hv_{(0)}, \tag{16.1.3}$$

$$\delta_V(xv) = xv_{(-1)} \otimes v_{(0)} + gv_{(-1)} \otimes xv - gv_{(-1)}g^{-1}x \otimes gv \tag{16.1.4}$$

for any $v \in V$, $h \in G$. Therefore $\delta'_V(hv) = f \otimes hv$ and $\delta'_V(xv) = gf \otimes xv$ for any $f, h \in G$, $v \in V_f$, that is,

$$GV_f = V_f, \quad xV_f \subseteq V_{gf}.$$

It remains to prove the formula on $\delta_V(v)$, $v \in V_f$, $f \in G$.

Let $f \in G$ and $v \in V_f$. By Lemma 16.1.5 there exist $v_n \in V$, $n \geq 0$, such that

$$\delta_V(v) = \sum_{n\geq 0} x^n fg^{-n} \otimes v_n,$$

where $v_0 = v$. Then

$$\begin{aligned}
\sum_{n\geq 0} x^n fg^{-n} \otimes \delta'_V(v_n) &= (\mathrm{id} \otimes \pi \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\delta_V(v) \\
&= \sum_{n\geq 0} (\mathrm{id} \otimes \pi)\Delta(x^n fg^{-n}) \otimes v_n \\
&= \sum_{n\geq 0} x^n fg^{-n} \otimes fg^{-n} \otimes v_n
\end{aligned}$$

because of $(\mathrm{id}\otimes\pi)\Delta(x) = x \otimes 1$ and $(\mathrm{id}\otimes\pi)\Delta(h) = h \otimes h$ for any $h \in G$. Therefore $v_n \in V_{fg^{-n}}$ for any $n \geq 0$. □

Remark 16.1.7. Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ be a weight module and let $h \in G$. Since $gV_h = V_h$, the restriction of the action of g to V_h is diagonalizable.

Definition 16.1.8. Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ be a weight module. For any $h \in G$, $\lambda \in \Bbbk^\times$ let

$$V_{h;\lambda} = \{v \in V_h \mid gv = \chi(h)^{-1}\lambda v\}.$$

The scalars λ with $V_{h;\lambda} \neq 0$ for some $h \in G$ are called the **weights of** V. For any weight λ, the sum $\bigoplus_{h\in G} V_{h;\lambda}$ is called the **weight space** of λ and the elements of such a weight space are called **weight vectors**.

Lemma 16.1.9. *Let* $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$, $v \in V$, $n \in \mathbb{N}_0$ *and* $h \in G$. *Assume that* $\delta_V(v) = h \otimes v$. *Then*

$$\delta_V(x^n v) = \sum_{i=0}^{n} \binom{n}{i}_{\chi(g)} x^{n-i}g^i h \otimes x^i \prod_{k=i}^{n-1}(1 - \chi(h)\chi(g)^k g)v.$$

PROOF. Let $q = \chi(g)$. For $n = 0$ the claim holds by assumption. For $n > 0$ it follows by induction using (16.1.4):

$$\begin{aligned}\delta_V(x^n v) =& \delta_V(x(x^{n-1}v)) \\ =& \sum_{i=0}^{n-1} \binom{n-1}{i}_q x x^{n-1-i} g^i h \otimes x^i \prod_{k=i}^{n-2} (1 - \chi(h) q^k g) v \\ &+ \sum_{i=0}^{n-1} \binom{n-1}{i}_q g x^{n-1-i} g^i h \otimes x x^i \prod_{k=i}^{n-2} (1 - \chi(h) q^k g) v \\ &- \sum_{i=0}^{n-1} \binom{n-1}{i}_q g x^{n-1-i} g^i h g^{-1} x \otimes g x^i \prod_{k=i}^{n-2} (1 - \chi(h) q^k g) v.\end{aligned}$$

By using the commutation rules in $\Bbbk[x;\chi,g]$ this yields that

$$\begin{aligned}\delta_V(x^n v) =& \sum_{i=0}^{n-1} \binom{n-1}{i}_q x^{n-i} g^i h \otimes x^i \prod_{k=i}^{n-2} (1 - \chi(h) q^k g) v \\ &+ \sum_{i=1}^{n} q^{n-i} \binom{n-1}{i-1}_q x^{n-i} g^i h \otimes x^i \prod_{k=i-1}^{n-2} (1 - \chi(h) q^k g) v \\ &- \sum_{i=0}^{n-1} q^{n-1+i} \chi(h) \binom{n-1}{i}_q x^{n-i} g^i h \otimes x^i g \prod_{k=i}^{n-2} (1 - \chi(h) q^k g) v.\end{aligned}$$

Now use that

$$\prod_{k=i-1}^{n-2} (1 - \chi(h) q^k g) v = (1 - \chi(h) q^{i-1} g) \prod_{k=i}^{n-2} (1 - \chi(h) q^k g) v$$

and the formulas in Lemma 1.9.3 on the q-binomial numbers. □

LEMMA 16.1.10. *Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ and let $h \in G$, $v \in V_h$ and $\lambda \in \Bbbk^\times$. Assume that v has weight λ. Then $x^n v$ has weight $\chi(g)^{2n}\lambda$ for any $n \in \mathbb{N}_0$.*

PROOF. Let $n \in \mathbb{N}_0$. Then $x^n v \in V_{hg^n}$ by Proposition 16.1.6. Moreover,

$$g x^n v = \chi(g)^n x^n g v = \chi(g)^n x^n \chi(h)^{-1} \lambda v = \chi(hg^n)^{-1} \chi(g)^{2n} \lambda x^n v.$$

Thus the weight of $x^n v$ is $\chi(g)^{2n}\lambda$. □

LEMMA 16.1.11. *Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ and let $h \in G$, $v \in V$, $\lambda \in \Bbbk^\times$ and $r \in \mathbb{N}$. Assume that $(g - \chi(h)^{-1}\lambda)^r v = 0$. For any $n \in \mathbb{N}_0$ let $v_n \in V$ such that $\delta_V(v) = \sum_{n=0}^\infty x^n h g^{-n} \otimes v_n$. Then $v_n \in V_{hg^{-n}}$ and*

$$(g - \chi(h)^{-1} \chi(g)^{-n} \lambda)^r v_n = 0$$

for any $n \in \mathbb{N}_0$.

PROOF. The existence of v_n for $n \in \mathbb{N}_0$ follows from Lemma 16.1.5. Moreover, Proposition 16.1.6 implies that $v_n \in V_{hg^{-n}}$ for any $n \in \mathbb{N}_0$. By induction on s we obtain from the Yetter-Drinfeld condition on V that

$$\delta_V((g - \chi(h)^{-1}\lambda)^s v) = \sum_{n=0}^{\infty} x^n h g^{-n} \otimes (\chi(g)^n g - \chi(h)^{-1}\lambda)^s v_n$$

for any $s \in \mathbb{N}_0$. Since $(g - \chi(h)^{-1}\lambda)^r v = 0$, by comparision of the terms for each $n \in \mathbb{N}_0$ on the right hand side with 0 we obtain the claim. $\square$

Recall from Example 1.4.2 that any simple Yetter-Drinfeld module U over $\Bbbk G$ is a simple $\Bbbk G$-module and there exists a unique $h \in G$ such that $\delta_U(u) = h \otimes u$ for any $u \in U$. Conversely, for any simple $\Bbbk G$-module U and any $h \in G$, the left coaction $\delta_U : U \to \Bbbk G \otimes U$, $u \mapsto h \otimes u$, turns U into a simple Yetter-Drinfeld module over $\Bbbk G$.

In Proposition 4.5.1 we discussed induced Yetter-Drinfeld modules in general. Now we look at a special case of this construction.

LEMMA 16.1.12. *Assume that $g \in G$ has infinite order. Let $U \in {}^{G}_{G}\mathcal{YD}$ be a simple object and let $W = \Bbbk[x;\chi,g] \otimes_{\Bbbk G} U \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$. Let $h \in G$ such that $\delta_U(u) = h \otimes u$ for all $u \in U$.*

(1) *Let $X \subseteq W$ in ${}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ with $X \neq 0$. Then $X = \Bbbk[x]x^n \otimes U$ for some $n \in \mathbb{N}_0$.*
(2) *If W is not simple then there exists $n \in \mathbb{N}_0$ such that $1 \otimes u \in W$ has weight $\chi(g)^{-n}$ for any $u \in U$.*

PROOF. By Proposition 4.5.1 we know that $W \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$. Since $1 \otimes U \subseteq W_h$ and since $g \in G$ has infinite order, Proposition 16.1.6 tells that $W = \bigoplus_{n \in \mathbb{N}_0} W_{g^n h}$, where $W_{g^n h} = \Bbbk x^n \otimes U$ for any $n \in \mathbb{N}_0$.

Let now $X \subseteq W$ in ${}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ with $X \neq 0$. Then there exist $u \in U \setminus \{0\}$ and a smallest $n \in \mathbb{N}_0$ such that $x^n \otimes u \in X$. Lemma 16.1.9 and the minimality of n imply that $\delta_W(x^n \otimes u) = g^n h \otimes (x^n \otimes u)$. In particular, the summand of $\delta_W(x^n \otimes u)$ in Lemma 16.1.9 for $i = 0$ vanishes. Hence

$$x^n h \otimes \prod_{k=0}^{n-1} (1 - \chi(h)\chi(g)^k g)u = 0.$$

Therefore there exist $u' \in U \setminus \{0\}$ and an integer $k \in \{0, 1, \ldots, n-1\}$ such that $(1 - \chi(h)\chi(g)^k g)u' = 0$. Since $U = \Bbbk Gu = \Bbbk Gu'$ and G is abelian, we conclude that $gv = \chi(h)^{-1}\chi(g)^{-k}v$ and that $x^n \otimes v \in X$ for any $v \in U$. This implies both (1) and (2). $\square$

PROPOSITION 16.1.13. *Assume that $g \in G$ has infinite order. Let $U \in {}^{G}_{G}\mathcal{YD}$, $h \in G$, $n \in \mathbb{N}_0$, and let $W = \Bbbk[x;\chi,g] \otimes_{\Bbbk G} U \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$. Assume that U is simple, $\delta_U(u) = h \otimes u$, $gu = \chi(h)^{-1}\chi(g)^{-n}u$ for any $u \in U$, and that $\chi(g)^k \neq \chi(g)^n$ for any $0 \leq k < n$.*

(1) *W is a weight module with weights $\chi(g)^{2m-n}$, $m \geq 0$.*
(2) *$\Bbbk[x]x^{n+1} \otimes U$ is the only maximal Yetter-Drinfeld submodule of W.*

PROOF. By Proposition 4.5.1, $W \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$. Since $W = \Bbbk[x] \otimes U$ and $1 \otimes u$ has weight $\chi(g)^{-n}$ for any $u \in U$, (1) follows from Lemma 16.1.10.

(2) By assumption, $(1 - \chi(h)\chi(g)^n g)u = 0$ for any $u \in U$. Thus Lemma 16.1.9 implies that $\delta_W(x^{n+1} \otimes u) = g^{n+1}h \otimes (x^{n+1} \otimes u)$ for any $u \in U$. Using again Lemma 16.1.9 with $v = x^{n+1} \otimes u$ we conclude that $\Bbbk[x]x^{n+1} \otimes U$ is a Yetter-Drinfeld submodule of W.

Let X be a non-zero Yetter-Drinfeld submodule of W with $X \neq W$. By Lemma 16.1.12(1), there exists an $m \in \mathbb{N}_0$ such that $X = \Bbbk[x]x^m \otimes U$. Moreover, $m > 0$ since $X \neq W$. By the previous paragraph, it suffices to prove that $m \geq n+1$.

Assume that $0 < m \leq n$. Let $u \in U \setminus \{0\}$. Then $x^m \otimes u \in X$ and hence $\delta_W(x^m \otimes u) \in \Bbbk[x;\chi,g] \otimes X$. Therefore the summand of this expression for $i = 0$ in Lemma 16.1.9 vanishes, that is,

$$0 = x^m h \otimes \prod_{k=0}^{m-1} (1 - \chi(h)\chi(g)^k g)u = \prod_{k=0}^{m-1} (1 - \chi(g)^{k-n}) x^m h \otimes u.$$

Since $\chi(g)^k \neq \chi(g)^n$ for any $0 \leq k \leq n-1$, we obtain a contradiction. This proves (2). □

Corollary 16.1.14. *Assume that $g \in G$ has infinite order. Let $U \in {}^G_G\mathcal{YD}$, $h \in G$, $n \in \mathbb{N}_0$, and let $W = \Bbbk[x;\chi,g] \otimes_{\Bbbk G} U \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$. Assume that U is simple, $\delta_U(u) = h \otimes u$, $gu = \chi(h)^{-1}\chi(g)^{-n}u$ for any $u \in U$, and that $\chi(g)^k \neq \chi(g)^n$ for any $0 \leq k < n$.*

(1) *Assume that $\chi(g)$ is not a root of 1. Then $x^{n+1}W$ is the unique non-trivial Yetter-Drinfeld submodule of W.*
(2) *Assume that $\chi(g)$ is a primitive root of 1 of order $p \geq 1$. Then the non-zero Yetter-Drinfeld submodules of W are $x^{n+1+mp}W$ and $x^{mp}W$ with $m \in \mathbb{N}_0$.*

Proof. By Proposition 16.1.13, $x^{n+1}W = \Bbbk[x;\chi,g]x^{n+1} \otimes U$ is the unique maximal Yetter-Drinfeld submodule of W and $\Bbbk x^{n+1} \otimes U$ is a subspace of weight $\chi(g)^{2n+2}$. Moreover, $\Bbbk x^{n+1} \otimes U$ is a simple $\Bbbk G$-module. Lemma 16.1.9 implies that

$$\delta_W(x^{n+1} \otimes u) = hg^{n+1} \otimes (x^{n+1} \otimes u)$$

for any $u \in U$. We conclude that

$$x^{n+1}W \simeq \Bbbk[x;\chi,g] \otimes_{\Bbbk G} (\Bbbk x^{n+1} \otimes U).$$

(1) By assumption, the weight $\chi(g)^{2n+2}$ of $\Bbbk x^{n+1} \otimes U$ differs from $\chi(g)^{-l}$ for any $l \in \mathbb{N}_0$. Hence the Yetter-Drinfeld module $x^{n+1}W$ is simple by Lemma 16.1.12(2). Thus the claim follows from Proposition 16.1.13(2).

(2) By assumption, $n < p$. Assume that $n = p-1$. Then

$$\chi(g)^{-n+(2n+2)} = \chi(g)^{-n}.$$

It follows from Proposition 16.1.13(2) by induction on m that $x^{(m+1)p}W$ is the unique maximal Yetter-Drinfeld submodule of $x^{mp}W$ for any $m \in \mathbb{N}_0$. This proves the claim in this case.

Assume that $0 \leq n < p-1$. Then $\Bbbk x^{n+1} \otimes U$ has weight $\chi(g)^{-(p-2-n)}$. By induction on m it follows that $x^{n+1+mp}W$ is the unique maximal Yetter-Drinfeld submodule of $x^{mp}W$ and that $x^{(m+1)p}W$ is the unique maximal Yetter-Drinfeld submodule of $x^{n+1+mp}W$ for any $m \in \mathbb{N}_0$. □

In the next Proposition, for any $\Bbbk G$-module U, for any $n, l \in \mathbb{N}_0$ with $0 \leq n \leq l$, and for any $u \in U$ we write u_n for the element in U^{l+1} which has u in the $n+1$-st entry and 0 elsewhere. Then $(u', u'', \dots, u''') \in U^{l+1}$ is nothing but $u'_0 + u''_1 + \cdots + u'''_l$. We use the convention $u_{l+1} = 0$ for any $u \in U$.

PROPOSITION 16.1.15. *Assume that $g \in G$ has infinite order. Let U be a simple $\Bbbk G$-module, $h \in G$, and $l \in \mathbb{N}_0$ such that $gu = \chi(h)^{-1}\chi(g)^{-l}u$ for any $u \in U$. Then U^{l+1} is a Yetter-Drinfeld weight module over $\Bbbk[x;\chi,g]$ with left $\Bbbk[x;\chi,g]$-module structure*

$$f \cdot u_n = \chi(f)^n (fu)_n, \quad x \cdot u_n = u_{n+1}$$

and left $\Bbbk[x;\chi,g]$-comodule structure

$$\delta_V(u_n) = \sum_{i=0}^{n} \binom{n}{i}_{\chi(g)} \Big(\prod_{m=i-l}^{n-1-l} (1-\chi(g)^m) \Big) x^{n-i} g^i h \otimes u_i \tag{16.1.5}$$

for any $f \in G$, $u \in U$, and any integer $0 \le n \le l$. We write $M(U,h,l)$ for this Yetter-Drinfeld module. It is simple if and only if $\chi(g)$ is not a root of 1 of order $p \in \{1,2,\dots,l\}$.

PROOF. Consider U as a Yetter-Drinfeld module over $\Bbbk G$ with $\delta_U(u) = h \otimes u$ for any $u \in U$. Then $\Bbbk[x;\chi,g] \otimes_{\Bbbk G} U \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ by Proposition 4.5.1.

For any $n \in \mathbb{N}_0$ and $u \in U$ let $u^n = x^n \otimes u \in \Bbbk[x;\chi,g] \otimes_{\Bbbk G} U$. Then

$$fu^n = \chi(f)^n (fu)^n$$

for any $f \in G$. Moreover, Lemma 16.1.9 implies that $\delta(u^n)$ is given by (16.1.5) with u_i replaced by u^i for all $1 \le i \le n$. Since $gu = \chi(hg^l)^{-1}u$ for any $u \in U$, Proposition 16.1.13(2) implies that $\Bbbk[x;\chi,g]x^{l+1} \otimes U$ is a Yetter-Drinfeld submodule of $\Bbbk[x;\chi,g] \otimes_{\Bbbk G} U$. Hence $M(U,h,l)$ exists and

$$M(U,h,l) \simeq \Bbbk[x;\chi,g] \otimes_{\Bbbk G} U/\Bbbk[x;\chi,g]x^{l+1} \otimes U.$$

By Proposition 16.1.13(2), $M(U,h,l)$ is simple if and only if $\chi(g)^{-k} \ne \chi(g)^{-l}$ for any $0 \le k < l$. This happens if and only if $\chi(g)$ is not a root of 1 of order $p \in \{1,2,\dots,l\}$. $\square$

REMARK 16.1.16. Assume that $\chi(g)^p \ne 1$ for any $1 \le p \le l$. Then the proof of Proposition 16.1.15 also shows that $M(U,h,l)$ is isomorphic to the unique simple quotient of $\Bbbk[x;\chi,g] \otimes_{\Bbbk G} U$.

REMARK 16.1.17. Let U be a simple $\Bbbk G$-module, $h \in G$, and $l \in \mathbb{N}_0$ with $gu = \chi(hg^l)^{-1}u$ for any $u \in U$. The weights of $M(U,h,l)$ in Proposition 16.1.15 are the scalars $\chi(g)^{-l+2m}$ with $0 \le m \le l$. In particular, $\chi(g)^n$ for $n \in \mathbb{Z}$ is a weight of $M(U,h,l)$ if and only if $\chi(g)^{-n}$ is. Moreover, the weight spaces of $\chi(g)^n$ and of $\chi(g)^{-n}$ have the same dimension.

COROLLARY 16.1.18. *Assume that $\chi(g)$ is not a root of 1. Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ and let $v \in V$, $h \in G$, be such that $v \ne 0$, $\delta_V(v) = h \otimes v$, and that $\Bbbk Gv$ is a simple $\Bbbk G$-module and $\dim \Bbbk[x]v < \infty$. Then there exists a unique $l \in \mathbb{N}_0$ such that $gu = \chi(hg^l)^{-1}u$ for any $u \in \Bbbk Gv$. Moreover, $\Bbbk[x;\chi,g]v$ is simple and isomorphic to $M(\Bbbk Gv,h,l)$ in ${}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$.*

PROOF. Since $\chi(g)$ is not a root of 1, the integer l is unique and $g \in G$ has infinite order. Since $v \ne 0$, $\delta_V(v) = h \otimes v$, and $\dim \Bbbk[x]v < \infty$, we conclude from Lemma 16.1.9 that $\Bbbk[x;\chi,g]v$ is isomorphic to a non-trivial Yetter-Drinfeld module quotient of $\Bbbk[x;\chi,g] \otimes_{\Bbbk G} \Bbbk Gv$. Then Lemma 16.1.12 implies the existence of $l \in \mathbb{N}_0$ such that $gu = \chi(hg^l)^{-1}u$ for any element $u \in \Bbbk Gv$. By Corollary 16.1.14(1), $\Bbbk[x;\chi,g] \otimes_{\Bbbk G} \Bbbk Gv$ has a uniqe non-trivial quotient which is then necessarily simple. By Remark 16.1.16, this quotient is isomorphic to $M(\Bbbk Gv,h,l)$. $\square$

PROPOSITION 16.1.19. *Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ be a simple object. Assume that $\dim \Bbbk[x]v < \infty$ for all $v \in V$ and that $g \in G$ has infinite order. Then there exist $h \in G$, a simple $\Bbbk G$-module U and $l \in \mathbb{N}_0$ with $\chi(g)^n \neq 1$ for all $0 \leq n < l$ such that $V \simeq M(U,h,l)$. Moreover, h and l are uniquely determined and U is unique up to isomorphism of $\Bbbk G$-modules.*

PROOF. Since $\Bbbk[x;\chi,g]$ is pointed with coradical $\Bbbk G$ by Lemma 16.1.1, all simple subcoalgebras of $\Bbbk[x;\chi,g]$ are of the form $\Bbbk h$ for some $h \in G$. Since $V \neq 0$, Proposition 2.2.13 implies that there exist $v \in V \setminus \{0\}$ and $h \in G$ such that $\delta_V(v) = h \otimes v$. In particular, $\delta_V(fv) = h \otimes fv$ for any $f \in G$ and hence $\Bbbk Gv \subseteq V_h$. Lemma 16.1.9 implies that $\Bbbk[x;\chi,g]v$ is a Yetter-Drinfeld submodule of V. Since V is simple, $\Bbbk[x;\chi,g]v$ is isomorphic to a simple quotient of $\Bbbk[x;\chi,g] \otimes_{\Bbbk G} \Bbbk Gv$, where the isomorphism maps v to $1 \otimes v$. Moreover, $x^n u \in V_{g^n h}$ for any $n \in \mathbb{N}_0$ and any $u \in \Bbbk Gv$ by Proposition 16.1.6. Since $g \in G$ has infinite order, we conclude that h is uniquely determined and that $\Bbbk[x;\chi,g]U$ is a Yetter-Drinfeld submodule of V for any $\Bbbk G$-submodule U of $\Bbbk Gv$. Hence the simplicity of V implies that $\Bbbk Gv$ is a simple $\Bbbk G$-module and as such it is uniquely determined up to isomorphism. Since $\dim \Bbbk[x]v < \infty$, Lemma 16.1.12(2) implies that there exists a unique integer $l \geq 0$ such that $\chi(g)^n \neq 1$ for all $0 \leq n < l$ and that $\Bbbk Gv$ has weight $\chi(g)^{-l}$ in V. By Proposition 16.1.13(2), $\Bbbk[x;\chi,g] \otimes_{\Bbbk G} \Bbbk Gv$ has a unique maximal Yetter-Drinfeld submodule, and by the assumption on l and by Remark 16.1.16 the unique simple quotient of $\Bbbk[x;\chi,g] \otimes_{\Bbbk G} \Bbbk Gv$ is isomorphic to $M(\Bbbk Gv, h, l)$. $\square$

LEMMA 16.1.20. *Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ and let $W \subseteq V$ be a subobject which is a weight module. Let $h \in G$ and $\lambda \in \Bbbk^\times$. Assume that*

$$v \in V_h \setminus W_h, \quad gv - \chi(h)^{-1}\lambda v \in W, \quad \delta_V(v) - h \otimes v \in W.$$

If $\lambda \notin \{\chi(g)^k \mid k \geq 2\}$ then $\delta_V(v+w) = h \otimes (v+w)$ for some $w \in W_h$.

PROOF. Since $v \in V_h$, Proposition 16.1.6 yields that $gv \in V_h$. Therefore $gv - \chi(h)^{-1}\lambda v \in W_h$ by assumption. Since W is a weight module, there exist pairwise distinct scalars $\mu_1, \dots, \mu_r$, $r \geq 0$, and vectors $w_{\mu_i} \in W_{h;\mu_i}$, $1 \leq i \leq r$, such that $gv - \chi(h)^{-1}\lambda v = \sum_{i=1}^r w_{\mu_i}$. Therefore there exists $w \in W_h$ such that $g(v+w) - \chi(h)^{-1}\lambda(v+w) \in W_{h;\lambda}$. Thus in order to prove the claim we may assume that $gv - \chi(h)^{-1}\lambda v$ is a weight vector of W of weight λ.

By Lemma 16.1.5, $\delta_V(v) = \sum_{n\in\mathbb{N}_0} x^n g^{-n} h \otimes v_n$ for some $v_n \in V$ with $v_0 = v$ and $v_n = 0$ for all but finitely many n. Since $(g - \chi(h)^{-1}\lambda 1)^2 v = 0$, Lemma 16.1.11 implies that $(g - \chi(h)^{-1}\chi(g)^{-n}\lambda 1)^2 v_n = 0$ for all $n \in \mathbb{N}_0$. Since $v_n \in W$ for any $n > 0$ by assumption and since W is a weight module, we conclude that $v_n \in W_{hg^{-n};\lambda\chi(g)^{-2n}}$ for any $n > 0$.

Let $m \in \mathbb{N}_0$ maximal with $v_m \neq 0$. Assume that $m > 0$. The comodule axiom for δ_V applied to v implies that $\delta_V(v_m) = hg^{-m} \otimes v_m$. Then Lemma 16.1.9 implies that

$$\begin{aligned}\delta_V(x^m v_m) &= \sum_{i=0}^m \binom{m}{i}_{\chi(g)} x^{m-i} g^{i-m} h \otimes x^i \prod_{k=i}^{m-1} (1 - \chi(h)\chi(g)^{k-m} g) v_m \\ &= \sum_{i=0}^m \binom{m}{i}_{\chi(g)} x^{m-i} g^{i-m} h \otimes x^i \prod_{k=i}^{m-1} (1 - \chi(g)^{k-2m} \lambda) v_m\end{aligned}$$

since $gv_m = \chi(h)^{-1}\chi(g)^{-m}\lambda v_m$. Now, if $\lambda \notin \{\chi(g)^k \mid k \geq 2\}$ then the coefficient ζ of $x^m g^{-m} h \otimes v_m$ in $\delta_V(x^m v_m)$ is

$$\zeta = \prod_{k=0}^{m-1} (1 - \chi(g)^{k-2m}\lambda) \neq 0.$$

Thus $\delta_V(v - \zeta^{-1}x^m v_m) \in \sum_{n=0}^{m-1} x^n g^{-n} h \otimes V$. Note that $x^m v_m \in W_{h;\lambda}$. Now replace v by $v - \zeta^{-1}x^m v_m$ and apply the arguments of the proof to this element. After finitely many iterations we arrive at an element $v \in V_h$ with $\delta_V(v) = h \otimes v$. $\square$

THEOREM 16.1.21. *Assume that $\chi(g)$ is not a root of 1. Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ be such that V is a semisimple $\Bbbk G$-module and $\dim \Bbbk[x]v < \infty$ for any $v \in V$. Then V is a semisimple Yetter-Drinfeld module and any simple subobject of V is isomorphic to $M(U,h,l)$ for some simple $\Bbbk G$-module U, some $h \in G$ and some $l \in \mathbb{N}_0$.*

PROOF. By Proposition 16.1.19, all simple subobjects of V are isomorphic to $M(U,h,l)$ for some simple $\Bbbk G$-module U, some $h \in G$ and some $l \in \mathbb{N}_0$. Let W be the sum of all simple Yetter-Drinfeld submodules of V.

Assume that $V \neq W$. By Lemma 16.1.1, $\Bbbk[x;\chi,g]$ is pointed with coradical $\Bbbk G$. Hence all simple subcoalgebras of $\Bbbk[x;\chi,g]$ are of the form $\Bbbk h$ for some $h \in G$. Since $V/W \neq 0$, Proposition 2.2.13 implies that there exists $v \in V \setminus W$ and $h \in G$ such that $\delta_V(v) - h \otimes v \in \Bbbk[x;\chi,g] \otimes W$. In particular, $v + W \in V_h + W$. Proposition 16.1.6 implies that we may choose this v such that $v \in V_h$. Since

$$\delta_V(fv) - h \otimes fv \in \Bbbk[x;\chi,g] \otimes W$$

and $fv \in V_h$ for any $f \in G$, by the semisimplicity of V as a $\Bbbk G$-module we may additionally choose v to be in a simple $\Bbbk G$-module. Since $\dim \Bbbk[x]v < \infty$, by Corollary 16.1.18 there exists a unique $l \in \mathbb{N}_0$ such that $gu - \chi(hg^l)^{-1}u \in W$ for any $u \in \Bbbk Gv$. Then, since $\chi(g)$ is not a root of 1, by Lemma 16.1.20 we may choose the representative v of $v + W$ such that $\delta_V(v) = h \otimes v$. Then $\Bbbk[x;\chi,g]v$ is a simple subobject of V by Corollary 16.1.18 which is a contradiction to the choice of v and W. This proves the theorem. $\square$

COROLLARY 16.1.22. *Assume that $\chi(g)$ is not a root of 1. Let $V \in {}^{\Bbbk[x;\chi,g]}_{\Bbbk[x;\chi,g]}\mathcal{YD}$ be such that V is a semisimple $\Bbbk G$-module and $\dim \Bbbk[x]v < \infty$ for all $v \in V$. Then V is a weight module, the weights of V are of the form $\chi(g)^m$ with $m \in \mathbb{Z}$, and for any $m \in \mathbb{Z}$ the dimension of the weight space of any weight $\chi(g)^m$ coincides with the dimension of the weight space of $\chi(g)^{-m}$.*

PROOF. This follows immediately from Theorem 16.1.21, Proposition 16.1.19 and Remark 16.1.17. $\square$

In the remaining part of the section let $t \in \mathbb{N}$ with $t \geq 2$. If $\chi(g) \neq 1$ is a primitive t-th root of 1, then there is another class of Yetter-Drinfeld modules which plays a similarly important role, in particular in the description of $u_{\boldsymbol{q}}^+$ in Section 16.3. Let $\Bbbk_{\mathrm{red}}[x;\chi,g] = \Bbbk[x;\chi,g]/(x^t)$. Since x^t is $(g^t,1)$-primitive in $\Bbbk[x;\chi,g]$ by Proposition 2.4.2(5), $\Bbbk_{\mathrm{red}}[x;\chi,g]$ is a quotient Hopf algebra of $\Bbbk[x;\chi,g]$ by Proposition 2.4.4. Since x^t is homogeneous of degree t in $\Bbbk[x;\chi,g]$ with respect to the grading in Lemma 16.1.1, the Hopf algebra $\Bbbk_{\mathrm{red}}[x;\chi,g]$ is $\mathbb{N}_0$-graded with $\deg x^m h = m$ for all $0 \leq m < t$, $h \in \Bbbk G$. Let $\pi_{\mathrm{red}} : \Bbbk_{\mathrm{red}}[x;\chi,g] \to \Bbbk G = \Bbbk 1 \# \Bbbk G$ be the homogeneous Hopf algebra projection.

There is no analogue to Theorem 16.1.21 for the category ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$, but we are able to describe all simple objects. We proceed as for $\Bbbk[x;\chi,g]$.

Definition 16.1.23. A Yetter-Drinfeld module $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ is called a **weight module** if the action of g on V is diagonalizable.

Lemma 16.1.24. *Let $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{M}$ and let $v \in V$ and $h \in G$. Assume that $(\pi_{\mathrm{red}} \otimes \mathrm{id})\delta_V(v) = h \otimes v$. Then*

$$\delta_V(v) = h \otimes v + \sum_{n=1}^{t-1} x^n g^{-n} h \otimes v_n$$

for some $v_n \in V$, $1 \le n \le t-1$.

Proof. Similar to the proof of Lemma 16.1.5. □

Proposition 16.1.25. *Let $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$. For all $h \in G$ let*

$$V_h = \{v \in V \mid (\pi_{\mathrm{red}} \otimes \mathrm{id})\delta_V(v) = h \otimes v\}.$$

Then $V = \bigoplus_{h\in G} V_h$, $GV_f = V_f$, $xV_f \subseteq V_{gf}$, and

$$\delta_V(v) \in \sum_{n=0}^{t-1} x^n \Bbbk G \otimes V_{fg^{-n}}$$

for all $f \in G$, $v \in V_f$.

Proof. Similar to the proof of Proposition 16.1.6. □

Definition 16.1.26. Let $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ be a weight module. For any $h \in G$, $\lambda \in \Bbbk^\times$ let

$$V_{h;\lambda} = \{v \in V_h \mid gv = \chi(h)^{-1}\lambda v\}.$$

The scalars λ with $V_{h;\lambda} \neq 0$ for some $h \in G$ are called the **weights of** V. For any weight λ, the sum $\bigoplus_{h\in G} V_{h;\lambda}$ is called the **weight space** of λ.

Lemma 16.1.27. *Let $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$, $v \in V$, $h \in G$ and $n \in \mathbb{N}_0$ with $n < t$. Assume that $\delta_V(v) = h \otimes v$. Then*

$$\delta_V(x^n v) = \sum_{i=0}^{n} \binom{n}{i}_{\chi(g)} x^{n-i} g^i h \otimes x^i \prod_{k=i}^{n-1} (1 - \chi(h)\chi(g)^k g) v.$$

Proof. Literally the same as the proof of Lemma 16.1.9. □

Lemma 16.1.28. *Let $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ and let $h \in G$, $v \in V_h$ and $\lambda \in \Bbbk^\times$. Assume that v has weight λ. Then $x^n v$ has weight $\chi(g)^{2n}\lambda$ for any $n \in \mathbb{N}_0$.*

Proof. Analogous to the proof of Lemma 16.1.10. □

Lemma 16.1.29. *Let $U \in {}^G_G\mathcal{YD}$ and $W = \Bbbk_{\mathrm{red}}[x;\chi,g] \otimes_{\Bbbk G} U \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ such that U is simple. Let $h \in G$ such that $\delta_U(u) = h \otimes u$ for all $u \in U$.*

(1) *$W = \bigoplus_{n=0}^{t-1} W_{g^n h}$ and $W_{g^n h} = x^n \otimes U$ for any $0 \le n < t$.*
(2) *Let $X \subseteq W$ in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ with $X \neq 0$. Then $X = \Bbbk[x]x^n \otimes U$ for some $n \in \mathbb{N}_0$, $n < t$.*

(3) *If W is not simple then there exists $n \in \mathbb{N}_0$, $n < t-1$, such that $1 \otimes u \in W$ has weight $\chi(g)^{-n}$ for any $u \in U$.*

PROOF. By Proposition 4.5.1 we know that $W \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$. Since the order of $\chi(g) \in \Bbbk^\times$ is t, the order of $g \in G$ is at least t. Since $1 \otimes U \subseteq W_h$, we obtain (1) from Proposition 16.1.25.

Let now $X \subseteq W$ in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ with $X \neq 0$. Similarly to the previous paragraph we conclude that there exist $u \in U \setminus \{0\}$ and a smallest $n \in \mathbb{N}_0$ such that $n < t$ and $x^n \otimes u \in X$. Lemma 16.1.27 and the minimality of n imply that $\delta_W(x^n \otimes u) = g^n h \otimes (x^n \otimes u)$. In particular, the summand of $\delta_W(x^n \otimes u)$ in Lemma 16.1.27 for $i = 0$ vanishes. Hence

$$x^n h \otimes \prod_{k=0}^{n-1} (1 - \chi(h)\chi(g)^k g)u = 0.$$

Thus there exist $u' \in U \setminus \{0\}$ and $k \in \{0, 1, \dots, n-1\}$ with $(1 - \chi(h)\chi(g)^k g)u' = 0$. Since $U = \Bbbk G u = \Bbbk G u'$ and G is abelian, we conclude that $gv = \chi(h)^{-1}\chi(g)^{-k} v$ and that $x^n \otimes v \in X$ for all $v \in U$. This implies both (2) and (3). □

PROPOSITION 16.1.30. *Let $h \in G$, $n \in \mathbb{N}_0$ with $n < t$ and $U \in {}^G_G\mathcal{YD}$. Assume that $\delta_U(u) = h \otimes u$ and that $gu = \chi(h)^{-1}\chi(g)^{-n} u$ for all $u \in U$. Moreover, let $W = \Bbbk_{\mathrm{red}}[x;\chi,g] \otimes_{\Bbbk G} U \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$.*

(1) *W is a weight module with weights $\chi(g)^{2m-n}$, $0 \le m < t$.*
(2) *If U is simple then $\Bbbk[x]x^{n+1} \otimes U$ is the only maximal Yetter-Drinfeld submodule of W.*

PROOF. (1) By assumption, W is spanned by the elements $x^m \otimes u$ with $u \in U$ and $0 \le m < t$. Moreover, $1 \otimes u$ has weight $\chi(g)^{-n}$ for any $u \in U$. Thus the claim follows from Lemma 16.1.28.

(2) Lemma 16.1.29 implies that there is a unique maximal Yetter-Drinfeld submodule W' of W, and it is of the form $\Bbbk[x]x^m \otimes U$ for some $1 \le m \le t$.

By assumption, $(1 - \chi(h)\chi(g)^n g)u = 0$ for all $u \in U$. Thus, by Lemma 16.1.27,

$$\delta_W(x^{n+1} \otimes u) = g^{n+1} h \otimes x^{n+1} u$$

for all $u \in U$. Then Lemma 16.1.27 implies that $\Bbbk[x]x^{n+1} \otimes U \subseteq W'$ and hence $m \le n+1$.

Let $u \in U$ with $u \neq 0$. By Lemma 16.1.27, the coefficient of $x^n h \otimes (1 \otimes u)$ in $\delta_W(x^n \otimes u)$ is $\prod_{k=0}^{n-1}(1 - \chi(g)^{k-n})(1 \otimes u)$, which is non-zero since $\mathrm{ord}(\chi(g)) = t$. Thus $x^n \otimes u \notin W'$ and $m = n+1$. □

THEOREM 16.1.31. *Assume that $\chi(g)$ is a primitive root of 1 of order t. For any $U \in {}^G_G\mathcal{YD}$ let $W(U) = \Bbbk_{\mathrm{red}}[x;\chi,g] \otimes_{\Bbbk G} U \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$.*

(1) *Let $U_1, U_2 \in {}^G_G\mathcal{YD}$ be simple objects. Then $U_1 \cong U_2$ in ${}^G_G\mathcal{YD}$ if and only if $W(U_1) \cong W(U_2)$ in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$.*
(2) *Let U be a simple Yetter-Drinfeld module over $\Bbbk G$ and let $h \in G$. Assume that $\delta_U(u) = h \otimes u$ and that $gu \notin \Bbbk u$ for any non-zero element $u \in U$. Then $W(U)$ is simple in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$.*
(3) *Let $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ be a simple object. For any $h \in G$ let*

$$V_{(h)} = \{v \in V \mid \delta_V(v) = h \otimes v\}.$$

Then there exists a unique element $h \in G$ such that $V_{(h)} \neq 0$. Moreover, $V_{(h)} \in {}^G_G\mathcal{YD}$ is simple.

(4) *Let $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ be a simple object and let $h \in G$ with $V_{(h)} \neq 0$. Assume that $gv \notin \Bbbk v$ for any $v \in V_{(h)} \setminus \{0\}$. Then $V \cong W(V_{(h)})$.*

Proof. (1) An isomorphism $f : U_1 \to U_2$ in ${}^G_G\mathcal{YD}$ induces an isomorphism $\Bbbk_{\mathrm{red}}[x;\chi,g] \otimes_{\Bbbk G} U_1 \to \Bbbk_{\mathrm{red}}[x;\chi,g] \otimes_{\Bbbk G} U_2$ in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ by the functoriality of the construction of induced Yetter-Drinfeld modules.

Assume now that there is an isomorphism $f : W(U_1) \to W(U_2)$ in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$. Let $h_1, h_2 \in G$ such that $\delta_{U_i}(u_i) = h_i \otimes u_i$ for any $i \in \{1,2\}$ and $u_i \in U_i$. Since $W(U_i) = \bigoplus_{n=0}^{t-1} W(U_i)_{g^n h_i}$ and $W(U_i)_{g^n h} = x^n \otimes U_i$ for any $0 \le n < t$ and any $i \in \{1,2\}$ by Lemma 16.1.29(1), there exists $0 \le k < t$ such that

$$f(1 \otimes u) \in x^k \otimes U_2, \quad h_1 = g^k h_2$$

for any $u \in U_1$. Then $f(W(U_1)) \subseteq \Bbbk[x]x^k \otimes U_2$, and the surjectivity of f implies that $k = 0$ and $f(1 \otimes U_1) = 1 \otimes U_2$. Thus $U_i \cong 1 \otimes U_i$ for $i \in \{1,2\}$ are isomorphic in ${}^G_G\mathcal{YD}$ via restriction of f to $1 \otimes U_1$.

(2) Let W' be a Yetter-Drinfeld submodule of $W(U)$. Lemma 16.1.29(1) implies that

$$W' = \bigoplus_{n=0}^{t-1} (W' \cap W(U)_{g^n h}).$$

Let $0 \le n < t$ and $0 \neq v \in W' \cap W(U)_{g^n h}$. Then $v = x^n \otimes u = x^n(1 \otimes u)$ for some $u \in U \setminus \{0\}$ by Lemma 16.1.29(1). By assumption, $gu' \notin \Bbbk u'$ for any $u' \in U \setminus \{0\}$. Thus, by Lemma 16.1.27, the summand of $\delta_{W(U)}(v)$ in $x^n h \otimes U$ is

$$x^n h \otimes \Big(1 \otimes \prod_{k=0}^{n-1} (1 - \chi(hg^k)g) \cdot u\Big) \neq 0.$$

Thus $W' \cap U \neq 0$. Since U is simple, we conclude that $W' \cap U = U$ and hence $W' = W(U)$.

(3) Since $\Bbbk_{\mathrm{red}}[x;\chi,g]$ is pointed (as a quotient of $\Bbbk[x;\chi,g]$), $V_{(h)} \neq 0$ for some $h \in G$ by Proposition 2.2.13. Since $\Bbbk G V_{(h)} = V_{(h)}$ and since V is simple, we conclude that $V = \Bbbk_{\mathrm{red}}[x;\chi,g]V_{(h)} = \Bbbk[x]V_{(h)}$. Let $h' \in G$ with $V_{(h')} \neq 0$. Assume that $h' \neq h$. Then $V_{(h')} \subseteq V_{h'}$. Hence $h' = g^n h$ and $V_{(h')} \subseteq x^n V_{(h)}$ for some $1 \le n < t$. Similarly, $h = g^m h'$ and $V_{(h)} \subseteq x^m V_{(h')}$ for some $1 \le m < t$. Then $h = g^{m+n}h$, and hence $m + n \ge t$ since $\mathrm{ord}(\chi(g)) = t$. Thus $V_{(h)} \subseteq x^{m+n}V_{(h)} = 0$, a contradiction. It follows that $h' = h$.

For any $\Bbbk G$-submodule $U \neq 0$ of $V_{(h)}$, $\Bbbk[x]U$ is a Yetter-Drinfeld submodule of V by Lemma 16.1.27. Since $x^n U \subseteq V_{g^n h}$ for any $0 \le n < t$ and since $\mathrm{ord}(g) \ge t$, it follows that $\Bbbk[x]U \cap V_{(h)} = U$. Thus the simplicity of V implies that $V_{(h)}$ is simple in ${}^G_G\mathcal{YD}$.

(4) Since $V_{(h)} \in {}^G_G\mathcal{YD}$ and V is simple in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$, Lemma 16.1.27 implies that $V = \Bbbk[x]V_{(h)}$. Thus V is isomorphic to a quotient of $W(V_{(h)})$. Hence the claim follows from (2). □

Theorem 16.1.32. *Assume that $\chi(g)$ is a primitive root of 1 of order t. For any $U \in {}^G_G\mathcal{YD}$ let $W(U) = \Bbbk_{\mathrm{red}}[x;\chi,g] \otimes_{\Bbbk G} U \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$.*

(1) *Let U be a simple Yetter-Drinfeld module over $\Bbbk G$ and let $h \in G$ and $\lambda \in \Bbbk^\times$. Assume that*

$$\delta_U(u) = h \otimes u, \quad gu = \chi(h)^{-1}\lambda u$$

for any $u \in U$. Let

$$W(U)_{\mathrm{red}} = \begin{cases} W(U) & \text{if } \lambda \notin \{\chi(g)^{-m} \mid 0 \le m < t-1\}, \\ W(U)/(\Bbbk[x]x^{n+1} \otimes U) & \text{if } \lambda = \chi(g)^{-n},\ 0 \le n < t-1. \end{cases}$$

Then $W(U)_{\mathrm{red}}$ is simple in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$.

(2) *Let $U_1, U_2 \in {}^G_G\mathcal{YD}$ be simple objects. Assume that g acts on U_1 and on U_2 by a multiple of the identity. Then $U_1 \cong U_2$ in ${}^G_G\mathcal{YD}$ if and only if $W(U_1)_{\mathrm{red}} \cong W(U_2)_{\mathrm{red}}$ in ${}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$.*

(3) *Let $h \in G$ and $V \in {}^{\Bbbk_{\mathrm{red}}[x;\chi,g]}_{\Bbbk_{\mathrm{red}}[x;\chi,g]}\mathcal{YD}$ with $V_{(h)} \neq 0$ and $gu = \chi(h)^{-1}\lambda u$ for some $u \in V_{(h)} \setminus \{0\}$, $\lambda \in \Bbbk^\times$. Then $V_{(h)} \in {}^G_G\mathcal{YD}$ is simple, $gv = \chi(h)^{-1}\lambda v$ for any $v \in V_{(h)}$, and $V \cong W(V_{(h)})_{\mathrm{red}}$.*

Proof. (1) If $\lambda = \chi(g)^{-n}$ for some $0 \le n < t-1$, then $W(U)_{\mathrm{red}}$ is simple by Proposition 16.1.30(2). Otherwise the proof is analogous to the proof of Theorem 16.1.31(2).

(2) Analogous to the proof of Theorem 16.1.31(1).

(3) By Theorem 16.1.31(3), $V_{(h)} \in {}^G_G\mathcal{YD}$ is simple. Hence $V_{(h)}$ is a simple $\Bbbk G$-module. Since G is abelian and $gu = \chi(h)^{-1}\lambda u$, it follows that $gv = \chi(h)^{-1}\lambda v$ for all $v \in V_{(h)}$. Moreover,

$$V = \Bbbk_{\mathrm{red}}[x;\chi,g]V_{(h)} = \Bbbk[x]V_{(h)}.$$

Thus V is isomorphic to a quotient of $W(V_{(h)})$. If $\lambda = \chi(g)^{-n}$ for some $0 \le n < t$, then $V \cong W(V_{(h)})_{\mathrm{red}}$ by Proposition 16.1.30(2). Otherwise $W(V_{(h)}) = W(V_{(h)})_{\mathrm{red}}$ is simple by (1) and hence $V \cong W(V_{(h)})_{\mathrm{red}}$. □

16.2. On the structure of $U_{\boldsymbol{q}}^+$

Let $\theta \ge 1$, $\mathbb{I} = \{1, \dots, \theta\}$, and let $\boldsymbol{q} = (q_{ij})_{i,j\in\mathbb{I}}$ be a family of non-zero elements in $\Bbbk$. We choose a realization of $\boldsymbol{q}$ as the braiding matrix of a Yetter-Drinfeld module as follows. Let G be an abelian group, $H = \Bbbk G$ its group algebra, and let $K_1, \dots, K_\theta \in G$ and $\chi_1, \dots, \chi_\theta \in \mathrm{Alg}(\Bbbk G, \Bbbk)$ be such that $\chi_j(K_i) = q_{ij}$ for all $i, j \in \mathbb{I}$. (Elements $K_1, \dots, K_\theta$ and maps $\chi_1, \dots, \chi_\theta$ as required exist for example if $G = \mathbb{Z}^\theta$.) For all $j \in \mathbb{I}$, let $M_j \in {}^H_H\mathcal{YD}$ be a one-dimensional object in ${}^H_H\mathcal{YD}$, $E_j \in M_j \setminus \{0\}$ with

$$\delta_{M_j}(E_j) = K_j \otimes E_j, \quad h \cdot E_j = \chi_j(h)E_j \tag{16.2.1}$$

for all $h \in H$, and $M = (M_1, \dots, M_\theta)$. The existence of M is guaranteed by Example 1.4.3.

Assume that the matrix $\boldsymbol{q}$ is quasi-generic in the sense of Definition 8.2.1. Then by Lemma 15.1.4, M is i-finite for all i if and only if $\boldsymbol{q}$ is of Cartan type, that is, there is a Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$ with

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}} \text{ for all } i, j \in \mathbb{I}. \tag{16.2.2}$$

In this case, $a_{ij} = a_{ij}^M$ for all $i, j \in \mathbb{I}$.

Thus $\mathcal{D} = \mathcal{D}(G, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ is a quasi-generic YD-datum of Cartan type with braiding matrix $\boldsymbol{q}$.

Assume (16.2.2) for the rest of the section. By Theorem 15.1.14, M admits all reflections, and its Cartan graph is standard. By Example 1.10.1, $\mathcal{B}(M_i) = \Bbbk[E_i]$ is a polynomial algebra for all $i \in \mathbb{I}$. Moreover, for any $N \in \mathcal{F}_\theta^H(M)$, the set $\mathrm{Hom}(\mathcal{W}(M), [N])$ carries a natural group structure isomorphic to the Weyl group of A by Proposition 9.3.15.

If A is of finite type, then we know already a basis of $\mathcal{B}(M)$.

Theorem 16.2.1. *Assume that A is of finite type. Let w be the longest element of the Weyl group of A and let $\kappa = (i_1, \ldots, i_l)$ be a reduced decomposition of w. Let $x_1, \ldots, x_l$ be a root vector sequence for κ in $\mathcal{B}(M)$. Then $\mathcal{B}(M) = E^{\mathcal{B}(M)}(\kappa)$, and the monomials*

$$x_l^{n_l} \cdots x_1^{n_1}, \quad n_1, \ldots, n_l \geq 0,$$

form a basis of $\mathcal{B}(M)$.

At the end of the section, see Remark 16.2.6, we relate the root vector sequences in Theorem 16.2.1 for braiding matrices $\boldsymbol{q} = (q^{d_i a_{ij}})_{i,j\in\mathbb{I}}$ to the root vectors of quantized enveloping algebras defined by Lusztig.

Proof. By the above, M admits all reflections. By Theorem 9.3.5, κ is $[M]$-reduced. Since $(i, i_1, \ldots, i_l)$ is not $[R_i(M)]$-reduced by assumption and by Theorem 9.3.5, $\alpha_i \in \Lambda^{[M]}(\kappa)$ for all $i \in \mathbb{I}$. Hence the claim follows from Theorem 15.2.7 and Example 1.10.1. Indeed, for any $\alpha \in \boldsymbol{\Delta}_+^{[M]\,\mathrm{re}}$, $q_{\alpha\alpha} = q_{jj}$ for some $j \in \mathbb{I}$ and q_{jj} is not a root of unity or $q_{jj} = 1$, $\mathrm{char}(\Bbbk) = 0$. $\square$

Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Thus S is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ generated by N, and the canonical map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ is a surjective morphism of Hopf algebras inducing the isomorphism $N_i \xrightarrow{f_i} M_i$ in ${}^H_H\mathcal{YD}$ with $\deg(M_i) = \deg(N_i) = \alpha_i$ for all $i \in \mathbb{I}$.

For all $i \in \mathbb{I}$, let $0 \neq x_i \in N_i$. Note that the first axiom of a Nichols system is satisfied for $\mathcal{N}$, that is, $p^{\mathcal{N}}$ induces an isomorphism $\Bbbk[x_i] \to \Bbbk[E_i] = \mathcal{B}(M_i)$, both being isomorphic to the polynomial ring in one indeterminate by Example 1.10.1. Since the Yetter-Drinfeld modules N_i are one-dimensional for all i, $\mathcal{N}$ is a Nichols system of (M, i) for all $i \in \mathbb{I}$ if and only if

$$(\mathrm{ad}_S x_i)^{1-a_{ij}^M}(x_j) = 0 \quad \text{for all } i, j \in \mathbb{I} \text{ with } i \neq j. \tag{16.2.3}$$

A tool to verify (16.2.3) was formulated in Lemma 13.5.6.

Recall that for all $i \in \mathbb{I}$, the diagram

$$\begin{tikzcd} & \Bbbk[E_i]\#H \arrow[dl, "\gamma_i^{\mathcal{N}}\#\mathrm{id}"'] \arrow[d, "="] \\ S\#H \arrow[r, "\pi_i^{\mathcal{N}}\#\mathrm{id}"] & \Bbbk[E_i]\#H \end{tikzcd}$$

commutes. By definition, $K_i^{\mathcal{N}}$ is the set of right coinvariant elements of the projection $\pi_i^{\mathcal{N}}\#\mathrm{id} : S\#H \to \Bbbk[E_i]\#H$. Hence $K_i^{\mathcal{N}} \in {}^{\Bbbk[E_i]\#H}_{\Bbbk[E_i]\#H}\mathcal{YD}$, where $K_i^{\mathcal{N}}$ is a left $\Bbbk[E_i]\#H$-module via the adjoint action. Recall that $K_i^{\mathcal{N}}$ is an $\mathbb{N}_0^\theta$-graded subalgebra of S.

Now we fix $i \in \mathbb{I}$. We want to apply the theory of weight modules over $\Bbbk[E_i]\#H$ in Section 16.1 with $x = E_i$, $g = K_i$ and $\chi = \chi_i$.

For all $\alpha = \sum_{j=1}^{\theta} a_j\alpha_j$, $a_1, \dots, a_\theta \in \mathbb{Z}$, we define

$$K_\alpha = \prod_{j=1}^{\theta} K_j^{a_j}, \quad \chi_\alpha = \prod_{j=1}^{\theta} \chi_j^{a_j}.$$

Note that for all $\alpha \in \mathbb{N}_0^\theta$ and $x \in S(\alpha)$,

$$g \cdot x = \chi_\alpha(g)x \quad \text{for all } g \in G. \tag{16.2.4}$$

In particular, $K_i^{\mathcal{N}}$ is a semisimple H-module, and a weight module for $\Bbbk[E_i]\#H$.

LEMMA 16.2.2. *Let $\alpha \in \mathbb{Z}^\theta$ and $i \in \mathbb{I}$. Then*

$$\chi_i(K_{s_i^M(\alpha)}) = \chi_\alpha(K_i)^{-1}, \quad \chi_{s_i^M(\alpha)}(K_i) = \chi_i(K_\alpha)^{-1}.$$

PROOF. Let $\alpha = \sum_{j=1}^{\theta} a_j\alpha_j$, where $a_1, \dots, a_\theta \in \mathbb{Z}$. By definition of the reflection s_i^M, $s_i^M(\alpha) = \alpha - (\sum_{j=1}^{\theta} a_j a_{ij})\alpha_i$. Hence

$$\chi_i(K_{s_i^M(\alpha)}) = \prod_{j=1}^{\theta} q_{ji}^{a_j} \prod_{j=1}^{\theta} q_{ii}^{-a_j a_{ij}} = \prod_{j=1}^{\theta} q_{ji}^{a_j} \prod_{j=1}^{\theta} (q_{ij}q_{ji})^{-a_j} = \chi_\alpha(K_i)^{-1},$$

and the second equation follows from the first, since $(s_i^M)^2 = \mathrm{id}$. □

Note that in the notation of Proposition 16.1.6, $(K_i^{\mathcal{N}})_{K_\alpha} = K_i^{\mathcal{N}}(\alpha)$.

Let $V \subseteq K_i^{\mathcal{N}}$ be a subobject in ${}^{\Bbbk[E_i]\#H}_{\Bbbk[E_i]\#H}\mathcal{YD}$, and $\lambda \in \Bbbk$. For all $\alpha \in \mathbb{N}_0^\theta$ let

$$V(\alpha)_\lambda = \{v \in V(\alpha) \mid K_i \cdot v = \chi_i^{-1}(K_\alpha)\lambda v\}. \tag{16.2.5}$$

Recall from Definitions 16.1.3 and 16.1.8 that $\lambda \in \Bbbk$ is a weight of V, if $V(\alpha)_\lambda \neq 0$ for some α. If λ is a weight of V, then $V_\lambda = \bigoplus_{\alpha \in \mathbb{N}_0^\theta} V(\alpha)_\lambda$ is the weight space of V of weight λ.

The next theorem mainly follows from the theory of Yetter-Drinfeld modules over a Hopf algebra of polynomials from Section 16.1.

THEOREM 16.2.3. *Let $\alpha \in \mathbb{N}_0^\theta$, and $i \in \mathbb{I}$. Assume that q_{ii} is not a root of unity. Then*

$$\dim K_i^{\mathcal{N}}(\alpha) = \dim K_i^{\mathcal{N}}(s_i^M(\alpha)).$$

PROOF. We separate the α_i-part of α and write $\alpha = \beta + m\alpha_i$, where $m \geq 0$, $\beta = \sum_{j=1}^{\theta} b_j\alpha_j$ with $b_1, \dots, b_\theta \geq 0$, $b_i = 0$. Let

$$V = \bigoplus_{p \geq 0} K_i^{\mathcal{N}}(\beta + p\alpha_i).$$

(1) We claim that $V \subseteq K_i^{\mathcal{N}}$ is a subobject in ${}^{\Bbbk[E_i]\#H}_{\Bbbk[E_i]\#H}\mathcal{YD}$.

For all $\gamma \in \mathbb{N}_0^\theta$, $\operatorname{ad} E_i(K_i^{\mathcal{N}}(\gamma)) \subseteq K_i^{N}(\gamma + \alpha_i)$. In particular, $V \subseteq K_i^{\mathcal{N}}$ is a $\Bbbk[E_i]\#H$-submodule.

We denote the comultiplication of S by $\Delta_S(x) = x^{(1)} \otimes x^{(2)}$ for all $x \in S$. Then the $\Bbbk[E_i]\#H$-comodule structure of $K_i^{\mathcal{N}}$ is

$$K_i^{\mathcal{N}} \xrightarrow{\delta} \Bbbk[E_i]\#H \otimes K_i^{\mathcal{N}}, \; x \mapsto \pi_i^{\mathcal{N}}(x^{(1)})\#x^{(2)}{}_{(-1)} \otimes x^{(2)}{}_{(0)}.$$

For all $p \geq 0$, $x \in K_i^{\mathcal{N}}(\beta + p\alpha_i)$,

$$\begin{aligned}\pi_i^{\mathcal{N}}(x^{(1)}) \otimes x^{(2)} \in & \bigoplus_{\substack{\gamma+\lambda=\beta+p\alpha_i \\ \gamma,\lambda\in\mathbb{N}_0^\theta}} \pi_i^{\mathcal{N}}(S(\gamma)) \otimes K_i^{\mathcal{N}}(\lambda) \\ = & \bigoplus_{\substack{\gamma=r\alpha_i, 0\leq r\leq p \\ \lambda=\beta+(p-r)\alpha_i}} \pi_i^{\mathcal{N}}(S(\gamma)) \otimes K_i^{\mathcal{N}}(\lambda),\end{aligned}$$

since $\pi_i^{\mathcal{N}}(E_j) = 0$ for all $j \neq i$. Hence $V \subseteq K_i^{\mathcal{N}}$ is a $\Bbbk[E_i]\#H$-subcomodule.

(2) We next show that for all $p \geq 0$, $K_i^{\mathcal{N}}(\beta + p\alpha_i)$ is 0 or the weight space of V of weight $\lambda_p = \chi_{\beta+p\alpha_i}(K_i)\chi_i(K_{\beta+p\alpha_i})$.

For any $p \geq 0$,

$$K_i \cdot v = \chi_{\beta+p\alpha_i}(K_i)v \text{ for all } v \in K_i^{\mathcal{N}}(\beta + p\alpha_i) \tag{16.2.6}$$

by (16.2.4). Now (16.2.5) and (16.2.6) imply that

$$K_i^{\mathcal{N}}(\beta + p\alpha_i) = K_i^{\mathcal{N}}(\beta + p\alpha_i)_{\lambda_p}$$

with $\lambda_p = \chi_{\beta+p\alpha_i}(K_i)\chi_i(K_{\beta+p\alpha_i}) = \chi_\beta(K_i)\chi_i(K_\beta)q_{ii}^{2p}$. Then the claim in (2) follows from the definition of V, since q_{ii} is not a root of unity.

(3) Note that the assumptions in Corollary 16.1.22 are satisfied for V, since V is a semisimple H-module, and $K_i^{\mathcal{N}}$ is a rational $\Bbbk[E_i]$-module under the adjoint action by Lemma 13.5.11 and by the assumption that M is i-finite. Let $p \geq 0$. We prove the theorem for $\alpha = \beta + p\alpha_i$.

(a) Assume that $K_i^{\mathcal{N}}(\beta + p\alpha_i) \neq 0$. By (2), $K_i^{\mathcal{N}}(\beta + p\alpha_i) = V_{\lambda_p}$. Hence by Corollary 16.1.22, λ_p^{-1} is a weight of V, and $\dim V_{\lambda_p} = \dim V_{\lambda_p^{-1}}$. By (2), there is an integer $r \geq 0$ such that $V_{\lambda_p^{-1}} = K_i^{\mathcal{N}}(\beta + r\alpha_i)$, and

$$\lambda_p^{-1} = \chi_{\beta+r\alpha_i}(K_i)\chi_i(K_{\beta+r\alpha_i}) = \chi_\beta(K_i)\chi_i(K_\beta)q_{ii}^{2r}. \tag{16.2.7}$$

On the other hand, by Lemma 16.2.2,

$$\begin{aligned}\lambda_p^{-1} =& \chi_{\beta+p\alpha_i}(K_i)^{-1}\chi_i(K_{\beta+p\alpha_i})^{-1} \\ =& \chi_{s_i^M(\beta+p\alpha_i)}(K_i)\chi_i(K_{s_i^M(\beta+p\alpha_i)}).\end{aligned} \tag{16.2.8}$$

Let $t = -\sum_{j=1}^\theta b_j a_{ij} - p$. Then $s_i^M(\beta + p\alpha_i) = \beta + t\alpha_i$, and it follows from (16.2.8) that $\lambda_p^{-1} = \chi_\beta(K_i)\chi_i(K_\beta)q_{ii}^{2t}$. Since q_{ii} is not a root of 1, and $t \geq 0$ by Theorem 13.5.12(4), (16.2.7) implies $t = r$. Thus $\beta + r\alpha_i = s_i^M(\beta + p\alpha_i)$, and

$$\dim K_i^{\mathcal{N}}(\beta + p\alpha_i) = \dim V_{\lambda_p} = \dim V_{\lambda_p^{-1}} = \dim K_i^{\mathcal{N}}(s_i^M(\beta + p\alpha_i)).$$

(b) Assume that $K_i^{\mathcal{N}}(\beta+p\alpha_i) = 0$. Then $K_i^{\mathcal{N}}(s_i^M(\beta+p\alpha_i)) = 0$ by (a) applied to $K_i^{\mathcal{N}}(s_i^M(\beta + p\alpha_i))$ and since $(s_i^M)^2 = \mathrm{id}$. □

DEFINITION 16.2.4. Let $T(M) = T(M_1 \oplus \cdots \oplus M_\theta)$ be the tensor algebra as a Hopf algebra in ${}_H^H\mathcal{YD}$. By Proposition 4.3.12, the elements $(\mathrm{ad}_{T(M)}E_i)^{1-a_{ij}}(E_j)$, $i \neq j$, are primitive in $T(M)$. Hence the quotient algebra

$$U_{\boldsymbol{q}}^+ = T(M)/\big((\mathrm{ad}_{T(M)}E_i)^{1-a_{ij}^M}(E_j), 1 \leq i, j \leq \theta,\ i \neq j\big)$$

is a Hopf algebra in ${}_H^H\mathcal{YD}$. We also use the notation

$$U_{\boldsymbol{q}}^+ = \Bbbk\langle E_1, \ldots, E_\theta \mid (\mathrm{ad}\, E_i)^{1-a_{ij}}(E_j) = 0 \text{ for all } i, j \in \mathbb{I},\ i \neq j\rangle. \tag{16.2.9}$$

Note that $U_{\boldsymbol{q}}^+ = U(\mathcal{D})$, where $\mathcal{D} = \mathcal{D}(G, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ (see Definition 8.3.1).

An explicit form of the elements $(\operatorname{ad} E_i)^{1-a_{ij}}(E_j)$ of the tensor algebra $T(M)$ was given in Lemma 15.1.3.

THEOREM 16.2.5. *Let $\boldsymbol{q} = (q_{ij})_{i,j\in\mathbb{I}}$ be a family of non-zero elements in $\Bbbk$, and assume that $\boldsymbol{q}$ is quasi-generic and of Cartan type with Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$. Let G be an abelian group, $H = \Bbbk G$, $K_1, \dots, K_\theta \in G$, and $\chi_1, \dots, \chi_\theta \in \operatorname{Alg}(H, \Bbbk)$ such that $\chi_j(K_i) = q_{ij}$ for all $i, j \in \mathbb{I}$. For all $j \in \mathbb{I}$, let $M_j \in {}^H_H\mathcal{YD}$ be a one-dimensional object in ${}^H_H\mathcal{YD}$ and let $E_j \in M_j\setminus\{0\}$ satisfying* (16.2.1), *and let $M = (M_1, \dots, M_\theta)$.*

(1) *Let $\mathcal{N}$ be a pre-Nichols system of M such that $(\operatorname{ad}\mathcal{N}_i)^{1-a_{ij}}(\mathcal{N}_j) = 0$ for any $i, j \in \mathbb{I}$ with $i \neq j$. Then $\mathcal{N}$ admits all reflections.*
(2) *Assume that the Cartan matrix $(a_{ij})_{i,j\in\mathbb{I}}$ is of finite type. Then*

$$\mathcal{B}(M) \cong \Bbbk\langle E_1, \dots, E_\theta \mid (\operatorname{ad} E_i)^{1-a_{ij}}(E_j) = 0 \text{ for all } i, j \in \mathbb{I},\ i \neq j\rangle.$$

PROOF. (1) Let $\mathcal{N} = \mathcal{N}(S, N, f)$ and let $i \in \mathbb{I}$. As argued below Theorem 16.2.1, $\mathcal{N}$ is a Nichols system of (M, i). Then $R_i(\mathcal{N}) = \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f})$ is a Nichols system of (M, i) by Proposition 13.5.14. For all $j \in \mathbb{I}$ let $\widetilde{E}_j \in \widetilde{N}_j$.

By Lemma 15.1.8, the braiding matrix $(q'_{ij})_{i,j\in\mathbb{I}}$ of $R_i(M)$ satisfies

$$\begin{aligned} q'_{jj} &= q_{jj} && \text{for all } j \in \mathbb{I},\\ q'_{jk}q'_{kj} &= q_{jk}q_{kj} = {q'_{jj}}^{a_{jk}} && \text{for all } j, k \in \mathbb{I}. \end{aligned}$$

Hence it is enough to prove that

$$(\operatorname{ad}_{\widetilde{S}} \widetilde{E}_j)^{1-a_{jk}}(\widetilde{E}_k) = 0 \text{ for all } j, k \in \mathbb{I} \text{ with } i \neq j \neq k.$$

(We know already from Remark 13.5.15 that the same equation for $i = j \neq k$ holds.) We distinguish two cases.

(a) $j \neq i$, $k = i$.
(b) $j \neq i$, $k \neq i$, $j \neq k$.

(a) Let $j \in \mathbb{I}$ with $j \neq i$. If $q_{ii} = 1$ and $\operatorname{char}(\Bbbk) = 0$, then $q_{ij}q_{ji} = 1$ and $a_{ij} = a_{ji} = 0$. Hence $\operatorname{ad}_{\widetilde{S}} \widetilde{E}_i(\widetilde{E}_j) = 0$ as mentioned before and thus $\operatorname{ad}_{\widetilde{S}} \widetilde{E}_j(\widetilde{E}_i) = 0$.

Assume now that q_{ii} is not a root of unity. By Lemma 13.5.6 it is enough to show that for any $m \geq 0$, $\dim \widetilde{S}(\alpha_i + m\alpha_j) = \dim S(\alpha_i + m\alpha_j)$.

Let $m \geq 0$. We first claim that

$$\dim S(\alpha_i + m\alpha_j) = \dim K_i^{\mathcal{N}}(\alpha_i + m\alpha_j) + 1, \tag{16.2.10}$$

$$\dim \widetilde{S}(\alpha_i + m\alpha_j) = \dim \Omega(K_i^{\mathcal{N}})(\alpha_i + m\alpha_j) + 1. \tag{16.2.11}$$

Since $S \cong K_i^{\mathcal{N}} \# \Bbbk[E_i]$, we compute

$$\begin{aligned} \dim S(\alpha_i + m\alpha_j) &= \sum_{\gamma\in\mathbb{N}_0^\theta} \dim K_i^{\mathcal{N}}(\gamma) \cdot \dim \Bbbk[E_i](\alpha_i + m\alpha_j - \gamma)\\ &= \dim K_i^{\mathcal{N}}(\alpha_i + m\alpha_j) + \dim K_i^{\mathcal{N}}(m\alpha_j), \end{aligned}$$

where the last equality follows, since for any $\gamma \in \mathbb{N}_0^\theta$, the following are equivalent.

(i) $\dim \Bbbk[E_i](\alpha_i + m\alpha_j - \gamma) \neq 0$,
(ii) $\alpha_i + m\alpha_j - \gamma = t\alpha_i$ for some $t \geq 0$,
(iii) $\gamma = (1-t)\alpha_i + m\alpha_j$ with $t = 0$ or $t = 1$.

This finishes the proof of (16.2.10), since $\dim K_i^{\mathcal{N}}(m\alpha_j) = 1$, and (16.2.11) follows in the same way, since $\widetilde{S} = \Omega(K_i^{\mathcal{N}}) \# \Bbbk[E_i^*]$.

Now we can prove our claim.

$$\begin{aligned}
\dim \widetilde{S}(\alpha_i + m\alpha_j) &= \dim \Omega(K_i^{\mathcal{N}})(\alpha_i + m\alpha_j) + 1 && \text{(by (16.2.11))}\\
&= \dim K_i^{\mathcal{N}}(s_i^M(\alpha_i + m\alpha_j)) + 1 && \text{(by Thm. 13.5.12(4))}\\
&= \dim K_i^{\mathcal{N}}(\alpha_i + m\alpha_j) + 1 && \text{(by Thm. 16.2.3)}\\
&= \dim S(\alpha_i + m\alpha_j). && \text{(by (16.2.10))}
\end{aligned}$$

(b) Let $j, k \in \mathbb{I}$. Assume that i, j, k are pairwise distinct. Again it is enough to show that for all $m \geq 0$, $\dim \widetilde{S}(\alpha_k + m\alpha_j) = \dim S(\alpha_k + m\alpha_j)$. We argue as in (a).

$$\begin{aligned}
\dim \widetilde{S}(\alpha_k + m\alpha_j) &= \dim \Omega(K_i^{\mathcal{N}})(\alpha_k + m\alpha_j)\\
&= \dim K_i^{\mathcal{N}}(s_i^M(\alpha_k + m\alpha_j)) && \text{(by Thm. 13.5.12(4))}\\
&= \dim K_i^{\mathcal{N}}(\alpha_k + m\alpha_j) && \text{(by Thm. 16.2.3)}\\
&= \dim S(\alpha_k + m\alpha_j).
\end{aligned}$$

In fact, if $q_{ii} = 1$ and $\operatorname{char}(\Bbbk) = 0$, in the second last step we cannot use Theorem 16.2.3. However, then $a_{ij} = a_{ik} = 0$, and hence $s_i^M(\alpha_k + m\alpha_j) = \alpha_k + m\alpha_j$.

(2) Since the quantum Serre relations are homogeneous, $U_{\boldsymbol{q}}^+$ is $\mathbb{N}_0^\theta$-graded, where $\deg(E_i) = \alpha_i$ for all $i \in \mathbb{I}$. Hence $\mathcal{N} = \mathcal{N}(U_{\boldsymbol{q}}^+, M, \mathrm{id})$ is a pre-Nichols system of M and $(\operatorname{ad} \mathcal{N}_i)^{1-a_{ij}}(\mathcal{N}_j) = 0$ for all $i, j \in \mathbb{I}$ with $i \neq j$. By Theorem 15.1.14, the Cartan graph of M is finite. Hence (2) follows from (1) and Theorem 14.5.4. $\square$

We note that the second part of the above Theorem holds without the finiteness assumption on the Cartan matrix. However, the proof of the general case requires other techniques.

Remark 16.2.6. This remark is based on formulas and facts which are not proven in this book. It is intended to prove that Lusztig's root vectors form a root vector sequence in the sense of Definition 15.2.1.

Assume that $\Bbbk = \mathbb{Q}(v)$, A is of finite type, and $q_{ij} = v^{d_i a_{ij}}$ for all $i, j \in \mathbb{I}$, where $d_i \in \{1, 2, 3\}$ and $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in \mathbb{I}$. Let W be the Weyl group of A. In Section 1.1 and Theorem 3.1 in [**Lus90b**] Lusztig defines the quantized enveloping algebra $\mathbf{U}$ attached to the pair $(A, (d_i)_{i \in \mathbb{I}})$, and automorphisms T_i, $i \in \mathbb{I}$, of $\mathbf{U}$. We follow these definitions without spelling them out explicitly. Lusztig proves that for any reduced decomposition $\kappa = (i_1, \dots, i_l)$ of an element $w \in W$ and any $i \in \mathbb{I}$ with $w(\alpha_i) > 0$ the element $T_\kappa(E_i) = T_{i_1} \cdots T_{i_l}(E_i)$, called a root vector, is in the positive part $\mathbf{U}^+$ of $\mathbf{U}$. Moreover, $T_\kappa(E_i)$ is homogeneous of degree $w(\alpha_i)$ and does not depend on the choice of the reduced decomposition of w. Let us prove that the root vectors

$$E_{i_1},\, T_{i_1}(E_{i_2}),\, \dots,\, T_{i_1} T_{i_2} \cdots T_{i_{l-1}}(E_{i_l}) \tag{16.2.12}$$

for a reduced decomposition

$$\kappa = (i_1, \dots, i_l)$$

of an element $w \in W$ form a root vector sequence for κ in $\mathbf{U}^+$ in the sense of Definition 15.2.1. Note that the conditions on the degrees of the root vectors are satisfied. Moreover, Lusztig's root vectors satisfy Levendorskii-Soibelman type commutation relations as in Theorem 14.1.12, and hence their ordered products (in reverse ordering) form a subalgebra of $\mathbf{U}^+$. Let us write Δ for the (braided) comultiplication of $\mathbf{U}^+$. Then it remains to show for each $1 \le k \le l$ that

$$\Delta(E) - E \otimes 1 \in C_{k-1} \otimes \mathbf{U}^+, \tag{16.2.13}$$

where E is the k-th member of the sequence (16.2.12) and C_{k-1} is the subalgebra of $\mathbf{U}^+$ generated by the first $k-1$ members of the sequence (16.2.12). To do so, we use the braided commutators from Definition 6.2.16. Moreover, we may assume that the submatrix of A formed by the rows and columns $i_1, \dots, i_l$ is indecomposable. Recall from [**Lus90b**] the notation

$$[n]_d = \frac{v^{nd} - v^{-nd}}{v^d - v^{-d}}, \qquad [m]_d^! = \prod_{k=1}^m [k]_d$$

for all $n \in \mathbb{Z}$, $m \in \mathbb{N}_0$, and $d > 0$.

Before starting, it will be helpful to collect some formulas. Define for each $i, j \in \mathbb{I}$ with $i \ne j$ and for each $k \ge 0$ inductively

$$E_{i^0,j} = E_j, \qquad E_{i^{k+1},j} = [E_i, E_{i^k,j}]_c, \tag{16.2.14}$$

$$E_{j,i^0} = E_j, \qquad E_{j,i^{k+1}} = [E_{j,i^k}, E_i]_c. \tag{16.2.15}$$

In particular, we have $E_{i^1,j} = [E_i, E_j]_c$, $E_{i^2,j} = [E_i, [E_i, E_j]_c]_c$, $E_{j,i^1} = [E_j, E_i]_c$, and $E_{j,i^2} = [[E_j, E_i]_c, E_i]_c$. By induction on k one obtains that

$$\frac{(-1)^k}{[k]_{d_i}^!} E_{i^k,j} = \sum_{r+s=k} (-1)^r \frac{v^{d_i s(a_{ij}+k-1)}}{[r]_{d_i}^! [s]_{d_i}^!} E_i^r E_j E_i^s, \tag{16.2.16}$$

$$\frac{(-1)^k}{[k]_{d_i}^!} E_{j,i^k} = \sum_{r+s=k} (-1)^s \frac{v^{d_i r(a_{ij}+k-1)}}{[r]_{d_i}^! [s]_{d_i}^!} E_i^r E_j E_i^s \tag{16.2.17}$$

for all $i, j \in \mathbb{I}$ with $i \ne j$ and all $k \in \mathbb{N}_0$. Moreover,

$$F_i E_{i^k,j} - E_{i^k,j} F_i = [1 - a_{ij} - k]_{d_i} [k]_{d_i} E_{i^{k-1},j} K_i^{-1} \tag{16.2.18}$$

for all $i, j \in \mathbb{I}$ with $i \ne j$ and all $k \in \mathbb{N}_0$.

Setting $k = -a_{ij}$ in (16.2.16) one obtains that

$$T_i(E_j) = \frac{(-1)^{-a_{ij}}}{[-a_{ij}]_{d_i}^!} E_{i^{-a_{ij}},j}.$$

With this and (16.2.18) one obtains quickly by induction on k that

$$T_i(E_{j,i^k}) = \frac{(-1)^{-a_{ij}} [k]_{d_i}^!}{[-a_{ij} - k]_{d_i}^!} E_{i^{-a_{ij}-k},j} \tag{16.2.19}$$

for all $i, j \in \mathbb{I}$ with $i \ne j$ and for all $k \in \mathbb{N}_0$.

In order to check (16.2.13), we will also use the following formulas for $\Delta(E_{i^k,j})$ and $\Delta(E_{j,i^k})$, $k \geq 0$, which again can be obtained by induction on k:

$$\begin{aligned}\Delta(E_{i^k,j}) &= E_{i^k,j} \otimes 1 \\ &\quad + \sum_{r=0}^{k} v^{d_i r(k-r)} \prod_{s=r}^{k-1} (1 - v^{2d_i(a_{ij}+s)}) \frac{[k]^!_{d_i}}{[r]^!_{d_i}[k-r]^!_{d_i}} E_i^{k-r} \otimes E_{i^r,j},\end{aligned} \tag{16.2.20}$$

$$\begin{aligned}\Delta(E_{j,i^k}) &= 1 \otimes E_{j,i^k} \\ &\quad + \sum_{r=0}^{k} v^{d_i r(k-r)} \prod_{s=r}^{k-1} (1 - v^{2d_i(a_{ij}+s)}) \frac{[k]^!_{d_i}}{[r]^!_{d_i}[k-r]^!_{d_i}} E_{j,i^r} \otimes E_i^{k-r}.\end{aligned} \tag{16.2.21}$$

Step 1*: There exist* $i, j \in \mathbb{I}$, $i \neq j$, *with* $\{i_1, \dots, i_l\} \subseteq \{i, j\}$. Then either $a_{ij} = a_{ji} = 0$ or $a_{ij}a_{ji} \in \{1, 2, 3\}$. Moreover, in the second case we may assume that $a_{ji} = -1$ and $a_{ij} \in \{-1, -2, -3\}$. If $a_{ij} = -3$ then let $m = 6$, and let $m = 2 - a_{ij}$ otherwise. It suffices to look at the sequences $\kappa_1 = (i, j, i, j, \dots)$ and $\kappa_2 = (j, i, j, i, \dots)$ of length m.

Case 1.1*:* $a_{ij} = a_{ji} = 0$. Then $m = 2$, $T_i(E_j) = E_j$, and (16.2.13) is trivial.

Case 1.2*:* $a_{ij} = a_{ji} = -1$. Then $m = 3$ and the root vectors for κ_1 and κ_2 are

$$E_i, \quad T_i(E_j) = -[E_i, E_j]_c, \quad T_iT_j(E_i) = E_j, \tag{16.2.22}$$

and

$$E_j, \quad T_j(E_i) = -[E_j, E_i]_c, \quad T_jT_i(E_j) = E_i, \tag{16.2.23}$$

respectively, by (16.2.19). Then (16.2.13) follows for both sequences from (16.2.20).

Case 1.3*:* $a_{ij} = -2$, $a_{ji} = -1$, $m = 4$. The root vectors for κ_1 and κ_2 are

$$E_i, \quad T_i(E_j) = \frac{1}{[2]_{d_i}}[E_i, [E_i, E_j]_c]_c, \quad T_iT_j(E_i) = -[E_i, E_j]_c, \quad E_j,$$

and

$$E_j, \quad T_j(E_i) = -[E_j, E_i]_c, \quad T_jT_i(E_j) = \frac{1}{[2]_{d_i}}[[E_j, E_i]_c, E_i]_c, \quad E_i,$$

respectively, because of (16.2.19). Thus (16.2.13) for the root vectors in the first sequence follows from (16.2.20), and for the root vectors in the second sequence from (16.2.21).

Case 1.4*:* $a_{ij} = -3$, $a_{ji} = -1$, $m = 6$. The root vectors for κ_1 and κ_2 are

$$\begin{aligned} &E_i, \\ T_i(E_j) &= \frac{-1}{[3]^!_{d_i}} E_{i^3,j}, \\ T_iT_j(E_i) &= \frac{1}{[2]_{d_i}} E_{i^2,j}, \\ T_iT_jT_i(E_j) &= \frac{1}{[3]^!_{d_i}} [E_{i^2,j}, [E_i, E_j]_c]_c, \\ T_iT_jT_iT_j(E_i) &= -[E_i, E_j]_c, \\ &E_j, \end{aligned}$$

and

$$\begin{aligned}
&E_j,\\
T_j(E_i) =& -[E_j, E_i]_c,\\
T_jT_i(E_j) =& \frac{1}{[3]^!_{d_i}}[[E_j, E_i]_c, E_{j,i^2}]_c,\\
T_jT_iT_j(E_i) =& \frac{1}{[2]_{d_i}}E_{j,i^2},\\
T_jT_iT_jT_i(E_j) =& \frac{-1}{[3]^!_{d_i}}E_{j,i^3},\\
&E_i,
\end{aligned}$$

respectively, because of (16.2.19). Again, (16.2.13) follows from (16.2.20), (16.2.21), and an explicit calculation of the coproduct of the root vectors $T_1T_2T_1(E_2)$ and $T_2T_1(E_2)$.

Step 2: $\theta \geq 1$, $l \geq 2$, and there exists $0 \leq p \leq l-2$ such that $s_{i_1} \cdots s_{i_p}(\alpha_j) > 0$ for all $j \in \{i_l, i_{l-1}\}$, and $i_n \in \{i_l, i_{l-1}\}$ for all $p < n \leq l$. Assume that (16.2.13) holds for all sequences of length at most $p+1$. Let $\lambda = (i_1, \dots, i_p)$, $i = i_{p+1}$, and $j = i_{p+2}$. As above, for each $p \leq k \leq l$ let C_k be the subalgebra of $\mathbf{U}^+$ generated by the first k root vectors in (16.2.12). Then

$$\Delta(T_\lambda(E_i)) - T_\lambda(E_i) \otimes 1, \Delta(T_\lambda(E_j)) - T_\lambda(E_j) \otimes 1 \in C_p \otimes \mathbf{U}^+ \tag{16.2.24}$$

by assumption on the sequences $(i_1, \dots, i_p, i)$ and $(i_1, \dots, i_p, j)$, respectively. Moreover,

$$[x, T_\lambda(E_j)]_c \in C_p \otimes \mathbf{U}^+ \tag{16.2.25}$$

for all $x \in C_p$ by the Levendorskii-Soibelman type commutation relations, and hence

$$[x' \otimes x'', T_\lambda(E_j) \otimes 1]_c \in C_p \otimes \mathbf{U}^+ \tag{16.2.26}$$

for all $x' \otimes x'' \in C_p \otimes \mathbf{U}^+$.

Since type G_2 was already discussed in Step 1, we may assume additionally that $a_{ij}a_{ji} \in \{0, 1, 2\}$.

Case 2.1*: $a_{ij} = a_{ji} = 0$.* Then $l = p+2$ and $T_\lambda T_i(E_j) = T_\lambda(E_j)$. Thus (16.2.13) holds by (16.2.24).

Case 2.2*: $a_{ij} = a_{ji} = -1$.* Then $p+2 \leq l \leq p+3$. Let

$$E = -T_\lambda T_i(E_j) = [T_\lambda(E_i), T_\lambda(E_j)]_c.$$

Then

$$\begin{aligned}
\Delta(E) =& \Delta([T_\lambda(E_i), T_\lambda(E_j)]_c)\\
&\in [T_\lambda(E_i) \otimes 1 + C_p \otimes \mathbf{U}^+, T_\lambda(E_j) \otimes 1 + C_p \otimes \mathbf{U}^+]_c\\
&\subseteq [T_\lambda(E_i), T_\lambda(E_j)]_c \otimes 1 + [C_p \otimes \mathbf{U}^+, T_\lambda(E_j) \otimes 1]_c + C_{p+1} \otimes \mathbf{U}^+
\end{aligned}$$

by (16.2.24). Hence $\Delta(T_\lambda T_i(E_j)) - T_\lambda T_i(E_j) \otimes 1 \in C_{p+1} \otimes \mathbf{U}^+$ by (16.2.26).

Note that $T_\lambda T_i T_j(E_i) = T_\lambda(E_j)$. Thus, if $l = p+3$ then (16.2.13) holds for $k = l$ by (16.2.24).

Case 2.3*: $a_{ij} = -2$, $a_{ji} = -1$.* Then $p+2 \leq l \leq p+4$. Let

$$E' = -T_\lambda T_i T_j(E_i) = [T_\lambda(E_i), T_\lambda(E_j)]_c, \quad E'' = [2]_{d_i} T_\lambda T_i(E_j) = [T_\lambda(E_i), E']_c,$$

see Case 1.3. Thus

$$\begin{aligned}\Delta(E') =&\Delta([T_\lambda(E_i), T_\lambda(E_j)]_c)\\ &\in [T_\lambda(E_i)\otimes 1 + C_p\otimes \mathbf{U}^+, T_\lambda(E_j)\otimes 1 + C_p\otimes \mathbf{U}^+]_c\\ &\subseteq [T_\lambda(E_i), T_\lambda(E_j)]_c\otimes 1 + C_{p+1}\otimes \mathbf{U}^+,\\ \Delta(E'') =&\Delta([T_\lambda(E_i), E']_c)\\ &\in [T_\lambda(E_i)\otimes 1 + C_p\otimes \mathbf{U}^+, E'\otimes 1 + C_{p+1}\otimes \mathbf{U}^+]_c\\ &\subseteq [T_\lambda(E_i), E']_c\otimes 1 + C_{p+1}\otimes \mathbf{U}^+\end{aligned}$$

by (16.2.26). This implies (16.2.13) for $k\le p+3$. If $l=p+4$ then (16.2.13) holds for $k=l$ by (16.2.24), since $T_iT_jT_i(E_j)=E_j$.

Case 2.4*:* $a_{ij}=-1$, $a_{ji}=-2$. Then $p+2\le l\le p+4$. Let

$$E' = -T_\lambda T_i(E_j) = [T_\lambda(E_i), T_\lambda(E_j)]_c,\quad E'' = [2]_{d_i}T_\lambda T_iT_j(E_i) = [E', T_\lambda(E_j)]_c,$$

see Case 1.3. Thus

$$\begin{aligned}\Delta(E') =&\Delta([T_\lambda(E_i), T_\lambda(E_j)]_c)\\ &\in [T_\lambda(E_i)\otimes 1 + C_p\otimes \mathbf{U}^+, T_\lambda(E_j)\otimes 1 + C_p\otimes \mathbf{U}^+]_c\\ &\subseteq [T_\lambda(E_i), T_\lambda(E_j)]_c\otimes 1 + C_{p+1}\otimes \mathbf{U}^+,\\ \Delta(E'') =&\Delta([E', T_\lambda(E_j)]_c)\\ &\in [E'\otimes 1 + C_{p+1}\otimes \mathbf{U}^+, T_\lambda(E_j)\otimes 1 + C_p\otimes \mathbf{U}^+]_c\\ &\subseteq [E', T_\lambda(E_j)]_c\otimes 1 + C_{p+3}\otimes \mathbf{U}^+\end{aligned}$$

by (16.2.26), since $T_\lambda(E_j)=T_\lambda T_iT_jT_i(E_j)$. This implies (16.2.13) for $k\le p+3$. If $l=p+4$ then (16.2.13) holds for $k=l$ again by (16.2.24).

Step 3*: General setting.* Let $\theta\in\mathbb{N}$, $\mathbb{I}=\{1,2,\dots,\theta\}$, and proceed by induction on the length l of the sequence κ. The claim for $l\le 1$ is trivial.

Assume that $l\ge 2$ and that the claim is proven for elements of W of length at most $l-1$. Then, by induction hypothesis, it remains to prove (16.2.13) for $k=l$. Note that the algebra C_{l-1} in (16.2.13) generated by the first $l-1$ root vectors is independent of the choice of the reduced decomposition of $ws_{i_l}=s_{i_1}\cdots s_{i_{l-1}}$. Indeed, if $1\le k\le l-4$, $i_{k+2}=i_k$, $i_{k+3}=i_{k+1}$, and $s_{i_k}s_{i_{k+1}}s_{i_k}s_{i_{k+1}}=s_{i_{k+1}}s_{i_k}s_{i_{k+1}}s_{i_k}$, then (by Step 1, Case 3,) C_{l-1} is generated as an algebra by the root vectors $T_{i_1}\cdots T_{i_{n-1}}(E_{i_n})$ with $1\le n\le l-1$, $n\notin\{k+1,k+2\}$, and the same algebra is generated by the root vectors corresponding to the reduced decomposition

$$(i_1,\dots,i_{k-1},\, i_{k+1},i_k,i_{k+1},i_k,\, i_{k+4},\dots,i_{l-1}).$$

The argument for the other Coxeter relations is analogous by the other cases in Step 1.

By the previous paragraph, and by Kostant's decomposition of ws_{i_l}, see Corollary 9.4.17 with $J=\{i_l,i_{l-1}\}$, we may assume that there exists $0\le p\le l-2$ such that $i_n\in\{i_l,i_{l-1}\}$ whenever $p<n\le l$, and $s_{i_1}\cdots s_{i_p}(\alpha_j)>0$ for all $j\in\{i_l,i_{l-1}\}$. Then (16.2.13) for $k=l$ follows from Step 2.

Remark 16.2.7. We keep the notation of the previous remark. Let G be a free abelian group with basis $(K_i)_{i\in\mathbb{I}}$, and for all $j\in\mathbb{I}$, let $M_j=\Bbbk E_j\in {}^G_G\mathcal{YD}$ the one-dimensional object with $E_j\in (M_j)^{\chi_j}_{K_j}$, where $\chi_j(E_i)=q_{ij}=v^{d_ia_{ij}}$ for all

$i, j \in \mathbb{I}$. Then $\mathbf{U}^+ = \mathcal{B}(M)$ by Theorem 16.2.5. We want to relate the Hopf algebra isomorphism

$$T_i^{\mathcal{B}(M)} : L_i^{\mathcal{B}(R_i(M))} \to K_i^{\mathcal{B}(M)}, \quad i \in \mathbb{I},$$

defined in Definition 14.3.3, with the Lusztig automorphism $T_i : \mathbf{U} \to \mathbf{U}$.

Let $i \in \mathbb{I}$, and $E_i^* \in (\Bbbk E_i)^*$ with $E_i^*(E_i) = 1$. We have shown in [**HS13a**, Section 7] that there is an isomorphism $\varphi_i : \mathcal{B}(M) \to \mathcal{B}(R_i(M))$ of $\mathbb{N}_0$-graded algebras and coalgebras with

$$\varphi_i(E_j) = \begin{cases} \operatorname{ad}(E_i^{(-a_{ij})})E_j, & \text{if } j \neq i, \\ (v_i^{-3} - v_i^{-1})^{-1} E_i^*, & \text{if } j = i, \end{cases}$$

and an injective algebra map $\iota_i : K_i^{\mathcal{B}(M)} \# \mathcal{B}(M_i^*) \to \mathbf{U}$ such that the composition

$$\mathcal{B}(M) \xrightarrow{\varphi_i} \mathcal{B}(R_i(M)) \xrightarrow{\widetilde{\Theta}} K_i^{\mathcal{B}(M)} \# \mathcal{B}(M_i^*) \xrightarrow{\iota_i} \mathbf{U}$$

is the restriction of T_i to $\mathbf{U}^+ = \mathcal{B}(M)$. The Hopf algebra isomorphism $\widetilde{\Theta}$ is the map defined in Corollary 13.4.10. The restriction of φ_i defines an algebra isomorphism $\varphi_i : {}^{\operatorname{co}\mathcal{B}(M_i)}\mathcal{B}(M) \to {}^{\operatorname{co}\mathcal{B}(M_i^*)}\mathcal{B}(R_i(M)) = L_i^{\mathcal{B}(R_i(M))}$. The restriction of $\widetilde{\Theta}$ defines the isomorphism $T_i^{\mathcal{B}(M)} : L_i^{\mathcal{B}(R_i(M))} \to K_i^{\mathcal{B}(M)}$. The map ι_i restricted to $K_i^{\mathcal{B}(M)}$ is the inclusion $K_i^{\mathcal{B}(M)} \subseteq \mathcal{B}(M) \subseteq \mathbf{U}$. It follows that T_i defines by restriction an algebra isomorphism T_i^+ between the subalgebras ${}^{\operatorname{co}\mathcal{B}(M_i)}\mathcal{B}(M)$ and $\mathcal{B}(M)^{\operatorname{co}\mathcal{B}(M_i)}$ of $\mathbf{U}^+$ such that the following diagram commutes.

$$\begin{tikzcd}
{}^{\operatorname{co}\mathcal{B}(M_i)}\mathcal{B}(M) \arrow[rr, "T_i^+"] \arrow[dr, "\varphi_i"'] & & K_i^{\mathcal{B}(M)} = \mathcal{B}(M)^{\operatorname{co}\mathcal{B}(M_i)} \\
& L_i^{\mathcal{B}(R_i(M))} \arrow[ur, "T_i^{\mathcal{B}(M)}"'] &
\end{tikzcd}$$

16.3. On the structure of $u_{\boldsymbol{q}}^+$

In this section we study a setting similar to Section 16.2, however the braiding matrix is now non-generic. Let $\theta \geq 1$, $\mathbb{I} = \{1, \dots, \theta\}$, and let $\boldsymbol{q} = (q_{ij})_{i,j \in \mathbb{I}}$ be a family of non-zero elements in $\Bbbk$. We choose a realization of $\boldsymbol{q}$ as the braiding matrix of a Yetter-Drinfeld module. Let G be an abelian group, $H = \Bbbk G$ its group algebra, and let $K_1, \dots, K_\theta \in G$ and $\chi_1, \dots, \chi_\theta \in \operatorname{Alg}(H, \Bbbk)$ such that $\chi_j(K_i) = q_{ij}$ for all $i, j \in \mathbb{I}$. For all $j \in \mathbb{I}$, let $M_j \in {}^H_H\mathcal{YD}$ be a one-dimensional object in ${}^H_H\mathcal{YD}$ and let $E_j \in M_j \setminus \{0\}$ satisfying (16.2.1) for all $h \in H$. Let

$$M = (M_1, \dots, M_\theta)$$

and $V = \bigoplus_{i=1}^\theta M_i$. Then $V \in {}^H_H\mathcal{YD}$ and $(V, c_{V,V})$ is a braided vector space of diagonal type with braiding matrix $\boldsymbol{q}$, see Example 1.5.3 and Remark 1.5.4.

Assume that $q_{ii} \neq 1$ is a root of unity for all $i \in \mathbb{I}$ and that $\boldsymbol{q}$ is of Cartan type, that is, there is a Cartan matrix $A = (a_{ij})_{i,j \in \mathbb{I}}$ with

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}}, \text{ where } 0 \leq -a_{ij} < \operatorname{ord}(q_{ii}) \text{ for all } i \neq j. \tag{16.3.1}$$

In this case, $a_{ij} = a_{ij}^M$ for all $i, j \in \mathbb{I}$ by Lemma 15.1.12. Finally, we assume that the Cartan matrix A is of finite type. Then Lemma 8.2.4 applies. Moreover, M admits all reflections by Theorem 15.1.14, and $R_{i_1}(\cdots(R_{i_k}(M)))$ is of Cartan type with Cartan matrix A for all $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in \mathbb{I}$. Let W denote the Weyl group of A.

DEFINITION 16.3.1. Let $\boldsymbol{q} = (q_{ij})_{i,j \in \mathbb{I}}$ and $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta^H$ as above in the beginning of this section. We define

$$\begin{aligned} u_{\boldsymbol{q}}^+ &= \mathcal{B}(M), \\ U_{\boldsymbol{q}}^+ &= T(M)/\big((\mathrm{ad}_{T(M)} E_i)^{1-a_{ij}^M}(E_j) \mid i,j \in \mathbb{I}, i \neq j\big). \end{aligned}$$

The tensor algebra $T(M)$ is a Hopf algebra in ${}^H_H\mathcal{YD}$, and by Proposition 4.3.12, the elements $(\mathrm{ad}_{T(M)} E_i)^{1-a_{ij}^M}(E_j)$, $i \neq j$, are primitive in $T(M)$. Therefore $U_{\boldsymbol{q}}^+$ is a Hopf algebra in ${}^H_H\mathcal{YD}$. Moreover, $\mathcal{N}(U_{\boldsymbol{q}}^+, M, \mathrm{id})$ is a pre-Nichols system of M. In some settings, also the notation $U^+(M)$ will be used for $U_{\boldsymbol{q}}^+$.

REMARK 16.3.2. The Hopf algebra $U_{\boldsymbol{q}}^+$ is a variant of the positive part of the quantized enveloping algebra of the complex Lie algebra associated to the Cartan matrix A. The positive parts of the small quantum groups are special cases of $u_{\boldsymbol{q}}^+$, see Notes to Section 16.3. Our notation is very close to the usual notation in the theory of quantum groups. However, in our context the notations of $u_{\boldsymbol{q}}^+$ and $U_{\boldsymbol{q}}^+$ are somewhat sloppy, since the objects depend on the Yetter-Drinfeld module M rather than on the matrix $\boldsymbol{q}$. This is one of the reasons why we introduce two different notations. The second reason is that occasionally we will need the above construction for reflections of M. Note that the matrix $\boldsymbol{q}$ can be recovered from $M \in {}^H_H\mathcal{YD}$. Indeed, if e_i and e_j are basis vectors of M_i and M_j, respectively, where $i, j \in \mathbb{I}$, then

$$c_{M_i,M_j}(e_i \otimes e_j) = q_{ij} e_j \otimes e_i.$$

The braiding matrix $\boldsymbol{q}' = (q'_{jk})_{j,k \in \mathbb{I}}$ of $R_i(M)$ with $i \in \mathbb{I}$ was determined in Lemma 15.1.8(1).

By Example 1.10.1, $N(q)$ is the order of q for all $1 \neq q \in \Bbbk$ of finite order. Recall that for any $\alpha = \sum_{i \in \mathbb{I}} a_i \alpha_i$ in $\mathbb{Z}^\theta$, $g_\alpha = \prod_{i \in \mathbb{I}} g_i^{a_i} \in G, \chi_\alpha = \prod_{i \in \mathbb{I}} \chi_i^{a_i} \in \widehat{G}$, and $q_{\alpha\alpha} = \chi_\alpha(g_\alpha)$. Since $\boldsymbol{q}$ is of Cartan type, it is easy to see that for all $\alpha \in \mathbb{Z}^\theta$ and all $i \in \mathbb{I}$,

$$q_{\alpha\alpha} = q_{s_i(\alpha)s_i(\alpha)}. \tag{16.3.2}$$

THEOREM 16.3.3. *Let $\boldsymbol{q} = (q_{ij})_{i,j \in \mathbb{I}}$ and W be as above in the beginning of this section. Let $(i_1, \dots, i_t)$ be a reduced decomposition of the longest element of W. Then there is a root vector sequence $x_1, \dots, x_t$ in $\mathcal{B}(M)$ for $(i_1, \dots, i_t)$, and the elements*

$$x_t^{n_t} \cdots x_1^{n_1}, \quad 0 \leq n_k < N(q_{i_k i_k}) \text{ for all } 1 \leq k \leq t,$$

form a basis of $u_{\boldsymbol{q}}^+ = \mathcal{B}(M)$.

PROOF. By Theorem 15.1.14, M admits all reflections. Hence by Proposition 15.2.6, there is a root vector sequence $x_1, \dots, x_t$ in $\mathcal{B}(M)$ for $(i_1, \dots, i_t)$. Let $\kappa = (i_1, \dots, i_t)$, and for all $1 \leq k \leq t$, let $\beta_k = \beta_k^{[M],\kappa}$. By (16.3.2), $q_{i_k i_k} = q_{\beta_k \beta_k}$ for all k, since M is of Cartan type. Hence the claim on the basis of $u_{\boldsymbol{q}}^+$ follows from Theorem 15.2.7. □

The relevance of $U_{\boldsymbol{q}}^+$ is indicated already by the following lemma.

LEMMA 16.3.4. *Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M and for all $j \in \mathbb{I}$ let $e_j \in N_j \setminus \{0\}$. Then (Sys4) holds for all $i \in \mathbb{I}$ in S if and only if $(\mathrm{ad}_S e_i)^{1-a_{ij}}(e_j) = 0$ for all $i, j \in \mathbb{I}$ with $i \neq j$.*

PROOF. Assume that (Sys4) holds in S for all $i \in \mathbb{I}$. Since the element $(\mathrm{ad}_{T(M)} E_i)^{1-a_{ij}}(E_j)$ is primitive in $T(M)$ by Proposition 4.3.12 for any $i, j \in \mathbb{I}$ with $i \neq j$, it is mapped to zero in $\mathcal{B}(M)$ by the canonical map. Therefore (Sys4) implies that $(\mathrm{ad}_S e_i)^{1-a_{ij}}(e_j) = 0$ in S for all $i, j \in \mathbb{I}$ with $i \neq j$.

Conversely, assume that $(\mathrm{ad}_S e_i)^{1-a_{ij}}(e_j) = 0$ for all $i, j \in \mathbb{I}$ with $i \neq j$. Since $\mathcal{N}$ is a pre-Nichols system of M, $e_j \neq 0$ for any $j \in \mathbb{I}$. Moreover,

$$(\mathrm{ad}_S N_i)^m(N_j) = \Bbbk(\mathrm{ad}_S e_i)^m(e_j)$$

for all $i, j \in \mathbb{I}$ with $i \neq j$ and for all $m \in \mathbb{N}_0$. Now

$$\Delta_{1,m}\big((\mathrm{ad}_S e_i)^m(e_j)\big) = (m)_{q_{ii}}(1 - q_{ii}^{m-1} q_{ij} q_{ji}) e_i \otimes (\mathrm{ad}_S e_i)^{m-1}(e_j)$$

by Proposition 4.3.12 for all $i, j \in \mathbb{I}$ with $i \neq j$ and for all $m > 0$. Therefore $(\mathrm{ad}_S e_i)^m(e_j) \neq 0$ in S for all $i, j \in \mathbb{I}$ with $i \neq j$ and for all $0 \leq m \leq -a_{ij}$. Hence (Sys4) holds in S for all $i \in \mathbb{I}$ because of $a_{ij}^M = a_{ij}$ for all $i, j \in \mathbb{I}$. $\square$

REMARK 16.3.5. Let $i \in \mathbb{I}$. By Example 1.10.1 and since $q_{ii} \neq 1$, the validity of (Sys3) for a pre-Nichols system $\mathcal{N}$ of M is equivalent to $e_i^n = 0$, where $e_i \in \mathcal{N}_i \setminus \{0\}$ and $n = \mathrm{ord}(q_{ii})$.

Our goal in this section is to provide a basis of $U_{\boldsymbol{q}}^+$, see Theorem 16.3.14, and to define the Nichols algebra quotient $u_{\boldsymbol{q}}^+$ of $U_{\boldsymbol{q}}^+$ by generators and relations, see Theorem 16.3.17. Our claims require an additional technical assumption on the set $\{q_{ii} \mid i \in \mathbb{I}, d_i = 1\}$ which leads us to the notion of a braiding matrix which is genuinely of finite Cartan type. The necessity of this assumption is discussed at the end of Remark 16.3.19.

We say that $\boldsymbol{q}$ is **genuinely of finite Cartan type** if for all $i \in \mathbb{I}$ with $d_i = 1$ one of the following holds:

(1) the component containing i has a Cartan matrix of type A_1, A_2, or B_2,
(2) $\mathrm{ord}(q_{ii}) \geq 3$, and the component containing i has a Cartan matrix of type A_θ, $\theta \geq 3$, or D_θ, $\theta \geq 4$, or E_θ with $6 \leq \theta \leq 8$,
(3) $\mathrm{ord}(q_{ii}) \geq 5$, and the component containing i has a Cartan matrix of type B_θ, $\theta \geq 3$, or C_θ, $\theta \geq 3$, or F_4,
(4) $\mathrm{ord}(q_{ii}) \notin \{1, 2, 3, 4, 6\}$, and the component containing i has a Cartan matrix of type G_2.

By Lemma 8.2.4, the scalars q_{ii} with $d_i = 1$ depend only on the component containing i. Hence the above conditions have to be checked only once for each component.

The definition of a braiding matrix of genuinely finite Cartan type has a technical interpretation in Lemma 16.3.7 below which will be crucial for the proof of the main results of this section. The next lemma will be used to prove this interpretation.

LEMMA 16.3.6. *Let $j \in \mathbb{I}$, $i, k \in \mathbb{I} \setminus \{j\}$, and $b_{ijk} = a_{ij}a_{jk} - a_{ij} - a_{ik} \in \mathbb{Z}$. Then the following hold.*

(1) *$b_{ijk} \geq 0$ if and only if $i \neq k$ or $i = k$, $a_{ij} < 0$.*
(2) *$b_{ijk} \leq \max\{-a_{ij}, -a_{ik}\}$ except the following cases:*
   – *$i = k$, $a_{ij}a_{ji} = 3$; then $b_{ijk} = 1 - a_{ij}$.*

– *$i \neq k$, $a_{ij} = -1$, $a_{ik} < 0$; then $b_{ijk} = 1 - a_{ik}$.*
– *$i \neq k$, $a_{ij} = -1$, $a_{ik} = 0$, $a_{jk} < 0$; then $b_{ijk} = 1 - a_{jk}$.*
– *$i \neq k$, $a_{ij} = -2$, $a_{jk} = 0$, $a_{ik} = -1$; then $b_{ijk} = 3$.*
– *$i \neq k$, $a_{ij} = -2$, $a_{jk} = -1$, $a_{ik} = 0$; then $b_{ijk} = 4$.*

PROOF. (1) If $i = k$ then $b_{ijk} = a_{ij}a_{ji} - a_{ij} - 2$. If moreover $a_{ij} = 0$, then $b_{ijk} = -2$, and otherwise $a_{ij}, a_{ji} \leq -1$ and $b_{ijk} \geq 0$.

If $i \neq k$ then $a_{ij}, a_{jk}, a_{ik} \leq 0$ and hence $b_{ijk} \geq 0$.

(2) Assume first that $i = k$. Then

$$b_{ijk} = a_{ij}a_{ji} - a_{ij} - 2 \leq 3 - a_{ij} - 2 = 1 - a_{ij}$$

since A is of finite type. Moreover, $b_{ijk} = 1 - a_{ij}$ if and only if $a_{ij}a_{ji} = 3$.

Assume now that $i \neq k$. If $a_{ij} = 0$ then $b_{ijk} = -a_{ik}$ and we are done. If $a_{ij} = -3$ then $a_{ik} = a_{jk} = 0$ since A is of finite type. In this case, $b_{ijk} = -a_{ij}$ and the lemma is again proven. If $a_{ij} = -2$ then $a_{ik} + a_{jk} \in \{0, -1\}$ since A is of finite type. Hence $\max\{-a_{ij}, -a_{ik}\} = 2$ and $b_{ijk} = -2a_{jk} + 2 - a_{ik}$. If $a_{jk} = -1$ then $b_{ijk} = 4$, if $a_{ik} = -1$ then $b_{ijk} = 3$, and if $a_{jk} = a_{ik} = 0$ then $b_{ijk} = 2$.

Assume now that $i \neq k$ and $a_{ij} = -1$. Then $b_{ijk} = -a_{jk} - a_{ik} + 1$. Since A is of finite type, we conclude that $a_{ik}a_{jk} = 0$ and $a_{ik}, a_{jk} \in \{0, -1, -2\}$. If $a_{ik} < 0$ then $a_{jk} = 0$ and $b_{ijk} = 1 - a_{ik}$. Finally, if $a_{ik} = 0$ then $b_{ijk} = 1 - a_{jk}$. Hence $b_{ijk} \leq -a_{ij}$ if and only if $a_{jk} = 0$. □

LEMMA 16.3.7. *For all $i, j, k \in \mathbb{I}$ with $i \neq j$ and $j \neq k$ let*

$$b_{ijk} = a_{ij}a_{jk} - a_{ij} - a_{ik}.$$

Then $\boldsymbol{q}$ is genuinely of finite Cartan type if and only if $\operatorname{ord}(q_{ii}) > b_{ijk}$ for all $i, j, k \in \mathbb{I}$ with $i \neq j$ and $j \neq k$.

PROOF. Assume first that $\operatorname{ord}(q_{ii}) > b_{ijk}$ for all $i, j, k \in \mathbb{I}$ with $i \neq j$, $j \neq k$. We show that $\boldsymbol{q}$ is genuinely of finite Cartan type. Let $i \in \mathbb{I}$ with $d_i = 1$ and let $\hat{A}$ be the Cartan matrix of the component of i.

Assume that $\hat{A}$ is of type G_2. Let j be the second entry of the component of i. Then

$$b_{iji} = 1 - a_{ij} = 4, \quad b_{jij} = 1 - a_{ji} = 2$$

and hence $\operatorname{ord}(q_{ii}) > 4$ and $\operatorname{ord}(q_{jj}) > 2$. Since $q_{jj} = q_{ii}^3$, we conclude that $\operatorname{ord}(q_{ii}) \neq 6$.

Assume that $\hat{A}$ is of type A_m, $m \geq 3$, or D_m, $m \geq 4$, or E_m, $m \in \{6, 7, 8\}$. Let j, k be two other entries in the component of i such that $a_{ij} = -1$ and $a_{ik} + a_{jk} = -1$. Then $b_{ijk} = 2$ and hence $\operatorname{ord}(q_{ii}) > 2$.

Assume that $\hat{A}$ is of type B_m, $m \geq 3$. There are unique entries j, k in the component of i such that $d_j = d_k = 2$ and $a_{ij} = -2$, $a_{jk} = -1$, $a_{ik} = 0$. Then $b_{ijk} = 4$ and hence $\operatorname{ord}(q_{ii}) \geq 5$.

Assume that $\hat{A}$ is of type C_m, $m \geq 3$, or F_4. There are unique entries l, j, k in the component of i such that $d_l = 1$, $d_j = 1$, $d_k = 2$, $a_{lj} = -1$, $a_{jk} = -2$, $a_{lk} = 0$. Then $b_{ljk} = 3$ and hence $\operatorname{ord}(q_{ii}) \geq 4$. Moreover, $b_{kjl} = 2$ and hence $\operatorname{ord}(q_{kk}) > 2$. Since $q_{kk} = q_{ii}^2$, we conclude that $\operatorname{ord}(q_{ii}) \geq 5$. Thus $\boldsymbol{q}$ is genuinely of finite Cartan type.

We proved the first half of the claim. To proceed with the other half, assume that $\boldsymbol{q}$ is genuinely of finite Cartan type. We have to show that $\operatorname{ord}(q_{ii}) > b_{ijk}$ for all $i, j, k \in \mathbb{I}$ with $i \neq j$, $j \neq k$. We assumed already below Equation (16.3.1) that

$0 \le -a_{il} < \mathrm{ord}(q_{ii})$ for all $i, l \in \mathbb{I}$ with $i \ne l$. Therefore we only have to consider the triples (i, j, k) with

$$b_{ijk} > \max\{-a_{ij}, -a_{ik}\}. \tag{16.3.3}$$

These are described in detail in Lemma 16.3.6.

Let $(i, j, k) \in \mathbb{I}^3$ with $i \ne j$, $j \ne k$, such that (16.3.3) holds. Lemma 16.3.6 implies that i, j, k belong to the same component. Let $\hat{A}$ be the Cartan matrix of this component. Again from Lemma 16.3.6 we conclude that the type of $\hat{A}$ is none of A_1, A_2, B_2. Moreover, $i = k$ if and only if $\hat{A}$ is of type G_2.

Assume that $\hat{A}$ is of type G_2. Then $k = i$. If $d_i = 1$ then $\mathrm{ord}(q_{ii}) \notin \{1, 2, 3, 4, 6\}$ by assumption, $b_{iji} = 4$, and hence $\mathrm{ord}(q_{ii}) > b_{iji}$. On the other hand, if $d_i = 3$ then $d_j = 1$, $a_{ij} = -1$, $q_{ii} = q_{jj}^3$, $\mathrm{ord}(q_{jj}) \notin \{3, 6\}$ and hence $\mathrm{ord}(q_{ii}) > 2 = b_{iji}$.

Assume that $\hat{A}$ is of type A_m, $m \ge 3$, or D_m, $m \ge 4$, or E_m, $m \in \{6, 7, 8\}$. Then $i \ne k$, $a_{ij} = -1$, $a_{ik} + a_{jk} = -1$ and $b_{ijk} = 2$ by Lemma 16.3.6. Moreover, $\mathrm{ord}(q_{ii}) \ge 3$ since $\boldsymbol{q}$ is genuinely of finite Cartan type. Therefore $\mathrm{ord}(q_{ii}) > b_{ijk}$.

Assume that $\hat{A}$ is of type B_m, $m \ge 3$, or C_m, $m \ge 3$, or F_4. If $d_i = 1$ then $\mathrm{ord}(q_{ii}) \ge 5$ by assumption, $b_{ijk} \le 4$ by Lemma 16.3.6, and hence $\mathrm{ord}(q_{ii}) > b_{ijk}$. On the other hand, if $d_i = 2$ then $a_{ij} = -1$, $d_j = 1$, $a_{ji} = -2$, and hence $a_{ik}a_{jk} = 0$, $a_{ik}, a_{jk} \in \{0, -1\}$, and $b_{ijk} = 2$ by Lemma 16.3.6. Moreover, $\mathrm{ord}(q_{ii}) > 2$ by assumption. Thus $\mathrm{ord}(q_{ii}) > b_{ijk}$. This finishes the proof of the lemma. $\square$

Recall the definition of $R_i(\mathcal{N})$ from Definition 13.5.13, where $i \in \mathbb{I}$ and $\mathcal{N}$ is a Nichols system of (M, i). The following Proposition is fundamental for the definition and study of $u_{\boldsymbol{q}}^+$.

PROPOSITION 16.3.8. *Assume that $\theta \ge 2$ and that the braiding matrix $\boldsymbol{q}$ is genuinely of finite Cartan type. Let $i \in \mathbb{I}$, $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system of (M, i) and let $R_i(\mathcal{N}) = \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f})$. Let $j, k \in \mathbb{I}$ with $j \ne k$ and let $x_j \in N_j \setminus \{0\}$, $x_k \in N_k \setminus \{0\}$, $y_j \in \widetilde{N}_j$, and $y_k \in \widetilde{N}_k$. Assume that $(\mathrm{ad}_S x_j)^{1-a_{jk}}(x_k) = 0$. Then $(\mathrm{ad}_{\widetilde{S}} y_j)^{1-a_{jk}}(y_k) = 0$.*

PROOF. Assume first that $j = i$. By Proposition 13.5.14, $R_i(\mathcal{N})$ is a Nichols system of $(R_i(M), i)$. Since $a_{ik}^{R_i(M)} = a_{ik}^M = a_{ik}$ by Proposition 13.5.19(2), we conclude from (Sys4) that $(\mathrm{ad}_{\widetilde{S}} y_j)^{1-a_{jk}}(y_k) = 0$.

Secondly, assume that $i = k$ and $a_{ij} = 0$. Then $a_{ji} = 0$. Let q'_{ij}, q'_{ji} be as in Lemma 15.1.13. Then $q'_{ij}q'_{ji} = 1$ since $a_{ij} = 0$. Hence

$$\mathrm{ad}_{\widetilde{S}} y_j(y_i) = y_j y_i - q'_{ji} y_i y_j = -q'_{ji}(y_i y_j - q'_{ij} y_j y_i) = -q'_{ji} \mathrm{ad}_{\widetilde{S}} y_i(y_j),$$

and $\mathrm{ad}_{\widetilde{S}} y_i(y_j) = 0$ by the last paragraph. Therefore the Proposition is proven in this case.

Assume for the rest of the proof that $i \ne j$ and that $a_{ij} < 0$ if $i = k$. If $i \ne k$ then let $x_i \in N_i \setminus \{0\}$. Let $\mathcal{N}_{\mathrm{ini}} = \mathcal{N}(\widehat{S}, \widehat{N}, \widehat{f})$ be a Nichols system of (M, i) as in Proposition 13.5.24. Let $p : \mathcal{N}_{\mathrm{ini}} \to \mathcal{N}$ be the unique morphism from Proposition 13.5.24. We identify x_i, x_j, and x_k with their unique preimage in $\widehat{N}_i$, $\widehat{N}_j$, and $\widehat{N}_k$, respectively, with respect to p. Let

$$s_{jk} = (\mathrm{ad}_{\widehat{S}} x_j)^{1-a_{jk}}(x_k) \in \widehat{S}(\alpha_k + (1 - a_{jk})\alpha_j).$$

Then $s_{jk} \in \ker(p)$ by assumption.

Next we show that $s_{jk} \neq 0$. By the construction in Proposition 13.5.24, the defining ideal of $\widehat{S}$ consists of elements of degree $m\alpha_i$ for some $m \geq 2$ and of elements of degree $\alpha_l + (1 - a_{il})\alpha_i$, $l \in \mathbb{I} \setminus \{i\}$. Since $k \neq i$ or $k = i$, $a_{ij} < 0$, we conclude that for any $m \geq 0$ the elements

$$x_j^n x_k x_j^{m-n}, \quad 0 \leq n \leq m,$$

form a basis in $\widehat{S}(\alpha_k + m\alpha_j)$. Hence, $s_{jk} \neq 0$ in $\widehat{S}$.

By Proposition 4.3.12, $s_{jk} \in \widehat{S}$ is primitive. Further, $j \neq k$ implies that $\alpha_k + (1 - a_{jk})\alpha_j \notin \mathbb{N}_0\alpha_i$, and hence

$$s_{jk} \in \widehat{S}^{\mathrm{co}\,\Bbbk[N_i]}(\alpha_k + (1 - a_{jk})\alpha_j).$$

Recall the definition of $\Bbbk_{\mathrm{red}}[x; \chi_i, K_i]$ from Section 16.1. There is a unique injective Hopf algebra map $\varphi_i : \Bbbk_{\mathrm{red}}[x; \chi_i, K_i] \to \widehat{S}\#H$ which is the identity on H and sends x to x_i. Clearly, $\varphi_i(\Bbbk_{\mathrm{red}}[x; \chi_i, K_i]) = \Bbbk[N_i]\#H$. Since $\widehat{S}^{\mathrm{co}\,\Bbbk[N_i]} \in {}^{\Bbbk[N_i]\#H}_{\Bbbk[N_i]\#H}\mathcal{YD}$, we may regard $\widehat{S}^{\mathrm{co}\,\Bbbk[N_i]}$ as a Yetter-Drinfeld module over $\Bbbk_{\mathrm{red}}[x; \chi_i, K_i]$ via φ_i. Since s_{jk} is primitive in $\widehat{S}$ and

$$\delta_{\widehat{S}}(s_{jk}) = K_j^{1-a_{jk}} K_k \otimes s_{jk}, \quad K \cdot s_{jk} = \chi_j^{1-a_{jk}} \chi_k(K) s_{jk}$$

for all $K \in G$, we conclude from Proposition 4.5.1(2) that there is a unique morphism F_i in ${}^{\Bbbk_{\mathrm{red}}[x;\chi_i,K_i]}_{\Bbbk_{\mathrm{red}}[x;\chi_i,K_i]}\mathcal{YD}$ from $\Bbbk_{\mathrm{red}}[x; \chi_i, K_i] \otimes_H \Bbbk s_{jk}$ to $\widehat{S}^{\mathrm{co}\,\Bbbk[N_i]}$ which sends s_{jk} to s_{jk}.

We record that

$$\begin{aligned}
\chi_i(K_j^{1-a_{jk}} K_k)^{-1} =& q_{ji}^{a_{jk}-1} q_{ki}^{-1} \\
=& (q_{ij}q_{ji})^{a_{jk}-1}(q_{ik}q_{ki})^{-1} q_{ij}^{1-a_{jk}} q_{ik} \\
=& q_{ii}^{a_{ij}a_{jk}-a_{ij}-a_{ik}} q_{ij}^{1-a_{jk}} q_{ik}, \\
K_i \cdot s_{jk} =& q_{ij}^{1-a_{jk}} q_{ik} s_{jk} \\
=& \chi_i(K_j^{1-a_{jk}} K_k)^{-1} q_{ii}^{a_{ij}+a_{ik}-a_{ij}a_{jk}} s_{jk},
\end{aligned}$$

and hence $\Bbbk s_{jk}$ is a weight vector of weight $q_{ii}^{-b_{ijk}}$ in the sense of Definition 16.1.8, where

$$b_{ijk} = a_{ij}a_{jk} - a_{ij} - a_{ik}.$$

Note that $b_{ijk} \geq 0$ by Lemma 16.3.6(1), since either $i \neq k$ or $i = k$, $a_{ij} < 0$ by assumption. Moreover, $\mathrm{ord}(q_{ii}) > b_{ijk}$ by Lemma 16.3.7 since $\boldsymbol{q}$ is genuinely of finite Cartan type. Therefore Proposition 16.1.30 implies that

$$0 \neq F_i(x^{b_{ijk}} \otimes s_{jk}) = (\mathrm{ad}_{\widehat{S}} x_i)^{b_{ijk}}(s_{jk})$$

in $\widehat{S}^{\mathrm{co}\,\Bbbk[N_i]}$. Moreover, $(\mathrm{ad}_{\widehat{S}} x_i)^{b_{ijk}}(s_{jk}) \in \ker(p)$ since $s_{jk} \in \ker(p)$, and hence $\ker(p) \cap \widehat{S}^{\mathrm{co}\,\Bbbk[N_i]}(\alpha_k + (1 - a_{jk})\alpha_j + b_{ijk}\alpha_i)$ is non-zero. Note that

$$s_i(\alpha_k + (1 - a_{jk})\alpha_j + b_{ijk}\alpha_i) = \alpha_k + (1 - a_{jk})\alpha_j.$$

Thus $\ker(R_i(p))$ contains a non-zero element in degree $\alpha_k + (1 - a_{jk})\alpha_j$ by Theorem 13.5.12(4) and Lemma 13.5.27(1). This and Lemma 13.5.6 imply the claim. □

Proposition 16.3.8 and Lemma 16.3.4 imply directly the following claim.

COROLLARY 16.3.9. *Assume that* $\boldsymbol{q}$ *is genuinely of finite Cartan type. Let* $k \in \mathbb{I}$ *and let* $\mathcal{N}$ *be a Nichols system of* (M,k) *for which (Sys4) holds for all* $i \in \mathbb{I}$. *Then (Sys4) holds for* $R_k(\mathcal{N})$ *for all* $i \in \mathbb{I}$.

We also conclude an important information about reflections of particular Nichols systems.

COROLLARY 16.3.10. *Assume that* $\boldsymbol{q}$ *is genuinely of finite Cartan type. Let* $k \in \mathbb{I}$ *and let* $n = \operatorname{ord}(q_{kk})$. *Let* J *be the (Hopf) ideal of* $U_{\boldsymbol{q}}^+$ *generated by* E_k^n.

(1) $\mathcal{N} = \mathcal{N}(U_{\boldsymbol{q}}^+/J, M, \mathrm{id})$ *is a Nichols system of* (M,k).
(2) *Let* $e_k \in M_k^* \setminus \{0\}$ *and let* J' *be the (Hopf) ideal of* $U^+(R_k(M))$ *generated by* e_k^n. *Then the Nichols systems* $R_k(\mathcal{N})$ *and*
$$\mathcal{N}(U^+(R_k(M))/J', R_k(M), \mathrm{id})$$
of $(R_k(M), k)$ *are isomorphic.*

PROOF. (1) Since E_k^n is homogeneous and primitive by Proposition 2.4.2(5), the ideal J is a Hopf ideal in ${}^H_H\mathcal{YD}$ and a graded subsapce of S in the sense of the definition in Section 5.1. Moreover, $J \cap \bigoplus_{i\in\mathbb{I}} M_i = 0$, and hence $\mathcal{N}$ is a pre-Nichols system of M. Finally, (Sys4) holds for k by Lemma 16.3.4, and (Sys3) is valid by Remark 16.3.5.

(2) Let $\mathcal{N}'' = \mathcal{N}(U^+(R_k(M))/J'', R_k(M), \mathrm{id})$. Then $\mathcal{N}''$ is a Nichols system of $(R_k(M), k)$ for the same reason as for $\mathcal{N}$. Let $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_k(\mathcal{N})$. Note that $\boldsymbol{q}$ is genuinely of finite Cartan type by assumption. Hence (Sys4) holds in $\widetilde{S}$ for any $i \in \mathbb{I}$ by Corollary 16.3.9. Moreover, (Sys3) holds in $\widetilde{S}$ by construction. Hence there is a morphism $p : \mathcal{N}'' \to R_k(\mathcal{N})$ of Nichols systems of $(R_k(M), k)$.

By Proposition 13.5.19, $R_k^2(M)$ and M are isomorphic in $\mathcal{F}_\theta^H$. Hence there is an isomorphism $f : U^+(M) \to U^+(R_k^2(M))$ of $\mathbb{N}_0^\theta$-graded Hopf algebras in ${}^H_H\mathcal{YD}$. Let $\mathcal{N}' = \mathcal{N}(U^+(R_k^2(M))/f(J), R_k^2(M), \mathrm{id})$. By the arguments of the previous paragraph there exists a morphism $p' : \mathcal{N}' \to R_k(\mathcal{N}'')$ of Nichols systems of $(R_k^2(M), k)$. The composition
$$R_k(p)p' : \mathcal{N}' \to R_k^2(\mathcal{N})$$
is an isomorphism by Proposition 13.5.25. Since p' is surjective, $R_k(p)$ is an isomorphism, and then so is p. $\square$

PROPOSITION 16.3.11. *Assume that* $\boldsymbol{q}$ *is genuinely of finite Cartan type. Let* $\mathcal{N} = \mathcal{N}(S, N, f)$ *be a pre-Nichols system of* M *such that axiom (Sys4) holds for all* $1 \le i \le \theta$. *Let* $w \in W$ *and* $\kappa = (i_1, \dots, i_t)$ *be a reduced decomposition of* w, *where* $t = \ell(w)$.

(1) *There exists a root vector sequence for* κ *in* S.
(2) *Let* $x_1, \dots, x_t$ *and* $y_1, \dots, y_t$ *be root vector sequences for* κ *in* S. *Then there exist* $\lambda_1, \dots, \lambda_t \in \Bbbk^\times$ *with* $y_l = \lambda_l x_l$ *for all* $1 \le l \le t$.

PROOF. We proceed by induction on t. For $t = 0$ the claim is trivial. Assume that $t \ge 1$ and that the Proposition holds for all words of length at most $t-1$. Let J be the Hopf ideal of S generated by x_1^n, where $x_1 \in N_{i_1} \setminus \{0\}$ and $n = \operatorname{ord}(q_{i_1 i_1})$. Then $\mathcal{N}' = \mathcal{N}(S/J, N, f)$ is a Nichols system of (M, i_1). Let $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_{i_1}(\mathcal{N}')$. Because of Corollary 16.3.9, (Sys4) holds in $\widetilde{S}$ for all $i \in \{1, \dots, \theta\}$. Since $(i_2, \dots, i_t)$ is a reduced decomposition of $s_{i_1} w$, by induction hypothesis there exists a root vector sequence $x_2, \dots, x_t$ for $(i_2, \dots, i_t)$ in $\widetilde{S}$. Moreover, any other root vector sequence for $(i_2,\dots,i_t)$ in $\widetilde{S}$ is of the form $\lambda_2 x_2,\dots,\lambda_t x_t$ with $\lambda_2,\dots,\lambda_t \in \Bbbk^\times$. (We call this uniqueness up to scaling.) Then from Proposition 15.2.5 it follows that there exists a root vector sequence $x'_1,\dots,x'_t$ for $(i_1,\dots,i_t)$ in S/J. Moreover, this root vector sequence is unique up to scaling because of Proposition 15.2.4 and since $\dim N_{i_1}=1$. By Lemma 15.2.3, $x'_l \in N_{i_1} \cup K_{i_1}^{\mathcal{N}'}$ for any $1\le l\le t$. Thus (1) and (2) hold by Proposition 15.2.9(2) and by Lemma 15.2.3(1). □

COROLLARY 16.3.12. *Assume that $\boldsymbol{q}$ is genuinely of finite Cartan type. Let $p:\mathcal{N}\to\mathcal{N}'$ be a morphism of pre-Nichols systems of M, where*

$$\mathcal{N}=\mathcal{N}(S,N,f),\quad \mathcal{N}'=\mathcal{N}(S',N',f')$$

such that (Sys4) holds in S for any $i\in\{1,\dots,\theta\}$. Let $\kappa=(i_1,\dots,i_t)$ be a reduced decomposition of an element $w\in W$, where $t=\ell(w)$. Then for any root vector sequence $x'_1,\dots,x'_t$ for κ in S', there is a unique root vector sequence $x_1,\dots,x_t$ for κ in S such that $p(x_l)=x'_l$ for all $1\le l\le t$. Moreover, $\Bbbk x_l\cong \Bbbk x'_l$ for all $1\le l\le t$ in ${}^H_H\mathcal{YD}$.

PROOF. Since (Sys4) holds in S for all i, it also holds in S' by Lemma 16.3.4. Let $x'_1,\dots,x'_t$ be a root vector sequence for $(i_1,\dots,i_t)$ in S', and let $(\tilde{x}_1,\dots,\tilde{x}_t)$ be a root vector sequence for $(i_1,\dots,i_t)$ in S. These exist by Proposition 16.3.11(1). By (1), $p(\tilde{x}_1),\dots,p(\tilde{x}_t)$ is a root vector sequence for $(i_1,\dots,i_t)$ in S', too. Hence by Proposition 16.3.11(2) there exist $\lambda_1,\dots,\lambda_t\in\Bbbk^\times$ such that $x'_l=\lambda_l p(\tilde{x}_l)$ for all $1\le l\le t$. Let $x_l=\lambda_l\tilde{x}_l$ for all $1\le l\le t$. Then $x_1,\dots,x_t$ is the desired root vector sequence for $(i_1,\dots,i_t)$ in S (see Remark 15.2.2(1)). The uniqueness follows again from Proposition 16.3.11(2). The last claim of the Corollary follows from Remark 15.1.2 by degree reasons, since p is a morphism in ${}^H_H\mathcal{YD}$. □

LEMMA 16.3.13. *Assume that the braiding matrix $\boldsymbol{q}$ is genuinely of finite Cartan type. Let $\mathcal{N}=\mathcal{N}(S,N,f)$ be a pre-Nichols system of M such that (Sys4) holds for any $i\in\{1,\dots,\theta\}$. Let $\kappa=(i_1,\dots,i_t)$ be a reduced decomposition of an element $w\in W$, where $t=\ell(w)$. Let $x_1,\dots,x_t$ be a root vector sequence for κ in S, and for all $1\le j\le t$ let b_j be the multiplicative order of $q_{i_j i_j}$. Then the ideal J of S generated by $x_1^{b_1},\dots,x_t^{b_t}$ is a graded Hopf ideal in ${}^H_H\mathcal{YD}$.*

In the Lemma, we consider S as an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ and J as a graded subspace of S as introduced in Section 5.1.

PROOF. Induction on t. If $t=0$, then the claim is trivial. Assume that $t=1$. Then $x_1^{b_1}$ is primitive by Proposition 2.4.2(5), and $\Bbbk x_1^{b_1}$ is a graded subspace of S in ${}^H_H\mathcal{YD}$. This implies the claim.

Assume now that $t\ge 2$ and that the claim holds for all words in W of length at most $t-1$. Let J be the ideal of S generated by $x_1^{b_1}$. Then J is a subobject of S in ${}^H_H\mathcal{YD}$, a graded subspace of S and has trivial intersection with N_i for any $1\le i\le\theta$. Hence $\mathcal{N}'=\mathcal{N}(S/J,N,f)$ is a Nichols system of (M,i_1), and $x_1,\dots,x_t$ is a root vector sequence for $(i_1,\dots,i_t)$ in S/J by Remark 15.2.2(2). Let

$$\mathcal{N}(\widetilde{S},\widetilde{N},\widetilde{f})=R_{i_1}(\mathcal{N}').$$

Because of Corollary 16.3.9, (Sys4) holds in $\widetilde{S}$ for any $i\in\{1,\dots,\theta\}$. Moreover, $T_{i_1}^{-1}(x_2),\dots T_{i_1}^{-1}(x_t)$ is a root vector sequence for $(i_2,\dots,i_t)$ in $\widetilde{S}$ by Proposition 15.2.4. Hence the ideal $\widetilde{J}$ of $\widetilde{S}$ generated by $T_{i_1}^{-1}(x_l)^{b_l}=T_{i_1}^{-1}(x_l^{b_l})$ with

$2 \le l \le t$ is a Hopf ideal of $\widetilde{S}$ by induction hypothesis. Since

$$T_{i_1}^{-1}(x_l^{b_l}) \in L_{i_1}^{R_{i_1}(\mathcal{N}')}$$

for any $2 \le l \le t$, Proposition 13.5.29 implies that there is a morphism $p : \mathcal{N}' \to \mathcal{N}''$ of Nichols systems of (M, i_1) such that $\ker(p)$ is generated by $x_l^{b_l}$, $2 \le l \le t$. This yields the claim. □

THEOREM 16.3.14. *Assume that $\boldsymbol{q}$ is genuinely of finite Cartan type. Let $x_1, \dots, x_t$, $t \in \mathbb{N}_0$, be a root vector sequence in $U_{\boldsymbol{q}}^+$ for a reduced decomposition $(i_1, \dots, i_t)$ of an element $w \in W$ with $\ell(w) = t$.*

(1) *The elements*

$$x_t^{n_t} \cdots x_1^{n_1}, \quad n_1, \dots, n_t \in \mathbb{N}_0,$$

form a vector space basis of the (right coideal) subalgebra of $U_{\boldsymbol{q}}^+$ generated by $x_1, \dots, x_t$.

(2) *Assume that w is the longest element of W. Then the elements*

$$x_t^{n_t} \cdots x_1^{n_1}, \quad n_1, \dots, n_t \in \mathbb{N}_0,$$

form a vector space basis of $U_{\boldsymbol{q}}^+$.

REMARK 16.3.15. Assume that $q_{ij} = \epsilon^{d_i a_{ij}}$ for some root ϵ of 1. Then, using the observation in Remark 16.2.6, one can specialize Lusztig's root vectors in the generic case to get a root vector sequence in $U_{\boldsymbol{q}}^+$.

PROOF OF THEOREM 16.3.14. (1) Induction on t. If $t = 0$, then the claim is trivial.

Assume that $t \ge 1$. Let $i = i_1$, let n be the multiplicative order of q_{ii}, and let J be the (Hopf) ideal of $U_{\boldsymbol{q}}^+$ generated by x_1^n. Then $\mathcal{N} = \mathcal{N}(U_{\boldsymbol{q}}^+, M, \mathrm{id})$ is a pre-Nichols system of M and $\overline{\mathcal{N}} = \mathcal{N}(U_{\boldsymbol{q}}^+/J, M, \mathrm{id})$ is a Nichols system of (M, i). Let $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_i(\overline{\mathcal{N}})$. Corollary 16.3.10 implies that there is a morphism

$$p : \mathcal{N}(U^+(R_i(M)), R_i(M), \mathrm{id}) \to R_i(\overline{\mathcal{N}})$$

with $\ker(p) = (E_i^n)$. Moreover, $x_1, \dots, x_t$ is a root vector sequence for $(i_1, \dots, i_t)$ in $U_{\boldsymbol{q}}^+/J$ by Remark 15.2.2(2). Hence $T_i^{-1}(x_2), \dots, T_i^{-1}(x_t)$ is a root vector sequence for $(i_2, \dots, i_t)$ in $\widetilde{S}$ by Proposition 15.2.4, and by Corollary 16.3.12 there is a unique root vector sequence $y_2, \dots, y_t$ for $(i_2, \dots, i_t)$ in $U^+(R_i(M))$ with $p(y_l) = T_i^{-1}(x_l)$ for all $2 \le l \le t$. In particular, the monomials $y_t^{n_t} \cdots y_2^{n_2}$ with $n_2, \dots, n_t \in \mathbb{N}_0$ form a vector space basis of a right coideal subalgebra C of $U^+(R_i(M))$ by induction hypothesis. Note that

$$C \subseteq {}^{\mathrm{co}\,\Bbbk[M_i^*]}U^+(R_i(M))$$

and that $J \cap {}^{\mathrm{co}\,\Bbbk[M_i^*]}U^+(R_i(M)) = 0$, by Lemma 15.2.8(1) and using that J is a Hopf ideal. In particular, $p|C$ is injective. Hence the monomials $T_i^{-1}(x_t^{n_t}) \cdots T_i^{-1}(x_2^{n_2})$ with $n_2, \dots, n_t \in \mathbb{N}_0$ form a vector space basis of the right coideal subalgebra $p(C)$ of $\widetilde{S}$. Since $p(C) \subseteq L_i^{R_i(\overline{N})}$, the monomials $x_t^{n_t} \cdots x_2^{n_2} x_1^{n_1}$ with $n_1, n_2, \dots, n_t \in \mathbb{N}_0$, $n_1 < n$, form a vector space basis of the right coideal subalgebra $T_i(C)\Bbbk[x_1]$ of $U_{\boldsymbol{q}}^+/J$ by Theorem 12.4.5. This and Proposition 15.2.9 imply the claim.

(2) follows directly from (1), since the subalgebra of $U_{\boldsymbol{q}}^+$ generated by $x_1, \dots, x_t$ contains a non-zero element of degree α_i for any $1 \le i \le \theta$, and hence it coincides with $U_{\boldsymbol{q}}^+$. □

We also have a variant of Theorem 14.1.12 for $U_{\boldsymbol{q}}^+$.

THEOREM 16.3.16. *Assume that $\boldsymbol{q}$ is genuinely of finite Cartan type. Let $x_1, \dots, x_t$, $t \in \mathbb{N}_0$, be a root vector sequence in $U_{\boldsymbol{q}}^+$ for a reduced decomposition $(i_1, \dots, i_t)$ of an element of W. Then for any $1 \le i < j \le t$,*

$$x_i x_j - (g \cdot x_j)x_i \in \Bbbk[x_{j-1}] \cdots \Bbbk[x_{i+1}],$$

where $g \in G(H)$ such that $\delta_{U_{\boldsymbol{q}}^+}(x_i) = g \otimes x_i$.

PROOF. Let $\kappa = (i_1, \dots, i_t)$, $1 \le i < j \le t$, $y = x_i x_j - (g \cdot x_j)x_i$, and let $C_j = \Bbbk[x_j] \cdots \Bbbk[x_1]$. Let $K = (U_{\boldsymbol{q}}^+)^{\mathrm{co}\,\Bbbk[x_1]}$ with respect to the $\mathbb{N}_0^\theta$-graded projection $\pi : U_{\boldsymbol{q}}^+ \to \Bbbk[x_1]$. We prove by induction on i that $y \in \Bbbk[x_{j-1}] \cdots \Bbbk[x_{i+1}]$.

Assume first that $i = 1$. By assumption, C_j is a right coideal subalgebra of $U_{\boldsymbol{q}}^+$. In particular, $y \in C_j$. Moreover, $x_j \in K$ by Lemma 15.2.3 and hence $y \in K$. This and Lemma 15.2.3 imply that y is a linear combination of the monomials $x_j^{n_j} \cdots x_2^{n_2}$ with $n_2, \dots, n_j \ge 0$ and

$$\sum_{l=2}^{j} n_l \beta_l^{[M],\kappa} = \alpha_{i_1} + \beta_j^{[M],\kappa}.$$

Thus $y \in \Bbbk[x_{j-1}] \cdots \Bbbk[x_2]$ by degree reasons.

Assume that $i \ge 2$. Then $x_i, x_j \in K$, and hence y is a linear combination of the monomials $x_j^{n_j} \cdots x_2^{n_2} \in K$ with $n_2, \dots, n_j \ge 0$ by Lemma 15.2.3. Let J be the Hopf ideal of $U_{\boldsymbol{q}}^+$ generated by x_1^n with $n = \mathrm{ord}(q_{i_1 i_1})$, and let $p : U_{\boldsymbol{q}}^+ \to U_{\boldsymbol{q}}^+/J$ be the canonical map. Then $p : \mathcal{N} \to \mathcal{N}'$ is a morphism of pre-Nichols systems of M, where

$$\mathcal{N} = \mathcal{N}(U_{\boldsymbol{q}}^+, M, \mathrm{id}_M), \quad \mathcal{N}' = \mathcal{N}(U_{\boldsymbol{q}}^+/J, M, \mathrm{id}_M).$$

Moreover, $p(x_1), \dots, p(x_t)$ is a root vector sequence for κ in $U_{\boldsymbol{q}}^+/J$ by Remark 15.2.2 and Lemma 15.2.8. By the same references it suffices to show that $p(y)$ is contained in $\Bbbk[p(x_{j-1})] \cdots \Bbbk[p(x_{i+1})]$.

Let $T_{i_1} = T_{i_1}^{\mathcal{N}'}$. By Corollary 16.3.10, $\mathcal{N}'$ is a Nichols system of (M, i_1). Hence, by Proposition 15.2.4, $T_{i_1}^{-1}(p(x_2)), \dots, T_{i_1}^{-1}(p(x_t))$ is a root vector sequence for $(i_2, \dots, i_t)$ in $\widetilde{S}$, where $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_{i_1}(\mathcal{N}')$. Moreover, Corollary 16.3.10(2) implies that there is a morphism

$$p' : \mathcal{N}(U^+(R_{i_1}(M)), R_{i_1}(M), \mathrm{id}) \to R_{i_1}(\mathcal{N}').$$

By Corollary 16.3.12 there is a root vector sequence $y_2, \dots, y_t$ for $(i_2, \dots, i_t)$ in $U^+(R_{i_1}(M))$ such that $p'(y_l) = T_{i_1}^{-1}(p(x_l))$ and $\Bbbk y_l \cong \Bbbk T_{i_1}^{-1}(p(x_l))$ in ${}_H^H\mathcal{YD}$ for any $2 \le l \le t$. By induction hypothesis,

$$y_i y_j - (g \cdot y_j)y_i \in \Bbbk[y_{j-1}] \cdots \Bbbk[y_{i+1}].$$

Since p' and $T_{i_1}^{-1}$ are algebra maps, this implies the claim. $\square$

THEOREM 16.3.17. *Assume that $\boldsymbol{q}$ is genuinely of finite Cartan type. Let $x_1, \dots, x_t$, $t \in \mathbb{N}_0$, be a root vector sequence in $U_{\boldsymbol{q}}^+$ for a reduced decomposition $(i_1, \dots, i_t)$ of the longest element of W. Let J be the ideal of $U_{\boldsymbol{q}}^+$ generated by $x_1^{b_1}, \dots, x_t^{b_t}$, where for any $1 \le j \le t$, $b_j = \mathrm{ord}(q_{i_j i_j})$. Then $U_{\boldsymbol{q}}^+/J$ is isomorphic as a Hopf algebra in ${}_H^H\mathcal{YD}$ to the Nichols algebra $u_{\boldsymbol{q}}^+ = \mathcal{B}(M)$.*

Proof. Let $p' : U_{\boldsymbol{q}}^+ \to U_{\boldsymbol{q}}^+/J$ and $p : U_{\boldsymbol{q}}^+/J \to \mathcal{B}(M)$ be the canonical maps. By Remark 15.2.2(2) and by Theorem 15.2.7, the monomials $pp'(x_t)^{n_t} \cdots pp'(x_1)^{n_1}$ with $0 \le n_k < \operatorname{ord}(q_{i_k i_k})$ for all $1 \le k \le t$ form a vector space basis of $\mathcal{B}(M)$. Indeed, $q_{i_k i_k} = q_{\beta_k \beta_k}$ for all $1 \le k \le t$ in Theorem 15.2.7, since M is of Cartan type. Moreover, the monomials $p'(x_t)^{n_t} \cdots p'(x_1)^{n_1}$ with $0 \le n_k < \operatorname{ord}(q_{i_k i_k})$ for all $1 \le k \le t$ span $U_{\boldsymbol{q}}^+/J$ by Theorem 16.3.14(2). Hence $\dim U_{\boldsymbol{q}}^+/J \le \dim \mathcal{B}(M)$. Moreover, J contains no primitive elements of degree 1. Hence $U_{\boldsymbol{q}}^+/J \cong \mathcal{B}(M)$. $\square$

Remark 16.3.18. The relations $x_j^{b_j} = 0$, $1 \le j \le t$, in $u_{\boldsymbol{q}}^+$ are usually called the **root vector relations**.

In other approaches to $u_{\boldsymbol{q}}^+$ in the literature one constructs root vector sequences explicitly and uses certain normalization to achieve uniqueness. In our approach the root vector sequences are only unique up to scaling and are defined by characterizing properties instead of ad hoc constructions.

Remark 16.3.19. Angiono determines for any finite-dimensional Nichols algebra of diagonal type over a field of characteristic 0 the defining relations. His result implies (whenever $\operatorname{char}(\Bbbk) = 0$) that $\mathcal{B}(M)$ is the quotient of $U_{\boldsymbol{q}}^+$ by root vector relations if and only if $\boldsymbol{q}$ is genuinely of finite Cartan type.

Remark 16.3.20. In the literature there exist various definitions of (plus parts of) quantum groups at roots of unity ϵ, mostly under some restrictions on the order of ϵ. A usual way is to take an integral form and specialize it to ϵ. Another way is to write down the (Hopf) algebra by generators and relations. Interestingly, it seems that before the study of Nichols algebras of diagonal type by generators and relations it was unnoticed that the Lusztig automorphisms are not well-defined in the second approach for particular, very small orders of ϵ, that is, if the braiding matrix $(\epsilon^{d_i a_{ij}})_{i,j \in \mathbb{I}}$ is not genuinely of finite Cartan type. This concerns among others the examples of type B_θ and C_θ, $\theta \ge 3$, at third roots of 1.

16.4. A characterization of Nichols algebras of finite Cartan type

Our aim in this section is to discuss pre-Nichols systems where the braided Hopf algebra is a domain of finite Gelfand-Kirillov dimension generated by one-dimensional Yetter-Drinfeld modules. In Theorem 16.4.23 we relate these braided Hopf algebras to $U_{\boldsymbol{q}}^+$. As a special case, we provide in Corollary 16.4.24 a characterization of finite-dimensional braided vector spaces V of diagonal type such that the Nichols algebra of V is a domain of finite Gelfand-Kirillov dimension.

Recall that a ring R is a domain if $ab \neq 0$ for any $a, b \in R \setminus \{0\}$. In this section we consider algebras in the category of vector spaces over the field $\Bbbk$. After some preliminaries we will prove in Propositions 16.4.5 and 16.4.6 that $U_{\boldsymbol{q}}^+$ is a domain.

Lemma 16.4.1. *Let A be an algebra with a filtration $\mathcal{F}(A) = (F_\alpha(A))_{\alpha \in \mathbb{N}_0}$. If $\operatorname{gr} A$ is a domain, then A is a domain.*

Proof. Assume that $\operatorname{gr} A$ is a domain. Let $a, b \in A \setminus \{0\}$. Let $m, n \in \mathbb{N}_0$ such that $a \in F_m(A) \setminus F_{m-1}(A)$ and $b \in F_n(A) \setminus F_{n-1}(A)$. Then

$$ab + F_{m+n-1}(A) = (a + F_{m-1}(A))(b + F_{n-1}(A)) \neq 0$$

since $\operatorname{gr} A$ is a domain. Hence $ab \neq 0$. $\square$

Ore extensions have been discussed in Remark 2.6.14.

LEMMA 16.4.2. *Let A be a domain. Then any Ore extension $A[x;\sigma,\delta]$ with $\sigma \in \mathrm{Aut}(A)$ is a domain.*

PROOF. Let σ be an automorphism of A and let $\delta : A \to A$ be a (σ, id_A)-derivation. By Remark 2.6.14, the elements of $A[x;\sigma,\delta]$ are polynomials of the form $\sum_{i=0}^n a_i x^i$ with $n \geq 0$ and $a_0, \dots, a_n \in A$. By (2.6.5).

$$x^k a - \sigma^k(a) x^k \in \sum_{i=0}^{k-1} A x^i$$

for any $k \geq 0$ and $a \in A$. An element $\sum_{i=0}^n a_i x^i = 0$ with $a_0, \dots, a_n \in A$ in $A[x;\sigma,\delta]$ is zero if and only if $a_i = 0$ for all $0 \leq i \leq n$. Let now

$$\bar{a} = \sum_{i=0}^m a_i x^i, \quad \bar{b} = \sum_{j=0}^n b_j x^j \in A[x;\sigma,\delta]$$

with $a_m, b_n \neq 0$. Then $\bar{a}\bar{b} - a_m \sigma^m(b_n) x^{m+n} \in \sum_{i=0}^{m+n-1} A x^i$. Since σ is invertible, $a_m, b_n \neq 0$, and A is a domain, it follows that $a_m \sigma^m(b_n) \neq 0$ and hence $\bar{a}\bar{b} \neq 0$. ☐

PROPOSITION 16.4.3. *Let $n \in \mathbb{N}_0$ and for any $1 \leq j < i \leq n$ let $q_{ij} \in \Bbbk^\times$. Then the algebra*

$$\mathcal{Q}_{\boldsymbol{q}}[x_1, \dots, x_n] = \Bbbk\langle x_1, \dots, x_n \rangle / (x_i x_j - q_{ij} x_j x_i \mid 1 \leq j < i \leq n)$$

of **quantum polynomials**, *where $\boldsymbol{q} = (q_{ij})_{1 \leq j < i \leq n}$, is a domain and the monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $m_1, \dots, m_n \in \mathbb{N}_0$ form a basis of $\mathcal{Q}_{\boldsymbol{q}}[x_1, \dots, x_n]$.*

PROOF. Let us write $\mathcal{Q}_{\boldsymbol{q}}$ for $\mathcal{Q}_{\boldsymbol{q}}[x_1, \dots, x_n]$. Let A denote the polynomial ring $\Bbbk[X_1, \dots, X_n]$. For any $1 \leq i \leq n$ let $\xi_i \in \mathrm{End}(A)$ such that

$$\xi_i(X_1^{m_1} \cdots X_n^{m_n}) = \left(\prod_{j=1}^{i-1} q_{ij}^{m_j} \right) X_i X_1^{m_1} \cdots X_n^{m_n}$$

for any $1 \leq i \leq n$ and $m_1, \dots, m_n \in \mathbb{N}_0$. Then $\xi_i \xi_j(a) = q_{ij} \xi_j \xi_i(a)$ for any $a \in A$ and $1 \leq j < i \leq n$. Thus there is a unique algebra map $\rho : \mathcal{Q}_{\boldsymbol{q}} \to \mathrm{End}(A)$ with $\rho(x_i) = \xi_i$ for any $1 \leq i \leq n$. Since

$$\rho(x_1^{m_1} \cdots x_n^{m_n})(1) = \xi_1^{m_1} \cdots \xi_n^{m_n}(1) = X_1^{m_1} \cdots X_n^{m_n},$$

we conclude that the elements $x_1^{m_1} \cdots x_n^{m_n}$ with $m_1, \dots, m_n \in \mathbb{N}_0$ are linearly independent in $\mathcal{Q}_{\boldsymbol{q}}$. Hence it follows from the defining relations of $\mathcal{Q}_{\boldsymbol{q}}$ that these elements form a basis of $\mathcal{Q}_{\boldsymbol{q}}$.

The defining relations of $\mathcal{Q}_{\boldsymbol{q}}$ imply that for any $k_1, \dots, k_n, l_1, \dots, l_n \in \mathbb{N}_0$ there exists $\lambda \in \Bbbk^\times$ such that

$$x_1^{k_1} \cdots x_n^{k_n} x_1^{l_1} \cdots x_n^{l_n} = \lambda x_1^{k_1+l_1} \cdots x_n^{k_n+l_n}. \tag{16.4.1}$$

For any

$$a = \sum_{m_1, \dots, m_n \geq 0} a_{m_1, \dots, m_n} x_1^{m_1} \cdots x_n^{m_n} \in \mathcal{Q}_{\boldsymbol{q}} \setminus \{0\},$$

let $N(a)$ denote the set of all tuples $(m_1, \dots, m_n) \in \mathbb{N}_0^n$ such that $a_{m_1, \dots, m_n} \neq 0$, and let

$$\mathrm{lm}(a) = x_1^{t_1} \cdots x_n^{t_n},$$

where $(t_1, \dots, t_n) \in N(a)$ is maximal with respect to the total order on $\mathbb{N}_0^n$ introduced in Example 5.2.1. Axiom (M2) in Section 5.2, which is valid for the total order above, and Equation (16.4.1) imply that $\mathrm{lm}(ab) = \mathrm{lm}(a)\mathrm{lm}(b)$ for any $a, b \in \mathcal{Q}_{\boldsymbol{q}} \setminus \{0\}$. Hence $\mathcal{Q}_{\boldsymbol{q}}$ is a domain. $\square$

Since $\mathcal{Q}_{\boldsymbol{q}}$ is an iterated Ore extension, where the skew derivation is zero in each step, another proof of Proposition 16.4.3 can be given using Lemma 16.4.2.

LEMMA 16.4.4. *Let A be an algebra, $l \geq 0$ and $y_1, \dots, y_l \in A$. Assume that the elements $y_l^{n_l} \cdots y_1^{n_1}$ with $n_1, \dots, n_l \geq 0$ form a basis of A. Let $h_1, \dots, h_l$ be positive integers and let $\lambda_{ij} \in \Bbbk^\times$ for all $1 \leq i < j \leq l$. Assume that for any $1 \leq i < j \leq l$, $y_i y_j - \lambda_{ij} y_j y_i$ is a linear combination of monomials $y_l^{n_l} \cdots y_{i+1}^{n_{i+1}}$ such that $n_{i+1}, \dots, n_l \geq 0$ and $h_{i+1}n_{i+1} + \cdots + h_l n_l \leq h_i + h_j$. Then A is a domain.*

PROOF. The main idea of the proof is to use the filtration introduced in the proof of Corollary 14.1.13.

Let $\Gamma = \mathbb{N}_0^l$ together with the weighted lexicographic ordering $<$:

$$\begin{aligned}(k_1, \dots, k_l) < (m_1, \dots, m_l) \Leftrightarrow\ & h_1 k_1 + \cdots h_l k_l < h_1 m_1 + \cdots h_l m_l \text{ or}\\ & h_1 k_1 + \cdots h_l k_l = h_1 m_1 + \cdots + h_l m_l,\ k_1 = m_1, \dots,\\ & k_{i-1} = m_{i-1},\ k_i < m_i \text{ for some } 1 \leq i \leq l.\end{aligned}$$

Then Γ is a totally ordered abelian monoid satisfying axioms (M1) and (M2) in Section 5.2.

We introduce a filtration $\mathcal{F}$ of A by Γ. For any $\alpha \in \Gamma$, let $F_\alpha(A)$ be the span of all monomials $y_{j_1} \cdots y_{j_m}$ with $m \geq 0$ and $j_1, \dots, j_m \in \{1, \dots, l\}$, such that $(n_1, \dots, n_l) \leq \alpha$, where for any $1 \leq k \leq l$ the number n_k counts the appearances of k in $(j_1, \dots, j_m)$. Then $\mathcal{F}$ is an algebra filtration because of Axiom (M2) for Γ.

By assumption, in the graded algebra $\mathrm{gr}\, A$ associated to the filtration $\mathcal{F}$ of A the relation

$$y_i y_j = \lambda_{ij} y_j y_i \tag{16.4.2}$$

holds for any $1 \leq i < j \leq l$. Let $Q = \mathcal{Q}_\lambda[x_1, \dots, x_l]$, where $\lambda = (\lambda_{ji}^{-1})_{1 \leq j < i \leq l}$. For any $\alpha \in \mathbb{N}_0^l$ let $F_\alpha(Q)$ be the linear span of all monomials $x_l^{m_l} \cdots x_1^{m_1}$ with $(m_1, \dots, m_l) \leq \alpha$. The elements $x_l^{n_l} \cdots x_1^{n_1}$ with $n_1, \dots, n_l \geq 0$ form a basis of Q_λ by Proposition 16.4.3. Since the elements $y_l^{n_l} \cdots y_1^{n_1}$ with $n_1, \dots, n_l \geq 0$ form a basis of A, there is an isomorphism f of the filtered vector spaces Q and A sending any monomial $x_l^{m_l} \cdots x_1^{m_1}$ to $y_l^{m_l} \cdots y_1^{m_1}$. Thus $\mathrm{gr}\, f : Q \to \mathrm{gr}\, A$ is an isomorphism. Moreover, $\mathrm{gr}\, f$ is an algebra map by (16.4.2). Then $\mathrm{gr}\, A$ is a domain by Proposition 16.4.3. Hence A is a domain by Lemma 16.4.1. $\square$

Recall the definition of $U_{\boldsymbol{q}}^+$ for quasi-generic $\boldsymbol{q}$ from (16.2.9).

PROPOSITION 16.4.5. *Let M and $\boldsymbol{q}$ be as in Section 16.2. Assume that $\boldsymbol{q}$ is quasi-generic and of finite Cartan type. Then $U_{\boldsymbol{q}}^+$ is a domain.*

PROOF. By Theorem 16.2.5, $U_{\boldsymbol{q}}^+ \cong \mathcal{B}(M)$. Thus it suffices to show that $\mathcal{B}(M)$ is a domain.

Let A be the Cartan matrix of finite type such that $\boldsymbol{q}$ is of Cartan type with Cartan matrix A. Let w_0 be the longest element of the Weyl group of A. Let $\kappa = (i_1, \dots, i_l)$ with $l = \ell(w)$ be a reduced decomposition of w. Then κ is $[M]$-reduced by Theorem 9.3.5. Let $y_1, \dots, y_l$ be a root vector sequence for κ in $\mathcal{B}(M)$. This exists by Proposition 15.2.6. Then $\mathcal{B}(M) = E^{\mathcal{B}(M)}(\kappa)$, and the monomials $y_l^{n_l} \cdots y_1^{n_1}$ with $n_1, \dots, n_l \geq 0$ form a basis of $\mathcal{B}(M)$ by Theorem 16.2.1.

Let $h : \mathbb{N}_0^\theta \to \mathbb{N}_0$ be an additive map with $h(\beta) > 0$ for any $\beta \neq 0$. For any $1 \le i \le l$ let $h_i = h\big(\beta_i^{[M],\kappa}\big)$. Since $\mathcal{B}(M)$ is $\mathbb{N}_0^\theta$-graded, Theorem 14.1.12 implies that for any $1 \le i < j \le l$ there exists a scalar $\lambda_{ij} \in \Bbbk^\times$ such that $y_i y_j - \lambda_{ij} y_j y_i$ is a linear combination of monomials $y_{j-1}^{n_{j-1}} \cdots y_{i+1}^{n_{i+1}}$ with

$$n_1, \dots, n_l \ge 0, \quad h_{i+1}n_{i+1} + \cdots + h_{j-1}n_{j-1} = h_i + h_j.$$

Hence $\mathcal{B}(M)$ is a domain by Lemma 16.4.4. $\square$

Proposition 16.4.6. *Let M and $\boldsymbol{q}$ be as in Section 16.3. Assume that $\boldsymbol{q}$ is genuinely of finite Cartan type. Then $U_{\boldsymbol{q}}^+$ is a domain.*

Proof. Let A be the Cartan matrix of finite type such that $\boldsymbol{q}$ is of Cartan type with Cartan matrix A. Let w_0 be the longest element of the Weyl group of A. Let $\kappa = (i_1, \dots, i_l)$ with $l = \ell(w)$ be a reduced decomposition of w. Then κ is $[M]$-reduced by Theorem 9.3.5. By Proposition 16.3.11 for $\mathcal{N} = \mathcal{N}(U_{\boldsymbol{q}}^+, M, \mathrm{id}_M)$, there exists a root vector sequence $y_1, \dots, y_l$ for κ in $U_{\boldsymbol{q}}^+$. The monomials $y_l^{n_l} \cdots y_1^{n_1}$ with $n_1, \dots, n_l \ge 0$ form a basis of $U_{\boldsymbol{q}}^+$ by Theorem 16.3.14.

Let $h : \mathbb{N}_0^\theta \to \mathbb{N}_0$ be an additive map with $h(\beta) > 0$ for any $\beta \neq 0$. For any $1 \le i \le l$ let $h_i = h\big(\beta_i^{[M],\kappa}\big)$. Since $U_{\boldsymbol{q}}^+$ is $\mathbb{N}_0^\theta$-graded, Theorem 16.3.16 implies that for any $1 \le i < j \le l$ there exists a scalar $\lambda_{ij} \in \Bbbk^\times$ such that $y_i y_j - \lambda_{ij} y_j y_i$ is a linear combination of monomials $y_{j-1}^{n_{j-1}} \cdots y_{i+1}^{n_{i+1}}$ with

$$n_1, \dots, n_l \ge 0, \quad h_{i+1}n_{i+1} + \cdots + h_{j-1}n_{j-1} = h_i + h_j.$$

Hence $U_{\boldsymbol{q}}^+$ is a domain by Lemma 16.4.4. $\square$

Now we discuss the Gelfand-Kirillov dimension of algebras. Recall that the limes superior of a real sequence $(x_m)_{m\ge 0}$ is defined by

$$\limsup_{m\to\infty} x_m = \inf_{k\ge 0} \sup_{m \ge k} x_m \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

Remark 16.4.7. Let $(x_m)_{m\ge 0}$ be a real sequence which is bounded below. If the sequence is not bounded above, then $\limsup_{m\to\infty} x_m = \infty$. If it is bounded above, let $s_k = \sup_{m\ge k} x_m$ for all $k \ge 0$. Then $(s_k)_{k\ge 0}$ is decreasing, hence convergent, and $\limsup_{m\to\infty} x_m = \lim_{k\to\infty} s_k$. We note the following easy rules for sequences $(x_m)_{m\ge 0}$ and $(y_m)_{m\ge 0}$ which are bounded below.

(1) If $x_m \le y_m$ for all $m \ge 0$, then $\limsup_{m\to\infty} x_m \le \limsup_{m\to\infty} y_m$.
(2) $\limsup_{m\to\infty}(x_m + y_m) \le \limsup_{m\to\infty} x_m + \limsup_{m\to\infty} y_m$.
(3) If $(y_m)_{m\ge 0}$ is convergent, then $\limsup_{m\to\infty} y_m = \lim_{m\to\infty} y_m$, and $\limsup_{m\to\infty}(x_m + y_m) = \limsup_{m\to\infty} x_m + \lim_{m\to\infty} y_m$.

Definition 16.4.8. Let A be an algebra. For any finite subset V of A containing $1 = 1_A$ and for any $m \ge 0$ let

$$g_m^{(V)} = \mathrm{span}_\Bbbk\{v_1 \cdots v_m \mid v_1, \dots, v_m \in V\}, \quad d_V = \limsup_{m\to\infty} \frac{\log \dim g_m^{(V)}}{\log m}.$$

Proposition 16.4.9. *Let A be an algebra. Let V, W be finite subsets of A containing 1 such that $\bigcup_{m\ge 0} g_m^{(V)} \subseteq \bigcup_{m\ge 0} g_m^{(W)}$. Then $d_V \le d_W$.*

PROOF. Let $n \in \mathbb{N}$ such that $V \subseteq g_n^{(W)}$. Such n exists by assumption. Then $g_m^{(V)} \subseteq g_{mn}^{(W)}$ for any $m \in \mathbb{N}_0$, and hence

$$\begin{aligned} d_V = \limsup_{m\to\infty} \frac{\log \dim g_m^{(V)}}{\log m} &\leq \limsup_{m\to\infty} \frac{\log \dim g_{mn}^{(W)}}{\log m} \\ &= \limsup_{m\to\infty} \frac{\log \dim g_{mn}^{(W)}}{\log mn} = d_W. \end{aligned}$$

This proves the claim. □

Proposition 16.4.9 directly yields the following claim.

COROLLARY 16.4.10. *Let A be a finitely generated algebra. Then $d_V = d_W$ for any two finite generating sets V, W of A containing* 1.

DEFINITION 16.4.11. Let A be an algebra. Then

$$\operatorname{GKdim} A = \sup\{d_V \mid V \text{ is a finite subset of } A \text{ containing } 1\}$$

is called the **Gelfand-Kirillov dimension of** A.

REMARK 16.4.12. Let A be a finitely generated algebra. Then

$$\operatorname{GKdim} A = d_V$$

for any finite generating subset V of A containing 1 because of Corollary 16.4.10.

The Gelfand-Kirillov dimension of a finitely generated graded algebra can be obtained from its Hilbert series.

LEMMA 16.4.13. *Let $A = \bigoplus_{m=0}^{\infty} A(m)$ be a finitely generated $\mathbb{N}_0$-graded algebra with $A(0) = \Bbbk$. Then*

$$\operatorname{GKdim} A = \limsup_{m\to\infty} \frac{\log \dim A_m}{\log m},$$

where for any $m \in \mathbb{N}_0$, $A_m = \sum_{i=0}^{m} A(i)$.

PROOF. Let $d = \limsup_{m\to\infty} (\log \dim A_m)/\log m$. Let V be a finite set of homogeneous generators of A containing 1. Then for all $m \in \mathbb{N}_0$, $A_m \subseteq g_m^{(V)}$. Hence $d \leq \operatorname{GKdim} A$.

On the other hand, let $n \in \mathbb{N}_0$ such that A is generated by A_n. Let V be a homogeneous basis of A_n. Then for any $m \in \mathbb{N}_0$, $g_m^{(V)} \subseteq A_{mn}$ and hence

$$\begin{aligned} \operatorname{GKdim} A = \limsup_{m\to\infty} \frac{\log \dim g_m^{(V)}}{\log m} &\leq \limsup_{m\to\infty} \frac{\log \dim A_{mn}}{\log m} \\ &= \limsup_{m\to\infty} \frac{\log \dim A_{mn}}{\log mn} \leq d. \end{aligned}$$

This proves the lemma. □

LEMMA 16.4.14. *Let A, B be finitely generated $\mathbb{N}_0$-graded algebras such that $A(0) = B(0) = \Bbbk$. Assume that for all $m \in \mathbb{N}_0$, $\dim A(m) = \dim B(m)$. Then $\operatorname{GKdim} A = \operatorname{GKdim} B$.*

PROOF. This follows directly from Lemma 16.4.13. □

EXAMPLE 16.4.15. Let $\theta \in \mathbb{N}$, let H be the group algebra of $\mathbb{Z}^\theta$, and let $M \in \mathcal{F}_\theta^H$. Assume that the matrix $\boldsymbol{q}$ of M is quasi-generic of finite Cartan type, or genuinely of finite Cartan type. Let A^M be the Cartan matrix of M and let $\beta_1, \dots, \beta_t$ with $t \in \mathbb{N}_0$ be the positive roots attached to a reduced decomposition of the longest element of the Weyl group of A^M.

Let B be the polynomial ring in t indeterminates $X_1, \dots, X_t$. Define a grading on B such that for all i, $\deg(X_i)$ is the height of β_i.

Regard $U_{\boldsymbol{q}}^+$ as a graded algebra such that for all i, $\deg E_i = 1$. By Theorems 16.2.1 and 16.2.5 (if $\boldsymbol{q}$ is quasi-generic of finite Cartan type) and by Theorem 16.3.14(2) (if $\boldsymbol{q}$ is genuinely of finite Cartan type), respectively, $U_{\boldsymbol{q}}^+$ and B have the same Hilbert series. Hence

$$\operatorname{GKdim} U_{\boldsymbol{q}}^+ = \operatorname{GKdim} B = t$$

by Lemma 16.4.14.

The following lemma is of general interest. We will use it in the proof of Corollary 16.4.24.

LEMMA 16.4.16. *Let A be an algebra generated by elements*

$$e_1 \dots, e_k, f_1, \dots, f_l \quad \textit{where } k, l \geq 1.$$

Assume that for any $1 \leq i \leq k$ and $1 \leq j \leq l$ there exists $q_{ji} \in \Bbbk^\times$ such that

$$f_j e_i = q_{ji} e_i f_j. \tag{16.4.3}$$

Let B and C be the subalgebras of A generated by $e_1, \dots, e_k$ and $f_1, \dots, f_l$, respectively. Then $A = BC$ and

$$\operatorname{GKdim} A \leq \operatorname{GKdim} B + \operatorname{GKdim} C.$$

PROOF. Let $V = \{1, e_1, \dots, e_k, f_1, \dots, f_l\}$. Since A is spanned by the monomials $a_1 \cdots a_m$ with $a_i \in V$, $m \geq 0$, we conclude from Equations (16.4.3) that $A = BC$. Let

$$V_B = \{1, e_1, \dots, e_k\}, \quad V_C = \{1, f_1, \dots, f_l\}.$$

Then

$$g_m^{(V)} = \sum_{n=0}^m g_n^{(V_B)} g_{m-n}^{(V_C)} \subseteq g_m^{(V_B)} g_m^{(V_C)}$$

because of Equations (16.4.3). Thus

$$\begin{aligned} \operatorname{GKdim} A = d_V &\leq \limsup_{m \to \infty} \frac{\log(\dim g_m^{(V_B)} \cdot \dim g_m^{(V_C)})}{\log m} \\ &\leq \limsup_{m \to \infty} \left(\frac{\log \dim g_m^{(V_B)}}{\log m} + \frac{\log \dim g_m^{V_C}}{\log m} \right) \\ &\leq \operatorname{GKdim} B + \operatorname{GKdim} C. \end{aligned}$$

Hence the Lemma is proven. □

In what follows let H be the group algebra of an abelian group.

LEMMA 16.4.17. *Let $\theta \geq 2$ and let S be an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Let $\alpha, \beta \in \mathbb{N}_0^\theta \setminus \{0\}$ with $\mathbb{Q}\alpha \neq \mathbb{Q}\beta$ and let $e \in S(\alpha)$, $f \in S(\beta)$. Assume that $\Bbbk e$ and*

$\Bbbk f$ are one-dimensional objects in ${}^H_H\mathcal{YD}$ and that e and f are primitive in S. For all $m \geq 0$ let $y_m = (\mathrm{ad}_S e)^m(f)$. If $y_m \neq 0$ for all $m \geq 0$, then the monomials

$$y_{m_1} \cdots y_{m_k}, \quad k \geq 0,\ 0 \leq m_1 < m_2 < \cdots < m_k \tag{16.4.4}$$

are linearly independent in S.

Proof. Assume that $y_m \neq 0$ for all $m \geq 0$ and that the monomials in (16.4.4) are linearly dependent. The $\mathbb{N}_0^\theta$-degree of the monomial $y_{m_1} \cdots y_{m_k}$ with $k, m_1, \dots, m_k \in \mathbb{N}_0$ is $(m_1 + \cdots + m_k)\alpha + k\beta$. Since S is an $\mathbb{N}_0^\theta$-graded algebra and $\mathbb{Q}\alpha \neq \mathbb{Q}\beta$, there exist $m, k \in \mathbb{N}_0$, $k \geq 2$, and a scalar $\lambda_{m_1,\dots,m_k}$ for any tuple $(m_1, \dots, m_k) \in \mathbb{N}_0^k$ with $0 \leq m_1 < \cdots < m_k$, $m_1 + \cdots + m_k = m$, such that

$$\sum_{m_1,\dots,m_k} \lambda_{m_1,\dots,m_k} y_{m_1} \cdots y_{m_k} = 0$$

and not all $\lambda_{m_1,\dots,m_k}$ are zero. Further we may assume that the monomials in (16.4.4) with $k-1$ factors are linearly independent. Let $n \in \mathbb{N}_0$ be the smallest integer such that there exists $(m_1, \dots, m_k) \in \mathbb{N}_0^k$ with $m_1 = n$ and $\lambda_{m_1,\dots,m_k} \neq 0$. Since S is an $\mathbb{N}_0^\theta$-graded coalgebra, the homogeneous summand of

$$\sum_{m_1,\dots,m_k} \lambda_{m_1,\dots,m_k} \Delta(y_{m_1} \cdots y_{m_k})$$

in $S(n\alpha + \beta) \otimes S((m-n)\alpha + (k-1)\beta)$ has to vanish. Since $\mathbb{Q}\alpha \neq \mathbb{Q}\beta$, the latter and Proposition 4.3.12 imply that

$$\sum_{m_2,\dots,m_k} \lambda_{n,m_2,\dots,m_k} y_n \otimes y_{m_2} \cdots y_{m_k} = 0.$$

This violates the assumption that the monomials in (16.4.4) with $k-1$ factors are linearly independent. Thus the Lemma is proven. □

Proposition 16.4.18. *Let $\theta \geq 2$ and let S be an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Let $\alpha, \beta \in \mathbb{N}_0^\theta \setminus \{0\}$ with $\mathbb{Q}\alpha \neq \mathbb{Q}\beta$ and let $e \in S(\alpha)$, $f \in S(\beta)$. Assume that $\Bbbk e$ and $\Bbbk f$ are one-dimensional objects in ${}^H_H\mathcal{YD}$ and that e and f are primitive in S. If $\mathrm{GKdim}\, S < \infty$ then there exists $m \geq 0$ such that $(\mathrm{ad}_S e)^m(f) = 0$.*

Proof. Assume that $y_m = (\mathrm{ad}_S e)^m(f) \neq 0$ for all $m \geq 0$, and let

$$V = \{1, e, f\} \subseteq S.$$

Note that $y_m \in g_{m+1}^V$ for any $m \geq 0$. Thus for any $n \in \mathbb{N}_0$ the monomials $y_{m_1} \cdots y_{m_k}$ with $0 \leq m_1 < \cdots < m_k < n$ are contained in $g_{n(n+1)/2}^{(V)}$ and hence in $g_{n^2}^{(V)}$. Since there are 2^n such monomials and they are linearly independent by Lemma 16.4.17, we conclude that

$$\dim g_{n^2}^{(V)} \geq 2^n, \quad \frac{\log \dim g_{n^2}^{(V)}}{\log n^2} \geq \frac{n \log 2}{2 \log n}$$

for any $n \in \mathbb{N}_0$. This is a contradiction to $\mathrm{GKdim}\, S < \infty$. □

Corollary 16.4.19. *Let $\theta \geq 1$ and let $M \in \mathcal{F}_\theta^H$ such that $\dim M_k = 1$ for any $1 \leq k \leq \theta$. Let $\mathcal{N}(S, N, f)$ be a pre-Nichols system of M such that $\mathrm{GKdim}\, S < \infty$. Assume one of the following.*

1. *S is a domain.*
2. *M is quasi-generic.*

Then M is of Cartan type.

PROOF. For any $1 \leq i \leq \theta$ let $e_i \in N_i \setminus \{0\}$, $g_i \in G(H)$, $\chi_i \in \mathrm{Alg}(H, \Bbbk)$ such that for all $h \in H$,

$$h \cdot e_i = \chi_i(h)e_i, \quad \delta_{N_i}(e_i) = g_i \otimes e_i.$$

For all $1 \leq i,j \leq \theta$ let $q_{ij} = \chi_j(g_i)$, hence $c_{S,S}(e_i \otimes e_j) = q_{ij}e_j \otimes e_i$. Thus by Proposition 4.3.12, for all $m \in \mathbb{N}_0$,

$$\begin{aligned}\Delta_S((\mathrm{ad}_S e_i)^m(e_j)) =& (\mathrm{ad}_S e_i)^m(e_j) \otimes 1 \\ &+ \sum_{k=0}^{m} \binom{m}{k}_{q_{ii}} \Big(\prod_{l=k}^{m-1} (1 - q_{ii}^l q_{ij} q_{ji}) \Big) e_i^{m-k} \otimes (\mathrm{ad}_S e_i)^k(e_j).\end{aligned}$$

By Proposition 16.4.18, for any $1 \leq i,j \leq \theta$ with $i \neq j$ there exists $m_{ij} > 0$ such that $(\mathrm{ad}_S e_i)^{m_{ij}}(e_j) = 0$ and $(\mathrm{ad}_S e_i)^{m_{ij}-1}(e_j) \neq 0$. In particular, the homogeneous summands of $\Delta_S((\mathrm{ad}_S e_i)^{m_{ij}}(e_j))$ contained in

$$S(m_{ij}\alpha_i) \otimes S(\alpha_j) \,\oplus\, S(\alpha_i) \otimes S((m_{ij}-1)\alpha_i + \alpha_j),$$

that is, the summands with $k = 0$ and with $k = m_{ij} - 1$, are zero. We conclude that

(a) $\prod_{l=0}^{m_{ij}-1}(1 - q_{ii}^l q_{ij} q_{ji}) e_i^{m_{ij}} = 0$,
(b) $\binom{m_{ij}}{m_{ij}-1}_{q_{ii}} (1 - q_{ii}^{m_{ij}-1} q_{ij} q_{ji}) = 0$.

Assume (1). Then $e_i^{m_{ij}} \neq 0$. By (a), $1 - q_{ii}^l q_{ij} q_{ji} = 0$ for some $0 \leq l \leq m_{ij} - 1$. Assume (2). Then $\binom{m_{ij}}{m_{ij}-1}_{q_{ii}} \neq 0$. By (b), $1 - q_{ii}^{m_{ij}-1} q_{ij} q_{ji} = 0$. In both cases we have shown that the braiding of M is of Cartan type. $\square$

In the following remark we discuss two examples which indicate potential difficulties regarding a general classification of pre-Nichols systems $\mathcal{N}(S, N, f)$, where S is a domain of finite Gelfand-Kirillov dimension.

REMARK 16.4.20. (1) The entries of the Cartan matrix of the braiding of M and the quantum Serre relations of S in Proposition 16.4.18 are not necessarily directly related. Assume that H is the group algebra of the trivial group and that $\theta = 2$. Let $M \in \mathcal{F}_2^H$ and $e_1 \in M_1 \setminus \{0\}$, $e_2 \in M_2 \setminus \{0\}$. Let U be the universal enveloping algebra of the Heisenberg Lie algebra

$$\mathrm{span}_\Bbbk\{e_1, e_2, e_{12}\}, \quad e_{12} = [e_1, e_2], [e_1, e_{12}] = [e_2, e_{12}] = 0.$$

Then U is a domain with $\mathrm{GKdim}\, U = 3$, $\mathcal{N}(U, M, \mathrm{id})$ is a pre-Nichols system of Cartan type $A_1 \times A_1$, but $\mathrm{ad}\, e_1(e_2) \neq 0$.

(2) Let $M \in \mathcal{F}_2^H$ such that $\dim M_1 = \dim M_2 = 1$. Let $x_1 \in M_1 \setminus \{0\}$ and $x_2 \in M_2 \setminus \{0\}$, and for all $i, j \in \{1, 2\}$ let $q_{ij} \in \Bbbk^\times$ such that

$$c_{M_i,M_j}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i.$$

Let $q \in \Bbbk^\times$ with $\mathrm{ord}(q) = 3$ and assume that

$$q_{12}q_{21} = q_{11}^{-1} = q_{22}^{-1} = q.$$

Then $\mathcal{N} = \mathcal{N}(S, M, \mathrm{id}_M)$ with

$$S = T(M)/((\mathrm{ad}_{T(M)} x_1)^2(x_2), (\mathrm{ad}_{T(M)} x_2)^3(x_1))$$

is a pre-Nichols system of M. The braiding of M is of Cartan type A_2, and hence $\mathcal{N}$ is not a Nichols system of $(M, 2)$. One can show that S is a domain with $\mathrm{GKdim}\, S = 4$.

PROPOSITION 16.4.21. *Let $\theta \in \mathbb{N}$ and let $M \in \mathcal{F}_\theta^H$ such that $\dim M_k = 1$ for any $1 \le k \le \theta$. Let $1 \le i \le \theta$ and let $p : \mathcal{N} = \mathcal{N}(S, N, f) \to \mathcal{N}'$ be a morphism of pre-Nichols systems of M such that $\ker(p)$ is generated by $\ker(p) \cap \Bbbk[N_i]$. Let $E \subseteq S$ be an $\mathbb{N}_0^\theta$-graded right coideal subalgebra which is a domain containing N_i. Assume that $\mathcal{N}$ is i-finite, (Sys4) holds for $\mathcal{N}$ and i, and that $\mathcal{N}'$ is a Nichols system of (M, i). Let $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_i(\mathcal{N}')$ and $\widetilde{E} = (t_i^{\mathcal{N}'})^{-1}(p(E))$. Then $\widetilde{E}$ is a right coideal subalgebra of $\widetilde{S}$ and $\widetilde{E}$ is a domain.*

PROOF. Let $\mathcal{N}' = \mathcal{N}(S', N', f')$. Since p is an $\mathbb{N}_0^\theta$-graded Hopf algebra map, $p(E)$ is an $\mathbb{N}_0^\theta$-graded right coideal subalgebra of S'. Since $N_i \subseteq E$, it follows that $N_i' \subseteq p(E)$, and hence $\widetilde{E}$ is an $\mathbb{N}_0^\theta$-graded right coideal subalgebra of $\widetilde{S}$.

Let $\boldsymbol{q} = (q_{jk})_{1 \le j,k \le \theta}$ be the braiding matrix of M. Let $\pi : S \to \Bbbk[N_i]$ be the $\mathbb{N}_0^\theta$-graded projection, let $x \in N_i \setminus \{0\}$, $n = \operatorname{ord}(q_{ii})$, and let $y = 0$ if $n = \infty$ and $y = x^n$ otherwise. Let $\overline{J}$ be the Hopf ideal of $\Bbbk[N_i]$ generated by y. Then $\ker(p)$ is generated by y by construction and the assumption on $\ker(p)$, since $\mathcal{N}'$ satisfies (Sys3) for i. Moreover, $\mathrm{ad}_S y(x') = 0$ for any $x' \in S$ since $\mathcal{N}$ satisfies (Sys4) for i. Thus, by Lemma 15.2.8 for $J = \ker(p)$, $\bar{S} = \Bbbk[N_i]$ and by the surjectivity of p, p induces an algebra isomorphism $p_0 : S^{\mathrm{co}\,\Bbbk[N_i]} \to S'^{\mathrm{co}\,\Bbbk[N_i']}$ in ${}^H_H\mathcal{YD}$. In particular, $p_0(E \cap S^{\mathrm{co}\,\Bbbk[N_i]})$ is a domain. Therefore,

$$\widetilde{E} = (T_i^{\mathcal{N}'})^{-1}(p(E) \cap S'^{\mathrm{co}\,\Bbbk[N_i']}) = (T_i^{\mathcal{N}'})^{-1}(p_0(E \cap S^{\mathrm{co}\,\Bbbk[N_i]}))$$

is a domain since $T_i^{\mathcal{N}'}$ is an algebra isomorphism. $\square$

COROLLARY 16.4.22. *Let $\theta \in \mathbb{N}$ and $M \in \mathcal{F}_\theta^H$ be such that $\dim M_k = 1$ for any $1 \le k \le \theta$. Assume that the braiding matrix of $M_1 \oplus \cdots \oplus M_\theta$ is genuinely of finite Cartan type. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M such that (Sys4) holds for S for all $1 \le i \le \theta$. Let $t \ge 0$, $1 \le i_1, \dots, i_t \le \theta$, and $x_1, \dots, x_t \in S$ such that $\kappa = (i_1, \dots, i_t)$ is an $[M]$-reduced sequence and $x_1, \dots, x_t$ is a root vector sequence for κ in S. Assume that the right coideal subalgebra E of S generated by $x_1, \dots, x_t$ is a domain. Then the monomials $x_t^{n_t} \cdots x_1^{n_1}$ with $n_1, \dots, n_t \ge 0$ form a basis of E.*[1]

PROOF. We proceed by induction on t. If $t = 0$ then $E = \Bbbk 1$ and the claim is trivial.

Assume that $t > 0$. Let $\boldsymbol{q} = (q_{ij})_{1 \le i,j \le \theta}$ be the braiding matrix of M and let $n = \operatorname{ord}(q_{i_1 i_1})$. Then $N_{i_1} = \Bbbk x_1 \subseteq E$ and $n < \infty$. Let J be the ideal of S generated by x_1^n. Then J is a Hopf ideal and $\mathcal{N}' = \mathcal{N}(S/J, N, f)$ is a Nichols system of (M, i_1). Moreover, $p : S \to S/J$ is a morphism of pre-Nichols systems $p : \mathcal{N} \to \mathcal{N}'$ of M. Let $\mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}) = R_{i_1}(\mathcal{N}')$. According to Proposition 16.4.21, $\widetilde{E} = (t_{i_1}^{\mathcal{N}'})^{-1}(p(E))$ is a right coideal subalgebra of $\widetilde{S}$ and $\widetilde{E}$ is a domain. Moreover, $p(x_1), \dots, p(x_t)$ is a root vector sequence for κ in S/J by Remark 15.2.2(2), and $y_2, \dots, y_t$, where $y_i = (T_{i_1}^{\mathcal{N}'})^{-1}(p(x_i))$ for any $2 \le i \le t$, is a root vector sequence for $(i_2, \dots, i_t)$ in $\widetilde{S}$ by Proposition 15.2.4. By assumption and by Lemma 15.1.13(2), the braiding matrix of $R_{i_1}(M)$ is genuinely of finite Cartan type. By assumption and by Corollary 16.3.9, $R_{i_1}(\mathcal{N}')$ is a Nichols system of $(R_{i_1}(M), i_1)$ for which (Sys4) holds for all $1 \le i \le \theta$. Thus, by induction hypothesis, the monomials $y_t^{n_t} \cdots y_2^{n_2}$ with $n_2, \dots, n_t \ge 0$ form a basis of $\widetilde{E}$. Then Theorem 14.1.4 implies that the monomials $p(x_t)^{n_t} \cdots p(x_2)^{n_2} p(x_1)^{n_1}$ with $n_2, \dots, n_t \ge 0$ and $0 \le n_1 < n$

[1] If $t = 0$ then by convention the basis consists of the single monomial 1.

form a basis of $p(E) = t_{i_1}^{\mathcal{N}'}(\widetilde{E})$. Since $x_2, \dots, x_t \in K_{i_1}^{\mathcal{N}}$, the claim follows from Proposition 15.2.9(1). $\square$

THEOREM 16.4.23. *Let $\theta \in \mathbb{N}$, and let $M \in \mathcal{F}_\theta^H$ such that $\dim M_k = 1$ for any $1 \le k \le \theta$. Let $\boldsymbol{q}$ be the braiding matrix of $M_1 \oplus \cdots \oplus M_\theta$, and let $\mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Assume that $\operatorname{GKdim} S < \infty$.*

(1) *If S is a domain, then the braiding matrix $\boldsymbol{q}$ is of Cartan type.*
(2) *If $\boldsymbol{q}$ is quasi-generic, then it is of finite Cartan type and $S \cong U_{\boldsymbol{q}}^+$.*
(3) *If S is a domain, $\boldsymbol{q}$ is genuinely of finite Cartan type and $\mathcal{N}(S, N, f)$ satisfies (Sys4) for all $1 \le i \le \theta$, then $S \cong U_{\boldsymbol{q}}^+$.*

PROOF. (1) holds by Corollary 16.4.19.

(2) By Corollary 16.4.19, the braiding of M is of Cartan type. For all $1 \le i \le \theta$ let $x_i \in N_i \setminus \{0\}$. Let A be the Cartan matrix such that $q_{ii}^{a_{ij}} = q_{ij}q_{ji}$ for all $1 \le i, j \le \theta$. By Lemma 15.1.12, $a_{ij}^M = a_{ij}$ for all $1 \le i, j \le \theta$.

By Proposition 16.4.18, for any $1 \le i, j \le \theta$ with $i \ne j$ there exists an integer $m \ge 1$ such that $(\operatorname{ad}_S x_i)^m(x_j) = 0$ and $(\operatorname{ad}_S x_i)^{m-1}(x_j) \ne 0$. Then by the proof of Corollary 16.4.19, $1 - q_{ii}^{m-1}q_{ij}q_{ji} = 0$. Hence $m = 1 - a_{ij}$. Consequently, $\mathcal{N}(S, N, f)$ admits all reflections by Theorem 16.2.5(1). Let $\kappa = (i_1, \dots, i_t)$ be an $[M]$-reduced sequence. For any $1 \le k \le t$ let $\beta_k = \beta_k^{[M],\kappa}$. By Lemma 15.1.13 and Theorem 15.2.7(1), $q_{\beta_k\beta_k} = q_{i_k i_k}$ is not a root of unity or equal to 1 (if $\operatorname{char}(\Bbbk) = 0$) for any $1 \le k \le t$. Hence the elements $x_t^{n_t} \cdots x_1^{n_1}$ with $n_1, \dots, n_t \ge 0$ form a basis of $E^{\mathcal{B}(M)}(\kappa)$ by Theorem 15.2.7(2). Thus

$$\operatorname{GKdim} E^{\mathcal{B}(M)}(\kappa) = \operatorname{GKdim} \Bbbk[x_1, \dots, x_t] = t$$

by Lemma 16.4.14, see also Example 16.4.15. We conclude that

$$t = \operatorname{GKdim} E^{\mathcal{B}(M)}(\kappa) \le \operatorname{GKdim} \mathcal{B}(M) \le \operatorname{GKdim} S < \infty.$$

In particular, $\mathcal{G}(M)$ is finite by Proposition 9.2.25. Hence the small Cartan graph $\mathcal{G}_{\mathrm{s}}(M)$ of M defined in Proposition 15.1.10 is finite by Lemma 10.1.4 and therefore A is of finite type by Example 9.1.17.

Since $\mathcal{N}(S, N, f)$ is a Nichols system of (M, i) for all $1 \le i \le \theta$, S is isomorphic as a graded Hopf algebra to a quotient of $U_{\boldsymbol{q}}^+$, and there is a natural graded surjection from S to $\mathcal{B}(M)$. On the other hand, $\mathcal{B}(M)$ and $U_{\boldsymbol{q}}^+$ are isomorphic graded Hopf algebras by Theorem 16.2.5(2). Thus $S \cong U_{\boldsymbol{q}}^+$.

(3) Let A be the Cartan matrix corresponding to $\boldsymbol{q}$. Let κ be a reduced decomposition of the longest element w_0 of the Weyl group of A. By Proposition 16.3.11 there exists a root vector sequence $x_1, \dots, x_t$ for κ in S, where $t = \ell(w_0)$. In particular, the monomials

$$x_t^{n_t} \cdots x_1^{n_1}, \quad n_1, \dots, n_t \ge 0 \tag{16.4.5}$$

span S. Because of Axiom (Sys4) for S there exists a surjective Hopf algebra map $f : U_{\boldsymbol{q}}^+ \to S$ in ${}^H_H\mathcal{YD}$. Hence, in view of Theorem 16.3.14(2), it suffices to prove that the monomials in (16.4.5) form a basis of S. This is true by Corollary 16.4.22 with $E = S$. $\square$

COROLLARY 16.4.24. *Let $(V, c_{V,V})$ be a finite-dimensional braided vector space. The following are equivalent.*

(1) *V is of diagonal type and $\mathcal{B}(V)$ is a domain with* $\operatorname{GKdim}\mathcal{B}(V) < \infty$.
(2) *V is quasi-generic of finite Cartan type.*

In this case, $\mathcal{B}(V) \cong U_{\boldsymbol{q}}^+$.

Proof. Assume that (1) holds. Let H be the group algebra of $\mathbb{Z}^\theta$ and let $M \in \mathcal{F}_\theta^H$ such that $\dim M_k = 1$ for any $1 \le k \le \theta$ and that V and $\bigoplus_{k=1}^\theta M_k$ are isomorphic as braided vector spaces. Since $\mathcal{B}(V)$ is a domain, Example 1.10.1 implies that the diagonal entries of the braiding are 1 (if $\operatorname{char}(\Bbbk) = 0$) or not roots of 1. Thus V is quasi-generic. Since $\operatorname{GKdim}\mathcal{B}(V) < \infty$ by (1), V is of finite Cartan type by Theorem 16.4.23(2).

Assume now that (2) holds. Then V is of diagonal type by definition. Moreover, $\mathcal{B}(V) \cong U_{\boldsymbol{q}}^+$ by Theorem 16.2.5. Hence $\mathcal{B}(V)$ is a domain by Proposition 16.4.5 and $\operatorname{GKdim}\mathcal{B}(V) < \infty$ by Example 16.4.15. □

The structure of $\mathcal{B}(V)$ when the equivalent conditions of Corollary 16.4.24 hold, is discussed in Section 16.2.

Corollary 16.4.25. *Let $R = \bigoplus_{n\in\mathbb{N}_0} R(n)$ be a locally finite $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Assume that $R(0) = \Bbbk$ and $R(1) = M_1 \oplus \cdots \oplus M_\theta$ with $M_i \in {}^H_H\mathcal{YD}$ and $\dim M_i = 1$ for each $1 \le i \le \theta$. Let $\boldsymbol{q}$ be a braiding matrix of $R(1)$. Assume that $\boldsymbol{q}$ is quasi-generic and* $\operatorname{GKdim} R < \infty$. *The following are equivalent.*

(1) *$P(R) = R(1)$, that is, R is a strictly graded coalgebra.*
(2) *R is generated as an algebra by $R(1)$, that is, R is a pre-Nichols algebra of M.*
(3) *R is a Nichols algebra of M, the braiding matrix $\boldsymbol{q}$ is of finite Cartan type, and $R \cong U_{\boldsymbol{q}}^+$.*

Proof. (2) $\Rightarrow$ (3). Let $\mathcal{N}(\operatorname{gr} R, N, f)$ be the pre-Nichols system of M described in Lemma 13.5.8, where $\operatorname{gr} R$ is the $\mathbb{N}_0^\theta$-graded Hopf algebra constructed from R in Proposition 5.2.21. By Theorem 16.4.23(2), $\operatorname{gr} R$ is a Nichols algebra of M, $\boldsymbol{q}$ is of finite Cartan type, and $\mathcal{B}(M) \cong U_{\boldsymbol{q}}^+$. Hence $R \cong \mathcal{B}(M)$ is a Nichols algebra by Lemma 13.5.8.

(3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). Recall from Corollary 4.2.9 that there exists a braided monoidal equivalence $((\ )^{*\mathrm{gr}}, \varphi_0, \varphi) : (\mathbb{N}_0\text{-}\mathrm{Gr}({}^H_H\mathcal{YD})^{\mathrm{lf}})^{\mathrm{op}} \to \mathbb{N}_0\text{-}\mathrm{Gr}({}^H_H\mathcal{YD})^{\mathrm{lf}}$. By (1) and Corollary 4.2.10(1), $R^{*\mathrm{gr}}$ is a pre-Nichols algebra. Thus $\operatorname{GKdim} R^{*\mathrm{gr}} = \operatorname{GKdim} R < \infty$ by Lemma 16.4.14 and $R^{*\mathrm{gr}}$ is strictly graded by (2) $\Rightarrow$ (3) for $R^{*\mathrm{gr}}$. Then Corollary 4.2.10(2) implies that R is a pre-Nichols algebra. □

Remark 16.4.26. Corollary 16.4.25 should be compared with Theorem 15.5.1. There we assumed that R is finite-dimensional and $\operatorname{char}(\Bbbk) = 0$ instead of $\boldsymbol{q}$ is quasi-generic and $\operatorname{GKdim} R$ is finite.

16.5. Application to the Hopf algebras $U(\mathcal{D}, \lambda)$

In this section we study the Hopf algebras $U(\mathcal{D}, \lambda)$ of Section 8.3 when $\mathcal{D}$ is generic and of finite Cartan type. In Theorem 16.5.5 we compute a PBW basis, the coradical filtration, the associated graded Hopf algebra, and the Gelfand-Kirillov dimension of $U(\mathcal{D}, \lambda)$. Recall that the quantum groups $U_q(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, and q not a root of unity, are special cases in this class of Hopf algebras.

In the second half of the section we look at the **lifting problem**: Given a coradically graded pointed Hopf algebra $\mathcal{H}$, determine all pointed Hopf algebras A with $\operatorname{gr} A \cong \mathcal{H}$ as coradically graded Hopf algebras. In Theorem 16.5.10 we assume that $\Bbbk$ is algebraically closed. We show that a pointed Hopf algebra with abelian coradical and finite Gelfand-Kirillov dimension is isomorphic to $U(\mathcal{D}, \lambda)$ as above, if its infinitesimal braiding is generic.

We begin with some general results on the Gelfand-Kirillov dimension of a class of pointed Hopf algebras.

LEMMA 16.5.1. *Let $(A, \mathcal{F}(A))$ be an $\mathbb{N}_0$-filtered algebra. Then*

$$\operatorname{GKdim} \operatorname{gr} A \leq \operatorname{GKdim} A.$$

PROOF. Let $V \subseteq \operatorname{gr} A$ be a finite subset containing $1_{\operatorname{gr} A}$. Let

$$U = \{b_i \mid i \in I\} \subseteq \operatorname{gr} A$$

be the subset of all homogeneous components of elements in V, where I is a finite index set. For all $i \in I$ we choose an element $a_i \in F_{d_i}(A)$, where $d_i = \deg(b_i)$, and $b_i = a_i + F_{d_i - 1}(A)$. Let $W = \{a_i \mid i \in I\} \subseteq A$.

Let $m \geq 0$. Then $g_m^{(V)} \subseteq g_m^{(U)} \subseteq \operatorname{gr} A$.

To prove that $\dim g_m^{(U)} \leq \dim g_m^{(W)}$, let $X = g_m^{(U)} \subseteq \operatorname{gr} A$, $Y = g_m^{(W)} \subseteq A$. Let $F_d(Y) = F_d(A) \cap Y$, $d \geq 0$, be the induced filtration on Y. Let $d \geq 0$. The inclusion $F_d(Y) \subseteq F_d(A)$ defines a linear map

$$\varphi : F_d(Y)/F_{d-1}(Y) \to F_d(A)/F_{d-1}(A) = (\operatorname{gr} A)(d).$$

For all $i_1, \ldots, i_m \in I$ with $d_{i_1} + \cdots + d_{i_m} = d$,

$$\varphi(a_{i_1} \cdots a_{i_m} + F_{d-1}(Y)) = b_{i_1} \cdots b_{i_m}.$$

Hence the restriction $\varphi^{-1}(X(d)) \xrightarrow{\varphi} X(d)$ of φ is surjective. It follows that

$$\dim X = \sum_{d \geq 0} \dim X(d) \leq \sum_{d \geq 0} \dim F_d(Y)/F_{d-1}(Y) = \dim Y.$$

Here, $F_{-1}(Y) = 0$. We have shown that $\dim g_m^{(V)} \leq \dim g_m^{(U)} \leq \dim g_m^{(W)}$, which implies that $d_V \leq d_W$, and the lemma follows. $\square$

PROPOSITION 16.5.2. *Let G be an abelian group, and A a left $\Bbbk G$-module algebra. Assume that A is finitely generated as an algebra and locally finite as a $\Bbbk G$-module. Then*

$$\operatorname{GKdim} A \# \Bbbk G = \operatorname{GKdim} A + \operatorname{GKdim} \Bbbk G.$$

If G is finitely generated, then $\operatorname{GKdim} \Bbbk G$ *is the rank of the group G.*

PROOF. Any finite subset of $A \# \Bbbk G$ is contained in $A \# \Bbbk G_0$ for a finitely generated subgroup G_0 of G. Thus, by definition of the Gelfand-Kirillov dimension, we may assume that G is finitely generated.

(1) We first assume that $G = \langle g \rangle$ is infinite cyclic. Let $X \subseteq A$ be a finite-dimensional G-stable subspace which generates the algebra A and contains the unit element 1 of A. Such a subspace exists by our assumptions. Let

$$V = X + Xg + Xg^{-1} \subseteq A \# \Bbbk G.$$

Then for all $n \geq 1$, $V^n = \bigoplus_{k=-n}^{n} X^n g^k$, and $\dim V^n = (2n+1)\dim X^n$. Hence

$$\operatorname{GKdim} A\#\Bbbk G = \limsup_{n\to\infty}\Big(\frac{\log(2n+1)}{\log n} + \frac{\log(\dim X^n)}{\log n}\Big) = \operatorname{GKdim} A + 1,$$

since the sequence $\frac{\log(2n+1)}{\log n}$ converges to 1. In particular, $\operatorname{GKdim} \Bbbk G = 1$, by taking $A = \Bbbk$.

(2) Now we assume that $G = \langle g\rangle$ is a finite cyclic group of order N. Let X as in (1), and define $V = X + Xg$. Then $V^n = \bigoplus_{k=0}^{N-1} X^n g^k$, and $\dim V^n = N \dim X^n$, for all $n \geq N-1$. Hence $\operatorname{GKdim} A\#\Bbbk G = \operatorname{GKdim} A$, and $\operatorname{GKdim} \Bbbk G = 0$.

(3) If G_1, G_2 are abelian groups, then $A\#\Bbbk(G_1 \times G_2) \cong (A\#\Bbbk G_1)\#\Bbbk G_2$, where the G_2-action on $A\#\Bbbk G_1$ is defined by

$$g_2 \cdot (a\#g_1) = g_2 \cdot a\#g_1 \text{ for all } g_1 \in G_1,\ g_2 \in G_2,\ a \in A.$$

Hence the general case of the proposition follows by induction from (1) and (2). $\square$

LEMMA 16.5.3. *Let H be a Hopf algebra with bijective antipode, and R an $\mathbb{N}_0$-graded connected coalgebra in ${}^H_H\mathcal{YD}$. Then $A = R\#H$ is an $\mathbb{N}_0$-graded coalgebra with $A(n) = R(n)\#H$ for all $n \geq 0$. Let $C \subseteq A$ be an $\mathbb{N}_0$-graded subcoalgebra. Then $C \subseteq R\#(C\cap H)$.*

PROOF. Let $\pi = \varepsilon \otimes \mathrm{id}_H : A \to H$ be the projection onto degree 0. It follows from the definition of Δ_A, that

$$(\mathrm{id}_A \otimes \pi)\Delta_A = \mathrm{id}_R \otimes \Delta_H : A \to A \otimes H,$$

and $\mathrm{id}_A = (\mathrm{id}_R \otimes \varepsilon \otimes \mathrm{id}_H)(\mathrm{id}_A \otimes \pi)\Delta_A$. Since $C \subseteq A$ is a graded subcoalgebra, $C(0) = C \cap H$, and $(\mathrm{id}_C \otimes \pi|C)\Delta_C : C \to C \otimes (C\cap H)$. Hence C is contained in $R\#(C\cap H)$. $\square$

THEOREM 16.5.4. *Let A be a pointed Hopf algebra. Assume that $G = G(A)$ is abelian and A is generated by G and by finitely many skew-primitive elements. Let $R = (\operatorname{gr} A)^{\operatorname{co}\Bbbk G}$ with respect to coradical filtration of A and the projection* $\operatorname{gr} A \to \Bbbk G$ *onto degree* 0*, and assume that $R(1)$ is finite-dimensional. Then*

$$\operatorname{GKdim} A = \operatorname{GKdim} \operatorname{gr} A = \operatorname{GKdim} R + \operatorname{GKdim} \Bbbk G.$$

PROOF. By Corollary 5.3.16, $\operatorname{gr} A \cong R\#\Bbbk G$, R is strictly graded, and by Proposition 1.3.14, $\dim R(n) < \infty$ for all $n \geq 1$. Thus we know from Lemma 16.5.1 and Proposition 16.5.2 that

$$\operatorname{GKdim} R + \operatorname{GKdim} \Bbbk G = \operatorname{GKdim} \operatorname{gr} A \leq \operatorname{GKdim} A.$$

Hence it suffices to show the inequality

$$\operatorname{GKdim} A \leq \operatorname{GKdim} R + \operatorname{GKdim} \Bbbk G. \tag{16.5.1}$$

By assumption, any finite subset of A is contained in a subalgebra of A generated by finitely many skew-primitive and group-like elements. Thus for the proof of (16.5.1) we may assume that G is finitely generated.

By assumption there is a finite set S of skew-primitive elements in A and a finite subset $T \subseteq G$ such that A is generated by $S \cup T$. We may assume that for all $x \in S$, $\Delta(x) = g \otimes x + x \otimes h$, where $g, h \in T$, and that $1 \in T$. Then $C = \sum_{x\in S} \Bbbk x + \sum_{g\in T} \Bbbk g$ is a subcoalgebra of A.

(1) Let $n \geq 1$, and C^n the $\Bbbk$-span of all products $a_1 \cdots a_n$, $a_1, \dots, a_n \in C$. We claim that $C^n \cap \Bbbk G = g_n^{(T)}$.

We define a coalgebra filtration $F_0(C^n) \subseteq \cdots \subseteq F_n(C^n) = C^n$, where for all $0 \le i \le n$, $F_i(C^n)$ is the $\Bbbk$-span of all products $a_1 \cdots a_n$ of elements in $S \cup T$ such that at most i elements of the $(a_j)_{1 \le j \le n}$ are in S. By Proposition 1.3.2, $g_n^{(T)} = F_0(C^n) = \mathrm{Corad}(C^n)$. This proves our claim, since $\mathrm{Corad}(C^n) = C^n \cap \Bbbk G$ by Corollary 5.3.5.

(2) Let $n \ge 1$. Then $\mathrm{gr}\,(C^n) \subseteq \mathrm{gr}\, A$ is a graded subcoalgebra by Theorem 5.4.5. Note that $C^n \subseteq A_n$, since $C \subseteq A_1$. Hence $\mathrm{gr}\,(C^n) \subseteq \oplus_{k=0}^n (\mathrm{gr}\, A)(k) = R_n \# \Bbbk G$ by Corollary 5.4.6, where $R_n = \oplus_{i=0}^n R(i)$. Hence it follows from (1) and Lemma 16.5.3 that $\mathrm{gr}\, C^n \subseteq R_n \# g_n^{(T)}$, and

$$\dim C^n = \dim \mathrm{gr}\,(C^n) \le \dim R_n \dim g_n^{(T)}.$$

Using Lemma 16.4.13 we conclude that

$$\begin{aligned} \mathrm{GKdim}\, A = \limsup_{n\to\infty} \frac{\log \dim C^n}{\log n} &\le \limsup_{n\to\infty} \frac{\log(\dim R_n \dim g_n^{(T)})}{\log n} \\ &\le \limsup_{n\to\infty} \frac{\log \dim R_n}{\log n} + \limsup_{n\to\infty} \frac{\log \dim g_n^{(T)}}{\log n} \\ &\le \mathrm{GKdim}\, R + \mathrm{GKdim}\, \Bbbk G. \end{aligned}$$

□

Let G be an abelian group, $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ a generic YD-datum of finite Cartan type with Cartan matrix A, and λ a linking parameter for $\mathcal{D}$. Choose a decomposition $I = I^- \cup I^+$, $I^- \cap I^+ = \emptyset$, as in Section 8.3. Let X be the Yetter-Drinfeld module in ${}^G_G\mathcal{YD}$ with basis $(x_i)_{i\in I}$, and $x_i \in X_{g_i}^{\chi_i}$ for all $i \in I$. Recall that $U(\mathcal{D},\lambda) = (T(X)\#\Bbbk G)/I(\mathcal{D},\lambda)$. Let $X^- \subseteq X$ (respectively $X^+ \subseteq X$) be the subobject in ${}^G_G\mathcal{YD}$ with basis $(x_i)_{i\in I^-}$ (respectively $(x_i)_{i\in I^+}$). We denote the n-th term of the coradical filtration of $U(\mathcal{D},\lambda)$ by $U(\mathcal{D},\lambda)_n$, $n \ge 0$.

THEOREM 16.5.5. *Let G be an abelian group, $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ a generic YD-datum of finite Cartan type with Cartan matrix A, and λ a linking parameter for $\mathcal{D}$. Use the notation above.*

(1) *There are injective Hopf algebra maps*

$$\mathcal{B}(X^-)\#\Bbbk G \to U(\mathcal{D},\lambda),\ x_i \mapsto x_i,\ g \mapsto g,\ \text{for all } i \in I^-,\ g \in G,$$
$$\mathcal{B}(X^+)\#\Bbbk G \to U(\mathcal{D},\lambda),\ x_i \mapsto x_i,\ g \mapsto g,\ \text{for all } i \in I^+,\ g \in G,$$

which we view as inclusions. The algebras $\mathcal{B}(X^-)$ and $\mathcal{B}(X^+)$ have PBW-bases constructed in Theorem 16.2.1.

(2) *The multiplication map*

$$(\mathcal{B}(X^-)\otimes\mathcal{B}(X^+))\#\Bbbk G \to U(\mathcal{D},\lambda)$$

is an isomorphism of coalgebras, and $(\mathcal{B}(X^-)\otimes\mathcal{B}(X^+))\#\Bbbk G$, the smash coproduct coalgebra, is coradically graded.

(3) *For all $n \ge 0$, $U(\mathcal{D},\lambda)_n$ is the $\Bbbk$-span of*

$$\{x_{i_1}\cdots x_{i_k} g \mid i_1,\dots,i_k \in I, k \le n, g \in G\}.$$

(4) $U(\mathcal{D},\lambda)_1 = \Bbbk G \oplus \bigoplus_{(i,g)\in I\times G} \Bbbk x_i g$, *and $x_i g \ne 0$ for all $i \in I$, $g \in G$.*

(5) *There is an isomorphism of Hopf algebras*

$$U(\mathcal{D},0) \to \mathrm{gr}\, U(\mathcal{D},\lambda),\ \ x_i \mapsto \overline{x_i},\ g \mapsto g, \quad \text{for all } i \in I,\ g \in G.$$

(6) $U(\mathcal{D}, \lambda)$ *is isomorphic to a two-cocycle deformation of* $\operatorname{gr} U(\mathcal{D}, \lambda)$.
(7) $\operatorname{GKdim} U(\mathcal{D}, \lambda) = t + \operatorname{GKdim} \Bbbk G$, *where* t *is the number of positive roots attached to a reduced decomposition of the longest element of the Weyl group of the Cartan matrix* A.

PROOF. (1), (2). Recall the definition of $U(\mathcal{D}^-)$ and $U(\mathcal{D}^+)$ from Section 8.3. By Theorem 16.2.5, $U(\mathcal{D}^-) = \mathcal{B}(X^-)$, and $U(\mathcal{D}^+) = \mathcal{B}(X^+)$. By Proposition 1.3.17 and Proposition 5.3.18, $(\mathcal{B}(X^-) \otimes \mathcal{B}(X^+))\#\Bbbk G$ is coradically graded. Hence (1) and (2) follow from Theorem 8.3.9 and Theorem 16.2.1.

(3) and (4) follow from (2).

(5) By (3), there is a well-defined surjective map of Hopf algebras in ${}^G_G\mathcal{YD}$

$$\varphi : U(\mathcal{D}, 0) \to \operatorname{gr} U(\mathcal{D}, \lambda), \quad x_i \mapsto \overline{x_i},\ g \mapsto g,$$

for all $i \in I$ and $g \in G$. Hence φ is an isomorphism by Theorem 5.4.5, since the restriction of φ to $U(\mathcal{D}, 0)_1$ is injective by (4).

(6) follows from (4), Lemma 8.3.8 and Theorem 8.3.9.

(7) follows from Theorem 16.5.4 and Example 16.4.15. □

LEMMA 16.5.6. *Let* G *be a free abelian group,* R *a domain and a left* $\Bbbk G$*-module algebra. Then* $R\#\Bbbk G$ *is a domain.*

PROOF. Assume that G has rank 1 with basis element g. Let $x, y \in R\#\Bbbk G$ be non-zero elements, and write $x = \sum_{a\le i\le b} r_i g^i$, $y = \sum_{c\le j\le d} s_j g^j$, where a, b, c, d are integers, $r_i, s_j \in R$ for all i, j, $r_b \neq 0$, $s_d \neq 0$. Then $xy = \sum_{a+c\le k\le b+d} t_k g^k$, where $t_k \in R$ for all k, and $t_{b+d} = r_b(g^b \cdot s_d) \neq 0$.

Then the lemma follows by induction, since we may assume that G has finite rank, and since $R\#\Bbbk(G_1 \times G_2) \cong (R\#\Bbbk G_1)\#\Bbbk G_2$ for all abelian groups G_1, G_2. □

COROLLARY 16.5.7. *Let* G *be a free abelian group,* $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$ *a quasi-generic YD-datum of finite Cartan type with Cartan matrix* A*, and* λ *a linking parameter for* $\mathcal{D}$*. Then* $U(\mathcal{D}, \lambda)$ *is a domain.*

PROOF. In the notation of Section 16.4, $U^+_{\boldsymbol{q}} = U(\mathcal{D})$. Recall that by Proposition 8.3.2(4), $U(\mathcal{D}, 0) \cong U(\mathcal{D})\#\Bbbk G$. Hence $U(\mathcal{D}, 0)$ is a domain by Proposition 16.4.5 and Lemma 16.5.6, and $U(\mathcal{D}, \lambda)$ is a domain by Theorem 16.5.5(5) and Lemma 16.4.1. □

LEMMA 16.5.8. *Let* G *be an abelian group, and* R *an* $\mathbb{N}_0$*-graded Hopf algebra in* ${}^H_H\mathcal{YD}$ *with* $R(0) = \Bbbk 1$*. Assume that* $V = R(1)$ *is finite-dimensional with basis* $(x_i)_{i\in I}$*, where for all* $i \in I$*,* $x_i \in V^{\chi_i}_{g_i}$*,* $g_i \in G$*, and* $\chi_i \in \widehat{G}$*. Let* $(q_{ij})_{i,j\in I}$ *be the braiding matrix, where* $q_{ij} = \chi_j(g_i)$ *for all* i, j*, and assume that* $(q_{ij})_{i,j\in I}$ *is generic of Cartan type. Let* $R\#\Bbbk G$ *be the bosonization. For all* $g, h \in G$*,* $\chi \in \widehat{G}$ *we define* $P^\chi_{g,h}(V\#\Bbbk G) = P^\chi_{g,h}(R\#\Bbbk G) \cap (V\#\Bbbk G)$*. Then*

(1) $V\#\Bbbk G = \bigoplus_{(\chi,g,h)\in\widehat{G}\times G\times G} P^\chi_{g,h}(V\#\Bbbk G)$.
(2) *For all* $i \in I$*,* $g \in G$*,* $P^{\chi_i}_{g_i g,g}(V\#\Bbbk G)$ *is one-dimensional with basis* $x_i \otimes g$.
(3) *Let* $\chi \in \widehat{G}$*,* $a, b \in G$*. If* $P^\chi_{a,b}(V\#\Bbbk G) \neq 0$*, then there is an element* $i \in I$ *with* $(\chi, a, b) = (\chi_i, g_i b, b)$.

By letting G to be the trivial group one can easily see that Lemma 16.5.8(2) does not hold if the braiding matrix is assumed to be quasi-generic instead of generic.

Proof. We first note that for all $i, j \in I$,

$$\text{if } i \neq j, \text{ then } (g_i, \chi_i) \neq (g_j, \chi_j). \tag{16.5.2}$$

Indeed, if $(g_i, \chi_i) = (g_j, \chi_j)$, then $q_{ij} = q_{ji} = q_{ii}$, hence $q_{ii}^2 = q_{ii}^{a_{ij}}$, where $(a_{ij})_{i,j\in I}$ is the Cartan matrix of $(q_{ij})_{i,j\in I}$. Then (16.5.2) follows, since q_{ii} is not a root of unity.

The elements $(x_i \otimes g)_{i\in I, g\in G}$ form a basis of $V \otimes \Bbbk G$, and

$$x_i \otimes g \in P^{\chi_i}_{g_i g, g}(V \# \Bbbk G) \text{ for all } g, i.$$

Let $0 \neq x = \sum_{i\in I, g\in G} \alpha_{i,g} x_i \otimes g$, where $\alpha_{i,g} \in \Bbbk$. Let $a, b \in G$, $\chi \in \widehat{G}$, and assume that $x \in P^{\chi}_{a,b}(V\#\Bbbk G)$, where $a, b \in G$, and $\chi \in \widehat{G}$. Since $x \in P_{a,b}(V\#\Bbbk G)$, there is a finite non-empty subset $J \subseteq I$ with

$$x = \sum_{i\in J} \alpha_{i,b} x_i \otimes b, \quad \text{and for all } i \in J,\ g_i b = a,\ \alpha_{i,b} \neq 0.$$

Since $g \cdot x = \chi(g)x$ for all $g \in G$, it follows that $\chi_i = \chi_j$ for all $i, j \in J$. Hence $|J| = 1$ by (16.5.2), and

$$x = \alpha_{i,b} x_i \otimes b \in P^{\chi_i}_{g_i b, b}, \text{ where } a = g_i b,\ \chi = \chi_i.$$

The lemma is proved. □

Proposition 16.5.9. *Let $\Bbbk$ be algebraically closed, and A a pointed Hopf algebra with coradical filtration $(A_n)_{n\geq 0}$, and abelian group $G = G(A)$. Let $R = (\operatorname{gr} A)^{\operatorname{co} \Bbbk G}$ with respect to the projection of $\operatorname{gr} A$ onto degree 0. Assume that $V = R(1) \in {}^G_G\mathcal{YD}$ is finite-dimensional with basis $(x_i)_{i\in I}$, where for all $i \in I$, $x_i \in V^{\chi_i}_{g_i}$, $g_i \in G$, and $\chi_i \in \widehat{G}$. Let $(q_{ij})_{i,j\in I}$ be the braiding matrix, where $q_{ij} = \chi_j(g_i)$ for all i, j, and assume that $(q_{ij})_{i,j\in I}$ is generic of Cartan type. Then*

1. $A_1 = A_0 \oplus \bigoplus_{(g,i)\in G\times I} P^{\chi_i}_{g_i g, g}(A)$.
2. *For each $i \in I$, there is a non-zero element $a_i \in A_1$ such that x_i is the residue class of a_i in A_1/A_0, and*
$$\Delta(a_i) = g_i \otimes a_i + a_i \otimes 1, \quad g a_i g^{-1} = \chi_i(g) a_i \text{ for all } g \in G.$$
For all $g \in G$, $i \in I$, $P^{\chi_i}_{g_i g, g}(A)$ is one-dimensional with basis $a_i g$.
3. *Let $\varepsilon \neq \chi \in \widehat{G}$, $a, b \in G$. If $P^{\chi}_{a,b}(A) \neq 0$, then there is an element $i \in I$ with $(\chi, a, b) = (\chi_i, g_i b, b)$.*

Proof. This follows from Proposition 5.4.16(2) and Lemma 16.5.8. □

Let A be a pointed Hopf algebra with abelian group $G(A)$. As in Corollary 5.3.16, there is a decomposition $\operatorname{gr} A \cong R\#\Bbbk G$. We say that the **infinitesimal braiding** of A is generic, if the Yetter-Drinfeld module $V = R(1)$ has a finite basis $(x_i)_{i\in I}$ with $x_i \in V^{\chi_i}_{g_i}$, $g_i \in G$, $\chi_i \in \widehat{G}$, such that $\chi_i(g_i)$ is not a root of unity for all $i \in I$.

Theorem 16.5.10. *Assume that $\Bbbk$ is algebraically closed. Let A be a pointed Hopf algebra such that $G = G(A)$ is abelian and $\operatorname{GKdim} \Bbbk G < \infty$. Then the following are equivalent.*

1. *The infinitesimal braiding of A is generic, and $\operatorname{GKdim} A < \infty$.*

(2) *There are a generic YD-datum $\mathcal{D}$ of finite Cartan type with group G, and a linking datum λ for $\mathcal{D}$ with*

$$A \cong U(\mathcal{D}, \lambda).$$

Assume (2) *and that $G(A)$ is finitely generated. Then A is a domain if and only if $G(A)$ is free abelian.*

PROOF. (1) $\Rightarrow$ (2). Let $\operatorname{gr} A \cong R\#\Bbbk G$ be the decomposition of Corollary 5.3.16, and let $V = R(1) \in {}^H_H\mathcal{YD}$. By assumption, V has a finite basis $(x_i)_{i\in I}$, where $x_i \in V^{\chi_i}_{g_i}$, $g_i \in G$, $\chi_i \in \widehat{G}$, such that $\chi_i(g_i)$ is not a root of unity for all $i \in I$. Let $q_{ij} = \chi_j(g_i)$ for all i, j. Since R is strictly graded, $R(n)$ is finite-dimensional for all $n \geq 0$ by Proposition 1.3.14.

Note that $\operatorname{GKdim} R \leq \operatorname{GKdim} \operatorname{gr} A \leq \operatorname{GKdim} A < \infty$ by Lemma 16.5.1. Hence by Corollary 16.4.25, R is the Nichols algebra $\mathcal{B}(V)$, and the braiding matrix $(q_{ij})_{i,j\in I}$ is of finite Cartan type with Cartan matrix $(a_{ij})_{i,j\in I}$. By Proposition 16.5.9(2), for all $i \in I$, we can choose preimages $a_i \in P^{\chi_i}_{g_i,1}(A)$ of x_i under the canonical map $A_1 \to A_1/A_0$.

Let $i, j \in I$, $i \neq j$. We claim that

(a) There is no $l \in I$ with $g_i^{1-a_{ij}} g_j = g_l$, and $\chi_i^{1-a_{ij}} \chi_j = \chi_l$.
(b) If $i \sim j$, then $\chi_i^{1-a_{ij}} \chi_j \neq \varepsilon$.
(c) If $i \sim j$, then $(\operatorname{ad} a_i)^{1-a_{ij}}(a_j) = 0$.
(d) If $i \not\sim j$, then $a_i a_j - q_{ij} a_j a_i = \lambda_{ij}(g_i g_j - 1)$, where $\lambda_{ij} \in \Bbbk$, and $\chi_j \chi_j = \varepsilon$ if $\lambda_{ij}(g_i g_j - 1) \neq 0$.

To prove (a), assume that $g_i^{1-a_{ij}} g_j = g_l$, and $\chi_i^{1-a_{ij}} \chi_j = \chi_l$ for some l. Then

$$q_{ii}^{a_{il}} = \chi_l(g_i)\chi_i(g_l) = \chi_i^{1-a_{ij}}(g_i)\chi_j(g_i)\chi_i(g_i)^{1-a_{ij}}\chi_i(g_j) = q_{ii}^{2-a_{ij}},$$

hence $a_{ij} + a_{il} = 2$, since q_{ii} is not a root of unity. Then $i = l$ and $a_{ij} = 0$, which implies $g_j = 1$. This is imposible, since $q_{jj} \neq 1$.

To prove (b), assume that $i \sim j$ and $\chi_i^{1-a_{ij}} \chi_j = \varepsilon$. Then

$$1 = \chi_i(g_i)^{1-a_{ij}}\chi_j(g_i) = q_{ii} q_{ji}^{-1}, \quad 1 = \chi_i(g_j)^{1-a_{ij}}\chi_j(g_j) = q_{ji}^{1-a_{ij}} q_{jj},$$

hence $q_{jj} = q_{ii}^{a_{ij}-1}$. By Lemma 8.2.4, there are an element $q \in \Bbbk$ (depending on the connected component containing i, j), and $d_i, d_j \in \{1, 2, 3\}$ with $q_{ii} = q^{d_i}$, $q_{jj} = q^{d_j}$. Since q is not a root of unity, we obtain the contradiction $d_j + (1 - a_{ij})d_i = 0$.

By Proposition 4.3.12, $(\operatorname{ad} a_i)^{1-a_{ij}}(a_j) \in P^{\chi_i^{1-a_{ij}}\chi_j}_{g_i^{1-a_{ij}}g_j,1}(A)$. Hence (c) follows from (a),(b) and Proposition 16.5.9(3).

To prove (d), assume that $i \not\sim j$. Then $a_{ij} = 0$, and

$$a_i a_j - q_{ij} a_j a_i = (\operatorname{ad} a_i)(a_j) \in P^{\chi_i\chi_j}_{g_i g_j,1}(A).$$

Suppose that $(\operatorname{ad} a_i)(a_j) \neq 0$. Then it follows from (a) and Proposition 16.5.9(3) that $\chi_i \chi_j = \varepsilon$. Thus $(\operatorname{ad} a_i)(a_j) \in \Bbbk G$, and (d) follows, since for all $g \in G$, $P_{g,1}(\Bbbk G) = \Bbbk(g - 1)$.

Let $\mathcal{D} = \mathcal{D}(G, (g_i)_{i\in I}, (\chi_i)_{i\in I})$, and define $\lambda = (\lambda_{ij})_{i,j\in I, i\not\sim j}$ by (d). Then $\mathcal{D}$ is a generic YD-datum of finite Cartan type, and λ is a linking parameter for $\mathcal{D}$. By (c) and (d), we have constructed a Hopf algebra map

$$\varphi : U(\mathcal{D}, \lambda) \to A, \ \ x_i \mapsto a_i, \, g \mapsto g \quad \text{for all } i \in I, \, g \in G.$$

The induced Hopf algebra map

$$\mathcal{B}(V)\#\Bbbk G \cong \operatorname{gr} U(\mathcal{D},\lambda) \xrightarrow{\operatorname{gr}\varphi} \operatorname{gr} A \cong \mathcal{B}(V)\#\Bbbk G$$

is the identity, where the first isomorphism follows from Theorem 16.5.5(4). Hence $\operatorname{gr}\varphi$ is an isomorphism. Then φ is an isomorphism by Lemma 5.2.14.

(2) $\Rightarrow$ (1) follows from Theorem 16.5.5.

Assume (2). If $G(A)$ is free then A is a domain by Corollary 16.5.7. If A is a domain, then $\Bbbk G(A)$ is a domain, and $G(A)$ must be free if it is finitely generated. $\square$

16.6. Notes

16.1. The theory of Yetter-Drinfeld modules over the Hopf algebra $\Bbbk[x;\chi,g]$, developed in Section 16.1, is a variation of the standard representation theory of $U_q(\mathfrak{sl}_2)$. Although the obtained results are very similar to those in the classical setting, there also exist essential differences.

We refer to the books [**Lus93**], [**Kas95**], [**Jan96**], [**KS97**] for the representation theory of $U_q(\mathfrak{sl}_2)$ when q is not a root of 1. The irreducible finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ for q a root of 1 have been determined first in [**RA89**].

16.2. In [**Lus93**] Lusztig defined the positive part $\mathbf{U}^+$ of his quantum group $\mathbf{U}$ over the rational function field $\mathbb{Q}(v)$ by a universal property (modding out the radical of a bilinear form). It is not difficult to see that this universal property defines $\mathbf{U}^+$ as a Nichols algebra, see [**Sch96**], and Proposition 2.7 in [**AS04**]. Lusztig proves in [**Lus93**], Theorem 33.1.3, that $\mathbf{U}^+$ is given by the Serre relations. Thus U_q^+, defined in Proposition 8.1.3, is a Nichols algebra for any symmetrizable Cartan matrix. Here q is transcendental, and $\Bbbk = \mathbb{Q}(q)$. This was noted independently in [**Ros95**], [**Ros98**]. In [**Lus93**], Corollary 40.2.2, a PBW-basis of U_q^+ over $\Bbbk = \mathbb{Q}(q)$, q transcendental, with Cartan matrix of finite type was constructed. In another approach, the algebra U_q^+ was constructed by Ringel in [**Rin95**] from the Hall algebra of the path algebra of a Dynkin quiver over finite fields. The Hopf algebra structure in this approach was found by Green [**Gre95**].

In the special case of $U_q(\mathfrak{g})$, q not a root of unity, $\mathfrak{g}$ a semisimple Lie algebra, Theorem 16.2.5(2) follows from Corollary 8.30 in [**Jan96**], and Theorem 16.2.1 is shown in Theorem 8.24 in [**Jan96**]. The proofs of these results in [**Jan96**] are long and technical using the explicit relations and case by case considerations (referring to [**Lus93**] for the case of G_2).

Independently of [**Lus93**], Theorem 16.2.5(2) and Theorem 16.2.1 were shown in [**Ang09**] (over algebraically closed fields of characteristic zero). Angiono's work is based on the theory of Lyndon words, the construction of a PBW-basis in [**Kha99**], and on the Weyl groupoid in [**Hec06**]. He discusses the Cartan matrices of finite type case by case.

16.3. In [**Lus90a**], [**Lus90b**] Lusztig defined a new class of finite-dimensional Hopf algebras, the so-called small quantum groups or Frobenius-Lusztig kernels $\mathfrak{u}$. This was a break-through in the theory of finite-dimensional pointed Hopf algebras.

Let $(a_{ij})_{i,j\in\mathbb{I}}$ be a finite Cartan matrix, $d_i a_{ij} = d_j a_{ji}$ for all $i,j \in \mathbb{I}$, where $d_i \in \{1,2,3\}$ for all i. Assume that $\operatorname{char}(\Bbbk) = 0$. Let $1 \neq q \in \Bbbk$ be a primitive root of 1 of order N. We consider the braiding matrix $\boldsymbol{q} = (q^{d_i a_{ij}})_{i,j\in\mathbb{I}}$. Let $\mathbf{B}$ be the quotient of $\mathbb{Q}[v,v^{-1}]$ by the ideal generated by the N-th cyclotomic polynomial (where v is an indeterminate). In [**Lus90b**, Section 8] Lusztig defines the algebra $\mathbf{u}^+$ over the field $\mathbf{B}$. Let $\mathbf{B} \to \Bbbk$ be the field homomorphism given by $v \mapsto q$, and define $\mathbf{u}^+_\Bbbk = \mathbf{u}^+ \otimes_\mathbf{B} \Bbbk$ by specialization. Then by [**Lus90b**, Theorem 8.3], $\mathbf{u}^+_\Bbbk$ is a pre-Nichols algebra of M (defined in the beginning of Section 16.3) of dimension $\prod_{i=1}^t N_i$, where N_i is the order of $q^{2d_i} = q_{ii}$ for all $1 \leq i \leq t$ in the notation of Theorem 16.3.3. Since by Theorem 16.3.3, the Nichols algebra $\mathcal{B}(M)$ has the same dimension, it follows that $\mathbf{u}^+_\Bbbk \cong u^+_{\boldsymbol{q}} = \mathcal{B}(M)$.

It was observed independently in [**Ros92**] and in [**Mue98**], Section 2, that the positive part of the small quantum group is a Nichols algebra (under some restrictions on the order of the root of 1).

Let us go back to the situation of M in the beginning of Section 16.3 of a braiding matrix $\boldsymbol{q}$ of finite Cartan type, where the $q_{ii} \neq 1$ are roots of 1. In [**AD05**, Theorem 3.9], Andruskiewitsch and Dăscălescu gave a presentation of type A Nichols algebras of diagonal type by generators and explicit relations under the assumption that each entry of the braiding matrix is ± 1. They noticed that for the presentation of these Nichols algebras the quantum Serre relations and the root vector relations are not sufficient. Then Angiono in [**Ang09**, Theorem 5.25] described the Nichols algebra $\mathcal{B}(M)$, where M is as above, by generators and relations. ([**AA17**] contains a more explicit list but also some unfortunate mistakes in types F_4 and C_θ, $\theta \geq 3$, when q has order 4, and in type G_2 when q has order 6. Additional relations in these cases are given in [**Ang13**, Theorem 3.1].) Angiono introduced root vectors x_α for all positive roots α in the tensor algebra $T(V)$ as iterated commutators coming from the theory of Lyndon words. In his list the relations consist of only the Serre relations and the root vector relations if and only if $\boldsymbol{q}$ is genuinely of finite Cartan type.

In view of Remark 16.3.15, the algebras in Theorem 16.3.14(1) for $q_{ij} = \epsilon^{d_i a_{ij}}$ for some root ϵ of 1, and their analogs for generic parameters have been studied in detail already in [**DCP93**] and were denoted by U^w_ϵ and U^w, respectively.

16.4. The filtration in the proof of Proposition 16.4.5 goes back to De Concini and Kac, see 10.1 in [**DCP93**]. The main part of the proof are the Levendorskii-Soibelman relations which we derived over any field as a special case of Theorem 14.1.12. They were shown in [**DCP93**], Theorem 9.3 and Appendix, over the field of rational functions $\mathbb{C}(q)$, q transcendental, by reduction to rank two and going through all cases in rank two.

The first classification results on the braiding of Nichols algebras with diagonal braiding and finite Gelfand-Kirillov dimension were obtained in [**Ros98**] over the field $\Bbbk = \mathbb{C}$, where the q_{ii} are positive real numbers.

16.5. Lemma 16.5.1 is Lemma 6.5 in [**KL00**], and the following results on the Gelfand-Kirillov dimension are special cases of the theory of Zhuang in [**Zhu13**]. In Theorem 5.4 he proves the following. Let G be a group, A a pointed Hopf algebra with $G = G(A)$, and $R = (\operatorname{gr} A)^{\operatorname{co} \Bbbk G}$. Assume that R is a finitely generated algebra, and $R(1)$ is finite-dimensional. Then

$$\operatorname{GKdim} R + \operatorname{GKdim} \Bbbk G = \operatorname{GKdim} \operatorname{gr} A = \operatorname{GKdim} A.$$

His proof depends heavily on Takeuchi's construction of free Hopf algebras in [**Tak71**]. Our proof of Theorem 16.5.4 is a modification of Zhuang's proof. We avoid the use of [**Tak71**] by giving a direct argument under the additional assumptions of Theorem 16.5.4.

The coradical filtration of $U_q(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, and $\Bbbk = \mathbb{Q}(q)$, q transcendental, was determined in [**CM00**] without the theory of Nichols algebras.

The rest of Section 16.5 is essentially taken from [**AS04**], where the field is algebraically closed of characteristic 0, and where the braiding was assumed to be positive depending on a result of Rosso in [**Ros98**]. The results from [**AS04**] were then extended in [**AA08**] to the case of generic braidings using [**Hec06**]. Our proof (in arbitrary characteristic) follows instead from the previous theory in Chapter 8, Theorem 8.3.9, and Chapter 16, in particular from Corollary 16.4.25. We give a detailed exposition of the ideas of [**AS04**], where the arguments have been sketchy and partly unclear (in Lemma 4.4).

Let us consider the class of Hopf algebras A over an algebraically closed field satisfying the following axioms:

- A is a pointed Hopf algebra with free abelian group $G(A)$ of finite rank,
- A is a domain with $\operatorname{GKdim} A < \infty$,
- A is reductive (i.e., all finite-dimensional A-modules are semisimple),
- the infinitesimal braiding of A is generic.

By Theorem 16.5.10 together with Theorem 5.3 in [**ARS10**], the Hopf algebras in this class are up to isomorphism the Hopf algebras $U(\mathcal{D}_{\mathrm{red}}, \ell)$, where $\mathcal{D}_{\mathrm{red}}$ is a generic, reduced YD-datum of finite Cartan type with free abelian group G of finite rank, linking parameter ℓ, and finite quotient group G/G^2 (see Notes to Section 8.4). The relations in Example 8.4.7 show that they are very close to the classical quantum groups $U_q(\mathfrak{g})$, q not a root of unity, $\mathfrak{g}$ semisimple, and to their multiparameter versions.

CHAPTER 17

Nichols algebras over non-abelian groups

Let G be any finite non-abelian group. In this chapter we focus on applications of the reflection theory to the structure of Nichols algebras over G. In particular, we prove that the Nichols algebra of a direct sum of at least two irreducible Yetter-Drinfeld modules over a finite simple group is infinite-dimensional. A more surprising application concerns the structure of Nichols algebras of irreducible Yetter-Drinfeld modules, which is possible due to the functoriality of the Nichols algebra and the independence of the defining group.

In Sections 17.2 and 17.3 we collect the outcomes of certain classification results without proofs in order to provide more examples. We end the Chapter with a discussion of further main research directions which are not covered in the book.

17.1. Finiteness criteria for Nichols algebras over non-abelian groups

Let G be a finite non-abelian group. Assume that the characteristic of the field $\Bbbk$ does not divide the order of G. Let $H = \Bbbk G$.

DEFINITION 17.1.1. Let $\mathcal{O}', \mathcal{O}''$ be conjugacy classes of G. We say that $\mathcal{O}'$ and $\mathcal{O}''$ **commute** if $st = ts$ for any $s \in \mathcal{O}'$, $t \in \mathcal{O}''$.

PROPOSITION 17.1.2. *Let $\mathcal{O}', \mathcal{O}''$ be conjugacy classes of G and $V = \bigoplus_{s \in \mathcal{O}'} V_s$ and $W = \bigoplus_{t \in \mathcal{O}''} W_t$ be irreducible objects in ${}^G_G\mathcal{YD}$.*

(1) *If $\operatorname{ad} V(W) = 0$ in $\mathcal{B}(V)$ then $\mathcal{O}'$ and $\mathcal{O}''$ commute.*
(2) *If $(\operatorname{ad} V)^2(W) = 0$ in $\mathcal{B}(V)$ then $\mathcal{O}'$ commutes with $\mathcal{O}'$ or with $\mathcal{O}''$.*

PROOF. (1) By Theorem 13.3.1, $\operatorname{ad} V(W)$ is isomorphic in ${}^H_H\mathcal{YD}$ to

$$X_1^{V,W} = T_1(V \otimes W) = (\mathrm{id} - c_{W,V}c_{V,W})(V \otimes W).$$

Let $g \in \mathcal{O}'$, $h \in \mathcal{O}''$ and assume that $gh \neq hg$. Then

$$\begin{aligned} c_{W,V}c_{V,W}(V_g \otimes W_h) &= c_{W,V}(W_{ghg^{-1}} \otimes V_g) \\ &= V_{ghgh^{-1}g^{-1}} \otimes W_{ghg^{-1}} \neq V_g \otimes W_h \end{aligned}$$

since $ghg^{-1} \neq h$. Hence $c_{W,V}c_{V,W} \neq \mathrm{id}_{V \otimes W}$ and $\operatorname{ad} V(W) \neq 0$.

(2) Let $c_1 = c_{V,V} \otimes \mathrm{id}_W$ and $c_2^2 = \mathrm{id}_V \otimes c_{W,V}c_{V,W}$ in $\mathrm{Aut}(V \otimes V \otimes W)$. By Theorem 13.3.1, $(\operatorname{ad} V)^2(W)$ is isomorphic in ${}^H_H\mathcal{YD}$ to

$$\begin{aligned} X_2^{V,W} &= (S_2 \otimes \mathrm{id})T_2(V \otimes V \otimes W) \\ &= (\mathrm{id} + c_1)(\mathrm{id} - c_2^2 c_1)(\mathrm{id} - c_2^2)(V \otimes V \otimes W). \end{aligned}$$

Assume that $(\operatorname{ad} V)^2(W) = 0$ and that $\mathcal{O}'$ and $\mathcal{O}''$ do not commute with $\mathcal{O}'$. Let $g \in \mathcal{O}'$ and let $f \in \mathcal{O}'$, $h \in \mathcal{O}''$ with $fg \neq gf$, $gh \neq hg$.

Let $v_1 \in V_f$, $v_2 \in V_g$, and $w \in W_h$ be non-zero. Then $X_2^{V,W}(v_1 \otimes v_2 \otimes w)$ is the sum of non-zero tensors t_i, $1 \le i \le 8$, where $t_i \in V_r \otimes V_s \otimes W_t$, $(r,s,t) = Y_i$ for any $1 \le i \le 8$, and

$$\begin{aligned}(Y_i)_{1\le i\le 8} = \big(&(f,g,h),(f,gh\triangleright g, g\triangleright h),(f\triangleright g, fh\triangleright f, f\triangleright h),\\ &(fgh\triangleright g, fghg^{-1}\triangleright f, fg\triangleright h),(f\triangleright g,f,h),(fgh\triangleright g,f,g\triangleright h),\\ &(fgh\triangleright f, f\triangleright g, f\triangleright h),(fgh\triangleright f, fgh\triangleright g, fg\triangleright h)\big),\end{aligned}$$

and $\triangleright$ means left adjoint action in G. Since $X_2^{V,W}(v_1 \otimes v_2 \otimes w) = 0$, the triple $Y_1 = (f,g,h)$ (like any Y_i with $1 \le i \le 8$) has to coincide with one of the other seven triples. Since $fg \ne gf$ and $gh \ne hg$, only $Y_1 = Y_4$ or $Y_1 = Y_8$ is possible, and hence $h = fg \triangleright h$. Thus Y_1, Y_4, Y_5 and Y_8 have h as the third entry. Moreover, $g \triangleright h \ne h$ and hence $h = fg \triangleright h \ne f \triangleright h$, that is, f and h do not commute. Hence $Y_4 = Y_i$ for some $i \in \{1,5,8\}$. By comparing the first entries, only $Y_4 = Y_1$ remains possible, hence $f = gh \triangleright g$.

We started with an arbitrary $f \in \mathcal{O}'$ and $h \in \mathcal{O}''$ with $fg \ne gf$ and $gh \ne hg$, and obtained that $f = gh \triangleright g$. Hence precisely one element of $\mathcal{O}'$ does not commute with g, which is absurd. This implies the claim. □

Let $C_f(G)$ denote the set of conjugacy classes $\mathcal{O}$ of G such that $\mathcal{B}(V)$ is finite-dimensional for some $V \in {}^G_G\mathcal{YD}$ with $V = \bigoplus_{s\in\mathcal{O}} V_s \ne 0$.

REMARK 17.1.3. Assume that the characteristic of $\Bbbk$ is 0. Then the conjugacy class $\{1\}$ is not contained in $C_f(G)$. Indeed, Let $V = V_1 \in {}^G_G\mathcal{YD}$ with $V \ne 0$ and let $v \in V \setminus \{0\}$. Then $c_{V,V}(v \otimes v) = v \otimes v$, and hence $\mathcal{B}(\Bbbk v) = \Bbbk[v]$ by Example 1.10.1 and since the characteristic of $\Bbbk$ is 0. Hence $\mathcal{B}(\Bbbk v)$ and $\mathcal{B}(V)$ are infinite-dimensional.

THEOREM 17.1.4. *Assume that any two conjugacy classes in $C_f(G)$ do not commute. Let $U \in {}^G_G\mathcal{YD}$. If $\mathcal{B}(U)$ is finite-dimensional, then $U = 0$ or U is irreducible in ${}^G_G\mathcal{YD}$.*

PROOF. By Proposition 1.4.20, U is the direct sum of irreducible subobjects. By Remark 1.6.19, any injection $f : V \to W$ with $V, W \in {}^G_G\mathcal{YD}$ induces an injection $\mathcal{B}(f) : \mathcal{B}(V) \to \mathcal{B}(W)$. Hence it suffices to prove that $\mathcal{B}(V \oplus W)$ is infinite-dimensional for any two irreducible objects $V, W \in {}^G_G\mathcal{YD}$.

Let $V, W \in {}^G_G\mathcal{YD}$ be irreducible objects and let $\mathcal{O}', \mathcal{O}''$ be conjugacy classes of G such that $V = \bigoplus_{s\in\mathcal{O}'} V_s$, $W = \bigoplus_{t\in\mathcal{O}''} W_t$. Assume that $\mathcal{B}(V \oplus W)$ is finite-dimensional. Then $\mathcal{B}(V)$ and $\mathcal{B}(W)$ are finite-dimensional by the above and hence $\mathcal{O}', \mathcal{O}'' \in C_f(G)$. Let $M = (V, W) \in \mathcal{F}_2^H$. By Corollary 14.5.3, M admits all reflections and $\mathcal{G}(M)$ is a finite Cartan graph. By Theorem 10.2.18, there exists $P \in \mathcal{F}_2^H(M)$ such that $A^{[P]}$ is of finite type. Since $\dim \mathcal{B}(P) = \dim \mathcal{B}(M)$ by Proposition 13.6.4, we may assume that $P = M$ (and hence A^M is of finite type).

By assumption, $\mathcal{O}'$ and $\mathcal{O}''$ neither commute with themselves nor with each other. Hence

$$(\operatorname{ad} V)^2(W) \ne 0, \quad (\operatorname{ad} W)^2(V) \ne 0$$

by Proposition 17.1.2. Therefore $a_{12}^M, a_{21}^M < -1$. Then A^M is not of finite type, a contradiction. This finishes the proof of the theorem. □

COROLLARY 17.1.5. *Assume that G is a non-abelian finite simple group and that the characteristic of $\Bbbk$ is 0. Let $U \in {}^G_G\mathcal{YD}$. If $\mathcal{B}(U)$ is finite-dimensional, then $U = 0$ or U is irreducible in ${}^G_G\mathcal{YD}$.*

PROOF. Let $\mathcal{O} \in C_f(G)$. Then $\mathcal{O} \neq \{1\}$ by Remark 17.1.3 and since the characteristic of $\Bbbk$ is 0. The subgroup $\langle\mathcal{O}\rangle$ of G generated by $\mathcal{O}$ is normal in G. Hence $\langle\mathcal{O}\rangle = G$, since G is simple and $\mathcal{O} \neq \{1\}$. Since G is non-abelian, it follows that any two conjugacy classes of G in $C_f(G)$ do not commute. Hence the Corollary follows from Theorem 17.1.4. $\square$

We prepare another corollary of Theorem 17.1.4 with two lemmas.

LEMMA 17.1.6. *Assume that $\Bbbk$ contains a primitive third root of 1 and the characteristic of $\Bbbk$ is 0. Then the conjugacy class of $(1\,2\,3)$ is not in $C_f(\mathbb{S}_3)$.*

PROOF. Let $g = (1\,2\,3)$ and let $V \in {}^G_G\mathcal{YD}$ with $V = V_g \oplus V_{g^{-1}}$, $V \neq 0$. Since $g^3 = 1$ in G, by assumption there exists $\zeta \in \Bbbk$ and $v \in V_g \setminus \{0\}$ such that $\zeta^3 = 1$ and $gv = \zeta v$. If $\zeta = 1$, then $\Bbbk v \in {}^{G'}_{G'}\mathcal{YD}$ for some group G' by Remark 1.5.4. Thus $\dim \mathcal{B}(\Bbbk v) = \infty$ by Example 1.10.1, and hence $\dim \mathcal{B}(V) = \infty$. Assume now that $\zeta \neq 1$. Let $w = (1\,2)v$. Then $w \in V_{g^{-1}}$ and $gw = (1\,2)g^{-1}v = \zeta^{-1}w$. Hence $V' = \Bbbk v \oplus \Bbbk w$ is a braided vector space of diagonal type with braiding matrix $(q_{ij})_{1\le i,j\le 2}$, where

$$q_{11} = q_{22} = \zeta, \quad q_{12} = q_{21} = \zeta^{-1}.$$

Again, $\Bbbk v, \Bbbk w \in {}^{G'}_{G'}\mathcal{YD}$ for some group G'. Moreover, V' is of Cartan type with Cartan matrix $A = (a_{ij})_{1\le i,j\le 2}$, $a_{12} = a_{21} = -2$. Thus $\mathcal{B}(V')$ and $\mathcal{B}(V)$ are infinite dimensional by Theorem 15.1.14(6). This proves the Lemma. $\square$

LEMMA 17.1.7. *Assume that the characteristic of $\Bbbk$ is 0. Then the conjugacy class of $(1\,2)(3\,4)$ is not in $C_f(\mathbb{S}_4)$.*

PROOF. Let $G = \mathbb{S}_4$, $g = (1\,2)(3\,4)$, $h = (1\,3)(2\,4) \in G$ and let

$$V = V_g \oplus V_h \oplus V_{gh} \in {}^G_G\mathcal{YD}$$

be an irreducible object. The centralizer G_0 of g in $\mathbb{S}_4$ is generated by $(1\,2)$ and h and has order 8. Moreover, $g = (1\,2)h(1\,2)h$.

Assume first that $\dim V_g = 1$ and let $v \in V_g \setminus \{0\}$. Then $(1\,2)v = \varepsilon v$, $hv = \eta v$ for some $\varepsilon, \eta \in \{1, -1\}$. Thus

$$gv = (1\,2)h(1\,2)hv = \epsilon^2\eta^2 v = v$$

and hence $\dim \mathcal{B}(\Bbbk v) = \infty$ by Example 1.10.1.

Assume that $\dim V_g > 1$. Since $g^2 = h^2 = 1$ and $gh = hg$, it follows that $\dim V_g = 2$ and V_g is the $\Bbbk G_0$-module induced by a one-dimensional representation of the abelian subgroup of G_0 generated by g and h. Let $v \in V_g \setminus \{0\}$ and let $\varepsilon, \eta \in \{1, -1\}$ such that $gv = \varepsilon v$, $hv = \eta v$. If $\varepsilon = 1$, then again $\dim \mathcal{B}(\Bbbk v) = \infty$ by Example 1.10.1. If $\varepsilon = -1$, then let

$$v_1 = (1\,2)v \in V_g, \quad v_2 = (1\,3)v \in V_{gh}, \quad v_3 = (1\,4)v \in V_h.$$

Then $V' = \Bbbk v_1 + \Bbbk v_2 + \Bbbk v_3$ is a three-dimensional braided vector space of diagonal type with braiding matrix

$$\begin{pmatrix} -1 & -\eta & \eta \\ \eta & -1 & -\eta \\ -\eta & \eta & -1 \end{pmatrix}.$$

Since $\eta^2 = 1$, this braiding is of Cartan type with Cartan matrix $(a_{ij})_{1\le i,j\le 3}$, where $a_{ij} = -1$ for all $1 \le i,j \le 3$, $i \ne j$. Thus $\mathcal{B}(V')$ and $\mathcal{B}(V)$ are infinite dimensional by Theorem 15.1.14(6). This proves the Lemma. □

COROLLARY 17.1.8. *Assume that $G = \mathbb{S}_n$ is the symmetric group with $n \ge 3$, the characteristic of $\Bbbk$ is 0, and if $n = 3$ then $\Bbbk$ contains a primitive third root of 1. Let $U \in {}^G_G\mathcal{YD}$. If $\mathcal{B}(U)$ is finite-dimensional, then $U = 0$ or U is irreducible in ${}^G_G\mathcal{YD}$.*

PROOF. Assume first that $n \ge 5$. Then the alternating group $\mathbb{A}_n$ is simple and is the only non-trivial normal subgroup of G. Let $\mathcal{O} \in C_f(G)$ and let G_0 be the subgroup of G generated by $\mathcal{O}$. Then $G_0 \ne \{1\}$ by Remark 17.1.3, since the characteristic of $\Bbbk$ is 0. Moreover, G_0 is a normal subgroup of $\mathbb{S}_n$, and hence $\mathbb{A}_n \subseteq G_0$. Since $\mathbb{A}_n$ is non-abelian, it follows that any two conjugacy classes of G in $C_f(G)$ do not commute. Hence the Corollary follows from Theorem 17.1.4.

Assume that $n = 3$ or $n = 4$. Again, $\{1\} \notin C_f(G)$ by Remark 17.1.3. Moreover, the class of $(1\,2\,3)$ is not in $C_f(\mathbb{S}_3)$ by Lemma 17.1.6, and the class of $(1\,2)(3\,4)$ is not in $C_f(\mathbb{S}_4)$ by Lemma 17.1.7. It follows that any two conjugacy classes of G in $C_f(G)$ do not commute. Hence the Corollary follows from Theorem 17.1.4. □

We also formulate an application of Corollary 14.5.1(5) for $H = \Bbbk G$.

PROPOSITION 17.1.9. *Assume that $\Bbbk$ is algebraically closed and the characteristic of $\Bbbk$ does not divide the order of G. Let $\mathcal{O}'$ and $\mathcal{O}''$ be conjugacy classes of G and let $V = \bigoplus_{g\in\mathcal{O}'} V_g$ and $W = \bigoplus_{h\in\mathcal{O}''} W_h$ be irreducible Yetter-Drinfeld modules over G. Assume that $\operatorname{ad} V(W) \subseteq \mathcal{B}(V \oplus W)$ is irreducible in ${}^G_G\mathcal{YD}$. Then $(gh)^2 = (hg)^2$ for any $g \in \mathcal{O}'$, $h \in \mathcal{O}''$.*

PROOF. By Theorem 13.3.1, $\operatorname{ad} V(W) \subseteq \mathcal{B}(V \oplus W)$ is isomorphic in ${}^H_H\mathcal{YD}$ to $(\mathrm{id}_{V\otimes W} - c_{W,V}c_{V,W})(V \otimes W)$. Let now $g \in \mathcal{O}'$, $h \in \mathcal{O}''$, $v \in V_g$ and $w \in W_h$, and let $y = (\mathrm{id}_{V\otimes W} - c_{W,V}c_{V,W})(v \otimes w)$. Then

$$y = v \otimes w - ghg^{-1} \cdot v \otimes g \cdot w \in (V \otimes W)_{gh}.$$

By Proposition 1.4.21 there exist q_V, q_W and q in $\Bbbk^\times$ such that

$$g \cdot v = q_V v, \quad h \cdot w = q_W w, \quad gh \cdot y = qy.$$

In particular,

$$q_W gh \cdot v \otimes g \cdot w - q_V^{-1} ghgh \cdot v \otimes ghg \cdot w = qv \otimes w - qq_V^{-1} gh \cdot v \otimes g \cdot w.$$

Since

$$\begin{gathered} gh \cdot v \otimes g \cdot w \in V \otimes W_{ghg^{-1}}, \\ ghgh \cdot v \otimes ghg \cdot w \in V \otimes W_{ghgh(ghg)^{-1}}, \\ v \otimes w \in V \otimes W_h, \end{gathered}$$

we conclude that $ghgh(ghg)^{-1} = h$. This implies the claim. □

COROLLARY 17.1.10. *Assume that $\Bbbk$ is algebraically closed and the characteristic of $\Bbbk$ does not divide the order of G. Let $\mathcal{O}'$ and $\mathcal{O}''$ be conjugacy classes of G and let $V = \bigoplus_{g\in\mathcal{O}'} V_g$ and $W = \bigoplus_{h\in\mathcal{O}''} W_h$ be irreducible Yetter-Drinfeld modules over G. If $M = (V, W)$ admits all reflections and $\mathcal{G}(M)$ is finite, then $(gh)^2 = (hg)^2$ for any $g \in \mathcal{O}'$, $h \in \mathcal{O}''$.*

Reference	$\dim V$	$\dim \mathcal{B}(V)$
Example 17.2.1	1	$N(q)$
Example 17.2.2	3	12
Example 17.2.4	4	72
Example 17.2.6	4	5184
Example 17.2.7	5	1280
Example 17.2.7	5	1280
Example 17.2.2	6	576
Example 17.2.3	6	576
Example 17.2.5	6	576
Example 17.2.7	7	326592
Example 17.2.7	7	326592
Example 17.2.2	10	8294400
Example 17.2.3	10	8294400

TABLE 17.1. Examples of finite-dimensional Nichols algebras of simple Yetter-Drinfeld modules over groups

PROOF. If M admits all reflections and $\mathcal{G}(M)$ is finite, then $\operatorname{ad} V(W)$ is zero or irreducible by Corollary 14.5.1(5). Hence the Corollary follows from Propositions 17.1.2(1) and 17.1.9. $\square$

COROLLARY 17.1.11. *Assume that $\Bbbk$ is algebraically closed and the characteristic of $\Bbbk$ does not divide the order of G. Let $V \in {}^G_G\mathcal{YD}$ and assume that $\mathcal{B}(V)$ is finite-dimensional. Then $(gh)^2 = (hg)^2$ for all $g, h \in G$ with $V_g \neq 0$, $V_h \neq 0$ such that g is not conjugate to h in the subgroup $\langle g, h\rangle$.*

PROOF. Let $S = \langle g, h\rangle$, and $W = \bigoplus_{s\in S} V_s$. Then $W \in {}^S_S\mathcal{YD}$, and $\mathcal{B}(W)$ is embedded into $\mathcal{B}(V)$ by Lemma 7.1.5. Hence the claim follows from Corollary 17.1.10 and Corollary 14.5.3. $\square$

17.2. Finite-dimensional Nichols algebras of simple Yetter-Drinfeld modules

Assume that the field $\Bbbk$ is algebraically closed and its characteristic is not two. In this section we list all known irreducible Yetter-Drinfeld modules $V = \bigoplus_{g\in G} V_g$ over a group G such that G is generated by

$$\operatorname{supp}(V) = \{g \in G \mid V_g \neq 0\}$$

and $\mathcal{B}(V)$ is finite-dimensional. The results rely on [**Gn$^+$11**], [**HLV12**] and the references therein. In Table 17.1 we list some basic data of the examples in this section.

EXAMPLE 17.2.1. Let G be an abelian group and let V be an irreducible Yetter-Drinfeld module over G. Example 1.4.2 implies that $\dim V = 1$. Let $v \in V \setminus \{0\}$ and let $g \in G$ and $q \in \Bbbk^\times$ such that

$$\delta_V(v) = g \otimes v, \quad c_{V,V}(v \otimes v) = qv \otimes v.$$

If G is generated by $\mathrm{supp}(V)$ then G is cyclic. By Example 1.10.1, the Nichols algebra $\mathcal{B}(V)$ is finite-dimensional if and only if $N(q)$ is an integer. In that case, $\mathcal{B}(V) \cong \Bbbk[x]/(x^{N(q)})$.

EXAMPLE 17.2.2. Let $n \geq 3$ and $G = \mathbb{S}_n$. Let $\mathcal{O}_2 = \{(i\,j) \mid 1 \leq i < j \leq n\}$ be the conjugacy class of transpositions in $\mathbb{S}_n$. As in Example 1.4.7 let V_n be the Yetter-Drinfeld module in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$ with basis x_t, $t \in \mathcal{O}_2$, such that

$$\delta_{V_n}(x_t) = t \otimes x_t, \quad s \cdot x_t = \mathrm{sign}(s) x_{s \triangleright t} \ \text{ for all } t \in \mathcal{O}_2,\ s \in \mathbb{S}_n.$$

Then V_n is irreducible in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$.

As mentioned in Example 1.10.3,

$$\dim \mathcal{B}(V_3) = 12, \quad \dim \mathcal{B}(V_4) = 576, \quad \dim \mathcal{B}(V_5) = 8.294.400,$$

and for none of the integers $n \geq 6$ it is known whether $\mathcal{B}(V_n)$ is finite-dimensional. The defining relations of $\mathcal{B}(V_n)$ for $3 \leq n \leq 5$ are the following quadratic relations.

$$\begin{aligned} x_t^2 &= 0 \text{ for all } t \in \mathcal{O}_2, \\ x_s x_t + x_t x_s &= 0 \text{ for all } s, t \in \mathcal{O}_2 \text{ with } st = ts,\ s \neq t, \\ x_s x_t + x_t x_{t \triangleright s} + x_{t \triangleright s} x_s &= 0 \text{ for all } s, t \in \mathcal{O}_2 \text{ with } st \neq ts. \end{aligned}$$

These Nichols algebras appeared first in [**MS00**, §5].

EXAMPLE 17.2.3. We discuss another family of examples related to those in Example 17.2.2. They appeared first in [**MS00**, §5] and in [**FK99**]. Let $n \geq 3$ and $G = \mathbb{S}_n$. Let $\mathcal{O}_2 = \{(i\,j) \mid 1 \leq i < j \leq n\}$ be the conjugacy class of transpositions in $\mathbb{S}_n$. Let W_n be the Yetter-Drinfeld module in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$ with basis x_{ij} with $1 \leq i < j \leq n$, such that

$$\delta_{W_n}(x_{ij}) = (i\,j) \otimes x_{ij}, \quad s \cdot x_{ij} = x_{s(i)s(j)} \ \text{ for all } 1 \leq i < j \leq n,\ s \in \mathbb{S}_n,$$

where $x_{ji} = -x_{ij}$ for any $1 \leq i < j \leq n$. Then W_n is irreducible in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$. Proposition 1.4.17 implies that the Yetter-Drinfeld modules V_n in Example 17.2.2 and W_n are isomorphic for $n = 3$ but non-isomorphic for $n > 3$.

As in Example 17.2.2,

$$\dim \mathcal{B}(W_3) = 12, \quad \dim \mathcal{B}(W_4) = 576, \quad \dim \mathcal{B}(W_5) = 8.294.400,$$

and for none of the integers $n \geq 6$ it is known whether $\mathcal{B}(W_n)$ is finite-dimensional. The defining relations of $\mathcal{B}(W_n)$ for $3 \leq n \leq 5$ are the following quadratic relations.

$$\begin{aligned} x_{ij}^2 &= 0 \text{ for all } 1 \leq i < j \leq n, \\ x_{ij} x_{kl} - x_{kl} x_{ij} &= 0 \text{ for all } i, j, k, l \in \{1, \dots, n\},\ \#\{i, j, k, l\} = 4, \\ x_{ij} x_{jk} + x_{jk} x_{ki} + x_{ki} x_{ij} &= 0 \text{ for all } i, j, k \in \{1, \dots, n\},\ \#\{i, j, k\} = 3. \end{aligned}$$

The remaining examples are presented in terms of racks and two-cocycles. All of them can be realized as Yetter-Drinfeld modules over finite groups.

For the sake of completeness, next we recall Example 1.10.4.

EXAMPLE 17.2.4. Let $X = \{1, 2, 3, 4\}$ and let φ_i with $i \in X$ be the permutations

$$\varphi_1 = (2\,3\,4), \quad \varphi_2 = (1\,4\,3), \quad \varphi_3 = (1\,2\,4), \quad \varphi_4 = (1\,3\,2).$$

Then $(X, \triangleright)$ is a quandle with $x \triangleright y = \varphi_x(y)$ for all $x, y \in X$. Let $\boldsymbol{q}$ be the constant 2-cocycle with $\boldsymbol{q}_{x,y} = -1$ for all $x, y \in X$. Then $(\Bbbk X, c^{\boldsymbol{q}})$ with

$$c^{\boldsymbol{q}}(x \otimes y) = -x \triangleright y \otimes x \quad \text{for all } x, y \in X$$

is a braided vector space of group type, and $\dim \mathcal{B}(\Bbbk X) = 72$. This Nichols algebra appeared first in [**Gn00b**]. A description of $\mathcal{B}(V)$ by generators and relations is given in Example 1.10.4.

EXAMPLE 17.2.5. Let $(X, \triangleright)$ be the conjugacy class of 4-cycles in $\mathbb{S}_4$ considered as a rack. Using the enumeration

$$x_1 = (1\,2\,3\,4), \qquad x_2 = (1\,3\,4\,2), \qquad x_3 = (1\,4\,2\,3),$$
$$x_4 = (1\,3\,2\,4), \qquad x_5 = (1\,2\,4\,3), \qquad x_6 = (1\,4\,3\,2),$$

the corresponding maps $\varphi_i : X \to X$, $x_j \mapsto x_i \triangleright x_j$, can be written in cycle notation as

$$\varphi_1 = (x_2\,x_4\,x_5\,x_3), \qquad \varphi_2 = (x_1\,x_3\,x_6\,x_4), \qquad \varphi_3 = (x_1\,x_5\,x_6\,x_2),$$
$$\varphi_4 = (x_1\,x_2\,x_6\,x_5), \qquad \varphi_5 = (x_1\,x_4\,x_6\,x_3), \qquad \varphi_6 = (x_2\,x_3\,x_5\,x_4).$$

Then $(\Bbbk X, c^{\boldsymbol{q}})$ with the constant 2-cocycle $c^{\boldsymbol{q}} = -1$ is a braided vector space of group type, and $\dim \mathcal{B}(\Bbbk X) = 576$. The algebra $\mathcal{B}(\Bbbk X)$ can be presented by generators $x_1, \dots, x_6$ and relations

$$\begin{gathered}
x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 0,\\
x_1x_6 + x_6x_1 = 0, \quad x_2x_5 + x_5x_2 = 0, \quad x_3x_4 + x_4x_3 = 0,\\
x_1x_2 + x_2x_4 + x_4x_1 = 0, \quad x_1x_3 + x_3x_2 + x_2x_1 = 0,\\
x_1x_4 + x_4x_5 + x_5x_1 = 0, \quad x_1x_5 + x_5x_3 + x_3x_1 = 0,\\
x_2x_3 + x_3x_6 + x_6x_2 = 0, \quad x_2x_6 + x_6x_4 + x_4x_2 = 0,\\
x_3x_5 + x_5x_6 + x_6x_3 = 0, \quad x_4x_6 + x_6x_5 + x_5x_4 = 0.
\end{gathered}$$

The Nichols algebra $\mathcal{B}(\Bbbk X)$ appeared first in [**AGn03**].

EXAMPLE 17.2.6. Let X and φ_i with $i \in X$ be as in Example 17.2.4. Again, $(X, \triangleright)$ is a quandle with $x \triangleright y = \varphi_x(y)$ for all $x, y \in X$. Let $q \in \Bbbk$. Assume that $q^2 + q + 1 = 0$. Let $\boldsymbol{q}$ be the 2-cocycle given by the matrix

$$(\boldsymbol{q}_{x,y})_{x,y \in X} = \begin{pmatrix} q & q & q & q \\ q & q & -q & -q \\ q & -q & q & -q \\ q & -q & -q & q \end{pmatrix} \tag{17.2.1}$$

Then $(\Bbbk X, c^{\boldsymbol{q}})$ with

$$c^{\boldsymbol{q}}(x \otimes y) = \boldsymbol{q}_{x,y} x \triangleright y \otimes x \quad \text{for all } x, y \in X$$

is a braided vector space of group type, and $\dim \mathcal{B}(\Bbbk X) = 5184$. This example appeared first in [**HLV12**, §7]. We write a, b, c, and d for the standard basis vectors of $\Bbbk X$. Then $\mathcal{B}(V)$ has the following presentation by generators and relations:

$$\begin{gathered}
a^3 = b^3 = c^3 = d^3 = 0,\\
-q^2ab - qbc + ca = -q^2ac - qcd + da = 0,\\
qad - q^2ba + db = qbd + q^2cb + dc = 0,\\
a^2bcb^2 + abcb^2a + bcb^2a^2 + cb^2a^2b + b^2a^2bc + ba^2bcb\\
+bcba^2c + cbabac + cb^2aca = 0.
\end{gathered}$$

EXAMPLE 17.2.7. Let $(X, \triangleright)$ be one of the affine quandles $\mathrm{Aff}(5,2)$, $\mathrm{Aff}(5,3)$, $\mathrm{Aff}(7,3)$, $\mathrm{Aff}(7,5)$ in Example 1.5.14. Let $\boldsymbol{q}$ be the constant 2-cocycle -1. Then $(\Bbbk X, c^{\boldsymbol{q}})$ with

$$c^{\boldsymbol{q}}(x \otimes y) = -x \triangleright y \otimes x \quad \text{for all } x, y \in X$$

is a braided vector space of group type with finite-dimensional Nichols algebra.

The Nichols algebra of $(\Bbbk X, c^{\boldsymbol{q}})$ for $\mathrm{Aff}(5,i)$ with $i \in \{2,3\}$ has dimension 1280. The Nichols algebra for $i=2$ can be presented by generators x_0, x_1, x_2, x_3, x_4 and relations

$$\begin{aligned} x_i^2 &= 0, \quad i \in X,\\ x_0x_1 + x_1x_3 + x_3x_2 + x_2x_0 &= 0,\\ x_0x_2 + x_2x_1 + x_1x_4 + x_4x_0 &= 0,\\ x_0x_3 + x_3x_4 + x_4x_1 + x_1x_0 &= 0,\\ x_0x_4 + x_4x_2 + x_2x_3 + x_3x_0 &= 0,\\ x_1x_2 + x_2x_4 + x_4x_3 + x_3x_1 &= 0,\\ x_1x_0x_1x_0 + x_0x_1x_0x_1 &= 0 \end{aligned}$$

if $\Bbbk$ has characteristic zero. Similarly, the Nichols algebra for $i=3$ can be presented by generators x_0, x_1, x_2, x_3, x_4 and relations

$$\begin{aligned} x_i^2 &= 0, \quad i \in X,\\ x_0x_1 + x_1x_4 + x_4x_3 + x_3x_0 &= 0,\\ x_0x_2 + x_2x_3 + x_3x_1 + x_1x_0 &= 0,\\ x_0x_3 + x_3x_2 + x_2x_4 + x_4x_0 &= 0,\\ x_0x_4 + x_4x_1 + x_1x_2 + x_2x_0 &= 0,\\ x_1x_3 + x_3x_4 + x_4x_2 + x_2x_1 &= 0,\\ x_1x_0x_1x_0 + x_0x_1x_0x_1 &= 0 \end{aligned}$$

if $\Bbbk$ has characteristic zero. These examples appeared in [**AGn03**]. The Nichols algebra of $(\Bbbk X, c^{\boldsymbol{q}})$ for $\mathrm{Aff}(7,i)$ with $i \in \{3,5\}$ has dimension 326592. These examples appeared first on the web page of M. Graña. The Nichols algebra for $i=3$ can be presented by generators x_0, x_1, x_2, x_3, x_4, x_5, x_6 and relations

$$\begin{gathered} x_0^2 = x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 0,\\ x_0x_1 + x_1x_3 + x_3x_0 = 0, \quad x_0x_2 + x_2x_6 + x_6x_0 = 0,\\ x_0x_3 + x_3x_2 + x_2x_0 = 0, \quad x_0x_4 + x_4x_5 + x_5x_0 = 0,\\ x_0x_5 + x_5x_1 + x_1x_0 = 0, \quad x_0x_6 + x_6x_4 + x_4x_0 = 0,\\ x_1x_2 + x_2x_4 + x_4x_1 = 0, \quad x_1x_4 + x_4x_3 + x_3x_1 = 0,\\ x_1x_5 + x_5x_6 + x_6x_1 = 0, \quad x_1x_6 + x_6x_2 + x_2x_1 = 0,\\ x_2x_3 + x_3x_5 + x_5x_2 = 0, \quad x_2x_5 + x_5x_4 + x_4x_2 = 0,\\ x_3x_4 + x_4x_6 + x_6x_3 = 0, \quad x_3x_6 + x_6x_5 + x_5x_3 = 0,\\ x_0x_1x_2x_0x_1x_2 + x_1x_2x_0x_1x_2x_0 + x_2x_0x_1x_2x_0x_1 = 0 \end{gathered}$$

if $\Bbbk$ has characteristic zero. (The Gröbner basis calculation for this algebra over the rationals runs with the GBNP package of GAP using the ordering

$$x_0, x_1, x_2, x_3, x_6, x_4, x_5$$

of generators in a reasonable time.) Similarly, the Nichols algebra for $i = 5$ can be presented by generators x_0, x_1, x_2, x_3, x_4, x_5, x_6 and relations

$$\begin{gathered} x_0^2 = x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = x_6^2 = 0, \\ x_0x_1 + x_1x_5 + x_5x_0 = 0, \quad x_0x_2 + x_2x_3 + x_3x_0 = 0, \\ x_0x_3 + x_3x_1 + x_1x_0 = 0, \quad x_0x_4 + x_4x_6 + x_6x_0 = 0, \\ x_0x_5 + x_5x_4 + x_4x_0 = 0, \quad x_0x_6 + x_6x_2 + x_2x_0 = 0, \\ x_1x_2 + x_2x_6 + x_6x_1 = 0, \quad x_1x_3 + x_3x_4 + x_4x_1 = 0, \\ x_1x_4 + x_4x_2 + x_2x_1 = 0, \quad x_1x_6 + x_6x_5 + x_5x_1 = 0, \\ x_2x_4 + x_4x_5 + x_5x_2 = 0, \quad x_2x_5 + x_5x_3 + x_3x_2 = 0, \\ x_3x_5 + x_5x_6 + x_6x_3 = 0, \quad x_3x_6 + x_6x_4 + x_4x_3 = 0, \\ x_0x_1x_2x_0x_1x_2 + x_1x_2x_0x_1x_2x_0 + x_2x_0x_1x_2x_0x_1 = 0 \end{gathered}$$

if $\Bbbk$ has characteristic zero. The Gröbner basis calculation of GAP with the ordering $x_0, x_1, x_2, x_3, x_4, x_5, x_6$ terminates. A slightly better performance can be achieved using the ordering $x_0, x_1, x_6, x_2, x_3, x_5, x_4$.

17.3. Nichols algebras with finite root system of rank two

Assume that the field $\Bbbk$ is algebraically closed and its characteristic is neither two nor three. Let G be a finite non-abelian group. Let $\mathcal{O}'$ and $\mathcal{O}''$ be conjugacy classes of G and $V = \bigoplus_{g\in\mathcal{O}'} V_g$, $W = \bigoplus_{h\in\mathcal{O}''} W_h$ be irreducible Yetter-Drinfeld modules over G. Assume that the group G is generated by $\mathcal{O}' \cup \mathcal{O}''$ and that $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$. Without proofs we give a sufficient and necessary condition for the pair (V, W) such that $M = (V, W)$ admits all reflections and $\mathcal{G}(M)$ is finite. The results are based on the series of papers [**HS10a**, **HV14**, **HV15**, **HV17b**]. For the notation we refer mainly to Section 1.4.

For any $n \geq 2$ let Γ_n be the group given by generators a, b, ν and relations

$$ba = \nu ab, \quad \nu a = a\nu^{-1}, \quad \nu b = b\nu, \quad \nu^n = 1.$$

Following [**HV17b**], for $n = 3$ we will use another presentation of Γ_n. Let Γ_3' be the group given by generators γ, ζ, ν and relations

$$\gamma\nu = \nu^{-1}\gamma, \quad \gamma\zeta = \zeta\gamma, \quad \zeta\nu = \nu\zeta, \quad \nu^3 = 1.$$

Then there is a group isomorphism $e : \Gamma_3 \to \Gamma_3'$ with $e(a) = \gamma$, $e(b) = \zeta\nu^{-1}$, $e(\nu) = \nu$. Its inverse is given by $e^{-1}(\gamma) = a$, $e^{-1}(\zeta) = b\nu$, $e^{-1}(\nu) = \nu$.

Let T be the group given by generators ζ, χ_1, χ_2 and relations

$$\zeta\chi_1 = \chi_1\zeta, \quad \zeta\chi_2 = \chi_2\zeta, \quad \chi_1\chi_2\chi_1 = \chi_2\chi_1\chi_2, \quad \chi_1^3 = \chi_2^3.$$

An epimorphic image of Γ_n is non-abelian if and only if the image of ν is not 1. An epimorphic image of T is non-abelian if and only if the image of $\chi_1\chi_2^{-1}$ is not 1.

The following Theorem was proven in [**HV15**, Th. 7.3] and [**HV17b**, Th. 2.1]. As in the previous section, we use the notation

$$\mathrm{supp}(V) = \{g \in G \mid V_g \neq 0\}$$

for any group G and any Yetter-Drinfeld module $V \in {}^G_G\mathcal{YD}$.

THEOREM 17.3.1. *Assume that $\Bbbk$ is an algebraically closed field and its characteristic is neither two nor three. Let G be a non-abelian group and let V and W be finite-dimensional irreducible Yetter-Drinfeld modules over G. Assume that*

$c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$ and that the group G is generated by $\mathrm{supp}(V\oplus W)$. *Then G is an epimorphic image of one of the groups $\Gamma_2, \Gamma_3, \Gamma_4$ and T. Moreover, the following are equivalent.*

(1) *The pair $M = (V,W) \in \mathcal{F}_2^{\Bbbk G}$ admits all reflections and $\mathcal{G}(M)$ is finite.*
(2) *The Nichols algebra $\mathcal{B}(V\oplus W)$ is finite-dimensional.*
(3) *One of the pairs (V,W), (W,V) appears in Examples* 17.3.2*,* 17.3.3*,* 17.3.4*,* 17.3.5*,* 17.3.6*,* 17.3.7*, or* 17.3.8*.*

We now list the examples in Theorem 17.3.1 one by one.

Example 17.3.2. Let $f:\Gamma_2\to G$ be a group epimorphism and let $g=f(a)$, $h=f(b)$, and $\epsilon=f(\nu)$. Assume that $\epsilon\neq 1$. Let $V,W\in {}^G_G\mathcal{YD}$. Assume that $V\cong M(g,\rho)$ and $W\cong M(h,\sigma)$, where ρ is a character of $G^g=\langle \epsilon, g, h^2\rangle$ and σ is a character of $G^h=\langle\epsilon,h,g^2\rangle$. Assume that $\rho(\epsilon h^2)\sigma(\epsilon g^2)=1$ and $\rho(g)=\sigma(h)=-1$. Then $\dim V=\dim W=2$ and $\dim\mathcal{B}(V\oplus W)=64$. This example appeared first in [**HS10a**, Th. 4.6] and in a special case in [**MS00**, Ex. 6.5].

Example 17.3.3. Let $f:\Gamma_3'\to G$ be a group epimorphism and let $g=f(\gamma)$, $z=f(\zeta)$, and $\epsilon=f(\nu)$. Let $V,W\in {}^G_G\mathcal{YD}$. Assume that $V\cong M(g,\rho)$ and $W\cong M(\epsilon z,\sigma)$, where ρ is a character of $G^g=\langle g,z\rangle$ and σ is a character of $G^{\epsilon z}=\langle\epsilon,z,g^2\rangle$. Assume that

$$\rho(g)=\sigma(\epsilon z)=-1,\quad \rho(z^2)\sigma(\epsilon g^2)=1,\quad 1+\sigma(\epsilon)+\sigma(\epsilon)^2=0.$$

Then $\dim V=3$, $\dim W=2$, and $\dim\mathcal{B}(V\oplus W)=10368$. This example appeared first in [**HV17b**, Ex. 1.9].

Example 17.3.4. As in Example 17.3.3, let $f:\Gamma_3'\to G$ be a group epimorphism and let $g=f(\gamma)$, $z=f(\zeta)$, and $\epsilon=f(\nu)\neq 1$. Let $V,W\in {}^G_G\mathcal{YD}$. Assume that $V\cong M(g,\rho)$ and $W\cong M(\epsilon z,\sigma)$, where ρ is a character of $G^g=\langle g,z\rangle$ and σ is a character of $G^{\epsilon z}=\langle\epsilon,z,g^2\rangle$. Differently from Example 17.3.3, assume that

$$\rho(g)=\sigma(\epsilon z)=-1,\quad \rho(z^2)\sigma(\epsilon g^2)=1,\quad \sigma(\epsilon)=1.$$

Then $\dim V=3$, $\dim W=2$, and $\dim\mathcal{B}(V\oplus W)=2304$. This example also appeared first in [**HV17b**, Ex. 1.9].

Example 17.3.5. Let $f:\Gamma_3'\to G$ be a group epimorphism and let $g=f(\gamma)$, $z=f(\zeta)$, and $\epsilon=f(\nu)\neq 1$. Let $V,W\in {}^G_G\mathcal{YD}$. Assume that $V\cong M(g,\rho)$ and $W\cong M(z,\sigma)$, where ρ is a character of $G^g=\langle g,z\rangle$ and σ is a character of $G^z=G$. Assume that

$$\rho(g)=-1,\quad 1-\sigma(z)+\sigma(z)^2=0,\quad \rho(z)\sigma(gz)=1.$$

Then $\dim V=3$, $\dim W=1$, and $\dim\mathcal{B}(V\oplus W)=10368$. This example appeared first in [**HV17b**, Ex. 1.10].

Example 17.3.6. Let $f:\Gamma_3'\to G$ be a group epimorphism and let $g=f(\gamma)$, $z=f(\zeta)$, and $\epsilon=f(\nu)\neq 1$. Let $V,W\in {}^G_G\mathcal{YD}$. Assume that $V\cong M(g,\rho)$ and $W\cong M(z,\sigma)$, where ρ is a character of $G^g=\langle g,z\rangle$ and σ is an irreducible representation of $G^z=G$ of degree two. Then $\sigma(1+\epsilon+\epsilon^2)=0$, and the isomorphism class of σ is uniquely determined by the constants $\sigma(g^2)$ and $\sigma(z)$. (Note that g^2 and z are in the center of G.) Assume that

$$\rho(g)=\sigma(z)=-1,\quad \rho(z^2)\sigma(g^2)=1.$$

Then $\dim V = 3$, $\dim W = 2$, and $\dim \mathcal{B}(V \oplus W) = 2304$. This example appeared first in [**HV17b**, Ex. 1.11].

EXAMPLE 17.3.7. Let $f : \Gamma_4 \to G$ be a group epimorphism and let $g = f(a)$, $h = f(b)$, and $\epsilon = f(\nu)$. Let $V, W \in {}^G_G\mathcal{YD}$. Assume that $V \cong M(h, \rho)$ and $W \cong M(g, \sigma)$, where ρ is a character of $G^h = \langle \epsilon, h, g^2 \rangle$ and σ is a character of $G^g = \langle \epsilon^2, \epsilon^{-1}h^2, g \rangle$. Assume that

$$\rho(h) = \sigma(g) = -1, \quad \rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2), \quad \rho(\epsilon^2) = -1.$$

Then $\dim V = 2$, $\dim W = 4$, and $\dim \mathcal{B}(V \oplus W) = 262144$. This example appeared first in [**HV15**, Th. 5.4].

EXAMPLE 17.3.8. Let $f : T \to G$ be a group epimorphism and let $z = f(\zeta)$, $x_1 = f(\chi_1)$, and $x_2 = f(\chi_2)$. Then z is a central element of G. Let $V, W \in {}^G_G\mathcal{YD}$. Assume that $V \cong M(z, \rho)$ and $W \cong M(x_1, \sigma)$, where ρ is a character of $G^z = G$ and σ is a character of $G^{x_1} = \langle x_1, x_2^2 x_1 x_2^{-1}, z \rangle$. Assume that

$$\begin{aligned} \sigma(x_1) &= -1, & \sigma(x_2^2 x_1 x_2^{-1}) &= 1, \\ (\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 &= 0, & \rho(x_1 z)\sigma(z) &= 1. \end{aligned}$$

Then $\dim V = 1$, $\dim W = 4$, and $\dim \mathcal{B}(V \oplus W) = 80621568$. This example appeared first in [**HV15**, Th. 2.8].

17.4. Outlook

Besides the theory presented in our book, many problems on Nichols and related algebras have been studied in the literature. Let us describe some of those results, which are closely related to the theory discussed here.

In Section 15.3 we classified rank two braided vector spaces of diagonal type which have a finite Cartan graph and finite-dimensional Nichols algebra, respectively. The corresponding classification of higher rank braided vector spaces of diagonal type is much more involved and has been done in [**Hec09**], over fields of characteristic 0, based on the theory of reflections and the Weyl groupoid.

For a better understanding of finite-dimensional Nichols algebras of diagonal type it is desirable to provide a presentation by generators and relations. Such an explicit presentation was obtained in [**Hec07**, Section 8] for rank two Nichols algebras using Stern-Brocot trees and in [**Ang15**] and [**Ang13**] in general for each braided vector space from the list in [**Hec09**]. In the approach of Angiono, reflection theory and the structure theory of coideal subalgebras of Nichols algebras of diagonal type are fundamental ingredients.

Given a tensor decomposition of a Nichols algebra $\mathcal{B}(V)$ of diagonal type in the sense of Definition 14.4.1, it is easy to deduce the Gelfand-Kirillov dimension of $\mathcal{B}(V)$. Using this idea, in [**Hec06**] it was pointed out that any Nichols algebra of diagonal type admitting a finite Cartan graph has finite Gelfand-Kirillov dimension. The converse statement, that is, if the Nichols algebra of a finite-dimensional braided vector space V of diagonal type has finite Gelfand-Kirillov dimension, then V admits a finite Cartan graph, is an open problem. By [**AAH19**], the answer is positive in characteristic 0 if $\dim V = 2$.

In [**Hec06**], to any Nichols algebra $\mathcal{B}(V)$ of diagonal type a root system was attached, based on the theory of Lyndon words [**Kha15**]. The real roots of this root system can be explained by the theory of reflections. However, only very little is known about imaginary roots. If $\mathcal{B}(V)$ is the free algebra, then all roots and their multiplicities, which depend on the braiding, can be determined. In [**HZ18**] all V of diagonal type with $\mathcal{B}(V) = T(V)$ have been classified in terms of polynomial equations for the entries of the braiding matrix. These equations appeared before in a variation in [**Fd$^+$01**].

Any braided vector space of diagonal type can be realized as a Yetter-Drinfeld module over an abelian group. The converse is not true: a Yetter-Drinfeld module over an abelian group is not necessarily a braided vector space of diagonal type. Finite Gelfand-Kirillov dimensional Nichols algebras of such examples appeared in [**CLW09**] in positive characteristic, and later many more in [**AAH16**] in characteristic 0.

One of the main motivations and applications of the theory of Nichols algebras of diagonal type is the classification of finite-dimensional complex pointed Hopf algebras with abelian coradical by the lifting method, as explained first in [**AS98**], see also [**And14**]. By now this project can be considered to be completed. For a survey with emphasis on the calculation of the liftings we refer to [**AI18**]. A generalization of the lifting method to other types of Hopf algebras was presented in [**AC13**].

Finite-dimensional pre-Nichols algebras with some emphasis on braidings of diagonal type have been studied recently from the perspective of geometric invariant theory in [**Mei19**].

Another direction of research related to Nichols algebras of diagonal type was initiated by Kolb and Yakimov in [**KY19**] with their study of symmetric pair coideal subalgebras with Iwasawa decomposition.

Finite-dimensional Nichols algebras over non-abelian groups are much less understood. The main reason for this is that the braided vector space structure of a Yetter-Drinfeld module is typically very complicated. Conjecturally, non-abelian finite simple groups have no non-trivial finite-dimensional complex Nichols algebra. For the alternating groups this was proven in [**AF$^+$11a**]. Unexpectedly, the proof relies among others on the reflection theory applied to specific braided subspaces. Partial results for other non-abelian finite simple groups have been obtained in a series of papers such as [**F$^+$10**], [**AF$^+$11b**], [**ACG15**], [**ACG16**], and [**ACG17**]. Typically, in these papers for all simple Yetter-Drinfeld modules not appearing in a specific list it is shown that its Nichols algebra is infinite dimensional. Related results for symmetric and dihedral groups appeared in [**AFZ09**] and [**FG11**].

Albeit only little is known about finite-dimensional Nichols algebras of irreducible Yetter-Drinfeld modules over non-simple non-abelian groups, the classification of those semisimple non-simple Yetter-Drinfeld modules over any non-abelian group, which have a finite Cartan graph, has been obtained in [**HV17b**] and [**HV17a**]. (In fact, in the precise claim some natural technical assumptions on the group and on the Yetter-Drinfeld modules appear.) The outcome in rank two is presented in Section 17.3. It turned out that all the corresponding Nichols algebras are finite-dimensional. Note that the latter is false for abelian groups. Up to few exceptions, the finite-dimensional Nichols algebras in the classification have been constructed much earlier from Nichols algebras of diagonal type in [**Len14**].

Not much is known about Nichols algebras over Hopf algebras which are not group algebras. An interesting nontrivial example was studied in [**Xio19**]. Among others, it is shown there that the Nichols algebra of any non-semisimple Yetter-Drinfeld module over the 12-dimensional Hopf algebra without the dual Chevalley property is infinite dimensional. In [**AA18**], structure theory and examples of Nichols algebras over basic Hopf algebras are studied. The example in [**Xio19**] is a particular case, and again non-semisimple Yetter-Drinfeld modules have infinite-dimensional Nichols algebras. This indicates that reflection theory is likely to become a useful tool in very general settings.

The theory of Nichols algebras is potentially also crucial for the representation theory of pointed Hopf algebras. Fairly general problems have been studied among others in [**RS08b**], [**ARS10**], [**HY10**], and [**AYY15**].

Other areas of applications of the theory of Nichols algebras include Schubert calculus on Coxeter groups [**Baz06**], [**Liu15**], [**Bär19**] and quasi-quantum groups [**Ang10**], [**HL$^+$17**], [**BHK17**], [**GLO18**]. The former is based on the observation that the classical coinvariant ring can be embedded into a Fomin-Kirillov algebra. The latter is a non-associative version of the theory of Nichols algebras originated in the theory of tensor categories.

Nichols algebras appear to be an important algebraic tool in the representation theory of vertex operator algebras realized in non-semisimple logarithmic conformal field theory models. The key point here is that the algebra generated by screening operators, regarded as a braided Hopf algebra, is a Nichols algebra of diagonal type. The analysis of this structure enjoys increasing interest [**ST12**], [**ST13**], [**Sem14**], [**FL18**], [**Len17**], [**FL19**].

More recently, Nichols algebras over groups have been used to prove a conjecture of Malle on the number of extensions of a global field [**ETW17**]. In a related work [**KS19**] the Nichols algebra of an object V in a $\Bbbk$-linear abelian braided monoidal category is interpreted as the collection of the intersection cohomology extensions of the local systems on the open configuration spaces associated to the tensor powers of V.

Cartan graphs are closely related to simplicial arrangements via their sets of real roots. The classification of finite Cartan graphs was performed algorithmically in [**CH15**]. In contrast to the classification in rank two, in each rank only finitely many isomorphism classes of finite Cartan graphs exist. In recent research papers on the topic, additional properties of the arrangements of Cartan graphs, generalizations, and associated algebraic structures are studied. For more details we refer to [**BC12**], [**CMW17**], [**CL17**], [**AY18**], [**DW19**].

17.5. Notes

17.1. The results in Section 17.1 are taken essentially completely from [**HS10b**, Sect. 8]. Corollary 17.1.11 and variations of it have been used among others in [**AF$^+$11a**] and [**AF$^+$11b**] to prove infinite dimensionality of most of the Nichols algebras over certain groups.

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