

Besov regularity of solutions to the p -Poisson equation in the vicinity of a vertex of a polygonal domain ^{*}

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Abstract

We study the regularity of solutions to the p -Poisson equation, $1 < p < \infty$, in the vicinity of a vertex of a polygonal domain. In particular, we are interested in smoothness results in the adaptivity scale of Besov spaces $B_\tau^{\sigma_\tau}(L_\tau(\Omega))$, $1/\tau = \sigma_\tau/2 + 1/p$, since the regularity in this scale is known to determine the maximal approximation rate that can be achieved by adaptive and other nonlinear approximation methods. We prove that under quite mild assumptions on the right-hand side f , solutions to the p -Poisson equation possess Besov regularity σ_τ for all $0 < \sigma_\tau < 2$. In case f vanishes in a small neighborhood of the corner, the solutions even admit arbitrary high Besov smoothness. The proofs are based on singular expansion results and continuous embeddings of intersections of Babuska-Kondratiev spaces $\mathcal{K}_{p,a}^\ell(\Omega)$ and Besov spaces $B_p^s(L_p(\Omega))$ into the specific scale of Besov spaces we are interested in. In regard of these embeddings, we extend the existing results to the limit case $\ell \rightarrow \infty$ by showing that the Fréchet spaces $\cap_{\ell=1}^\infty \mathcal{K}_{p,a}^\ell(\Omega) \cap B_p^s(L_p(\Omega))$ are continuously embedded into the metrizable complete topological vector space $\cap_{\sigma_\tau > 0} B_\tau^{\sigma_\tau}(L_\tau(\Omega))$.

Keywords: p -Poisson equation, regularity of solutions, Besov spaces, nonlinear and adaptive approximation.

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1 Introduction

This paper is concerned with the Besov regularity of solutions to the p -Poisson equation

$$-\Delta_p(u) := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f \quad \text{in } \Omega, \quad (1)$$

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where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^2$ denotes some polygonal domain, in the vicinity of a corner $x_0 \in \partial\Omega$ of the domain. Thereby we investigate the regularity of solutions u to (1) in a small cone $\mathfrak{C} \subset \Omega$ with vertex x_0 , measured in the adaptivity scale $B_\tau^{\sigma_\tau}(L_\tau(\mathfrak{C}))$, $1/\tau = \sigma_\tau/2 + 1/p$, of Besov spaces.

The variational formulation of (1) is given by

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle dx = \int_{\Omega} f v dx \quad \text{for all } v \in C_0^\infty(\Omega).$$

Problems of this type arise in many applications, e.g., in non-Newtonian fluid theory [26], non-Newtonian filtering, turbulent flows of a gas in porous media, rheology, radiation of heat and many others. Moreover, the p -Laplacian has a similar model character for nonlinear problems as the ordinary Laplace equation for linear problems [27].

The numerical computation of solutions to (nonlinear elliptic) partial differential equations in practice often leads to linear systems of many unknowns. To increase the algorithmic efficiency, a natural and established approach is *adaptivity*. Essentially, adaptive algorithms are based on an updating strategy, where additional degrees of freedom are only spent in regions where the numerical approximation is still “far away” from the exact solution. Nevertheless, these schemes are hard to analyze and to implement, so that some theoretical foundations that justify the use of adaptive strategies are highly desirable.

The analysis in this paper is motivated by this problem, in particular in connection with adaptive wavelet algorithms. In this setting there exists a natural benchmark scheme - the best n -term wavelet approximation - in the sense that the best we can expect for an adaptive wavelet algorithm is that it (asymptotically) realizes the approximation order of this benchmark scheme. Thus, the use of adaptive wavelet methods is theoretically justified if the best n -term wavelet approximation realizes a significantly higher convergence order when compared to more conventional, uniform approximation schemes. In the wavelet setting, it is known that the convergence order of uniform schemes with respect to L_p depends on the regularity of the object one wants to approximate in the scale $W^s(L_p(\Omega))$ of L_p -Sobolev spaces, whereas the order of best n -term wavelet approximation in L_p depends on the regularity in the *adaptivity scale* $B_\tau^{\sigma_\tau}(L_\tau(\Omega))$, $1/\tau = \sigma_\tau/d + 1/p$, of Besov spaces [6, 10, 19]. Moreover, recently a similar relation was established in the context of finite element approximations [17]. Therefore, the use of adaptive algorithms for (1) would be justified if the Besov smoothness σ_τ of the solution in the adaptivity scale of Besov spaces is higher than its Sobolev regularity s .

In [7] Dahlke et al. gave a positive answer for the p -Poisson equation (1) in regard of this question, justifying the use of adaptive algorithms in many relevant cases. In particular for the setting of polygonal domains and homogeneous Dirichlet boundary conditions, they showed that for a large range of parameters the Besov smoothness σ_τ of solutions to (1) is much higher than their Sobolev smoothness s , which is (in general) limited to $3/2$ and

$1 + 1/p$ for $1 < p \leq 2$ and $2 < p < \infty$, respectively. E.g., for right-hand sides $f \in L_\infty(\Omega)$, Dahlke et al. proved that $\sigma^* = 2$ and $\sigma^* = 1 + 1/(p - 1)$ for $1 < p \leq 2$ and $2 < p < \infty$, respectively [7, Thm. 4.20]. However, their proofs are designed for general Lipschitz domains treating all boundary points as “equally bad”. On the other hand, for solutions of PDEs on polygonal domains, it is known that the critical singularities typically occur only near corners of the domain. Indeed, for nonnegative solutions of the p -Poisson equation on finite cones \mathfrak{C} , there exist singular expansion results with respect to the vertex [15, 32]. Essentially, (the derivatives of) the solution can be estimated by some power of the distance to the vertex. Hence, by exploiting these stronger (local) results, one might expect better Besov smoothness estimates on polygonal domains.

The purpose of this paper is to make the first step in improving some of the Besov regularity results derived in [7] for polygonal domains. Therefore, as outlined above, a natural first step is to investigate the regularity of solutions in the vicinity of the corners of the domain. We prove that under some local growth condition on the right-hand side, solutions to (1) possess Besov regularity $\sigma^* = 2$ in a small neighborhood of the vertices for the full range $1 < p < \infty$. Moreover, if $f = 0$ in \mathfrak{C} , we even obtain arbitrary high Besov regularity of the solution in the vicinity of the corner, again for all $1 < p < \infty$. We remark that - from a local point of view - these regularity results are indeed stronger than those of [7, Thm. 4.20] in several aspects.

Our proofs are based on a singular expansion of the solution u in the vicinity of a conical boundary point, as well as on embeddings of the type

$$\mathcal{K}_{p,a}^\ell(\Omega) \cap B_p^s(L_p(\Omega)) \hookrightarrow B_\tau^{\sigma_\tau}(L_\tau(\Omega)), \quad (2)$$

where $\mathcal{K}_{p,a}^\ell$ denote the Babuska-Kondratiev spaces and $1/\tau = \sigma_\tau/2 + 1/p$. In addition, we show that the topological vector spaces (TVSs) $H_a^{\infty,s}(L_p(\Omega)) := \cap_{\ell=1}^\infty \mathcal{K}_{p,a}^\ell(\Omega) \cap B_p^s(L_p(\Omega))$ and $B_{\text{NL}}^\infty(L_p(\Omega)) := \cap_{\sigma_\tau > 0} B_\tau^{\sigma_\tau}(L_\tau(\Omega))$ are metrizable and complete, and that $H_a^{\infty,s}(L_p(\Omega))$ is locally convex, i.e., a Fréchet space. Finally, we prove the continuity of the embedding $H_a^{\infty,s}(L_p(\Omega)) \hookrightarrow B_{\text{NL}}^\infty(L_p(\Omega))$ which can be seen as the limit case of (2) for $\ell \rightarrow \infty$.

This paper is organized as follows: In Section 2, we introduce all function spaces that will be used in the sequel. In Subsection 2.2, after introducing the general embedding (2), we treat the case $\ell \rightarrow \infty$ and prove the topological properties mentioned above (Theorem 2.10). After a detailed description of the p -Poisson problems treated in this paper, in Subsection 3.1 we first collect some results concerning the singular expansion of solutions to the p -Poisson equation on finite cones. Then, in Subsection 3.2 and 3.3, our main Besov regularity results are derived (Theorem 3.7 and Corollary 3.9, as well as Theorem 3.10 and Corollary 3.12, respectively). Finally, some auxiliary assertions which are needed in our proofs are contained in Appendix A.

Notation: For families $\{a_{\mathcal{J}}\}_{\mathcal{J}}$ and $\{b_{\mathcal{J}}\}_{\mathcal{J}}$ of nonnegative real numbers over a common index set we write $a_{\mathcal{J}} \lesssim b_{\mathcal{J}}$ if there exists a constant $c > 0$ (independent of the context-dependent parameters \mathcal{J}) such that

$$a_{\mathcal{J}} \leq c \cdot b_{\mathcal{J}}$$

holds uniformly in \mathcal{J} . Consequently, $a_{\mathcal{J}} \sim b_{\mathcal{J}}$ means $a_{\mathcal{J}} \lesssim b_{\mathcal{J}}$ and $b_{\mathcal{J}} \lesssim a_{\mathcal{J}}$.

2 Function spaces

In the first part of this section we fix some notation related to several types of function spaces that will be needed in what follows. Afterwards, we present the embedding which forms the basis of our regularity proofs in Section 3. In addition, here we investigate some topological properties of the spaces involved.

2.1 Basic definitions

Let $\Omega \subseteq \mathbb{R}^d$ denote either \mathbb{R}^d itself, or some bounded domain (open and connected set). By $C^\ell(\overline{\Omega})$, $\ell \in \mathbb{N}_0$, we denote the space of all $g: \Omega \rightarrow \mathbb{R}$ with (classical) derivatives $\partial^\nu g := \partial^{|\nu|} g / (\partial x_1^{\nu_1} \dots \partial x_d^{\nu_d})$ being uniformly continuous and bounded on Ω for every multi-index $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d$ with $0 \leq |\nu| \leq \ell$, endowed with the usual norm.

Given $0 < p \leq \infty$ the *Lebesgue spaces* $L_p(\Omega)$ consist of all (equivalence classes of real-valued) measurable functions g on Ω for which the usual L_p -(quasi-)norm is finite. Moreover, for $1 \leq p < \infty$ and $\ell \in \mathbb{N}_0$, let

$$W^\ell(L_p(\Omega)) := \left\{ g \in L_p(\Omega) \mid \|g\|_{W^\ell(L_p(\Omega))} := \sum_{|\nu| \leq \ell} \|D^\nu g\|_{L_p(\Omega)} < \infty \right\}$$

denote the classical *Sobolev spaces* on Ω , with D^ν being the weak partial derivatives of order $\nu \in \mathbb{N}_0^d$. We extend this definition in the usual way to fractional smoothness parameters $s = \ell + \beta \notin \mathbb{N}$ by adding the common *Sobolev-Slobodeckij* semi-norm:

$$\|g\|_{W^s(L_p(\Omega))} := \|g\|_{W^\ell(L_p(\Omega))} + |g|_{W^s(L_p(\Omega))}.$$

Furthermore, for $s > 0$ and $1 < p < \infty$, let us denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^s(L_p(\Omega))$ by $W_0^s(L_p(\Omega))$. Then by $W^{-s}(L_{p'}(\Omega))$ we denote the dual space of $W_0^s(L_p(\Omega))$, where p' is determined by the relation $1/p + 1/p' = 1$. Sobolev spaces $W^s(L_p(\partial\Omega))$ for $0 \leq s < 1$ and $1 \leq p < \infty$ on the boundary $\partial\Omega$ of bounded Lipschitz domains (see, e.g., [35, Def. 1.103] for a proper definition) can be defined likewise using the usual surface measure.

For a detailed discussion of (unweighted) Sobolev spaces on domains and their surfaces we refer to standard textbooks such as [1, 34] and the references given therein, as well as to [18, Ch. 1.3.3] and [23]. Let us only mention the following result which is crucial for the proper treatment of boundary value problems:

Proposition 2.1 ([18, Sect. 1.5]). *Let $\Omega \subset \mathbb{R}^d$ denote a bounded Lipschitz domain and let $1 < p < \infty$. Then the trace operator $\text{tr } g := g|_{\partial\Omega}$, $g \in C(\overline{\Omega})$, has a unique continuous extension to an operator from $W^1(L_p(\Omega))$ onto $W^{1-1/p}(L_p(\partial\Omega))$. This operator has a continuous right inverse E independent of p , i.e.,*

$$E : W^{1-1/p}(L_p(\partial\Omega)) \rightarrow W^1(L_p(\Omega)) \quad \text{with} \quad \text{tr} \circ E = \text{id}.$$

Furthermore, we have $g \in W_0^1(L_p(\Omega))$ if and only if $g \in W^1(L_p(\Omega))$ and $\text{tr } g = 0$.

Besides these classical function spaces, we will make use of two classes of *weighted Sobolev spaces*. For $\emptyset \neq S \subsetneq \overline{\Omega}$ let $\varrho = \varrho_S : \Omega \rightarrow [0, 1]$ denote the smooth distance to S , meaning that ϱ is a smooth function and in the vicinity of S it is equivalent to the distance to that set. Then, for $\ell \in \mathbb{N}_0$, $1 < p < \infty$, and $a \geq 0$, the *Babuska-Kondratiev spaces* $\mathcal{K}_{p,a}^\ell(\Omega)$ consist of all $g \in L_p(\Omega)$ for which the norm

$$\|g\|_{\mathcal{K}_{p,a}^\ell(\Omega)} := \left(\sum_{|\nu| \leq \ell} \int_{\Omega} |\varrho(x)^{|\nu|-a} D^\nu g(x)|^p \, dx \right)^{1/p}$$

is finite. For further reading, we refer to [3] and the references given therein. Our second, closely related class $\mathcal{W}_{p,\beta}^\ell(\Omega)$ was used, e.g., in [14]. For $\ell \in \mathbb{N}$, $1 < p < \infty$, and some weight parameter $\beta \in \mathbb{R}$ it is given by $\mathcal{W}_{p,\beta}^\ell(\Omega) := \{g : \Omega \rightarrow \mathbb{R} \mid \|g\|_{\mathcal{W}_{p,\beta}^\ell(\Omega)} < \infty\}$, where

$$\|g\|_{\mathcal{W}_{p,\beta}^\ell(\Omega)} := \left(\sum_{|\nu| \leq \ell} \|D^\nu(\varrho^\beta g)\|_{L_p(\Omega)}^p \right)^{1/p} + \|\varrho^{\beta-\ell} g\|_{L_p(\Omega)}.$$

In regularity theory for elliptic PDEs, the set S is typically chosen to be the so-called *singular set* of Ω , i.e., the collection of non-smooth boundary points. In the sequel, we will only be concerned with (non-degenerated) cones or polygonal domains in \mathbb{R}^2 and choose S to be the apex or the collection of vertices, respectively. We may then assume that $\varrho(\cdot) = \text{dist}(\cdot, S)$ in the vicinity of S .

Remark 2.2. For the range of parameters as stated above, the spaces $\mathcal{K}_{p,a}^\ell(\Omega)$ are Banach spaces, see [25, Thm. 1.11 & Rem. 4.10], and we have $\mathcal{K}_{p,a}^{\ell+1}(\Omega) \hookrightarrow \mathcal{K}_{p,a}^\ell(\Omega)$ with

$$\|\cdot\|_{\mathcal{K}_{p,a}^\ell(\Omega)} \leq \|\cdot\|_{\mathcal{K}_{p,a}^{\ell+1}(\Omega)}.$$

That is, the Babuska-Kondratiev spaces $\mathcal{K}_{p,a}^\ell(\Omega)$ are ordered w.r.t the smoothness index ℓ . Moreover, if $a \geq \ell \in \mathbb{N}_0$, then $\mathcal{K}_{p,a}^\ell(\Omega) \hookrightarrow W^\ell(L_p(\Omega))$ with

$$\left\| \cdot \right\|_{W^\ell(L_p(\Omega))} \lesssim \left\| \cdot \right\|_{\mathcal{K}_{p,a}^\ell(\Omega)}.$$

In particular, all $\mathcal{K}_{p,a}^\ell(\Omega)$ are continuously embedded into $L_p(\Omega)$.

Finally, yet another way to measure the smoothness of functions is provided by the framework of *Besov spaces* which essentially generalizes the concept of Sobolev spaces. Besov spaces can be defined in various ways which (for a large range of the parameters involved) lead to equivalent descriptions; cf. [4, 8, 34, 35]. For our purposes the approach based on *iterated differences* Δ_h^r seems to be the most reasonable one, since it provides an entirely *intrinsic* definition when dealing with Lipschitz domains. For $0 < p, q \leq \infty$ and $\sigma_p := d \max\{0, 1/p - 1\} < s < r \in \mathbb{N}$ the Besov space $B_q^s(L_p(\Omega))$ is defined as the collection of all $g \in L_p(\Omega)$ for which the semi-norm

$$|g|_{B_q^s(L_p(\Omega))} := \begin{cases} \left(\int_0^\infty \left[t^{-s} \sup_{h \in \mathbb{R}^d, |h| \leq t} \|\Delta_h^r(g, \cdot)\|_{L_p(\Omega_{r,h})} \right]^q \frac{dt}{t} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{t > 0} t^{-s} \sup_{h \in \mathbb{R}^d, |h| \leq t} \|\Delta_h^r(g, \cdot)\|_{L_p(\Omega_{r,h})} & \text{if } q = \infty, \end{cases} \quad (3)$$

is finite, where $\Omega_{r,h}$ denotes the set of all $x \in \Omega$ such that the whole line segment $[x, x + rh]$ belongs to Ω . We endow $B_q^s(L_p(\Omega))$ with the canonical (quasi-) norm

$$\left\| g \right\|_{B_q^s(L_p(\Omega))} = \|g\|_{L_p(\Omega)} + |g|_{B_q^s(L_p(\Omega))}.$$

Roughly speaking, with this term we can control all (weak) partial derivatives $D^\nu g$ up to the order s , measured in $L_p(\Omega)$. Since the influence of the additional *fine index* q is neglectable for many applications, we will mainly focus on the *smoothness parameter* s , as well as on the *integrability index* p , and simply set $q = p$ in what follows.

Remark 2.3. Some comments are in order:

- (i) We note that different choices of $r \geq \lfloor s \rfloor + 1$ in (3) lead to equivalent (quasi-)norms. The same is true when we restrict the range for t in (3) to the interval $(0, 1)$. The spaces are quasi-Banach spaces (Banach spaces if and only if $\min\{p, q\} \geq 1$).
- (ii) The scale of Besov spaces as defined above is well-studied. In particular, sharp assertions on embeddings, interpolation and duality properties, characterizations in terms of various building blocks (e.g., atoms, local means, quarks, or wavelets), and best n -term approximation results are known; see, e.g., [8, 10, 13, 20].

(iii) The demarcation line for embeddings of Besov spaces into $L_p(\Omega)$, $1 < p < \infty$, is given by

$$\frac{1}{\tau} = \frac{\sigma_\tau}{d} + \frac{1}{p}. \quad (4)$$

Every Besov space with smoothness and integrability indices corresponding to a point above that line is continuously embedded into $L_p(\Omega)$ (regardless of the fine index q). The points below this line never embed into $L_p(\Omega)$. For spaces $B_q^{\sigma_\tau}(L_\tau(\Omega))$ with (σ_τ, τ) that satisfy (4) some care is needed. However, if $q = \tau$, then the embedding still holds. Note that (4) also defines the so-called *adaptivity scale of Besov spaces* we are interested in and we have

$$B_{\tau_2}^{\sigma_{\tau_2}}(L_{\tau_2}(\Omega)) \hookrightarrow B_{\tau_1}^{\sigma_{\tau_1}}(L_{\tau_1}(\Omega)) \hookrightarrow L_p(\Omega)$$

if $0 < \sigma_{\tau_1} < \sigma_{\tau_2}$ and $1/\tau_i = \sigma_{\tau_i}/d + 1/p$, $i = 1, 2$.

(iv) Besov spaces are closely related to Sobolev spaces. Indeed, it has been shown that for bounded Lipschitz domains Ω , $1 \leq p < \infty$, and $0 < s \notin \mathbb{N}$ the space $B_p^s(L_p(\Omega))$ coincides with $W^s(L_p(\Omega))$ in the sense of equivalent norms; see, e.g., [13, Thm. 6.7]. Using the fact that $X^s(L_p(\Omega)) \hookrightarrow X^{s-\varepsilon}(L_p(\Omega))$ for $X \in \{B_p, W\}$ and arbitrary small $\varepsilon > 0$ we thus have

$$W^{s+\varepsilon}(L_p(\Omega)) \hookrightarrow B_p^s(L_p(\Omega)) \hookrightarrow W^{s-\varepsilon}(L_p(\Omega))$$

for all $1 \leq p < \infty$ and every $s \geq \varepsilon > 0$.

For further reading we refer to [5, 10, 11, 12, 13].

2.2 Embeddings and the topology of infinite intersections

The derivation of our Besov regularity results for solutions to the p -Poisson equation in Section 3 is based on embeddings from Babuska-Kondratiev spaces $\mathcal{K}_{p,a}^\ell(\Omega)$, intersected with some Besov space $B_p^s(L_p(\Omega))$, into Besov spaces $B_\tau^{\sigma_\tau}(L_\tau(\Omega))$, $1/\tau = \sigma_\tau/d + 1/p$, within the adaptivity scale. Such kind of embeddings - for slightly different scales of Besov spaces - have been shown by Hansen [19]. One of his results for polyhedral domains, specialized to the case $d = 2$, reads as follows.

Proposition 2.4 ([19, Thm. 3]). *Let $\Omega \subset \mathbb{R}^2$ be some bounded polygonal domain. Furthermore, let $\ell \in \mathbb{N}$, $1 < p < \infty$, $a > 0$, as well as $s > 0$, and set $1/\tau_* := \ell/2 + 1/p$. Then there exists some $\tau_0 \in (\tau_*, p]$, such that the chain of continuous embeddings*

$$\mathcal{K}_{p,a}^\ell(\Omega) \cap B_\infty^s(L_p(\Omega)) \hookrightarrow B_\infty^\ell(L_\tau(\Omega)) \hookrightarrow L_p(\Omega)$$

holds true for all $\tau \in (\tau_, \tau_0)$.*

Note that the smoothness ℓ of the Kondratiev space leads to Besov smoothness ℓ measured in the scale $B_\infty^\ell(L_\tau(\Omega))$, but excluding the case $1/\tau = \ell/2 + 1/p$ we are interested in. However, if we relax the smoothness parameter of the target space slightly to $\sigma < \ell$, standard embeddings show that the above statement still holds true with $B_\infty^\ell(L_\tau(\Omega))$ replaced by $B_\tau^{\sigma\tau}(L_\tau(\Omega))$, where $1/\tau = \sigma\tau/2 + 1/p$. Furthermore, by a similar argument we may certainly impose the slightly more restrictive condition $q = p$ for the fine index parameter on the left-hand side. For an explicit proof including all necessary references we refer to [21].

Corollary 2.5. *Let $\Omega \subset \mathbb{R}^2$ be some bounded polygonal domain. Moreover, let $\ell \in \mathbb{N}$, $1 < p < \infty$, as well as $a > 0$, and $s > 0$. Then for all*

$$0 < \sigma_\tau < \ell \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma_\tau}{2} + \frac{1}{p}$$

we have the continuous embedding

$$H_a^{\ell,s}(L_p(\Omega)) := \mathcal{K}_{p,a}^\ell(\Omega) \cap B_p^s(L_p(\Omega)) \hookrightarrow B_\tau^{\sigma_\tau}(L_\tau(\Omega)), \quad (5)$$

i.e., for all $u \in H_a^{\ell,s}(L_p(\Omega))$ it holds

$$\|u\|_{B_\tau^{\sigma_\tau}(L_\tau(\Omega))} \lesssim \|u\|_{H_a^{\ell,s}(L_p(\Omega))} := \|u\|_{\mathcal{K}_{p,a}^\ell(\Omega)} + \|u\|_{B_p^s(L_p(\Omega))}. \quad (6)$$

Remark 2.6.

- (i) In view of Remark 2.3(iv) the space $B_p^s(L_p(\Omega))$ in (5) and (6) can be replaced by $W^s(L_p(\Omega))$. Moreover, Remark 2.2 shows that under the additional assumption $a \geq 1$ we can even drop it completely, because then $\mathcal{K}_{p,a}^\ell(\Omega) \hookrightarrow \mathcal{K}_{p,a}^1(\Omega) \hookrightarrow W^1(L_p(\Omega))$.
- (ii) All stated assertions also hold true for the more general setting of Lipschitz domains with *polyhedral structure* [19, Sect. 2.3]. For a precise definition of such domains we refer to [9, 29]. The finite cones we will use in Section 3 (defined by (10)) are in particular included by this class of domains.

Due to the ordering of the Kondratiev spaces (see Remark 2.2) it is obvious that also the Banach spaces $H_a^{\ell,s}(L_p(\Omega))$ get smaller with increasing ℓ . In turn, by Corollary 2.5 we then gain more and more smoothness in the adaptivity scale of Besov spaces. Therefore, the question arises what happens to the embedding (5) if we let $\ell \rightarrow \infty$. To clarify this question, fix some bounded polygonal domain $\Omega \subset \mathbb{R}^2$ and let $a, s > 0$, as well as $1 < p < \infty$. Then we consider the vector space

$$H_a^{\infty,s}(L_p(\Omega)) := \bigcap_{\ell=1}^{\infty} H_a^{\ell,s}(L_p(\Omega))$$

endowed with the family $\mathcal{N} := \{n_\ell \mid \ell \in \mathbb{N}\}$ of norms $n_\ell := \|\cdot\|_{H_a^{\ell,s}(L_p(\Omega))}$ as defined in (6). Analogously, for $1 < p < \infty$, we define the vector space

$$B_{\text{NL}}^\infty(L_p(\Omega)) := \bigcap_{\sigma_\tau > 0} B_\tau^{\sigma_\tau}(L_\tau(\Omega)), \quad \text{where} \quad \frac{1}{\tau} = \frac{\sigma_\tau}{2} + \frac{1}{p}, \quad (7)$$

equipped with the family $\mathcal{Q} := \{q_{\sigma_\tau} \mid \sigma_\tau > 0\}$ of quasi-norms $q_{\sigma_\tau} := \|\cdot\|_{B_\tau^{\sigma_\tau}(L_\tau(\Omega))}$, where $1/\tau = \sigma_\tau/2 + 1/p$. In a first step, we investigate properties of the topologies generated by \mathcal{N} and \mathcal{Q} , respectively. At this point, we assume that the reader is familiar with general topological concepts such as local bases, metrizable and completeness of (locally convex) topological vector spaces ((LC)TVSs), as well as with linear continuous mappings between them. A detailed introduction to the topic can be found in the textbooks [22, 24, 31, 33].

Proposition 2.7. *The space $H_a^{\infty,s}(L_p(\Omega))$ equipped with the topology induced by \mathcal{N} is a Hausdorff LCTVS. It is metrizable and complete, i.e., $H_a^{\infty,s}(L_p(\Omega))$ is a Fréchet space.*

Proof. The topology induced by \mathcal{N} is generated by the local neighborhood basis

$$U_{L_0,r}(g) := \bigcap_{\ell \in L_0} V_{\ell,r}(g) \quad \text{with} \quad L_0 \subset \mathbb{N} \text{ finite, } r > 0, g \in H_a^{\infty,s}(L_p(\Omega)),$$

where

$$V_{\ell,r}(g) := \{f \in H_a^{\infty,s}(L_p(\Omega)) \mid n_\ell(f - g) < r\} = g + V_{\ell,r}(0).$$

Since all $n_\ell \in \mathcal{N}$ are actually norms, this topology is locally convex and Hausdorff [21].

To show metrizable, it is enough to verify the existence of a countable neighborhood basis of the origin [31, Ch. I, §6]. We claim that such a basis is given by the collection

$$V_{\ell,1/m}(0), \quad \ell, m \in \mathbb{N}. \quad (8)$$

To see this, let U denote an arbitrary open neighborhood of $0 \in H_a^{\infty,s}(L_p(\Omega))$. Then there exists a local basis element $U_{L_0,r}(0) \subseteq U$. Fixing $\ell \in \mathbb{N}$ with $\ell \geq \max\{\ell \mid \ell \in L_0\}$, it follows from (6) and Remark 2.2 that we can find some $m \in \mathbb{N}$ such that $V_{\ell,1/m}(0) \subseteq U_{L_0,r}(0)$. Hence,

$$0 \in V_{\ell,1/m}(0) \subseteq U_{L_0,r}(0) \subseteq U$$

for some $\ell, m \in \mathbb{N}$ depending on U , i.e., (8) indeed forms the desired neighborhood basis and $H_a^{\infty,s}(L_p(\Omega))$ is metrizable.

We are left with proving completeness. Thanks to its metrizable it suffices to show *sequential* completeness of $H_a^{\infty,s}(L_p(\Omega))$. So, let $(g_j)_{j \in \mathbb{N}} \subset H_a^{\infty,s}(L_p(\Omega))$ be an arbitrary

Cauchy sequence. At first, we show that for all $\ell \in \mathbb{N}$ the sequence $(g_j)_{j \in \mathbb{N}}$ admits a limit in $H_a^{\ell,s}(L_p(\Omega))$. To this end, fix $\ell \in \mathbb{N}$ and note that $H_a^{\infty,s}(L_p(\Omega)) \subset H_a^{\ell,s}(L_p(\Omega))$. Let U_ℓ denote some arbitrary neighborhood of the origin in $H_a^{\ell,s}(L_p(\Omega))$. Consequently, there exists $m \in \mathbb{N}$ such that the ball $B_{\ell,1/m}(0) := \{f \in H_a^{\ell,s}(L_p(\Omega)) \mid n_\ell(f) < 1/m\}$ satisfies $B_{\ell,1/m}(0) \subseteq U_\ell$. Then, clearly, $V_{\ell,1/m}(0) \subseteq B_{\ell,1/m}(0)$, and since $(g_j)_{j \in \mathbb{N}}$ is a Cauchy sequence in $H_a^{\infty,s}(L_p(\Omega))$, there exists $J \in \mathbb{N}$ such that

$$g_j - g_k \in V_{\ell,1/m}(0) \subseteq B_{\ell,1/m}(0) \subseteq U_\ell \quad \text{for all } j, k \geq J,$$

i.e., $(g_j)_{j \in \mathbb{N}}$ is also a Cauchy sequence in $H_a^{\ell,s}(L_p(\Omega))$. By standard arguments we know that $H_a^{\ell,s}(L_p(\Omega))$ is complete [4] so that there exists $g^\ell \in H_a^{\ell,s}(L_p(\Omega))$ with $g_j \rightarrow g^\ell$, as $j \rightarrow \infty$. That is, for all $\varepsilon > 0$ there exists $J(\varepsilon, \ell) \in \mathbb{N}$ such that

$$n_\ell(g_j - g^\ell) < \varepsilon \quad \text{for all } j \geq J(\varepsilon, \ell). \quad (9)$$

Now, since $\|\cdot\|_{L_p(\Omega)} \lesssim n_1(\cdot) \lesssim n_\ell(\cdot)$, for sufficiently large j and $\hat{g} := g^1$ it holds

$$\|\hat{g} - g^\ell\|_{L_p(\Omega)} \lesssim n_1(g^1 - g_j) + n_\ell(g^\ell - g_j) < 2\varepsilon,$$

and we conclude that $g^1 = g^\ell$ a.e. on Ω . As ℓ was arbitrary and $g^\ell \in H_a^{\ell,s}(L_p(\Omega))$, this shows $\hat{g} \in H_a^{\infty,s}(L_p(\Omega))$. In order to prove that $g_j \rightarrow \hat{g}$ in $H_a^{\infty,s}(L_p(\Omega))$ it now suffices to note that, for an arbitrary element $V_{\ell,1/m}(0)$ of the neighborhood basis of the origin in $H_a^{\infty,s}(L_p(\Omega))$, (9) and $\hat{g} = g^\ell$ imply $\hat{g} - g_j \in V_{\ell,1/m}(0)$. Thus, the proof is complete. ■

Let us turn to the space $B_{\text{NL}}^\infty(L_p(\Omega))$ as defined in (7) and note that for $\sigma_\tau > 2(p-1)/p$ it holds $\tau < 1$ such that then $q_{\sigma_\tau} = \|\cdot\|_{B_\tau^{\sigma_\tau}(L_\tau(\Omega))} \in \mathcal{Q}$ is only a *quasi*-norm. However, by essentially the same arguments as before, \mathcal{Q} induces a topology on $B_{\text{NL}}^\infty(L_p(\Omega))$ such that this space becomes a metrizable and complete Hausdorff TVS.

Proposition 2.8. *The space $B_{\text{NL}}^\infty(L_p(\Omega))$ equipped with the topology induced by \mathcal{Q} is a Hausdorff TVS. It is metrizable and complete.*

Proof. It is straightforward to show that a family of quasi-norms defined on a vector space turns this space into a Hausdorff TVS. For details we refer to [21] and Appendix A.2. Moreover, metrizability and completeness are shown literally like in the case of $H_a^{\infty,s}(L_p(\Omega))$, where we use Remark 2.3(iii) instead of Remark 2.2. ■

Remark 2.9. According to Banach metrizable and complete TVSs are called (F)-spaces.

From Corollary 2.5 we immediately conclude that $H_a^{\infty,s}(L_p(\Omega)) \subset B_{\text{NL}}^\infty(L_p(\Omega))$. Now, with respect to the topologies induced by the corresponding families of (quasi-) norms, this embedding is in fact continuous.

Theorem 2.10. *The Fréchet space $H_a^{\infty,s}(L_p(\Omega))$ is continuously embedded into the (F)-space $B_{\text{NL}}^\infty(L_p(\Omega))$, i.e.,*

$$H_a^{\infty,s}(L_p(\Omega)) \hookrightarrow B_{\text{NL}}^\infty(L_p(\Omega)).$$

Proof. Since $X := H_a^{\infty,s}(L_p(\Omega)) \subset B_{\text{NL}}^\infty(L_p(\Omega)) =: Y$, it suffices to prove that the linear map given by $\text{id}: X \rightarrow Y$ is continuous at the origin 0_X ; cf. [22, Ch. 2, §5, Prop. 1]. For this purpose, let U be an arbitrary neighborhood of 0_Y . Then there exists an element $V_{\ell,1/m}(0_Y) = \{f \in B_{\text{NL}}^\infty(L_p(\Omega)) \mid q_\ell(f) < 1/m\}$ of the (countable) local basis of 0_Y with $V_{\ell,1/m}(0_Y) \subseteq U$. Now Corollary 2.5 yields that $u \in V_{\ell+1,r}(0_X) = \{f \in H_a^{\infty,s}(L_p(\Omega)) \mid n_{\ell+1}(f) < r\}$ implies

$$q_\ell(u) \leq C n_{\ell+1}(u) < C r.$$

Thus, choosing $r < 1/(Cm)$ we have found a neighborhood $V := V_{\ell+1,r}(0_X)$ of 0_X which satisfies

$$\text{id}(V) \subset V_{\ell,1/m}(0_Y) \subseteq U.$$

Since U was arbitrary, the proof is complete. ■

Remark 2.11. More generally, standard arguments show that the continuity of linear maps between TVSs endowed with families of (quasi-/semi-) norms can be characterized in terms of (quasi-/semi-) norm estimates. We refer to [21] for details.

Remark 2.12. Note that by continuous embeddings of TVSs, convergence of a sequence in the source space implies convergence in the target space. E.g., each Cauchy sequence in $H_a^{\infty,s}(L_p(\Omega))$ is also a Cauchy sequence in $B_{\text{NL}}^\infty(L_p(\Omega))$, and since $H_a^{\infty,s}(L_p(\Omega))$ is complete (Proposition 2.7), it converges to an element in $B_{\text{NL}}^\infty(L_p(\Omega))$. This observation leads to the following approach to derive regularity assertions for solutions to general (nonlinear) PDEs: Assume that little is known about the exact solution u to some specific PDE in terms of classical (weighted) Hölder or Sobolev regularity, except that it is contained in $L_p(\Omega)$. Now, if an L_p -convergent approximation scheme is known, where the approximants u_n form a Cauchy sequence in some complete subspace $X \subset L_p$ (e.g., $X = H_a^{\infty,s}(L_p(\Omega))$) which is continuously embedded into some other function space Y of interest (e.g., $Y = B_{\text{NL}}^\infty(L_p(\Omega))$), then we know that the exact solution, i.e., the limit of u_n , is contained in Y . The advantage of this concept - compared to directly proving that $u \in Y$ - is that often the approximants u_n are solutions to linear (sub-) problems, for which in general much more is known in regard of regularity compared to the solution of the nonlinear problem itself. In this way, the embedding Theorem 2.10 provides a tool for such kind of regularity proofs.

However, the derivation of our Besov regularity results in the next section is not based on this approach. Instead, we will be able to apply the corresponding embeddings to solutions of the p -Poisson equation directly.

3 Besov regularity

In this section we study the Besov smoothness of solutions to the p -Poisson equation (1), measured in the adaptivity scale of Besov spaces, in the vicinity of a corner x_0 of a polygonal domain $\Omega \subset \mathbb{R}^2$.

To describe the scope of problems considered here, let us introduce the following notation first. For a function f (in Cartesian coordinates x) on Ω , by \tilde{f} we denote its representation in polar coordinates, i.e., $\tilde{f}(r, \phi) := f(\Psi^{-1}(r, \phi))$, where Ψ denotes the corresponding transformation of coordinates, see (34). When appropriate, we will omit the transformation Ψ or Ψ^{-1} , and just write f or \tilde{f} for the representation in Cartesian or polar coordinates, respectively. Analogously, the same applies to domains, i.e., $\tilde{\Omega} := \Psi(\Omega)$.

If $R_0 > 0$ is sufficiently small, then

$$\mathfrak{C}_{R_0}(x_0) := B_{R_0}(x_0) \cap \Omega$$

with $B_{R_0}(x_0)$ being the open Euclidean ball centered at x_0 with radius R_0 , is congruent to some cone

$$\tilde{C}(R_0, \omega) = \{(r, \phi) \in (0, R_0) \times (0, \omega)\} \quad (10)$$

of sidelength R_0 and inner angle $0 < \omega < 2\pi$. By $S_{R_0}(x_0)$ we denote the straight sides of $\partial\mathfrak{C}_{R_0}(x_0)$, i.e.,

$$S_{R_0}(x_0) := B_{R_0}(x_0) \cap \partial\Omega,$$

see Figure 1.

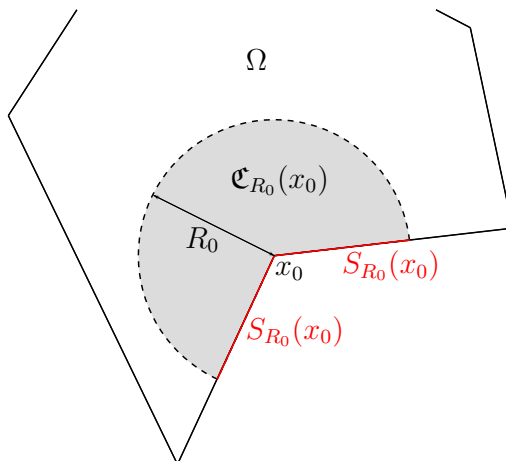


Figure 1: Polygonal domain Ω and cone $\mathfrak{C}_{R_0}(x_0)$ with sides $S_{R_0}(x_0)$.

Then, for $1 < p < \infty$, we consider the problem of finding $u \in W^1(L_p(\Omega))$ with

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{11}$$

We assume that

$$g \in W^{1-1/p}(L_p(\partial\Omega)) \quad \text{and} \quad f \in L_q(\Omega) \quad \text{with} \quad \begin{cases} \frac{2p}{3p-2} < q \leq \infty & \text{if } 1 < p \leq 2, \\ 1 \leq q \leq \infty & \text{if } 2 < p < \infty, \end{cases} \tag{12}$$

satisfy

$$f \geq 0 \quad \text{and} \quad g \geq 0 \quad \text{pointwise almost everywhere.} \tag{13}$$

Remark 3.1.

- (i) Problem (11)–(13) admits a uniquely determined weak solution $u \in W^1(L_p(\Omega))$. To see this, first note that (12) implies $L_q(\Omega) \hookrightarrow W^{-1}(L_{p'}(\Omega))$, see [16, Thm. 1.7]. Furthermore, by Proposition 2.1 there exists a continuous extension operator $E : W^{1-1/p}(L_p(\partial\Omega)) \rightarrow W^1(L_p(\Omega))$. Now, for $f \in W^{-1}(L_{p'}(\Omega))$ and Dirichlet boundary conditions given by $g \in W^1(L_p(\Omega))$ in the sense that $u - g \in W_0^1(L_p(\Omega))$, it is well-known that the p -Poisson equation indeed admits a unique weak solution [28, Ch. 2].
- (ii) Clearly, $q = 2$ would be a sufficient condition for (12) to hold true for all $1 < p < \infty$. Hence, for simplicity, one may impose the (less general) condition $f \in L_2(\Omega)$.
- (iii) Using the so-called *weak comparison principle* [14, Sect. 1] the assumption (13) assures that $u \geq 0$ in Ω . We treat only nonnegative solutions because the type of results - concerning the (growth-) behavior in the vicinity of a conical boundary point - needed for our proofs below are known only for solutions which are nonnegative in $\mathfrak{C}_{R_0}(x_0)$. Hence, all results remain valid if we drop assumption (13) and instead require the solution u to be nonnegative a.e. in $\mathfrak{C}_{R_0}(x_0)$.

In Subsection 3.1 we continue with collecting some known facts about the singular expansion of solutions to the p -Poisson equation in a cone. Afterwards, in Subsection 3.2 we treat problem (11)–(13) under the additional assumption that $f = 0$ in $\mathfrak{C}_{R_0}(x_0)$ and $g = 0$ on $S_{R_0}(x_0)$. Our main result here (Corollary 3.9) will show that then the solution u possesses *arbitrary high* Besov regularity in $\mathfrak{C}_R(x_0)$ for some $0 < R < R_0$. Finally, in Subsection 3.3 we prove that for the more general case of local growth conditions on f in $\mathfrak{C}_{R_0}(x_0)$ it holds $u \in B_\tau^{\sigma_\tau}(L_\tau(\mathfrak{C}_R(x_0)))$, $1/\tau = \sigma_\tau/2 + 1/p$, at least for all $0 < \sigma_\tau < 2$ (see Corollary 3.12).

3.1 Singular expansions

In the following, for notational convenience, we consider the unit cone $C(1, \omega) \subset \mathbb{R}^2$ with opening angle $0 < \omega < 2\pi$, see (10). However, note that all results are valid as well for cones $C(R_0, \omega)$ of arbitrary sidelength $R_0 > 0$. We consider the p -Laplace equation

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) &= 0 & \text{in } & C(1, \omega), \\ u &= 0 & \text{on } & S(1, \omega), \end{aligned} \quad (14)$$

where (in polar coordinates) $\tilde{S}(1, \omega) = [0, 1) \times \{0, \omega\}$. First of all, let us look for solutions of the form $u = s$ with

$$\tilde{s}(r, \phi) := r^\alpha t(\phi), \quad (r, \phi) \in \overline{\tilde{C}(1, \omega)}. \quad (15)$$

These solutions satisfy the nonlinear eigenvalue problem

$$\begin{aligned} \frac{\partial}{\partial \phi} \left\{ \left[\alpha^2 t(\phi)^2 + \left(\frac{\partial t}{\partial \phi}(\phi) \right)^2 \right]^{(p-2)/2} \frac{\partial t}{\partial \phi}(\phi) \right\} \\ + \alpha(\alpha(p-1) + 2 - p) \left[\alpha^2 t(\phi)^2 + \left(\frac{\partial t}{\partial \phi}(\phi) \right)^2 \right]^{(p-2)/2} t(\phi) &= 0, \quad \phi \in (0, \omega), \\ t(0) = t(\omega) &= 0. \end{aligned} \quad (16)$$

And, vice versa, for solutions $(\alpha, t(\cdot))$ of (16) $u = s$ given by (15) solves the primary problem (14), see [14, Eq. (1.5)]. The first positive eigenvalue α of (16) is given by the following lemma.

Lemma 3.2 ([15, Thm. 1], [32, Thm. 2.1.1 & Cor. 2.1]). *Let $1 < p < \infty$ and $\omega \in (0, 2\pi]$. Moreover, let*

$$\mu(\Gamma, p) := \frac{(\Gamma - 1)p - 2\Gamma}{2\Gamma(p - 1)}, \quad \text{where} \quad \Gamma(\omega) := \left(\frac{\omega}{\pi} - 1 \right)^2 - 1.$$

Then, for

$$\alpha := \alpha(\omega, p) := \begin{cases} \mu(\Gamma, p) + \sqrt{\mu(\Gamma, p)^2 + \Gamma^{-1}} & \text{if } \omega \leq \pi, \\ \mu(\Gamma, p) - \sqrt{\mu(\Gamma, p)^2 + \Gamma^{-1}} & \text{if } \pi \leq \omega < 2\pi, \\ (p-1)/p & \text{if } \omega = 2\pi, \end{cases} \quad (17)$$

there exists a solution $t(\cdot)$ of (16) with $t(\phi) > 0$ for all $\phi \in (0, \omega)$. Further, it holds

$$\alpha > \max\{0, (p-2)/(p-1)\} \quad (18)$$

and $t \in C^\infty((0, \omega))$, as well as $t \in C^k([0, \omega])$ for all $k \in \mathbb{N}$.

Remark 3.3. A short computation yields that for all $1 < p < \infty$ it holds

$$\begin{aligned} \alpha(\omega, p) > 1 &\iff 0 < \omega < \pi, \\ 0 < \alpha(\omega, p) < 1 &\iff \pi < \omega \leq 2\pi, \end{aligned}$$

and $\alpha(\omega, p) = 1$ if and only if $\omega = \pi$. Moreover, for $\omega \neq \pi$ it holds

$$\lim_{p \rightarrow 1} \alpha(\omega, p) = \begin{cases} \infty & \text{if } 0 < \omega < \pi, \\ 0 & \text{if } \pi < \omega \leq 2\pi, \end{cases} \quad \lim_{p \rightarrow \infty} \alpha(\omega, p) = \begin{cases} -1/\Gamma & \text{if } 0 < \omega < \pi, \\ 1 & \text{if } \pi < \omega \leq 2\pi. \end{cases}$$

Now the singular expansion result for solutions to (14) reads as follows.

Proposition 3.4 ([32, Thm. 1.3]). *For $1 < p < \infty$ and $\omega \in (0, 2\pi)$ let s be given by (15) with $(\alpha, t(\cdot))$ according to Lemma 3.2. Then for every nonnegative solution $u \in W^1(L_p(C(1, \omega)))$ to (14) there exists some $0 < R < 1$, such that either*

$$u = 0 \quad \text{in} \quad C(R, \omega),$$

or there exists $\kappa > 0$ such that for all $\nu \in \mathbb{N}_0^2$ it holds

$$D^\nu u(x) = \kappa D^\nu s(x) + w_\nu(x) \quad \text{for a.e. } x \in C(R, \omega),$$

with

$$|w_\nu(x)| < c_\nu |x|^{\alpha-|\nu|} \quad \text{for some } c_\nu > 0 \quad \text{and all } x \in C(R, \omega). \quad (19)$$

In regard of the inhomogeneous problem

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) &= f & \text{in} & \quad C(1, \omega), \\ u &= 0 & \text{on} & \quad S(1, \omega), \end{aligned} \quad (20)$$

we will use the following expansion result which can be derived from the proofs in [15]. For details we refer to [21].

Proposition 3.5. For $1 < p < \infty$ and $\omega \in (0, 2\pi)$ let s be given by (15) with $(\alpha, t(\cdot))$ according to Lemma 3.2. Assume that for some $c > 0$ and $\gamma > \gamma_0 := (\alpha - 1)(p - 1) - 1$ it holds

$$0 \leq f(x) \leq c|x|^\gamma \quad \text{for a.e. } x \in C(1, \omega). \quad (21)$$

Then for every nonnegative solution $u \in W^1(L_p(C(1, \omega)))$ to (20) there exists some $0 < R < 1$ such that either

$$u = 0 \quad \text{in } C(R, \omega),$$

or there exist constants $\kappa, c', \eta > 0$ such that u admits the singular expansion

$$u(x) = \kappa s(x) + w(x) \quad \text{for a.e. } x \in C(R, \omega) \quad (22)$$

with a remainder w that satisfies

$$|w(x)| \leq c'|x|^{\alpha+\eta} \quad \text{and} \quad |\nabla w(x)| \leq c'|x|^{\alpha-1+\eta} \quad \text{for all } x \in C(R, \omega),$$

as well as $w \in \mathcal{W}_{q,\beta}^2(C(R, \omega))$ for all $2 < q < \infty$ and some $\beta < 2 - (\alpha + 2/q)$.

Remark 3.6.

- (i) For re-entrant corners, i.e. $\omega > \pi$, it holds $\gamma_0 < -1$, because then $0 < \alpha < 1$, see Remark 3.3.
- (ii) Note that the local growth condition (21) implies $f \in L_q(C(1, \omega))$ for all

$$\begin{cases} 1 \leq q < -2/\gamma & \text{if } \gamma < 0, \\ 1 \leq q \leq \infty & \text{if } \gamma \geq 0. \end{cases} \quad (23)$$

In case $\gamma < 0$, we can use (18) to estimate

$$-\frac{2}{\gamma} > \frac{-2}{(\alpha - 1)(p - 1) - 1} > \begin{cases} 2/p & \text{if } 1 < p \leq 2, \\ 1 & \text{if } 2 < p < \infty. \end{cases}$$

Since $2/p \geq 2p/(3p - 2)$ for all $1 < p \leq 2$, we see that in all cases we can find some q with (23) which satisfies condition (12). Hence, by Remark 3.1(i) it particularly follows that problem (20), (21) admits at least one solution $u \in W^1(L_p(C(1, \omega)))$, e.g., for the boundary condition $u = 0$ on the whole boundary $\partial C(1, \omega)$.

3.2 Besov regularity: Locally vanishing source term

In order to discuss regularity properties of solutions to the p -Poisson equation (11)–(13) with right-hand sides f, g that vanish in the vicinity of a corner, we first consider the corresponding problem (14) on the unit cone. Our first Besov regularity assertion then reads as follows.

Theorem 3.7. *For $1 < p < \infty$ and $\omega \in (0, 2\pi)$ let $u \in W^1(L_p(C(1, \omega)))$ be a nonnegative solution of (14). Then there exists some $0 < R < 1$, such that*

$$u \in \mathcal{K}_{p,a}^\ell(C(R, \omega)) \quad \text{for all} \quad \ell \in \mathbb{N} \quad \text{and} \quad 0 \leq a < \alpha + \frac{2}{p}, \quad (24)$$

where $\alpha = \alpha(\omega, p) > \max\{0, (p-2)/(p-1)\}$ is specified by (17). Moreover, it holds $u \in B_{\text{NL}}^\infty(L_p(C(R, \omega)))$ and for all $0 < \sigma_\tau < \ell \in \mathbb{N}$ and $0 < a < \alpha + 2/p$ we have

$$\|u\|_{B_\tau^{\sigma_\tau}(L_\tau(C(R, \omega)))} \lesssim \|u\|_{\mathcal{K}_{p,a}^\ell(C(R, \omega))} + \|u\|_{W^1(L_p(C(1, \omega)))}, \quad \frac{1}{\tau} = \frac{\sigma_\tau}{2} + \frac{1}{p}.$$

Proof. Step 1. We show (24). W.l.o.g. we may assume that $u \neq 0$. Then from Proposition 3.4 we know that there exists some $0 < R < 1$ and $\kappa > 0$ such that $D^\nu u(x) = \kappa D^\nu s(x) + w_\nu(x)$ for a.e. $x \in C(R, \omega)$ and all $\nu \in \mathbb{N}_0^2$, where w_ν satisfies (19). Since $s(x) = \tilde{s}(\Psi_r(x), \Psi_\phi(x))$ with $\tilde{s}(r, \phi) = r^\alpha t(\phi)$, this means

$$D^\nu u(x) = \kappa D^\nu (|x|^\alpha t(\Psi_\phi(x))) + w_\nu(x). \quad (25)$$

Straightforward computations using Leibniz' rule for higher order (weak) partial derivatives and Lemma A.1 then show that for arbitrary $\nu \in \mathbb{N}_0^2$ with $|\nu| \leq \ell$ we can estimate

$$\begin{aligned} |D^\nu (|x|^\alpha t(\Psi_\phi(x)))| &\lesssim \sum_{\beta \leq \nu} |D^\beta (|x|^\alpha)| |D^{\nu-\beta} t(\Psi_\phi(x))| \\ &\lesssim \sum_{\beta \leq \nu} |x|^{\alpha-|\beta|} |D^{\nu-\beta} t(\Psi_\phi(x))|, \end{aligned} \quad (26)$$

see [21] for details. Furthermore, with the help of Lemma A.2 we find

$$\begin{aligned} |D^{\nu-\beta} t(\Psi_\phi(x))| &\leq \sum_{k=1}^{|\nu-\beta|} \sum_{j_1+j_2=|\nu-\beta|} |c_{\nu-\beta, k, j_1, j_2} t^{(k)}(\Psi_\phi(x))| \frac{|x|^{j_1+j_2}}{|x|^{2|\nu-\beta|}} \\ &\lesssim \|t\|_{C^{|\nu-\beta|}([0, \omega])} |x|^{-|\nu|+|\beta|}. \end{aligned} \quad (27)$$

Thus, the estimates (26), (27), and Lemma 3.2 yield

$$|D^\nu(|x|^\alpha t(\Psi_\phi(x)))| \lesssim \|t\|_{C^{|\nu|}([0, \omega])} \sum_{\beta \leq \nu} |x|^{\alpha-|\nu|} \lesssim |x|^{\alpha-|\nu|},$$

so that from (19) and (25) we conclude that

$$|D^\nu u(x)| \lesssim |x|^{\alpha-|\nu|} \quad \text{holds a.e. on } C(R, \omega). \quad (28)$$

Now let $\ell \in \mathbb{N}$ be arbitrarily fixed and decompose $C(R, \omega)$ into $U := C(\delta, \omega)$ and $U^c := C(R, \omega) \setminus C(\delta, \omega)$, where $\delta \in (0, R]$ is chosen such that $\varrho(x) \sim |x|$ on U . Then (28) and $\varrho(x) \sim 1$ on U^c imply

$$\begin{aligned} \|u\|_{\mathcal{K}_{p,a}^\ell(C(R, \omega))}^p &= \sum_{|\nu| \leq \ell} \int_{C(R, \omega)} |\varrho(x)^{|\nu|-a} D^\nu u(x)|^p \, dx \\ &\lesssim \sum_{|\nu| \leq \ell} \left(\int_{C(\delta, \omega)} |x|^{(\alpha-a)p} \, dx + \int_{C(R, \omega) \setminus C(\delta, \omega)} |x|^{(\alpha-|\nu|)p} \, dx \right) < \infty, \end{aligned} \quad (29)$$

provided that $(\alpha - a)p > -2$, i.e., if $a < \alpha + 2/p$.

Step 2. By Remark 2.6 the continuous embedding stated in Corollary 2.5 can be applied to $u \in \mathcal{K}_{p,a}^\ell(C(R, \omega)) \cap W^1(L_p(C(R, \omega)))$ which proves the claimed (quasi-) norm estimate. Finally, as ℓ was arbitrary, this also shows $u \in B_{\text{NL}}^\infty(L_p(\Omega))$. \blacksquare

Remark 3.8. Note that

- (i) the assertion of Theorem 3.7 particularly holds for the singular function $u = s$, since s is a (special) solution of (14).
- (ii) relation (24) implies $u \in W^\ell(L_p(C(R, \omega)))$, provided that $\alpha = \alpha(\omega, p) > \ell - 2/p = \ell - 2 + 2/p'$ which happens, e.g., for small values of p and small angles ω , see Remark 3.3 and Remark 2.2.

By means of an easy transformation of coordinates Theorem 3.7 implies the announced Besov regularity result for our primary global problem (11)–(13).

Corollary 3.9. *Let x_0 be a vertex of some bounded polygonal domain $\Omega \subset \mathbb{R}^2$. For $1 < p < \infty$ let $u \in W^1(L_p(\Omega))$ be the unique solution to (11)–(13). If, for some $R_0 > 0$, it holds $f = 0$ in $\mathfrak{C}_{R_0}(x_0) = B_{R_0}(x_0) \cap \Omega$ and $g = 0$ on $S_{R_0}(x_0) = B_{R_0}(x_0) \cap \partial\Omega$, then there exists some $0 < R < R_0$ such that*

$$u|_{\mathfrak{C}_R(x_0)} \in B_{\text{NL}}^\infty(L_p(\mathfrak{C}_R(x_0))).$$

Proof. Note that problem (11)–(13) and all our function spaces are invariant with respect to congruence transformations in the following sense. We can find a translation T , some rotation Q such that the cone of interest is mapped onto $C(R_0, \omega) = (Q \circ T)(\mathfrak{C}_{R_0}(x_0))$ with ω being the interior angle at the vertex x_0 . Then it is easily seen that u solves (11) if and only if $\hat{u} := u((Q \circ T)^{-1}(\cdot))$ solves the transformed problem on $\hat{\Omega} := (Q \circ T)(\Omega)$ with corresponding data \hat{f}, \hat{g} , and their respective (quasi-) norms of Sobolev, Besov and Kondratiev type coincide, see [21] for details. Moreover, we may rescale the problem such that $R_0 = 1$. Since $f, g \geq 0$, the unique solution u (and hence also \hat{u}) is nonnegative according to the weak comparison principle. Thus, we can apply Theorem 3.7 to \hat{u} giving locally unlimited Besov smoothness. Finally, an application of the respective inverse coordinate transformations then proves the assertion for u . \blacksquare

3.3 Besov regularity: The inhomogeneous equation

Here we treat the p -Poisson equation (11)–(13), where g locally vanishes and f satisfies a local growth condition in the vicinity of a vertex of the polygonal domain. Like in the previous subsection, at first we derive a corresponding regularity result on the unit cone.

Theorem 3.10. *For $1 < p < \infty$ and $0 < \omega < 2\pi$ let $\alpha > \max\{0, (p-2)/(p-1)\}$ be given by (17). Assume that for some $c > 0$ and $\gamma > \gamma_0 := (\alpha-1)(p-1) - 1$ it holds $0 \leq f(x) \leq c|x|^\gamma$ for a.e. $x \in C(1, \omega)$. Let $u \in W^1(L_p(C(1, \omega)))$ be a nonnegative solution of (20). Then there exists some $0 < R < 1$ such that*

$$u \in \mathcal{K}_{p,a}^2(C(R, \omega)) \quad \text{for all} \quad 0 \leq a < \alpha + \frac{2}{p}. \quad (30)$$

Moreover, it holds $u \in B_{\tau}^{\sigma\tau}(L_{\tau}(C(R, \omega)))$ for all $0 < \sigma_{\tau} < 2$ with $1/\tau = \sigma_{\tau}/2 + 1/p$, and

$$\|u\|_{B_{\tau}^{\sigma\tau}(L_{\tau}(C(R, \omega)))} \lesssim \|u\|_{\mathcal{K}_{p,a}^{\ell}(C(R, \omega))} + \|u\|_{W^1(L_p(C(1, \omega)))}$$

provided that $\sigma_{\tau} < \ell \in \{1, 2\}$ and $0 < a < \alpha + 2/p$.

Proof. Step 1. Again, we start with proving (30) and assume w.l.o.g. that $u \neq 0$. From Proposition 3.5 we know that u can be expanded into $u(x) = \kappa s(x) + w(x)$ with $\kappa > 0$ and a remainder w which satisfies a couple of properties. In regard of the singular part s , it holds

$$s \in \mathcal{K}_{p,a}^{\ell}(C(R, \omega)) \quad \text{for all} \quad \ell \in \mathbb{N} \quad \text{and} \quad 0 \leq a < \alpha + \frac{2}{p},$$

see Remark 3.8. Hence, it remains to show that $w \in \mathcal{K}_{p,a}^2(C(R, \omega))$.

Substep 1a. At first, let us establish

$$w \in \mathcal{K}_{q,a}^2(C(R,\omega)) \quad \text{for all} \quad q > 2, \quad 0 \leq a < \alpha + \frac{2}{q}.$$

To this end, we have to bound

$$\|w\|_{\mathcal{K}_{q,a}^2(C(R,\omega))}^q = \|w\|_{\mathcal{K}_{q,a}^1(C(R,\omega))}^q + \sum_{|\nu|=2} \int_{C(R,\omega)} |\varrho(x)^{2-a} D^\nu w(x)|^q dx. \quad (31)$$

Thanks to the bounds on $|w|$ and $|\nabla w|$ from Proposition 3.5 the first summand in (31) can be treated exactly like in Theorem 3.7, see (29), showing that

$$\|w\|_{\mathcal{K}_{q,a}^1(C(R,\omega))}^q < \infty. \quad (32)$$

In order to handle also the terms involving the second order derivatives of w , we will use that $w \in \mathcal{W}_{q,\beta}^2(C(R,\omega))$. Due to the properties of β and a we know that $\beta < 2 - (\alpha + 2/q) < 2 - a$, i.e., $2 - a - \beta > 0$. Together with Leibniz' rule and the fact that $0 \leq \varrho(x) \leq 1$ this yields that for fixed $\nu \in \mathbb{N}_0^2$ with $|\nu| = 2$ it holds

$$\begin{aligned} |\varrho(x)^{2-a} D^\nu w(x)| &= |\varrho(x)^{2-a-\beta}| |\varrho(x)^\beta D^\nu w(x)| \\ &= |\varrho(x)^{2-a-\beta}| \left| D^\nu(\varrho^\beta w)(x) - \sum_{0 \neq \mu \leq \nu} \binom{\nu}{\mu} D^\mu(\varrho^\beta)(x) D^{\nu-\mu} w(x) \right| \\ &\lesssim |D^\nu(\varrho^\beta w)(x)| + \sum_{0 \neq \mu \leq \nu} |\varrho(x)^{2-a-\beta} D^\mu(\varrho^\beta)(x) D^{\nu-\mu} w(x)| \end{aligned}$$

a.e. on $C(R,\omega)$. For $x \in C(R,\omega)$ with $|x| < \delta$ we can use that $\varrho(x) = |x|$ so that $|\varrho(x)^{2-a-\beta} D^\mu(\varrho^\beta)(x)| \lesssim \varrho(x)^{2-|\mu|-a}$, see Lemma A.1. However, due to properties of ϱ , this also holds if $|x| \geq \delta$. Therefore, on the whole of $C(R,\omega)$ we may estimate

$$|\varrho(x)^{2-a-\beta} D^\mu(\varrho^\beta)(x) D^{\nu-\mu} w(x)| \lesssim \begin{cases} \varrho(x)^{-a} |w(x)| & \text{if } \mu = \nu, \\ \varrho(x)^{|\nu-\mu|-a} |D^{\nu-\mu} w(x)| & \text{if } 0 \neq \mu \leq \nu. \end{cases}$$

Using (32), as well as

$$\|w\|_{\mathcal{W}_{q,\beta}^2(C(R,\omega))}^q \sim \int_{C(R,\omega)} |\varrho(x)^{\beta-2} w(x)|^q dx + \sum_{|\nu| \leq 2} \int_{C(R,\omega)} |D^\nu(\varrho^\beta w)(x)|^q dx,$$

this gives

$$\int_{C(R,\omega)} \left| \varrho(x)^{2-a} D^\nu w(x) \right|^q dx \lesssim \|w\| \mathcal{W}_{q,\beta}^2(C(R,\omega))\|^q + \|w\| \mathcal{K}_{q,a}^1(C(R,\omega))\|^q < \infty.$$

Hence, by (31) and (32), we indeed have $w \in \mathcal{K}_{q,a}^2(C(R,\omega))$ for $q > 2$ and $0 \leq a < \alpha + 2/q$.

Substep 1b. To conclude the proof of (30) it suffices to show that for $1 < p \leq 2 < q < \infty$, $\ell \in \mathbb{N}_0$, and all $a, \tilde{a} \geq 0$ with $a < \tilde{a} - 2/q + 2/p$ the embedding

$$\mathcal{K}_{q,\tilde{a}}^\ell(C(R,\omega)) \hookrightarrow \mathcal{K}_{p,a}^\ell(C(R,\omega)) \quad (33)$$

holds true. Indeed, if $p > 2$, we simply can take $q = p$ in Substep 1a. On the other hand, if $1 < p \leq 2$ and $0 \leq a < \alpha + 2/p$, we can choose any $q > 2$ and \tilde{a} with

$$\max\{0, a - 2/p + 2/q\} < \tilde{a} < \alpha + 2/q$$

to conclude that $u \in \mathcal{K}_{q,\tilde{a}}^2(C(R,\omega)) \subset \mathcal{K}_{p,a}^2(C(R,\omega))$ using Substep 1a and (33).

So let $1 < p \leq 2 < q < \infty$, $\ell \in \mathbb{N}_0$, and $a, \tilde{a} \geq 0$ with $a < \tilde{a} - 2/q + 2/p$ be given. Then Hölder's inequality applied with $r := q/p > 1$ and $r' = q/(q-p)$ shows

$$\begin{aligned} \|g\| \mathcal{K}_{p,a}^\ell(C(R,\omega))\|^p &= \sum_{|\nu| \leq \ell} \int_{C(R,\omega)} \varrho(x)^{(\tilde{a}-a)p} \left| \varrho(x)^{|\nu|-\tilde{a}} D^\nu g \right|^p dx \\ &\leq \sum_{|\nu| \leq \ell} \left[\int_{C(R,\omega)} \left(\varrho(x)^{(\tilde{a}-a)p} \right)^{q/(q-p)} dx \right]^{1-p/q} \left[\int_{C(R,\omega)} \left| \varrho(x)^{|\nu|-\tilde{a}} D^\nu g \right|^q dx \right]^{p/q}. \end{aligned}$$

Since $\varrho(x) \in [0, 1]$ equals $|x|$ for small x and is smooth otherwise, it only approaches zero in the vicinity of the origin. Thus, using the equivalence of $a < \tilde{a} - 2/q + 2/p$ and $(\tilde{a} - a)pq/(q-p) > -2$, we conclude that the first integral is bounded, such that

$$\|g\| \mathcal{K}_{p,a}^\ell(C(R,\omega))\|^p \lesssim \sum_{|\nu| \leq \ell} \left(\int_{C(R,\omega)} \left| \varrho(x)^{|\nu|-\tilde{a}} D^\nu g \right|^q dx \right)^{p/q} \sim \|g\| \mathcal{K}_{q,\tilde{a}}^\ell(C(R,\omega))\|^p,$$

as claimed.

Step 2. It remains to prove the membership of u in the Besov space $B_\tau^{\sigma\tau}(L_\tau(C(R,\omega)))$ together with the corresponding (quasi-) norm estimate. This can be done exactly like in Step 2 of the proof of Theorem 3.7. \blacksquare

Remark 3.11. We stress that (30) implies $u \in W^2(L_p(C(R,\omega)))$ provided that $\alpha(\omega, p)$ is larger than $2 - 2/p = 2/p'$ such that we can choose $a = 2$, see also Remark 3.8.

By the same arguments as before we can transfer the local smoothness estimate from the unit cone to our primary global problem (11)–(13) and obtain our second main Besov regularity result.

Corollary 3.12. *Let x_0 be a vertex of some bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with interior angle $0 < \omega < 2\pi$. For $1 < p < \infty$ let $u \in W^1(L_p(\Omega))$ be the unique solution to (11)–(13). Moreover, assume that for some $R_0 > 0$ there exists $c > 0$ and $\gamma > (\alpha(\omega, p) - 1)(p - 1) - 1$ with $\alpha(\omega, p) > 0$ given by (17) such that $0 \leq f(x) \leq c|x - x_0|^\gamma$ for a.e. $x \in \mathfrak{C}_{R_0}(x_0) = B_{R_0}(x_0) \cap \Omega$ and $g = 0$ on $S_{R_0}(x_0) = B_{R_0}(x_0) \cap \partial\Omega$. Then there exists $0 < R < R_0$ with*

$$u|_{\mathfrak{C}_R(x_0)} \in B_\tau^{\sigma_\tau}(L_\tau(\mathfrak{C}_R(x_0))) \quad \text{for all} \quad 0 < \sigma_\tau < 2, \quad \frac{1}{\tau} = \frac{\sigma_\tau}{2} + \frac{1}{p}.$$

Proof. The proof follows almost exactly the lines of the proof of Corollary 3.9, where here we apply Theorem 3.10 instead of Theorem 3.7. ■

Remark 3.13. In [15, p. 188] Dobrowolski notes that for the regular part w of the expansion (22) it holds $|\nabla^2 w(x)| \leq c|x - x_0|^{\alpha - 2 + \eta}$ etc. in a neighborhood of x_0 if the right-hand side f and $\partial\Omega \setminus \{x_0\}$ are sufficiently smooth. In this case, as can be seen from the proofs of Theorem 3.10 and Corollary 3.12, also the Besov regularity of the solution, measured in the adaptivity scale, would improve correspondingly.

A Appendix

A.1 Polar coordinates

Let us introduce the following notation. For $d = 2$, let Ψ denote the transformation of Cartesian coordinates to polar coordinates, i.e.,

$$\Psi: \mathbb{R}^2 \rightarrow [0, \infty) \times [0, 2\pi), \quad (x, y) \mapsto (r, \phi), \quad (34)$$

where $r := \Psi_r(x, y) := |(x, y)| = \sqrt{x^2 + y^2}$ and

$$\phi := \Psi_\phi(x, y) := \begin{cases} \arctan(y/x) & \text{if } x > 0, y \geq 0, \\ \arctan(y/x) + 2\pi & \text{if } x > 0, y < 0, \\ \arctan(y/x) + \pi & \text{if } x < 0, \\ \pi/2 & \text{if } x = 0, y > 0, \\ 3\pi/2 & \text{if } x = 0, y < 0, \\ 0 & \text{if } x = 0, y = 0. \end{cases}$$

Furthermore, we will need the derivative of Ψ ,

$$D\Psi(x, y) = \begin{pmatrix} \frac{\partial \Psi_r}{\partial x}(x, y) & \frac{\partial \Psi_r}{\partial y}(x, y) \\ \frac{\partial \Psi_\phi}{\partial x}(x, y) & \frac{\partial \Psi_\phi}{\partial y}(x, y) \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{pmatrix}, \quad (x, y) \in \mathbb{R}^2 \setminus \{0\}.$$

Now, for a function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, we denote its representation in polar coordinates by \tilde{u} , i.e.,

$$\tilde{u} : [0, \infty) \times [0, 2\pi) \rightarrow \mathbb{R}, \quad \tilde{u}(r, \phi) := u(\Psi^{-1}(r, \phi)).$$

Hence, $\tilde{u} \circ \Psi = u$.

Lemma A.1. *Let $r = r(x, y)$ as defined above and $\beta \in \mathbb{R}$. Then for all $\nu \in \mathbb{N}_0^2$ it holds*

$$\left| D^\nu(r^\beta)(x, y) \right| \lesssim r(x, y)^{\beta-|\nu|} \quad \text{for all } (x, y) \in \mathbb{R}^2 \setminus \{0\}.$$

Proof. The proof is based on induction and straightforward computations. We omit the details and refer to [21]. \blacksquare

Lemma A.2. *Let $1 < p < \infty$ and $0 < \omega < 2\pi$. Let $(\alpha, t(\cdot))$ be the solution of the eigenvalue problem (16) from Lemma 3.2. We set $T(x, y) := t(\Psi_\phi(x, y))$ for $(x, y) \in C(1, \omega)$. Then, for all $\nu \in \mathbb{N}_0^2$ there exist constants $c_{\nu, k, j_1, j_2} \in \mathbb{R}$, such that*

$$D^\nu T(x, y) = \sum_{k=1}^{|\nu|} \sum_{j_1+j_2=|\nu|} c_{\nu, k, j_1, j_2} t^{(k)}(\Psi_\phi(x, y)) \frac{x^{j_1} y^{j_2}}{r^{2|\nu|}}, \quad (x, y) \in C(1, \omega). \quad (35)$$

Proof. At first, since

$$\begin{aligned} \frac{\partial}{\partial x} (t(\Psi_\phi(x, y))) &= -t'(\Psi_\phi(x, y)) \frac{y}{r^2}, \\ \frac{\partial}{\partial y} (t(\Psi_\phi(x, y))) &= t'(\Psi_\phi(x, y)) \frac{x}{r^2}, \end{aligned}$$

we see that (35) holds true for $|\nu| \leq 1$. Now, let $\ell \in \mathbb{N}$ and assume that (35) holds for all $\nu \in \mathbb{N}_0^2$ with $|\nu| \leq \ell$. Let $\tilde{\nu} \in \mathbb{N}_0^2$ be arbitrary with $|\tilde{\nu}| = \ell + 1$. W.l.o.g. we assume that $D^{\tilde{\nu}} = \partial/\partial x \circ D^\nu$ for some $\nu \in \mathbb{N}_0^2$ with $|\nu| = \ell$. Then

$$\begin{aligned} D^{\tilde{\nu}} T(x, y) &= \frac{\partial}{\partial x} \left(\sum_{k=1}^{|\nu|} \sum_{j_1+j_2=|\nu|} c_{\nu, k, j_1, j_2} t^{(k)}(\Psi_\phi(x, y)) \frac{x^{j_1} y^{j_2}}{r^{2|\nu|}} \right) \\ &= \sum_{k=1}^{|\nu|} \sum_{j_1+j_2=|\nu|} c_{\nu, k, j_1, j_2} \left(-t^{(k+1)}(\Psi_\phi(x, y)) \frac{y}{r^2} \frac{x^{j_1} y^{j_2}}{r^{2|\nu|}} \right. \\ &\quad \left. + t^{(k)}(\Psi_\phi(x, y)) \frac{j_1 x^{j_1-1} y^{j_2} r^{2|\nu|} - 2|\nu| x^{j_1} y^{j_2} r^{2|\nu|-2} x}{r^{4|\nu|}} \right). \end{aligned}$$

With $j_1 x^{j_1-1} y^{j_2} r^{2|\nu|} = j_1 (x^{j_1+1} y^{j_2} + x^{j_1-1} y^{j_2+2}) r^{2|\nu|-2}$ we arrive at

$$\begin{aligned} D^{\tilde{\nu}} T(x, y) &= \sum_{k=1}^{|\nu|} \sum_{j_1+j_2=|\nu|} c_{\nu,k,j_1,j_2} \left(-t^{(k+1)}(\Psi_\phi(x, y)) \frac{x^{j_1} y^{j_2+1}}{r^{2(|\nu|+1)}} \right. \\ &\quad \left. + t^{(k)}(\Psi_\phi(x, y)) \frac{j_1 (x^{j_1+1} y^{j_2} + x^{j_1-1} y^{j_2+2}) - 2|\nu| x^{j_1+1} y^{j_2}}{r^{2(|\nu|+1)}} \right) \\ &= \sum_{k=1}^{|\tilde{\nu}|} \sum_{j_1+j_2=|\tilde{\nu}|} c_{\tilde{\nu},k,j_1,j_2} t^{(k)}(\Psi_\phi(x, y)) \frac{x^{j_1} y^{j_2}}{r^{2(|\tilde{\nu}|)}}, \end{aligned}$$

as claimed. ■

A.2 Topologies induced by families of semi-, quasi-, and p -norms

Here we show that, like in the case of semi-norms, any family of *quasi*-norms defined on a vector space turns this space into a Hausdorff TVS. For this purpose, let us recall some basic definitions. Let Y be a \mathbb{K} -vector space, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

A *quasi-norm* on Y is a mapping $q: Y \rightarrow [0, \infty)$ with the following properties:

- (i) If $q(y) = 0$, then $y = 0$,
- (ii) $q(\lambda y) = |\lambda| q(y)$ for all $y \in Y, \lambda \in \mathbb{K}$,
- (iii) There exists a constant $C > 0$ such that

$$q(y+z) \leq C(q(y) + q(z)) \quad \text{for all } y, z \in Y. \quad (36)$$

Note that (ii) implies that the constant C in (36) satisfies $C \geq 1$. Indeed from property (ii) it follows $q(0) = 0$ so that (iii) gives $q(y) = q(y+0) \leq C q(y)$ for all $y \in Y$. We set

$$C_Y := \min \{C \geq 1 \mid C \text{ satisfies (36) for all } y, z \in Y\}.$$

If $C_Y = 1$, then q is just a *norm*. Moreover, a mapping q which satisfies only (ii) and (iii) with $C_Y = 1$ is called *semi-norm*. A p -norm on Y , where $0 < p \leq 1$, is a mapping $\|\cdot\|_p: Y \rightarrow [0, \infty)$ such that

- (i) If $\|y\|_p = 0$, then $y = 0$,
- (ii) $\|\lambda y\|_p = |\lambda| \|y\|_p$ for all $y \in Y, \lambda \in \mathbb{K}$,

(iii) $\|y + z\|_p^p \leq \|y\|_p^p + \|z\|_p^p$ for all $y, z \in X$.

Of course, a p -norm with $p = 1$ is just a norm and it is easily seen that every p -norm is a quasi-norm. On the other hand, for every quasi-norm there is an equivalent p -norm.

Lemma A.3 (Aoki-Rolewicz [2, 30]). *Let (Y, q) be a quasi-normed vector space. Choose $p \in (0, 1]$ such that the constant of the quasi-triangle inequality (36) equals*

$$C = 2^{1/p-1}.$$

Then there exists a p -norm $\|\cdot\|_p$ on Y , which is equivalent to q . In detail, it holds

$$\|y\|_p \leq q(y) \leq 4^{1/p} \|y\|_p \quad \text{for all } y \in Y.$$

In a first step, we note that a family of quasi-norms indeed induces a topology.

Proposition A.4. *Let Y be a \mathbb{K} -vector space and let $\{q_j \mid j \in J\}$ be a family of quasi-norms on Y . For $j \in J$, $r > 0$, and $y \in Y$ we set*

$$V_{j,r}(y) := \{z \in Y \mid q_j(z - y) < r\} = y + V_{j,r}(0). \quad (37)$$

Then the sets

$$U_{J_0,r}(y) := \bigcap_{j \in J_0} V_{j,r}(y), \quad J_0 \subset J \text{ finite}, \quad r > 0, \quad y \in Y, \quad (38)$$

form a local basis on Y , and hence generate a topology \mathcal{O} on Y .

Proof. Let $y \in Y$. Clearly, $y \in U_{J_0,r}(y)$ for all finite $J_0 \subset J$ and $r > 0$. Next, let $U_{J_0,r_0}(y)$ and $U_{J_1,r_1}(y)$ be two arbitrary sets as defined in (38). With $J_2 := J_0 \cup J_1$ and $r_2 := \min\{r_0, r_1\}$ we have

$$U_{J_2,r_2}(y) = \bigcap_{j \in J_2} V_{j,r_2}(y) \subseteq \left(\bigcap_{j \in J_0} V_{j,r_0}(y) \right) \cap \left(\bigcap_{j \in J_1} V_{j,r_1}(y) \right) = U_{J_0,r_0}(y) \cap U_{J_1,r_1}(y).$$

So, the sets defined in (38) form a local basis on Y and thus generate a topology. ■

Note that the local basis elements $U_{J_0,r}(y)$ as defined in (38) are not necessarily open in the sense that they are contained in \mathcal{O} . Nevertheless, according to the next Lemma A.5, they always contain an open neighborhood of y , i.e., the sets $U_{J_0,r}(y)$ are in fact neighborhoods.

Lemma A.5. *Let the assumptions of Proposition A.4 hold. Then, for each of the sets $U_{J_0,r}(y)$ as defined in (38), there exists an open set U such that $y \in U$ and $U \subseteq U_{J_0,r}(y)$.*

Proof. It suffices to show that each of the sets $V_{j,r}(y)$ defined in (37) contains an open neighborhood V of y . Thus, in the following we fix $j \in J$, $r > 0$, and $y \in Y$. From Lemma A.3 we know that there exists a p -norm $\|\cdot\|_{p_j}$ which is equivalent to q_j . Based on this we define the balls

$$\tilde{V}_{j,r}(y) := \left\{ z \in Y \mid \|z - y\|_{p_j} < r \right\}.$$

Then, from the estimates stated in Lemma A.3, it follows that

$$\tilde{V}_{j,\varepsilon}(y) \subseteq V_{j,r}(y) \subseteq \tilde{V}_{j,r}(y) \quad \text{for all} \quad 0 < \varepsilon < 4^{-1/p_j} r. \quad (39)$$

Since obviously $y \in V := \tilde{V}_{j,\varepsilon}(y)$, it is therefore enough to prove that V is open, where $\varepsilon > 0$ is arbitrarily fixed. For this purpose, we show that each $x \in V$ is an inner point, i.e., there exists an element $U_{J_0,\mu}(x)$ of the local basis of x which is contained in V . Hence, fix $x \in V$ and set $\delta := \|x - y\|_{p_j} < \varepsilon$. Then $\mu^p := \varepsilon^p - \delta^p > 0$ and for each $z \in \tilde{V}_{j,\mu}(x)$ there holds

$$\|z - y\|_{p_j}^p \leq \|z - x\|_{p_j}^p + \|x - y\|_{p_j}^p < \mu^p + \delta^p = \varepsilon^p.$$

Thus $z \in \tilde{V}_{j,\varepsilon}(y)$ and therefore, by (39), $U_{\{j\},\mu}(x) = V_{j,\mu}(x) \subseteq \tilde{V}_{j,\mu}(x) \subseteq \tilde{V}_{j,\varepsilon}(y) = V$. ■

Proposition A.6. *The topology \mathcal{O} from Proposition A.4 is Hausdorff.*

Proof. Let $x, y \in Y$ with $x \neq y$. Moreover, for arbitrarily fixed $j \in J$ let $0 < r < \delta_j/(2C_j)$, where $\delta_j := q_j(x - y)$ and C_j denotes the constant of the quasi-triangle inequality of q_j . Then by Lemma A.5 the sets $U_{\{j\},r}(x) = V_{j,r}(x)$ and $U_{\{j\},r}(y) = V_{j,r}(y)$ describe disjoint neighborhoods of x and y . ■

Finally, it remains to check that (Y, \mathcal{O}) indeed forms a TVS, i.e., that the vector space operations are continuous with respect to \mathcal{O} .

Proposition A.7. *Under the assumptions of Proposition A.4 (Y, \mathcal{O}) is a TVS.*

Proof. Let α and β denote the vector addition and scalar multiplication on Y , respectively.

Step 1. We consider $\alpha : Y \times Y \rightarrow Y$ and prove continuity in $(x, y) \in Y \times Y$. Thus, we have to show that for all open neighborhoods $W(z)$ of $z := \alpha(x, y) = x + y \in Y$ there exists an open neighborhood $W(x, y)$ of $(x, y) \in Y \times Y$ such that $\alpha(W(x, y)) \subseteq W(z)$. Now fix (x, y) and $W(z)$. Since $W(z)$ is open, there exists a local basis element $U_{J_0,r}(z)$ of z with

$$U_{J_0,r}(z) \subseteq W(z).$$

For $j \in J_0$, let C_j denote the constant from the quasi-triangle inequality of q_j and set $\varepsilon_j := r/(2C_j)$. Then, for $(\tilde{x}, \tilde{y}) \in V_{j,\varepsilon_j}(x) \times V_{j,\varepsilon_j}(y)$ we have

$$q_j(\alpha(\tilde{x}, \tilde{y}) - z) = q_j(\tilde{x} + \tilde{y} - (x + y)) \leq C_j (q_j(\tilde{x} - x) + q_j(\tilde{y} - y)) < r.$$

Thus, $\alpha(\tilde{x}, \tilde{y}) \in V_{j,r}(z)$ and therefore $\alpha(V_{j,\varepsilon_j}(x) \times V_{j,\varepsilon_j}(y)) \subseteq V_{j,r}(z)$. Next, from Lemma A.5 we know that there exist open neighborhoods $W_j(x)$ and $W_j(y)$ of x and y , respectively, which are contained in $V_{j,\varepsilon_j}(x)$ and $V_{j,\varepsilon_j}(y)$, respectively. Hence, $W_j(x) \times W_j(y)$ is an open neighborhood of (x, y) with $\alpha(W_j(x) \times W_j(y)) \subseteq V_{j,r}(z)$. Setting

$$W(x, y) := \bigcap_{j \in J_0} W_j(x) \times W_j(y)$$

thus yields

$$\alpha(W(x, y)) \subseteq \bigcap_{j \in J_0} V_{j,r}(z) = U_{J_0,r}(z) \subseteq W(z).$$

Step 2. It remains to prove the continuity of $\beta: \mathbb{K} \times Y \rightarrow Y$ in $(\lambda, y) \in \mathbb{K} \times Y$, i.e., to show that for all open neighborhoods $W(z)$ of $z := \beta(\lambda, y) = \lambda y$ there exists an open neighborhood $W(\lambda, y)$ of (λ, y) such that $\alpha(W(\lambda, y)) \subseteq W(z)$. Let (λ, y) and $W(z)$ be fixed. Since $W(z)$ is open, there exists a local basis element $U_{J_0,r}(z)$ of z with $U_{J_0,r}(z) \subseteq W(z)$. For $j \in J_0$, let again C_j denote the constant from the quasi-triangle inequality of q_j and set

$$\varepsilon_j := \frac{r}{2C_j|\lambda|} \quad \text{and} \quad \delta_j := \frac{r}{2C_j^2(\varepsilon_j + q_j(y))}.$$

Then, for $\tilde{y} \in V_{j,\varepsilon_j}(y)$ and $\tilde{\lambda} \in W_j(\lambda) := \{\xi \in \mathbb{K} \mid |\lambda - \xi| < \delta_j\}$ we have

$$\begin{aligned} q_j(\beta(\tilde{\lambda}, \tilde{y}) - z) &= q_j(\lambda(\tilde{y} - y) + (\tilde{\lambda} - \lambda)\tilde{y}) \\ &\leq C_j (|\lambda| q_j(\tilde{y} - y) + |\tilde{\lambda} - \lambda| q_j(\tilde{y})) \\ &< C_j (|\lambda| \varepsilon_j + \delta_j q_j(\tilde{y} - y + y)) \\ &\leq \frac{r}{2} + C_j^2 \delta_j (q_j(\tilde{y} - y) + q_j(y)) \\ &< \frac{r}{2} + C_j^2 \delta_j (\varepsilon_j + q_j(y)) \\ &= r. \end{aligned}$$

Thus, $\beta(\tilde{\lambda}, \tilde{y}) \in V_{j,r}(z)$ and therefore $\beta(W_j(\lambda) \times V_{j,\varepsilon_j}(y)) \subseteq V_{j,r}(z)$. By Lemma A.5 there exists an open neighborhood $W_j(y)$ of y which is contained in $V_{j,\varepsilon_j}(y)$, i.e., $W_j(\lambda) \times W_j(y)$ is

an open neighborhood of (λ, y) with $\beta(W_j(\lambda) \times W_j(y)) \subseteq V_{j,r}(z)$. Setting

$$W(\lambda, y) := \bigcap_{j \in J_0} W_j(\lambda) \times W_j(y)$$

then yields

$$\beta(W(\lambda, y)) \subseteq \bigcap_{j \in J_0} V_{j,r}(z) = U_{J_0,r}(z) \subseteq W(z)$$

and completes the proof. ■

In conclusion, given a vector space Y and a family of quasi-norms $\{q_j \mid j \in J\}$ on Y , we have seen that the q_j induce a topology \mathcal{O} on Y which is Hausdorff, and that (Y, \mathcal{O}) is a topological vector space. But, in contrast to the case of seminorms, the resulting topology is not necessarily locally convex.

References

- [1] R. A. Adams. *Sobolev Spaces*. Academic Press, New York, 1975.
- [2] T. Aoki. Locally bounded linear topological spaces. *Proc. Imp. Acad.*, 18(10):588–594, 1942.
- [3] C. Bakuta, A. Mazzucato, V. Nistor, and L. Zikatanov. Interface and mixed boundary value problems on n -dimensional polyhedral domains. *Doc. Math.*, 15:687–745, 2010.
- [4] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction*, volume 223 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1976.
- [5] A. Cohen. *Numerical Analysis of Wavelet Methods*, volume 32 of *Studies in Mathematics and its Applications*. North-Holland, Amsterdam, 2003.
- [6] S. Dahlke, W. Dahmen, and R. DeVore. Nonlinear approximation and adaptive techniques for solving elliptic operator equations. In W. Dahmen, A. Kurdila, and P. Oswald, editors, *Multiscale Wavelet Methods for Partial Differential Equations*, pages 237–283. Academic Press, San Diego, 1997.
- [7] S. Dahlke, L. Diening, C. Hartmann, B. Scharf, and M. Weimar. Besov regularity of solutions to the p -poisson equation. *Nonlinear Anal.*, 130:298 – 329, 2016.
- [8] S. Dahlke, E. Novak, and W. Sickel. Optimal approximation of elliptic problems by linear and nonlinear mappings II. *J. Complexity*, 22:549–603, 2006.
- [9] M. Dauge. *Elliptic boundary value problems on corner domains*, volume 1341 of *Lecture Notes in Mathematics*. Springer, Berlin, 1988.

- [10] R. A. DeVore. Nonlinear approximation. *Acta Numer.*, 7:51–150, 1998.
- [11] R. A. DeVore and V. A. Popov. Interpolation of Besov spaces. *Trans. Amer. Math. Soc.*, 305(1):397–414, 1988.
- [12] R. A. DeVore and R. C. Sharpley. *Maximal Functions Measuring Smoothness*, volume 47(293) of *Mem. Am. Math. Soc.* AMS, Providence, RI, 1984.
- [13] R. A. DeVore and R. C. Sharpley. Besov spaces on domains in \mathbb{R}^d . *Trans. Amer. Math. Soc.*, 335(2):843–864, 1993.
- [14] M. Dobrowolski. On finite element methods for nonlinear elliptic problems on domains with corners. In P. Grisvard, W. Wendland, and J. Whiteman, editors, *Singularities and Constructive Methods for Their Treatment*, volume 1121 of *Lecture Notes in Mathematics*, pages 85–103. Springer Berlin Heidelberg, 1985.
- [15] M. Dobrowolski. On quasilinear elliptic equations in domains with conical boundary points. *J. Reine Angew. Math.*, 394:186–195, 1989.
- [16] E. Feireisl and D. Pražák. *Asymptotic Behavior of Dynamical Systems in Fluid Mechanics*, volume 4 of *AIMS series on applied mathematics*. American Institute of Mathematical Sciences, 2010.
- [17] F. Gaspoz and P. Morin. Approximation classes for adaptive higher order finite element approximation. *Math. Comp.*, 83(289):2127–2160, 2014.
- [18] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman Publishing, 1985.
- [19] M. Hansen. Nonlinear approximation rates and besov regularity for elliptic pdes on polyhedral domains. *J. Found. Comput. Math.*, 15(2):561–589, 2015.
- [20] M. Hansen and W. Sickel. Best m -term approximation and Lizorkin-Triebel spaces. *J. Approx. Theory*, 163(8):923–954, 2011.
- [21] C. Hartmann. *The p -Poisson Equation: Regularity Analysis and Adaptive Wavelet Frame Approximation*. Dissertation, Philipps-Universität Marburg. In preparation.
- [22] J. Horváth. *Topological Vector Spaces and Distributions. Vol. I*. Addison-Wesley Publishing Co., London, 1966.
- [23] A. Jonnson and H. Wallin. *Function Spaces on Subsets of \mathbb{R}^n* , volume 2 of *Mathematical Reports*. Harwood Academic Publishers, 1984.
- [24] G. Köthe. *Topological Vector Spaces I*, volume 159 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1969.
- [25] A. Kufner and B. Opic. How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carolin.*, 25(3):537–554, 1984.

- [26] O. A. Ladyzhenskaya. New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. *Proc. Stek. Inst. Math.*, 102:95–118, 1967.
- [27] P. Lindqvist. *Notes on the p -Laplace equation*. University of Jyväskylä, Department of Mathematics and Statistics, 2006.
- [28] J.-L. Lions. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*. Dunod, Paris, 1969.
- [29] A. L. Mazzucato and V. Nistor. Well-posedness and regularity for the elasticity equation with mixed boundary conditions on polyhedral domains and domains with cracks. *Arch. Ration. Mech. Anal.*, 195(1):25–73, 2010.
- [30] S. Rolewicz. On a certain class of linear metric spaces. *Bull. Acad. Polon. Sci.*, 5: 471–473, 1957.
- [31] H. H. Schaefer and M. P. Wolff. *Topological Vector Spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 1999.
- [32] P. Tolksdorf. On the dirichletproblem for quasilinear equations in domains with conical boundary points. *Comm. in Partial Differential Equations*, 8(7):773–817, 1983.
- [33] F. Trèves. *Topological Vector Spaces, Distributions and Kernels*. Academic Press, New York, London, 1967.
- [34] H. Triebel. *Theory of Function Spaces*. Birkhäuser, Basel/Boston/Stuttgart, 1983.
- [35] H. Triebel. *Theory of Function Spaces III*. Birkhäuser, Basel, 2006.

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