Numerical Center Manifold Methods

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dedicated to the 60^{th} birthday of my good friend
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Abstract This paper summarizes the first available proof and results for general full, so space and time discretizations for center manifolds of nonlinear parabolic problems. They have to admit a local time dependent solution (a germ) near the bifurcation point. For the linearization (A, B) of the nonlinear elliptic part and the boundary condition we require: A has a trivial unstable manifold, to be sectorial and for (A, B) the complementing condition is valid. The two last condition hold for $A : W_0^{m,p}(\Omega, \mathbb{R}^q) \to W^{-m,p}, 1 \leq m, q$, satisfying the Legendre-Hadamard condition with Amann [3] criteria for both conditions, and A in appropriate divergence form for $m > 1$. By the active research, the class of problems satisfying these conditions is strongly growing. The generalized Agmon *e.al.* systems require other techniques to prove these conditions. Then essentially all the up-to-date space discretizations, except meshfree methods, with geometric time discretization yield converging numerical center manifolds. Here I summarize results of my upcoming monograph and strongly generalize my earlier papers.

0.1 Introduction

The aim of this paper is a summary of the proof for convergence for general full, so space and time discretizations of center manifolds. It applies to the up to date numerical methods, finite elements without and with crimes and adaptivity, discontinuous Galerkin, difference, spectral and wavelet methods and a large class of nonlinear parabolic problems. With more than 500 papers on center manifolds for partial differential equations, we only give a short survey on the history and analytical aspects and then turn to their numerical realization.

Essential for the analysis of local bifurcations and the originating dynamical scenarios of parabolic problems is the reduction to low-dimensional ordinary differential equations for the center manifolds. They were introduced in the sixties by Pliss [45] and Kelley [33]. Due to Lanford's [37] contributions, this theory has been applied extensively to the study of bifurcation problems and dynamical systems of ordinary differential equations, in particular, in connection with the normal form theory. The extension to ordinary differential equations in infinite dimensional spaces, so to parabolic equations started with Carr [17] and Henry [31]. There the elliptic part is the sum of a linear and a nonlinear operator A and R. [17] allows an A generating different semigroups, e.g. strongly continuous, R has a uniformly continuous derivative and $R(0) = 0, R'(0) = 0$. Under the usual conditions for the spectrum of A he proves the existence of a center manifold. [31] discusses analytical semigroups with the sectorial generator A and its fractional powers. His approach to center manifolds is motivated by many interesting applications, e.g., reaction diffusion and Naviér-Stokes systems.

In the meantime there are different approaches: Generalizing [17], Bates and Jones [5] prove invariant manifold theorems for a similar problem with a continuous semigroup and Lipschitz continuous R with applications to the nonlinear Klein-Gordon and to Fitz Hugh-Nagumo equations. Vanderbauwhede [50] and with Iooss [51] generalize the center manifold theory in finite-dimensional systems in [50] to infinite-dimensional systems. Instead of the usual sectorial operators they consider some elementary spectral theory of closed linear operators and avoid the use of semigroups and semiflows. They apply it to the classical Naviér-Stokes equations and these equations in a cylinder. In the first problem the part to the right of the imaginary axis of the spectrum of the associated linear operator is bounded and the Cauchy problem is well posed for $t > 0$, however for the second the spectrum of the linear operator is unbounded as well to the left as to the right of the imaginary axis and the Cauchy problem has no meaning.

Combinations of center manifolds with stable and unstable invariant manifolds are studied for ordinary differential equations in Guckenheimer/Holmes [26,27], Iooss/Adelmayer [32] and Kuznetsov [36]. Chow and Hale [20], Hale and Koçak [29] study partial differential equations. Finally Haragus and Iooss [30] present a very up-to-date book on local bifurcations, the originating center manifolds, and normal forms in infinite-dimensional dynamical systems.

Many of these problems cannot be solved exactly. So appropriate discretization methods are mandatory. The implications of discretization for ordinary differential equations is well understood, cf. e.g., Beyn, Lorenz and Zou, [6–11, 21, 54, 55], Ma, [42], Sieber and Krauskopf, [48], Lynch, [41], Choe and Guckenheimer, [18]. [11] have shown the existence of an invariant manifold of the discretization close to the center manifold of the differential equation without, however, studying smoothness properties of this manifold, which are needed for the analysis of bifurcations.

Beyond many other related results of Fiedler in this direction, a particularly interesting example is due to him and Scheurle [24]. They discretize homoclinic orbits by one-step discretizations of order p and stepsize ϵ. This can be viewed as time-ϵ maps for the autonomous ordinary differential equations $x(t) = f(\lambda, x(t)) + \epsilon^p g(\epsilon, \lambda, t/\epsilon, x(t)), x \in \mathbb{N}, \lambda \in \Lambda$, with analytic f, g and ϵ-periodic g in t. This is a rapidly forced nonautonomous system. The authors study the behavior of a homoclinic orbit Γ for $\epsilon = 0, \lambda = 0$, under discretization. Under generic assumptions their Γ becomes transverse for positive ϵ. The transversality effects are estimated from above to be exponentially small in ϵ. For example, the length $l(\epsilon)$ of the parameter interval of λ for which Γ persists can be estimated by $l(\epsilon) \leq Cexp(2\pi\eta/\epsilon)$, where C, η are positive constants. The coefficient η is related to the minimal distance from the real axis of the poles of $\Gamma(t)$ in the complex time domain. Likewise, the region where complicated, chaotic dynamics prevail is estimated to be exponentially small, provided the saddle quantity of the associated equilibrium is nonzero. The results are visualized by high precision numerical experiment, showing that, due to exponential smallness, homoclinic transversality becomes practically invisible under normal circumstances, already for only moderately small discretization steps.

For parabolic problems the first systematic study of time discretizations of center manifolds is due to Lubich/Ostermann [38]. They study for a sectorial A the case of a trivial unstable manifold and an "approximate" center manifold with eigen values μ with $|\mathrm{Re}\ \mu| < \delta$, instead of the usual $|\mathrm{Re}\ \mu| = 0$. They give a new and simpler proof than Henry [8], directly generalized to numerical time, but not to space discretizations. This treatment is conceptually more closely related to Vanderbauwhede and Iooss [50, 51].

But there are still many problems open. We want to present some of the necessary answers in this paper, summarizing relevant results of my [16] and strongly generalizing my earlier papers [13, 14]. This paper is organized as follows.

In Section 0.2 we discuss linear and nonlinear elliptic equations or systems of order 2 or $2m$. Essential for center manifolds is the linearization (A, B) of the nonlinear elliptic part and the boundary condition there. A has to be sectorial, to have a trivial unstable manifold and for (A, B) the complementing condition has to be satisfied. This holds for $A : W^{m,p}(\Omega, \mathbb{R}^q) \to W^{-m,p}, 1 \leq m, q$, satisfying the Legendre-Hadamard, the Dirichlet or other boundary conditions with new criteria for the complementing condition and some generalized Agmon *e.al.* [1] systems. Since research in these areas is very active, the class of problems satisfying these conditions is strongly growing. We extend the elliptic to linear and nonlinear parabolic operators in Section 0.3. Here different solutions, semigroups and generators are discussed. Now we are ready for their local dynamics via center manifolds introduced in Section 0.4. The parabolic systems have to admit a local time dependent solution (a germ) near the bifurcation point and satisfy the above (A, B) conditions. The splitting of their local dynamics due to Lubich/Ostermann [38] plays a central role. We determine the asymptotic expansion via the homologic equation and the recursive definition of the systems for solving this equation. Next we define the essential part, the space discretization in Section 0.5. It yields for the asymptotic expansion of the parameterization of the center manifold convergence for the necessary discrete to the exact terms for essentially all up-to-date space discretizations, except meshfree methods, with geometric time discretization. Then a careful monitoring of the the discrete normal forms until determinacy, so for sufficiently many terms, allows the classification of the local dynamical scenarios and the final time discretization and its convergence in Section 0.6. For the latter the geometric integration methods in Hairer/Lubich/Wanner [28] are appropriate and applied to the small dimensional resulting system (36). Equivariance maintaining methods are mentioned.

This approach for full discretizations is formulated for the first time for general operators in Böhmer [13, 14]. It applies to all the nonlinear parabolic problems and their discretizations presented in [15, 16] under appropriate conditions for the nonlinearity. Consequently, the many results for the dynamics of, e.g., saddle node, transcritical, pitchfork, cusp, Hopf bifurcations, documented, e.g., in Govaerts [25], Guckenheimer/Holmes [27], Kuznetsov [35], Mei [43] or the many original papers on these subjects, apply to parabolic equations as well.

0.2 Linear and Nonlinear Elliptic Operators

0.2.1 Definitions of linear elliptic operators

We define linear and nonlinear elliptic operators essentially for its weak form and give some results, c.f. [15]. The $u(x)$ are $\in \mathbb{R}^q$ or $\in \mathbb{C}^q$ for $q \geq 1$, and usually we omit the vector symbol $\vec{u}$ or the $u^T = (u_1, .., u_q)$. Evaluated in $x \in \Omega$ the following $\partial^\alpha u, \partial^\alpha v$, $A_{\alpha\beta}\partial^\beta u, A_\alpha \partial^\alpha u$ are $\in \mathbb{R}^q$ and the Euclidean product $(r, s)_q$ in $\mathbb{R}^q$, here and below e.g. $(A_{\alpha\beta}\partial^\beta u, \partial^\alpha v)_q$ are well defined under the conditions (1) - (2) in $u \in W^{m,p}$ or $W^{2m,p}$ with $1 < p < \infty$. $W^{m,p=2} = H^m$. Mind in $\mathbb{C}^q$ the scalar product $(r, s)_q = \sum_{j=1}^q r_j \overline{s}_j \in \mathbb{C}$, is linear in r and sesquilinear in s, used in the following definitions.

$$\begin{aligned} &\text{Open bounded } \Omega \in \mathbb{R}^n, n \geq 1, \partial\Omega \in C^{0,1}, m, q \geq 1,\ 1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1,\ \textit{with} \\ &\alpha = (\alpha^1, \dots, \alpha^n), \alpha, \beta \in \mathbb{N}_0^n, |\alpha| = \sum_{j=1}^n \alpha^j, |\beta| \leq m, \partial^\alpha u = (\partial^1)^{\alpha^1} \dots (\partial^n)^{\alpha^n} u,\ \textit{for} \qquad (1) \\ &u \in \mathcal{U} := \mathcal{U}_p := W_0^{m,p}(\Omega, \mathbb{R}^q), v \in \mathcal{V} := \mathcal{U}_{p'} :\ \textit{the standard } \|u\|_{\mathcal{U}},\ \textit{and } \langle u, v \rangle_{\mathcal{U}\times\mathcal{V}}. \end{aligned}$$

We define the *differential operator* A and its A^p by the highest order terms, with

$$\begin{aligned} &A_{\alpha\beta} \in L^\infty(\Omega, \mathbb{R}^{q\times q}),\ \textit{a.e.}\ \forall x \in \Omega : |A_{\alpha\beta}(x)| \leq \Phi_0, \forall u \in \mathcal{U}, v \in \mathcal{V}, \mathcal{V}' := W^{-m,p}, \\ &A : \mathcal{D}(A) := \mathcal{U} \to \mathcal{V}', \langle Au, v \rangle_{\mathcal{V}'\times\mathcal{V}} = a(u, v) = \int_\Omega \sum_{|\alpha|,|\beta|\leq m} (A_{\alpha\beta}\partial^\beta u, \partial^\alpha v)_q dx \qquad (2) \\ &\textit{with principal parts } a^p(u, v) = \langle A^p u, v \rangle_{\mathcal{V}'\times\mathcal{V}} = \int_\Omega \sum_{|\alpha|,|\beta|=m} (A_{\alpha\beta}\partial^\beta u, \partial^\alpha v)_q dx. \end{aligned}$$

$$\begin{aligned} &\textit{Then solve for } u_0 \in \mathcal{U} :\ \ Au_0 = f\ \textit{ or } a(u_0, v) = \langle Au_0, v \rangle_{\mathcal{V}'\times\mathcal{V}} = \langle f, v \rangle_{\mathcal{V}'\times\mathcal{V}} \qquad (3) \\ &:= \int_\Omega \sum_{|\alpha|\leq m} (f_\alpha, \partial^\alpha v)_q dx\ \ \forall v \in \mathcal{V} := \mathcal{U}_{p'}\ \textit{for } f := (f_\alpha)_{|\alpha|\leq m} \in W^{-m,p}(\Omega, \mathbb{R}^q), \\ &\textit{below with parameters } \lambda \in \Lambda \subset \mathbb{R}^d \textit{ in } a_{ij}, A_{\alpha\beta} : \Omega \times \Lambda \to \mathbb{R}^{q\times q}, q \geq 1, \\ &A_{\alpha\beta} \text{ continuous in } \lambda \text{ and satisfying (7),(8) uniformly in } \lambda \in \Lambda. \qquad (4) \end{aligned}$$

This (4) is important for bifurcation and local dynamics of center manifolds.

We always consider pairs of differential and boundary operators (A, B), mainly Dirichlet, often included in $\mathcal{U}, \mathcal{V}$, cf. [15, 16], Amann [3] for general cases. So $\forall x \in \partial\Omega$ or in a subset $\partial\Omega_1$ with nonempty interior or different conditions in subsets we require

$$\textit{Dirichlet or general operator} \quad B_D u := (B_j u := \partial^j u/\partial\nu^j)_{j=0}^{m-1} = f_D \textit{ or in [3] } Bu = f. \tag{5}$$

Obviously all *bilinear forms in (1) - (3) are continuous.* More generally a continuous bilinear form induces a unique bounded linear operator A s.t.

$$A \in \mathcal{L}(\mathcal{U}, \mathcal{V}') : \ a(u, v) = \langle Au, v\rangle_{\mathcal{V}'\times\mathcal{V}} \ \ \forall \ \ u \in \mathcal{U}, \ v \in \mathcal{V}, \ \ \|A\|_{\mathcal{V}'\leftarrow\mathcal{U}} \le C_b. \tag{6}$$

Sometimes we replace the above partials $\partial^j u, \partial^\alpha u, u = \nabla^0 u, \partial u = (\partial^1, \dots, \partial^n)u = \nabla u, \nabla^k u = \{\partial^\alpha u, |\alpha| = k\}, \nabla^{\le k} u, \dots$, by reals or vectors. Whenever in a formula the $\nabla^0, \partial^j, \partial^\alpha, \partial = \nabla^1, \nabla^k$ are not applied to a function or a term, we interpret the $\nabla^0, \partial^j, \partial^\alpha, ..$ as $\in \mathbb{R}^n$, the $\partial = (\partial^1, \dots, \partial^n) = \nabla, \nabla^2, \dots,$ as $\in \mathbb{R}^{n\times q}, \mathbb{R}^{n^2\times q}, ..$ c.f. (7).

An *elliptic (we omit uniformly) operator* satisfies for $q = 1$ the *strong Legendre* (delete the $\eta-$terms in (7)) and for $q > 1$ the *strong Legendre-Hadamard condition,* so

$$\exists 0 < \psi_0 < \Psi_0 \in \mathbb{R}_+ : \forall x \in \Omega \text{ a.e. } \forall \partial = (\partial^1, \dots, \partial^n) \in \mathbb{R}^n, \ \partial^\alpha = \prod_{j=1}^n (\partial^j)^{\alpha_j} \in \mathbb{R}, \tag{7}$$

$$\eta \in \mathbb{C}^q, q, m \ge 1 : \psi_0 |\partial|^{2m} |\eta|^2 \le \overline{\eta}^T A^p(x, \partial)\eta := \overline{\eta}^T \sum_{|\alpha|=|\beta|=m} A_{\alpha,\beta} \partial^\alpha \partial^\beta \eta \le \Psi_0 |\partial|^{2m} |\eta|^2.$$

For a coercive principal part A^p we impose for $m > 1$ beyond (1) - (3), (7), mind (4):

$$\text{assume for } |\alpha| = |\beta| = m > 1 \text{ for } q \ge 1 : \ A_{\alpha,\beta} \in C(\overline{\Omega}, \mathbb{R}^{q\times q}). \tag{8}$$

0.2.2 Linear elliptic operators are sectorial

We study center manifolds via *analytic semigroups* generating sectorial operators and vice versa. So we introduce and restrict the further discussion to them. Theorem 0.5 discusses weak elliptic sectorial operators A. Most parabolic solutions satisfy the strong, and consequently the weak form as well, c.f. Section 0.3.

Definition 0.1. Analytic semigroup: *For Banach spaces, $\mathcal{U}, \mathcal{V}$, a family $S(t) : \mathcal{U} \to \mathcal{U}$ $\forall t \in \mathbb{R}_+$, defined on a sector $\Sigma := \Sigma_\epsilon := \{\lambda \in \mathbb{C} : \lambda \ne 0, |\arg(\lambda)| < \epsilon\}$, with $0 < \epsilon < \pi/2$, c.f. Figure 1, is called a* (linear strongly continuous) analytic semigroup *iff*

$$\begin{aligned} &S(0) = I : \mathcal{U} \to \mathcal{U}, \ \ \forall s, t \ge 0 : \ S(s + t) = S(s)S(t), \ \ S(t) \in \mathcal{L}(\mathcal{U}, \mathcal{U}), \\ &\forall u \in \mathcal{U} : \lim_{t\to 0_+} S(t)u = u, \quad \forall t \ge 0 \ \ \forall u \in \mathcal{U} : \ \ t \to S(t)u \textit{ is analytic.} \end{aligned} \tag{9}$$

For $\{S(t)\}_{t\ge 0}$ its (infinitesimal) generator $-A : \mathcal{D}(A) \subset \mathcal{U} \to \mathcal{U}$, *is defined by the following limit, e.g., for a weak elliptic $A : \mathcal{D}(A) \subset \mathcal{V} \subset \mathcal{U} \to \mathcal{U}$, as*

$$A : \mathcal{D}(A) \subset (\mathcal{U} \textit{ or } \mathcal{V}) \to \mathcal{U} \ \forall \ u \in \mathcal{D}(A) : -Au := \lim_{t\to+0} \frac{S(t)u - u}{t} \in \mathcal{U} \textit{ exists.} \tag{10}$$

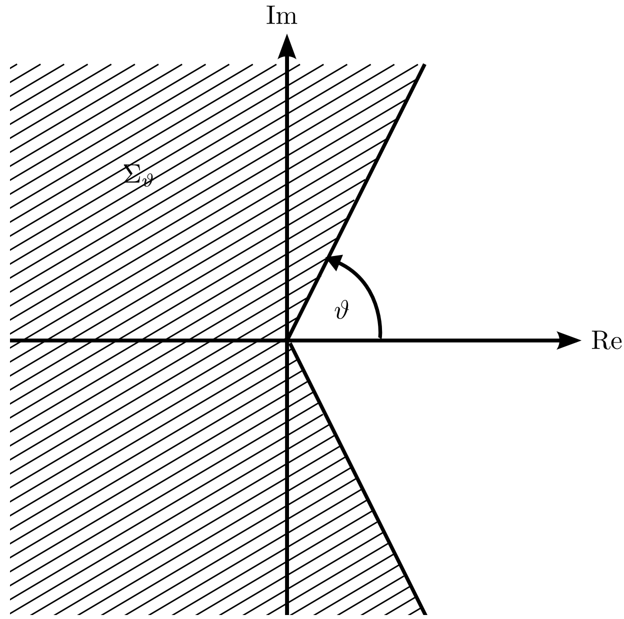

Figure 1. *Shaded sector contained in the resolvent set of A_{λ_0}.*

Definition 0.2. *cf. Figure 1:* Resolvent, resolvent set and sectorial operators *are defined for $A : \mathcal{D}(A) \subset (\mathcal{U}$ or $\mathcal{V}) \to \mathcal{U}$ as $(A - \lambda I)^{-1}$ and in $\rho(A) := \{\lambda \in \mathbb{C} : \exists (A - \lambda I)^{-1}\}$ these resolvents do exist. The operator A for a real ([31] p 18, pp 21) or a complex ([53] p 1096) Banach space, $\mathcal{U}$ or $\mathcal{V}$, is called* sectorial, *if and only if*

1. *A is linear, graph closed and densely defined on $\mathcal{U}$, hence $\overline{\mathcal{D}(A)}^{\|\cdot\|_{\mathcal{U}} \text{ or } \mathcal{V}} = \mathcal{U}$ or $\mathcal{V}$.*
2. *There exist real $c, M, \vartheta \in \mathbb{R}, M \geq 1, 0 < \vartheta < \pi/2$ s.t. the open sector* [1]

$$\Sigma_{c,\vartheta} := \{\lambda \in \mathbb{C} : c \neq \lambda, \quad \vartheta < |\arg(\lambda - c)| \leq \pi, \ \} \subset \rho(A), \ \textit{hence} \tag{11}$$
$$\forall \lambda \in \Sigma_{c,\vartheta} : \exists \ (A - \lambda I)^{-1} \ \textit{and additionally} \ \|(A - \lambda I)^{-1}\| \leq M/|\lambda - c|. \tag{12}$$

For parabolic problems a combination of Theorems 0.3-0.5, [22,31,40,47,53] is important.

Theorem 0.3. Sectorial operators generate analytic semigroups and "vice versa":

[1] The following results are, with minor changes, e.g., the curve C in (13), correct for the open [53] and closed [31] sectors: $\Sigma_{c,\vartheta} := \{\vartheta < |\arg(\lambda - c)| \cdots\}$ and $\Sigma_{c,\vartheta} := \{\vartheta \leq |\arg(\lambda - c)| \cdots\}$.

1. *A sectorial operator, $A : \mathcal{D}(A) \subset (\mathcal{U}$ or $\mathcal{V}) \to \mathcal{U}$, on a real or complex $\mathcal{U}$, generates an* analytic semigroup, *denoted as and given by the* Dunford integral

$$\{S(t) = e^{-At}\}_{t>0}, \text{ and } \quad e^{-At} \;=\; \frac{1}{2\pi i}\int_C (A+\lambda I)^{-1}e^{\lambda t}d\lambda \text{ independent of the} \tag{13}$$

$C \subset -\Sigma_{c,\vartheta} \ni \lambda$, $\arg \lambda_{|\lambda|\to\infty} \to \pm(\pi-\vartheta)$, *analytically extendable to* $-\Sigma_{c=0,\vartheta}$.

2. *For each $x \in \mathcal{U}, t > 0$, the $e^{-At}x \in \mathcal{D}(-A)$, so smoothing $w_0 \in \mathcal{U} \mapsto u(t) \in \mathcal{D}(A)$.*

3. *Conversely, let $-A$ be the generator of an analytic semigroup, $\{S(t)\}_{t>0}$. Then A is sectorial, and $-A$ uniquely generates the original $\{S(t)\}_{t>0}$.*

$A : W_0^{m,p}(\Omega,\mathbb{R}^q) \to W^{-m,p}$ is closed, sectorial, proved in [16] for $p = 2, m, q \geq 1, 1 < p < \infty, 1 = q \leq m$ and (7). Amann [3][2] considers $1 < p < \infty, 1 = m \leq q$ and uses the notations: $A_p \in \mathcal{H}(L^p)$ if $-A$ is the infinitesimal generator of a strongly continuous analytic semigroup $\{S(t) = e^{-At}\}_{t>0}$, on L^p. On $\partial\Omega$ he combines Dirichlet B_D and Neumann type B_N conditions for different equations in the system and components of $\partial\Omega$. So with the matrix $\mathrm{diag}(\delta^1 \dots \delta^q), \delta^j \in \{0,1\}$, his B is $= \delta B_N + (1-\delta)B_D$. He calls (A,B) *normally elliptic* if B satisfies the complementing condition for A, cf. [3] for good criteria, and its principal part A^p is elliptic (7). He claims, p.20, that these results remain correct as well for $1 \leq m, q$, and A in appropriate divergence form structure. With the spectrum σ he proves under these conditions Theorems 0.5 and 0.4.

$$\sigma\big\{\big((A_{\alpha\beta}(x)\partial^\beta,\partial^\alpha)_q\big)_{|\alpha|,|\beta|=m}\big\} \subset \big[\mathrm{Re}\ z > 0\big] := \{z \in \mathbb{C}; \mathrm{Re}\ z > 0\} \quad \forall x \in \overline{\Omega}, 0 \neq \partial \in \mathbb{R}^n. \tag{14}$$

Theorem 0.4. Normally elliptic pairs (A,B) : [3],p.20,21. *Under the previous conditions the pair (A,B) is normally elliptic, according to Amann [3], here and in Theorem 0.5 even for $1 \leq m, q$, and A in appropriate divergence form structure.*

Theorem 0.5. Sectorial A, [3] p.27: *Normally elliptic (A_p, B_p) imply $A_q \in \mathcal{H}(L^q), q \in (p,p')$, so A_q is sectorial. For $A_p \in \mathcal{H}(L^p)$, the A_p is normally elliptic with the B_p.*

This result does apply to the wide class of problems in (2) ff., but not to the more general elliptic systems in the sense of Agmon/Douglis/Nirenberg [1]. For those, e.g., the linearization of the Naviér–Stokes operator and other equations in in fluid dynamics other techniques have to be used, c.f. Kirchgässner and Kielhöfer [34] and Denk, Hieber and Prüss [46].

0.2.3 Nonlinear elliptic operators

Bifurcation and the interesting (local) dynamics only exist for nonlinear PDEs. So we extend the previous linear to nonlinear operators in weak and mention strong forms for

[2] thanks to Robert Denk I found this paper strongly related to my problems

the order $2m, m \geq 1$ and for systems $q \geq 1$, cf. [15]. With (2) - (7) we consider for the weak form nonlinear functions G_m^w for $n > 1$.

$$\begin{aligned} G_m^w &:= \big(A_\alpha^w, |\alpha| \leq m\big): \ w = (x, \nabla^{\leq m}) \ \in \ \mathcal{D}(G_m^w) \to \mathbb{R}^q, \\ \mathcal{D}(G_m^w) &= \mathcal{U}_o = \Omega \times \mathcal{U}_o^m, \ \mathcal{U}_o^m \subset \mathbb{R}^{N_m \times q}, \ N_m := (n^{m+1} - 1)/(n-1). \end{aligned} \tag{15}$$

Then the weak and divergence form for $w_u(\cdot) = (\cdot, \nabla^{\leq m} u(\cdot)) \in \mathcal{D}(G_m^w)$ a.e. are

$$\begin{aligned} (G(u), v)_{L^2} &:= \sum_{|\alpha| \leq m} \int_\Omega A_\alpha(u) \partial^\alpha v dx, \ A_\alpha(u) := A_\alpha^w(w_u) \in L^2 \ \textit{the weak and} \\ G_d(u)(\cdot) &:= \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha\big(A_\alpha(u)\big), \ A_\alpha \in C^{|\alpha|} \ \textit{the divergence form.} \end{aligned} \tag{16}$$

Replace $G_m^w, w_u, \mathcal{U}_0^m$ by $G_{2m}^w, w_u(\cdot) := (\cdot, \nabla^{\leq 2m} u(\cdot)), \mathcal{U}_0^{2m} \subset \mathbb{R}^{N_{2m} \times q}$ to define the strong

$$G_s(u)(\cdot) := G_{2m}^w(w_u(\cdot)), \quad \mathcal{D}(G_s) := \{u : \Omega \to \mathbb{R}^q \forall x \in \Omega \ a.e. w_u(x) \in \mathcal{D}(G_{2m}^w)\}. \tag{17}$$

Definition 0.6. Nonlinear differential operators, ellipticity, coercivity: cf. [15], p 80.

1. *A nonlinear PDE of order* 2 *or* $2m$ *– indicated as* 2 / $2m$ *– for* u *has the form form (15),(16) or (17). For a* fully nonlinear $G_s(u)$ *in (17) its* G_{2m}^w *is non linear in* ∇^2 / ∇^{2m}*. The form (15),(16) for* G_m^w, G, G_d *is called a* quasilinear. *If all the* A_α *are independent of* u, *but* $A_0(u)(\cdot) = A_0(\cdot, u(\cdot))$ *then it is called* semilinear.

2. *These* G, *similarly* G_d, G_s *are called* elliptic *in* $u \in \mathcal{D}(G)$ *if* G *is differentiable near* u *and* $G'(u)$ *is elliptic, So only the principal part of* $G'(u)$ *matters. This implies the coercivity and ellipticity under appropriate Gårding conditions.*

So we admit these nonlinear elliptic operators for the parabolic equations in (18), despite existence and uniqueness results only exist under very specific conditions.

0.3 Linear and Nonlinear Parabolic Operators

We start them with Ω, A, G in (1)- (3), (15) - Definition 0.6, u_0, u in the *state space* $\mathcal{U}$, with often included boundary conditions for $\forall t \in (t_0, T)$ with initial condition in t_0 as

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} + Au_0 = f(t) \text{ or } \frac{\mathrm{d}u_0}{\mathrm{d}t} + G(u_0) = \frac{\mathrm{d}u_0}{\mathrm{d}t} + G'(u_0)(u - u_0) + R(u - u_0) = f(t, u), \tag{18}$$

special $f, u_0(t_0) = w_0 \in \mathcal{V} \subset \mathcal{U}$ on Ω with elliptic $A, G'(u_0)$ and $R(v) = \mathcal{O}(\|v\|^2)$, *so*

$u_0 = u(w_0, t)$. These $A, G'(u_0)$ are assumed independent of the time, so we consider *(slightly generalized) autonomous systems* (18). We say that (18) generates an *evolution operator* Φ_t if for appropriate $w_0 \in \mathcal{V}$ $\exists T = T(w_0) > 0$ s.t. $u_0(t + t_0) := \Phi_t w_0$ solves (18) for $0 \leq t < T = T(w_0)$. The family $\{\Phi_t\}_t$ is called *flow* for (18).

0.3.1 Solutions of parabolic problems and analytic semigroups

We introduce different types of solutions for parabolic problems. Much less is known for parabolic than for elliptic problems. We interpret (18) and (19) as *linear or nonlinear ordinary differential equations for elements and operators in Banach function spaces,* with $u, A, G, f, \ldots, f(u) \in \mathcal{U}, \mathcal{V}, C(\Omega), \ldots,$ defined on Ω. We usually omit or indicate by $\cdot$ the reference to $x \in \Omega$, but include the time t in $u(t), f(t), (t, u(t)),\ t \in (0, T)$, in our *generalized autonomous systems* (18), studied in Henry [31], Carr [17], Pazy [44], Amann [3], Zeidler [53], Lunardi [40], Evans [23], Engel,Nagel [22], Raasch [47].

For linear, semi-, quasi- and fully non- linear problems many results for different types of solutions, their relation to semigroups and generators are known. We cite some related to the *generator* $-A$, *of an analytic semigroup* $\{S(t)\}_{t>0}$, [16], and start with:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + Au(t) = f(t) \textit{ or } = f(t, u(t)) \text{ for } t \in (t_0, T),\ u(t_0) = w_0 \text{ on } \Omega \textit{ for} \tag{19}$$

$$A_s : \mathcal{D}(A_s) =: W^{2m,p}(\Omega, \mathbb{R}^q) \cap W_0^{m,p} \to \mathcal{U} := L^p \to \mathcal{U},\ 1/p + 1/p' = 1, q \geq 1,$$

$$A = A_w : \mathcal{D}(A) := W_0^{m,p} \subset \mathcal{V} := L^p \subset \mathcal{U} := W^{-m,p} \to \mathcal{U}, \textit{often } p = 2. \tag{20}$$

For analytic semigroups $\{S(t) = e^{-At}\}_{t>0}$ by (13) we introduce a generalized formula of *variation of constants,* cf. [52], defining different types of solution by

$$u_0(t) := S(t - t_0)w_0 + \int_{t_0}^{t} S(t - s)f(s)ds = e^{-A(t-t_0)}w_0 + \int_{t_0}^{t} e^{-A(t-s)}f(s)ds \tag{21}$$

and replace the $f(s)$ here and in (22), (23) e.g. by $f(s, u(s))$, cf. (19) or other choices. More precisely the $f : (t_0, T) \to \mathcal{U}$ is replaced by $f : (t_0, T) \times \mathcal{O}_U \to \mathcal{U}$, with $\mathcal{O}_U$ either $= \mathcal{U}$ or an open subset of $\mathcal{U}$. We require additionally $u_0(t) \in \mathcal{O}_U\ \forall\, t \in [t_0, T)$. Or choose the $R(u - u_0)$ in (18) or $R(u, \lambda)$ in (25). The different conditions for f are listed below.

The technique for elliptic is employed for parabolic problems, their bifurcation and numerical methods as well, cf. Thomée [49] and Section 0.5. So for the weak equation in (20) determine a weak solution $u_0 \in \mathcal{D}(A)$ and $u_0(t_0) = w_0 \in \mathcal{V}$ in (22), (23) s.t.

$$\forall v \in W_0^{m,p'}(\Omega, \mathbb{R}^q) \text{ a.e.} \forall t \in (t_0, T) : \frac{\mathrm{d}\langle u_0(t), v\rangle_{\mathcal{D}(A) \times W_0^{m,p'}}}{\mathrm{d}t} + a(u_0(t), v) = \langle f(t), v\rangle_{\ldots}. \tag{22}$$

The Carr [17] solutions use the adjoint operator $A^* : \mathcal{D}(A^*) \subset \mathcal{U}^* \to \mathcal{U}^*$ of $A : \mathcal{D}(A) \subset \mathcal{U} \to \mathcal{U}$ and the pairing $\langle\cdot,\cdot\rangle$ between $\mathcal{U}$ and its dual. [17] defines u_0 by

$$\forall v \in \mathcal{D}(A^*) \text{ a.e. } t \in (t_0, T) : \frac{\mathrm{d}\langle u_0(t), v\rangle}{\mathrm{d}t} + \langle u_0(t), A^* v\rangle = \langle f(t), v\rangle,\ u_0(t_0) = w_0. \tag{23}$$

Definition 0.7. Mild, classical, weak solutions: *Assume* $S : (t_0, T) \to \mathcal{U}$ *with generator* $-A, t_0 < T < \infty, u_0 : [t_0, T) \to \mathcal{U},\ w_0 \in \mathcal{V}$ *and* $f(s)$ *or generalizations in (19), (21) ff.*

1. *Then the function* u_0 *in (21) is called a* mild solution *for (19).*

2. *A* $u_0 \in C^1((t_0,T];\mathcal{U}) \cap C((t_0,T],\mathcal{D}(A))$ *satisfying (19) for all* $t \in (t_0,T)$ *and* $u_0(t_0) = w_0$ *it is called a* classical (or strong) solution.

3. *A* $u_0 \in C[t_0,T)$, *with absolutely continuous, so differentiable,* $\langle u_0(t),v\rangle_{\mathcal{U}\times W_0^{m,p'}} \forall v \in W_0^{m,p'}$ *on* $[t_0,T]$ *with* $d\langle u_0(t),v\rangle_{\mathcal{D}(A)\times W_0^{m,p'}}/dt \in L^1(t_0,T)$ *satisfying (22) is called a* weak solution. *[17] calls* u_0 *satisfying (23) weak solution as well.*

4. *A* $u_0 \in C[t_0,T)$, *with* $\langle u_0(t),v\rangle \forall v \in \mathcal{D}(A^*)$ *absolutely continuous, so differentiable s.t.* $d < u_0(t),v > /dt \in L^1(t_0,T)$ *satisfies (23) is called a* distributional solution.

Theorem 0.8. Existence, uniqueness of different solutions for linear equations and analytic semigroups $S(t)$: *Assume* $u_0 : [t_0,T) \to \mathcal{U}, T < \infty, w_0 \in \mathcal{U}$. *Then*

1. *for the strong* $A_s = A$ *in (20) a mild solution* u_0 *satisfies* $u_0(t) \in \mathcal{D}(A_s)$ *for all* $t \in (t_0,T)$, *hence* u_0 *is a weak solution for* A_w *as well and we only use* A *here.*

2. $d(e^{-At})/dt = -Ae^{-At}$ *and there exists at most one classical solution* u_0 *of (19). Each classical is a mild and a weak solution.*

3. *for continuous* $f : [t_0,T) \to \mathcal{U}$ *the* u_0 *in (21) with* $\lim_{t\to t_0+0} u_0(t) = w_0 \in \mathcal{V}$ *is the well defined unique mild solution for (19).*

4. *for* $f \in C^1[t_0,T)$, $w_0 \in \mathcal{D}(A)$ *the uniquely existing mild solution* u_0 *in (21) satisfies* $u_0(t) \in \mathcal{D}(A)\ \forall t > t_0$, *and is a classical solution of (19).*

5. *for* $f \in L^1(t_0,T)$ *the weak and the distributional solution* u_0 *for (22) and (23), resp., uniquely exist and are obtained by (21).*

6. *for* $f \in L^2([t_0,T])$, *a unique weak solution* $u_0 : [t_0,T] \to H^1(\Omega)$, *hence* $m = 1$, *for (22) exists.*

7. *for a locally Hölder continuous* f *in* $[t_0,T]$ *with* $\int_{t_0}^{\rho} \|f(s)\|ds < \infty$ *for some* $\rho > 0$ *and* $w_0 \in \mathcal{U}$, *the mild solution* u_0 *in (21) uniquely exists, is a classical and weak solution with* $u_0 \in C[t_0,T) \cap C^1(t_0,T)$. *This* $u_0(t) \in \mathcal{D}(A)\ \forall t \in (t_0,T)$ *represents the well known smoothing from* $w_0 \in \mathcal{U}$ *into* $u(t) \in \mathcal{D}(A)$ *for* $t > t_0$.

8. *for the special* $A \in \mathcal{L}(\mathcal{U},\mathcal{U})$, $w_0 \in \mathcal{D}(A) = \mathcal{U}$, *and continuous* f, *the mild solution in (21) is a classical solution of (21) as well.*

The following two examples of semilinear parabolic equations show an essential difference between elliptic and parabolic nonlinear problems. For elliptic and some nonlinear parabolic problems still Hilbert, Sobolev and Hölder spaces are appropriate as in Theorem 0.9. For most parabolic cases these do not fit any more, cf. Theorem 0.10, *2.* Then for fractional powers $(A+aI)^\alpha, 0 \le \alpha$ with invertible sectorial $A : \mathcal{D}(A) \subset \mathcal{U} \to \mathcal{U}$, bounded $a \ge 0$, so Re $\sigma(A + aI) > 0$, often spaces with equivalent $||u||_\alpha$ are introduced

$$\mathcal{U}^\alpha := \mathcal{D}((A + aI)^\alpha), \textit{the new } ||u|| := ||u||_\alpha := \|(A + aI)^\alpha u\|_{\mathcal{U}}, \tag{24}$$

cf. e.g. [3, 22, 39, 40, 53]. We only mention that, but do not define them explicitly

Theorem 0.9. *[17], p 116:* Distributional solutions for semilinear equations: *Let $-A$ generate an analytic semigroup, replace $f(t)$ in (23) by $f(t, u) : \mathcal{U} \to \mathcal{U}$, Lipschitz continuous in u. Then a unique distributional solution u_0 exists with $u_0 \in C([t_0, T); \mathcal{U})$.*

Theorem 0.10. *[31], p. 53:* Solutions for semilinear equations: *Let $-A$ generate an analytic semigroup $S(t)$, $f = f(t, v)$ in lines below (21), be locally Hölder and Lipschitz continuous for small $t_0 - t_j, u_0 - u_j, j = 1, 2$, in a neighborhood $\mathcal{O}' \subset \mathcal{O}_U$ of u_0, s.t.*

$$\exists L > 0, \theta > 0 : \quad \forall (t, v), (s, w) \in \mathcal{O}' : \quad ||f(t, v) - f(s, w)|| \le L(|t - s|^\theta + ||w - v||_\alpha).$$

1. *Then for any $(t_0, w_0) \in \mathcal{O}$, there exists $T = T(t_0, w_0) > 0$ such that (19) has a unique strong, simultaneously mild solution u_0 on $(t_0, t_0 + T)$.*
2. *Conversely, if $u_0 \in C(t_0, t_1) \to \mathcal{U}^\alpha$, for some $\rho > 0 : \int_{t_0}^{t_0+\rho} ||f(t, u_0(t))||dt < \infty$, and if (21) holds for $t_0 < t < t_1$, then u_0 is a strong solution of (19) in $t_0 < t < t_1$.*
3. *Under these conditions each strong is a weak solution as well.*

0.4 Center Manifold for the Local Dynamics

0.4.1 Splitting according to local dynamics

The goal of this paper, center manifolds and their numerical approximation, originate in a stationary bifurcation point $(u_0 \equiv 0, \lambda_0)$ of the elliptic part, so $G(0, \lambda_0) = 0$ of nonlinear parabolic problems. An *equilibrium*, $G(u_0) = 0$, is called *stable,* if for every neighborhood $\mathcal{V}_0$ of u_0 there exists a neighborhood $\mathcal{V}_1 \subset \mathcal{V}_0$, s.t. for every $u_1 \in \mathcal{V}_1$ the solution $u(u_1, t)$ of (18) with w_0 replaced by u_1 stays in $\mathcal{V}_0$ as long as it exists. If $\mathcal{V}_1$ can be chosen s.t. all these solutions exist for all $t > 0$ and $u(u_1, t) \to u_0$ for $t \to \infty$ we call u_0 *asymptotically stable.* A *local invariant manifold* of (18) is a subset $\mathcal{M}$ of $\mathcal{U}$ in which the solution $u(t)$ of (18) remains for some $0 < t \le T = T_{u(0)}$ and $\forall u(0) \in \mathcal{M}$. If $T = \infty$ $\forall u(0) \in \mathcal{M}$, then $\mathcal{M}$ is called a *(global) invariant manifold.* A *local or global center manifold* is tangential to $\mathcal{N}^c(\lambda_0)$, c.f. (27), at $(u_0 \equiv 0, \lambda_0)$ and locally or globally invariant. Naviér-Stokes problems admit only local center manifolds.

Lubich and Ostermann develop in [39] Section 2, a new analytical approach for center manifolds for their time discretizations. We summarize the results without and with parameters in this and the end of the last Section. This yields a solid basis for the proof of the full, so space and time discretization of center manifolds in Section 0.5.

For the abstract evolution equation (25) in a Banach space $\mathcal{U}$ with $A(\lambda)$, uniformly sectorial in λ, and $\mathcal{D}(A)$ independent of λ, we consider, below again omitting the λ,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + G(u,\lambda) = \frac{\mathrm{d}u}{\mathrm{d}t} + A(\lambda)u + R(u,\lambda) = 0, \lambda \in \Lambda \subset \mathbb{R}^d, u \in \mathcal{D}(A) \cap \mathcal{D}_u(R) \subset \mathcal{U}.$$
$$\forall\ \lambda\ A(\lambda) = \partial_u G(u_0 \equiv 0, \lambda) \text{ and } R(0,\lambda) = 0,\ R_u(0,\lambda) = \partial_u R(0,\lambda) = 0. \tag{25}$$

Λ is a neighborhood of λ_0 for a stationary bifurcation point $(u_0 \equiv 0, \lambda_0)$ of the elliptic part. (25) is modified in case studies in [16] by replacing du/dt by Sdu/dt with a boundedly invertible elliptic operator $S(\lambda)$. For parameter dependent problems the following $\lambda-$independent $A, P, Q, G, \ldots$ in (26) and below have to be replaced again by $A(\lambda), P(\lambda), Q(\lambda), G(u,\lambda), \cdots$ So for $A : \mathcal{D}(A) \to \mathcal{V}'$, eg $\mathcal{D}(A) = W^{m,p}(\Omega, \mathbb{R}^q), u \in \mathcal{D}(A) \cap \mathcal{D}_u(R) \subset \mathcal{U}$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + G(u) = \frac{\mathrm{d}u}{\mathrm{d}t} + Au + R(u) = 0, A = G_u(0), R(0) = 0,\ R_u(0) = 0. \tag{26}$$

[39] have proved for (25), (26) existence, properties and the dominant role of center manifolds $\mathcal{W}^c$, if $\sigma(A) = (\sigma_u = \emptyset) \cup \sigma_c \cup \sigma_s$ with $A\mathcal{N}^s \subset \mathcal{N}^s$, its stable manifold $\mathcal{W}^c$ and

$$\sigma_u = \emptyset,\ \mathcal{N}^u = \{0\} \text{ and } \mathcal{N}^c = \mathcal{N}^c(\lambda) = \mathcal{N}, \kappa = \dim \mathcal{N}^c \text{ small}, \tag{27}$$

so a *non-hyperbolic bifurcation point.*

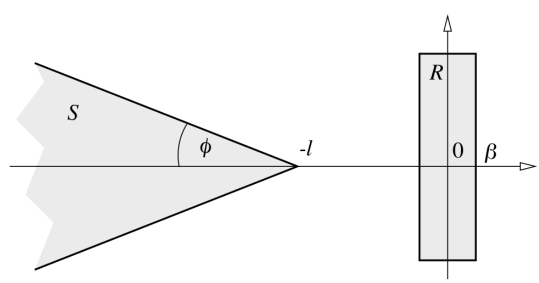

Figure 2. *Spectrum of $-A$ contained in the shaded area.*

Numerical approximation requires generalizing the spectrum σ_c to eigenvalues with $|\mathrm{Re}\ \mu| \leq \beta$ small instead of $\beta = 0$. This $\mathcal{N}$ is spanned by the basis of generalized eigenfunctions φ_i for A and $\mu \in \sigma_c$. With the conjugate eigen-values and -vectors φ_i' of its dual A^d s.t. with usually omitted indices in, e.g., $\langle \varphi_i', \varphi_j \rangle_{\mathcal{U}' \times \mathcal{U}} = \langle \varphi_i', \varphi_j \rangle$ we obtain

$$\mathcal{N} = span[\varphi_1, \dots, \varphi_\kappa] \subset \mathcal{U},\ \varphi_1', \dots, \varphi_\kappa' \in \mathcal{U}',\ \langle \varphi_i', \varphi_j \rangle = \delta_{i,j},\ i, j = 1, \dots, \kappa, \tag{28}$$

So $\sigma_c \subset \mathcal{R}$ and $\sigma_s \subset \mathcal{S}$ are subsets of a rectangle $\mathcal{R}$ and a sector $\mathcal{S}$ and a gap between, cf. Figure 2. We associate $\mathcal{R}, \mathcal{S}$, with the spectral projectors $P : \mathcal{U} \to \mathcal{N},\ Q : \mathcal{U} \to \mathcal{M}$

$$\begin{aligned} P = \frac{1}{2\pi i} \int_{\partial \mathcal{R}} (z + A)^{-1}\, \mathrm{d}z,\ Q = I - P,\ \mathcal{U} = \big(\mathcal{N} = P(\mathcal{U})\big) \oplus \big(\mathcal{M} := Q(\mathcal{U})\big), \\ \forall u \in \mathcal{U} :\ v := Pu := \sum_{i=1}^{\kappa} \langle \varphi_i', u \rangle \varphi_i =: \sum_{i=1}^{\kappa} v_i \varphi_i =: (\mathbf{v}, \Phi) \in \mathcal{N} \cong \mathbb{R}^\kappa \text{ and} \\ \mathbb{R}^\kappa \ni \mathbf{v} = (\mathbf{v}_i = v_i)_{i=1}^n, \mathbf{P}u := \mathbf{v},\ w = Qu \in \mathcal{M} \cong \mathbf{U},\ u = v + w \in \mathcal{N} \oplus \mathcal{M}. \end{aligned} \tag{29}$$

The standard notations for multi-index, factorial, power, derivative, c.f. (1),(2), yields

$$\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_\kappa),\ \mathbf{v}^k = \mathbf{v}_1^{k_1} \dots \mathbf{v}_\kappa^{k_\kappa}, \frac{\partial \mathbf{v}^k}{\partial \mathbf{v}_1} = k_1 \mathbf{v}_1^{k_1 - 1} \mathbf{v}_2^{k_2} \dots \mathbf{v}_\kappa^{k_\kappa} =: k_1 \frac{\mathbf{v}^k}{\mathbf{v}_1}, \cdots,\ |\mathbf{v}| \tag{30}$$

norm on $\mathbb{R}^\kappa$. Splitting (25) according to $P, \mathbf{P}, Q$, the sectorial A is block diagonalized as

$$A \cong \begin{pmatrix} B & 0 \\ 0 & L \end{pmatrix}, \tag{31}$$

with $L = QA_{|\mathcal{M}}$ sectorial on $\mathcal{M} \cong \mathbf{U}$ and $B \in \mathbb{R}^{\kappa \times \kappa}$ on $\mathbb{R}^\kappa$. B is a diagonal matrix of Jordan blocks one for each eigen value $\mu \in \mathcal{R}$. By Definition 0.2, the resolvents satisfy

$$\begin{aligned} |(z + B)^{-1}| &\le \frac{K}{\mathrm{dist}(z, \mathcal{R})} \qquad \text{for } z \notin \mathcal{R} \\ |(z + L)^{-1}| &\le \frac{K}{\mathrm{dist}(z, \mathcal{S})} \qquad \text{for } z \notin \mathcal{S} \text{ and a constant } K > 0. \end{aligned} \tag{32}$$

For the above α, $\mathbf{U}^\alpha = \mathcal{D}(L^\alpha)$ and norm we decompose the nonlinearity into

$$R(u) = \mathbf{P}R(u) + QR(u) \cong \begin{pmatrix} \mathbf{P}R(u) \\ QR(u) \end{pmatrix} = -\begin{pmatrix} f(\mathbf{v}, w) \\ g(\mathbf{v}, w) \end{pmatrix} \in \begin{pmatrix} \mathbb{R}^\kappa \\ \mathcal{M} \end{pmatrix}, \tag{33}$$

where $f : \mathbb{R}^\kappa \times \mathbf{U}^\alpha \to \mathbb{R}^\kappa, g : R^\kappa \times \mathbf{U}^\alpha \to \mathbf{U}$ are as differentiable as R near $(0, 0)$ s.t.

$$\begin{gathered} f(0,0) = 0\ , \quad g(0,0) = 0 \\ \partial_{\mathbf{v}} f(0,0) = 0\ , \quad \partial_w f(0,0) = 0\ , \quad \partial_{\mathbf{v}} g(0,0) = 0\ , \quad \partial_w g(0,0) = 0\ . \end{gathered} \tag{34}$$

For studying the local center manifold for (25), [39] modify these f, g with a problem depending sufficiently small $\rho > 0$ by a *smooth cutting function*

$$\begin{gathered} \chi : \mathbb{R}^\kappa \to [0,1], \chi(\mathbf{v}) = 1 \textit{ for } |\mathbf{v}| \le \rho \textit{ and } \chi(\mathbf{v}) = 0 \textit{ for } |\mathbf{v}| \ge 2\rho,\ \textit{replace} \\ f(\mathbf{v}, w), g(\mathbf{v}, w) \textit{ by } f(\chi(\mathbf{v})\mathbf{v}, w),\ g(\chi(\mathbf{v})\mathbf{v}, w),\ \textit{maintaining (34).} \end{gathered} \tag{35}$$

Then [39] prove the existence of germs for a local center manifold for

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -(B\mathbf{v} + \mathbf{P}R(\mathbf{v}+w)) = -B\mathbf{v} + f(\mathbf{v}, w) \tag{36}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -(Lw + QR(\mathbf{v}+w)) = -Lw + g(\mathbf{v}, w), \text{ with} \tag{37}$$

$$w = W(\mathbf{v}) = W^\rho(\mathbf{v}) \approx W^\iota(\mathbf{v}) = \sum_{2\le|k|}^{\iota} w_k \mathbf{v}^k, \iota \le \rho,\ w_k = 0 \textit{ for } |k| < 2. \tag{38}$$

Theorem 0.11. Exponentially attracting center manifold: *Assume a non hyperbolic bifurcation point $u_0 \equiv 0$ of the elliptic part with a sectorial linearized operator A and a spectrum split by the projectors P, Q into $\mathcal{R}$ and $\mathcal{S}$ with a gap $\beta < \ell$, and nonlinear terms R, f, g, Lipschitz continuous w.r.t. $(\mathbf{v}, w) \in \mathbb{R}^\kappa \times \mathbf{U}^\alpha$, cf. Theorem 0.5, (24),(26),(27). Finally define for a sufficiently small ρ, the cutting function χ in (35). Then (36),(37) admits a* local exponentially attracting invariant center manifold, *given as the graph of a Lipschitz continuous map $W : \mathbb{R}^\kappa \to \mathcal{V}^\alpha$ with $W(0) = 0$, cf. (38).*

0.4.2 Asymptotic expansion in the homological equation

It is folklore that for center manifolds of finitely determined problems the determining terms are uniquely determined, cf. Ashwin/Böhmer/Mei [4]. This is proved in Haragus/Iooss in Chapter 2, Remark 2.14: Local center manifolds are in general not unique even though the Taylor expansion at the origin is unique. This is due to the occurrence in the proof of a smooth cut-off function χ in (35), which is not unique. So our full discretization methods studied here only converge for these restricted problems, [16].

Theorem 0.12. Uniquely determined terms of a center manifold: *Assume the conditions of Theorem 0.11 for a $\rho-$determined problem. Then the coefficients w_k of the asymptotic expansion of the center manifold in (38) are uniquely determined for $|k| \le \rho$.*

The local dynamics of (26) would be known with the parameterization in (38)

$$\mathcal{W}^c := \{(\mathbf{v}, w) \in \mathbb{R}^\kappa \times \mathcal{M}, W, W_{\mathbf{v}} : \mathbb{R}^\kappa \to \mathcal{M}, w = W(\mathbf{v}), W(0) = 0, W_{\mathbf{v}}(0) = 0\}. \tag{39}$$

Then we could split (26) into two subproblems: Insert $w = W(\mathbf{v})$ into (36) and

$$\text{solve } \dot{\mathbf{v}} = -(B\mathbf{v} + \mathbf{P}R(\mathbf{v} + W(\mathbf{v}))), \text{ with this } \mathbf{v} \text{ solve (37).} \tag{40}$$

This unrealistic procedure is avoided by recursively computing the approximation $W^\iota(\mathbf{v})$ in (38) for $W(\mathbf{v})$ by the terms up to the order $\iota \le \rho$ of its asymptotic expansion. Here the Lipschitz continuity of the map $W : \mathbb{R}^\kappa \to \mathcal{V}^\alpha$ is essential. Due to the cut-off function in (35) we only get local results for $\mathbf{v} \approx 0$. So we recursively determine, starting with $|k| = 2$, the w_k, then $= 3, \ldots$: We formulate for $W(\mathbf{v})$ a characterizing, so called *homological equation* $C(W(\mathbf{v})) = 0$ in (41): Insert $\frac{\mathrm{d}}{\mathrm{d}t}\, w = W_{\mathbf{v}}(\mathbf{v})\frac{\mathrm{d}}{\mathrm{d}t}\, \mathbf{v}$ with $\frac{\mathrm{d}}{\mathrm{d}t}\, \mathbf{v}$ from (36) into (37) and obtain with (29) - (33) this $C(W(\mathbf{v}))$ and solve approximately the

$$C(W(\mathbf{v})) := -W_{\mathbf{v}}(\mathbf{v})\big(B\mathbf{v} + \mathbf{P}R(\mathbf{v} + W(\mathbf{v}))\big) + \big(LW(\mathbf{v}) + QR(\mathbf{v} + W(\mathbf{v}))\big) = 0. \quad (41)$$

For $W(\mathbf{v}) = W^{\iota}(\mathbf{v}))+\mathcal{O}(|\mathbf{v}|^{\iota+\mathbf{1}})$ we combine appropriate $|k|-$linear forms $w_k : (\mathbb{R}^{\kappa})^{|k|} \to \mathcal{M}$ applied to $|k| \in \mathbb{N}$ identical arguments $\mathbf{v}$. We want to determine the $w_k \in \mathcal{M}$

$$W(\mathbf{v}) \approx W^{\iota}(\mathbf{v}) = \sum_{2\le|k|}^{\iota} w_k\mathbf{v}^k \text{ aymptotically satisfying (41) as } C(W^{\iota}(\mathbf{v})) = \mathcal{O}(|\mathbf{v}|^{\iota+1}). \quad (42)$$

Collecting the different terms for the same powers of $\mathbf{v}^k$ in $C(W(\mathbf{v}))$ we obtain

$$C(W^{\iota}(\mathbf{v})) = \sum_{|k|\ge 2}^{\iota} \beta_k\mathbf{v}^k + \mathcal{O}(|\mathbf{v}|^{\iota+1}) = 0, \text{ with } \beta_k = 0 \text{ for } |k| < 2 \text{ by (39).} \quad (43)$$

Starting with $\iota = 2$ we determine the $w_k, |k| = 2$, in (42) from (43) by equating these $\beta_k = 0$. Then we continue with $\iota = 3, \ldots,$ inserting $W^{\iota}(\mathbf{v})$ with the known $w_k, |k| < \iota$ and the still unknown $w_k, |k| = \iota$ into $C(W(\mathbf{v}))$, Then (41), (43) are satisfied up to $\mathcal{O}(\|\mathbf{v}\|^{|k|+1})$. This implies as an immediate consequence Theorem 0.13 *1., 3.*. *2.* essentially is proved below at (52), cf. [16]. Here we formulate the $\lambda-$dependent case:

Theorem 0.13. A unique asymptotic expansion exists for the center manifold $W(\mathbf{v}, \lambda)$: *Under the conditions (25) - (27), cf. [39], we assume an approximate operator* W^{ι} : $\mathbb{R}^{\kappa} \times \Lambda \to \mathcal{M}$, *with* $W^{\iota}(0, \lambda) = 0, (W^{\iota})_{\mathbf{v}}(0, \lambda) = 0,$ *let* $C(W^{\iota}(v, \lambda)) = \mathcal{O}(\|(\mathbf{v}, \lambda-\lambda_0)\|^{\iota+1})$ *for small* $|\mathbf{v}|$, *all* λ *near* λ_0, $\|(\mathbf{v}, \lambda - \lambda_0)\| \to 0$ *and* $2 \le \iota \le \rho$.

1. *This implies, with unique k- linear operators* $w_k(\lambda)\mathbf{v}^k$, *for* $\|(\mathbf{v}, \lambda - \lambda_0)\| \to 0$

$$W(\mathbf{v}, \lambda) + \mathcal{O}(\|(\mathbf{v}, \lambda - \lambda_0)\|^{\iota+1} = W^{\iota}(\mathbf{v}, \lambda) := \sum_{2\le|k|}^{\iota} w_k(\lambda)\mathbf{v}^k \quad \textit{hence} \quad (44)$$

W^{ι} *represents the* unique asymptotic expansion *for* $W(\mathbf{v}, \lambda)$ *up to the order* ι.

2. *In particular, each system starting* $\iota = 2, \ldots,$ *is the compact perturbation of a coercive principal part and is uniquely solvable, hence is nonsingular.*

3. *With* $B(\lambda), f(\mathbf{v}, w, \lambda)$ *in (31),(33) we solve the* asymptotically reduced equations (36),(37) for the center manifold *for* $\iota = 2, \ldots, \rho$, *to* determinacy,

$$\frac{d\mathbf{v}}{dt} + B(\lambda)\mathbf{v} = f(\mathbf{v}, W^{\iota}(\mathbf{v}, \lambda), \lambda) + \mathcal{O}(\|(\mathbf{v}, \lambda - \lambda_0)\|^{\iota+1}). \quad (45)$$

The second parabolic differential equation (37) can then be (often numerically) solved with this $w = W^{\iota}(\mathbf{v}, \lambda)$ *up to* $\mathcal{O}(\|(\mathbf{v}, \lambda - \lambda_0)\|^{\iota+1})$.

This result allows the transformation of (40) into normal form: We can stop computing the $w_k, |k| = 2, 3, \ldots, \iota$ when we have reached *determinacy* $\iota = \rho$.

If the normal form is fully determined by second order terms, then $W(\mathbf{v})$ in (40) can be neglected. In some sense, this is not only numerically the easiest possible case. Indeed the equation for the center manifold has the form, cf. (36), (37),

$$\dot{\mathbf{v}} = -(B\mathbf{v} + \mathbf{P}R(\mathbf{v})) + \mathcal{O}(|\mathbf{v}|^3), \dot{w} = -Lw + \mathcal{O}(|\mathbf{v}|^3) \text{ and we are done!} \tag{46}$$

0.4.3 Recursive systems for solving the homological equation

To solve (43), we insert the components $\mathbf{v}_j$ in (29) into (45)

$$\dot{\mathbf{v}}_i = -\sum_{j=1}^{\kappa} \langle \varphi_i', A\varphi_j \rangle \mathbf{v}_j - \langle \varphi_i', R(\sum_{j=1}^{\kappa} \mathbf{v}_j \varphi_j + W(\mathbf{v})) \rangle, \; i = 1, \ldots, \kappa. \tag{47}$$

This generates the matrix J and the ODE for the center manifold, cf. (29), (33), (42),

$$J = (J_{ij})_{i,j=1}^{\kappa} := \langle \varphi_i', A\varphi_j \rangle_{i,j=1}^{\kappa}, \dot{\mathbf{v}} = -J\mathbf{v} - \mathbf{P}R((\mathbf{v}, \Phi) + W(\mathbf{v})), W(\mathbf{v}) = \sum_{2 \le |k|}^{\rho} w_k \mathbf{v}^k. \tag{48}$$

A combination with (29), (33), (37), (39), (41), $Q = I - P$, yields for $\dot{w}$ and $C(W(\mathbf{v}))$

$$\begin{aligned} \dot{w} &= \frac{d}{dt} W(\mathbf{v}) = \sum_{i=1}^{\kappa} \frac{\partial W}{\partial \mathbf{v}_i}(\mathbf{v}) \dot{\mathbf{v}}_i = -\sum_{i=1}^{\kappa} \frac{\partial W}{\partial \mathbf{v}_i}(\mathbf{v}) \big(J\mathbf{v} + \mathbf{P}R((\mathbf{v}, \Phi) + W(\mathbf{v})) \big)_i, \text{and} \\ C(W(\mathbf{v})) &= -\sum_{i=1}^{\kappa} \sum_{2 \le |k|}^{\rho} \frac{k_i}{\mathbf{v}_i} w_k \mathbf{v}^k \big(\sum_{j=1}^{\kappa} J_{ij} \mathbf{v}_j + \langle \varphi_i', R(\sum_{j=1}^{\kappa} \mathbf{v}_j \varphi_j + \sum_{2 \le |\nu|}^{\rho} w_\nu \mathbf{v}^\nu \rangle \big) \\ &+ \Big(I \bullet - \sum_{i=1}^{\kappa} \langle \varphi_i', \bullet \rangle \varphi_i \Big) \Big(A \sum_{2 \le |k|}^{\rho} w_k \mathbf{v}^k + R(\sum_{j=1}^{\kappa} \mathbf{v}_j \varphi_j + \sum_{2 \le |\nu|}^{\rho} w_\nu \mathbf{v}^\nu) \Big) + \mathcal{O}(\|\mathbf{v}\|^{\rho+1}) = 0. \end{aligned} \tag{49}$$

Collecting for $C(W(\mathbf{v}))$ the powers $\mathbf{v}^k$ yields the explicit form of their coefficients β_k in (43) and (50). It is important that we have a specific structure for $\beta_k := 0 \; \forall |k| < 2$. The terms, linear in the w_ν only depend upon the unknown w_ℓ with $|\ell| = |k|$, those nonlinear in the w_ν, the G_k in (50), only upon the known w_ν with $|\nu| < |k|$. This will be important for the numerical methods below. Collecting all the linear and nonlinear terms we obtain, c.f. [16],

$$C(W(\mathbf{v})) = \sum_{2 \le |k|}^{\rho} \mathbf{v}^k \big(-\sum_{|\ell| = |k|} w_\ell \beta_{\ell,k} + (I - P)A|_{\mathcal{M}} w_k + G_k(w_\nu, \; |\nu| < |k|) \big) + \mathcal{O}(h^{\rho+1})$$

$$\textit{with } k_{ij}^- := (k_1, ., k_i + 1, ., k_j - 1, ., k_\kappa), \quad \sum_{|\ell| = |k|} w_\ell \beta_{\ell,k} = \sum_{\substack{k_i > 0 \\ i=1}}^{\kappa} k_i \big(w_k J_{ii} + \sum_{\substack{k_j < |k| \\ 1 = j \ne i}}^{\kappa} J_{ij} w_{k_{ij}^-} \big). \tag{50}$$

According to the strategy indicated above, we annihilate the $\beta_k \forall\ |k| \leq \rho$, so up to determinacy, hence solve by (49),(50) the following equations for the $w_k \in \mathcal{M}$:

$$\beta_k = (I-P)A|_{\mathcal{M}} w_k - \sum_{|\ell|=|k|} w_\ell \beta_{\ell,k} + G_k(w_\nu,\ |\nu| < |k|) = 0,\ \forall\ 2 \leq |k| \leq \rho. \tag{51}$$

In contrast to Liapunov-Schmidt methods these equations (51) do not allow to compute each w_k separately, but only from a sequence of bordered coupled systems starting with $\forall k : |k| = 2$, then for $\forall k : |k| = 3$, a.s.o. The systems have to enforce the two conditions $C(W(\mathbf{v})) \in \mathcal{M}$ or $\beta_k \in \mathcal{M}$ and $w_k \in \mathcal{M}$. The w_k is coupled to the w_ℓ still to be determined with $|\ell| = |k|$, and to already known w_ν with $|\nu| < |k|$. The coupling for different k is realized via the terms $\sum_{|\ell|=|k|} w_\ell \beta_{\ell,k}$ and $G_k(w_\nu,\ |\nu| < |k|)$, the latter a compact perturbations of Aw_ℓ and known functions, resp. The following terms $\langle \varphi'_j, w_k \rangle = 0$ enforce a $w_k \in \mathcal{U}$ into $w_k \in \mathcal{M}$, the $\sum_{j=1}^{\kappa} \alpha_{k,j}\varphi_j$ compensates replacing $(I-P)A|_{\mathcal{M}} w_k$ in (51) by Aw_k. The systems for $|k| = 2, 3, \ldots, \rho \leq q$, are linear in the unknowns $w_\ell \in \mathcal{U}, \vec{\alpha}_k := (\alpha_{k,j})_{j=1}^{\kappa} \in \mathbb{R}^\kappa, |k| = |\ell|$. So we obtain

$$\begin{aligned} &F_k(\cdot) := Aw_k - \sum_{|\ell|=|k|} w_\ell \beta_{\ell,k} + \sum_{j=1}^{\kappa} \alpha_{k,j}\varphi_j + G_k(w_\nu,\ |\nu| < |k|) = 0, \quad \textit{and} \\ &\langle \varphi'_j, w_k \rangle = 0, \quad j = 1, \ldots, \kappa, \quad \textit{with known } \beta_{\ell,k}, \text{ recursively solve for } |k| = 2, 3, \ldots, \rho. \end{aligned} \tag{52}$$

This system is a compact perturbation of a coercive elliptic system with imposed side conditions, for details cf.(60). By Theorem 0.13, we do know that the system (49), or equivalently one of the (50) - (52) is uniquely solvable for the $w_k, \vec{\alpha}_k, |k| = 2$, then $= 3, \cdots \leq \rho$. The β_k with $|k| = 2$ are obtained by neglecting the $W(\mathbf{v})$ term in $R(\cdot)$ and uniquely define the $w_k, \vec{\alpha}_k$ for $|k| = 2$. If we procede to the β_k with $|k| = 3$ the w_k with $|k| = 2$ have already been computed, so the $G_k(w_\nu, |\nu| < |k|)\mathbf{v}^k$ are known. So the w_k with $|k| = 3$ are uniquely determined by the β_k with $|k| = 3$ a.s.o. So the corresponding operators are all boundedly invertible. We summarize our discussion in

Theorem 0.14. Bounded invertible linear systems imply a unique asymptotic expansion until determinacy for the center manifold: *Under the conditions for center manifolds in (26), (27), (28) and Theorem 0.13, the system (49), guarantees the unique approximate asymptotic expansion* $W^\iota : \mathbb{R}^\kappa \to \mathcal{M}$, *with* $W^\iota(0) = 0, (W^\iota)_{\mathbf{v}}(0) = 0$ *and* $C(W^\iota(\mathbf{v})) = \mathcal{O}(\|(\mathbf{v})\|^{\iota+1})$ *for all sufficiently small* $|\mathbf{v}| \to \mathbf{0}$ *and up to determinacy, so* $\iota \leq \rho$ *for the center manifold of (26) by Theorem 0.13. Therefore the systems (52), linear in the* $w_k, \vec{\alpha}_k\ \forall k$ *are uniquely solvable, recursively for* $|k| = 2, 3, \ldots, \rho$ *for these* $w_k, \vec{\alpha}_k$. *Hence they define, recursively for* $|k| = 2, 3, \ldots,$ *boundedly invertible systems of linear elliptic operators.*

0.5 Space Discretization for Center Manifolds

The summarized results in Section 0.4 and at the end of this Section yield a solid foundation for the convergence proof of the full, so space and time discretization of local center manifolds. The information about the structure of their local dynamics is concentrated in the asymptotic solution of (36), so of elliptic problems. Again we first omit the fixed λ, and start solving the stationary elliptic problem, c.f. (3), (52) with numerical methods. Only very seldom the solution of the complete problem (36), (37) for the parabolic (18) is interesting. So time discretization is only indicated at the end.

The equation in (52) for each $w_k, |k| = 2, \dots$ and the other w_ν with $\nu = |k|$ is linear in these unknowns, but nonlinear only in the already known $w_\nu, |\nu| < |k|$ via the nonlinear term G_k. Therefore we reduce the discretization here to linear operators.

(52) does not live in $\mathcal{U} = W_0^{m,p}(\Omega, \mathbb{R}^q)$, but in product spaces. For $|k| = \iota$ we have to compute $s_\iota := \sum_{|k|=\iota} 1$ different $w_k(x) \in \mathbb{R}^q$ with $\iota = |k| = 2, 3, \dots, \rho$. Furthermore, there are $s_\iota \kappa$ different $\alpha_{k,j}$, and $\langle \varphi'_j, w_k \rangle = 0$. We denote the linear operator for the full system as $\mathbf{A} = \mathbf{A}_\iota$, based upon $A := G_u^0 = G'(u_0)$.

We only give a short summary for simplified discretization methods and linear operators, for a general theory with nonlinear $G : \mathcal{D}(G) \subset \mathcal{U} \to \mathcal{V}'$ see [15, 16]. Often we have $\mathcal{U} = \mathcal{V}$ including boundary conditions. On a sequence of subspaces $\mathcal{U}^h, \mathcal{V}'^h$ for $h \to 0$ we consider *Petrov-Galerkin approximations* for a linear $A : \mathcal{U} \to \mathcal{V}'$. Choose $\mathcal{U}^h \subset \mathcal{U}, \mathcal{V}^h \subset \mathcal{V}$, e.g. for variational crimes only approximating $\mathcal{U}^h \not\subset \mathcal{U}, \mathcal{V}^h \not\subset \mathcal{V}$,

$$\mathcal{U}^h \subset \mathcal{U}, \mathcal{V}^h \subset \mathcal{V} \textit{ with } \dim \mathcal{U}^h = \mathit{dim}\, \mathcal{V}^h, \lim_{h \to 0} \mathit{dist}(u, \mathcal{U}^h) = 0 \ \forall u \in \mathcal{U}, (v \in \mathcal{V}). \tag{53}$$

Then interpolation and approximation operators I^h and P^h exist and projection operators Q'^h or quadrature approximations $\tilde{Q}^h \approx Q^h$ and discretizations are defined by

$$\forall u \in \mathcal{U} : \ \|P^h u - u\|_{\mathcal{U}^h} \ \mathit{or} \ \|Q^h u - u\|_{\mathcal{U}^h} \to 0 \ \mathit{for}\ h \to 0, \ \mathit{and} \tag{54}$$

$$Q'^h \in \mathcal{L}(\mathcal{V}', \mathcal{V}'^h) \ \text{ by } \ Q'^h f := f|_{\mathcal{V}^h} \in \mathcal{V}^{h'} \Leftrightarrow \langle Q'^h f - f, v^h \rangle^h_{\mathcal{V}' \times \mathcal{V}^h} = 0 \ \forall\ v^h \in \mathcal{V}^h,$$

$$\begin{array}{ccccccc} & \mathcal{U} & \xrightarrow{A} & & \mathcal{V}' & \overset{\mathit{tested\ by}}{\longleftrightarrow} & \mathcal{V} \\ I^h, P^h & \Big\downarrow & \Phi^h \Big\downarrow & Q'^h(\tilde{Q}^h) & \Big\downarrow & & \\ & \mathcal{U}^h & \xrightarrow{A^h} & & \mathcal{V}^{h'} & \overset{\mathit{tested\ (approximately)\ by}}{\longleftrightarrow} & \mathcal{V}^h \end{array} \tag{55}$$

The solutions $u_0^h \in \mathcal{U}^h$ are determined, possibly with appropriate extensions $A_h, \tilde{A}_h$ for variational crimes, quadrature formulas and collocation methods, from $\tilde{A}^h = \tilde{Q}^h \tilde{A}_h$ or

$$A^h u_0^h := Q'^h A_h u_0^h = Q'^h A_h|_{\mathcal{U}^h} u_0^h = Q'^h f \Leftrightarrow \langle A_h u_0^h - f, v^h \rangle^h_{\mathcal{V}' \times \mathcal{V}^h} = 0 \ \forall v^h \in \mathcal{V}^h. \tag{56}$$

Basic concepts are consistency in u and of order $p > 0$, satisfied by (55) and stability:

$$\|Q'^h A_h P^h u - Q'^h A u\|_{\mathcal{V}^{h'}} \to 0 \ \mathit{and}\ = \mathcal{O}(h^\ell) \ \mathit{e.g.} = C\|u\|_{W^{\ell,p}(\Omega;\mathbb{R}^q)} h^\ell \ \mathit{for}\ h \to 0 \tag{57}$$

$$\text{and } \exists h_0, S \in \mathbb{R}_+ \ \mathit{s.t.} \ \forall u_i^h \in \mathcal{U}^h, i = 1, 2 \Rightarrow \|u_1^h - u_2^h\|_{\mathcal{U}^h} \le S \|A^h u_1^h - A^h u_2^h\|_{\mathcal{V}^{h'}}. \tag{58}$$

All methods mentioned above are shown to be consistent, stable and satisfy Theorems 0.15 - 0.17, cf. [15, 16].

Theorem 0.15. Unique converging solution: *Let u_0 be the exact solution of $Au_0 = f$, $A \in \mathcal{L}(\mathcal{U}, \mathcal{V}')$ for a boundedly invertible A. Let its discretization $A^h : \mathcal{U}^h \to \mathcal{V}^{h'}$, satisfying (54)-(58), be consistent in $P^h u_0$ and stable. Then $\forall 0 < h \leq h_0$ there exists a unique discrete solution $u_0^h \in \mathcal{U}^h$ for $A^h u^h = f$ or $= Q^{'h} f$ s.t. u_0^h converge according to*

$$\|u_0^h - u_0\|_{\mathcal{U}^h} \leq S\|A^h P^h u_0 - Q^{'h} f\|_{\mathcal{V}'^h} \textit{ and } \leq \mathcal{O}(h^\ell) \text{ e.g.} = C\|u\|_{W^{\ell,p}(\Omega;\mathbb{R}^q)} h^\ell, \quad (59)$$

so of order ℓ. This holds, e.g., for a coercive A, hence with a stable discretization A^h.

Theorem 0.16. Compact linear perturbations of invertible B remain stable: *For $A, B, C \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ with boundedly invertible, A, B, $A = B + C$, stable B^h, and compact (low order) perturbation C, the A^h is stable as well.*

Very important for nonlinear problems is their possible equivariance. It strongly influences and determines the structure of the solutions. So the discretization methods have to inherit this equivariance. In [16] this is studied in two chapters on spectral methods for infinite groups and in 'Numerical Exploitation of Equivariance for Finite Groups'. We summarize for all the following results:

Theorem 0.17. Inherit equivariance in numerical methods: *If the chosen discretization methods inherits the equivariance of the elliptic or parabolic problem, for all the listed methods and following numerical results the corresponding equivariance results remain valid.*

We apply Theorems 0.15-0.17 to the systems (51) and the bordered forms (52). By Theorem 0.14 the matrix operator for (52) is boundedly invertible. In [16] we have proved that the equation (52) for the $w_k, \alpha_{k,j}$, with the known functions G_k, represents a modified compact perturbation of a coercive, hence stable system defined by its main diagonal in (61). So by Theorems 0.16, modified, and 0.15, all these discretization of (52) are stable and consistent with the results on convergence of eigenvalue problems of discretized elliptic problems in [15]. So we get convergent approximations $w_\nu^h, \alpha_{\nu,j}^h$ for $|\nu| = |k| = \iota, j = 1, \ldots, \kappa$. We go into the details:

To formulate (52) as a matrix equation we need an ordering for the $w_k, \vec{\alpha}_k = (\alpha_{k,j})_{j=1}^{\kappa}$, e.g., with $w_{k^1} = w_{(\iota,\ldots,0)}$, $w_{k^2} = w_{(\iota-1,1,0\ldots,0)}, w_{k^{s_\iota}}$ with $s_\iota = \sum_{|k|=\iota} 1$. With

$\mathcal{N} = \text{span}[\Phi = (\varphi_j)_{j=1}^{\kappa}]$ and $\mathcal{N}^d = \text{span}[\Phi' = (\varphi'_j)_{j=1}^{\kappa}]$ the operator matrices are

$$\mathbf{B}_\iota + \mathbf{C}_\iota = \mathbf{A}_\iota := \tag{60}$$
$$\begin{pmatrix} A_{e,k^1} & -\beta_{k^2,k^1} & -\beta_{k^3,k^1} & \dots & -\beta_{k^{s_\iota},k^1} & \Phi & 0 & 0 & 0 \\ -\beta_{k^1,k^2} & A_{e,k^2} & -\beta_{k^3,k^2} & \dots & -\beta_{k^{s_\iota},k^2} & 0 & \Phi & 0 & 0 \\ & \ddots & \ddots & \dots & & \ddots & \ddots & & \\ -\beta_{k^1,k^{s_\iota}} & -\beta_{k^2,k^{s_\iota}} & -\beta_{k^3,k^{s_\iota}} & \dots & A_{e,k^{s_\iota}} & 0 & 0 & 0 & \Phi \\ \Phi' & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \Phi' & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ & \ddots & \ddots & & & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & \Phi' & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $A_{e,k^i} w_{k^i} := (A - \beta_{k^i,k^i}) w_{k^i}$, the identity I_κ in $\mathbb{R}^{\kappa\times\kappa}$, the principle part B of A, so

$$\mathbf{B}_\iota := \begin{pmatrix} B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_\kappa & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_\kappa \end{pmatrix} \text{ is coercive and } \mathbf{A}_\iota - \mathbf{B}_\iota = \mathbf{C}_\iota. \tag{61}$$

So (52) for the unknowns $w_k, \vec{\alpha}_k := (\alpha_{k,j})_{j=1}^{\kappa}$ for each $\iota = |k| = 2, 3, \dots, \rho$ has the form

$$\mathbf{A}_\iota \times u_{\iota,0} := \big(\mathbf{B}_\iota + \mathbf{C}_\iota\big) \times \begin{pmatrix} w_{k^1} \\ w_{k^2} \\ \ddots \\ w_{k^{s_\iota}} \\ \vec{\alpha}_{k^1} \\ \vec{\alpha}_{k^2} \\ \ddots \\ \vec{\alpha}_{k^{s_\iota}} \end{pmatrix} = -f_\iota := -\begin{pmatrix} G_{k^1} \\ G_{k^2} \\ \ddots \\ G_{k^{s_\iota}} \\ 0 \\ 0 \\ \ddots \\ 0 \end{pmatrix} \tag{62}$$

Now we are able to formulate the discrete counterparts of our above systems. To this end we simply have to go through Section 0.4 and replace all the exact data by their discrete approximate counterparts. So e.g. in (60) – (62) the $\mathbf{A}_\iota = \mathbf{B}_\iota + \mathbf{C}_\iota$ $w_k, \vec{\alpha}_k = (\alpha_{k,j})_{j=1}^{\kappa}$, $\mathcal{N} = \text{span}[\Phi = (\varphi_j)_{j=1}^{\kappa}], \mathcal{N}^d = \text{span}[\Phi' = (\varphi'_j)_{j=1}^{\kappa}]$ are replaced by $\mathbf{A}_\iota^h = \mathbf{B}_\iota^h + \mathbf{C}_\iota^h$ $w_k^h, \vec{\alpha}_k^h = (\alpha_{k,j}^h)_{j=1}^{\kappa}$, $\mathcal{N}^h = \text{span}[\Phi^h = (\varphi_j^h)_{j=1}^{\kappa}], (\mathcal{N}^h)^d = \text{span}[\Phi'^h =$

$(\varphi_j^{'h})_{j=1}^{\kappa}]$. So we obtain e.g, instead of the (47), (48) with $W^h(\mathbf{v}^h) = \sum_{2\le|k|}^{\rho} w_k^h(\mathbf{v}^h)^k$,

$$\dot{\mathbf{v}}_i^h = -\sum_{j=1}^{\kappa}\langle\varphi_i^{'h}, A^h\varphi_j^h\rangle\mathbf{v}_j^h - \langle\varphi_i^{'h}, R(\sum_{j=1}^{\kappa}\mathbf{v}_j^h\varphi_j^h + W^h(\mathbf{v}^h))\rangle,\ i = 1,\ldots,\kappa., \tag{63}$$

$$J^h = (J_{ij}^h)_{i,j=1}^{\kappa} := \langle\varphi_i^h, A^h\varphi_j^h\rangle_{i,j=1}^{\kappa}, \dot{\mathbf{v}}^h = -J^h\mathbf{v}^h - \mathbf{P}^h R((\mathbf{v}^h, \Phi^h) + W^h(\mathbf{v}^h)). \tag{64}$$

Modifying (60)-(62), we obtain with $u_\iota^h = (w_{k^1}^h, \ldots, w_{k^{s_\iota}}^h, \vec{\alpha}_{k^1}^h, \ldots, \vec{\alpha}_{k^{s_\iota}}^h)^t$, the equation

$$\mathbf{A}_\iota^h u_{\iota,0}^h = -f_\iota^h = -Q_\iota^{'h} f_\iota \in \mathcal{U}_\iota^{'h}. \tag{65}$$

Our modified Theorems 0.16 in [16] for a *generalized compact perturbation* $\mathbf{C}_\iota$ of $\mathbf{A}_\iota$ applied to a boundedly invertible $\mathbf{B}_\iota$ with stable $\mathbf{B}_\iota^h$ yields stability for $\mathbf{A}_\iota^h$ and with an extension of the above consistency the convergence of

$u_\iota^h = (w_{k^1}^h, \ldots, w_{k^{s_\iota}}^h, \vec{\alpha}_{k^1}^h, \ldots, \vec{\alpha}_{k^{s_\iota}}^h) \to u_\iota = (w_{k^1}, \ldots, w_{k^{s_\iota}}, \vec{\alpha}_{k^1}, \ldots, \vec{\alpha}_{k^{s_\iota}})$.

The equations (49)-(52) require Taylor formulas for the $W(\vec{v})$ and $C(W(\vec{v}))$. If we discretize center manifold equations we need

Definition 0.18. *A general discretization method is called* linear *and k* times differentiably consistent, *if it is classically consistent and for a k times Frechet differentiable nonlinear operator G it satisfies (66) and (67), so with an interpolation operator I^h,*

$$(G_1 + G_2)^h = G_1^h + G_2^h, \quad (\alpha\, G)^h = \alpha\, G^h\ \forall\alpha\in\mathbb{R}, \tag{66}$$

$$\|(G^h)^{(j)}(I^h u)I^h v - Q^{'h}(G^{(j)}(u)v)\|_{\mathcal{V}'^h} \to 0,\ \textit{usually}\ \|v\|_{\mathcal{U}_s} = \text{ e.g.}\|u\|_{W^{\ell,p}(\Omega;\mathbb{R}^q)}$$

$$\begin{aligned}&\|(G^h)^{(j)}(I^h u)I^h u_1\cdot\ldots\cdot I^h u_j - Q^{'h}(G^{(j)}(u)u_1\cdot\ldots\cdot u_j)\|_{\mathcal{V}'^h}\\ &\quad= \mathcal{O}\big(\|u_1 - I^h u_1\|_{\mathcal{U}^h}\cdot\ldots\|u_j - I^h u_j\|_{\mathcal{U}^h}\big)(1 + \|u - I^h u\|_{\mathcal{U}^h})\ \textit{for}\ h\to 0.\end{aligned} \tag{67}$$

Theorem 0.19. Linear differentiably consistent discretization methods: *All space discretization methods for elliptic problems in [15, 16] have this property.*

We summarize the corresponding results in [16], yielding stable and convergent discretization methods for $w_k^h \approx w_k$, with appropriate perturbations in [13, 14].

Theorem 0.20. Stability and convergence for the center manifold discretizations: *We assume the conditions of Theorems 0.13 - 0.17, 0.19 guaranteeing existence and smoothness of a center manifold, and the unique existence of the asymptotic expansion expansion $\Phi_\iota, \iota = 2, 3, \ldots, \rho$, until determinacy. Finally, use approximations $\Phi^h, \Phi^{'h}$ for Φ, Φ' with the same order ℓ as in the above consistency.*

1. *Then the unique discrete solutions of $\mathbf{A}_\iota^h u_{\iota,0}^h = Q_\iota^{'h} f_\iota$, (and $\tilde{\mathbf{A}}_\iota^h \tilde{u}_{\iota,0}^h = \tilde{Q}_\iota^{'h} f_\iota$) erxist and converge (and of order ℓ) to the exact solution $u_{\iota,0}$ of $\mathbf{A}_\iota u_{\iota,0}^h = f_\iota$:*

$$\|u_{\iota,0}^h - I_\iota^h u_{\iota,0}\|_{\mathcal{U}_\iota^h} \le S\|\mathbf{A}_\iota^h(I_\iota^h u_{\iota,0}) - Q_\iota^{'h}\mathbf{A}_\iota u_{\iota,0}\|_{\mathcal{U}_\iota^{'h}}(\textit{and}\ \le \mathcal{O}(h^\ell))\textit{for}\ h\to 0. \tag{68}$$

2. *By (62) and (68) these unique solutions $u^h_{\iota,0}$ immediately yield the asymptotic expansion with the coefficients $w^h_k \approx w_k$. This implies the convergence for the asymptotic expansion for the center manifold, hence, for $\iota = 2, \ldots, \rho$, determinacy,*

$$(W^\iota)^h(\mathbf{v}) := \sum_{2\le|k|}^{\iota} w^h_k \mathbf{v}^k \in \mathcal{U}^h \text{ converge to } W^\iota(\mathbf{v}) := \sum_{2\le|k|}^{r} w_k \mathbf{v}^k \text{ and} \quad (69)$$
$$\|{Q'}^h W^\iota(\mathbf{v}) - (W^\iota)^h(\mathbf{v})\|_{\mathcal{U}^h} \le \mathcal{O}(h^\ell) + \mathcal{O}(\|\mathbf{v}\|^{r+1}) \text{ for } h, |\mathbf{v}| \to 0.$$

3. *There are interesting cases for which the Φ, Φ' are exactly available. Then they are used instead of the numerical approximations $\Phi^h, {\Phi'}^h$.*

0.6 Converging Normal Forms and Time Discretization, Parameter Dependence and Numerical Equivariance

We aim for a *normal form* for the small dimensional system of ODEs in (45). The transformations here have to account for the time derivative as well. So the recognition problem for center manifolds is still not fully solved. The standard reduction to normal form is a constructive process. It inductively eliminates or transforms to a special form as many of the lower order terms starting with the original $f_2 := f(\mathbf{v}, W^\iota(\mathbf{v}))$ again omitting λ in (45). To this end we define the ℓ_k in a sequence of transformations as

$$\begin{aligned}
&\mathbf{v} = \ell_k(z) = I + P_k'(z),\ P_k(z) \in H_k, \ell_k : \mathcal{D}(\ell_k) \subset \mathbb{R}^\kappa \to \mathbb{R}^\kappa,\ \ell_k'(0) \text{ regular, yielding} \\
&\ell_k'(z)\dot{z} = \dot{\mathbf{v}} = f_k(\ell_k(z)) \Longleftrightarrow \dot{z} = (\ell_k'(z))^{-1} f_k(\ell(z)) \text{ for small } \|z\| \textit{ and} \\
&H_k := \{\text{homog. polyn. in } z \text{ of degree } k \text{ with } z, H_k(z) \in \mathbb{C}^\kappa\},\ k = 2, 3, ... \qquad (70)
\end{aligned}$$

Since B resembles the spectral and sectorial information for the center manifold, we leave B and the lower order transformed terms unchanged. With the Lie bracket operator ad B and unique orthogonal complement $O_k \subset H_k$, [19,25], we obtain Theorem 0.21

$$\begin{aligned}
&\text{ad } B : H_k \to H_k,\quad P(z) \to \text{ad } B(P(z)) := BP(z) - P'(z)Bz \qquad (71) \\
&O_k \subseteq H_k = O_k \perp \text{ad } B(H_k) \Leftrightarrow (P_g(z), \overline{P_b(z)}) = 0 \forall P_g \in O_k, P_b \in \text{ad } B(H_k).
\end{aligned}$$

Theorem 0.21. Normal forms and convergence: *For a (center manifold) dynamical system $\dot{\mathbf{v}} = -B\mathbf{v} + f_n(\mathbf{v})$ of the form (36),(70) let $f, f_n \in C^r$,* ad B *with complements O_k as in (71). Then there exists a sequence of transformations of the form (36),(70) such that the original $\dot{\mathbf{v}} = -B\mathbf{v} + f(\mathbf{v})$ is transformed for $2 \le k \le \rho$, into a system*

$$\dot{z} = -Bz + f^{(2)}(z) + \cdots + f^{(\rho)}(z) + R_\iota,\ f^{(k)} \in O_k,\ 2 \le k \le \rho,\ R_\iota = o(\|z\|^\rho). \qquad (72)$$

For the above discretizations the discrete counterparts $\mathbf{v}^h, B^h, f^h, f^h_k, O^h_k, 2 \le k \le \rho$. converge (of order p) to the preceeding exact $\mathbf{v}, B, f, f_k, O_k$. So do the normal forms.

Remark 0.22. Different strategies for the time discretizat
ion of center manifolds: *1.) We have mainly discussed the normal form of the center manifold equation. This can be numerically solved for short times with the standard, e.g. Runge-Kutta methods with good stability and yield good enough approximations. Only if the full solution $\mathbf{v} + W(\mathbf{v})$ has to be determined, elliptic problems have to be solved.*
2.) For longer time intervals the geometric numerical integration methods and the corresponding structure-preserving algorithms for ODEs in Hairer/Lubich/Wanner, [28], can be modified for this case.
3.) Huge time intervals hardly interest for the strictly local structure of center manifolds.
4.) Compared to Lubich/Ostermann's [39] time discretization, our start with space discretization yields applicable numerical methods.

We return from the previously studied problem (18) to the parameter version

$$\frac{\mathrm{d}u}{\mathrm{d}t} + G(u, \lambda) = \frac{\mathrm{d}u}{\mathrm{d}t} + Au + R(u, \lambda), \quad \lambda \in \mathbb{R}^k, \; R(0,0) = 0, \; R'(0,0) = 0. \tag{73}$$

In the standard approach, cf. [27], the λ has to be added everywhere, e.g., in $u = v + w$, with $(v, \lambda), w = W_*(v, \lambda)$ and $W_*(0,0) = 0, \; W_*'(0,0) = 0$ and A is replaced by $G'(0,0) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, a.s.o. The invariant manifold $\mathcal{M}_\lambda^c$ for (36) is now the graph of a mapping $W_* : (v, \lambda) \in \mathcal{N}(\lambda) \times \mathbb{R}^d \to \mathcal{V}^\alpha$ or we consider W as a function of the two variables

$$W : (\mathbf{v}, \lambda) \in \mathbb{R}^\kappa \times \Lambda \to \mathcal{V}^\alpha \quad \text{with} \quad W(\mathbf{v}, \lambda) := W_*(v, \lambda). \tag{74}$$

Theorem 0.23. Center manifolds, smooth in parameters and initial values, cf. [31], Theorem 6.1.7: *Suppose the assumptions of Theorem 0.11 hold smoothly and uniformly in $\lambda \subset \Lambda \subset \mathbb{R}^d$ with fixed $\mathcal{R} \cup \mathcal{S}$ in Figure 2 and $A, R \in C^{k,1}$ with $0 < (k+1)\beta < \ell$. Then, the mapping W in (74), defining* locally *the center manifold $\mathcal{M}_\lambda^c$ for (73) near the stationary bifurcation point $(u_0 \equiv 0, \lambda_0)$, and $(\mathbf{v}, \lambda)$ for $\mathbf{v} \in \mathbb{R}^\kappa \cong \mathcal{N} = \mathcal{N}(\lambda)$, satisfies*

$$W \in C^{k,1}(\mathbb{R}^\kappa \times \Lambda, \mathcal{V}^\alpha), \quad \textit{and } \mathcal{M}_\lambda^c = \{(\mathbf{v}, w, \lambda) \in \mathbb{R}^\kappa \times \mathcal{V}^\alpha \times \Lambda | w = W(\mathbf{v}, \lambda)\}.$$

This center manifold is tangential to $\mathcal{N}(\lambda)$ near $(0, \lambda_0)$, but not unique and satisfies

$$W(0, \lambda) = 0, \partial_{\mathbf{v}} W(0, \lambda) = 0 \;\; \forall \lambda \in \Lambda.$$

Instead of (73) we could include the equation $\partial\lambda/\partial t = 0$. This yields a nonlinear term $\lambda(v + w)$, while for fixed λ such a term is linear. This allows the center manifold to capture dynamics of the original problem in neighborhoods of the parameter $\lambda = \lambda_0$. The techniques for computing center manifolds for both ways and the original problem are identical. The transformations to normal form have to be updated. The normal forms may change their character, when λ passes through the origin. The convergence results remain correct.

If the equivariance adapted numerical methods for finite and infinite groups in Chapters 10 and 11 in [16] are applied the corresponding equivariance, stability, convergence results hold, cf. [2, 12–14].

Theorem 0.24. Equivariance for center manifolds and their discretizations: *Let under the conditions of Theorem 0.14 the original problem (26) be equivariant under some group. Then the reduced center manifold ODE (40) exhibits the same equivariance, the discrete center manifold is even invariance under the group. This equivariance can be saved to the recursive systems (52) and their discretization in Section 0.5.*

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