MASTER THESIS

PBW DEFORMATIONS OF ALGEBRAS

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0 Introduction

While a lot of classes of Nichols algebras are well understood today, there exist some classes where there is not really much known about and that are only accessible superficially through computer calculations. The main interest of this thesis is one of them, the class of finite dimensional Nichols algebras over braided vector spaces of non-abelian group type. A list of such examples can be found in [8]. In a recent attempt to find out more about this specific class of finite dimensional Hopf algebras, PBW deformations of such algebras were studied in [5]. A meaningful question to ask is, when a PBW deformation of such an algebra is semisimple, which seams to be a generic property (generic meaning true for a dense subset of deformations).

We basically continue the work in [5] and take a look at the next smallest dimensional examples, the three 576-dimensional Nichols algebras of this type. One of these three belongs to the family of Fomin-Kirillov algebras, which is where we start in section 2. While giving some assertions about all Fomin-Kirillov algebras, we in particular with very few computer calculations almost classify, when the PBW deformations of this 576-dimensional algebra are semisimple, which is a previously unknown result. For the summary, refer to subsection 2.4. In section 3 we will take a look at some of the reoccurring traits in all of the solved examples. We will use this in subsection 3.3, where we discuss the other two 576-dimensional examples, to give some conjectures about the semisimplicity of the PBW deformations. These are Conjecture 3.12 and Conjecture 3.13. Since there is no known go-to approach to handle these kind of algebras, the results presented here are the outcome of some very time consuming experimentation. Therefore sadly there was no time left to handle the two examples in detail and check if the conjectures hold.

1 Preliminaries

Let $\Bbbk$ denote a field. All our algebras will be associative and unital over the field $\Bbbk$. If A is an algebra, we associate $\Bbbk 1_A$ with $\Bbbk$.

Definition 1.1. Let A be an algebra. An element $e \in A$ that satisfies $e^2 = e$ is called **idempotent**. Two idempotents $e_1, e_2 \in A$ are called **orthogonal**, if $e_1 e_2 = 0 = e_2 e_1$. They are called **isomorphic**, if there exists elements $e_{12}, e_{21} \in A$, such that

$$\begin{aligned} e_1 e_{12} e_2 &= e_{12}, & e_2 e_{21} e_1 &= e_{21}, \\ e_{12} e_{21} &= e_1, & e_{21} e_{12} &= e_2. \end{aligned}$$

A subset $\{e_{ij} \mid 1 \leq i, j \leq m\} \subset A$, $m \in \mathbb{N}$ is called a set of **matrix units** in A, if $\sum_{i=1}^{m} e_{ii} = 1$ and $e_{ij} e_{kl} = \delta_{jk} e_{il}$ for all $1 \leq i, j, k, l \leq m$.

Lemma 1.2. *Let A be an algebra and let $e_1, e_2 \in A$ be idempotents. If e_1 and e_2 are conjugate, i.e. if there exists a unit $u \in A$, such that $ue_1 = e_2 u$, then e_1 and e_2 are isomorphic idempotents.*

Proof. Setting $e_{12} = e_1 u^{-1}$ and $e_{21} = ue_1$, the four relations are elementary. □

Remark 1.3. Observe that if e_1, e_2 are isomorphic idempotents, then $e_1 e_{12} = e_1 e_1 e_{12} e_2 = e_1 e_{12} e_2 = e_{12}$ and similarly $e_{12} e_2 = e_{12}$ and we also get those relations for e_{21}. Moreover for idempotents being isomorphic is an equivalence relation: It is clear that it is reflective and symmetric. If e_1, e_2, e_3 are idempotents such that e_1, e_2 and e_1, e_3 are isomorphic, and $e_{12}, e_{21}, e_{13}, e_{31}$ are elements that yield those isomorphisms, then e_2 and e_3 are isomorphic with $e_{23} := e_{21} e_{13}$ and $e_{32} := e_{31} e_{12}$.

Proposition 1.4. *Let A be an algebra, $m \in \mathbb{N}$ and let $e_1, \dots e_m, \in A$ be a set of pairwise orthogonal and isomorphic idempotents, such that $\sum_{i=1}^{m} e_i = 1$. Then there exists a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq m\} \subset A$, where $e_{ii} = e_i$.*

Proof. Define $e_{ii} = e_i$ for all $1 \leq i \leq m$ and for $1 \leq i < j \leq m$ let e_{ij} and e_{ji} be elements in A with whom e_i and e_j become isomorphic. Considering Remark 1.3, observe that those e_{ij} and e_{ji} can be chosen, such that $e_{ij} e_{jk} = e_{ik}$ for all $1 \leq i, j, k \leq m$. Finally for $1 \leq i, j, k, l \leq m$, $j \neq k$ we have $e_{ij} e_{kl} = e_{ij} e_j e_k e_{kl} = 0$, since e_j and e_k are orthogonal. □

Remark 1.5. If A is an algebra with an idempotent $e \in A \setminus \{0\}$, than eAe becomes an algebra with $1_{eAe} = e$ (in general not a subalgebra of A).

Proposition 1.6. *Let A be an algebra, and e_1, e_2 idempotent elements in A. The following are equivalent:*

(1) e_1 and e_2 are isomorphic idempotents.

(2) $e_1 A$ and $e_2 A$ are isomorphic right A modules.

(3) Ae_1 and Ae_2 are isomorphic left A modules.

In that case $e_1 A e_1$ and $e_2 A e_2$ are isomorphic algebras.

Proof. We show that (1) and (2) are equivalent. The equivalency of (1) and (3) is obtained analogously. Suppose (1) holds, and let e_{12}, $e_{21} \in A$ be elements with whom e_1 and e_2 become isomorphic. Define the following right A module morphisms:

$$\phi : e_1A \to e_2A,\ e_1a \mapsto e_{21}a$$
$$\psi : e_2A \to e_1A,\ e_2a \mapsto e_{12}a.$$

ϕ is well defined, since for $a \in A$ we have $e_{21}a = e_2e_{21}a \in e_2A$ and if $e_1a = 0$, then $e_{21}a = e_{21}e_1a = 0$. Similarly ψ is well defined. Since $e_{12}e_{21} = e_1$ and $e_{21}e_{12} = e_2$, ϕ and ψ are inverse to each other. Hence (2) holds.

Now suppose (2) holds, and let $\phi : e_1A \to e_2A$ be a right A module isomorphism. Define $e_{21} := \phi(e_1)$ and $e_{12} := \phi^{-1}(e_2)$. Then $e_2e_{21} = e_{21}$, since $e_{21} \in e_2A$ and $e_1e_{12} = e_{12}$ since $e_{12} \in e_1A$. Moreover $e_{21}e_1 = \phi(e_1)e_1 = \phi(e_1e_1) = \phi(e_1) = e_{21}$, since ϕ is a right A module morphism and similarly $e_{12}e_2 = e_{12}$. Finally $e_{21}e_{12} = \phi(e_1)e_{12} = \phi(e_1e_{12}) = \phi(e_{12}) = \phi(\phi^{-1}(e_2)) = e_2$ and similarly $e_{12}e_{21} = e_1$. Hence (1) holds.

Now if (1) holds and e_{12}, $e_{21} \in A$ are elements with whom e_1 and e_2 become isomorphic, define the linear maps

$$\phi : e_1Ae_1 \to e_2Ae_2,\ e_1ae_1 \mapsto e_{21}ae_{12}$$
$$\psi : e_2Ae_2 \to e_1Ae_1,\ e_2ae_2 \mapsto e_{12}ae_{21}.$$

These are well defined, since for $a \in A$ we have $e_{21}ae_{12} = e_2e_{21}ae_{12}e_2 \in e_2Ae_2$ and if $e_1ae_1 = 0$ then $e_{21}ae_{12} = e_{21}e_1ae_1e_{12} = 0$ and similarly for ψ. Moreover for $a \in A$ we have $\psi(\phi(e_1ae_1)) = e_{12}e_{21}ae_{12}e_{21} = e_1ae_1$ and similarly $\phi(\psi(e_2ae_2)) = e_2ae_2$, hence ϕ and ψ are inverse to each other. Now $\phi(e_1) = \phi(e_11e_1) = e_{21}e_{12} = e_2$ and if $a, b \in A$, then

$$\phi(e_1ae_1e_1be_1) = e_{21}ae_1be_{12} = e_{21}ae_{12}e_{21}be_{12} = \phi(e_1ae_1)\phi(e_1be_1).$$

Hence ϕ is an algebra isomorphism. □

Proposition 1.7. *Let A be an algebra, $m \in \mathbb{N}$. If there exists a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq m\} \subset A$, then*

$$A \cong M_m(R)$$

as algebras, where R is the subalgebra of A of all elements commuting with all e_{ij}, $1 \leq i, j \leq m$. Furthermore $e_{11}Ae_{11} \to R$, $e_{11}ae_{11} \mapsto \sum_{k=1}^{m} e_{k1}ae_{1k}$ is an algebra isomorphism.

Proof. The statement is proven in [6] Proposition 2.26 for rings. It is easy to see, that the two given ring isomorphisms are linear, i.e. algebra isomorphisms. □

Corollary 1.8. *Let A be an algebra, $m \in \mathbb{N}$ and let $e_1, \dots, e_m \in A$ be a set of pairwise orthogonal and isomorphic idempotents, such that $\sum_{i=1}^{m} e_i = 1$. Then*

$$A \cong M_m(e_1Ae_1)$$

as algebras.

Proof. Follows by combining Propositions 1.4 and 1.7. □

2 PBW deformations of Fomin-Kirillov algebras

In this section we will take a look at PBW deformations of Fomin-Kirillov algebras. We are in particular interested in the PBW deformations of the 576-dimensional Fomin-Kirillov algebra, which is one of the next smallest example of PBW deformations of a finite dimensional Nichols algebra of non-abelian group type that was not yet handled in [5]. We want to find out precisely which PBW deformation is semisimple and which is not. We almost succeed in doing so, as Theorem 2.24 and Theorem 2.30 almost characterize when a deformation is semisimple, with the exception of two cases. Those two cases are talked about in Conjecture 2.26.

Suppose that $\Bbbk$ is a field with characteristic $\neq 2$ and let $\alpha_1, \alpha_2 \in \Bbbk$. Also assume there exists an $\lambda \in \Bbbk$, such that $\lambda^2 = \alpha_1$. We will fix one such λ and denote it $\sqrt{\alpha_1}$.

Notation 2.1. We will often have multiple elements, that are indexed by one or more indexes, for example x_{ij}, $1 \leq i, j \leq n$. To avoid confusion in products of those elements we will sometimes denote

$$x_{12} \underset{\rightarrow}{\cdots} x_{1n} := x_{12}x_{13} \cdots x_{1(n-1)}x_{1n}$$

$$x_{1n} \underset{\leftarrow}{\cdots} x_{12} := x_{1n}x_{1(n-1)} \cdots x_{13}x_{12}$$

for ascending and descending sequences. In particular, if $n = 1$ then $x_{12} \underset{\rightarrow}{\cdots} x_{1n} = 1$ and $x_{1n} \underset{\leftarrow}{\cdots} x_{12} = 1$.

2.1 The general case

We begin with a general definition and by lining out some general properties.

Definition 2.2. For $n \in \mathbb{N}$, $n \geq 3$, let $\mathcal{D}_n(\alpha_1, \alpha_2)$ denote the algebra with generators x_{ij}, $1 \leq i, j \leq n$ and relations

$$\begin{aligned} x_{ii} &= 0, && \\ x_{ij} + x_{ji} &= 0, && \\ x_{ij}^2 &= \alpha_1 && \text{if } i \neq j, \\ x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} &= \alpha_2 && \text{if } \#\{i,j,k\} = 3, \\ x_{ij}x_{kl} - x_{kl}x_{ij} &= 0 && \text{if } \#\{i,j,k,l\} = 4 \end{aligned}$$

for all $1 \leq i, j, k, l \leq n$. Since $\operatorname{char} \Bbbk \neq 2$ the first row of relations are implied by the second row. For $\alpha_1 = \alpha_2 = 0$ we get $\mathcal{E}_n = \mathcal{D}_n(0,0)$, commonly called **Fomin-Kirillov algebra**.

Remark 2.3. There is a unique action of the symmetric group S_n on $\mathcal{D}_n(\alpha_1, \alpha_2)$ such that $\pi \cdot x_{ij} = x_{\pi(i)\pi(j)}$ and $\pi \cdot (xy) = (\pi \cdot x)(\pi \cdot y)$ for all $1 \leq i, j \leq n$, $\pi \in S_n$, $x, y \in \mathcal{D}_n(\alpha_1, \alpha_2)$.

Remark 2.4. The smallest case where $n = 3$ has already been solved in [5] in section 2, so we will not handle this case here. We are in particular interested in the second smallest case, that is $n = 4$. It is known from computer calculations, that $\mathcal{D}_4(\alpha_1, \alpha_2)$ has dimension 576 for all α_1, α_2. From [5], Proposition 1.2. it thus follows, that $\mathcal{D}_4(\alpha_1, \alpha_2)$ is indeed a PBW-Deformation of $\mathcal{E}_4 = \mathcal{D}_4(0,0)$.

Similar to [5], Theorem 2.13., we obtain that these also must be all PBW deformations. The reason we do not need a α_3 for the last relations is the following: Assume $x_{ij}x_{kl} - x_{kl}x_{ij} = \alpha_3$ if $\#\{i,j,k,l\} = 4$. Then we get

$$\begin{aligned}\alpha_3 =& x_{ij}x_{kl} - x_{kl}x_{ij} = -(x_{ji}x_{kl} - x_{kl}x_{ji})\\ =& -(ij)\cdot(x_{ij}x_{kl} - x_{kl}x_{ij}) = -(ij)\cdot\alpha_3 = -\alpha_3.\end{aligned}$$

Hence $\alpha_3 = 0$, since $\mathrm{char}(\Bbbk) \neq 2$.

Notation 2.5. In $\mathcal{D}_n(\alpha_1, \alpha_2)$, $n \geq 3$ let

$$y_{ij} := x_{ij} + \sqrt{\alpha_1}$$

for all $1 \leq i, j \leq n$. Note that $\pi \cdot y_{ij} = y_{\pi(i)\pi(j)}$ for all $\pi \in S_n$.

Remark 2.6. Note that $y_{ij}x_{ij} = x_{ij}y_{ij} = \sqrt{\alpha_1}y_{ij}$ for all $1 \leq i, j \leq n$. This means multiplying any $x \in \mathcal{D}_n(\alpha_1, \alpha_2)$ left by y_{ij} pulls out all of the x_{ij} that start monomials in x and exchanges them for scalars in $\Bbbk$. Similarly, multiplying x right by y_{ij} pulls out all of the x_{ij} that end monomials in x.

Lemma 2.7. *Let $n \geq 3$ and $1 \leq i, j, k \leq n$, such that $\#\{i, j, k\} = 3$. In $\mathcal{D}_n(\alpha_1, \alpha_2)$ the relations*

$$y_{ij} + y_{ji} = 2\sqrt{\alpha_1} \qquad y_{ij}y_{ji} = 0 \qquad y_{ij}^2 = 2\sqrt{\alpha_1}y_{ij}$$

hold. Moreover

$$\begin{aligned}x_{ij}x_{ik}x_{jk} &= x_{jk}x_{ik}x_{ij}\\ x_{ij}x_{kj}x_{ki} &= x_{ki}x_{kj}x_{ij}\end{aligned}$$

and

$$\begin{aligned}y_{ij}y_{ik}y_{jk} &= y_{jk}y_{ik}y_{ij}\\ y_{ij}y_{kj}y_{ki} &= y_{ki}y_{kj}y_{ij}.\end{aligned}$$

Proof. The relations in the first row are elementary. Now

$$\begin{aligned}x_{ij}x_{ik}x_{jk} &= (x_{ik}x_{kj} + x_{kj}x_{ji} - \alpha_2)\, x_{jk}\\ &= -\alpha_1 x_{ik} + x_{kj}\,(x_{jk}x_{ki} + x_{ki}x_{ij} - \alpha_2) - \alpha_2 x_{jk}\\ &= -\alpha_1\,(x_{ik} + x_{ki}) + x_{kj}x_{ki}x_{ij} - \alpha_2\,(x_{kj} + x_{jk})\\ &= x_{jk}x_{ik}x_{ij}\end{aligned}$$

and acting with (kji) on that relation yields

$$x_{ki}x_{kj}x_{ij} = x_{ij}x_{kj}x_{ki}.$$

Using this we get

$$\begin{aligned}& y_{ij}y_{ik}y_{jk} - y_{jk}y_{ik}y_{ij}\\ =\ & x_{ij}x_{ik}x_{jk} + \sqrt{\alpha_1}\,(x_{ij}x_{ik} + x_{ij}x_{jk} + x_{ik}x_{jk}) + \alpha_1\,(x_{ij} + x_{ik} + x_{jk} + \sqrt{\alpha_1})\\ & - x_{jk}x_{ik}x_{ij} - \sqrt{\alpha_1}\,(x_{jk}x_{ik} + x_{jk}x_{ij} + x_{ik}x_{ij}) - \alpha_1\,(x_{jk} + x_{ik} + x_{ij} + \sqrt{\alpha_1})\\ =& \sqrt{\alpha_1}\left(x_{ij}x_{ik} + \underline{x_{ij}x_{jk}} + x_{ik}x_{jk} - \underline{x_{jk}x_{ik}} - x_{jk}x_{ij} - \underline{x_{ik}x_{ij}}\right)\\ =& \sqrt{\alpha_1}\left(\underline{x_{ij}x_{jk}} + \underline{x_{jk}x_{ki}} + \underline{x_{ki}x_{ij}} - (x_{ji}x_{ik} + x_{ik}x_{kj} + x_{kj}x_{ji})\right)\\ =& \sqrt{\alpha_1}\,(\alpha_2 - \alpha_2) = 0\end{aligned}$$

and acting with (kji) yields

$$y_{ki}y_{kj}y_{ij} - y_{ij}y_{kj}y_{ki} = 0.$$

□

Lemma 2.8. *Let $n \geq 3$ and $2 \leq m \leq n-1$. Moreover let $1 \leq i \leq n$ and $1 \leq j_1, \dots, j_m \leq n$, such that $\#\{j_1, \dots, j_m\} = m$ and $i \notin \{j_1, \dots, j_m\}$. Finally let $1 \leq s < t \leq m$. Then the following relations hold in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

$$\begin{aligned} & y_{ij_1} \cdots y_{ij_m}\, y_{j_sj_{s+1}} \cdots y_{j_sj_t} \\ = \, & y_{j_sj_{s+1}} \cdots y_{j_sj_t}\, y_{ij_1} \underset{\to}{\cdots} y_{ij_{s-1}} y_{ij_{s+1}} \cdots y_{ij_t} y_{ij_s} y_{ij_{t+1}} \underset{\to}{\cdots} y_{ij_m} \\ = \, & y_{j_sj_{s+1}} \cdots y_{j_sj_t}\, (j_sj_{s+1} \cdots j_t) \cdot (y_{ij_1} \cdots y_{ij_m}) \end{aligned}$$

and

$$\begin{aligned} & y_{j_tj_s} \cdots y_{j_tj_{t-1}}\, y_{ij_1} \cdots y_{ij_m} \\ = \, & (j_tj_{t-1} \cdots j_s) \cdot (y_{ij_1} \cdots y_{ij_m})\, y_{j_tj_s} \cdots y_{j_tj_{t-1}}. \end{aligned}$$

Proof. The second relation follows by acting with $(j_tj_{t-1} \cdots j_s)$ on the first. We proof the first by induction on t: For $t = s+1$ the relation follows using Lemma 2.7 and the fact that $y_{ij}y_{kl} = y_{kl}y_{ij}$ if $\#\{i,j,k,l\} = 4$:

$$\begin{aligned} y_{ij_1} \cdots y_{ij_m}\, y_{j_sj_{s+1}} &= y_{ij_1} \underset{\to}{\cdots} y_{ij_{s-1}} \underline{y_{ij_s} y_{ij_{s+1}}\, y_{j_sj_{s+1}}}\, y_{ij_{s+2}} \underset{\to}{\cdots} y_{ij_m} \\ &\underset{2.7}{=} y_{ij_1} \underset{\to}{\cdots} y_{ij_{s-1}}\, \underline{y_{j_sj_{s+1}}\, y_{ij_{s+1}} y_{ij_s}}\, y_{ij_{s+2}} \underset{\to}{\cdots} y_{ij_m} \\ &= y_{j_sj_{s+1}}\, y_{ij_1} \underset{\to}{\cdots} y_{ij_{s-1}} y_{ij_{s+1}} y_{ij_s} y_{ij_{s+2}} \underset{\to}{\cdots} y_{ij_m} \end{aligned}$$

If $m = s+1$ then the only possibility for t is $t = s+1$ and the statement is shown. So suppose $s+1 < m$ and that the relation holds for some $s+1 \leq t \leq m-1$. Then we get using Lemma 2.7:

$$\begin{aligned} & y_{ij_1} \cdots y_{ij_m}\, y_{j_sj_{s+1}} \cdots y_{j_sj_t} y_{j_sj_{t+1}} \\ = \, & y_{j_sj_{s+1}} \cdots y_{j_sj_t}\, y_{ij_1} \underset{\to}{\cdots} y_{ij_{s-1}} y_{ij_{s+1}} \cdots y_{ij_t} y_{ij_s} y_{ij_{t+1}} \underset{\to}{\cdots} y_{ij_m}\, y_{j_sj_{t+1}} \\ = \, & y_{j_sj_{s+1}} \cdots y_{j_sj_t}\, y_{ij_1} \underset{\to}{\cdots} y_{ij_{s-1}} y_{ij_{s+1}} \cdots y_{ij_t} \underline{y_{ij_s} y_{ij_{t+1}} y_{j_sj_{t+1}}}\, y_{ij_{t+2}} \underset{\to}{\cdots} y_{ij_m} \\ \underset{2.7}{=} \, & y_{j_sj_{s+1}} \cdots y_{j_sj_t}\, y_{ij_1} \underset{\to}{\cdots} y_{ij_{s-1}} y_{ij_{s+1}} \cdots y_{ij_t} \underline{y_{j_sj_{t+1}} y_{ij_{t+1}} y_{ij_s}}\, y_{ij_{t+2}} \underset{\to}{\cdots} y_{ij_m} \\ = \, & y_{j_sj_{s+1}} \cdots y_{j_sj_t} y_{j_sj_{t+1}}\, y_{ij_1} \underset{\to}{\cdots} y_{ij_{s-1}} y_{ij_{s+1}} \cdots y_{ij_t} y_{ij_{t+1}} y_{ij_s} y_{ij_{t+2}} \underset{\to}{\cdots} y_{ij_m} \end{aligned}$$

This finishes the induction. □

Remark 2.9. To show Lemma 2.8, only relations for the y_{ij} were used that also hold for the generators x_{ij}. Hence we can exchange all y for x and the claim would still hold.

Lemma 2.10. *Let $n \geq 3$ and $1 \leq k \leq n-1$. Moreover let $1 \leq i \leq n$ and $1 \leq j_1, \dots, j_k \leq n$, such that $\#\{j_1, \dots, j_k\} = k$ and $i \notin \{j_1, \dots, j_k\}$. Then the following relation holds in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

$$\sum_{\pi \in <(ij_1 \cdots j_k)>} \pi \cdot (x_{ij_1} \cdots x_{ij_k}) = \begin{cases} (-1)^{\frac{k}{2}}\, (\alpha_2)^{\frac{k}{2}}, & \textit{if } k \textit{ is even,} \\ 0, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof. A similar relation was shown in [7], Lemma 3.2. for the case $\alpha_1 = 1, \alpha_2 = 0$. We do a similar proof. First the case $k = 1$:

$$x_{ij_1} + x_{j_1 i} = x_{ij_1} - x_{ij_1} = 0$$

The case $k = 2$ is just the defining relation:

$$x_{ij_1}x_{ij_2} + x_{j_1j_2}x_{j_1i} + x_{j_2i}x_{j_2j_1} = -\left(x_{j_1i}x_{ij_2} + x_{j_2j_1}x_{j_1i} + x_{ij_2}x_{j_2j_1}\right) = -\alpha_2$$

Now suppose $k \geq 3$ and the claim holds for $k-1$ and $k-2$. Denote $j_0 := i$ and for $a \in \mathbb{Z}$ denote $j_a := j_{a \bmod (k+1)}$, for example $j_{k+1} = j_0 = i$. Let $\sigma := (j_0 j_1 \cdots j_k)$. Then $\sigma^l(j_s) = j_{s+l}$ for all $0 \leq l \leq k$. Hence for $1 \leq l \leq k-1$ we have

$$\begin{aligned}
&\sigma^{k-l} \cdot (x_{j_0j_1} \cdots x_{j_0j_k}) = \sigma^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_{l-1}} x_{j_0j_l} x_{j_0j_{l+1}} x_{j_0j_{l+2}} \cdots x_{j_0j_k}\right) \\
=&\sigma^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_{l-1}} \left(x_{j_0j_{l+1}} x_{j_{l+1}j_l} - x_{j_{l+1}j_l} x_{j_0j_l} - \alpha_2\right) x_{j_0j_{l+2}} \cdots x_{j_0j_k}\right) \\
=&\sigma^{k-l} \cdot (x_{j_0j_1} \cdots x_{j_0j_{l-1}} x_{j_0j_{l+1}} x_{j_0j_{l+2}} \cdots x_{j_0j_k} x_{j_{l+1}j_l} \\
&- x_{j_{l+1}j_l} x_{j_0j_1} \cdots x_{j_0j_{l-1}} x_{j_0j_l} x_{j_0j_{l+2}} \cdots x_{j_0j_k} \\
&- \alpha_2 x_{j_0j_1} \cdots x_{j_0j_{l-1}} x_{j_0j_{l+2}} \cdots x_{j_0j_k}) \\
=&\sigma^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_{l-1}} x_{j_0j_{l+1}} \cdots x_{j_0j_k}\right) x_{j_0j_k} \\
&- x_{j_0j_k} \sigma^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_l} x_{j_0j_{l+2}} \cdots x_{j_0j_k}\right) \\
&- \alpha_2 \sigma^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_{l-1}} x_{j_0j_{l+2}} \cdots x_{j_0j_k}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
&\sigma^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_{l-1}} x_{j_0j_{l+1}} \cdots x_{j_0j_k}\right) \\
=&x_{j_{k-l}j_{k-l+1}} \cdots x_{j_{k-l}j_{k-1}} x_{j_{k-l}j_0} \cdots x_{j_{k-l}j_{k-l-1}} \\
=&(j_0j_1 \cdots j_{k-1})^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_{k-1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sigma^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_l} x_{j_0j_{l+2}} \cdots x_{j_0j_k}\right) \\
=&x_{j_{k-l}j_{k-l+1}} \cdots x_{j_{k-l}j_k} x_{j_{k-l}j_1} \cdots x_{j_{k-l}j_{k-l-1}} \\
=&(j_1j_2 \cdots j_k)^{k-l-1} \cdot (x_{j_1j_2} \cdots x_{j_1j_k})
\end{aligned}$$

as well as

$$\begin{aligned}
&\sigma^{k-l} \cdot \left(x_{j_0j_1} \cdots x_{j_0j_{l-1}} x_{j_0j_{l+2}} \cdots x_{j_0j_k}\right) \\
=&x_{j_{k-l}j_{k-l+1}} \cdots x_{j_{k-l}j_{k-1}} x_{j_{k-l}j_1} \cdots x_{j_{k-l}j_{k-l-1}} \\
=&(j_1j_2 \cdots j_{k-1})^{k-l-1} \cdot \left(x_{j_1j_2} \cdots x_{j_1j_{k-1}}\right)
\end{aligned}$$

and finally

$$\begin{aligned}
\sigma^k \left(x_{j_0j_1} \cdots x_{j_0j_k}\right) &= x_{j_kj_0} x_{j_kj_1} \cdots x_{j_kj_{k-1}} \\
&= -x_{j_0j_k} (j_1j_2 \cdots j_k)^{-1} \cdot (x_{j_1j_2} \cdots x_{j_1j_k}).
\end{aligned}$$

Piecing this all together we can calculate the sum from the claim:

$$
\begin{aligned}
&\sum_{\pi\in<(ij_1\cdots j_k)>}\pi\left(x_{ij_1}\cdots x_{ij_k}\right)=\sum_{l=0}^{k}\sigma^{k-l}\left(x_{j_0j_1}\cdots x_{j_0j_k}\right)\\
=&x_{j_0j_1}\cdots x_{j_0j_k}+\sigma^{k}\left(x_{j_0j_1}\cdots x_{j_0j_k}\right)+\sum_{l=1}^{k-1}\sigma^{k-l}\left(x_{j_0j_1}\cdots x_{j_0j_k}\right)\\
=&x_{j_0j_1}\cdots x_{j_0j_k}-x_{j_0j_k}(j_1j_2\cdots j_k)^{-1}\cdot\left(x_{j_1j_2}\cdots x_{j_1j_k}\right)\\
&+\left(\sum_{l=1}^{k-1}(j_0j_1\cdots j_{k-1})^{k-l}\left(x_{j_0j_1}\cdots x_{j_0j_{k-1}}\right)\right)x_{j_0j_k}\\
&-x_{j_0j_k}\sum_{l=1}^{k-1}(j_1j_2\cdots j_k)^{k-l-1}\cdot\left(x_{j_1j_2}\cdots x_{j_1j_k}\right)\\
&-\alpha_2\sum_{l=1}^{k-1}(j_1j_2\cdots j_{k-1})^{k-l-1}\cdot\left(x_{j_1j_2}\cdots x_{j_1j_{k-1}}\right)\\
=&\left(\sum_{l=1}^{k}(j_0j_1\cdots j_{k-1})^{k-l}\left(x_{j_0j_1}\cdots x_{j_0j_{k-1}}\right)\right)x_{j_0j_k}\\
&-x_{j_0j_k}\sum_{l=1}^{k}(j_1j_2\cdots j_k)^{k-l-1}\cdot\left(x_{j_1j_2}\cdots x_{j_1j_k}\right)\\
&-\alpha_2\sum_{l=1}^{k-1}(j_1j_2\cdots j_{k-1})^{k-l-1}\cdot\left(x_{j_1j_2}\cdots x_{j_1j_{k-1}}\right)
\end{aligned}
$$

Now for the first and second sum, we can use the induction hypothesis for $k-1$ and for the third sum for $k-2$. If k is even, $k-1$ is odd and $k-2$ is even, hence the above sums equal

$$0\cdot x_{j_0j_k}-x_{j_0j_k}\cdot 0-\alpha_2\left(-1\right)^{\frac{k-2}{2}}\left(\alpha_2\right)^{\frac{k-2}{2}}=\left(-1\right)^{\frac{k}{2}}\left(\alpha_2\right)^{\frac{k}{2}}.$$

If k is odd, $k-1$ is even and $k-2$ is odd, hence the above sums equal

$$\left(-1\right)^{\frac{k-1}{2}}\left(\alpha_2\right)^{\frac{k-1}{2}}\cdot x_{j_0j_k}-x_{j_0j_k}\cdot\left(-1\right)^{\frac{k-1}{2}}\left(\alpha_2\right)^{\frac{k-1}{2}}-\alpha_2\cdot 0=0.$$

This finishes the proof. □

Lemma 2.11. *Let $n\geq 3$ and $1\leq m\leq n-1$. Moreover let $1\leq i\leq n$ and $1\leq j_1,\ldots,j_m\leq n$, such that $\#\{j_1,\ldots,j_m\}=m$ and $i\notin\{j_1,\ldots,j_m\}$. Finally let $G=<(ij_1\cdots j_m)>$ be the subgroup of S_n of order $m+1$ generated by $(ij_1\cdots j_m)$. Then for $1\leq k\leq m$ the following relation holds in $\mathcal{D}_n(\alpha_1,\alpha_2)$:*

$$\sum_{1\leq s_1<\ldots<s_k\leq m}\sum_{\pi\in G}\pi\cdot\left(x_{ij_{s_1}}\cdots x_{ij_{s_k}}\right)=\begin{cases}\binom{m+1}{k+1}\left(-\alpha_2\right)^{\frac{k}{2}}, & \text{if } k \text{ is even,}\\ 0, & \text{if } k \text{ is odd.}\end{cases}$$

Proof. First denote $j_0:=i$ and for $a\in\mathbb{Z}$ we denote $j_a:=j_{a \bmod (m+1)}$, for example $j_{m+1}=j_0=i$. For $0\leq l\leq m$ denote $\pi_l:=(j_0j_1\cdots j_m)^l\in G$. Finally for the sake of readability, during this proof we denote $[st]:=x_{st}$ for all $1\leq s,t\leq n$.

Let $1 \leq s_1 < \ldots < s_k \leq m$. Then

$$\pi_l \cdot ([j_0 j_{s_1}] \cdots [j_0 j_{s_k}]) = [j_l j_{s_1+l}] \cdots [j_l j_{s_k+l}] \,.$$

Hence the sum in the claim is

$$\sum_{1 \leq s_1 < \ldots < s_k \leq m} \sum_{l=0}^{m} [j_l j_{s_1+l}] \cdots [j_l j_{s_k+l}]$$

We want to reorder that sum. Every summand can be indexed by an element in $\{(l, s_1, \ldots, s_k) : 0 \leq l \leq m, 1 \leq s_1 < \ldots < s_k \leq m\}$. The size of this set is $(m+1)\binom{m}{k}$. Now consider the following sum:

$$S := \sum_{l=0}^{m-k} \sum_{l+1 \leq s_1 < \ldots < s_k \leq m} \sum_{\pi \in <(j_l j_{s_1} \cdots j_{s_k})>} \pi \cdot ([j_l j_{s_1}] \cdots [j_l j_{s_k}])$$

First of all this sum has

$$(k+1) \sum_{l=0}^{m-k} \binom{m-l}{k} = (m+1)\binom{m}{k}$$

summands, i.e. the same amount as the sum from the claim. Now consider a summand $[j_l j_{s_1+l}] \cdots [j_l j_{s_k+l}]$, $0 \leq l \leq m, 1 \leq s_1 < \ldots < s_k \leq m$ from the first sum. If $l < s_1 + l \mod (m+1) < \ldots < s_k + l \mod (m+1)$ then $l < m - k$ and this summand corresponds to

$$\mathrm{id} \cdot \left([j_l j_{s'_1}] \cdots \left[j_l j_{s'_k} \right] \right) \qquad \text{where } s'_t := s_t + l,\ 1 \leq t \leq k.$$

from the second sum. If not then let $l', s'_1, \ldots, s'_k \in \{l\} \cup \{s_t + l \mod (m+1) : 1 \leq t \leq k\}$, such that $l' < s'_1 < \ldots < s'_k$. In particular $l' < m - k$. Since $l < s_1 + l < \ldots < s_k + l$ and $s_k + l \mod (m+1) < l$, there must exist a permutation $\pi \in <(j_{l'} j_{s'_1} \ldots j_{s'_k})>$ that reverses this process, i.e. $\pi(j_{l'}) = j_l$, $\pi(j_{s'_t}) = j_{s_t + l \mod (m+1)}$, $1 \leq t \leq k$. Hence we can correspond the original summand with one from the second sum:

$$\pi \cdot \left([j_{l'} j_{s'_1}] \cdots \left[j_{l'} j_{s'_k} \right] \right) = [j_l j_{s_1+l}] \cdots [j_l j_{s_k+l}]$$

Observe that this correspondence is injective on the indexes and since both sums have the same amount of indexes, the correspondence is bijective on indexes. Hence the sum from the claim is equal to S. Now observe that we can in fact use Lemma 2.10 to calculate S. If k is odd, then $S = 0$. If k is even, then $m \geq 2$ and

$$S = \sum_{l=0}^{m-k} \sum_{l+1 \leq s_1 < \cdots < s_k \leq m} (-\alpha_2)^{\frac{k}{2}} = \binom{m+1}{k+1} (-\alpha_2)^{\frac{k}{2}} \,.$$

This proofs the claim. □

Notation 2.12. Let $m \in \mathbb{N}$ and denote

$$\lambda_m := (m+1)\sqrt{\alpha_1}^{m} + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2k+1} \sqrt{\alpha_1}^{m-2k} (-\alpha_2)^k \in \Bbbk.$$

The first few values of λ_m are

$$\begin{aligned}
\lambda_1 &= 2\sqrt{\alpha_1}\\
\lambda_2 &= 3\alpha_1 - \alpha_2\\
\lambda_3 &= 4\sqrt{\alpha_1}\,(\alpha_1 - \alpha_2)\\
\lambda_4 &= 5\alpha_1^2 - 10\alpha_1\alpha_2 + \alpha_2^2\\
\lambda_5 &= 2\sqrt{\alpha_1}\left(3\alpha_1^2 - 10\alpha_1\alpha_2 + 3\alpha_2^2\right) = 2\sqrt{\alpha_1}\,(3\alpha_1 - \alpha_2)\,(\alpha_1 - 3\alpha_2)\,.
\end{aligned}$$

Lemma 2.13. *Let $n \geq 3$ and $1 \leq m \leq n-1$. Moreover let $1 \leq i \leq n$ and $1 \leq j_1, \ldots, j_m \leq n$, such that $\#\{j_1, \ldots, j_m\} = m$ and $i \notin \{j_1, \ldots, j_m\}$. Finally let $G =< (ij_1 \cdots j_m) >$ be the subgroup of S_n of order $m+1$ generated by $(ij_1 \cdots j_m)$. Then the following relations hold in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

(1) $\sum_{\pi \in G} \pi\left(y_{ij_1} \cdots y_{ij_m}\right) = \lambda_m$.

(2) $\sigma\left(y_{ij_1} \cdots y_{ij_m}\right) \pi\left(y_{ij_1} \cdots y_{ij_m}\right) = 0$ *for all* $\sigma, \pi \in G$, $\sigma \neq \pi$.

(3) $\left(y_{ij_1} \cdots y_{ij_m}\right)^2 = \lambda_m y_{ij_1} \cdots y_{ij_m}$.

Proof. (1): Let $\pi \in G$. Consider the summands of the product $\pi\left(y_{ij_1} \cdots y_{ij_m}\right) = y_{\pi(i)\pi(j_1)} \cdots y_{\pi(i)\pi(j_m)}$ that we get by simply multiplying the polynomials and without using any defining relations. The only summand of degree 0 is simply $\sqrt{\alpha_1}^m$. The summands of degree $1 \leq k \leq m$ are

$$\sqrt{\alpha_1}^{m-k}\, x_{\pi(i)\pi(j_{s_1})} \cdots x_{\pi(i)\pi(j_{s_k})} = \sqrt{\alpha_1}^{m-k}\, \pi \cdot \left(x_{ij_{s_1}} \cdots x_{ij_{s_k}}\right),$$

where $1 \leq s_1 < \ldots < s_k \leq m$. Hence using Lemma 2.11 we obtain

$$\begin{aligned}
&\sum_{\pi \in G} \pi\left(y_{ij_1} \cdots y_{ij_m}\right)\\
=&\sum_{\pi \in G}\left(\sqrt{\alpha_1}^m + \sum_{k=1}^{m} \sqrt{\alpha_1}^{m-k} \sum_{1 \leq s_1 < \ldots < s_k \leq m} \pi \cdot \left(x_{ij_{s_1}} \cdots x_{ij_{s_k}}\right)\right)\\
=&(m+1)\sqrt{\alpha_1}^m + \sum_{k=1}^{m} \sqrt{\alpha_1}^{m-k} \sum_{1 \leq s_1 < \ldots < s_k \leq m} \sum_{\pi \in G} \pi \cdot \left(x_{ij_{s_1}} \cdots x_{ij_{s_k}}\right)\\
=&(m+1)\sqrt{\alpha_1}^m + \sum_{k=1,\ k \text{ even}}^{m} \sqrt{\alpha_1}^{m-k} \binom{m+1}{k+1} (-\alpha_2)^{\frac{k}{2}}\\
=&(m+1)\sqrt{\alpha_1}^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2k+1} \sqrt{\alpha_1}^{m-2k} (-\alpha_2)^k = \lambda_m
\end{aligned}$$

(2): We show the relation for $\sigma = \mathrm{id}$. The other relations are obtained by acting with G on that relation. Let $\pi \in G \setminus \{\mathrm{id}\}$, i.e. $\pi = (ij_1 \cdots j_m)^s$ for some $1 \leq s \leq m$. Consider first the case where $s = m$. Then $\pi(i) = j_m$ and $\pi(j_1) = i$. Hence

$$\begin{aligned}
y_{ij_1} \cdots y_{ij_m} \pi\left(y_{ij_1} \cdots y_{ij_m}\right) &= y_{ij_1} \cdots y_{ij_m}\, \pi \cdot \left(y_{ij_1}\right)\, \pi \cdot \left(y_{ij_2} \cdots y_{ij_m}\right)\\
&= y_{ij_1} \cdots \underline{y_{ij_m} y_{j_m i}}\, \pi \cdot \left(y_{ij_2} \cdots y_{ij_m}\right) = 0.
\end{aligned}$$

Now consider the case where $s < m$. Observe that $\pi(i) = j_s$ and $\pi(j_k) = j_{k+s}$ for all $1 \leq k \leq m - s$. This means

$$\pi\left(y_{ij_1} \cdots y_{ij_{m-s}}\right) = y_{j_s j_{s+1}} \cdots y_{j_s j_m}$$

Hence we can use Lemma 2.8 with $t = m$:

$$\begin{aligned} & y_{ij_1} \cdots y_{ij_m} \pi\left(y_{ij_1} \cdots y_{ij_{m-s}}\right) \\ =& y_{ij_1} \cdots y_{ij_m}\, y_{j_s j_{s+1}} \cdots y_{j_s j_m} \\ =& y_{j_s j_{s+1}} \cdots y_{j_s j_m}\, y_{ij_1} \underset{\rightarrow}{\cdots} y_{ij_{s-1}} y_{ij_{s+1}} \underset{\rightarrow}{\cdots} y_{ij_m} y_{ij_s} \end{aligned}$$

Since $\pi(j_{m-s+1}) = i$, multiplying the above relation with $\pi \cdot (y_{ij_{m-s+1}}) = y_{j_s i}$ from the right yields

$$y_{ij_1} \cdots y_{ij_m} \pi\left(y_{ij_1} \underset{\rightarrow}{\cdots} y_{ij_{m-s}} y_{ij_{m-s+1}}\right) = 0.$$

If $s = 1$ then this is the claim, if not one can simply multiply this relation right by the remaining $\pi \cdot \left(y_{ij_{m-s+2}} \cdots y_{ij_m}\right)$ and get the claim.

(3): Using (1) and (2) we obtain

$$\begin{aligned} y_{ij_1} \cdots y_{ij_m}\, y_{ij_1} \cdots y_{ij_m} &= y_{ij_1} \cdots y_{ij_m} \left(\lambda_m - \sum_{\pi \in G \setminus \{\mathrm{id}\}} \pi \cdot (y_{ij_1} \cdots y_{ij_m})\right) \\ &= \lambda_m y_{ij_1} \cdots y_{ij_m}. \end{aligned}$$

□

2.2 The semisimple case

In this section we will fix $n \in \mathbb{N}$, $n \geq 3$ and assume $\lambda_1 \cdots \lambda_{n-1} \neq 0$. The main result of this section is Theorem 2.24, where we show (using some computer calculations), that in most cases the 576-dimensional algebra $\mathcal{D}_4(\alpha_1, \alpha_2)$ is semisimple and isomorphic to $M_{24}(\Bbbk)$.

Notation 2.14. In $\mathcal{D}_n(\alpha_1, \alpha_2)$, denote for all for all $1 \leq i \leq n-1$

$$\begin{aligned} v_i :=& \frac{1}{\lambda_{n-i}}\, y_{i(i+1)} \underset{\rightarrow}{\cdots} y_{in} \\ w_0 :=& 1 \\ w_i :=& w_{i-1} v_i = v_1 \cdots v_i \\ w :=& w_{n-1}. \end{aligned}$$

For example, if $n = 4$ then

$$w = v_1 v_2 v_3 = \frac{1}{\lambda_1 \lambda_2 \lambda_3} y_{12} y_{13} y_{14} y_{23} y_{24} y_{34}.$$

Observe that v_i is a special case of the elements handled in Lemma 2.13.

Remark 2.15. Let $1 \le m \le n-1$. Note that in $w_m = v_1 \cdots v_m$ we can shift any v_i, $1 \le i \le m$ to the left according to Lemma 2.8, i.e. $w_m \in v_i \mathcal{D}_n(\alpha_1, \alpha_2)$. Shifting $v_{i+1} \cdots v_m$ to the left exposes

$$\begin{aligned}&(m \cdots n)((m-1)\cdots n)\cdots((i+1)\cdots n)\cdot v_i\\ =&\frac{1}{\lambda_{n-i}} y_{i(m+1)} \underset{\rightarrow}{\cdots} y_{in} y_{im} \underset{\leftarrow}{\cdots} y_{i(i+1)} =: v_i'\end{aligned}$$

on the right, i.e. $w_m \in \mathcal{D}_n(\alpha_1, \alpha_2) v_i'$. This means, considering Lemma 2.13(3), that multiplying $a \in \mathcal{D}_n(\alpha_1, \alpha_2)$ left by w_m we can pull any v_i' that start products in a into w_m and thus ignoring them in a, and similarly by multiplying right by w_m we can ignore any v_i that end products in a. Another similar shifting is also possible: In w_m, first shift $y_{m(m+1)}$ that starts v_m to the left. This will cause $y_{(m-1)(m+1)}$ to be shifted to the front of v_{m-1} and thus can be shifted to the left of w_{m-1}. Continuing this way we obtain $w_m \in y_{m(m+1)} y_{(m-1)(m+1)} \underset{\leftarrow}{\cdots} y_{1(m+1)} \mathcal{D}_n(\alpha_1, \alpha_2)$.

Lemma 2.16. *For all for all $1 \le m \le n-1$ and $m \le s < t \le n$ the following relations hold in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

$$\begin{aligned}w_{m-1} y_{s(s+1)} \cdots y_{st} &= y_{s(s+1)} \cdots y_{st}\, (s(s+1)\cdots t) \cdot w_{m-1}\\ y_{ts} \cdots y_{t(t-1)} w_{m-1} &= (t(t-1)\cdots s) \cdot w_{m-1}\, y_{ts} \cdots y_{t(t-1)}.\end{aligned}$$

Proof. The statement is trivial for $m = 1$, since $w_0 = 1$. Suppose $m \ge 2$. Using Lemma 2.8 we obtain for all $1 \le i \le m-1$, since $i < s$:

$$\begin{aligned}v_i y_{s(s+1)} \cdots y_{st} &= y_{s(s+1)} \cdots y_{st}\, (s(s+1)\cdots t) \cdot v_i\\ y_{ts} \cdots y_{t(t-1)} v_i &= (t(t-1)\cdots s) \cdot v_i\, y_{ts} \cdots y_{t(t-1)}.\end{aligned}$$

Since w_{m-1} is a product of those v_i, the claim follows. □

Proposition 2.17. *Let $1 \le m \le n-1$. The element w_m is an idempotent in $\mathcal{D}_n(\alpha_1, \alpha_2)$. Moreover $w_{m-1} v_m w_{m-1} = w_m w_{m-1}$ is an idempotent isomorphic to w_m.*

Proof. We do induction on m. If $m = 1$, then $w_1 = v_1$ is an idempotent according to Lemma 2.13(3) and $w_0 v_1 w_0 = v_1 = w_1$. Now suppose $m \ge 2$ and w_{m-1} is an idempotent. Combining Lemmas 2.13(3) and 2.16 we obtain

$$\begin{aligned}w_m w_m =& w_{m-1} v_m w_{m-1} v_m = w_{m-1} v_m v_m\, (m \cdots n) \cdot w_{m-1}\\ =& w_{m-1} v_m\, (m \cdots n) \cdot w_{m-1} = w_{m-1} w_{m-1} v_m = w_{m-1} v_m = w_m.\end{aligned}$$

Hence w_m is an idempotent. Similarly

$$\begin{aligned}w_{m-1} v_m w_{m-1} w_{m-1} v_m w_{m-1} =& w_{m-1} v_m w_{m-1} v_m w_{m-1}\\ =& w_{m-1} v_m v_m\, (m \cdots n) \cdot w_{m-1} w_{m-1}\\ =& w_{m-1} v_m\, (m \cdots n) \cdot w_{m-1} w_{m-1}\\ =& w_{m-1} w_{m-1} v_m w_{m-1} = w_{m-1} v_m w_{m-1}\end{aligned}$$

and thus $w_m w_{m-1}$ is an idempotent. If we denote $e_1 := w_m$ and $e_2 := w_m w_{m-1}$, then define $e_{12} := e_2$, $e_{21} := e_1$. Since

$$\begin{aligned} e_1 e_2 =& w_m w_m w_{m-1} = w_m w_{m-1} = e_2 \\ e_2 e_1 =& w_m w_{m-1} w_m = w_m w_{m-1} w_{m-1} v_m = w_m w_{m-1} v_m = w_m w_m = w_m = e_1 \end{aligned}$$

we immediately obtain

$$\begin{aligned} e_1 e_{12} e_2 =& e_1 e_2 e_2 = e_1 e_2 = e_2 = e_{12} & \quad e_2 e_{21} e_1 =& e_2 e_1 e_1 = e_2 e_1 = e_1 = e_{21} \\ e_{12} e_{21} =& e_2 e_1 = e_1 & e_{21} e_{12} =& e_1 e_2 = e_2 \end{aligned}$$

and hence e_1 and e_2 are isomorphic idempotents. □

Lemma 2.18. *Let $1 \leq m \leq s < t \leq n$. In $\mathcal{D}_4(\alpha_1, \alpha_2)$ the following relation holds:*

$$\begin{aligned} & y_{st}(t \cdots n)(s \cdots n)(w_{m-1})(t \cdots n)(s \cdots m)(w_{m-1})(s \cdots m)(t \cdots m)(w_{m-1}) y_{ts} \\ & = 0. \end{aligned}$$

Proof. If $m = 1$, then the relation follows immediately since $w_0 = 1$ and $y_{st} y_{ts} = 0$. Suppose $m \geq 2$. Consider the following construction: Let $\lambda_0 = 1$ and for all $1 \leq i \leq m-1$ let

$$\begin{aligned} v'_i :=& \frac{\lambda_{m-i}}{\lambda_{n-i}} y_{i(m+1)} \cdots y_{in} & w'_i :=& v'_1 \cdots v'_i, \\ v''_i :=& \frac{\lambda_{m-i}}{\lambda_{m-1-i}\lambda_{n-i}} y_{i(i+1)} \underset{\rightarrow}{\cdots} y_{i(m-1)} y_{i(m+1)} \cdots y_{in} & w''_i :=& v''_1 \cdots v''_i, \\ v'''_i :=& \frac{\lambda^2_{m-i}}{\lambda_{m-1-i}\lambda_{n-i}} y_{i(m+1)} \underset{\rightarrow}{\cdots} y_{i(n-1)} & w'''_i :=& v'''_i \underset{\leftarrow}{\cdots} v'''_1. \end{aligned}$$

Below we will show the following three relations:

(1) $(s \cdots n)(w_{m-1})(s \cdots m)(w_{m-1}) = (s \cdots n)(w_{m-1})(s \cdots m)(w'_{m-1})$,

(2) $(s \cdots n)(w_{m-1})(s \cdots m)(w'_{m-1}) = (s \cdots n)(w_{m-1})(s \cdots m)(w''_{m-1})$,

(3) $(t \cdots n)(w''_{m-1})(t \cdots m)(w_{m-1}) = (t \cdots n)(w'''_{m-1})(t \cdots m)(w_{m-1})$.

Combining (1), (2) and (3), we obtain

$$\begin{aligned} &(t \cdots n)(s \cdots n)(w_{m-1})(t \cdots n)(s \cdots m)(w_{m-1})(s \cdots m)(t \cdots m)(w_{m-1}) \\ =&(t \cdots n)(s \cdots n)(w_{m-1})(t \cdots n)(s \cdots m)(w'''_{m-1})(s \cdots m)(t \cdots m)(w_{m-1}), \end{aligned}$$

since the permutations $(t \cdots n)$ and $(s \cdots m)$ commute. Now observe, that since $y_{st} y_{it} y_{is} = y_{is} y_{it} y_{st}$ for all $1 \leq i \leq m-1$ and since $(t \cdots n)(s \cdots n)(n-1) = t$, $(t \cdots n)(s \cdots n)(n) = s$ we can shift

$$y_{st}(t \cdots n)(s \cdots n)(w_{m-1}) = (st)(t \cdots n)(s \cdots n)(w_{m-1}) y_{st},$$

Moreover y_{st} and $(t \cdots n)(s \cdots m)(w'''_{m-1})$ commute, since w'''_{m-1} has no y with indexes in $\{m, n\}$, i.e. $(t \cdots n)(s \cdots m)(w'''_{m-1})$ has no y with indexes in $\{s, t\}$.

Similarly, since $(s\cdots m)(t\cdots m)(m)=t$, $(s\cdots m)(t\cdots m)(m+1)=s$ we can shift

$$y_{st}(s\cdots m)(t\cdots m)(w_{m-1})y_{ts}=(st)(s\cdots m)(t\cdots m)(w_{m-1})y_{st}y_{ts}=0,$$

which finishes the proof.

Regarding (1): Since $(m\cdots s)(s\cdots n)=(m\cdots n)$, (1) is equivalent to

$$(m\cdots n)(w_{m-1})w_{m-1}=(m\cdots n)(w_{m-1})w'_{m-1}$$

(via acting with $(m\cdots s)=(s\cdots m)^{-1}$). First recall Remark 2.15 and observe that $w_{m-1}\in\mathcal{D}_n(\alpha_1,\alpha_2)y_{in}y_{i(m-1)}\underset{\leftarrow}{\cdots}y_{i(i+1)}$ for all $1\le i\le m-1$. Lemma 2.13(3) implies $w_{m-1}y_{in}y_{i(m-1)}\underset{\leftarrow}{\cdots}y_{i(i+1)}=\lambda_{m-i}w_{m-1}$ for all $1\le i\le m-1$, and thus by acting with $(m\cdots n)$ we obtain

$$(m\cdots n)(w_{m-1})y_{im}y_{i(m-1)}\underset{\leftarrow}{\cdots}y_{i(i+1)}=\lambda_{m-i}(m\cdots n)(w_{m-1})$$

for all $1\le i\le m-1$. Hence for $i=m-1$ we obtain

$$\begin{aligned}
&(m\cdots n)(w_{m-1})w_{m-1}=(m\cdots n)(w_{m-1})v_1\cdots v_{m-1}\\
=&(m\cdots n)(w_{m-1})v_1\cdots v_{m-2}\frac{1}{\lambda_{n-m+1}}y_{(m-1)m}y_{(m-1)(m+1)}\cdots y_{(m-1)n}\\
=&(m\cdots n)(w_{m-1})y_{(m-1)m}((m-1)m)(v_1\cdots v_{m-2})\\
&\frac{1}{\lambda_{n-m+1}}y_{(m-1)(m+1)}\cdots y_{(m-1)n}\\
=&(m\cdots n)(w_{m-1})((m-1)m)(v_1\cdots v_{m-2})\frac{\lambda_1}{\lambda_{n-m+1}}y_{(m-1)(m+1)}\cdots y_{(m-1)n}\\
=&(m\cdots n)(w_{m-1})((m-1)m)(v_1\cdots v_{m-2})v'_{m-1}.
\end{aligned}$$

If $m=2$ we are finished, so suppose $m\ge 3$. Now for $1\le i\le m-2$ we have

$$\begin{aligned}
&((m-1)m)((m-2)(m-1)m)\cdots((i+1)\cdots m)(y_{i(i+1)}\cdots y_{im})\\
=&y_{im}\underset{\leftarrow}{\cdots}y_{i(i+1)}
\end{aligned}$$

and hence

$$\begin{aligned}
&(m\cdots n)(w_{m-1})((m-1)m)\cdots((i+1)\cdots m)(v_1\cdots v_i)\\
=&(m\cdots n)(w_{m-1})y_{im}\underset{\leftarrow}{\cdots}y_{i(i+1)}\\
&((m-1)m)\cdots(i\cdots m)\,(v_1\cdots v_{i-1})\,\frac{1}{\lambda_{n-i}}y_{i(m+1)}\cdots y_{in}\\
=&(m\cdots n)(w_{m-1})((m-1)m)\cdots(i\cdots m)\,(v_1\cdots v_{i-1})\,v'_i.
\end{aligned}$$

The last equality is implied by the relation at the beginning of the proof of (1). Thus inductively we obtain

$$\begin{aligned}
&(m\cdots n)(w_{m-1})w_{m-1}=(m\cdots n)(w_{m-1})v_1\cdots v_{m-1}\\
=&(m\cdots n)(w_{m-1})v'_1\cdots v'_{m-1}=(m\cdots n)(w_{m-1})w'_{n-1},
\end{aligned}$$

which is equivalent to (1).

Regarding (2): If $m = 2$ there is nothing to show, so suppose $m \geq 3$. It is equivalent to show

$$(m \cdots n)(w_{m-1})w'_{m-1} = (m \cdots n)(w_{m-1})w''_{m-1}$$

Therefore we reverse the process in (1), with leaving out the y_{im}. If one wishes to, one can check on some examples for $m \geq 4$, that it is indeed necessary to do this step before (3). So in (1) we already discussed, that $w_{m-1} \in \mathcal{D}_n(\alpha_1, \alpha_2)y_{i(m-1)}\underset{\leftarrow}{\cdots}y_{i(i+1)}$ for all $1 \leq i \leq m-2$. Lemma 2.13(3) implies $w_{m-1} = \frac{1}{\lambda_{m-1-i}}w_{m-1}y_{i(m-1)}\underset{\leftarrow}{\cdots}y_{i(i+1)}$ for all $1 \leq i \leq m-2$, and thus by acting with $(m \cdots n)$ we obtain

$$(m \cdots n)(w_{m-1}) = \frac{1}{\lambda_{m-1-i}}(m \cdots n)(w_{m-1})y_{i(m-1)}\underset{\leftarrow}{\cdots}y_{i(i+1)}$$

for all $1 \leq i \leq m-2$. Now we inductively pull those y in, first for $i = 1$, up to $i = m-2$ and get

$$\begin{aligned}(m \cdots n)(w_{m-1}) =&(m \cdots n)(w_{m-1})\frac{1}{\lambda_1 \cdots \lambda_{m-2}}\\ &y_{(m-2)(m-1)}y_{(m-3)(m-1)}y_{(m-3)(m-2)} \cdots y_{1(m-1)}\underset{\leftarrow}{\cdots}y_{12}\\ =&(m \cdots n)(w_{m-1})\frac{1}{\lambda_1 \cdots \lambda_{m-2}}\\ &y_{12}\underset{\rightarrow}{\cdots}y_{1(m-1)} \cdots y_{(m-3)(m-2)}y_{(m-3)(m-1)}y_{(m-2)(m-1)}\end{aligned}$$

The last equality just uses Lemma 2.8 multiple times to shift first the $y_{(m-1)(\cdot)}$ to the right, then $y_{(m-2)(\cdot)}$ up to the $y_{2(\cdot)}$. It basically reverses the process in Remark 2.15, but just for the last few y. Finally observe that for $2 \leq i \leq m-2$ the product $y_{i(i+1)} \cdots y_{i(m-1)}$ commutes with v'_j for all $1 \leq j < i$, and $\frac{1}{\lambda_{m-1-i}}y_{i(i+1)} \cdots y_{i(m-1)}v'_i = v''_i$. Hence, considering that $v'_{m-1} = v''_{m-1}$, we obtain

$$\begin{aligned}(m \cdots n)(w_{m-1})w'_{m-1} &= (m \cdots n)(w_{m-1})v'_1 \cdots v'_{m-1}\\ =(m \cdots n)(w_{m-1})v''_1 \cdots v''_{m-1} &= (m \cdots n)(w_{m-1})w''_{m-1},\end{aligned}$$

which proves (2).

Regarding (3): Acting with $(n \cdots t)$, observe that (3) is equivalent to

$$w''_{m-1}(n \cdots m)(w_{m-1}) = w'''_{m-1}(n \cdots m)(w_{m-1}).$$

Again consider Remark 2.15 and observe that $w_{m-1} \in y_{i(i+1)} \cdots y_{im}\mathcal{D}_n(\alpha_1, \alpha_2)$ for all $1 \leq i \leq m-1$. Lemma 2.13(3) implies $y_{i(i+1)} \cdots y_{im}w_{m-1} = \lambda_{m-i}w_{m-1}$ for all $1 \leq i \leq m-1$, and thus by acting with $(n \cdots m)$ we obtain

$$y_{i(i+1)}\underset{\rightarrow}{\cdots}y_{i(m-1)}y_{in}(n \cdots m)(w_{m-1}) = \lambda_{m-i}(n \cdots m)(w_{m-1})$$

for all $1 \leq i \leq m-1$. Hence for $i = m-1$:

$$\begin{aligned}
&w''_{m-1}(n \cdots m)(w_{m-1}) = v''_1 \cdots v''_{m-1}(n \cdots m)(w_{m-1}) \\
=&v''_1 \underset{\rightarrow}{\cdots} v''_{m-2} \frac{\lambda_1}{\lambda_{n-m+1}} y_{(m-1)(m+1)} \cdots y_{(m-1)n}(n \cdots m)(w_{m-1}) \\
=&v''_1 \underset{\rightarrow}{\cdots} v''_{m-2} \frac{\lambda_1^2}{\lambda_{n-m+1}} y_{(m-1)(m+1)} \cdots y_{(m-1)(n-1)}(n \cdots m)(w_{m-1}) \\
=&v'''_{m-1}((m-1)(m+1) \cdots (n-1)) \left(v''_1 \underset{\rightarrow}{\cdots} v''_{m-2}\right)(n \cdots m)(w_{m-1}).
\end{aligned}$$

If $m = 2$ we are finished, so suppose $m \geq 3$. Now if $1 \leq i \leq m-2$, then for

$$v''_i := \frac{\lambda_{m-i}}{\lambda_{m-1-i}\lambda_{n-i}} y_{i(i+1)} \underset{\rightarrow}{\cdots} y_{i(m-1)} y_{i(m+1)} \cdots y_{in}$$

observe that

$$\begin{aligned}
&((i+1)(m+1) \cdots (n-1)) \cdots ((m-1)(m+1) \cdots (n-1))(v''_i) \\
=&\frac{\lambda_{m-i}}{\lambda_{m-1-i}\lambda_{n-i}} y_{i(m+1)} \underset{\rightarrow}{\cdots} y_{i(n-1)} y_{i(i+1)} \underset{\rightarrow}{\cdots} y_{i(m-1)} y_{in}.
\end{aligned}$$

By construction we can use Lemma 2.8 to shift the first $n-1-m$ factors of v''_i in the product $v''_j v''_i$ to the left for all $1 \leq j < i \leq m-2$ (this would not have worked for the v'_i) and thus with the above we get

$$\begin{aligned}
&((i+1)(m+1) \cdots (n-1)) \cdots ((m-1)(m+1) \cdots (n-1)) \\
&(v''_1 \cdots v''_i)(n \cdots m)(w_{m-1}) \\
=&((i+1)(m+1) \cdots (n-1)) \cdots ((m-1)(m+1) \cdots (n-1)) \\
&(v''_1 \underset{\rightarrow}{\cdots} v''_{i-1}) \frac{\lambda_{m-i}^2}{\lambda_{m-1-i}\lambda_{n-i}} y_{i(m+1)} \underset{\rightarrow}{\cdots} y_{i(n-1)}(n \cdots m)(w_{m-1}) \\
=&v'''_i(i(m+1) \cdots (n-1)) \cdots ((m-1)(m+1) \cdots (n-1)) \\
&(v''_1 \underset{\rightarrow}{\cdots} v''_{i-1})(n \cdots m)(w_{m-1})
\end{aligned}$$

Hence inductively we obtain

$$\begin{aligned}
&w''_{m-1}(n \cdots m)(w_{m-1}) = v''_1 \cdots v''_{m-1}(n \cdots m)(w_{m-1}) \\
=&v'''_{m-1} \underset{\leftarrow}{\cdots} v'''_1(n \cdots m)(w_{m-1}) = w'''_{m-1}(n \cdots m)(w_{m-1}),
\end{aligned}$$

which proves (3). □

Proposition 2.19. *Assume $n \geq 4$ and let $1 \leq m \leq n-1$. Then*

$$e_i := w_{m-1}\left((m \cdots n)^i \cdot v_m\right) w_{m-1}, \qquad 0 \leq i \leq n-m$$

forms a set of orthogonal idempotents in the algebra $w_{m-1}\mathcal{D}_n(\alpha_1, \alpha_2)w_{m-1}$ and $\sum_{i=0}^{n-m} e_i = w_{m-1}$.

Proof. Note that $e_0 = w_m w_{m-1}$. We have shown that w_{m-1} is an idempotent in Proposition 2.17 (if $m = 1$ then $w_{m-1} = w_0 = 1$ is trivially idempotent). Using Lemma 2.13(1) we obtain:

$$\sum_{i=0}^{n-m} e_i = w_{m-1}\left(\sum_{i=0}^{n-m}(m \cdots n)^i \cdot v_m\right) w_{m-1} = w_{m-1}\, 1\, w_{m-1} = w_{m-1}.$$

Next we show, that e_i and e_j are orthogonal for $0 \leq i < j \leq n-m$. For readability, we ignore the factors $\frac{1}{\lambda_k}$, $n-m \leq k \leq n-1$, in this part (assuming $\lambda_k = 1$ for all k, so to speak), since they are not necessary for the orthogonality. Let $s = m+i$ and $t = m+j$. Then $m \leq s < t \leq n$ and

$$(m \cdots n)^i \cdot v_m = (m \cdots n)^i \left(y_{m(m+1)} \cdots y_{mn}\right) = y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{sn}\, y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)}$$
$$(m \cdots n)^j \cdot v_m = y_{t(t+1)} \underset{\rightarrow}{\cdots} y_{tn}\, y_{tm} \underset{\rightarrow}{\cdots} y_{t(t-1)}.$$

Observe that if $i = 0$, then $y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)} = 1$, and if $j = n-m$, then $y_{t(t+1)} \underset{\rightarrow}{\cdots} y_{tn} = 1$ and thus, setting $v_n = 1$, we get

$$\begin{aligned} e_i =& w_{m-1}\, y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{sn} y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)}\, w_{m-1} = w_{m-1}\, v_s y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)}\, w_{m-1} \\ e_j =& w_{m-1}\, v_t y_{tm} \underset{\rightarrow}{\cdots} y_{t(t-1)}\, w_{m-1}. \end{aligned}$$

Since according to Lemma 2.13(2) and $s < t$ we have

$$\begin{aligned} & y_{ts} \underset{\rightarrow}{\cdots} y_{t(t-1)} y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{st} \\ =& (t(t-1) \cdots s) \cdot \left(y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{st}\right) y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{st} = 0, \end{aligned}$$

and thus

$$\begin{aligned} & y_{tm} \underset{\rightarrow}{\cdots} y_{t(t-1)} v_s \\ =& y_{tm} \underset{\rightarrow}{\cdots} y_{t(s-1)} y_{ts} \underset{\rightarrow}{\cdots} y_{t(t-1)} y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{st} y_{s(t+1)} \underset{\rightarrow}{\cdots} y_{sn} = 0. \end{aligned}$$

With this, using Lemma 2.16, we obtain

$$\begin{aligned} e_j e_i &= w_{m-1} v_t y_{tm} \underset{\rightarrow}{\cdots} y_{t(t-1)} w_{m-1} v_s y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)} w_{m-1} \\ &= w_{m-1} v_t\, y_{tm} \underset{\rightarrow}{\cdots} y_{t(t-1)} v_s\, (s \cdots n) \cdot w_{m-1} y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)} w_{m-1} = 0 \end{aligned}$$

Now note that $y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)}$ and v_t commute. Using Lemma 2.8 we obtain

$$\begin{aligned} e_i e_j =& w_{m-1} v_s y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)} w_{m-1} v_t y_{tm} \underset{\rightarrow}{\cdots} y_{t(t-1)} w_{m-1} \\ =& v_t (t \cdots n)(w_{m-1} v_s)(t \cdots n)(s \cdots m)(w_{m-1}) \\ & (s \cdots m)(y_{tm} \underset{\rightarrow}{\cdots} y_{t(t-1)} w_{m-1}) y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)} \\ =& v_t (t \cdots n)(v_s)(t \cdots n)(s \cdots n)(w_{m-1})(t \cdots n)(s \cdots m)(w_{m-1}) \\ & (s \cdots m)(t \cdots m)(w_{m-1})(s \cdots m)(y_{tm} \underset{\rightarrow}{\cdots} y_{t(t-1)}) y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)} = 0. \end{aligned}$$

The last product is equal to 0 according to Lemma 2.18, since the last factor of $(t \cdots n)(v_s)$ is y_{st}, and the first factor of $(s \cdots m)(y_{tm} \cdots y_{t(t-1)})$ is y_{ts}.

Finally for $0 \leq i \leq m-n$ we have

$$e_i e_i = e_i \left(w_{m-1} - \sum_{j=0, j \neq i}^{n-m} e_j \right) = e_i w_{m-1} = e_i,$$

implying that e_i is idempotent. □

Assumption 2.20. *For all* $1 \leq m \leq n-1$ *the elements*

$$e_i := w_{m-1}\left((m\cdots n)^i \cdot v_m\right) w_{m-1}, \qquad 0 \leq i \leq n-m$$

form a set of isomorphic idempotents in the algebra $w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1}$*.*

Remark 2.21. In section 2.5 we will give some discussion to when Assumption 2.20 holds. The result of that is, that it holds in particular in the case where $n=4$ and $\alpha_1^2-\alpha_2^2\neq 0$, i.e. besides $\lambda_1\lambda_2\lambda_3\neq 0$ we also need $\alpha_1+\alpha_2\neq 0$. This does however rely on some computer calculations. The assumption is also not true for any α_1,α_2, if $n=3$.

Theorem 2.22. *Let* $1 \leq m \leq n-1$ *and suppose Assumption 2.20 holds. Then*

$$w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1} \cong M_{n-m+1}\left(w_m\mathcal{D}_n(\alpha_1,\alpha_2)w_m\right)$$

as algebras. Moreover

$$\mathcal{D}_n(\alpha_1,\alpha_2) \cong M_{n!}\left(w\mathcal{D}_n(\alpha_1,\alpha_2)w\right)$$

as algebras.

Proof. Let

$$e_i := w_{m-1}\left((m\cdots n)^i \cdot v_m\right) w_{m-1}, \qquad 0 \leq i \leq n-m.$$

From Proposition 2.19 and Assumption 2.20 we obtain that those e_i form a set orthogonal and isomorphic idempotents in the algebra $w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1}$ and $\sum_{i=0}^{n-m} e_i = w_{m-1}$. Using Corollary 1.8, we obtain the algebra isomorphy

$$w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1} \cong M_{n-m+1}\left(e_0w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1}e_0\right),$$

and $e_0w_{m-1} = w_mw_{m-1}w_{m-1} = w_mw_{m-1}$ as well as $w_{m-1}e_0 = w_mw_{m-1}$ and according to Proposition 2.17 w_mw_{m-1} is isomorphic to w_m, hence using Proposition 1.6 we obtain

$$e_0w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1}e_0 \cong w_m\mathcal{D}_n(\alpha_1,\alpha_2)w_m$$

as algebras. This proves the first part of the theorem. Now using this part inductively, starting with $m=1$ and iterating up to $m=n-1$ yields

$$\begin{aligned}\mathcal{D}_n(\alpha_1,\alpha_2) =& w_0\mathcal{D}_n(\alpha_1,\alpha_2)w_0 \cong M_n\left(w_1\mathcal{D}_n(\alpha_1,\alpha_2)w_1\right)\\ \cong& M_n\left(M_{n-1}\left(w_2\mathcal{D}_n(\alpha_1,\alpha_2)w_2\right)\right)\\ \cong& \cdots \cong M_n\left(M_{n-1}\left(\cdots\left(M_2\left(w_{n-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{n-1}\right)\right)\cdots\right)\right)\\ \cong& M_{n!}\left(w\mathcal{D}_n(\alpha_1,\alpha_2)w\right),\end{aligned}$$

which implies the second part of the theorem. □

Computer calculations show, that the following proposition holds.

Proposition 2.23. *If* $n=4$*, then*

$$w\mathcal{D}_4(\alpha_1,\alpha_2)w \cong \Bbbk.$$

It is rather simple to check that $wy_{ij}w \in \Bbbk w$ for all $1 \leq i, j \leq 4$, but so far I did not find a general proof.

Theorem 2.24. *In the case $n = 4$, if $\alpha_1 + \alpha_2 \neq 0$ (as well as $\lambda_1\lambda_2\lambda_3 \neq 0$, as presumed in this whole section), then*

$$\mathcal{D}_4(\alpha_1, \alpha_2) \cong M_{24}(\Bbbk)$$

as algebras. In particular $\dim \mathcal{D}_4(\alpha_1, \alpha_2) = 576$ and $\mathcal{D}_4(\alpha_1, \alpha_2)$ is simple and semisimple.

Proof. Assumption 2.20 holds in this case, as discussed in Remark 2.21. So the theorem follows by combining Theorem 2.22 and Proposition 2.23. □

Remark 2.25. Theorem 2.24 relies on Assumption 2.20 and Proposition 2.23, which both rely on computer calculations.

For the following, final conjecture of this section, we will revoke the requirement that $\lambda_1 \cdots \lambda_{n-1} \neq 0$.

Conjecture 2.26. *If $n = 4$ and $(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$, then $\mathcal{D}_4(\alpha_1, \alpha_2)$ is semisimple.*

To prove the Conjecture there would be only two cases left that were not handled in Theorem 2.24: The one where $\alpha_1 \neq 0$, $3\alpha_1 - \alpha_2 = 0$ and the one where $\alpha_1 = 0$, $\alpha_2 \neq 0$. What backs up this conjecture is on one hand that I calculated a lot of different subalgebras that definitely do not contain any elements from the radical in both these cases, which was true for multiple of those subalgebras in the cases $\alpha_1^2 = \alpha_2^2$. Moreover I partially calculated the trace forms in both these cases and at least the first 25 rows of its matrix form are linearly independent in the case where $3\alpha_1 = \alpha_2$ (that is not much, but also not true in the case where $\alpha_1^2 = \alpha_2^2$) and at least 118 rows in the case where $\alpha_1 = 0$, $\alpha_2 \neq 0$. On the other hand the behaviour discussed in section 3 backs up this conjecture as well.

2.3 The non-semisimple case

In this section we will restrict to the case where $n = 4$ and $(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) = 0$. This is precisely the case that was not proven (or conjectured) semisimple in the previous section for $n = 4$. In Theorem 2.30 we will show, that the algebra is indeed not semisimple in this case.

Notation 2.27. Let

$$\begin{aligned} c &= 2(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2) \\ t &= (x_{12} + x_{13})^2 + (x_{12} + x_{14})^2 + (x_{13} + x_{14})^2 \\ z_2 &= (tx_{12} - x_{12}t)^2 + c \\ z_3 &= (tx_{13} - x_{13}t)^2 + c \\ z_4 &= (tx_{14} - x_{14}t)^2 + c \end{aligned}$$

Observe that t is invariant under the subgroup $S(\{2, 3, 4\})$ of S_4, and that $z_3 = (23)z_2$, $z_4 = (24)z_2$ as well as $z_4 = (34)z_3$. Also observe that $tx_{12} - x_{12}t = x_{13}x_{14}x_{12} + x_{14}x_{13}x_{12} - x_{12}x_{13}x_{14} - x_{12}x_{14}x_{13}$.

Lemma 2.28. *The following relations hold for* $2 \leq i, j \leq 4$, $i \neq j$:

$$z_2 + z_3 + z_4 = c \qquad z_i z_j = 0 \qquad z_i^2 = c z_i.$$

Moreover $z_2 x_{ij} = x_{ij} z_2$ *for all* $(i,j) \in \{(1,2),(1,3),(1,4),(3,4)\}$.

Proof (relies on computer calculations). We will start with the second relations. First observe that

$$(tx_{12} - x_{12}t)x_{12} = -x_{12}(tx_{12} - x_{12}t),$$

which implies $z_2 x_{12} = x_{12} z_2$. Using Lemma 2.7 we obtain

$$\begin{aligned}(tx_{12} - x_{12}t)x_{34} &= (x_{13}x_{14}x_{12} + x_{14}x_{13}x_{12} - x_{12}x_{13}x_{14} - x_{12}x_{14}x_{13})\, x_{34} \\ &= x_{13}x_{14}x_{34}x_{12} - x_{14}x_{13}x_{43}x_{12} - x_{12}x_{13}x_{14}x_{34} + x_{12}x_{14}x_{13}x_{43} \\ &= x_{34}x_{14}x_{13}x_{12} - x_{43}x_{13}x_{14}x_{12} - x_{34}x_{12}x_{14}x_{13} + x_{43}x_{12}x_{13}x_{14} \\ &= x_{34}\,(x_{14}x_{13}x_{12} + x_{13}x_{14}x_{12} - x_{12}x_{14}x_{13} - x_{12}x_{13}x_{14}) = x_{34}(tx_{12} - x_{12}t),\end{aligned}$$

which implies $z_2 x_{34} = x_{34} z_2$. We now want to show that

$$(tx_{12} - x_{12}t)^2 x_{13} = x_{13}(tx_{12} - x_{12}t)^2$$

which implies $z_2 x_{13} = x_{13} z_2$. Now using that $tx_{12} - x_{12}t = x_{13}x_{14}x_{12} + x_{14}x_{13}x_{12} - x_{12}x_{13}x_{14} - x_{12}x_{14}x_{13}$ and calculating $(tx_{12} - x_{12}t)^2$ without using any relations except $x_{ij}^2 = \alpha_1$ we obtain the following 12 terms:

$$\begin{aligned}&(x_{13}x_{14}x_{12})^2 + (x_{12}x_{14}x_{13})^2 + (x_{14}x_{13}x_{12})^2 + (x_{12}x_{13}x_{14})^2 \\ &+ x_{13}x_{14}x_{12}x_{14}x_{13}x_{12} + x_{12}x_{14}x_{13}x_{12}x_{13}x_{14} - x_{12}(x_{14}x_{13})^2 x_{12} \\ &+ x_{14}x_{13}x_{12}x_{13}x_{14}x_{12} + x_{12}x_{13}x_{14}x_{12}x_{14}x_{13} - x_{12}(x_{13}x_{14})^2 x_{12} \\ &- \alpha_1(x_{13}x_{14})^2 - \alpha_1(x_{14}x_{13})^2 - 4\alpha_1^3\end{aligned}$$

The order in which I have written this sum is not arbitrary: First computer calculations that

$$(x_{13}x_{14}x_{12})^2 + (x_{12}x_{14}x_{13})^2 = 2\alpha_1\alpha_2^2.$$

By acting with (34) we obtain

$$(x_{14}x_{13}x_{12})^2 + (x_{12}x_{13}x_{14})^2 = 2\alpha_1\alpha_2^2,$$

hence the first row of terms is equal to $4\alpha_1\alpha_2^2$, hence x_{13} commutes with the first row. Now computer calculations also show, that x_{13} multiplied from the left to the second row is the same as multiplying the third row from the right with x_{13} and x_{13} multiplied from the left to the third row is the same as multiplying the second row from the right with x_{13}. Hence x_{13} commutes with the second and third row combined. Finally we have

$$\begin{aligned}x_{13}\ (x_{13}x_{14}x_{13}x_{14} + x_{14}x_{13}x_{14}x_{13}) &= \alpha_1 x_{14}x_{13}x_{14} + x_{13}x_{14}x_{13}x_{14}x_{13} \\ &= (x_{13}x_{14}x_{13}x_{14} + x_{14}x_{13}x_{14}x_{13})\ x_{13},\end{aligned}$$

thus x_{13} commutes with the 4th row, which in total implies that x_{13} commutes with $(tx_{12} - x_{12}t)^2$. It follows that $z_2 x_{13} = x_{13} z_2$. Acting with (34) we obtain $z_2 x_{14} = x_{14} z_2$.

We can conclude that z_2, z_3 and z_4 commute with each other. Hence for the orthogonality it is enough to show $z_2 z_3 = 0$, since $z_2 z_4 = 0$ is obtained from that by acting with (34) and $z_3 z_4 = 0$ is obtained by acting with (234). Computer calculations yield

$$z_2 z_3 = 0,$$

as well as

$$z_2 + z_3 + z_4 = c.$$

Similar to the proof of Lemma 2.13(3) we can conclude from the above, that $z_i^2 = c z_i$ for all $2 \leq i \leq 4$. □

Lemma 2.29. *Assume* $(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2) = 0$. *Then the following relations hold:*

$$\begin{aligned} z_2 x_{23} =& x_{23} z_3 + x_{14} z_4, \\ z_2 x_{24} =& x_{13} z_3 + x_{24} z_4. \end{aligned}$$

Moreover $z_2 \mathcal{D}_4(\alpha_1, \alpha_2) \subset \mathcal{D}_4(\alpha_1, \alpha_2) z_2 + \mathcal{D}_4(\alpha_1, \alpha_2) z_3 + \mathcal{D}_4(\alpha_1, \alpha_2) z_4$.

Proof (relies on computer calculations). The first relation is obtained via computer calculations (it does not hold if $(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2) \neq 0$). The second relation is obtained by acting with (34) on the first. Now combining these two relations with Lemma 2.28, we obtain that $z_2 x_{ij} \in \mathcal{D}_4(\alpha_1, \alpha_2) z_2 + \mathcal{D}_4(\alpha_1, \alpha_2) z_3 + \mathcal{D}_4(\alpha_1, \alpha_2) z_4$ for all $1 \leq i, j \leq 4$. Acting with (23) and (24) we obtain the same for $z_3 x_{ij}$ and $z_4 x_{ij}$. Hence $z_2 \mathcal{D}_4(\alpha_1, \alpha_2) \subset \mathcal{D}_4(\alpha_1, \alpha_2) z_2 + \mathcal{D}_4(\alpha_1, \alpha_2) z_3 + \mathcal{D}_4(\alpha_1, \alpha_2) z_4$. □

Theorem 2.30. *Assume* $(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2) = 0$. *Then* $(z_2 \mathcal{D}_4(\alpha_1, \alpha_2))^2 = 0$. *It follows that the ideal generated by the elements* $\pi \cdot z_2$, $\pi \in S_4$, *is a subset of the Jacobson radical of* $\mathcal{D}_4(\alpha_1, \alpha_2)$, *implying that* $\mathcal{D}_4(\alpha_1, \alpha_2)$ *is not semisimple.*

Proof. It is enough to show that $z_2 \mathcal{D}_4(\alpha_1, \alpha_2) z_2 = 0$. This follows from the fact that $z_2 \mathcal{D}_4(\alpha_1, \alpha_2) \subset \mathcal{D}_4(\alpha_1, \alpha_2) z_2 + \mathcal{D}_4(\alpha_1, \alpha_2) z_3 + \mathcal{D}_4(\alpha_1, \alpha_2) z_4$, as shown in Lemma 2.29, as well as $z_i z_2 = 0$ for all $2 \leq i \leq 4$ according to Lemma 2.28. □

Remark 2.31. Assume $(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2) = 0$. If $\alpha_1 = \alpha_2 = 0$, then computer calculations show that the Jacobson radical is generated by the generators x_{ij}, $1 \leq i < j \leq 4$, hence the quotient over the Jacobson radical has dimension 1. So assume $\alpha_1 \neq 0$. Computer calculations show that the ideal generated by the elements $\pi \cdot z_2$, $\pi \in S_4$ has dimension 288 if $\alpha_1 = \alpha_2$ and dimension 240 if $\alpha_1 = -\alpha_2$. This is however not the entire Jacobson radical: Computer calculations also show that in the case where $\alpha_1 = -\alpha_2$ the right ideal $I := (x_{12} x_{13} - x_{13} x_{12}) A$ satisfies $I^4 = 0$, hence the (two-sided) ideal generated by the elements $\pi(x_{12} x_{13} - x_{13} x_{12})$, $\pi \in S_4$ is contained in the Jacobson radical. This ideal has dimension 552. Thus the quotient over this ideal is commutative and has dimension 24. Calculating the trace form yields that this quotient is semisimple, which implies that this ideal coincides with the Jacobson radical.

For the other case, where $\alpha_1 = \alpha_2$, we obtain, that the right ideal $I := (x_{12} x_{13} + x_{12} x_{14} + x_{12} x_{23} + x_{13} x_{23} + x_{14} x_{12} + \alpha_1) A$ satisfies $I^5 = 0$. Hence the two-sided ideal generated by the same elements must lie in the Jacobson radical and as it turns out again has dimension 552. Computer calculations also show, that the 24 dimensional quotient over that ideal is semisimple, hence the Jacobson racial is equal to that ideal.

2.4 Conclusion

In section 2.2 we showed that $\mathcal{D}_4(\alpha_1, \alpha_2)$ is semisimple if $\alpha_1(3\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$ and conjectured that is also semisimple in the case where just $(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$. In section 2.3 we showed that $\mathcal{D}_4(\alpha_1, \alpha_2)$ is indeed not semisimple if $(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) = 0$. We summarize this in the following corollary.

Corollary 2.32. *Assuming Conjecture 2.26 to be true, the algebra $\mathcal{D}_4(\alpha_1, \alpha_2)$ is semisimple if and only if $(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$.*

2.5 Discussing Assumption 2.20

We find ourselves in the context of section 2.2, so we have a fixed $n \in \mathbb{N}$, $n \geq 3$ and assume $\lambda_1 \cdots \lambda_{n-1} \neq 0$. If needed, also recall Notation 2.14.

In this section we will discuss a specific assumption that is very easy to check for algebras with low dimension and which is true in the case $n = 4$ where $\alpha_1^2 \neq \alpha_2^2$ (see Remark 2.35). At the end of the section (Proposition 2.38) we will show that this assumption implies Assumption 2.20.

Assumption 2.33. *Suppose there exists an invertible element $u \in \mathcal{D}_n(\alpha_1, \alpha_2)$ satisfying*

$$u\,a = (12) \cdot a\,u$$

for all $a \in \mathcal{D}_n(\alpha_1, \alpha_2)$.

Proposition 2.34. *Suppose that Assumption 2.33 holds. Then for all $\pi \in S_n$ there exists an invertible element $u_\pi \in \mathcal{D}_n(\alpha_1, \alpha_2)$ satisfying*

$$u_\pi\,a = \pi \cdot a\,u_\pi$$

for all $a \in \mathcal{D}_n(\alpha_1, \alpha_2)$.

Proof. We define $V_\pi := \{u \in \mathcal{D}_4(\alpha_1, \alpha_2) \,|\, \forall a \in \mathcal{D}_4(\alpha_1, \alpha_2) :\ ua = \pi \cdot a\,u\}$ for all $\pi \in S_n$. Observe that V_{id} is the center of $\mathcal{D}_n(\alpha_1, \alpha_2)$, and $1 \in V_{\mathrm{id}}$ is invertible. If $u \in V_\pi$ and $v \in V_\sigma$ for some $\pi, \sigma \in S_n$, then for $a \in \mathcal{D}_n(\alpha_1, \alpha_2)$ we have

$$uva = u\,\sigma \cdot a\,v = (\pi\sigma) \cdot a\,uv,$$

hence $uv \in V_{\pi\sigma}$. We also obtain

$$\sigma \cdot u\,a = \sigma \cdot \big(u\,\sigma^{-1} \cdot a\big) = \sigma \cdot \big((\pi\sigma^{-1}) \cdot a\,u\big) = (\sigma\pi\sigma^{-1}) \cdot a\ \sigma \cdot u,$$

hence $\sigma \cdot u \in V_{\sigma\pi\sigma^{-1}}$. This means that we only need to calculate these elements for one element of every cycle type of S_n (since the cycle types are the conjugacy classes) and by acting with the group we obtain the other ones. By assumption there exists an invertible element $u \in V_{(12)}$. This implies that there exists an invertible element in $V_{(ij)}$ for every $1 \leq i < j \leq n$, by acting with the group, as shown above. If v is a unit in $V_{(ij)}$ and w is a unit in $V_{(kl)}$, then vw is a unit in $V_{(ij)(kl)}$. Since S_n is generated by the the 2-cycles (ij), it follows that there exists a unit in every V_π, $\pi \in S_n$, implying the proposition. □

Remark 2.35. Assume $n = 4$. We will use computer calculations to show, that Assumption 2.33 holds if $\alpha_1^2 - \alpha_2^2 \neq 0$. With some techniques of linear algebra, we can calculate the spaces $V_\pi := \{u \in \mathcal{D}_4(\alpha_1, \alpha_2) \,|\, \forall a \in \mathcal{D}_4(\alpha_1, \alpha_2) : \; ua = \pi \cdot a\, u\}$, for all $\pi \in S_4$. The result is that V_π is at least one dimensional (meaning the dimension is one if we assume that α_1 and α_2 are algebraically independent) for all $\pi \in S_4$ and for all $\alpha_1, \alpha_2 \in \Bbbk$. As it turns out, choosing a normalized $u \in V_{(12)}$, $u \neq 0$, such that $u \in V_{(12)}$ for all $\alpha_1, \alpha_2 \in \Bbbk$ (u is uniquely determined that way), calculations show $u^2 = -8(\alpha_1^2 - \alpha_2^2)^5$. This implies that u is invertible if and only if $\alpha_1^2 \neq \alpha_2^2$. We assumed $\lambda_3 \neq 0$, so we have that $\alpha_1 \neq \alpha_2$. Hence if we also assume $\alpha_1 \neq -\alpha_2$ we get that u is invertible, implying the assumption.

Another interesting conclusion is, that if $\alpha_1^2 = \alpha_2^2$, then $u^2 = 0$, hence $(ua)^2 = (12) \cdot a\, u^2 a = 0$ for all $a \in \mathcal{D}_4(\alpha_1, \alpha_2)$, implying that u lies in the Jacobson radical of $\mathcal{D}_4(\alpha_1, \alpha_2)$ and thus giving another argument that the algebra is not semisimple. The right ideal (which is equal to the two-sided ideal) generated by u has dimension 16, if $\alpha_1 = \alpha_2$ and dimension 8 if $\alpha_1 = -\alpha_2$.

Lemma 2.36. *For $1 \leq m \leq n-1$, setting $\lambda_0 = 1$, the following relations hold:*

$$w_{m-1}\,(m \cdots n) \cdot w_{m-1}\, w_{m-1} = \frac{\lambda_{n-m}\lambda_{m-1}}{\lambda_{n-1}} w_{m-1},$$
$$w_{m-1}\,(n \cdots m) \cdot w_{m-1}\, w_{m-1} = \frac{\lambda_{n-m}\lambda_{m-1}}{\lambda_{n-1}} w_{m-1}.$$

In particular, the left hand sides coincide.

Proof. The statement is trivial for $m = 1$. So assume $m \geq 2$. We start with the first relation. Therefore, consider the following construction: Let $\lambda_0 = 1$ and for all $1 \leq i \leq m-1$ let

$$v'_i := \frac{\lambda_{n-i-1}}{\lambda_{n-i}} y_{im} \qquad\qquad w'_i := v'_1 \cdots v'_i,$$
$$v''_i := \frac{\lambda_{n-i-1}}{\lambda_{m-1-i}\lambda_{n-i}} y_{i(i+1)} \underset{\rightarrow}{\cdots} y_{im} \qquad\qquad w''_i := v''_1 \cdots v''_i.$$

Below we will show the following three relations:

(1) $w_{m-1}\,(m \cdots n) \cdot w_{m-1} = w_{m-1}\, w'_{m-1}$,

(2) $w_{m-1}\, w'_{m-1} = w_{m-1}\, w''_{m-1}$,

(3) $w''_{m-1}\, w_{m-1} = \frac{\lambda_{n-m}\lambda_{m-1}}{\lambda_{n-1}} w_{m-1}$.

Combining (1), (2) and (3) implies the first relation of the claim. Regarding (1): First observe, that the factors of $(m \cdots n) \cdot w_{m-1}$ are

$$(m \cdots n) \cdot v_i = \frac{1}{\lambda_{n-i}} y_{i(i+1)} \underset{\rightarrow}{\cdots} y_{i(m-1)} y_{i(m+1)} \underset{\rightarrow}{\cdots} y_{in} y_{im} \qquad 1 \leq i \leq m-1$$

We start with shifting $y_{(m-1)(m+1)}\underset{\rightarrow}{\cdots}y_{(m-1)n}$ in the factor for $i = m-1$ to the left of $(m\cdots n)\cdot w_{m-1}$, according to Lemma 2.8:

$$\begin{aligned}
&(m\cdots n)\cdot w_{m-1} = (m\cdots n)\cdot(v_1\cdots v_{m-1})\\
=&\frac{1}{\lambda_{n-m+1}}(m\cdots n)\cdot(y_{(m-1)m}\underset{\rightarrow}{\cdots}y_{(m-1)(n-1)}\\
&((m-1)m\cdots(n-1))\cdot(v_1\cdots v_{m-2})\,y_{(m-1)n})\\
=&\frac{1}{\lambda_{n-m+1}}y_{(m-1)(m+1)}\underset{\rightarrow}{\cdots}y_{(m-1)n}\\
&(m\cdots n)((m-1)m\cdots(n-1))\cdot(v_1\cdots v_{m-2})\,y_{(m-1)m}.
\end{aligned}$$

Observe that $w_{m-1}\in\mathcal{D}_n(\alpha_1,\alpha_2)y_{(m-1)(m+1)}\cdots y_{(m-1)m}$, so we can exchange the $y_{(m-1)(m+1)}\underset{\rightarrow}{\cdots}y_{(m-1)n}$ that we pulled to the left for the scalar λ_{n-m}. In total we get

$$\begin{aligned}
&w_{m-1}\,(m\cdots n)\cdot w_{m-1}\\
=&w_{m-1}\,(m\cdots n)((m-1)m\cdots(n-1))\cdot(v_1\cdots v_{m-2})v'_{m-1}.
\end{aligned}$$

Now if $m=2$, we are done with this step. If $m\geq 3$, assume that there exists some $1\leq k\leq m-2$, such that

$$\begin{aligned}
&w_{m-1}\,(m\cdots n)\cdot w_{m-1}\\
=&w_{m-1}\,(m\cdots n)((m-1)\cdots(n-1))\cdots((k+1)\cdots(n-1))\\
&\cdot(v_1\cdots v_k)v'_{k+1}\cdots v'_{m-1}.
\end{aligned}$$

Acting with $(m\cdots n)((m-1)\cdots(n-1))\cdots((k+1)\cdots(n-1))$ on v_k we get

$$\frac{1}{\lambda_{n-k}}y_{k(m+1)}\underset{\rightarrow}{\cdots}y_{kn}y_{k(m-1)}\underset{\leftarrow}{\cdots}y_{k(k+1)}y_{km}.$$

We pull everything except y_{km} to the left:

$$\begin{aligned}
&(m\cdots n)((m-1)\cdots(n-1))\cdots((k+1)\cdots(n-1))\\
&\cdot(v_1\cdots v_k)v'_{k+1}\cdots v'_{m-1}\\
=&\frac{1}{\lambda_{n-k}}(m\cdots n)((m-1)\cdots(n-1))\cdots((k+1)\cdots(n-1))\\
&\cdot\left(y_{k(k+1)}\cdots y_{k(n-1)}(k\cdots(n-1))\cdot(v_1\cdots v_{k-1})\,y_{kn}\right)v'_{k+1}\cdots v'_{m-1}\\
=&\frac{1}{\lambda_{n-k}}y_{k(m+1)}\underset{\rightarrow}{\cdots}y_{kn}y_{k(m-1)}\underset{\leftarrow}{\cdots}y_{k(k+1)}\\
&(m\cdots n)((m-1)\cdots(n-1))\cdots(k\cdots(n-1))\cdot(v_1\cdots v_{k-1})\,y_{km}v'_{k+1}\cdots v'_{m-1}
\end{aligned}$$

As discussed in Remark 2.15, we can exchange $y_{k(m+1)}\underset{\rightarrow}{\cdots}y_{kn}y_{k(m-1)}\underset{\leftarrow}{\cdots}y_{k(k+1)}$ left into w_{m-1} for the scalar λ_{n-k-1}. This yields

$$\begin{aligned}
&w_{m-1}\,(m\cdots n)\cdot w_{m-1}\\
=&w_{m-1}\,(m\cdots n)((m-1)\cdots(n-1))\cdots(k\cdots(n-1))\\
&\cdot(v_1\cdots v_{k-1})v'_k\cdots v'_{m-1}
\end{aligned}$$

This finishes (1).

Regarding (2): This step is basically the same as step (2) in the proof of Lemma 2.18. Above we already used that $w_{m-1} \in \mathcal{D}_n(\alpha_1, \alpha_2) y_{i(m-1)} \underset{\leftarrow}{\cdots} y_{i(i+1)}$ for all $1 \leq i \leq m-2$. Hence $w_{m-1} = \frac{1}{\lambda_{m-1-i}} w_{m-1} y_{i(m-1)} \underset{\leftarrow}{\cdots} y_{i(i+1)}$ for all $1 \leq i \leq m-2$. Now we inductively pull those y in, first for $i = 1$, up to $i = m-2$ and get

$$\begin{aligned} w_{m-1} =& w_{m-1} \frac{1}{\lambda_1 \cdots \lambda_{m-2}} y_{(m-2)(m-1)} y_{(m-3)(m-1)} y_{(m-3)(m-2)} \cdots y_{1(m-1)} \underset{\leftarrow}{\cdots} y_{12} \\ =& w_{m-1} \frac{1}{\lambda_1 \cdots \lambda_{m-2}} y_{12} \underset{\rightarrow}{\cdots} y_{1(m-1)} \cdots y_{(m-3)(m-2)} y_{(m-3)(m-1)} y_{(m-2)(m-1)} \end{aligned}$$

The last equality just uses Lemma 2.8 multiple times to shift first the $y_{(m-2)(\cdot)}$ to the right, then $y_{(m-3)(\cdot)}$ down to the $y_{2(\cdot)}$. It basically reverses the process in Remark 2.15, but just for the last few y. Finally observe that for $2 \leq i \leq m-2$ the product $y_{i(i+1)} \cdots y_{i(m-1)}$ commutes with v'_j for all $1 \leq j < i$, and $\frac{1}{\lambda_{m-1-i}} y_{i(i+1)} \cdots y_{i(m-1)} v'_i = v''_i$. Hence, considering that $v'_{m-1} = v''_{m-1}$, we obtain

$$w_{m-1}\, w'_{m-1} = w_{m-1} v'_1 \cdots v'_{m-1} = w_{m-1} v''_1 \cdots v''_{m-1} = w_{m-1} w''_{m-1},$$

which proves (2).

Regarding (3): In Remark 2.15 it was also discussed that we have $w_{m-1} \in y_{i(i+1)} \cdots y_{im} \mathcal{D}_4(\alpha_1, \alpha_2)$ for all $1 \leq i \leq m-1$. Hence starting with $i = m-1$ down to $i = 1$, we can exchange the y in v''_i for the scalar λ_{m-i}, i.e. the whole w''_{m-1} vanishes:

$$w''_{m-1}\, w_{m-1} = \frac{\lambda_{n-m} \underset{\rightarrow}{\cdots} \lambda_{n-2}}{\lambda_{n-m+1} \underset{\rightarrow}{\cdots} \lambda_{n-1}} \frac{\lambda_1 \underset{\rightarrow}{\cdots} \lambda_{m-1}}{\lambda_1 \underset{\rightarrow}{\cdots} \lambda_{m-2}} w_{m-1} = \frac{\lambda_{n-m} \lambda_{m-1}}{\lambda_{n-1}} w_{m-1}.$$

This implies (3).

Now the second relation. Consider the following construction: Let $\lambda_0 = 1$ and for all $1 \leq i \leq m-1$ let

$$\begin{aligned} v'_i :=& \frac{\lambda_{n-i-1}}{\lambda_{n-i}} y_{in} & w'_i :=& v'_i \underset{\leftarrow}{\cdots} v'_1, \\ v''_i :=& \frac{\lambda_{n-i-1}}{\lambda_{m-1-i} \lambda_{n-i}} y_{in} y_{i(m-1)} \underset{\leftarrow}{\cdots} y_{i(i+1)} & w''_i :=& v''_i \underset{\leftarrow}{\cdots} v''_1. \end{aligned}$$

Below we will show the following three relations:

(1) $(n \cdots m) \cdot w_{m-1}\, w_{m-1} = w'_{m-1}\, w_{m-1}$,

(2) $w'_{m-1}\, w_{m-1} = w''_{m-1}\, w_{m-1}$,

(3) $w_{m-1}\, w''_{m-1} = \frac{\lambda_{n-m} \lambda_{m-1}}{\lambda_{n-1}} w_{m-1}$.

Combining these implies the second relation of the claim. Regarding (1): First observe, that the factors of $(n \cdots m) \cdot w_{m-1}$ are

$$(n \cdots m) \cdot v_i = \frac{1}{\lambda_{n-i}} y_{i(i+1)} \underset{\rightarrow}{\cdots} y_{i(m-1)} y_{in} y_{im} \underset{\rightarrow}{\cdots} y_{i(n-1)} \qquad 1 \leq i \leq m-1$$

The last factor where $i = m-1$ is $y_{(m-1)n}y_{(m-1)m}\underset{\rightarrow}{\cdots}y_{(m-1)(n-1)}$. Now $w_{m-1} \in y_{(m-1)m}\underset{\rightarrow}{\cdots}y_{(m-1)(n-1)}\mathcal{D}_4(\alpha_1, \alpha_2)$, hence we can exchange this for the scalar λ_{n-m}:

$$\begin{aligned}
&(n \cdots m) \cdot w_{m-1}\, w_{m-1} = (n \cdots m) \cdot (v_1 \cdots v_{m-1})\, w_{m-1}\\
=&\frac{1}{\lambda_{n-m+1}}(n \cdots m) \cdot (v_1 \cdots v_{m-2}\, y_{(m-1)m})y_{(m-1)m}\underset{\rightarrow}{\cdots}y_{(m-1)(n-1)}\, w_{m-1}\\
=&\frac{\lambda_{n-m}}{\lambda_{n-m+1}}(n \cdots m) \cdot (v_1 \cdots v_{m-2}\, y_{(m-1)m})\, w_{m-1}\\
=&\frac{\lambda_{n-m}}{\lambda_{n-m+1}}y_{(m-1)n}(n \cdots m)((m-1)m) \cdot (v_1 \cdots v_{m-2})\, w_{m-1}\\
=&v'_{m-1}(n \cdots (m-1)) \cdot (v_1 \cdots v_{m-2})\, w_{m-1}.
\end{aligned}$$

Now if $m = 2$, we are done with this step. If $m \geq 3$, assume that there exists some $1 \leq k \leq m-2$, such that

$$\begin{aligned}
&(n \cdots m) \cdot w_{m-1}\, w_{m-1}\\
=&v'_{m-1}\underset{\leftarrow}{\cdots}v'_{k+1}(n \cdots (k+1)) \cdot (v_1 \cdots v_k)\, w_{m-1}.
\end{aligned}$$

Acting with $(n \cdots (k+1))$ on v_k we get

$$\frac{1}{\lambda_{n-k}}y_{kn}y_{k(k+1)}\underset{\rightarrow}{\cdots}y_{k(n-1)}.$$

Now again we have $w_{m-1} \in y_{k(k+1)}\underset{\rightarrow}{\cdots}y_{k(n-1)}\mathcal{D}_4(\alpha_1, \alpha_2)$, hence we can exchange this for the scalar λ_{n-k-1}:

$$\begin{aligned}
&v'_{m-1}\underset{\leftarrow}{\cdots}v'_{k+1}(n \cdots (k+1)) \cdot (v_1 \cdots v_k)\, w_{m-1}\\
=&\frac{\lambda_{n-k-1}}{\lambda_{n-k}}v'_{m-1}\underset{\leftarrow}{\cdots}v'_{k+1}(n \cdots (k+1)) \cdot (v_1 \cdots v_{k-1}\, y_{k(k+1)})\, w_{m-1}\\
=&\frac{\lambda_{n-k-1}}{\lambda_{n-k}}v'_{m-1}\underset{\leftarrow}{\cdots}v'_{k+1}y_{kn}(n \cdots (k+1))(k(k+1)) \cdot (v_1 \cdots v_{k-1})\, w_{m-1}\\
=&v'_{m-1}\underset{\leftarrow}{\cdots}v'_{k+1}v'_k\,(n \cdots k) \cdot (v_1 \cdots v_{k-1})\, w_{m-1}.
\end{aligned}$$

In total we get $(n \cdots m) \cdot (v_1 \cdots v_{m-1})\, w_{m-1} = v'_{m-1}\underset{\leftarrow}{\cdots}v'_1\, w_{m-1} = w'_{m-1}\, w_{m-1}$.

Regarding (2): We again use that $w_{m-1} \in y_{i(i+1)}\underset{\rightarrow}{\cdots}y_{i(m-1)}\mathcal{D}_n(\alpha_1, \alpha_2)$ for all $1 \leq i \leq m-2$. Hence $w_{m-1} = \frac{1}{\lambda_{m-1-i}}y_{i(i+1)}\underset{\rightarrow}{\cdots}y_{i(m-1)}w_{m-1}$ for all $1 \leq i \leq m-2$. Now we inductively pull those y in, first for $i = 1$, up to $i = m-2$ and get

$$\begin{aligned}
w_{m-1} =&\frac{1}{\lambda_1 \cdots \lambda_{m-2}}y_{12}\underset{\rightarrow}{\cdots}y_{1(m-1)} \cdots y_{(m-3)(m-2)}y_{(m-3)(m-1)}y_{(m-2)(m-1)}w_{m-1}\\
=&\frac{1}{\lambda_1 \cdots \lambda_{m-2}}y_{(m-2)(m-1)}y_{(m-3)(m-1)}y_{(m-3)(m-2)} \cdots y_{1(m-1)}\underset{\leftarrow}{\cdots}y_{12}w_{m-1}
\end{aligned}$$

The last equality just uses Lemma 2.8 multiple times to shift first the $y_{(m-2)(\cdot)}$ to the left, then $y_{(m-3)(\cdot)}$ down to the $y_{2(\cdot)}$. It is the same process as in

Remark 2.15, but just for the first few y. Finally observe that for $2 \leq i \leq m-2$ the product $y_{i(m-1)} \underset{\leftarrow}{\cdots} y_{i(i+1)}$ commutes with v'_j for all $1 \leq j < i$, and $\frac{1}{\lambda_{m-1-i}} v'_i y_{i(m-1)} \underset{\leftarrow}{\cdots} y_{i(i+1)} = v''_i$. Hence, considering that $v'_{m-1} = v''_{m-1}$, we obtain

$$w'_{m-1}\, w_{m-1} = v'_{m-1} \underset{\leftarrow}{\cdots} v'_1 w_{m-1} = v''_{m-1} \underset{\leftarrow}{\cdots} v''_1 w_{m-1} = w''_{m-1} w_{m-1},$$

which proves (2).

Regarding (3): In Remark 2.15 it was also discussed that we have $w_{m-1} \in \mathcal{D}_n(\alpha_1, \alpha_2) y_{in} y_{i(m-1)} \underset{\leftarrow}{\cdots} y_{i(i+1)}$ for all $1 \leq i \leq m-1$. Hence starting with $i = m-1$ down to $i = 1$, we can exchange the y in v''_i for the scalar λ_{m-i}, i.e. the whole w''_{m-1} vanishes:

$$w_{m-1}\, w''_{m-1} = \frac{\lambda_{n-m} \underset{\rightarrow}{\cdots} \lambda_{n-2}}{\lambda_{n-m+1} \underset{\rightarrow}{\cdots} \lambda_{n-1}} \frac{\lambda_1 \underset{\rightarrow}{\cdots} \lambda_{m-1}}{\lambda_1 \underset{\rightarrow}{\cdots} \lambda_{m-2}} w_{m-1} = \frac{\lambda_{n-m} \lambda_{m-1}}{\lambda_{n-1}} w_{m-1}.$$

This implies (3) and finishes the proof. □

Lemma 2.37. *For $1 \leq m \leq n-1$ the following relation holds:*

$$\begin{aligned} &w_{m-1}(m \cdots n) \cdot w_{m-1}\, y_{nm}\, (nm) \cdot w_{m-1} \\ =&w_{m-1}(n \cdots m) \cdot w_{m-1}\, y_{nm}\, (nm) \cdot w_{m-1}. \end{aligned}$$

Proof. It is trivial for $m = 1$, so assume $m \geq 2$. Observe that the claim cannot be solved trivially from this previous Lemma, since y_{nm} cannot be shifted to the right of $(nm)w_{m-1}$, nor can it be shifted to the left of w_{m-1}. It can however be shifted to the left of $(m \cdots n) \cdot w_{m-1}$ and to the left of $(n \cdots m) \cdot w_{m-1}$. We will basically redo the first two steps of the previous lemma, but with leaving enough of the $(m \cdots n) \cdot w_{m-1}$ intact so that we can shift y_{nm} to the left of it. Then we handle the right hand side similar to the second relation of the previous lemma. Consider the following construction: Let $\lambda_0 = 1$ and for all $1 \leq i \leq m-1$ let

$$\begin{aligned} v'_i :=& \frac{\lambda_{n-i-1}}{\lambda_{n-i}} y_{im} & w'_i :=& v'_1 \cdots v'_i, \\ v''_i :=& \frac{\lambda_{n-i-1}}{\lambda_i \lambda_{n-i}} y_{i(i+1)} \underset{\rightarrow}{\cdots} y_{i(m-1)} y_{in} y_{im} & w''_i :=& v''_1 \cdots v''_i, \\ v'''_i :=& \frac{\lambda_{n-i-1}}{\lambda_{n-i}} y_{in} & w'''_i :=& v'''_i \underset{\leftarrow}{\cdots} v'''_1. \end{aligned}$$

We will divide the relation from the claim in the following steps:

(1) $w_{m-1}\, (m \cdots n) \cdot w_{m-1} = w_{m-1}\, w'_{m-1}$,

(2) $w_{m-1}\, w'_{m-1} = w_{m-1}\, w''_{m-1}$,

(3) $w''_{m-1} y_{nm} = y_{nm}(nm) \cdot w''_{m-1}$,

(4) $w''_{m-1} w_{m-1} = w'''_{m-1} w_{m-1}$,

(5) $w'''_{m-1} w_{m-1} = (n \cdots m) \cdot w_{m-1}\, w_{m-1}$,

(6) $y_{nm}(nm)(n \cdots m) \cdot w_{m-1} = (n \cdots m) \cdot w_{m-1}\, y_{nm}$.

Observe that (1) is precisely step (1) of the first relation in Lemma 2.36, and step (5) is precisely step (1) of the second relation of that lemma. Also (3) and (6) are trivial at this point.

Regarding (2): This step is basically the same as step (2) of the first relation of the preceding Lemma but with one significant difference. We again use that $w_{m-1} \in \mathcal{D}_n(\alpha_1, \alpha_2) y_{in} y_{i(m-1)} \underset{\leftarrow}{\cdots} y_{i(i+1)}$ for all $1 \leq i \leq m-1$. Hence $w_{m-1} = \frac{1}{\lambda_{m-i}} w_{m-1} y_{in} y_{i(m-1)} \underset{\leftarrow}{\cdots} y_{i(i+1)}$ for all $1 \leq i \leq m-1$. Now we inductively pull those y in, first for $i = 1$, up to $i = m-1$ and get

$$\begin{aligned} w_{m-1} =& w_{m-1} \frac{1}{\lambda_1 \cdots \lambda_{m-1}} y_{(m-1)n} y_{(m-2)n} y_{(m-2)(m-1)} \cdots y_{1n} y_{1(m-1)} \underset{\leftarrow}{\cdots} y_{12} \\ =& w_{m-1} \frac{1}{\lambda_1 \cdots \lambda_{m-1}} y_{12} \underset{\rightarrow}{\cdots} y_{1(m-1)} y_{1n} \cdots y_{(m-2)(m-1)} y_{(m-2)n} y_{(m-1)n} \end{aligned}$$

The last equality just uses Lemma 2.8 multiple times to shift first the $y_{(m-1)(\cdot)}$ to the right, then $y_{(m-2)(\cdot)}$ down to the $y_{2(\cdot)}$. It basically reverses the process in Remark 2.15, but just for the last few y. Finally observe that for $2 \leq i \leq m-1$ the product $y_{i(i+1)} \cdots y_{i(m-1)} y_{in}$ commutes with v'_j for all $1 \leq j < i$, and $\frac{1}{\lambda_i} y_{i(i+1)} \cdots y_{i(m-1)} y_{in} v'_i = v''_i$. Hence we obtain

$$w_{m-1}\, w'_{m-1} = w_{m-1} v'_1 \cdots v'_{m-1} = w_{m-1} v''_1 \cdots v''_{m-1} = w_{m-1} w''_{m-1},$$

which proves (2).

Regarding (4): The last factor of w''_{m-1}, where $i = m-1$, is $y_{(m-1)n} y_{(m-1)m}$. Now $w_{m-1} \in y_{(m-1)m} \mathcal{D}_4(\alpha_1, \alpha_2)$, hence we can exchange $y_{(m-1)m}$ for λ_1:

$$\begin{aligned} & w''_{m-1}\, w_{m-1} = (v''_1 \cdots v''_{m-1})\, w_{m-1} \\ =& \frac{\lambda_{n-m}}{\lambda_1 \lambda_{n-m+1}} \lambda_1 (v''_1 \cdots v''_{m-2}\, y_{(m-1)n})\, w_{m-1} \\ =& \frac{\lambda_{n-m}}{\lambda_{n-m+1}} y_{(m-1)n} ((m-1)n) \cdot (v''_1 \cdots v''_{m-2})\, w_{m-1} \\ =& v'''_{m-1} ((m-1)n) \cdot (v''_1 \cdots v''_{m-2})\, w_{m-1}. \end{aligned}$$

Now if $m = 2$, we are done with this step. If $m \geq 3$, assume that there exists some $1 \leq k \leq m-2$, such that

$$\begin{aligned} & w''_{m-1}\, w_{m-1} \\ =& v'''_{m-1} \underset{\leftarrow}{\cdots} v'''_{k+1} (n(m-1) \underset{\leftarrow}{\cdots} (k+1)) \cdot (v''_1 \cdots v''_k)\, w_{m-1}. \end{aligned}$$

Acting with $(n(m-1) \underset{\leftarrow}{\cdots} (k+1))$ on v''_k we get

$$\begin{aligned} & \frac{\lambda_{n-k-1}}{\lambda_k \lambda_{n-k}} (n(m-1) \underset{\leftarrow}{\cdots} (k+1)) \cdot (y_{k(k+1)} \underset{\rightarrow}{\cdots} y_{k(m-1)} y_{kn} y_{km}) \\ =& \frac{\lambda_{n-k-1}}{\lambda_k \lambda_{n-k}} y_{kn} y_{k(k+1)} \underset{\rightarrow}{\cdots} y_{k(m-1)} y_{km}. \end{aligned}$$

Now again we have $w_{m-1} \in y_{k(k+1)} \underset{\rightarrow}{\cdots} y_{km} \mathcal{D}_4(\alpha_1, \alpha_2)$, hence we can exchange this for the scalar λ_k:

$$
\begin{aligned}
&v'''_{m-1}\underset{\leftarrow}{\cdots}v'''_{k+1}(n(m-1)\underset{\leftarrow}{\cdots}(k+1))\cdot(v''_1\cdots v''_k)\,w_{m-1}\\
=&\frac{\lambda_{n-k-1}}{\lambda_k\lambda_{n-k}}\lambda_k v'''_{m-1}\underset{\leftarrow}{\cdots}v'''_{k+1}(n(m-1)\underset{\leftarrow}{\cdots}(k+1))\cdot(v''_1\cdots v''_{k-1}\,y_{k(k+1)})\,w_{m-1}\\
=&\frac{\lambda_{n-k-1}}{\lambda_{n-k}}v'''_{m-1}\underset{\leftarrow}{\cdots}v'''_{k+1}y_{kn}(n(m-1)\underset{\leftarrow}{\cdots}(k+1))(k(k+1))\cdot(v''_1\cdots v''_{k-1})\,w_{m-1}\\
=&v'''_{m-1}\underset{\leftarrow}{\cdots}v'''_{k+1}v'''_k\,(n(m-1)\underset{\leftarrow}{\cdots}k)\cdot(v''_1\cdots v''_{k-1})\,w_{m-1}.
\end{aligned}
$$

In total we get $(v''_1\cdots v''_{m-1})\,w_{m-1} = v'''_{m-1}\underset{\leftarrow}{\cdots}v'''_1\,w_{m-1} = w'''_{m-1}\,w_{m-1}$. This finishes (4) and the proof. □

We will now show that Assumption 2.33 implies Assumption 2.20.

Proposition 2.38. *Let $1 \le m \le n-1$ and suppose Assumption 2.33 holds. Then*

$$e_i := w_{m-1}\,\big((m\cdots n)^i\cdot v_m\big)\,w_{m-1}, \qquad 0\le i\le n-m$$

forms a set of isomorphic idempotents in the algebra $w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1}$.

Proof. We already discussed that these elements are idempotents in Proposition 2.19. Let $0\le i\le n-m-1$. We show that e_i is isomorphic to e_{i+1}. Using Lemma 1.2, it is enough to construct a unit $u\in w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1}$, such that $ue_i = e_{i+1}u$. Let $\pi=(m\cdots n)$ and let u_π be the invertible element obtained from Proposition 2.34. Now define $u := w_{m-1}u_\pi w_{m-1}$. Lemma 2.36 implies that u is indeed invertible in $w_{m-1}\mathcal{D}_n(\alpha_1,\alpha_2)w_{m-1}$:

$$
\begin{aligned}
u\,w_{m-1}u_\pi^{-1}w_{m-1} &= w_{m-1}u_\pi w_{m-1}u_\pi^{-1}w_{m-1} = w_{m-1}\,\pi\cdot w_{m-1}\,w_{m-1}\\
&=\frac{\lambda_{n-m}\lambda_{m-1}}{\lambda_{n-1}}w_{m-1} = w_{m-1}\,\pi^{-1}\cdot w_{m-1}\,w_{m-1}\\
&= w_{m-1}u_\pi^{-1}w_{m-1}u_\pi w_{m-1} = w_{m-1}u_\pi^{-1}w_{m-1}\,u.
\end{aligned}
$$

Now denote $s=m+i$ and observe that

$$(m\cdots n)^i\cdot v_m = \frac{1}{\lambda_{n-m}}y_{s(s+1)}\underset{\rightarrow}{\cdots}y_{sn}y_{sm}\underset{\rightarrow}{\cdots}y_{s(s-1)}.$$

Since $s\le n-1$, the first sequence $y_{s(s+1)}\underset{\rightarrow}{\cdots}y_{sn}$ is not 1 and in particular ends with y_{sn}. Now Lemma 2.37 says that

$$
\begin{aligned}
&w_{m-1}(m\cdots n)\cdot w_{m-1}\,y_{nm}\,(nm)\cdot w_{m-1}\\
=&w_{m-1}(n\cdots m)\cdot w_{m-1}\,y_{nm}\,(nm)\cdot w_{m-1}.
\end{aligned}
$$

First, since the factors of $(n\cdots m)\cdot w_{m-1}$ start with $y_{jn}y_{jm}$, $1\le j\le m-1$, we can shift y_{nm} on the right hand side as follows:

$$
\begin{aligned}
&w_{m-1}(m\cdots n)\cdot w_{m-1}\,y_{nm}\,(nm)\cdot w_{m-1}\\
=&w_{m-1}\,y_{nm}\,(nm)(n\cdots m)\cdot w_{m-1}\,(nm)\cdot w_{m-1}.
\end{aligned}
$$

Acting with $(s \underset{\leftarrow}{\cdots} mn) = (s \underset{\rightarrow}{\cdots} (n-1))(n \cdots m)$ on that equation yields

$$\begin{aligned} &(s \underset{\rightarrow}{\cdots} (n-1)) \cdot ((n \dots m) \cdot w_{m-1}\, w_{m-1})\; y_{sn}\, (s \underset{\leftarrow}{\cdots} m) \cdot w_{m-1} \\ =&(s \underset{\rightarrow}{\cdots} (n-1))(n \cdots m) \cdot w_{m-1}\, y_{sn}\, (s \underset{\leftarrow}{\cdots} m)\, ((n \cdots m) \cdot w_{m-1}\, w_{m-1})\,. \end{aligned}$$

Now we multiply this equation from the left side with $y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{s(n-1)}$ and from the right with $y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)}$. We observe that on both sides of the equation, we can shift all of these y_{kl} to y_{sn}, using Lemma 2.8, precisely canceling some permutations:

$$\begin{aligned} &(n \dots m) \cdot w_{m-1} w_{m-1}\, y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{sn} y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)}\, w_{m-1} \\ =&(n \cdots m) \cdot w_{m-1}\, y_{s(s+1)} \underset{\rightarrow}{\cdots} y_{sn} y_{sm} \underset{\rightarrow}{\cdots} y_{s(s-1)}\, (n \cdots m) \cdot w_{m-1}\, w_{m-1}. \end{aligned}$$

Note that this shifting from the right side is only possible since $s \leq n-1$ and would not be possible for $s = n$ (for the same reason it is not possible to shift in y_{sn} from the left, which is the reason we need Lemma 2.37 in the first place and can not rely on Lemma 2.36). Multiplying both sides with $\frac{1}{\lambda_{n-m}} u_\pi$ from the left gives

$$\begin{aligned} &u_\pi\, (n \dots m) \cdot w_{m-1} w_{m-1}\, (m \cdots n)^i \cdot v_m\, w_{m-1} \\ =&u_\pi\, (n \cdots m) \cdot w_{m-1}\, (m \cdots n)^i \cdot v_m\, (n \cdots m) \cdot w_{m-1}\, w_{m-1}. \end{aligned}$$

Finally, shift u_π to obtain

$$w_{m-1}\, u_\pi\, w_{m-1}\, (m \cdots n)^i \cdot v_m\, w_{m-1} = w_{m-1} (m \cdots n)^{i+1} \cdot v_m\, w_{m-1}\, u_\pi\, w_{m-1}$$

This is precisely $ue_i = e_{i+1} u$, which finishes the proof. □

3 Reoccurring traits

In this section we will take a look at PBW deformations of some other finite dimensional Nichols algebras over braided vector spaces of non-abelian group type that are defined by a quandle and a 2-cocycle. Interestingly, all of the solved examples of this type have something in common (see Remark 3.9), which leads to conjectures about the two other unsolved 576-dimensional examples at the end of the section (Conjecture 3.12 and Conjecture 3.13).

As always let $\Bbbk$ be a field with characteristic $\neq 2$.

3.1 Nichols algebras defined over quandles

We will give a quick overview of how a Nichols algebra is defined from a quandle and a 2-cocycle. This process is described in more detail in [9] or [1].

Definition 3.1. A **quandle** is a set X together with an operation $\triangleright : X \times X \to X$, such that $x \triangleright x = x$, $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$ and such that the map $X \to X$, $y \mapsto x \triangleright y$ is bijective for all $x \in X$. The **enveloping group** of X is the group given by generators g_x, $x \in X$ and relations $g_x g_y = g_{x \triangleright y} g_x$ for all $x, y \in X$. We denote it with G_X.

To a quandle X we associate the vector space V_X that has basis v_x, $x \in X$. A **2-cocycle** on X is a map $q : X \times X \to K^\times$, such that $q(y,z)q(x, y \triangleright z) = q(x,z)q(x \triangleright y, x \triangleright z)$ for all $x, y, z \in X$. Given such a 2-cocycle q, the map $c \in \mathrm{Aut}_\Bbbk(V_X \otimes V_X)$, defined linearly by

$$c(v_x \otimes v_y) = q(x,y)\, v_{x \triangleright y} \otimes v_x \qquad \text{for all } x, y \in X,$$

is a braiding on V_X, hence (V_X, c) is a braided vector space (the reverse is also true, i.e. if q is just a map, then c is a braiding if and only if q is a 2-cocycle). From a braided vector space we obtain the Nichols algebra $\mathcal{B}(V_X)$ (see [9], section 1), which is basically the algebra with generators v_x, $x \in X$ and relations involving the braiding c. We denote this Nichols algebra with $\mathcal{B}_X$.

Example 3.2. Let G be a group and let $X \subset G$ be the conjugacy class of one element in G. Then X becomes a quandle via $x \triangleright y = xyx^{-1}$ for all $x, y \in X$.

Remark 3.3. Let X be a quandle with a 2-cocylce q on X. There is an intrinsic group action of G_X on $\mathcal{B}$ defined by $g_x \cdot v_y = q(x,y)v_{x \triangleright y}$. This group action is invariant under the defining relations of the enveloping group. The reason this group action is well defined on $\mathcal{B}_X$ is because it commutes with the braiding: For $x, y, z \in X$ we have

$$\begin{aligned} g_x \cdot (c(v_y \otimes v_z)) =& g_x \cdot (q(y,z)v_{y \triangleright z} \otimes v_y) \\ =& q(y,z)q(x, y \triangleright z)q(x,y)v_{x \triangleright (y \triangleright z)} \otimes v_{x \triangleright y} \\ =& q(x,z)q(x \triangleright y, x \triangleright z)q(x,y)v_{(x \triangleright y) \triangleright (x \triangleright z)} \otimes v_{x \triangleright y} \\ =& c\left(q(x,y)q(x,z)v_{x \triangleright y} \otimes v_{x \triangleright z}\right) \\ =& c\left(g_x \cdot (v_y \otimes v_z)\right). \end{aligned}$$

Without going to much into detail, if $\mathcal{D}$ is a PBW deformation of $\mathcal{B}_X$, then the above defined action must also be a well defined action on $\mathcal{D}$. This drastically reduces the possibilities for actual PBW deformations in the set of all deformations of $\mathcal{B}_X$.

Remark 3.4. A list of known examples of such Nichols algebras defined by quandle and 2-cocycle can be found at [8]. It is known for $n = 3, 4, 5$ (and conjectured for $n \geq 6$), that the Fomin-Kirillov algebra $\mathcal{E}_n$ is isomorphic to the Nichols algebra given by the quandle $X = \{(ij) \,|\, 1 \leq i < j \leq n\} \subset S_n$, with $\sigma \triangleright \pi = \sigma\pi\sigma^{-1}$ for all $\pi, \sigma \in X$ and 2-cocycle q defined by

$$q(\pi, (ij)) = \begin{cases} 1, & \text{if } \pi(i) < \pi(j), \\ -1, & \text{otherwise,} \end{cases}$$

where $1 \leq i < j \leq n$ and $\pi \in X$. If we divide from G_X the relations $g_\pi^2 = \mathrm{id}$ for all $\pi \in X$ (which we can do since g_π^2 acts trivially on $\mathcal{B}_X$), we obtain S_n. Then the group action on $\mathcal{E}_n$ from Remark 3.3 coincides with the one from Remark 2.3.

3.2 Semisimplicity of PBW deformations

We take a look at some type of elements that seem to correlate with the semisimplicity of a PBW deformation of a Nichols algebra defined over quandle and 2-cocycle. The group action plays an important role here.

Definition 3.5. Let X be a quandle, q a 2-cocycle on X and let $\mathcal{D}$ be a PBW deformation of $\mathcal{B}_X$. Let $g \in G_X$ and denote

$$\mathcal{D}_g := \{a \in \mathcal{D} \mid ab = g \cdot b\, a \;\text{ for all } b \in \mathcal{D}\}.$$

If g is the neutral element, then V_g is the center of $\mathcal{D}$. Also observe that these defining relations are very similar to the defining relations of the enveloping group.

Proposition 3.6. *Let X be a quandle, q a 2-cocycle on X and let $\mathcal{D}$ be a PBW deformation of $\mathcal{B}_X$. Let $g, h \in G_X$. The following hold:*

$$\mathcal{D}_g\mathcal{D}_h \subset \mathcal{D}_{gh} \qquad\qquad h \cdot \mathcal{D}_g = \mathcal{D}_{hgh^{-1}}.$$

In particular if g and h are conjugate, than $\mathcal{D}_g$ and $\mathcal{D}_h$ are isomorphic vector spaces.

Proof. If $a \in D_g$, $b \in D_h$ then for all $c \in \mathcal{D}$ we have

$$abc = ah \cdot (c)b = (gh) \cdot (c)ab,$$

hence $ab \in D_{gh}$. Moreover we have

$$h \cdot a\, b = h \cdot (a\, h^{-1} \cdot b) = h \cdot ((gh^{-1}) \cdot b\, a) = (hgh^{-1}) \cdot b\, h \cdot a,$$

hence $h \cdot a \in D_{hgh^{-1}}$. Similarly we obtain $h^{-1} \cdot \mathcal{D}_{hgh^{-1}} \subset \mathcal{D}_{h^{-1}hgh^{-1}h} = \mathcal{D}_g$, i.e. $\mathcal{D}_{hgh^{-1}} \subset h \cdot \mathcal{D}_g$. □

Remark 3.7. Let X be a quandle, q a 2-cocycle on X and let $\mathcal{D}$ be a PBW deformation of $\mathcal{B}_X$. If $x, y \in X$ then there must exist a $z \in X$, such that $z \triangleright x = y$. Then $\mathcal{D}_{g_x}$ is isomorphic to $g_z \cdot \mathcal{D}_{g_x} = \mathcal{D}_{g_z g_x g_z^{-1}} = \mathcal{D}_{g_{z \triangleright x}} = \mathcal{D}_{g_y}$.

Proposition 3.8. *Let X be a quandle, q a 2-cocycle on X and let $\mathcal{D}$ be a PBW deformation of $\mathcal{B}_X$. Suppose there exists a $g \in G_X$ and a $u \in \mathcal{D}_g \setminus \{0\}$ that is nilpotent. Then $\mathcal{D}$ is not semisimple.*

Proof. It is enough to show that the right ideal generated by u (which is also a two-sided ideal by definition of u) consists only of nilpotent elements. So let $x \in \mathcal{D}$ and let $m \in \mathbb{N}$, such that $u^m = 0$. In $(ux)^m$ we can shift all the u to the right, obtaining

$$(ux)^m = g \cdot x\, g^2 \cdot x \cdots g^m \cdot x\, u^m = 0.$$

Hence ux is nilpotent. □

Remark 3.9. In [5], the generic semisimplicity (meaning there exists a dense subset of semisimple PBW deformations) were solved for the two smallest Nichols algebras, the 12-dimensional one and the 72-dimensional one. Moreover, the generic semisimplicity of PBW deformations of one of the next bigger Nichols algebras, the 576 dimensional Fomin-Kirillov algebra, was almost solved in section 2. All of those algebras correspond to a quandle X and a 2-cocycle q. In all of those cases there was a common theme: The spaces $\mathcal{D}_g$, where $\mathcal{D} = \mathcal{D}(\alpha_1, \ldots, \alpha_k)$ is a PBW deformation of $\mathcal{B}_X$, $g \in G_X$, $\alpha_1, \ldots \alpha_k \in \Bbbk$, always had a relatively small dimension in all of the semisimple cases, sometimes even 0. The dimension of the center corresponded to the amount of simple factors. Obviously non of this spaces contained a nilpotent element (since it would then not be a semisimple case according to the preceding proposition). But one could ask when do these elements in $\mathcal{D}_g$ become nilpotent. The interesting result was that there always existed a $g \in G_X$, and an element $u \in \mathcal{D}_g$ with the property $u^2 = pv$, where $v \in \mathcal{D} \setminus \{0\}$ and $p \in \Bbbk$ and such that $p = 0$ if and only if the algebra was not semisimple. Even more: In the non-semisimple case that space $\mathcal{D}_g$ degenerated and had a bigger dimension than in the semisimple case.

Take for example the 12-dimensional case. The PBW deformations $\mathcal{D} = \mathcal{D}(\alpha_1, \alpha_2)$ are semisimple if and only if $(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$. Corresponding to that there exists a $u \in \mathcal{D}_{g_{(12)}}$, such that $u^2 = (3\alpha_1 - \alpha_2)^2(\alpha_1 + \alpha_2)v$, where $v \neq 0$ in any case (also u is not invertible here). The space $\mathcal{D}_{g_{(12)}}$ has dimension 1 in the semisimple case and dimension 2 in the non-semisimple case if $\alpha_1 \neq 0$. If $\alpha_1 = \alpha_2 = 0$ it has dimension 3. The center $\mathcal{D}_{\mathrm{id}}$ has dimension 3 in both cases except if $\alpha_1 = \alpha_2 = 0$, where the dimension is 4.

In the 72-dimensional case, the PBW deformations $\mathcal{D} = \mathcal{D}(\alpha_1, \alpha_2, \alpha_3)$ are semisimple if and only if $\alpha_3 \left(\alpha_3 + (\alpha_1 + \alpha_2)(3\alpha_1 - \alpha_2)^2\right) \neq 0$. Here we even find an element u in the center $\mathcal{D}_{\mathrm{id}}$ with the property

$$u^2 = \alpha_3^k \left(\alpha_3 + (\alpha_1 + \alpha_2)(3\alpha_1 - \alpha_2)^2\right)^l v,$$

where $v \neq 0$ in any case, $k, l \geq 1$ (also u is invertible here). The center has dimension 2 in the semisimple case and dimension > 2 in the non-semisimple case.

Finally for the PBW deformations $\mathcal{D} = \mathcal{D}_4(\alpha_1, \alpha_2)$ of the 576-dimensional Fomin-Kirillov algebra, if we reduce G_X to S_4, any $\pi \in S_4$, $\pi \neq \mathrm{id}$ has an element $u \in \mathcal{D}_\pi$, such that $u^2 = (\alpha_1 - \alpha_2)^k(\alpha_1 + \alpha_2)^l v$, where $v \neq 0$ in any case, $k, l \geq 1$ (u is invertible here, see also Remark 2.35). In the (presumedly semisimple) case where $(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$, all $\mathcal{D}_\pi$, $\pi \in S_n$, have dimension 1 (including the center). In the other case, the center has dimension 1 if $\alpha_1 = -\alpha_2 \neq 0$, dimension 7 if $\alpha_1 = \alpha_2 \neq 0$ and dimension 14 if $\alpha_1 = \alpha_2 = 0$. The spaces for all other $\pi \neq \mathrm{id}$ have dimension 2 if $\alpha_1 = -\alpha_2 \neq 0$, dimension 3 if $\alpha_1 = \alpha_2 \neq 0$ and dimension 8 if $\alpha_1 = \alpha_2 = 0$.

From those results one could conjecture that the reverse of Proposition 3.8 is also true, i.e. a PBW deformation $\mathcal{D}$ is not semisimple if and only if there exists a $g \in G_X$ with a nilpotent element in $\mathcal{D}_g$. However, I did not found any real logical argument backing up that conjecture so far.

3.3 Two other 576-dimensional Nichols algebras

In [8] there are two other examples of Nichols algebras of dimension 576 apart from the Fomin-Kirillov algebra $\mathcal{E}_4$. We obtain the PBW deformations in a similar fashion as we did with $\mathcal{E}_4$.

Example 3.10. Let X be the quandle of the six 2-cycles of S_4 (which form a conjugacy class) and let q be the 2-cocycle on X that is constantly -1. The PBW deformations of $\mathcal{B}_X$ are the 576-dimensional algebras $\mathcal{D} = \mathcal{D}(\alpha_1, \alpha_2, \alpha_3)$ generated by v_x, $x \in X$ and relations

$$\begin{aligned} v_x^2 &= \alpha_1 && \text{for all } x \in X, \\ v_{(12)}v_{(13)} + v_{(13)}v_{(23)} + v_{(23)}v_{(12)} &= \alpha_2 \\ v_{(12)}v_{(14)} + v_{(14)}v_{(24)} + v_{(24)}v_{(12)} &= \alpha_2 \\ v_{(12)}v_{(23)} + v_{(23)}v_{(13)} + v_{(13)}v_{(12)} &= \alpha_2 \\ v_{(12)}v_{(24)} + v_{(24)}v_{(14)} + v_{(14)}v_{(12)} &= \alpha_2 \\ v_{(13)}v_{(14)} + v_{(14)}v_{(34)} + v_{(34)}v_{(13)} &= \alpha_2 \\ v_{(13)}v_{(34)} + v_{(34)}v_{(14)} + v_{(14)}v_{(13)} &= \alpha_2 \\ v_{(23)}v_{(24)} + v_{(24)}v_{(34)} + v_{(34)}v_{(23)} &= \alpha_2 \\ v_{(23)}v_{(34)} + v_{(34)}v_{(24)} + v_{(24)}v_{(23)} &= \alpha_2 \\ v_{(ij)}v_{(kl)} + v_{(kl)}v_{(ij)} &= \alpha_3 && \text{if } \#\{i,j,k,l\} = 4, \end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \Bbbk$. Observe that unlike in the Fomin-Kirillov case, here an α_3 is needed, since the 2-cocycle, and thus the group action, is different. We have $\mathcal{D}(0,0) = \mathcal{B}_X$. Computer calculations show that there exist elements $u \in \mathcal{D}_{g_x}$, $x \in X$ with the property

$$u^2 = (-2\alpha_1 + \alpha_3)^4(\alpha_1 - \alpha_2 + \alpha_3)^3(\alpha_1 + 3\alpha_2 + \alpha_3)v,$$

where $v \neq 0$ (and this is the only scalar that we obtain in such ways). Proposition 3.8 implies that the deformation is not semisimple if $(-2\alpha_1+\alpha_3)^4(\alpha_1-\alpha_2+\alpha_3)^3(\alpha_1+3\alpha_2+\alpha_3) = 0$. The generic dimension (i.e. if we assume that α_1 and α_2 are algebraically independent) of the center of $\mathcal{D}$ is 4. The generic dimension of $\mathcal{D}_{g_x}$ is 2. The dimension is > 2 if $(-2\alpha_1+\alpha_3)^4(\alpha_1-\alpha_2+\alpha_3)^3(\alpha_1+3\alpha_2+\alpha_3) = 0$.

Example 3.11. Let X be the quandle of conjugacy class of the six 4-cycles of S_4 and let q be the 2-cocycle on X that is constantly -1. The PBW deformations of $\mathcal{B}_X$ are the 576-dimensional algebras $\mathcal{D} = \mathcal{D}(\alpha_1, \alpha_2, \alpha_3)$ generated by v_x,

$x \in X$ and relations

$$
\begin{aligned}
v_x^2 &= \alpha_1 \qquad \text{for all } x \in X,\\
v_{(1234)}v_{(1423)} + v_{(1423)}v_{(1243)} + v_{(1243)}v_{(1234)} &= \alpha_2\\
v_{(1234)}v_{(1342)} + v_{(1342)}v_{(1423)} + v_{(1423)}v_{(1234)} &= \alpha_2\\
v_{(1234)}v_{(1243)} + v_{(1243)}v_{(1324)} + v_{(1324)}v_{(1234)} &= \alpha_2\\
v_{(1234)}v_{(1324)} + v_{(1324)}v_{(1342)} + v_{(1342)}v_{(1234)} &= \alpha_2\\
v_{(1423)}v_{(1342)} + v_{(1342)}v_{(1432)} + v_{(1432)}v_{(1423)} &= \alpha_2\\
v_{(1342)}v_{(1324)} + v_{(1324)}v_{(1432)} + v_{(1432)}v_{(1342)} &= \alpha_2\\
v_{(1423)}v_{(1432)} + v_{(1432)}v_{(1243)} + v_{(1243)}v_{(1423)} &= \alpha_2\\
v_{(1243)}v_{(1432)} + v_{(1432)}v_{(1324)} + v_{(1324)}v_{(1243)} &= \alpha_2\\
v_{(ijkl)}v_{(lkji)} + v_{(lkji)}v_{(ijkl)} &= \alpha_3 \qquad \text{if } \#\{i,j,k,l\} = 4,
\end{aligned}
$$

where $\alpha_1, \alpha_2, \alpha_3 \in \Bbbk$. We have $\mathcal{D}(0,0) = \mathcal{B}_X$. Computer calculations show that there exist elements in $u \in \mathcal{D}_{g_x}$, $x \in X$ with the property

$$u^2 = (-2\alpha_1 + \alpha_3)^6(4\alpha_1 - 2\alpha_2 + \alpha_3)^3(4\alpha_1 + 6\alpha_2 + \alpha_3)v,$$

where $v \neq 0$ (and this is the only scalar that we obtain in such ways). Proposition 3.8 implies that the deformation is not semisimple if this scalar is 0. The generic dimension of the center of $\mathcal{D}$ is 1. The generic dimension of $\mathcal{D}_{g_x}$ is also 1. If the above scalar is 0, then the dimension of $\mathcal{D}_{g_x}$ is at least 2.

The behaviour in the preceding two examples is the same as in the solved cases described in Remark 3.9. This justifies the following two conjectures.

Conjecture 3.12. *The algebra $\mathcal{D}(\alpha_1, \alpha_2, \alpha_3)$ from Example 3.10 is semisimple if and only if $(-2\alpha_1 + \alpha_3)(\alpha_1 - \alpha_2 + \alpha_3)(\alpha_1 + 3\alpha_2 + \alpha_3) \neq 0$. In the semisimple case we have an algebra isomorphy $\mathcal{D}(\alpha_1, \alpha_2, \alpha_3) \cong (M_{12}(\Bbbk))^4$.*

Conjecture 3.13. *The algebra $\mathcal{D}(\alpha_1, \alpha_2, \alpha_3)$ from Example 3.11 is semisimple if and only if $(-2\alpha_1+\alpha_3)(4\alpha_1-2\alpha_2+\alpha_3)(4\alpha_1+6\alpha_2+\alpha_3) \neq 0$. In the semisimple case we have an algebra isomorphy $\mathcal{D}(\alpha_1, \alpha_2, \alpha_3) \cong M_{24}(\Bbbk)$.*

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I hereby assure, that I have written this thesis myself and that I did not use any other means than the above listed sources and tools / Ich versichere hiermit, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die oben angegebenen Quellen und Hilfsmittel benutzt habe.

Marburg, September 13, 2018

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