INHOMOGENEOUS SHEARLET COORBIT SPACES

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Abstract. In this paper we establish inhomogeneous coorbit spaces related to the continuous shearlet transform and the weighted Lebesgue spaces $L_{p,v}, p \geq 1$, for certain weights v. We present a discretization of the underlying parameter space leading to atomic decompositions and Banach frames of the coorbit spaces.

1. Introduction

When analyzing a given signal, the decomposition of the signal into a certain set of building blocks is crucial. Which kinds of building blocks to choose depends on the information that one wants to extract from the signal. Very popular kinds of building blocks are wavelets, especially when dealing with signals with isolated singularities. Because of its isotropic nature, the wavelet transform cannot efficiently deal with anisotropic features, therefore several extensions of this framework were proposed, among those the shearlet transform. While the wavelets consist only of dilated and translated copies of a mother function, the shearlets are also sheared in each scale, thereby changing the orientation of the functions. This makes them especially well suited to deal with localized directional features in a signal. Indeed, it was shown in [18] that the shearlet transform can be used to resolve the wavefront set of a signal and in [16] that the approximation of cartoon-like images with shearlets is optimally sparse.

Another main advantage of shearlets, which sets them apart from other such frameworks like the ridgelets [2], curvelets [1] or contourlets [9] for example, is, that the continuous shearlet transform, introduced and investigated in [4, 5, 7, 15], stems from the action of a square-integrable representation of a topological group, the so-called full shearlet group $\mathbb{S}$. This property makes it possible to use the abstract coorbit theory, developed by Feichtinger and Gröchenig in [10, 11, 12], to define smoothness spaces related to the shearlet transform by measuring the decay of the voice transform. Shearlet coorbit spaces were investigated by Dahlke et al in a series of papers [3, 4, 5, 7, 8]. Since the shearlets being used to construct these spaces need to have vanishing moments, any polynomial part in a signal is ignored by the transform because for a polynomial g one has $\mathcal{SH}(f+g)(x) = \langle f+g, \psi_x\rangle = \langle f, \psi_x\rangle = \mathcal{SH}f(x)$. This leads to the resulting shearlet coorbit spaces being homogeneous spaces. However, in practice the smoothness spaces being used, for example to analyze the regularity of the solution space of an operator equation, are usually inhomogeneous. Therefore, inhomogeneous smoothness spaces related to the shearlet transform are also of interest. In this paper we introduce non-homogeneous shearlet coorbit spaces by using a generalization of the coorbit theory developed by Fornasier, Rauhut, Ullrich et al [14, 17, 20]. Their approach uses a more general parameter space for the transform, resulting in more design flexibility. Instead of the parameter space being a locally compact topological group, it is

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only assumed to be a locally compact topological Hausdorff space, thereby allowing the construction of inhomogeneous coorbit spaces.

After giving a short overview of the main definitions and results of this generalized coorbit theory in Section 2, we use their approach in Section 3 to define a new shearlet transform given by a continuous frame $\mathfrak{F} = \{\psi_x\}_{x\in X}$ through the action

$$\mathcal{SH}_{\mathfrak{F}} f(x) = \langle f, \psi_x \rangle, \quad x \in X,$$

where the frame is indexed by a topological Hausdorff space X (without group structure). We prove that an integrability condition on the kernel function

$$R_{\mathfrak{F}} : X \times X \to \mathbb{C},\ (x, y) \mapsto \langle \psi_y, \psi_x \rangle$$

holds so that the coorbit spaces

$$\mathcal{SC}^{r,n}_{\mathfrak{F},p} = \{f \,|\, \mathcal{SH}_{\mathfrak{F}} f \in L_{p,v_{r,n}}(X)\}, \quad p \geq 1, v_{r,n} \text{ weight function on } X,$$

classifying distributions by the decay of their transform, are well-defined Banach spaces. Furthermore, in Section 4, we construct a discrete covering of the parameter space X and prove that all necessary conditions can be satisfied, so that the abstract discretization results of the coorbit theory hold in this case. These discretization results yield Banach frames of $\mathcal{SC}^{r,n}_{\mathfrak{F},p}$ and atomic decompositions of the elements by sampling the continuous frame $\mathfrak{F}$ based on the covering of X. It should be noted that our proofs do assume specific moment conditions on all the elements of the frame $\mathfrak{F}$, which means that the Fourier transforms of the functions in $\mathfrak{F}$ have to satisfy certain smoothness conditions, thereby making the resulting spaces not entirely inhomogeneous in this sense.

We also note that there are other approaches, not based on coorbit space theory, to develop inhomogeneous shearlet smoothness spaces. In [19] Labate, Mantovani and Negi used the notion of decomposition spaces to define shearlet smoothness spaces, while in [21, 22] Vera applied the framework of the φ-transform, introduced by Frazier and Jawerth, for this purpose.

We finish this section by stating a few notational conventions. Throughout this paper $d \in \mathbb{N}$ with $d \geq 2$ is the space dimension. We usually treat elements $x \in \mathbb{R}^d$ as $x = (x_1, \tilde{x})$ with $\tilde{x} = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. For two elements $x, y \in \mathbb{R}^d$ we use the canonical inner product

$$x \cdot y = \sum_{i=1}^{d} x_i y_i.$$

The convention $\mathbb{R}^*$ is used for the set $\mathbb{R} \setminus \{0\}$.

For a measure space (X, Σ, μ) with a weight function $v : X \to (0, \infty)$ we denote the usual (weighted) Lebesgue spaces by $L_{p,v}(X, \mu)$ or just by $L_{p,v}$, if the respective measure space is clear from the context, while $L_1^{\mathrm{loc}}(X, \mu)$ is used for the space of locally integrable functions on X. For the unweighted Lebesgue spaces with $v \equiv 1$ we write $L_p(X, \mu)$ and L_p. We use the Hilbert space $L_2(\mathbb{R}^d)$ of complex-valued, square-integrable functions on $\mathbb{R}^d$ with the inner product

$$\langle f, g \rangle_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\, \mathrm{d}x.$$

For two functions $f, g \in L_2(\mathbb{R}^d)$ the convolution product $f * g$ is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y)\, \mathrm{d}y.$$

We write $\mathscr{C}^k, k \in \mathbb{N}_0$ for the space of functions $f : \mathbb{R}^d \to \mathbb{C}$, for which all (classical) partial derivatives $\partial^\alpha f$ for $\alpha \in \mathbb{N}_0^d, |\alpha| \leq k$ exist and are continuous. We also use $\mathscr{C}_0^\infty$ for the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support.

Concerning the Fourier transform of a function $f \in L_1(\mathbb{R}^d)$ we write $\hat{f} = \mathcal{F}(f)$ using the convention

$$\mathcal{F}(f)(\omega) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \omega \cdot x} \, \mathrm{d}x, \qquad \omega \in \mathbb{R}^d,$$

with the same symbol being used for the extension to functions $f \in L_2(\mathbb{R}^d)$.

Given a measure space (X, Σ, μ) we say that a Banach space Y of locally integrable, complex-valued functions on X satisfies Condition **($\boldsymbol{Y}$)**, if it is solid, i.e. if from $f \in L_1^{\mathrm{loc}}(X, \mu), g \in Y$ with $|f| \le |g|$ almost everywhere it follows that $f \in Y$ with $\|f|Y\| \le \|g|Y\|$. Lastly, for quantities a and b we write $a \lesssim b$ if there exists a finite constant $C > 0$ so that $a \le C \cdot b$, with the constant being independent of the relevant parameters. If both $a \lesssim b$ and $b \lesssim a$ holds we use the shorthand notation $a \sim b$.

2. Generalized coorbit theory

In this section we give a short overview of the generalized coorbit theory. We follow [20] in our exposition. To generalize the classical coorbit theory—which assumes a locally compact group as the underlying parameter space of the respective transform—the generalization of Fornasier and Rauhut allows for the parameter space to be of a more general nature. In this case the parameter space X is only assumed to be a locally compact Hausdorff space equipped with a positive Radon measure μ. In the following $\mathcal{H}$ denotes a separable Hilbert space (the signal space) and v is a weight function on X while Y is a Banach space of complex-valued functions on X. We start with a set of functions $\mathfrak{F} = \{\psi_x\}_{x \in X} \subset \mathcal{H}$, which is indexed by the parameter space, and constitutes a tight continuous frame. I.e., there exists a finite constant $A > 0$ such that

$$A\|f|\mathcal{H}\| = \int_X |\langle f, \psi_x \rangle|^2 \, \mathrm{d}\mu(x) \text{ for all } f \in \mathcal{H}. \tag{2.1}$$

Based on $\mathfrak{F}$, a signal transform on the space $\mathcal{H}$ is introduced in the following way.

Definition 2.1. Let $\mathfrak{F} = \{\psi_x\}_{x \in X} \subset \mathcal{H}$ be a tight continuous frame. Then the associated *voice transform* is defined as the mapping

$$V_{\mathfrak{F}} : \mathcal{H} \to L_2(X, \mu), \quad f \mapsto V_{\mathfrak{F}} f$$

with

$$V_{\mathfrak{F}} f : X \to \mathbb{C}, \quad x \mapsto \langle f, \psi_x \rangle.$$

The above transform is well defined due to (2.1).

2.1. Kernel spaces.

In order for the resulting smoothness spaces to be well defined and to state the discretization results, conditions on the voice transform $V_{\mathfrak{F}}$ and therefore conditions on $\mathfrak{F}$ are needed. In this approach the kernel function

$$R_{\mathfrak{F}} : X \times X \to \mathbb{C}, (x, y) \mapsto R_{\mathfrak{F}}(x, y) := \langle \psi_y, \psi_x \rangle \tag{2.2}$$

is used. Through the action of an operator given by $R_{\mathfrak{F}}$ a reproducing identity is established, which is a crucial tool in the proof of the discretization results. To formulate certain conditions on this kernel function the following spaces, classifying kernel functions in terms of integrability, are used. Let

$$\mathcal{A}_1 := \Big\{ K : X \times X \to \mathbb{C} : K \text{ is measurable}, \|K|\mathcal{A}_1\| < \infty \Big\}$$

with

$$\|K|\mathcal{A}_1\| := \max\left\{ \operatorname*{ess\,sup}_{x \in X} \int_X |K(x,y)| \, \mathrm{d}\mu(y), \operatorname*{ess\,sup}_{y \in X} \int_X |K(x,y)| \, \mathrm{d}\mu(x) \right\}.$$

Through a weight function v on X a kernel weight function is defined through

$$m_v : X \times X \to (0, \infty), (x, y) \mapsto \max\left\{ \frac{v(x)}{v(y)}, \frac{v(y)}{v(x)} \right\}.$$

Now the associated weighted space $\mathcal{A}_{m_v}$ is given by

$$\mathcal{A}_{m_v} := \Big\{ K : X \times X \to \mathbb{C} : K \cdot m_v \in \mathcal{A}_1 \Big\}$$

where

$$\|K|\mathcal{A}_{m_v}\| := \|K \cdot m_v|\mathcal{A}_1\|.$$

In the following, depending on the context, K will also denote the operator on Y induced by the kernel function through

$$K(F)(x) := \int_X K(x,y)F(y)\,\mathrm{d}\mu(y) \text{ for } F \in Y, x \in X,$$

which is well-defined if $K \in \mathcal{A}_1$ because of the solidity of Y. This makes it possible to define the kernel algebra

$$B_{Y,m_v} := \Big\{ K : X \times X \to \mathbb{C} : K \in \mathcal{A}_{m_v}, K : Y \to Y \text{ is bounded} \Big\}$$

endowed with the norm

$$\|K|B_{Y,m_v}\| := \max\Big\{ \|K|\mathcal{A}_{m_v}\|, \|K|Y \to Y\| \Big\}.$$

With this definition a condition on the set of functions $\mathfrak{F}$ can be stated, that is used in the construction and discretization of the coorbit spaces.

Definition 2.2. Let $\mathfrak{F} = \{\psi_x\}_{x \in X} \subset \mathcal{H}$ be a tight continuous frame on $\mathcal{H}$ and let $R_{\mathfrak{F}}$ denote the kernel function defined by (2.2) and the associated operator on Y. We say that $\mathfrak{F}$ satisfies the property $\boldsymbol{(F_{v,Y})}$ if the range $R_{\mathfrak{F}}(Y)$ is continuously embedded into $L_{\infty,v^{-1}}(X,\mu)$ and if $R_{\mathfrak{F}} \in B_{Y,m_v}$.

2.2. **Coorbit spaces.** Before introducing coorbit spaces the concept of signals can first be generalized from elements of the Hilbert space $\mathcal{H}$ to a suitable space of distributions. First of all, consider the following dense subset of $\mathcal{H}$, the space

$$\mathcal{H}_{1,v} := \big\{ f \in \mathcal{H} : V_{\mathfrak{F}} f \in L_{1,v}(X,\mu) \big\}$$

of test functions equipped with the norm

$$\|f|\mathcal{H}_{1,v}\| := \|V_{\mathfrak{F}} f|L_{1,v}\|.$$

This set of test functions then leads to the Gelfand triple setting of dense embeddings

$$\mathcal{H}_{1,v} \hookrightarrow \mathcal{H} \cong \mathcal{H}^{\sim} \hookrightarrow (\mathcal{H}_{1,v})^{\sim}$$

with $(\mathcal{H}_{1,v})^{\sim}$ being the anti-dual space (the space of all conjugate linear, continuous functionals) of $\mathcal{H}_{1,v}$. An element $h \in \mathcal{H}$ is hereby identified with the functional $f \mapsto \langle h, f \rangle$. With these embeddings it is possible to extend the notion of the voice transform in a canonical way to elements $f \in (\mathcal{H}_{1,v})^{\sim}$ by $V_{\mathfrak{F}} f(x) = f(\psi_x)$. For this to be well-defined it is necessary that $\mathfrak{F} \subset \mathcal{H}_{1,v}$, which we can assume if $R_{\mathfrak{F}} \in \mathcal{A}_{m_v}$, see Remark 2.11 in [17]. Coorbit spaces are then defined with respect to Y, the Banach space of functions on X measuring the decay of the extended voice transform.

Definition 2.3. Let Y satisfy Condition (Y), as stated at the end of Section 1, and let $\mathfrak{F}$ be a tight continuous frame on $\mathcal{H}$ satisfying property $(F_{v,Y})$, stated in Definition 2.2. Then the *coorbit space* $\mathrm{Co}(Y)$ of Y with respect to $\mathfrak{F}$ is given by

$$\mathrm{Co}(Y) := \big\{ f \in (\mathcal{H}_{1,v})^{\sim} : V_{\mathfrak{F}} f \in Y \big\} \quad \text{with} \quad \|f|\mathrm{Co}(Y)\| := \|V_{\mathfrak{F}} f|Y\|.$$

It was proven in [14] that $\|\cdot|\mathrm{Co}(Y)\|$ constitutes a norm and that the spaces $\mathrm{Co}(Y)$ are indeed Banach spaces with respect to this norm. Furthermore, it was also shown that a reproducing identity holds for elements of these spaces, which reads as follows. For a function $F \in L_{\infty,v^{-1}}$ there exists an element $f \in (\mathcal{H}_{1,v})^{\sim}$ with $F = V_{\mathfrak{F}} f$ if and only if $F = R_{\mathfrak{F}}(F)$. This reproducing identity is the main tool in the proof of the discretization results stated below.

2.3. **Discretization.** To get atomic decompositions of the coorbit space elements and Banach frames of the coorbit spaces, a discretization of the parameter space X is needed. This is accomplished by choosing a covering of X that satisfies the following particular properties.

Definition 2.4. A countable family $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets of X is called a *moderate admissible covering* if it fulfills the following conditions:

(i) For all $i \in I$ the following holds: $\mathring{U}_i \neq \emptyset$, U_i is relatively compact.
(ii) $\bigcup\limits_{i\in I} U_i = X$.
(iii) There exists a constant $N > 0$: $\sup\limits_{j\in I} \#\{i \in I : U_i \cap U_j \neq \emptyset\} \leq N < \infty$.
(iv) There exist constants $c, C > 0$: $c \leq \mu(U_i) \leq C$.

A moderate admissible covering is used to construct a discretization of X by choosing one element from every subset in the family $\mathcal{U}$. Given a moderate admissible covering $\mathcal{U}$, the associated sequence spaces used in the discretization results are stated in the following definition.

Definition 2.5. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a moderate admissible covering. The associated sequence space is defined as

$$y(\mathcal{U}) := \Big\{\{\lambda_i\}_{i\in I} : \|\{\lambda_i\}_{i\in I}|y(\mathcal{U})\| := \Big\|\sum_{i\in I}|\lambda_i|\chi_{U_i}\Big|Y\Big\| < \infty\Big\}.$$

Remark 1. The original setting in [14] and [20] is slightly more general in the sense that there are two transforms being considered (with the second transform being given through the canonical dual frame) so that for non-tight frames it can be necessary to distinguish between two different sets of smoothness spaces. However, since we will construct a tight shearlet frame in Section 3, we do not need to consider non-tight frames in this context. Also, it is possible to consider moderate admissible coverings for which there is no constant $C > 0$ so that $\mu(U_i) \leq C$ holds for all $i \in I$. In that case one would have to deal with two different associated sequence spaces. For the moderate admissible covering given by Definition 4.1 this is not necessary.

For the discretization results to hold, it is necessary to assume that the behavior of the operator $R_{\mathfrak{F}}$ can be controlled on each part of the covering. To this end, the following kernel function on X is used.

Definition 2.6. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a moderate admissible covering with $W_y := \bigcup_{U_i \ni y} U_i$ and $\mathfrak{F} = \{\psi_x\}_{x\in X}$ be a tight continuous frame. Then the *$\mathcal{U}$-oscillation* $\operatorname{osc}_{\mathcal{U}}$ is given by

$$\operatorname{osc}_{\mathcal{U}} : X \times X \to \mathbb{R}, (x, y) \mapsto \sup_{z\in W_y} |\langle\psi_x, \psi_y - \psi_z\rangle|.$$

Let also $\operatorname{osc}^*_{\mathcal{U}}(x, y) := \operatorname{osc}_{\mathcal{U}}(y, x)$.

As already stated, the central tool in the discretization is the reproducing identity. In detail, this means that, given an element $f \in \operatorname{Co}(Y)$, one has the identity $V_{\mathfrak{F}}f = R_{\mathfrak{F}}(V_{\mathfrak{F}}f)$. The action of the operator $R_{\mathfrak{F}}$ is now discretized through a partition of unity corresponding to a moderate admissible covering $\mathcal{U}$, thereby creating a decomposition of $V_{\mathfrak{F}}f$. By applying the inverse transform, this subsequently leads to a decomposition of f. For this to succeed, certain properties with respect to $R_{\mathfrak{F}}$ and the $\mathcal{U}$-oscillation need to be satisfied.

Definition 2.7. Let $\mathfrak{F}$ be a tight continuous frame, let $\delta > 0$ and let $m : X \times X \to (0, \infty)$ be a weight function. We say that $\mathfrak{F}$ satisfies the property $\boldsymbol{D[\delta, m, Y]}$ with respect to a moderate admissible covering $\mathcal{U}$ if the following three conditions are simultaneously fulfilled:

(i) $\sup_{i\in I} \sup_{x,y\in U_i} m(x, y) \leq C_{m,\mathcal{U}}$,
(ii) $R_{\mathfrak{F}} \in B_{Y,m}$ and
(iii) $\|\operatorname{osc}_{\mathcal{U}}|B_{Y,m}\| < \delta$, $\|\operatorname{osc}^*_{\mathcal{U}}|B_{Y,m}\| < \delta$.

The following lemma states sufficient conditions on the discretization, for which the first part of Property $(F_{v,Y})$ is satisfied. In Section 4 we use this result to check that for the shearlet transform defined in Section 3 Property $(F_{v,Y})$ is satisfied.

Lemma 2.1. *Let Y satisfy Condition (Y) and let $v \geq 1$ be a weight function on X with m_v being bounded in the sense of Definition 2.7(i). Then, if $\mathcal{U} = \{U_i\}_{i\in I}$ is a moderate admissible covering with $\|\chi_{U_i}|Y\| \gtrsim v_i^{-1}$, $v_i := \sup_{x\in U_i} v(x)$, and $\mathfrak{F}$ satisfies Property $D[1,1,Y]$, we get the embedding $R_{\mathfrak{F}}(Y) \hookrightarrow L_{\infty,v^{-1}}(X,\mu)$.*

Now we can state the main discretization theorem of the generalized coorbit theory, which was proven in [14]. We assume that a separable Hilbert space $\mathcal{H}$, a locally compact topological Hausdorff space X with a weight function $v \geq 1$, a Banach space Y of functions on X, a tight continuous frame $\mathfrak{F} = \{\psi_x\}_{x\in X} \subset \mathcal{H}$ and a moderate admissible covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X are given.

Theorem 2.2. *Let Y satisfy Condition (Y) and $\mathfrak{F}$ satisfy the properties $(F_{v,Y})$ and $D[\delta, m_v, Y]$ with*

$$\delta(\|R_{\mathfrak{F}}|B_{Y,m_v}\| + \max\{C_{m_v,\mathcal{U}}\|R_{\mathfrak{F}}|B_{Y,m_v}\|, \|R_{\mathfrak{F}}|B_{Y,m_v}\| + \delta\}) \leq 1, \tag{2.3}$$

where $C_{m_v,\mathcal{U}}$ is a finite constant satisfying condition (i) *from Property $D[\delta, m_v, Y]$. Then, by choosing points $x_i \in U_i$, the set $\mathfrak{F}_d := \{\psi_{x_i}\}_{i\in I}$ is an atomic decomposition and a Banach frame of the space* $\mathrm{Co}(Y)$*, i.e., there exists a dual frame $\{e_i\}_{i\in I}$ so that for every $f \in \mathrm{Co}(Y)$ the following statements hold.*

(i) $\|\{\langle f, \psi_{x_i}\rangle\}_{i\in I}|y(\mathcal{U})\| \sim \|f|\mathrm{Co}(Y)\| \sim \|\{\langle f, e_i\rangle\}_{i\in I}|y(\mathcal{U})\|$

(ii) *The series*

$$f = \sum_{i\in I} \langle f, e_i\rangle \psi_{x_i} = \sum_{i\in I} \langle f, \psi_{x_i}\rangle e_i$$

converge unconditionally in the norm of $\mathrm{Co}(Y)$*, if the finite sequences are dense in $y(\mathcal{U})$, and with weak-* convergence in general.*

3. Shearlet coorbit spaces

In this section we introduce an inhomogeneous version of the shearlet transform and define smoothness spaces associated to this transform. In order to accomplish this we use the generalized coorbit theory outlined in Section 2. Since our approach is based on the homogeneous shearlet transform and the resulting coorbit spaces (as treated in [3, 4, 5, 7]), we start by giving a short overview of the respective theory. By modifying the homogeneous shearlet transform, we then develop a new transform, given through the action of an (inhomogeneous) frame. For this new transform we then show that all the necessary conditions from the main theorems stated in Section 2 hold, so that we can introduce the associated (weighted) coorbit spaces with respect to the Lebesgue spaces. A discretization of these spaces is then given in Section 4.

Homogeneous shearlet transform. To define the shearlet transform, one starts with an *admissible* function $\psi \in L_2(\mathbb{R}^d)$, i.e. a function satisfying the condition

$$c_\psi := \int_{\mathbb{R}^d} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^d} \,\mathrm{d}\omega < \infty. \tag{3.1}$$

This condition is necessary for the transform to be square-integrable. The admissible function is then translated, dilated and sheared in order to change its localization, scale and orientation. For a parameter $a \in \mathbb{R}^*$ let

$$A_a = \begin{pmatrix} a & 0_{d-1}^T \\ 0_{d-1} & \mathrm{sign}(a)|a|^{\frac{1}{d}} I_{d-1} \end{pmatrix}$$

denote a generalized parabolic scaling matrix and for a parameter $s \in \mathbb{R}^{d-1}$ let

$$S_s = \begin{pmatrix} 1 & s^T \\ 0_{d-1} & I_{d-1} \end{pmatrix}$$

denote the so-called shear matrix. It is easy to see that $|\det S_s| = 1$ and $|\det A_a| = |a|^{2-\frac{1}{d}}$. Using these matrices one can then define the translated, dilated and sheared version of ψ through

$$\psi_{(a,s,t)}(x) = |\det A_a|^{-\frac{1}{2}}\psi(A_a^{-1}S_s^{-1}(x-t)).$$

In the homogeneous setting, the shearlet transform is then defined through the action of a unitary, irreducible and integrable representation of the full parameter group, the so-called shearlet group $\mathbb{S} = \mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$ with the group law

$$(a,s,t) \circ (a',s',t') = (aa', s + |a|^{1-\frac{1}{d}}s', t + S_sA_at').$$

Given the mapping $\pi : \mathbb{S} \to \mathcal{U}(L_2(\mathbb{R}^d))$ with $\pi(a,s,t)\psi = \psi_{(a,s,t)}$, which can be shown to be a unitary group representation, the shearlet transform is defined as

$$\mathcal{SH} : L_2(\mathbb{R}^d) \to L_2(\mathbb{S}), \quad f \mapsto \mathcal{SH}f$$

with

$$\mathcal{SH}f : \mathbb{S} \to \mathbb{C}, \quad (a,s,t) \mapsto \langle f, \pi(a,s,t)\psi\rangle_{L_2(\mathbb{R}^d)}.$$

Based on this notion of the shearlet transform Dahlke et al. introduced homogeneous shearlet coorbit spaces with respect to the Lebesgue spaces by using the coorbit space theory developed by Feichtinger and Gröchenig in [10, 11, 12].

Inhomogeneous shearlet frame. Similar to the wavelet approach in [20] we now introduce an inhomogeneous shearlet transform by restricting the dilation parameter to a closed subspace of the full parameter group, thereby only covering the higher-frequency content of a signal. To analyze the polynomial and lower-frequency part a second function is introduced to construct an inhomogeneous frame of functions in $L_2(\mathbb{R}^d)$ as the set of building blocks for our new transform. Therefore we choose the set

$$X := \Big(\{\infty\} \times \mathbb{R}^{d-1} \times \mathbb{R}^d\Big) \cup \Big([-1,1]^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d\Big)$$

as the new parameter space with "∞" representing an isolated point in $\mathbb{R}$ and $[-1,1]^* := [-1,1] \setminus \{0\}$. The right-hand side of the union is the aforementioned subspace of the shearlet group $\mathbb{S}$, which is closed under the group action. Obviously, this definition leads to a locally compact Hausdorff space. In the following definition we introduce a measure on the parameter space so that X, together with its Borel σ-algebra, becomes a measure space.

Definition 3.1. On the space X a measure μ is defined by

$$\int_X F(x)\,\mathrm{d}\mu(x) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} F(\infty,s,t)\,\mathrm{d}s\,\mathrm{d}t + \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-1}^{1} F(a,s,t)\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\,\mathrm{d}t \tag{3.2}$$

with F being a complex-valued function on X which is measurable with respect to the Borel σ-algebra.

The first summand in the definition above is composed of the point measure on $\mathbb{R}$ and the Lebesgue measure on $\mathbb{R}^{d-1} \times \mathbb{R}^d$, while the second summand is the restriction of the (left) Haar measure on the shearlet group to the subset $[-1,1]^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$. Therefore it is obvious that μ given by (3.2) is a positive Radon measure. Choosing the measure space $(X, \mathfrak{B}(X), \mu)$ as the underlying index space, we can introduce a continuous shearlet frame.

Definition 3.2. Let $a \in \mathbb{R}^*$, $s \in \mathbb{R}^{d-1}$ and $t \in \mathbb{R}^d$. Then
(i) $L_t : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ with $L_t\psi := \psi(\cdot - t)$ is called the *(left) translation operator*,
(ii) $D_{S_s} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ with $D_{S_s}\psi := \psi(S_s^{-1}\cdot)$ is called the *shearing operator*, and
(iii) $D_{A_a} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ with $D_{A_a}\psi := |\det A_a|^{-\frac{1}{2}}\psi(A_a^{-1}\cdot)$ is called the *(anisotropic) dilation operator*.

Using the above defined operators, we can define an *inhomogeneous shearlet frame*.

Definition 3.3. Let $\Phi, \Psi \in L_2(\mathbb{R}^d)$ with Ψ being an admissible shearlet. Then we define $\mathfrak{F} := \{\psi_x\}_{x\in X}$ with

$$\psi_{(\infty,s,t)} := D_{S_s} L_t \Phi = \Phi(S_s^{-1}(\cdot - t)) \text{ and} \tag{3.3}$$

$$\psi_{(a,s,t)} := D_{A_a} D_{S_s} L_t \Psi = |\det A_a|^{-\frac{1}{2}} \Psi(A_a^{-1} S_s^{-1}(\cdot - t)). \tag{3.4}$$

The main theorem of this section is that $\mathfrak{F}$, given by (3.3) and (3.4), constitutes a continuous Parseval frame under the conditions given in Theorem 3.3 below so that the transform based on $\mathfrak{F}$ is well defined. To this end we need two technical results that can also be found in [7].

Lemma 3.1. *For all $(\alpha, s, t) \in X$ with $\alpha = a$ or $\alpha = \infty$ and $f, \psi \in L_2(\mathbb{R}^d)$ the identity*

$$\langle f, \psi_{(\alpha,s,t)}\rangle_{L_2(\mathbb{R}^d)} = (f * \psi^*_{(\alpha,s,0)})(t)$$

holds true with $\psi^* := \overline{\psi(-\cdot)}$.

Lemma 3.2. *Let $\phi \in L_2(\mathbb{R}^d)$, $a \in \mathbb{R}^*$, $s \in \mathbb{R}^{d-1}$ and $\xi \in \mathbb{R}^d$. Then the following equations hold:*
(i) $\mathcal{F}(D_{S_s}\phi)(\xi) = \hat{\phi}(S_s^T \xi)$;
(ii) $\mathcal{F}(D_{A_a} D_{S_s}\phi)(\xi) = |\det A_a|^{\frac{1}{2}} \hat{\phi}(A_a S_s^T \xi)$.

We now state the main theorem of this section, which identifies conditions on Φ and Ψ for $\mathfrak{F}$ being a continuous Parseval frame.

Theorem 3.3. *Let $\Psi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ be an admissible shearlet and let $\Phi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ be such that*

$$\int\limits_{\mathbb{R}^{d-1}} \frac{|\hat{\Phi}(y,\sigma)|^2}{|y|^{d-1}} \, \mathrm{d}\sigma + \int\limits_{\mathbb{R}^{d-1}} \int\limits_{-|y|}^{|y|} \frac{|\hat{\Psi}(\xi_1, \tilde{\xi})|^2}{|\xi_1|^d} \, \mathrm{d}\xi_1 \, \mathrm{d}\tilde{\xi} = 1 \quad \text{for almost every } y \in \mathbb{R}. \tag{3.5}$$

Then the inhomogeneous shearlet frame $\mathfrak{F}$ is a continuous Parseval frame of $L_2(\mathbb{R}^d)$, i.e.,

$$\int\limits_X |\langle f, \psi_x\rangle|^2 \, \mathrm{d}\mu(x) = \|f|L_2(\mathbb{R}^d)\|^2, \quad f \in L_2(\mathbb{R}^d).$$

Proof. Applying (3.2), Fubini's and Plancherel's theorem we obtain

$$\begin{aligned}
\int\limits_X |\langle f, \psi_x\rangle|^2 \, \mathrm{d}\mu(x) &= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}} |\langle f, \psi_{(\infty,s,t)}\rangle|^2 \, \mathrm{d}s \, \mathrm{d}t + \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{-1}^{1} |\langle f, \psi_{(a,s,t)}\rangle|^2 \, \frac{\mathrm{d}a}{|a|^{d+1}} \, \mathrm{d}s \, \mathrm{d}t \\
&= \int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}^d} |\langle f, \psi_{(\infty,s,t)}\rangle|^2 \, \mathrm{d}t \, \mathrm{d}s + \int\limits_{\mathbb{R}^{d-1}}\int\limits_{-1}^{1}\int\limits_{\mathbb{R}^d} |\langle f, \psi_{(a,s,t)}\rangle|^2 \, \mathrm{d}t \, \frac{\mathrm{d}a}{|a|^{d+1}} \, \mathrm{d}s \\
&= \int\limits_{\mathbb{R}^{d-1}} \|\langle f, \psi_{(\infty,s,\cdot)}\rangle|L_2(\mathbb{R}^d)\|^2 \, \mathrm{d}s + \int\limits_{\mathbb{R}^{d-1}}\int\limits_{-1}^{1} \|\langle f, \psi_{(a,s,\cdot)}\rangle|L_2(\mathbb{R}^d)\|^2 \, \frac{\mathrm{d}a}{|a|^{d+1}} \, \mathrm{d}s \\
&= \int\limits_{\mathbb{R}^{d-1}} \|\mathcal{F}(\langle f, \psi_{(\infty,s,\cdot)}\rangle)|L_2(\mathbb{R}^d)\|^2 \, \mathrm{d}s + \int\limits_{\mathbb{R}^{d-1}}\int\limits_{-1}^{1} \|\mathcal{F}(\langle f, \psi_{(a,s,\cdot)}\rangle)|L_2(\mathbb{R}^d)\|^2 \, \frac{\mathrm{d}a}{|a|^{d+1}} \, \mathrm{d}s \\
&= \int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}^d} |\mathcal{F}(\langle f, \psi_{(\infty,s,\cdot)}\rangle)(t)|^2 \, \mathrm{d}t \, \mathrm{d}s + \int\limits_{\mathbb{R}^{d-1}}\int\limits_{-1}^{1}\int\limits_{\mathbb{R}^d} |\mathcal{F}(\langle f, \psi_{(a,s,\cdot)}\rangle)(t)|^2 \, \mathrm{d}t \, \frac{\mathrm{d}a}{|a|^{d+1}} \, \mathrm{d}s.
\end{aligned}$$

Using Lemma 3.1, Fubini's theorem, the fact that $\mathcal{F}(f*g)=\hat{f}\hat{g}$ and $|\mathcal{F}(f^*)|=|\mathcal{F}(f)|$ leads to

$$\begin{aligned}\int_X |\langle f,\psi_x\rangle|^2\,\mathrm{d}\mu(x) &= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\mathcal{F}(f*\psi^*_{(\infty,s,0)})(t)|^2\,\mathrm{d}t\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}\int_{\mathbb{R}^d} |\mathcal{F}(f*\psi^*_{(a,s,0)})(t)|^2\,\mathrm{d}t\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\\
&= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\hat{f}(t)|^2|\mathcal{F}(\psi^*_{(\infty,s,0)})(t)|^2\,\mathrm{d}t\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}\int_{\mathbb{R}^d} |\hat{f}(t)|^2|\mathcal{F}(\psi^*_{(a,s,0)})(t)|^2\,\mathrm{d}t\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\\
&= \int_{\mathbb{R}^d} |\hat{f}(t)|^2\Bigg(\int_{\mathbb{R}^{d-1}} |\mathcal{F}(\psi_{(\infty,s,0)})(t)|^2\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}|\mathcal{F}(\psi_{(a,s,0)})(t)|^2\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\Bigg)\,\mathrm{d}t.\end{aligned}$$

Thus, if we can prove that

$$\int_{\mathbb{R}^{d-1}} |\mathcal{F}(\psi_{(\infty,s,0)})(t)|^2\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}|\mathcal{F}(\psi_{(a,s,0)})(t)|^2\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s \stackrel{!}{=} 1 \quad \text{for almost every } t\in\mathbb{R}^d, \tag{3.6}$$

the assertion follows, since then

$$\int_X |\langle f,\psi_x\rangle|^2\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d} |\hat{f}(t)|^2\,\mathrm{d}t = \|\hat{f}|L_2(\mathbb{R}^d)\|^2 = \|f|L_2(\mathbb{R}^d)\|^2.$$

Hence, it remains to show (3.6). Assuming that $t_1\neq 0$ we use Lemma 3.2 to obtain

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}} |\mathcal{F}(\psi_{(\infty,s,0)})(t)|^2\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}|\mathcal{F}(\psi_{(a,s,0)})(t)|^2\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\\
&\qquad= \int_{\mathbb{R}^{d-1}} |\mathcal{F}(D_{S_s}\Phi)(t)|^2\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}|\mathcal{F}(D_{A_a}D_{S_s}\Psi)(t)|^2\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\\
&\qquad= \int_{\mathbb{R}^{d-1}} |\hat{\Phi}(S_s^Tt)|^2\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}|\det A_a||\hat{\Psi}(A_aS_s^Tt)|^2\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\\
&\qquad= \int_{\mathbb{R}^{d-1}} |\hat{\Phi}(t_1,\tilde{t}+t_1s)|^2\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}|\det A_a||\hat{\Psi}(at_1,\operatorname{sign}(a)|a|^{\frac{1}{d}}(\tilde{t}+t_1s))|^2\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s,\end{aligned}$$

with $t=(t_1,\tilde{t})^T$, $\tilde{t}\in\mathbb{R}^{d-1}$. Substituting $\sigma:=\tilde{t}+t_1s$ and $\xi=(\xi_1,\tilde{\xi}):=(at_1,\operatorname{sign}(a)|a|^{\frac{1}{d}}(\tilde{t}+t_1s))$, we end up with

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}} |\mathcal{F}(\psi_{(\infty,s,0)})(t)|^2\,\mathrm{d}s + \int_{\mathbb{R}^{d-1}}\int_{-1}^{1}|\mathcal{F}(\psi_{(a,s,0)})(t)|^2\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\\
&\qquad= \int_{\mathbb{R}^{d-1}} |t_1|^{-(d-1)}|\hat{\Phi}(t_1,\sigma)|^2\,\mathrm{d}\sigma + \int_{\mathbb{R}^{d-1}}\int_{-|t_1|}^{|t_1|} |t_1|^{-d}|a|^{-d}|\hat{\Psi}(\xi_1,\tilde{\xi})|^2\,\mathrm{d}\xi_1\,\mathrm{d}\tilde{\xi}\\
&\qquad= \int_{\mathbb{R}^{d-1}} \frac{|\hat{\Phi}(t_1,\sigma)|^2}{|t_1|^{d-1}}\,\mathrm{d}\sigma + \int_{\mathbb{R}^{d-1}}\int_{-|t_1|}^{|t_1|} \frac{|\hat{\Psi}(\xi_1,\tilde{\xi})|^2}{|\xi_1|^d}\,\mathrm{d}\xi_1\,\mathrm{d}\tilde{\xi},\end{aligned}$$

and (3.6) follows from assumption (3.5). $\square$

Remark 2. The proof of Theorem 3.3 can also be stated in a similar manner for the case of a tight frame with arbitrary frame constant $A < \infty$. The only difference is that Φ and Ψ have to satisfy

$$\int_{\mathbb{R}^{d-1}} \frac{|\hat{\Phi}(y,\sigma)|^2}{|y|^{d-1}} \, d\sigma + \int_{\mathbb{R}^{d-1}} \int_{-|y|}^{|y|} \frac{|\hat{\Psi}(\xi_1,\tilde{\xi})|^2}{|\xi_1|^d} \, d\xi_1 \, d\tilde{\xi} = A \quad \text{for almost every } y \in \mathbb{R}$$

instead of (3.5).

Remark 3. For a given shearlet Ψ it is still necessary to show that one can satisfy condition (3.5) for a function $\Phi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$. To this end choose some nontrivial function $\hat{\varphi}_2 \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ and define $\hat{\varphi}_1 : \mathbb{R} \to \mathbb{C}$ by

$$|\hat{\varphi}_1(\xi_1)| = \|\hat{\varphi}_2\|_{L_2}^{-1} \Big(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}\setminus[-|\xi_1|,|\xi_1|]} \frac{|\hat{\Psi}(\omega_1,\tilde{\omega})|^2}{|\omega_1|^d} \, d\omega_1 \, d\tilde{\omega} \Big)^{\frac{1}{2}}.$$

Then it is easy to see that $\hat{\Phi} : \mathbb{R}^d \to \mathbb{C}$ given through $\hat{\Phi}(\xi) := \hat{\varphi}_1(\xi_1)\hat{\varphi}_2(\xi_1^{-1}\tilde{\xi})$ satisfies (3.5). In the following we will assume $\hat{\Phi}$ to be chosen in this way.

Because of Theorem 3.3, we can now state the definition of the shearlet transform based on $\mathfrak{F}$.

Definition 3.4. Let $\Phi, \Psi \in L_2(\mathbb{R}^d)$ satisfy the assumptions of Theorem 3.3 and let $\mathfrak{F} = \{\psi_x\}_{x\in X}$ be given by Definition 3.3. Then the *shearlet transform* based on $\mathfrak{F}$ is defined as

$$\mathcal{SH}_{\mathfrak{F}} : L_2(\mathbb{R}^d) \to L_2(X,\mu), f \mapsto \mathcal{SH}_{\mathfrak{F}} f$$

with

$$\mathcal{SH}_{\mathfrak{F}} f : X \to \mathbb{C}, x \mapsto \langle f, \psi_x \rangle.$$

Conditions on the reproducing kernel. The main goal of this section is the definition of the coorbit spaces $\mathrm{Co}(L_{p,v}(X,\mu))$, $p \geq 1$, with v being a weight function on X, associated to the inhomogeneous shearlet transform introduced in the previous section. To prove that these spaces are well-defined Banach spaces, we first show that the conditions on $\mathfrak{F}$, as stated in Section 2, are satisfied, i.e. that $R_{\mathfrak{F}} \in B_{L_{p,v},m_v}$. For a kernel function $K \in \mathcal{A}_{m_v}$ the generalized Young inequality (cf. [6, Theorem A.1] or [13]) yields that the corresponding operator between the $L_{p,v}$-spaces is bounded, i.e. $K \in B_{L_{p,v},m_v}$ for all $p \geq 1$ with $\|K|B_{L_{p,v},m_v}\| \leq \|K|\mathcal{A}_{m_v}\|$. Since we are interested in the coorbit spaces of the Lebesgue spaces $L_{p,v}(X,\mu)$, it is therefore only necessary to check whether $R_{\mathfrak{F}} \in \mathcal{A}_{m_v}$. To this end we need the following auxiliary result.

Lemma 3.4. *Let* $a, a' \in [-1,1]^*$, $s, s' \in \mathbb{R}^{d-1}$, $t, t' \in \mathbb{R}^d$ *and* $\varphi_{(a,s,t)} := |\det A_a|^{-\frac{1}{2}} \Phi(A_a^{-1} S_s^{-1}(\cdot - t))$. *It follows that*

$$|\langle \psi_{(\infty,s,t)}, \psi_{(\infty,s',t')} \rangle| = |(\mathcal{SH}\Phi)(\infty, s-s', S_{s'}^{-1}(t-t'))|, \tag{3.7}$$

$$|\langle \psi_{(\infty,s,t)}, \psi_{(a',s',t')} \rangle| = |\langle \Psi, \varphi_{(a'^{-1},|a'|^{\frac{1}{d}-1}(s-s'),A_{a'}^{-1}S_{s'}^{-1}(t-t')} \rangle|, \tag{3.8}$$

$$|\langle \psi_{(a,s,t)}, \psi_{(\infty,s',t')} \rangle| = |(\mathcal{SH}\Phi)(a, s-s', S_{s'}^{-1}(t-t'))|, \tag{3.9}$$

$$|\langle \psi_{(a,s,t)}, \psi_{(a',s',t')} \rangle| = |(\mathcal{SH}\Psi)(aa'^{-1}, |a'|^{\frac{1}{d}-1}(s-s'), A_{a'}^{-1}S_{s'}^{-1}(t-t'))|. \tag{3.10}$$

Proof. We only state the proof for (3.10) in detail, (3.7)–(3.9) can be proven analogously. By the definition of $\psi_{(a,s,t)}$ we obtain

$$\begin{aligned}\langle \psi_{(a,s,t)}, \psi_{(a',s',t')}\rangle &= \int_{\mathbb{R}^d} \psi_{(a,s,t)}(x)\overline{\psi_{(a',s',t')}(x)}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} |\det A_a|^{-\frac{1}{2}}\Psi(A_a^{-1}S_s^{-1}(x-t))\overline{|\det A_{a'}|^{-\frac{1}{2}}\Psi(A_{a'}^{-1}S_{s'}^{-1}(x-t'))}\,\mathrm{d}x,\end{aligned}$$

which, by means of the substitution $y = A_{a'}^{-1}S_{s'}^{-1}(x-t')$, leads to

$$\begin{aligned}\langle \psi_{(a,s,t)}, \psi_{(a',s',t')}\rangle &= \int_{\mathbb{R}^d} |\det A_{aa'^{-1}}|^{-\frac{1}{2}}\Psi(A_a^{-1}S_s^{-1}(S_{s'}A_{a'}y+t'-t))\overline{\Psi(y)}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} |\det A_{aa'^{-1}}|^{-\frac{1}{2}}\Psi(A_a^{-1}S_s^{-1}S_{s'}A_{a'}(y-(A_{a'}^{-1}S_{s'}^{-1}(t-t'))))\overline{\Psi(y)}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} |\det A_{aa'^{-1}}|^{-\frac{1}{2}}\Psi(A_{aa'^{-1}}^{-1}S_{|a'|^{\frac{1}{d}-1}(s-s')}^{-1}(y-(A_{a'}^{-1}S_{s'}^{-1}(t-t'))))\overline{\Psi(y)}\,\mathrm{d}y \\ &= \langle \psi_{(aa'^{-1},|a'|^{\frac{1}{d}-1}(s-s'),A_{a'}^{-1}S_{s'}^{-1}(t-t'))}, \Psi\rangle.\end{aligned}$$

This yields

$$\begin{aligned}|\langle \psi_{(a,s,t)}, \psi_{(a',s',t')}\rangle| &= |\langle \psi_{(aa'^{-1},|a'|^{\frac{1}{d}-1}(s-s'),A_{a'}^{-1}S_{s'}^{-1}(t-t'))}, \Psi\rangle| \\ &= |\langle \Psi, \psi_{(aa'^{-1},|a'|^{\frac{1}{d}-1}(s-s'),A_{a'}^{-1}S_{s'}^{-1}(t-t'))}\rangle| \\ &= |(\mathcal{SH}\Psi)(aa'^{-1},|a'|^{\frac{1}{d}-1}(s-s'),A_{a'}^{-1}S_{s'}^{-1}(t-t'))|.\end{aligned}$$

□

Using the auxiliary result above, we can prove the following lemma concerning the $\mathcal{A}_{m_v}$-Norm of $R_{\mathfrak{F}}$.

Lemma 3.5. *Let $R_{\mathfrak{F}}$ be the kernel function associated to the inhomogeneous shearlet frame as defined by (2.2). Then the following identity holds:*

$$\begin{aligned}&\operatorname*{ess\,sup}_{(\alpha,\sigma,\tau)\in X}\int_X |R_{\mathfrak{F}}((\alpha,\sigma,\tau),(a,s,t))|m_v((\alpha,\sigma,\tau),(a,s,t))\,\mathrm{d}\mu(a,s,t) \\ &= \max\Bigg\{\operatorname*{ess\,sup}_{(\sigma,\tau)\in\mathbb{R}^{d-1}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\Bigg(\max\left\{\frac{v(\infty,\sigma,\tau)}{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})},\frac{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})}{v(\infty,\sigma,\tau)}\right\}|\langle\Phi,\psi_{(\infty,s',t')}\rangle| \\ &\qquad + \int_{-1}^{1}\max\left\{\frac{v(\infty,\sigma,\tau)}{v(a,\tilde{\sigma_2},\tilde{\tau_2})},\frac{v(a,\tilde{\sigma_2},\tilde{\tau_2})}{v(\infty,\sigma,\tau)}\right\}|\langle\Phi,\psi_{(a,s',t')}\rangle|\frac{\mathrm{d}a}{|a|^{d+1}}\Bigg)\,\mathrm{d}s'\,\mathrm{d}t', \\ &\operatorname*{ess\,sup}_{(\alpha,\sigma,\tau)\in[-1,1]^*\times\mathbb{R}^{d-1}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\Bigg(\max\left\{\frac{v(\alpha,\sigma,\tau)}{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})},\frac{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})}{v(\alpha,\sigma,\tau)}\right\}|\langle\Phi,\psi_{(\alpha,s',t')}\rangle| \\ &\qquad + \int_{-|\alpha|^{-1}}^{|\alpha|^{-1}}\max\left\{\frac{v(\alpha,\sigma,\tau)}{v(\tilde{\alpha},\tilde{\sigma_3},\tilde{\tau_3})},\frac{v(\tilde{\alpha},\tilde{\sigma_3},\tilde{\tau_3})}{v(\alpha,\sigma,\tau)}\right\}|\langle\Psi,\psi_{(a',s',t')}\rangle|\frac{\mathrm{d}a'}{|a'|^{d+1}}\Bigg)\,\mathrm{d}s'\,\mathrm{d}t'\Bigg\}\end{aligned}$$

with $\tilde{\sigma_1} = \sigma - s', \tilde{\tau_1} = \tau - S_{\tilde{\sigma_1}} t', \tilde{\sigma_2} = \sigma + s', \tilde{\tau_2} = \tau + S_\sigma t', \tilde{\alpha} = \alpha a', \tilde{\sigma_3} = \sigma + |\alpha|^{1-\frac{1}{d}} s', \tilde{\tau_3} = \tau + S_\sigma A_\alpha t'$.

Proof. Let $(\alpha,\sigma,\tau) \in X$ with $\alpha \in \{\infty\} \cup [-1,1]^*$. Using (3.7) and (3.9) we obtain

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(\infty,s,t)}, \frac{v(\infty,s,t)}{v(\alpha,\sigma,\tau)} \right\} |\langle \psi_{(\infty,s,t)}, \psi_{(\alpha,\sigma,\tau)} \rangle| \,\mathrm{d}s\,\mathrm{d}t \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(\infty,s,t)}, \frac{v(\infty,s,t)}{v(\alpha,\sigma,\tau)} \right\} |\langle \Phi, \psi_{(\alpha,\sigma-s,S_s^{-1}(\tau-t))} \rangle| \,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Substituting $s' = \sigma - s$ and $t' = S^{-1}_{\sigma-s'}(\tau - t)$ then leads to

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(\infty,s,t)}, \frac{v(\infty,s,t)}{v(\alpha,\sigma,\tau)} \right\} |\langle \psi_{(\infty,s,t)}, \psi_{(\alpha,\sigma,\tau)} \rangle| \,\mathrm{d}s\,\mathrm{d}t \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(\infty,\tilde{\sigma_1},t)}, \frac{v(\infty,\tilde{\sigma_1},t)}{v(\alpha,\sigma,\tau)} \right\} \left|\left\langle \Phi, \psi_{(\alpha,s',S^{-1}_{\sigma-s'}(\tau-t))} \right\rangle\right| \,\mathrm{d}s'\,\mathrm{d}t \qquad (3.11) \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})}, \frac{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})}{v(\alpha,\sigma,\tau)} \right\} |\langle \Phi, \psi_{(\alpha,s',t')} \rangle| \,\mathrm{d}s'\,\mathrm{d}t'.
\end{aligned}$$

Analogously we see that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{-1}^{1} \max\left\{ \frac{v(\infty,\sigma,\tau)}{v(a,s,t)}, \frac{v(a,s,t)}{v(\infty,\sigma,\tau)} \right\} |\langle \psi_{(a,s,t)}, \psi_{(\infty,\sigma,\tau)} \rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \,\mathrm{d}s\,\mathrm{d}t \qquad (3.12) \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{-1}^{1} \max\left\{ \frac{v(\infty,\sigma,\tau)}{v(a,\tilde{\sigma_2},\tilde{\tau_2})}, \frac{v(a,\tilde{\sigma_2},\tilde{\tau_2})}{v(\infty,\sigma,\tau)} \right\} |\langle \Phi, \psi_{(a,s',t')} \rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \,\mathrm{d}s'\,\mathrm{d}t',
\end{aligned}$$

for $\sigma \in \mathbb{R}^{d-1}$ and $\tau \in \mathbb{R}^d$. Now let $\alpha \in [-1,1]^*$. Then (3.10) yields

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{-1}^{1} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(a,s,t)}, \frac{v(a,s,t)}{v(\alpha,\sigma,\tau)} \right\} |\langle \psi_{(a,s,t)}, \psi_{(\alpha,\sigma,\tau)} \rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \,\mathrm{d}s\,\mathrm{d}t \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{-1}^{1} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(a,s,t)}, \frac{v(a,s,t)}{v(\alpha,\sigma,\tau)} \right\} \left|\left\langle \Psi, \psi_{(a\alpha^{-1},|\alpha|^{\frac{1}{d}-1}(s-\sigma),A_\alpha^{-1}S_\sigma^{-1}(t-\tau))} \right\rangle\right| \frac{\mathrm{d}a}{|a|^{d+1}} \,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}$$

which—by substituting $a' := a\alpha^{-1}$—leads to

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{-1}^{1} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(a,s,t)}, \frac{v(a,s,t)}{v(\alpha,\sigma,\tau)} \right\} |\langle \psi_{(a,s,t)}, \psi_{(\alpha,\sigma,\tau)} \rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \,\mathrm{d}s\,\mathrm{d}t \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{-|\alpha|^{-1}}^{|\alpha|^{-1}} \max\left\{ \frac{v(\alpha,\sigma,\tau)}{v(\tilde{\alpha},s,t)}, \frac{v(\tilde{\alpha},s,t)}{v(\alpha,\sigma,\tau)} \right\} \left|\left\langle \Psi, \psi_{(a',|\alpha|^{\frac{1}{d}-1}(s-\sigma),A_\alpha^{-1}S_\sigma^{-1}(t-\tau))} \right\rangle\right| \frac{1}{|\alpha|^d} \frac{\mathrm{d}a'}{|a'|^{d+1}} \,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Again, substituting with $s' := |\alpha|^{\frac{1}{d}-1}(s-\sigma)$ and $t' := A_\alpha^{-1}S_\sigma^{-1}(t-\tau)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-1}^{1} \max\left\{\frac{v(\alpha,\sigma,\tau)}{v(a,s,t)},\frac{v(a,s,t)}{v(\alpha,\sigma,\tau)}\right\}|\langle\psi_{(a,s,t)},\psi_{(\alpha,\sigma,\tau)}\rangle|\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\,\mathrm{d}t\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-|\alpha|^{-1}}^{|\alpha|^{-1}} \max\left\{\frac{v(\alpha,\sigma,\tau)}{v(\tilde{\alpha},\tilde{\sigma_3},t)},\frac{v(\tilde{\alpha},\tilde{\sigma_3},t)}{v(\alpha,\sigma,\tau)}\right\}\left|\left\langle\Psi,\psi_{(a',s',A_\alpha^{-1}S_\sigma^{-1}(t-\tau))}\right\rangle\right||\alpha|^{\frac{1}{d}-2}\,\frac{\mathrm{d}a'}{|a'|^{d+1}}\,\mathrm{d}s'\,\mathrm{d}t \qquad (3.13)\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-|\alpha|^{-1}}^{|\alpha|^{-1}} \max\left\{\frac{v(\alpha,\sigma,\tau)}{v(\tilde{\alpha},\tilde{\sigma_3},\tilde{\tau_3})},\frac{v(\tilde{\alpha},\tilde{\sigma_3},\tilde{\tau_3})}{v(\alpha,\sigma,\tau)}\right\}|\langle\Psi,\psi_{(a',s',t')}\rangle|\,\frac{\mathrm{d}a'}{|a'|^{d+1}}\,\mathrm{d}s'\,\mathrm{d}t'.
\end{aligned}$$

Using (3.11), (3.12), and (3.13), we now have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(\alpha,\sigma,\tau)\in X}\int_X |R_{\mathfrak{F}}((\alpha,\sigma,\tau),(a,s,t))|m_v((\alpha,\sigma,\tau),(a,s,t))\,\mathrm{d}\mu(a,s,t)\\
&\quad=\operatorname*{ess\,sup}_{(\alpha,\sigma,\tau)\in X}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\Bigg(\max\left\{\frac{v(\alpha,\sigma,\tau)}{v(\infty,s,t)},\frac{v(\infty,s,t)}{v(\alpha,\sigma,\tau)}\right\}|\langle\psi_{(\infty,s,t)},\psi_{(\alpha,\sigma,\tau)}\rangle|\\
&\qquad\qquad+\int_{-1}^{1}\max\left\{\frac{v(\alpha,\sigma,\tau)}{v(a,s,t)},\frac{v(a,s,t)}{v(\alpha,\sigma,\tau)}\right\}|\langle\psi_{(a,s,t)},\psi_{(\alpha,\sigma,\tau)}\rangle|\,\frac{\mathrm{d}a}{|a|^{d+1}}\Bigg)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad=\max\Bigg\{\operatorname*{ess\,sup}_{(\sigma,\tau)\in\mathbb{R}^{d-1}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\Bigg(\max\left\{\frac{v(\infty,\sigma,\tau)}{v(\infty,s,t)},\frac{v(\infty,s,t)}{v(\infty,\sigma,\tau)}\right\}|\langle\psi_{(\infty,s,t)},\psi_{(\infty,\sigma,\tau)}\rangle|\\
&\qquad\qquad+\int_{-1}^{1}\max\left\{\frac{v(\infty,\sigma,\tau)}{v(a,s,t)},\frac{v(a,s,t)}{v(\infty,\sigma,\tau)}\right\}|\langle\psi_{(a,s,t)},\psi_{(\infty,\sigma,\tau)}\rangle|\,\frac{\mathrm{d}a}{|a|^{d+1}}\Bigg)\,\mathrm{d}s\,\mathrm{d}t,\\
&\qquad\operatorname*{ess\,sup}_{(\alpha,\sigma,\tau)\in[-1,1]^*\times\mathbb{R}^{d-1}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\Bigg(\max\left\{\frac{v(\alpha,\sigma,\tau)}{v(\infty,s,t)},\frac{v(\infty,s,t)}{v(\alpha,\sigma,\tau)}\right\}|\langle\psi_{(\infty,s,t)},\psi_{(\alpha,\sigma,\tau)}\rangle|\\
&\qquad\qquad+\int_{-1}^{1}\max\left\{\frac{v(\alpha,\sigma,\tau)}{v(a,s,t)},\frac{v(a,s,t)}{v(\alpha,\sigma,\tau)}\right\}|\langle\psi_{(a,s,t)},\psi_{(\alpha,\sigma,\tau)}\rangle|\,\frac{\mathrm{d}a}{|a|^{d+1}}\Bigg)\,\mathrm{d}s\,\mathrm{d}t\Bigg\}\\
&=\max\Bigg\{\operatorname*{ess\,sup}_{(\sigma,\tau)\in\mathbb{R}^{d-1}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\Bigg(\max\left\{\frac{v(\infty,\sigma,\tau)}{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})},\frac{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})}{v(\infty,\sigma,\tau)}\right\}|\langle\Phi,\psi_{(\infty,s',t')}\rangle|\\
&\qquad+\int_{-1}^{1}\max\left\{\frac{v(\infty,\sigma,\tau)}{v(a,\tilde{\sigma_2},\tilde{\tau_2})},\frac{v(a,\tilde{\sigma_2},\tilde{\tau_2})}{v(\infty,\sigma,\tau)}\right\}|\langle\Phi,\psi_{(a,s',t')}\rangle|\,\frac{\mathrm{d}a}{|a|^{d+1}}\Bigg)\,\mathrm{d}s'\,\mathrm{d}t',\\
&\qquad\operatorname*{ess\,sup}_{(\alpha,\sigma,\tau)\in[-1,1]^*\times\mathbb{R}^{d-1}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\Bigg(\max\left\{\frac{v(\alpha,\sigma,\tau)}{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})},\frac{v(\infty,\tilde{\sigma_1},\tilde{\tau_1})}{v(\alpha,\sigma,\tau)}\right\}|\langle\Phi,\psi_{(\alpha,s',t')}\rangle|
\end{aligned}$$

$$+\int_{-|\alpha|^{-1}}^{|\alpha|^{-1}} \max\left\{\frac{v(\alpha,\sigma,\tau)}{v(\tilde\alpha,\tilde{\sigma_3},\tilde{\tau_3})},\frac{v(\tilde\alpha,\tilde{\sigma_3},\tilde{\tau_3})}{v(\alpha,\sigma,\tau)}\right\}|\langle\Psi,\psi_{(a',s',t')}\rangle|\,\frac{\mathrm{d}a'}{|a'|^{d+1}}\Big)\,\mathrm{d}s'\,\mathrm{d}t'\Big\}$$

□

We use Lemma 3.5 to prove $R_{\mathfrak{F}} \in \mathcal{A}_{m_v}$ for certain functions Φ and Ψ. Since it is not possible to construct functions $\Phi, \Psi \in L_2$ with compact support in the spatial domain satisfying the conditions in Remark 3, in the following we assume Ψ to be a *bandlimited* function, in particular

$$\operatorname{supp}\hat\Psi \subseteq ([-a_1,-a_0]\cup[a_0,a_1]\times Q_b)$$

with $0<a_0<a_1$ and $Q_b := \times_{i=1}^{d-1}[-b_i,b_i]$ for $b\in\mathbb{R}_+^{d-1}$. The function Φ is chosen in the same way as in Remark 3 with $\operatorname{supp}\hat\varphi_2 \subseteq Q_c,\ c\in\mathbb{R}_+^{d-1}$. It follows that $\operatorname{supp}\hat\varphi_1\subseteq[-a_1,a_1]$ and

$$\operatorname{supp}\hat\Phi \subseteq \{(\xi_1,\tilde\xi)\in\mathbb{R}^d|-a_1\le\xi_1\le a_1, \tilde\xi\in Q_{c|\xi_1|}\}.$$

As weight functions on X we consider

$$v_{r,n}(\alpha,s,t)=v_{r,n}(\alpha,s) := \begin{cases}(1+\|s\|)^n, & \alpha=\infty,\\ \left(\frac{1}{|\alpha|}\right)^r\left(\frac{1}{|\alpha|}+\|s\|\right)^n, & \alpha\in[-1,1]^*\end{cases} \tag{3.14}$$

with $r,n\in\mathbb{N}$, which satisfy all necessary conditions. Through simple calculations one can verify the following properties of the weights $v_{r,n}$ for $a,a'\in[-1,1]^*, |a|\le|a'|, s,s'\in\mathbb{R}^{d-1}$:

$$m_{v_{r,n}}((\infty,s),(\infty,s'))\le v_{r,n}(\infty,s-s'), \tag{3.15}$$

$$m_{v_{r,n}}((a,s),(\infty,s'))\le v_{r,n}(a,s-s'), \tag{3.16}$$

$$m_{v_{r,n}}((a,s),(a',s'))\le v_{r,n}\left(\frac{a}{a'},s-s'\right). \tag{3.17}$$

The following technical lemma concerns support properties of Φ and Ψ in the frequency domain, similar to [7, Lemma 3.1].

Lemma 3.6. *Let $0<a_0<a_1$ and $b\in\mathbb{R}_+^{d-1}$. Then with Φ and Ψ as defined above for $a\in\mathbb{R}^*, s\in\mathbb{R}^{d-1}$ we have:*

(i) $\hat\Phi\hat\Phi(S_s^T\cdot)\not\equiv 0$ *implies* $s\in Q_{2c}$

(ii) $\hat\Psi\hat\Phi(A_aS_s^T\cdot)\not\equiv0$ *implies* $a\in[-\frac{a_1}{a_0},\frac{a_1}{a_0}]$ *and* $s\in Q_{\tilde d}$ *with* $\tilde d := a_0^{-1}(a_1c+b)$

(iii) $\hat\Phi\hat\Psi(A_aS_s^T\cdot)\not\equiv0$ *implies* $a\in[-\infty,-\frac{a_0}{a_1}]\cup[\frac{a_0}{a_1},\infty]$ *and* $s\in Q_{\tilde d}$ *with* $\tilde d := a_0^{-(1+\frac{1}{d})}a_1^{\frac{1}{d}}b+c$

(iv) $\hat\Psi\hat\Psi(A_aS_s^T\cdot)\not\equiv0$ *implies* $a\in[-\frac{a_1}{a_0},-\frac{a_0}{a_1}]\cup[\frac{a_0}{a_1},\frac{a_1}{a_0}]$ *and* $s\in Q_{\tilde d}$ *with* $\tilde d := (a_0^{-1}+a_0^{-(1+\frac{1}{d})}a_1^{\frac{1}{d}})b$

Proof. The proof of (iv) can be found in [7, Lemma 3.1]. To prove (iii) we assume that there exists a $\xi\in\operatorname{supp}\hat\Phi\cap\operatorname{supp}\hat\Psi(A_aS_s^T\cdot)$ which means that $\xi\in\operatorname{supp}\hat\Phi$ and $A_aS_s^T\xi\in\operatorname{supp}\hat\Psi$. This leads to

$$|\xi_1|\le a_1, \tag{3.18}$$

$$-c_i|\xi_1|\le\xi_{i+1}\le c_i|\xi_1|, \tag{3.19}$$

$$a_0\le|a||\xi_1|\le a_1, \tag{3.20}$$

$$-b_i|a|^{-\frac{1}{d}}-\xi_1s_i\le\xi_{i+1}\le b_i|a|^{-\frac{1}{d}}-\xi_1s_i, \tag{3.21}$$

with $i=1,\dots,d-1$. By (3.18) and (3.20) it follows that $|a|\ge\frac{a_0}{a_1}$ which means that $a\in[-\infty,-\frac{a_0}{a_1}]\cup[\frac{a_0}{a_1},\infty]$. Using (3.20) and $|a|\le1$, it follows that $a_0\le|a||\xi_1|\le|\xi_1|$. Also, with (3.21) and (3.19) we obtain

$$-c_i|\xi_1|\le b_i|a|^{-\frac{1}{d}}-\xi_1s_i \text{ and } -b_i|a|^{-\frac{1}{d}}-\xi_1s_i\le c_i|\xi_1|,$$

which leads to

$$|s_i| \le |\xi_1|^{-1} b_i |a|^{-\frac{1}{d}} + c_i \le a_0^{-1} b_i \left(\frac{a_0}{a_1}\right)^{-\frac{1}{d}} + c_i$$

for $i = 1, \dots, d-1$ which proves (iii). To prove (ii), we assume that there exists a $\xi \in \operatorname{supp} \hat{\Psi} \cap \hat{\Phi}(A_a S_s^T \cdot)$, which means that $\xi \in \operatorname{supp} \hat{\Psi}$ and $A_a S_s^T \xi \in \operatorname{supp} \hat{\Phi}$. This yields

$$a_0 \le |\xi_1| \le a_1, \tag{3.22}$$

$$|\xi_j| \le b_{j-1}, \quad j = 2, \dots, d, \tag{3.23}$$

$$|a||\xi_1| \le a_1, \tag{3.24}$$

$$-c_{j-1}|\xi_1| \le \operatorname{sign}(a)|a|^{\frac{1}{d}}(\xi_1 s_{j-1} + \xi_j) \le c_{j-1}|\xi_1|, \quad j = 2, \dots, d. \tag{3.25}$$

Estimates (3.22) and (3.24) yield $|a| \le \frac{a_1}{a_0}$. This combined with (3.23) and (3.25) gives

$$|s_{j-1}| \le a_0^{-1}(c_{j-1} a_1 + b_{j-1}),$$

which proves (ii). The proof of (i) works analogously. □

Now we are able to prove that the integrability condition on the kernel function is satisfied, i.e. that $R_{\mathfrak{F}} \in \mathcal{A}_{m_{v_{r,n}}}$.

Theorem 3.7. *Let $\Psi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ be an admissible shearlet with*

$$\operatorname{supp} \hat{\Psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1] \times Q_b).$$

Let $\Phi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ be chosen as in Remark 3 so that condition (3.5) is satisfied with $\operatorname{supp} \hat{\varphi}_1 \subseteq [-a_1, a_1]$ and $\operatorname{supp} \hat{\varphi}_2 \subseteq Q_c$ for $0 < a_0 < a_1$ and $b, c \in \mathbb{R}_+^{d-1}$. Then, the kernel $R_{\mathfrak{F}}$ fulfills:

$$R_{\mathfrak{F}} \in \mathcal{A}_{m_{v_{r,n}}}.$$

Proof. Let $\tilde{\alpha} := \alpha a, \tilde{\sigma_1} := \sigma - s, \tilde{\sigma_2} := \sigma + s, \tilde{\sigma_3} := \sigma + |\alpha|^{1-\frac{1}{d}} s$. Using Lemma 3.5 leads to the equality

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in X} \int_X |R_{\mathfrak{F}}(x,y)| m_{v_{r,n}}(x,y)\,\mathrm{d}\mu(y) = \operatorname*{ess\,sup}_{x\in X} \int_X |\langle \psi_y, \psi_x\rangle| m_{v_{r,n}}(x,y)\,\mathrm{d}\mu(y) \\
&= \max\Bigg\{ \operatorname*{ess\,sup}_{(\sigma,\tau)\in\mathbb{R}^{d-1}\times\mathbb{R}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} \Bigg(\max\left\{ \frac{v_{r,n}(\infty,\sigma)}{v_{r,n}(\infty,\tilde{\sigma_1})}, \frac{v_{r,n}(\infty,\tilde{\sigma_1})}{v_{r,n}(\infty,\sigma)} \right\} |\langle \Phi, \psi_{(\infty,s,t)}\rangle| \\
&\qquad + \int_{-1}^{1} \max\left\{ \frac{v_{r,n}(\infty,\sigma)}{v_{r,n}(a,\tilde{\sigma_2})}, \frac{v_{r,n}(a,\tilde{\sigma_2})}{v_{r,n}(\infty,\sigma)} \right\} |\langle \Phi, \psi_{(a,s,t)}\rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \Bigg) \mathrm{d}s\,\mathrm{d}t, \\
&\quad \operatorname*{ess\,sup}_{(\alpha,\sigma,\tau)\in[-1,1]^*\times\mathbb{R}^{d-1}\times\mathbb{R}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} \Bigg(\max\left\{ \frac{v_{r,n}(\alpha,\sigma)}{v_{r,n}(\infty,\tilde{\sigma_1})}, \frac{v_{r,n}(\infty,\tilde{\sigma_1})}{v_{r,n}(\alpha,\sigma)} \right\} |\langle \Phi, \psi_{(\alpha,s,t)}\rangle| \\
&\qquad + \int_{-|\alpha|^{-1}}^{|\alpha|^{-1}} \max\left\{ \frac{v_{r,n}(\alpha,\sigma)}{v_{r,n}(\tilde{\alpha},\tilde{\sigma_3})}, \frac{v_{r,n}(\tilde{\alpha},\tilde{\sigma_3})}{v_{r,n}(\alpha,\sigma)} \right\} |\langle \Psi, \psi_{(a,s,t)}\rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \Bigg) \mathrm{d}s\,\mathrm{d}t \Bigg\}.
\end{aligned}$$

With (3.15) and since $\mathcal{F}(f^*) = \mathcal{F}(\overline{f(-\cdot)}) = \overline{\mathcal{F}(f)}$ and $\Phi * \psi^*_{(\infty,s,0)} \in L_1(\mathbb{R}^d)$ we can compute

$$\begin{aligned}
\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}} & \max\left\{\frac{v_{r,n}(\infty,\sigma)}{v_{r,n}(\infty,\tilde{\sigma_1})}, \frac{v_{r,n}(\infty,\tilde{\sigma_1})}{v_{r,n}(\infty,\sigma)}\right\} |\langle\Phi,\psi_{(\infty,s,t)}\rangle| \,\mathrm{d}s\,\mathrm{d}t \\
&\leq \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}} (1+\|s\|)^n \, |\langle\Phi,\psi_{(\infty,s,t)}\rangle| \,\mathrm{d}s\,\mathrm{d}t \\
&= \int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}^d} (1+\|s\|)^n \, |(\Phi * \psi^*_{(\infty,s,0)})(t)| \,\mathrm{d}t\,\mathrm{d}s \\
&= \int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}^d} (1+\|s\|)^n \, \left|\mathcal{F}^{-1}(\mathcal{F}(\Phi * \psi^*_{(\infty,s,0)}))(t)\right| \,\mathrm{d}t\,\mathrm{d}s \\
&= \int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}^d} (1+\|s\|)^n \, \left|\mathcal{F}^{-1}(\hat{\Phi}\mathcal{F}(\psi^*_{(\infty,s,0)}))(t)\right| \,\mathrm{d}t\,\mathrm{d}s \\
&= \int\limits_{\mathbb{R}^{d-1}} (1+\|s\|)^n \, \left\|\mathcal{F}^{-1}(\hat{\Phi}\overline{\mathcal{F}(\psi_{(\infty,s,0)})})\middle|L_1\right\| \,\mathrm{d}s.
\end{aligned}$$

Applying Lemma 3.6(i), we see that if $s \notin Q_{2c}$, then $\hat{\Phi}\mathcal{F}(\psi_{(\infty,s,0)}) \equiv 0$, which implies that

$$\mathcal{F}^{-1}(\hat{\Phi}\overline{\mathcal{F}(\psi_{(\infty,s,0)})}) \equiv 0$$

and therefore we have

$$\begin{aligned}
\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}} & \max\left\{\frac{v_{r,n}(\infty,\sigma)}{v_{r,n}(\infty,\tilde{\sigma_1})}, \frac{v_{r,n}(\infty,\tilde{\sigma_1})}{v_{r,n}(\infty,\sigma)}\right\} |\langle\Phi,\psi_{(\infty,s,t)}\rangle| \,\mathrm{d}s\,\mathrm{d}t \\
&\leq \int\limits_{Q_{2c}} (1+\|s\|)^n \, \left\|\mathcal{F}^{-1}(\hat{\Phi}\overline{\mathcal{F}(\psi_{(\infty,s,0)})})\middle|L_1\right\| \,\mathrm{d}s \\
&= \int\limits_{Q_{2c}} (1+\|s\|)^n \, \left\|\Phi * \psi^*_{(\infty,s,0)}\middle|L_1\right\| \,\mathrm{d}s < \infty.
\end{aligned} \tag{3.26}$$

Using the same arguments with (3.16), we obtain

$$\begin{aligned}
\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{-1}^{1} & \max\left\{\frac{v_{r,n}(\infty,\sigma)}{v_{r,n}(a,\tilde{\sigma_2})}, \frac{v_{r,n}(a,\tilde{\sigma_2})}{v_{r,n}(\infty,\sigma)}\right\} |\langle\Phi,\psi_{(a,s,t)}\rangle| \,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\,\mathrm{d}t \\
&\leq \int\limits_{\mathbb{R}^{d-1}}\int\limits_{-1}^{1} \left(\frac{1}{|a|}\right)^r \left(\frac{1}{|a|}+\|s\|\right)^n \left\|\mathcal{F}^{-1}(\hat{\Phi}\overline{\mathcal{F}(\psi_{(a,s,0)})})\middle|L_1\right\| \,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s
\end{aligned}$$

and analogously Lemma 3.6(iii) yields that if $a \notin [-\infty, -\frac{a_0}{a_1}] \cup [\frac{a_0}{a_1}, \infty]$ or $s \notin Q_{\tilde{d}}$ with $\tilde{d} = a_0^{-(1+\frac{1}{d})} a_1^{\frac{1}{d}} b + c$, then $\left\|\mathcal{F}^{-1}(\hat{\Phi}\overline{\mathcal{F}(\psi_{(a,s,0)})})\middle|L_1\right\| = 0$ which leads to

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{-1}^{1} \max\left\{\frac{v_{r,n}(\infty,\sigma)}{v_{r,n}(a,\tilde{\sigma_2})}, \frac{v_{r,n}(a,\tilde{\sigma_2})}{v_{r,n}(\infty,\sigma)}\right\} |\langle \Phi, \psi_{a,s,t}\rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \,\mathrm{d}s \,\mathrm{d}t \\
&\qquad\le \Bigg(\int\limits_{-1}^{-\frac{a_0}{a_1}} + \int\limits_{\frac{a_0}{a_1}}^{1}\Bigg) \int\limits_{Q_{\tilde{d}}} \left(\frac{1}{|a|}\right)^r \left(\frac{1}{|a|} + \|s\|\right)^n \left\|\mathcal{F}^{-1}(\hat{\Phi}\overline{\mathcal{F}(\psi_{(a,s,0)})})\middle|L_1\right\| \mathrm{d}s \,\frac{\mathrm{d}a}{|a|^{d+1}} \qquad (3.27)\\
&\qquad= \Bigg(\int\limits_{-1}^{-\frac{a_0}{a_1}} + \int\limits_{\frac{a_0}{a_1}}^{1}\Bigg) \int\limits_{Q_{\tilde{d}}} \left(\frac{1}{|a|}\right)^r \left(\frac{1}{|a|} + \|s\|\right)^n \|\Phi * \psi^*_{(a,s,0)}|L_1\| \,\mathrm{d}s \,\frac{\mathrm{d}a}{|a|^{d+1}} < \infty.
\end{aligned}$$

By Lemma 3.6(iii) and writing $I := [-1, -\frac{a_0}{a_1}] \cup [\frac{a_0}{a_1}, 1]$ it also follows that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\alpha\in[-1,1]^*} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}} \max\left\{\frac{v_{r,n}(\alpha,\sigma)}{v_{r,n}(\infty,\tilde{\sigma_1})}, \frac{v_{r,n}(\infty,\tilde{\sigma_1})}{v_{r,n}(\alpha,\sigma)}\right\} |\langle \Phi, \psi_{(\alpha,s,t)}\rangle| \,\mathrm{d}s \,\mathrm{d}t \\
&\qquad\le \operatorname*{ess\,sup}_{\alpha\in[-1,1]^*} \int\limits_{\mathbb{R}^{d-1}} \left(\frac{1}{|\alpha|}\right)^r \left(\frac{1}{|\alpha|} + \|s\|\right)^n \left\|\mathcal{F}^{-1}(\hat{\Phi}\overline{\mathcal{F}(\psi_{(\alpha,s,0)})})\middle|L_1\right\| \mathrm{d}s \qquad (3.28)\\
&\qquad= \operatorname*{ess\,sup}_{\alpha\in I} \int\limits_{Q_{\tilde{d}}} \left(\frac{1}{|\alpha|}\right)^r \left(\frac{1}{|\alpha|} + \|s\|\right)^n \left\|\mathcal{F}^{-1}(\hat{\Phi}\overline{\mathcal{F}(\psi_{(\alpha,s,0)})})\middle|L_1\right\| \mathrm{d}s \\
&\qquad= \operatorname*{ess\,sup}_{\alpha\in I} \int\limits_{Q_{\tilde{d}}} \left(\frac{1}{|\alpha|}\right)^r \left(\frac{1}{|\alpha|} + \|s\|\right)^n \|\Phi * \psi^*_{(a,s,0)}|L_1\| \,\mathrm{d}s < \infty.
\end{aligned}$$

Finally, Lemma 3.6(iv) and (3.17) lead to

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\alpha\in[-1,1]^*} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{-|\alpha|^{-1}}^{|\alpha|^{-1}} \max\left\{\frac{v_{r,n}(\alpha,\sigma)}{v_{r,n}(\tilde{\alpha},\tilde{\sigma_3})}, \frac{v_{r,n}(\tilde{\alpha},\tilde{\sigma_3})}{v_{r,n}(\alpha,\sigma)}\right\} |\langle \Psi, \psi_{(a,s,t)}\rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \,\mathrm{d}s \,\mathrm{d}t \\
&\qquad\le \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}} \left(\frac{1}{|a|}\right)^r \left(\frac{1}{|a|} + \|s\|\right)^n |\langle \Psi, \psi_{(a,s,t)}\rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \,\mathrm{d}s \,\mathrm{d}t \\
&\qquad= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{d-1}} \left(\frac{1}{|a|}\right)^r \left(\frac{1}{|a|} + \|s\|\right)^n \left\|\mathcal{F}^{-1}(\hat{\Psi}\overline{\mathcal{F}(\psi_{(a,s,0)})})\middle|L_1\right\| \mathrm{d}s \,\frac{\mathrm{d}a}{|a|^{d+1}} \qquad (3.29)\\
&\qquad= \Bigg(\int\limits_{-\frac{a_1}{a_0}}^{-\frac{a_0}{a_1}} + \int\limits_{\frac{a_0}{a_1}}^{\frac{a_1}{a_0}}\Bigg) \int\limits_{Q_{\tilde{d}}} \left(\frac{1}{|a|}\right)^r \left(\frac{1}{|a|} + \|s\|\right)^n \left\|\mathcal{F}^{-1}(\hat{\Psi}\overline{\mathcal{F}(\psi_{(a,s,0)})})\middle|L_1\right\| \mathrm{d}s \,\frac{\mathrm{d}a}{|a|^{d+1}} \\
&\qquad= \Bigg(\int\limits_{-\frac{a_1}{a_0}}^{-\frac{a_0}{a_1}} + \int\limits_{\frac{a_0}{a_1}}^{\frac{a_1}{a_0}}\Bigg) \int\limits_{Q_{\tilde{d}}} \left(\frac{1}{|a|}\right)^r \left(\frac{1}{|a|} + \|s\|\right)^n \|\Psi * \psi^*_{(a,s,0)}|L_1\| \,\mathrm{d}s \,\frac{\mathrm{d}a}{|a|^{d+1}} < \infty
\end{aligned}$$

with $\tilde{d} := (a_0^{-1} + a_0^{-(1+\frac{1}{d})} a_1^{\frac{1}{d}})b$. By (3.26) and (3.27),

$$\operatorname*{ess\,sup}_{(\sigma,\tau)\in\mathbb{R}^{d-1}\times\mathbb{R}^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}} \Bigg(\max\left\{ \frac{v_{r,n}(\infty,\sigma)}{v_{r,n}(\infty,\tilde{\sigma_1})}, \frac{v_{r,n}(\infty,\tilde{\sigma_1})}{v_{r,n}(\infty,\sigma)} \right\} |\langle \Phi, \psi_{(\infty,s,t)}\rangle|$$

$$+ \int\limits_{-1}^{1} \max\left\{ \frac{v_{r,n}(\infty,\sigma)}{v_{r,n}(a,\tilde{\sigma_2})}, \frac{v_{r,n}(a,\tilde{\sigma_2})}{v_{r,n}(\infty,\sigma)} \right\} |\langle \Phi, \psi_{(a,s,t)}\rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \Bigg) \mathrm{d}s\,\mathrm{d}t < \infty.$$

Also, using (3.28) and (3.29), we obtain

$$\operatorname*{ess\,sup}_{(\alpha,\sigma,\tau)\in[-1,1]^*\times\mathbb{R}^{d-1}\times\mathbb{R}^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}} \Bigg(\max\left\{ \frac{v_{r,n}(\alpha,\sigma)}{v_{r,n}(\infty,\tilde{\sigma_1})}, \frac{v_{r,n}(\infty,\tilde{\sigma_1})}{v_{r,n}(\alpha,\sigma)} \right\} |\langle \Phi, \psi_{(\alpha,s,t)}\rangle|$$

$$+ \int\limits_{-|\alpha|^{-1}}^{|\alpha|^{-1}} \max\left\{ \frac{v_{r,n}(\alpha,\sigma)}{v_{r,n}(\tilde{\alpha},\tilde{\sigma_3})}, \frac{v_{r,n}(\tilde{\alpha},\tilde{\sigma_3})}{v_{r,n}(\alpha,\sigma)} \right\} |\langle \Psi, \psi_{(a,s,t)}\rangle| \frac{\mathrm{d}a}{|a|^{d+1}} \Bigg) \mathrm{d}s\,\mathrm{d}t < \infty.$$

Using Lemma 3.5 now gives the statement. □

Now we are able to give a definition of the coorbit spaces associated to our inhomogeneous shearlet frame with respect to the weighted Lebesgue spaces $L_{p,v_{r,n}}(X,\mu)$. The $L_{p,v_{r,n}}(X,\mu)$-spaces obviously satisfy the condition (Y).

Definition 3.5. Let the shearlet frame $\mathfrak{F}$ be chosen so that it satisfies the conditions in Theorem 3.7. Then for $p \geq 1$ the shearlet coorbit space with respect to the Lebesgue space $L_{p,v_{r,n}}(X,\mu)$ is defined as

$$\mathcal{SC}^{r,n}_{\mathfrak{F},p} := \{f \in (\mathcal{H}_{1,v_{r,n}})^\sim : \mathcal{SH}_{\mathfrak{F}} f \in L_{p,v_{r,n}}(X,\mu)\}.$$

It is endowed with the norm

$$\|f|\mathcal{SC}^{r,n}_{\mathfrak{F},p}\| := \|\mathcal{SH}_{\mathfrak{F}} f | L_{p,v_{r,n}}(X,\mu)\|.$$

As stated earlier, for the coorbit spaces to be well-defined, we need the embedding

$$R_{\mathfrak{F}}(L_{p,v_{r,n}}) \hookrightarrow L_{\infty,v_{r,n}^{-1}}.$$

The proof of Lemma 3.8 will be given in Section 4.

Lemma 3.8. *Let Ψ be an admissible shearlet ($c_\Psi = 1$) with $\hat{\Psi} \in \mathscr{C}^{d+3}(\mathbb{R}^d)$ and*

$$\operatorname{supp}\hat{\Psi} \subseteq ([-a_1,-a_0] \cup [a_0,a_1]) \times Q_b,$$

where $a_1 > a_0 > 0$, $b \in \mathbb{R}^{d-1}_+$ and $Q_b = \times_{i=1}^{d-1}[-b_i,b_i]$. Let $\varphi_2 \in \mathcal{S}(\mathbb{R}^{d-1})$ satisfy $\operatorname{supp}\hat{\varphi_2} \subseteq Q_c$ with $c \in \mathbb{R}^{d-1}_+$ and $\|\hat{\varphi_2}|L_2\| = 1$. Let $\varphi_1 \in L_2(\mathbb{R})$ be such that

$$|\hat{\varphi_1}(\xi_1)| = \left(\int\limits_{\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}\setminus[-|\xi_1|,|\xi_1|]} \frac{|\hat{\Psi}(\omega)|^2}{|\omega_1|^d}\,\mathrm{d}\omega \right)^{\frac{1}{2}}$$

and let φ_2 and Ψ be such that $\hat{\Phi} \in \mathscr{C}^{d+3}(\mathbb{R}^d)$ with $\hat{\Phi}(\xi) := \hat{\varphi_1}(\xi_1)\hat{\varphi_2}(\xi_1^{-1}\tilde{\xi})$. Then it follows that

$$R_{\mathfrak{F}}(L_{p,v_{r,n}}) \hookrightarrow L_{\infty,v_{r,n}^{-1}}.$$

Theorem 3.9. *With the same assumptions as in Lemma 3.8 the spaces $\mathcal{SC}^{r,n}_{\mathfrak{F},p}$ are indeed well-defined Banach spaces.*

Proof. As remarked at the beginning of this section it follows from Theorem 3.7 and the generalized Young inequality that $R_{\mathfrak{F}} \in B_{L_{p,v_{r,n}},m_{v_{r,n}}}$. By Lemma 3.8, we then obtain the continuous embedding $R_{\mathfrak{F}}(L_{p,v_{r,n}}) \hookrightarrow L_{\infty,v_{r,n}^{-1}}(X)$ so that condition $(F_{v,Y})$ from Definition 2.2 is satisfied and therefore the coorbit spaces $\mathcal{SC}_{\mathfrak{F},p}^{r,n}$ are well defined Banach spaces. □

Remark 4. From Definition 3.5 it is not immediately obvious whether we get scales of smoothness spaces since the reservoirs $(\mathcal{H}_{1,v_{r,n}})^\sim$ seem to depend on the weight function $v_{r,n}$. However if we assume that for $r, \tilde{r}, n, \tilde{n} \in \mathbb{N}$ we have the estimate $v_{r,n} \leq C v_{\tilde{r},\tilde{n}}$ for some constant $C > 0$ (which is the case for $r \leq \tilde{r}$ and $n \leq \tilde{n}$), this of course means that the embedding $\mathcal{H}_{1,v_{\tilde{r},\tilde{n}}} \hookrightarrow \mathcal{H}_{1,v_{r,n}}$ holds, which in turn leads to the embedding $(\mathcal{H}_{1,v_{r,n}})^\sim \hookrightarrow (\mathcal{H}_{1,v_{\tilde{r},\tilde{n}}})^\sim$. Now we can write

$$\mathcal{SC}_{\mathfrak{F},p}^{r,n} := \{f \in (\mathcal{H}_{1,v_{\tilde{r},\tilde{n}}})^\sim : \mathcal{SH}_{\mathfrak{F}} f \in L_{p,v_{r,n}}(X,\mu)\},$$

since for an element $f \in (\mathcal{H}_{1,v_{\tilde{r},\tilde{n}}})^\sim$ with $\mathcal{SH}_{\mathfrak{F}} f \in L_{p,v_{r,n}}$ it already follows from the reproducing identity stated in Section 2 and Lemma 3.8 that $f \in (\mathcal{H}_{1,v_{r,n}})^\sim$.

4. Discretization

In this section we show that we can apply the abstract results in Section 2 to obtain a discretization of the new shearlet coorbit spaces introduced in the previous section through a discretization of the parameter space X. This will be the main result of this section. In order to discretize these new shearlet coorbit spaces a moderate admissible covering of

$$X = (\{\infty\} \times \mathbb{R}^{d-1} \times \mathbb{R}^d) \cup ([-1,1]^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d),$$

satisfying the conditions in Definition 2.4, which is compatible with the dilation, shearing and translation operators acting on Φ and Ψ, is needed. In the following let $\alpha > 1$, $\beta, \gamma > 0$.

Definition 4.1. For the parameter space X we define a family of subsets

$$\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$$

with $\Lambda = (\{\infty\} \times \mathbb{Z}^{d-1} \times \mathbb{Z}^d) \cup (\{-1,1\} \times \mathbb{N}_0 \times \mathbb{Z}^{d-1} \times \mathbb{Z}^d)$. The subsets U_λ are given by

$$U_{(\infty,k^0,m^0)} := \Big\{\infty\Big\} \times \beta\Big(k^0 + \Big[-\frac{1}{2},\frac{1}{2}\Big)^{d-1}\Big) \times \gamma S_{\beta k^0}\Big(m^0 + \Big[-\frac{1}{2},\frac{1}{2}\Big)^d\Big)$$

and

$$U_{(\varepsilon,j,k,m)} := \varepsilon\alpha^{-j}\Big(\alpha^{-1},1\Big] \times \alpha^{j(\frac{1}{d}-1)}\beta\Big(k + \Big[-\frac{1}{2},\frac{1}{2}\Big)^{d-1}\Big) \times \gamma S_{\alpha^{j(\frac{1}{d}-1)}\beta k} A_{\alpha^{-j}}\Big(m + \Big[-\frac{1}{2},\frac{1}{2}\Big)^d\Big)$$

with $\varepsilon \in \{-1,1\}, j \in \mathbb{N}_0, k^0, k \in \mathbb{Z}^{d-1}$ and $m^0, m \in \mathbb{Z}^d$.

We can now prove that this is indeed a moderate admissible covering.

Lemma 4.1. *The family of subsets $\mathcal{U}$, as stated in Definition 4.1, is a moderate admissible covering in the sense of Definition 2.4.*

Proof. The fact that every element in $\mathcal{U}$ is relatively compact and has non-void interior is obvious from the definition. Since $k^0 + [-\frac{1}{2},\frac{1}{2})^{d-1}$ is a partition of $\mathbb{R}^{d-1}$ for $k^0 \in \mathbb{Z}^{d-1}$, naturally $\beta(k^0 + [-\frac{1}{2},\frac{1}{2})^{d-1})$ is also a partition of $\mathbb{R}^{d-1}$. Using the same arguments we conclude that for a fixed $k^0 \in \mathbb{Z}^{d-1}$ the subsets $\gamma S_{\beta k^0}(m^0 + [-\frac{1}{2},\frac{1}{2})^d)$ constitute a partition of $\mathbb{R}^d$. It follows that

$$\bigcup_{k^0 \in \mathbb{Z}^{d-1}, m^0 \in \mathbb{Z}^d} U_{(\infty,k^0,m^0)} = \{\infty\} \times \mathbb{R}^{d-1} \times \mathbb{R}^d.$$

Similar arguments yield

$$\bigcup_{\varepsilon \in \{-1,1\}, j \in \mathbb{N}_0, k \in \mathbb{Z}^{d-1}, m \in \mathbb{Z}^d} U_{(\varepsilon,j,k,m)} = [-1,1]^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d.$$

Consequently, $\mathcal{U}$ is indeed a partition of X, i.e.,

$$\bigcup_{\lambda\in\Lambda} U_\lambda = X.$$

Since the partition is disjoint, condition (iii) from Definition 2.4 is also satisfied. Elementary calculations show that

$$\mu(U_{(\infty,k^0,m^0)}) = \beta^{d-1}\gamma^d$$

and

$$\mu(U_{(\varepsilon,j,k,m)}) = \frac{1}{d}(\alpha^d - 1)\beta^{d-1}\gamma^d.$$

Thus, for fixed α, β and γ there are constants $C, c > 0$ with $c \le \mu(U_\lambda) \le C$ for all $\lambda \in \Lambda$. □

In order to prove the discretization result it is sufficient to show that the $B_{L_{p,v_{r,n}},m_{v_{r,n}}}$-norm of the mapping $\mathrm{osc}_{\mathcal{U}}$ can be made arbitrarily small. The following theorem states that for certain choices of Φ and Ψ, one can use the "size parameters" α, β, γ to accomplish this. The proof of the theorem uses ideas that were also used in the proof of [5, Theorem 4.4]. Recall that c_Ψ is the admissibility constant from (3.1) and that the $\mathcal{U}$-oscillation is defined as

$$\mathrm{osc}_{\mathcal{U}}(x,y) := \sup_{z\in W_y} |\langle \psi_x, \psi_y\rangle - \langle \psi_x, \psi_z\rangle|, \quad x, y \in X,$$

with $W_y := \bigcup_{U_\lambda \ni y} U_\lambda$. We show later on that we can find functions Φ and Ψ that satisfy the assumptions stated in the following theorem.

Theorem 4.2. *Let $\Psi \in L_1 \cap L_2$ be an admissible shearlet ($c_\Psi = 1$) with $\hat{\Psi} \in \mathscr{C}^{d+3}(\mathbb{R}^d)$ and*

$$\operatorname{supp}\hat{\Psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times Q_b,$$

where $a_1 > a_0 > 0$, $b \in \mathbb{R}_+^{d-1}$ and $Q_b = \times_{i=1}^{d-1}[-b_i, b_i]$. Let $\varphi_2 \in \mathcal{S}(\mathbb{R}^{d-1})$ satisfy $\operatorname{supp}\hat{\varphi_2} \subseteq Q_c$ with $c \in \mathbb{R}_+^{d-1}$ and $\|\hat{\varphi_2}|L_2\| = 1$. Let $\varphi_1 \in L_2(\mathbb{R})$ be such that

$$|\hat{\varphi_1}(\xi_1)| = \left(\int\limits_{\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}\setminus[-|\xi_1|,|\xi_1|]} \frac{|\hat{\Psi}(\omega)|^2}{|\omega_1|^d}\, \mathrm{d}\omega \right)^{\frac{1}{2}}$$

and let φ_2 and Ψ be such that $\hat{\Phi} \in \mathscr{C}^{d+3}(\mathbb{R}^d)$ with $\hat{\Phi}(\xi) := \hat{\varphi_1}(\xi_1)\hat{\varphi_2}(\xi_1^{-1}\tilde{\xi})$. Fix an arbitrary $\delta > 0$. Then there exist $\alpha > 1$ and $\beta, \gamma > 0$, so that for the corresponding moderate admissible covering $\mathcal{U}$ stated in Definition 4.1 the estimates $\|osc_{\mathcal{U}}|\mathcal{A}_{m_{v_{r,n}}}\| < \delta$ and $\|osc^*_{\mathcal{U}}|\mathcal{A}_{m_{v_{r,n}}}\| < \delta$ hold.

Proof. Fix an arbitrary $\delta > 0$. Since $\|\mathrm{osc}_{\mathcal{U}}|\mathcal{A}_{m_{v_{r,n}}}\| = \|\mathrm{osc}^*_{\mathcal{U}}|\mathcal{A}_{m_{v_{r,n}}}\|$ we only need to prove that $\|\mathrm{osc}_{\mathcal{U}}|\mathcal{A}_{m_{v_{r,n}}}\| < \delta$ is satisfied. I.e., we show that there exist $\alpha > 1$, $\beta, \gamma > 0$, so that

$$\begin{aligned}\|\mathrm{osc}_{\mathcal{U}}|\mathcal{A}_{m_{v_{r,n}}}\| = \max\Bigg\{ &\operatorname*{ess\,sup}_{(a,s,t)\in X} \int_X |\mathrm{osc}_{\mathcal{U}}((a,s,t),(a',s',t'))| m_{v_{r,n}}((a,s,t),(a',s',t'))\,\mathrm{d}\mu(a',s',t'),\\ &\operatorname*{ess\,sup}_{(a',s',t')\in X} \int_X |\mathrm{osc}_{\mathcal{U}}((a,s,t),(a',s',t'))| m_{v_{r,n}}((a,s,t),(a',s',t'))\,\mathrm{d}\mu(a,s,t) \Bigg\} < \delta\end{aligned}$$

with $\mathcal{U}$ being the corresponding moderate admissible covering of X given by Definition 4.1. Recall that $W_y = \bigcup_{U_i \ni y} U_i$. With $z = (\infty, s'', t''), y = (\infty, s', t'), z' = (a'', s'', t''), y' = (a', s', t')$ and using (3.7)–(3.10) we have the estimate

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{(a,s,t)\in X} \int_X |\mathrm{osc}_{\mathcal{U}}((a,s,t),(a',s',t'))| m_{v_{r,n}}((a,s,t),(a',s',t'))\, \mathrm{d}\mu(a',s',t') \\
&\leq \max\Bigg\{ \operatorname*{ess\,sup}_{(s,t)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} \max\left\{\frac{v_{r,n}(\infty,s)}{v_{r,n}(\infty,s')}, \frac{v_{r,n}(\infty,s')}{v_{r,n}(\infty,s)}\right\} \sup_{z\in W_y} |(\mathcal{SH}\Phi)(\infty, s-s', S_{s'}^{-1}(t-t')) \\
&\qquad\qquad - (\mathcal{SH}\Phi)(\infty, s-s'', S_{s''}^{-1}(t-t''))|\, \mathrm{d}s'\, \mathrm{d}t' \\
&\quad + \operatorname*{ess\,sup}_{(s,t)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-1}^{1} \max\left\{\frac{v_{r,n}(\infty,s)}{v_{r,n}(a',s')}, \frac{v_{r,n}(a',s')}{v_{r,n}(\infty,s)}\right\} \sup_{z'\in W_{y'}} |\langle \Psi, \varphi_{((a')^{-1}, |a'|^{\frac{1}{d}-1}(s-s'), A_{a'}^{-1}S_{s'}^{-1}(t-t'))}\rangle \\
&\qquad\qquad - \langle \Psi, \varphi_{((a'')^{-1}, |a''|^{\frac{1}{d}-1}(s-s''), A_{a''}^{-1}S_{s''}^{-1}(t-t''))}\rangle| \frac{\mathrm{d}a'}{|a'|^{d+1}}\, \mathrm{d}s'\, \mathrm{d}t', \\
&\quad \operatorname*{ess\,sup}_{(a,s,t)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} \max\left\{\frac{v_{r,n}(a,s)}{v_{r,n}(\infty,s')}, \frac{v_{r,n}(\infty,s')}{v_{r,n}(a,s)}\right\} \sup_{z\in W_y} |(\mathcal{SH}\Phi)(a, s-s', S_{s'}^{-1}(t-t')) \\
&\qquad\qquad - (\mathcal{SH}\Phi)(a, s-s'', S_{s''}^{-1}(t-t''))|\, \mathrm{d}s'\, \mathrm{d}t' \\
&+ \operatorname*{ess\,sup}_{(a,s,t)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-1}^{1} \max\left\{\frac{v_{r,n}(a,s)}{v_{r,n}(a',s')}, \frac{v_{r,n}(a',s')}{v_{r,n}(a,s)}\right\} \sup_{z'\in W_{y'}} |(\mathcal{SH}\Psi)(a(a')^{-1}, |a'|^{\frac{1}{d}-1}(s-s'), A_{a'}^{-1}S_{s'}^{-1}(t-t')) \\
&\qquad\qquad - (\mathcal{SH}\Psi)(a(a'')^{-1}, |a''|^{\frac{1}{d}-1}(s-s''), A_{a''}^{-1}S_{s''}^{-1}(t-t''))| \frac{\mathrm{d}a'}{|a'|^{d+1}}\, \mathrm{d}s'\, \mathrm{d}t' \Bigg\}.
\end{aligned}
$$

Since each one of the essential suprema in the expression above can be treated analogously, we will only show that there exist $\alpha > 1, \beta, \gamma > 0$ so that

$$
\begin{aligned}
I := \operatorname*{ess\,sup}_{(s,t)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-1}^{1} &\max\left\{\frac{v_{r,n}(\infty,s)}{v_{r,n}(a',s')}, \frac{v_{r,n}(a',s')}{v_{r,n}(\infty,s)}\right\} \sup_{z'\in W_{y'}} |\langle \Psi, \varphi_{((a')^{-1}, |a'|^{\frac{1}{d}-1}(s-s'), A_{a'}^{-1}S_{s'}^{-1}(t-t'))}\rangle \\
&- \langle \Psi, \varphi_{((a'')^{-1}, |a''|^{\frac{1}{d}-1}(s-s''), A_{a''}^{-1}S_{s''}^{-1}(t-t''))}\rangle| \frac{\mathrm{d}a'}{|a'|^{d+1}}\, \mathrm{d}s'\, \mathrm{d}t' < \delta.
\end{aligned}
$$

After substituting $\eta := (a')^{-1}, \sigma := |a'|^{\frac{1}{d}-1}(s-s')$ and $\tau := A_{a'}^{-1}S_{s'}^{-1}(t-t')$ we obtain

$$
\begin{aligned}
I = \operatorname*{ess\,sup}_{(s,t)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}\setminus[-1,1]} &\max\left\{\frac{v_{r,n}(\infty,s)}{v_{r,n}(\eta^{-1}, s-|\eta|^{\frac{1}{d}-1}\sigma)}, \frac{v_{r,n}(\eta^{-1}, s-|\eta|^{\frac{1}{d}-1}\sigma)}{v_{r,n}(\infty,s)}\right\} \sup_{z'\in W_{\tilde{y}}} |\langle \Psi, \varphi_{(\eta,\sigma,\tau)}\rangle \\
&- \langle \Psi, \varphi_{((a'')^{-1}, |a''|^{\frac{1}{d}-1}(s-s''), A_{a''}^{-1}S_{s''}^{-1}(t-t''))}\rangle| \frac{\mathrm{d}\eta}{|\eta|}\, \mathrm{d}\sigma\, \mathrm{d}\tau
\end{aligned}
$$

with $\tilde{y} := (\eta^{-1}, s-|\eta|^{\frac{1}{d}-1}\sigma, t - S_{s-|\eta|^{\frac{1}{d}-1}\sigma} A_{\eta^{-1}}\tau)$. Since $\langle \Psi, \varphi_{(\eta,\sigma,\tau)}\rangle = (\Psi * \overline{\varphi_{(\eta,\sigma,0)}(-\cdot)})(\tau)$ and $\Psi * \overline{\varphi_{(\eta,\sigma,0)}(-\cdot)} \in L_2(\mathbb{R}^d)$ since $\Psi \in L_1, \Phi \in L_2$, the relationship

$$
\langle \Psi, \varphi_{(\eta,\sigma,\tau)}\rangle = (\Psi * \overline{\varphi_{(\eta,\sigma,0)}(-\cdot)})(\tau) = \mathcal{F}^{-1}(\mathcal{F}(\Psi * \overline{\varphi_{(\eta,\sigma,0)}(-\cdot)}))(\tau) = |\det A_\eta|^{\frac{1}{2}} \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\tau)
$$

holds true. Furthermore we get from (3.16) that

$$\max\left\{\frac{v_{r,n}(\infty,s)}{v_{r,n}(\eta^{-1},s-|\eta|^{\frac{1}{d}-1}\sigma)},\frac{v_{r,n}(\eta^{-1},s-|\eta|^{\frac{1}{d}-1}\sigma)}{v_{r,n}(\infty,s)}\right\}\le|\eta|^{r+n}\left(1+\|\sigma\|\right)^{n}.$$

It follows that

$$\begin{aligned}
I &\le \operatorname*{ess\,sup}_{(s,t)}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}\setminus[-1,1]}|\eta|^{r+n}\left(1+\|\sigma\|\right)^{n}\sup_{z'\in W_{\tilde y}}\Big||\det A_\eta|^{\frac12}\mathcal{F}^{-1}(\hat\Psi\overline{\hat\Phi(A_\eta S_\sigma^T\cdot)})(\tau)\\
&\qquad -|\det A_{(a'')^{-1}}|^{\frac12}\mathcal{F}^{-1}(\hat\Psi\overline{\hat\Phi(A_{(a'')^{-1}}S^T_{|a''|^{\frac1d-1}(s-s'')}\cdot)})(A^{-1}_{a''}S^{-1}_{s''}(t-t''))\Big|\frac{\mathrm{d}\eta}{|\eta|}\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&\le \operatorname*{ess\,sup}_{(s,t)}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}\setminus[-1,1]}|\eta|^{r+n}\left(1+\|\sigma\|\right)^{n}|\det A_\eta|^{\frac12}\sup_{z'\in W_{\tilde y}}\left|1-|\det A_{(a''\eta)^{-1}}|^{\frac12}\right|\\
&\qquad\cdot\left|\mathcal{F}^{-1}(\hat\Psi\overline{\hat\Phi(A_\eta S_\sigma^T\cdot)})(\tau)\right|\frac{\mathrm{d}\eta}{|\eta|}\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&+\operatorname*{ess\,sup}_{(s,t)}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}\setminus[-1,1]}|\eta|^{r+n}\left(1+\|\sigma\|\right)^{n}\sup_{z'\in W_{\tilde y}}|\det A_{(a'')^{-1}}|^{\frac12}\Big|\mathcal{F}^{-1}(\hat\Psi\overline{\hat\Phi(A_\eta S_\sigma^T\cdot)})(\tau)\\
&\qquad-\mathcal{F}^{-1}(\hat\Psi\overline{\hat\Phi(A_\eta S_\sigma^T\cdot)})(A^{-1}_{a''}S^{-1}_{s''}(t-t''))\Big|\frac{\mathrm{d}\eta}{|\eta|}\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&+\operatorname*{ess\,sup}_{(s,t)}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}\setminus[-1,1]}|\eta|^{r+n}\left(1+\|\sigma\|\right)^{n}\sup_{z'\in W_{\tilde y}}|\det A_{(a'')^{-1}}|^{\frac12}\Big|\mathcal{F}^{-1}(\hat\Psi\overline{\hat\Phi(A_\eta S_\sigma^T\cdot)})(A^{-1}_{a''}S^{-1}_{s''}(t-t''))\\
&\qquad-\mathcal{F}^{-1}(\hat\Psi\overline{\hat\Phi(A_{(a'')^{-1}}S^T_{|a''|^{\frac1d-1}(s-s'')}\cdot)})(A^{-1}_{a''}S^{-1}_{s''}(t-t''))\Big|\frac{\mathrm{d}\eta}{|\eta|}\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&=: I_1+I_2+I_3.
\end{aligned}$$

Our aim is to successively estimate each summand $I_j, j=1,2,3$ on the right hand side from above by $\frac{\delta}{3}$. Recalling the definition of the moderate admissible covering $\mathcal{U}$ and the fact that for every $y\in X$ there exists exactly one $\lambda\in\Lambda$ so that $W_{\tilde y}=U_\lambda$ (since we have a disjoint partition of X) we get the following results for some $(a'',s'',t'')\in W_{\tilde y}$. Let $u,u'\in(\alpha^{-1},1]$, $v,v'\in[-\frac12,\frac12)^{d-1}$, $w,w'\in[-\frac12,\frac12)^d$, $j\in\mathbb{N}_0$, $k\in\mathbb{Z}^{d-1}$, $m\in\mathbb{Z}^d$ and $\varepsilon\in\{-1,1\}$. Then we can write

$$\begin{aligned}
a''&=\varepsilon\alpha^{-j}u, & \eta^{-1}&=\varepsilon\alpha^{-j}u',\\
s''&=\alpha^{j(\frac1d-1)}\beta(k+v), & s-|\eta|^{\frac1d-1}\sigma&=\alpha^{j(\frac1d-1)}\beta(k+v'),\\
t''&=\gamma S_{\alpha^{j(\frac1d-1)}\beta k}A_{\alpha^{-j}}(m+w), & t-S_{s-|\eta|^{\frac1d-1}\sigma}A_{\eta^{-1}}\tau&=\gamma S_{\alpha^{j(\frac1d-1)}\beta k}A_{\alpha^{-j}}(m+w').
\end{aligned}$$

Simple calculations yield

$$(a''\eta)^{-1}=u'u^{-1}, \tag{4.1}$$

$$\sigma-|a''|^{\frac1d-1}(s-s'')=(1-|u'u^{-1}|^{1-\frac1d})\sigma-|u|^{\frac1d-1}\beta(v'-v), \tag{4.2}$$

$$A^{-1}_{a''}S^{-1}_{s''}(t-t'')-\tau=(A^{-1}_{u(u')^{-1}}S^{-1}_{(u')^{\frac1d-1}\beta(v-v')}-I)\tau+A^{-1}_{\varepsilon u}S^{-1}_{\beta v}\gamma(w'-w), \tag{4.3}$$

with $u'u^{-1}\in(\alpha^{-1},\alpha)$, $\beta(v'-v)\in(-\beta,\beta)^{d-1}$ and $\gamma(w'-w)\in(-\gamma,\gamma)^d$. Using Lemma 3.6(ii) we see that $\hat\Psi\overline{\hat\Phi(A_\eta S_\sigma^T\cdot)}\equiv0$ unless $\eta\in[-\frac{a_1}{a_0},\frac{a_1}{a_0}]$ and $\sigma\in Q_{a_0^{-1}(a_1c+b)}$. Considering (4.1) and (4.2) we can find an $\tilde a>1$ and $\tilde b\in\mathbb{R}^{d-1}_+$ so that $\hat\Psi\overline{\hat\Phi(A_\eta S_\sigma^T\cdot)}\equiv0$ and $\hat\Psi\overline{\hat\Phi(A_{(a'')^{-1}}S^T_{|a''|^{\frac1d-1}(s-s'')}\cdot)}\equiv0$ unless $\eta\in[-\tilde a,\tilde a]$

and $\sigma \in Q_{\tilde{b}}$. This means that we can restrict the integrals over η and σ to $\eta \in [-\tilde{a}, -1] \cup [1, \tilde{a}] =: \tilde{I}$ and $\sigma \in Q_{\tilde{b}}$ since

$$\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)} \equiv 0 \Longleftrightarrow \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)}) \equiv 0.$$

Since $|\eta|^{r+n}(1 + \|\sigma\|)^n$ is finite on the compact set $\tilde{I} \times Q_{\tilde{b}}$ we can estimate it by a constant in each summand.

First summand: We now show that for an arbitrary $\delta > 0$ we can choose $\alpha > 1$ and $\beta, \gamma > 0$ so that

$$I_1 \lesssim \operatorname*{ess\,sup}_{(s,t)} \int\limits_{\mathbb{R}^d} \int\limits_{Q_{\tilde{b}}} \int\limits_{\tilde{I}} |\det A_\eta|^{\frac{1}{2}} \sup_{z' \in W_{\tilde{y}}} \left|1 - |\det A_{(a''\eta)^{-1}}|^{\frac{1}{2}}\right| \left|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\tau)\right| \frac{\mathrm{d}\eta}{|\eta|} \,\mathrm{d}\sigma \,\mathrm{d}\tau < \delta.$$

Since $u' u^{-1} \in (\alpha^{-1}, \alpha)$, it follows from (4.1) that

$$|\det A_{(a''\eta)^{-1}}|^{\frac{1}{2}} = |(a''\eta)^{-1}|^{1-\frac{1}{2d}} = |u'u^{-1}|^{1-\frac{1}{2d}} \in (\alpha^{\frac{1}{2d}-1}, \alpha^{1-\frac{1}{2d}}).$$

Consequently, for every $\rho > 0$ there exists an $\alpha > 1$ so that $|1 - |\det A_{(a''\eta)^{-1}}|^{\frac{1}{2}}| < \rho$. This implies that

$$\begin{aligned} I_1 &\lesssim \rho \operatorname*{ess\,sup}_{(s,t)} \int\limits_{\mathbb{R}^d} \int\limits_{Q_{\tilde{b}}} \int\limits_{\tilde{I}} \sup_{z' \in W_{\tilde{y}}} |\eta|^{-\frac{1}{2d}} \left|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\tau)\right| \mathrm{d}\eta \,\mathrm{d}\sigma \,\mathrm{d}\tau \\ &= \rho \int\limits_{\mathbb{R}^d} \int\limits_{Q_{\tilde{b}}} \int\limits_{\tilde{I}} |\eta|^{-\frac{1}{2d}} \left|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\tau)\right| \mathrm{d}\eta \,\mathrm{d}\sigma \,\mathrm{d}\tau \\ &= \rho \int\limits_{\mathbb{R}^d} \int\limits_{Q_{\tilde{b}}} \int\limits_{\tilde{I}} |\eta|^{-\frac{1}{2d}} \left|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\tau)\right| \mathrm{d}\eta \,\mathrm{d}\sigma \,\mathrm{d}\tau \\ &< \rho\, C \int\limits_{\mathbb{R}^d} \left|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_{\tilde{\eta}} S_{\tilde{\sigma}}^T \cdot)})(\tau)\right| \mathrm{d}\tau \end{aligned} \tag{4.4}$$

with $\tilde{\eta} \in \tilde{I}$ and $\tilde{\sigma} \in Q_{\tilde{b}}$. Since $\hat{\Psi}, \hat{\Phi} \in \mathscr{C}^{d+3}(\mathbb{R}^d)$ with compact support, so is $\hat{\Psi}\overline{\hat{\Phi}(A_{\tilde{\eta}} S_{\tilde{\sigma}}^T \cdot)}$ and therefore $\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_{\tilde{\eta}} S_{\tilde{\sigma}}^T \cdot)})$ satisfies

$$\left|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_{\tilde{\eta}} S_{\tilde{\sigma}}^T \cdot)})(x)\right| \lesssim (1 + \|x\|_2)^{-N} \tag{4.5}$$

for all $N \in \mathbb{N}$ with $|N| \leq d+3$ with the constant in the estimate depending only on $N, \tilde{\eta}$ and $\tilde{\sigma}$ which means that the integral in (4.4) is finite. By choosing $\rho > 0$ small enough we get $I_1 < \frac{\delta}{3}$.

Second summand: To estimate I_2 we use the fact that, since $\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})$ is continuously differentiable, one can approximate $\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(x)$ by the first summand of its Taylor series. By the mean value theorem we see that for $x, \xi \in \mathbb{R}^d$ there exists a $\theta \in (0,1)$, such that

$$\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(x) = \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\xi) + (\nabla \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)}))(\xi + \theta(x - \xi)) \cdot (x - \xi).$$

This leads to

$$\begin{aligned} &\left|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\tau) - \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(A_{a''}^{-1} S_{s''}^{-1}(t - t''))\right| \\ &\qquad = \left|(\nabla \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\tau + \theta(A_{a''}^{-1} S_{s''}^{-1}(t - t'') - \tau))) \cdot (A_{a''}^{-1} S_{s''}^{-1}(t - t'') - \tau)\right| \\ &\qquad \leq \left\|\nabla \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T \cdot)})(\tau + \theta(A_{a''}^{-1} S_{s''}^{-1}(t - t'') - \tau))\right\|_2 \cdot \left\|A_{a''}^{-1} S_{s''}^{-1}(t - t'') - \tau\right\|_2. \end{aligned}$$

By (4.3), and since (4.5) also holds for the partial derivatives $\frac{\partial}{\partial x_j}\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_{\tilde{\eta}}S_{\tilde{\sigma}}^T\cdot)})(x)$ for all N with $|N| \leq d+2$, we obtain

$$\begin{aligned}
&\big|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(\tau) - \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(A_{a''}^{-1}S_{s''}^{-1}(t-t''))\big| \\
&\qquad\lesssim \frac{\big\|A_{a''}^{-1}S_{s''}^{-1}(t-t'')-\tau\big\|_2}{(1+\big\|\tau+\theta(A_{a''}^{-1}S_{s''}^{-1}(t-t'')-\tau)\big\|_2)^N} \\
&\qquad= \frac{\big\|(A_{u(u')^{-1}}^{-1}S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1}-I)\tau + A_{\varepsilon u}^{-1}S_{\beta v}^{-1}\gamma(w'-w)\big\|_2}{(1+\big\|((1-\theta)I+\theta A_{u(u')^{-1}}^{-1}S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1})\tau+\theta A_{\varepsilon u}^{-1}S_{\beta v}^{-1}\gamma(w'-w)\big\|_2)^N}.
\end{aligned}$$

Since $u(u')^{-1} \in (\alpha^{-1},\alpha)$, $(u')^{\frac{1}{d}-1}\beta(v-v') \in (-\alpha^{-1}\beta,\alpha^{-1}\beta)^{d-1}$, $\varepsilon u \in [-1,-\alpha^{-1})\cup(\alpha^{-1},1]$, $\beta v \in [-\frac{\beta}{2},\frac{\beta}{2})^{d-1}$ and $\gamma(w'-w) \in (-\gamma,\gamma)^d$ we have that $\lim\limits_{\alpha\to 1} A_{u(u')^{-1}}^{-1} = I$ and $\lim\limits_{\beta\to 0} S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1} = I$ (componentwise) so that for every $\rho > 0$, there exist $\alpha > 1$, $\beta,\gamma > 0$ small enough so that

$$\big\|A_{u(u')^{-1}}^{-1}S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1}-I\big\|_2 < \rho, \quad \big\|A_{\varepsilon u}^{-1}S_{\beta v}^{-1}\gamma(w'-w)\big\|_2 < \rho$$

and

$$\big\|((1-\theta)I+\theta A_{u(u')^{-1}}^{-1}S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1})\tau+\theta A_{\varepsilon u}^{-1}S_{\beta v}^{-1}\gamma(w'-w)\big\|_2 > \|\tau\|_2 - \frac{1}{2}.$$

This yields

$$\begin{aligned}
&\big|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(\tau) - \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(A_{a''}^{-1}S_{s''}^{-1}(t-t''))\big| \\
&\qquad= \frac{\big\|(A_{u(u')^{-1}}^{-1}S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1}-I)\tau + A_{\varepsilon u}^{-1}S_{\beta v}^{-1}\gamma(w'-w)\big\|_2}{(1+\big\|((1-\theta)I+\theta A_{u(u')^{-1}}^{-1}S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1})\tau+\theta A_{\varepsilon u}^{-1}S_{\beta v}^{-1}\gamma(w'-w)\big\|_2)^N} \\
&\qquad\leq \frac{\big\|(A_{u(u')^{-1}}^{-1}S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1}-I)\big\|_2\|\tau\|_2 + \big\|A_{\varepsilon u}^{-1}S_{\beta v}^{-1}\gamma(w'-w)\big\|_2}{(1+\big\|((1-\theta)I+\theta A_{u(u')^{-1}}^{-1}S_{(u')^{\frac{1}{d}-1}\beta(v-v')}^{-1})\tau+\theta A_{\varepsilon u}^{-1}S_{\beta v}^{-1}\gamma(w'-w)\big\|_2)^N} \\
&\qquad< \rho \quad \frac{1+\|\tau\|_2}{(\frac{1}{2}+\|\tau\|_2)^N}.
\end{aligned}$$

With this at hand, we can estimate

$$\begin{aligned}
I_2 &\lesssim \operatorname*{ess\,sup}_{(s,t)} \int_{\mathbb{R}^d}\int_{Q_{\tilde{b}}}\int_{\tilde{I}} \sup_{z'\in W_{\tilde{y}}} |\det A_{(a'')^{-1}}|^{\frac{1}{2}}\big|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(\tau) \\
&\qquad\qquad - \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(A_{a''}^{-1}S_{s''}^{-1}(t-t''))\big| \frac{\mathrm{d}\eta}{|\eta|}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&< \rho \operatorname*{ess\,sup}_{(s,t)} \int_{\mathbb{R}^d}\int_{Q_{\tilde{b}}}\int_{\tilde{I}} \sup_{z'\in W_{\tilde{y}}} |\det A_{(a'')^{-1}}|^{\frac{1}{2}} \frac{1+\|\tau\|_2}{(\frac{1}{2}+\|\tau\|_2)^N}\frac{\mathrm{d}\eta}{|\eta|}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&= \rho \operatorname*{ess\,sup}_{(s,t)} \int_{\mathbb{R}^d}\int_{Q_{\tilde{b}}}\int_{\tilde{I}} \sup_{z'\in W_{\tilde{y}}} |a''|^{\frac{1}{2d}-1}|\eta|^{-1}\frac{1+\|\tau\|_2}{(\frac{1}{2}+\|\tau\|_2)^N}\,\mathrm{d}\eta\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&= \rho \operatorname*{ess\,sup}_{(s,t)} \int_{\mathbb{R}^d}\int_{Q_{\tilde{b}}}\int_{\tilde{I}} \sup_{z'\in W_{\tilde{y}}} |a''|^{\frac{1}{2d}}|(a''\eta)^{-1}|\frac{1+\|\tau\|_2}{(\frac{1}{2}+\|\tau\|_2)^N}\,\mathrm{d}\eta\,\mathrm{d}\sigma\,\mathrm{d}\tau
\end{aligned}$$

$$\lesssim \rho \int\limits_{\mathbb{R}^d} \frac{1+\|\tau\|_2}{(\frac{1}{2}+\|\tau\|_2)^N}\,\mathrm{d}\tau = \rho\Bigg(\frac{1}{2}\int\limits_{\mathbb{R}^d}\frac{1}{(\frac{1}{2}+\|\tau\|_2)^N}\,\mathrm{d}\tau + \int\limits_{\mathbb{R}^d}\frac{1}{(\frac{1}{2}+\|\tau\|_2)^{N-1}}\,\mathrm{d}\tau\Bigg).$$

The integrals on the right hand side are finite for $N \geq d+2$. Again, choosing ρ small enough, I_2 can be estimated by $\frac{\delta}{3}$.

Third summand: For the last part we need to prove the estimate

$$I_3 \lesssim \operatorname*{ess\,sup}_{(s,t)} \int\limits_{\mathbb{R}^d}\int\limits_{Q_{\tilde{b}}}\int\limits_{\tilde{I}} \sup_{z'\in W_{\tilde{y}}} |\det A_{(a'')^{-1}}|^{\frac{1}{2}} \big|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(A_{a''}^{-1}S_{s''}^{-1}(t-t''))$$
$$- \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\cdot)})(A_{a''}^{-1}S_{s''}^{-1}(t-t''))\big| \frac{\mathrm{d}\eta}{|\eta|}\,\mathrm{d}\sigma\,\mathrm{d}\tau < \frac{\delta}{3}.$$

By the definition of the inverse Fourier transform,

$$\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(x) - \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\cdot)})(x)$$
$$= \int_{\mathbb{R}^d} \hat{\Psi}(\xi)(\overline{\hat{\Phi}(A_\eta S_\sigma^T\xi) - \hat{\Phi}(A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\xi)})e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi.$$

The submultiplicativity of the standard matrix norm yields

$$\begin{aligned}\big\|A_\eta S_\sigma^T\xi - A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\xi\big\|_2 &\leq \big\|A_\eta S_\sigma^T - A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\big\|_2\,\|\xi\|_2\\ &= \big\|A_\eta(S^T_{\sigma-|a''|^{\frac{1}{d}-1}(s-s'')} - A_{(a''\eta)^{-1}})S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\big\|_2\,\|\xi\|_2\\ &\leq \|A_\eta\|_2\,\big\|S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\big\|_2\,\|\xi\|_2\,\big\|S^T_{\sigma-|a''|^{\frac{1}{d}-1}(s-s'')} - A_{(a''\eta)^{-1}}\big\|_2.\end{aligned}$$

Since $\hat{\Psi}$ has compact support and η and σ can be restricted to compact sets we can estimate

$$\|A_\eta\|_2\big\|S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\big\|_2\|\xi\|_2 < C$$

with some constant $C<\infty$ and together with (4.1) and (4.2) we have

$$\big\|A_\eta S_\sigma^T\xi - A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\xi\big\|_2 \leq C\big\|S^T_{(1-|u'u^{-1}|^{1-\frac{1}{d}})\sigma-|u|^{\frac{1}{d}-1}\beta(v'-v)} - A_{u'u^{-1}}\big\|_2. \tag{4.6}$$

This can be made arbitrarily small by choosing α and β sufficiently close to one, respectively zero. Integration by parts yields

$$\begin{aligned}&|x_j|^N\big|\mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot)})(x) - \mathcal{F}^{-1}(\hat{\Psi}\overline{\hat{\Phi}(A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\cdot)})(x)\big|\\ &\quad= \Big|x_j^N\int_{\mathbb{R}^d}\hat{\Psi}(\xi)(\overline{\hat{\Phi}(A_\eta S_\sigma^T\xi) - \hat{\Phi}(A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\xi)})e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi\Big|\\ &\quad\leq (2\pi)^{-N}\int_{\mathbb{R}^d}\Big|\frac{\partial^N}{\partial\xi_j^N}(\hat{\Psi}(\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot) - \hat{\Phi}(A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\cdot)}))(\xi)\Big|\,\mathrm{d}\xi\\ &\quad= (2\pi)^{-N}\int_{\mathbb{R}^d}\Big|\sum_{i\leq N}\binom{N}{i}\frac{\partial^i}{\partial\xi_j^i}\hat{\Psi}(\xi)\frac{\partial^{N-i}}{\partial\xi_j^{N-i}}(\overline{\hat{\Phi}(A_\eta S_\sigma^T\cdot) - \hat{\Phi}(A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\cdot)})(\xi)\Big|\,\mathrm{d}\xi\\ &\quad\leq (2\pi)^{-N}\sum_{i\leq N}\binom{N}{i}\int_{\mathbb{R}^d}\Big|\frac{\partial^i\hat{\Psi}}{\partial\xi_j^i}(\xi)\Big|\,\Big|\det A_\eta\frac{\partial^{N-i}\hat{\Phi}}{\partial\xi_j^{N-i}}(A_\eta S_\sigma^T\xi)\\ &\qquad\qquad - \det A_{(a'')^{-1}}\frac{\partial^{N-i}\hat{\Phi}}{\partial\xi_j^{N-i}}(A_{(a'')^{-1}}S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\xi)\Big|\,\mathrm{d}\xi\end{aligned}$$

$$= (2\pi)^{-N} |\det A_\eta| \sum_{i\le N} \binom{N}{i} \int_{\mathbb{R}^d} \Big|\frac{\partial^i \hat\Psi}{\partial \xi_j^i}(\xi)\Big| \, \Big|\frac{\partial^{N-i}\hat\Phi}{\partial \xi_j^{N-i}}(A_\eta S_\sigma^T \xi)$$
$$- (u'u^{-1})^{2-\frac{1}{d}} \frac{\partial^{N-i}\hat\Phi}{\partial \xi_j^{N-i}}(A_{(a'')^{-1}} S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\xi)\Big| \, \mathrm{d}\xi$$

Since the partial derivatives of $\hat\Phi$ are continuous for all $N \le d+3$ and since $u'u^{-1} \in (\alpha^{-1}, \alpha)$, using (4.6) we can choose α and β small enough so that for a given $\rho > 0$ we get

$$\Big|\frac{\partial^{N-i}\hat\Phi}{\partial \xi_j^{N-i}}(A_\eta S_\sigma^T \xi) - (u'u^{-1})^{2-\frac{1}{d}} \frac{\partial^{N-i}\hat\Phi}{\partial \xi_j^{N-i}}(A_{(a'')^{-1}} S^T_{|a''|^{\frac{1}{d}-1}(s-s'')}\xi)\Big| < \rho$$

which leads to

$$|x_j|^N \big|\mathcal{F}^{-1}(\hat\Psi \overline{\hat\Phi(A_\eta S_\sigma^T \cdot)})(x) - \mathcal{F}^{-1}(\hat\Psi \overline{\hat\Phi(A_{(a'')^{-1}} S^T_{|a''|^{\frac{1}{d}-1}(s-s'')} \cdot)})(x)\big|$$
$$\le \rho \, (2\pi)^{-N} |\det A_\eta| \sum_{i\le N} \binom{N}{i} \int_{\mathbb{R}^d} \Big|\frac{\partial^i \hat\Psi}{\partial \xi_j^i}(\xi)\Big| \, \mathrm{d}\xi.$$

Thus, since $\hat\Psi \in \mathscr{C}_0^{d+3}$ the integrals in the estimation above are finite so that we obtain

$$\big|\mathcal{F}^{-1}(\hat\Psi \overline{\hat\Phi(A_\eta S_\sigma^T \cdot)})(x) - \mathcal{F}^{-1}(\hat\Psi \overline{\hat\Phi(A_{(a'')^{-1}} S^T_{|a''|^{\frac{1}{d}-1}(s-s'')} \cdot)})(x)\big| \le \rho \, C_\eta (1 + \|x\|)^{-N}$$

for all $N \in \mathbb{N}_0$ with $N \le d+3$. Now we have

$$I_3 \lesssim \operatorname*{ess\,sup}_{(s,t)} \int\limits_{\mathbb{R}^d} \int\limits_{Q_{\tilde b}} \int\limits_{\tilde I} \sup_{z' \in W_{\tilde y}} |\det A_{(a'')^{-1}}|^{\frac12} \big|\mathcal{F}^{-1}(\hat\Psi \overline{\hat\Phi(A_\eta S_\sigma^T \cdot)})(A_{a''}^{-1} S_{s''}^{-1}(t-t''))$$
$$- \mathcal{F}^{-1}(\hat\Psi \overline{\hat\Phi(A_{(a'')^{-1}} S^T_{|a''|^{\frac{1}{d}-1}(s-s'')} \cdot)})(A_{a''}^{-1} S_{s''}^{-1}(t-t''))\big| \frac{\mathrm{d}\eta}{|\eta|} \, \mathrm{d}\sigma \, \mathrm{d}\tau$$
$$\le \rho \operatorname*{ess\,sup}_{(s,t)} \int\limits_{\mathbb{R}^d} \int\limits_{Q_{\tilde b}} \int\limits_{\tilde I} C_\eta \sup_{z' \in W_{\tilde y}} |\det A_{(a'')^{-1}}|^{\frac12} (1 + \|A_{a''}^{-1} S_{s''}^{-1}(t-t'')\|)^{-N} \frac{\mathrm{d}\eta}{|\eta|} \, \mathrm{d}\sigma \, \mathrm{d}\tau$$
$$\le \rho \, C \operatorname*{ess\,sup}_{(s,t)} \int\limits_{\mathbb{R}^d} (\frac12 + \|\tau\|)^{-N} \, \mathrm{d}\tau.$$

The integral on the right hand side is finite for $N \ge d+1$, so that by choosing ρ small enough, we obtain the right estimate for I_3. The proof of

$$\operatorname*{ess\,sup}_{(a',s',t') \in X} \int_X |\mathrm{osc}_{\mathcal{U}}((a,s,t),(a',s',t'))| m_{v_{r,n}}((a,s,t),(a',s',t')) \, \mathrm{d}\mu(a,s,t) < \delta$$

is analogous. □

As stated earlier, we still need to give a proof of Lemma 3.8 which follows from Theorem 4.2 by using Lemma 2.1.

Proof of Lemma 3.8. Recall the assumptions in Lemma 2.1. Since we have already shown that $\mathcal{U}$ is a moderate admissible covering and it follows from the previous theorem that $\mathfrak{F}$ satisfies property $D[1, 1, L_{p,v_{r,n}}]$, it is left to show that $m_{v_{r,n}}$ is bounded in the sense that there exists a constant $C_{m_{v_{r,n}},\mathcal{U}}$ so that $\sup_{\lambda\in\Lambda} \sup_{x,y\in U_\lambda} m_{v_{r,n}}(x,y) \le C_{m_{v_{r,n}},\mathcal{U}}$ and that the condition $\|\chi_{U_\lambda} | L_{p,v_{r,n}}\| \gtrsim (\sup_{x\in U_\lambda} v_{r,n}(x))^{-1}$ is satisfied. Suppose that $\lambda = (\infty, k^0, m^0)$, $k^0 \in \mathbb{Z}^{d-1}$, $m^0 \in \mathbb{Z}^d$ and that $x, y \in U_\lambda$

with $x = (\infty, \beta(k^0+s_1), \gamma S_{\beta k^0}(m^0+t_1))$, $y = (\infty, \beta(k^0+s_2), \gamma S_{\beta k^0}(m^0+t_2))$, $s_1, s_2 \in [-\frac{1}{2}, \frac{1}{2})^{d-1}$, $t_1, t_2 \in [-\frac{1}{2}, \frac{1}{2})^d$. Then it holds that

$$\frac{v_{r,n}(x)}{v_{r,n}(y)} = \frac{(1+\|\beta(k^0+s_1)\|)^n}{(1+\|\beta(k^0+s_2)\|)^n} \lesssim \left(1 + \frac{\beta\|s_1-s_2\|}{1+\beta\|k^0+s_2\|}\right)^n \leq (1+\beta\|s_1-s_2\|)^n$$

which is bounded by a constant independent of x and y. An analogous estimate can be shown for the case $\lambda = (\varepsilon, j, k, m)$ with $\varepsilon \in \{-1, 1\}, j \in \mathbb{N}_0, k \in \mathbb{Z}^{d-1}, m \in \mathbb{Z}^d$. Concerning the second condition we see that $(\sup_{x\in U_\lambda} v_{r,n}(x))\|\chi_{U_\lambda}|L_{p,v_{r,n}}\| \geq \|\chi_{U_\lambda}|L_{p,v_{r,n}}\|$ since $v_{r,n}(x) \geq 1$ for all $x \in X$. For $\lambda = (\infty, k^0, m^0)$ we now get

$$\begin{aligned}
\|\chi_{U_\lambda}|L_{p,v_{r,n}}\|^p &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} |\chi_{U_\lambda}(\infty, s, t)|^p v_{r,n}(\infty, s)^p \,\mathrm{d}s\,\mathrm{d}t \\
&= \iint_{U_\lambda} v_{r,n}(\infty, s)^p \,\mathrm{d}s\,\mathrm{t} \\
&= \beta^{d-1}\gamma^d \int_{[-\frac{1}{2},\frac{1}{2})^{d-1}}\int_{[-\frac{1}{2},\frac{1}{2})^d} v_{r,n}(\infty, \beta(k^0+s'))^p \,\mathrm{d}s'\,\mathrm{d}t' \\
&\geq \beta^{d-1}\gamma^d \int_{[-\frac{1}{2},\frac{1}{2})^{d-1}}\int_{[-\frac{1}{2},\frac{1}{2})^d} 1 \,\mathrm{d}s'\,\mathrm{d}t'
\end{aligned}$$

which is a constant independent of λ. An analogous estimate can be made for the case $\lambda = (\varepsilon, j, k, m)$. This proves $(\sup_{x\in U_\lambda} v_{r,n}(x))\|\chi_{U_\lambda}|L_{p,v_{r,n}}\| \gtrsim 1$. □

Remark 5. We should note that, as already stated in the introduction, the assumptions on $\hat{\Phi}$ necessitate Φ having certain vanishing moments, thereby making the coorbit spaces $\mathcal{SC}^{r,n}_{\tilde{\mathfrak{F}},p}$ not entirely inhomogeneous in this sense.

We still have to show that there exist functions $\hat{\Phi}$ satisfying the assumptions of Theorem 4.2. Indeed we will show that we can find $\hat{\Psi}$ so that $\hat{\Phi} \in \mathscr{C}_0^\infty(\mathbb{R}^d)$.

Example 4.1. Let $\hat{\Psi}(\xi) := \hat{\psi_1}(\xi_1)\hat{\psi_2}(\xi_1^{-1}\tilde{\xi})$ with

$$\hat{\psi_1}(\xi_1) := \begin{cases} \sqrt{|\xi_1|}e^{\frac{1}{\xi_1{}^2-4\xi_1+3}}, & 1 < \xi_1 < 3 \\ \sqrt{|\xi_1|}e^{\frac{1}{\xi_1{}^2+4\xi_1+3}}, & -3 < \xi_1 < -1 \\ 0, & \text{otherwise} \end{cases}$$

and $\hat{\psi_2} \in \mathscr{C}_0^\infty(\mathbb{R}^{d-1})$ with $\|\hat{\psi_2}\|_{L_2} = 1$. Furthermore, we set

$$\hat{\varphi}_1(\xi_1) := \left(\int_{\mathbb{R}^{d-1}}\Big(\int_{-\infty}^{-|\xi_1|} + \int_{|\xi_1|}^{\infty}\Big)\frac{|\hat{\Psi}(\omega)|^2}{|\omega_1|^d}\,\mathrm{d}\omega\right)^{\frac{1}{2}} = \left(2\int_{|\xi_1|}^{3}\frac{|\hat{\psi_1}(\omega_1)|^2}{|\omega_1|}\,\mathrm{d}\omega_1\right)^{\frac{1}{2}} = \left(2\int_{|\xi_1|}^{3} e^{\frac{2}{\omega_1^2-4\omega_1+3}}\,\mathrm{d}\omega_1\right)^{\frac{1}{2}}$$

and $\hat{\varphi}_2 \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ with $\|\hat{\varphi}_2\|_{L_2} = 1$ and $D^\alpha\hat{\varphi}_2(0) = 0$ for all $\alpha \in \mathbb{N}_0^{d-1}$. Now we show that the function $\hat{\Phi}(\xi) := \hat{\varphi}_1(\xi_1)\hat{\varphi}_2(\xi_1^{-1}\tilde{\xi})$ satisfies the required assumptions. The fact that $\hat{\psi_1} \in \mathscr{C}_0^\infty(\mathbb{R})$ and therefore that $\hat{\Psi} \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ is immediately obvious. With the given construction, together with Remark 3, we see that the necessary condition from Theorem 3.3 is satisfied, i.e. the functions Φ and Ψ constitute a tight frame. We still want to show that $\hat{\Phi} \in \mathscr{C}_0^\infty(\mathbb{R}^d)$, which means that, given the assumptions on $\hat{\varphi}_2$, we need to show that $\hat{\varphi}_1$ is infinitely differentiable. To show this we need to prove that

$$\lim_{x\nearrow 3}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\hat{\varphi}_1(x)) = 0$$

for all $n \in \mathbb{N}$. For the remainder of this example we assume $2 < x < 3$. Since we have $\hat{\varphi_1}(x) = (f \circ g)(x)$ with $f(x) = \sqrt{x}$ and

$$g(x) = 2 \int_x^3 e^{\frac{2}{\omega^2 - 4\omega + 3}} \,\mathrm{d}\omega,$$

we can use Faà di Bruno's formula to get a closed expression for the n-th derivative. Recall that for two functions f and g the identity

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big((f \circ g)(x)\big) = \sum_{k=1}^{n} \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(g(x)) B_{n,k}\Big(\frac{\mathrm{d}g}{\mathrm{d}x}(x), \frac{\mathrm{d}^2 g}{\mathrm{d}x^2}(x), \ldots, \frac{\mathrm{d}^{(n-k+1)} g}{\mathrm{d}x^{(n-k+1)}}(x)\Big) \tag{4.7}$$

holds with $B_{n,k}$ being the Bell polynomials, i.e.

$$B_{n,k}(x_1, x_2, \ldots, x_{(n-k+1)}) = \sum \frac{n!}{j_1! \cdots j_{(n-k+1)}!} \Big(\frac{x_1}{1!}\Big)^{j_1} \cdots \Big(\frac{x_{(n-k+1)}}{(n-k+1)!}\Big)^{j_{(n-k+1)}}.$$

The sum in the above expression is taken over all $(j_1, \ldots, j_{(n-k+1)})$ with $j_1 + \cdots + j_{(n-k+1)} = k$ and $j_1 + 2j_2 + \cdots + (n-k+1)j_{(n-k+1)} = n$. The derivatives of the square root satisfy

$$\frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x) = c_k x^{-k+\frac{1}{2}}$$

with c_k being some constant and since we have

$$\frac{\mathrm{d}g}{\mathrm{d}x}(x) = -2e^{\frac{2}{x^2-4x+3}} \tag{4.8}$$

this means that for all $k \in \mathbb{N}$ the derivatives of g satisfy

$$\frac{\mathrm{d}^k g}{\mathrm{d}x^k}(x) = Q_k(x) e^{\frac{2}{x^2-4x+3}}$$

with Q_k being some rational function. Thus, using (4.7) we now have

$$\begin{aligned}
\frac{\mathrm{d}^n \hat{\varphi}_1}{\mathrm{d}x^n}(x) &= \sum_{k=1}^{n} c_k \big(g(x)\big)^{-k+\frac{1}{2}} \sum_{(j_1,\ldots,j_{(n-k+1)})} c_{n,k,j} \Big(Q_1(x) e^{\frac{2}{x^2-4x+3}}\Big)^{j_1} \cdots \Big(Q_{(n-k+1)}(x) e^{\frac{2}{x^2-4x+3}}\Big)^{j_{(n-k+1)}} \\
&= \sum_{k=1}^{n} Q(x) \big(g(x)\big)^{-k+\frac{1}{2}} \big(e^{\frac{2}{x^2-4x+3}}\big)^k \\
&= \sum_{k=1}^{n} \Bigg(\frac{Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}}}{g(x)}\Bigg)^{k-\frac{1}{2}}
\end{aligned}$$

where Q is a function that changes from line to line but is always a rational function. Since

$$\lim_{x \nearrow 3} Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}} = 0 \quad \text{and} \quad \lim_{x \nearrow 3} g(x) = 0$$

we use l'Hospital's rule to determine the limit of the fraction. For the derivative of the numerator we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big(Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}}\Big) &= \frac{\mathrm{d}}{\mathrm{d}x}Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}} + Q(x)\frac{\mathrm{d}}{\mathrm{d}x}\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}} \\
&= Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}}.
\end{aligned}$$

This, together with (4.8), yields

$$\lim_{x \nearrow 3} \frac{\frac{\mathrm{d}}{\mathrm{d}x}\Big(Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}}\Big)}{\frac{\mathrm{d}}{\mathrm{d}x}\big(g(x)\big)} = \lim_{x \nearrow 3} Q(x) e^{\frac{2}{(2k-1)(x^2-4x+3)}} = 0.$$

Thus, with l'Hospital's rule we get

$$\begin{aligned}\lim_{x\nearrow 3}\frac{\mathrm{d}^n\hat{\varphi}_1}{\mathrm{d}x^n}(x) &= \lim_{x\nearrow 3}\sum_{k=1}^{n}\left(\frac{Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}}}{g(x)}\right)^{k-\frac{1}{2}}\\ &= \sum_{k=1}^{n}\left(\lim_{x\nearrow 3}\frac{Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}}}{g(x)}\right)^{k-\frac{1}{2}}\\ &= \sum_{k=1}^{n}\left(\lim_{x\nearrow 3}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\Big(Q(x)\big(e^{\frac{2}{x^2-4x+3}}\big)^{1+\frac{1}{2k-1}}\Big)}{\frac{\mathrm{d}}{\mathrm{d}x}\big(g(x)\big)}\right)^{k-\frac{1}{2}} = 0.\end{aligned}$$

This proves that $\hat{\varphi}_1 \in \mathscr{C}_0^\infty(\mathbb{R})$ and therefore that $\hat{\Phi} \in \mathscr{C}_0^\infty(\mathbb{R}^d)$.

Using the abstract discretization result from Theorem 2.2, we can now state a discretization result for the shearlet coorbit spaces $\mathcal{SC}^{r,n}_{\mathfrak{F},p}$.

Theorem 4.3. *Let $p \geq 1$ and let $\Phi, \Psi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ be chosen as in Remark 3 so that the conditions of Theorem 3.7 and Theorem 4.2 are satisfied. Then $\alpha > 1$ and $\beta, \gamma > 0$ can be chosen small enough so that by choosing one element ψ_{x_λ} from every set U_λ of the moderate admissible covering given by Definition 4.1 the discretized set $\mathfrak{F}_d := \{\psi_{x_\lambda}\}_{\lambda\in\Lambda}$ is a Banach frame of the space $\mathcal{SC}^{r,n}_{\mathfrak{F},p}$ and yields an atomic decomposition of its elements, i.e. there exists a dual frame $\{e_\lambda\}_{\lambda\in\Lambda}$ so that the following statements hold:*

(i) $\|\{\langle f, \psi_{x_\lambda}\rangle\}_{\lambda\in\Lambda}|\ell_{p,v_{r,n}}(\Lambda)\| \sim \|f|\mathcal{SC}^{r,n}_{\mathfrak{F},p}\| \sim \|\{\langle f, e_\lambda\rangle\}_{\lambda\in\Lambda}|\ell_{p,v_{r,n}}(\Lambda)\|$

(ii) *The series*

$$f = \sum_{\lambda\in\Lambda}\langle f, e_\lambda\rangle\psi_{x_\lambda} = \sum_{\lambda\in\Lambda}\langle f, \psi_{x_\lambda}\rangle e_\lambda$$

converges unconditionally in the norm of $\mathcal{SC}^{r,n}_{\mathfrak{F},p}$ for $p < \infty$ and with weak- convergence in general.*

Proof. Obviously, the Lebesgue spaces $L_{p,v_{r,n}}(X, \mu)$ with $p \geq 1$ satisfy the condition (Y). Theorem 3.7 implies that $R_{\mathfrak{F}} \in \mathcal{A}_{m_{v_{r,n}}}$ which, as remarked earlier, together with the generalized Young inequality results in $R_{\mathfrak{F}} \in B_{L_{p,v_{r,n}},m_{v_{r,n}}}$. Since Lemma 3.8 yields the embedding of $R_{\mathfrak{F}}(L_{p,v_{r,n}})$ into $L_{\infty,v_{r,n}^{-1}}$, condition $(F_{v_{r,n},L_{p,v_{r,n}}})$ is satisfied. It remains to show that condition $D[\delta, m_{v_{r,n}}, L_{p,v_{r,n}}]$ is satisfied for a $\delta > 0$ satisfying the inequality

$$\delta \leq \big(1 + \|R_{\mathfrak{F}}|B_{L_{p,v_{r,n}},m_{v_{r,n}}}\|^2\big)^{\frac{1}{2}} - \|R_{\mathfrak{F}}|B_{L_{p,v_{r,n}},m_{v_{r,n}}}\|.$$

The boundedness of the weight function $m_{v_{r,n}}$ was already shown in the proof of Lemma 3.8. From Theorem 4.2 we get that for an arbitrary $\delta > 0$ we can choose $\alpha > 1$ and $\beta, \gamma > 0$ small enough so that $\|\mathrm{osc}_{\mathcal{U}}|\mathcal{A}_{m_{v_{r,n}}}\| < \delta$ and $\|\mathrm{osc}^*_{\mathcal{U}}|\mathcal{A}_{m_{v_{r,n}}}\| < \delta$. Again, using the generalized Young inequality this implies $\|\mathrm{osc}_{\mathcal{U}}|B_{L_{p,v_{r,n}},m_{v_{r,n}}}\| < \delta$ and $\|\mathrm{osc}^*_{\mathcal{U}}|B_{L_{p,v_{r,n}},m_{v_{r,n}}}\| < \delta$. ☐

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