# On Besov Regularity of Solutions to Nonlinear Elliptic Partial Differential Equations

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**Abstract**

In this paper, we study the regularity of the solutions of some nonlinear elliptic equations in Kondratiev spaces on certain domains of polyhedral type. General embedding theorems between Kondratiev spaces and Besov spaces will allow to avoid drawbacks to the standard Sobolev regularity theory for those nonsmooth domains. This will give us the opportunity to derive optimal rates for certain nonlinear approximation schemes.

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**Key Words:** (Nonlinear) elliptic equation, regularity of solutions, Kondratiev spaces, Besov spaces, polyhedral cones, domains of polyhedral type, linear and nonlinear approximation methods.

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# 1 Introduction

This paper is concerned with nonstandard regularity estimates of the solutions to semilinear elliptic partial differential equations of the form

$$-\nabla\big(A(x)\cdot\nabla u(x)\big)+g(x,u(x))=f(x)\quad\text{in}\quad D\,,\qquad u|_{\partial D}=0\,,\tag{1}$$

where $A=(a_{i,j})_{i,j=1}^d$ is symmetric and its coefficients satisfy certain smoothness and growth conditions, respectively. In particular, we study the regularity of the solution to (1) in the specific scale

$$B^{m+s}_{\tau,q}(D)\,,\qquad m=d\left(\frac{1}{\tau}-\frac{1}{2}\right),\quad \tau<2\,,\quad s\geq 0\,,\tag{2}$$

of Besov spaces. The motivation for these studies can be explained as follows. Equations of the form (1) frequently arise in science and technology when it comes to the modeling of nonlinear stationary phenomena. Very often, in practice much can be said concerning the existence and uniqueness of solutions to elliptic operator equations (although for nonlinear equations even these questions might become delicate, see [23] for details), but an analytic expression of the solution is usually not available. Therefore, numerical algorithms for the constructive approximation of the solution up to a prescribed tolerance are needed. Numerical studies clearly indicate that modern adaptive algorithms have a lot of potential. In an adaptive strategy, the choice of the underlying degrees of freedom is not a priori fixed but depends on the shape of the unknown solution, i.e., additional degrees of freedom are only spent in regions where the numerical approximation is still 'far away' from the exact solution. Although the basic idea is convincing, adaptive algorithms are hard to implement and to analyze, so that a rigorous mathematical analysis to justify the use of adaptive strategies is highly desirable.

The guideline to achieve this goal can be described as follows. Given an adaptive algorithm based on a dictionary for the solution space of the PDE, the best one can expect is an optimal performance in the sense that it realizes the convergence rate of best $n$-term approximation schemes. In this sense best $n$-term approximation serves as the benchmark scheme for adaptive algorithms. Given a dictionary $\Psi=\{\psi_\lambda\}_{\lambda\in\Lambda}$ of functions in a Banach space $X$, the error of best $N$-term approximation is defined as

$$\sigma_N\big(u;X\big) = \inf_{\Gamma\subset\Lambda:\#\Gamma\le N}\inf_{c_\lambda}\left\|u - \sum_{\lambda\in\Gamma} c_\lambda\psi_\lambda\big|X\right\|.$$

In the context of the numerical treatment of PDEs especially best $N$-term approximations with respect to the Sobolev spaces $H^s$ (which are usually the energy spaces) are important. For many dictionaries, in particular for wavelet bases and frames, it has been shown that in this case the order of convergence that can be achieved depends on the regularity of the object one wants to approximate in the specific scale (2). We refer, e.g., to [21] and [17] for details. Quite recently, it has also turned out that similar relations hold for finite element approximations, see [26]. On the other hand, the performance of nonadaptive (uniform) methods is determined by the $L_2$–Sobolev smoothness of the solution, see, e.g., Hackbusch [30] and [14] for details. Therefore, the use of adaptivity is justified if the Besov smoothness within this scale (2) of the exact solution to an operator equation is high enough compared to the classical Sobolev smoothness. These relations are clearly the reason why we are highly interested in regularity estimates in the scale (2).

It is nowadays classical knowledge that the Sobolev regularity of the solutions to elliptic problems depends not only on the properties of the coefficients and the right-hand side, but also on the regularity/roughness of the boundary of the underlying domain. While for smooth coefficients and smooth boundaries we have $u \in H^{s+2}(D)$ for $f \in H^s(D)$, it is well-known that this becomes false for more general domains. In particular, if we only assume $D$ to be a Lipschitz domain, then it was shown by Jerison and Kenig [35] that in general we only have $u \in H^s$ for all $s \le 3/2$ for the solution of the Poisson equation, even for smooth right-hand side $f$. This behaviour is caused by singularities near the boundary. Therefore, the $H^{3/2}$-Theorem implies that the optimal rate of convergence for nonadaptive methods of approximation is just $3/2d$ as long as we do not impose further restrictions on $\Omega$. Similar relationships also hold for more specific domains such as domains of polyhedral or polygonal type, see, e.g., [29, 28, 20]. However, the norms considered in (2) are weaker than the $L_2$-Sobolev norm with respect to $m$, and therefore, there is some hope that the boundary singularities do not influence the smoothness of the solution in the scale (2) too much.

Regularity estimates in quasi-Banach spaces according to (2) have only been developed quite recently. For linear elliptic operator equations a lot of positive results in this direction already exist, see, e.g. [13, 14, 15] (this list is clearly not complete). First studies for semilinear equations have been reported in [19]. This paper is clearly related to this work, but we generalize and modify the analysis presented in [19] in the following sense. First of all, in [19] only semilinear versions of the Poisson equation have been studied, whereas here much more general elliptic operators are treated. Secondly, in [19] only nonlinear terms $g(x,\xi)$ that satisfy a growth condition of the form $|g(x,\xi)| \leq a + b|\xi|^\delta$, $\delta \leq 1$ have been considered. Here this condition is removed to the greatest possible extent. In particular, the important special case

$$-\nabla\big(A \cdot \nabla u\big) + u^{2n+1} = f \quad \text{on } D\,, \qquad u\big|_{\partial D} = 0\,, \tag{3}$$

is covered by our analysis. Thirdly, in [19] general Lipschitz domains are considered, whereas here we are mainly concerned with polyhedral domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. When it comes to practical applications, these kind of domains are clearly the most important special cases and they have the following advantage. For polyhedral domains, not all points at the boundary are equally 'bad', only the singular set which forms a lower dimensional manifold causes trouble in the sense that it induces singularities. As a consequence, in this paper we achieve higher Besov regularity compared to [19]. Fourthly, the proof technique in this paper is different and more general. In [19], Besov regularity has been established by applying fixed point theorems directly in the spaces of the Besov scale (2). In contrast to this, in this paper we study the nonlinear equation (1) first in so-called *Kondratiev spaces* $\mathcal{K}^m_{p,a}(D)$ defined as the collection of all measurable functions which admit $m$ weak derivatives satisfying

$$\|u|\mathcal{K}^m_{p,a}(D)\|^p = \sum_{|\alpha|\leq m} \int_D |\rho(x)^{|\alpha|-a}\partial^\alpha u(x)|^p\, dx < \infty\,.$$

Therein, the weight function $\rho : D \longrightarrow [0,1]$ is the smooth distance to the singular set of $D$. The relevance of these spaces stems from the fact that within this scale of spaces we can prove shift theorems also on nonsmooth domains analog to those in the usual Sobolev scale on smooth domains. Then, in a second step, we employ embedding theorems of Kondratiev spaces into Besov spaces as e.g. established in [32]. This approach has the following advantages. Certain admissibility problems that might arise in the context of quasi-Banach spaces can be ameliorated. Moreover, regularity estimates in Kondratiev spaces exist for huge classes of operator equations, not only for the Poisson equations, see, e.g. [3, 47].

The main results of this paper are stated in Theorems 8 and 9 which deal with $n$–term approximation of the solution of (3). To be more precise, let $D \subset \mathbb{R}^d$ with $d = 2, 3$, be a bounded Lipschitz domain of polyhedral type. Then for a right–hand side $f \in \mathcal{K}^{m-1}_{a-1,2}(D)$ with $m \in \mathbb{N}$, $m \geq 2$, and certain technical restrictions on $a$, equation (3) has a solution $u$ satisfying the estimates

$$\sigma_N(u; H^1(D)) \lesssim N^{-m/d} \| f | \mathcal{K}^{m-1}_{a-1,2}(D) \|$$

and

$$\sigma_N(u; L_2(D)) \lesssim N^{-(m+1)/d} \| f | \mathcal{K}^{m-1}_{a-1,2}(D) \|, \tag{4}$$

respectively, for $n$–term wavelet approximation. Since it is well-known that (for nonconvex domains) the $L_2$-Sobolev smoothness of the solution is usually strictly smaller than 2 even for smooth right-hand sides (we refer once again to [29, 28] for details) and the Besov smoothness $m + 1$ we obtain in Theorem 7 (which gives rise to (4)) can be quite large, these results justify the use of adaptive algorithms. Concerning finite element approximation, corresponding results are stated in Theorem 10.

This paper is organized as follows. First of all, in Section 2, we define and collect some basic properties of Kondratiev spaces as far as they are needed for our purposes, in particular, results on pointwise multiplication and embeddings. Moreover, the mapping properties of nonlinear composition operators in Kontradiev spaces are studied. In Section 3 we briefly recall the fixed point theorem that will be used to establish our new regularity results. Then, in Section 4, we show that the Kondratiev spaces and (nonlinear) operators satisfy the requirements for the fixed point theorem to hold and present our main regularity results. The first fundamental result is Theorem 5, which implies that regularity estimates for linear elliptic operators in Kondratiev spaces mainly carry over to the semilinear equations (1). This fact is then used to establish Besov regularity in Theorem 7. Finally, in Section 6, we apply our findings to the analysis of adaptive numerical algorithms. It turns out that for $f \in \mathcal{K}^{m-1}_{a-1}(D)$ adaptive wavelet approximations as well as adaptive finite element approximations give rise to an approximation order $\mathcal{O}(N^{-(m-\varepsilon)/d})$, which justifies the use of adaptive strategies. In Appendix A, we briefly recall the basics we need throughout the paper about Besov and Triebel-Lizorkin spaces.

## Notation

We start by collecting some general notation used throughout the paper.

As usual, $\mathbb{N}$ stands for the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^d$, $d \in \mathbb{N}$, is the $d$-dimensional real Euclidean space with $|x|$, for $x \in \mathbb{R}^d$, denoting the Euclidean norm of $x$. Let $\mathbb{N}_0^d$, where $d \in \mathbb{N}$, be the set of all multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \mathbb{N}_0$ and $|\alpha| := \sum_{j=1}^d \alpha_j$.

Furthermore, $B_\varepsilon(x)$ is the open ball of radius $\varepsilon > 0$ centered at $x$.

We denote by $c$ a generic positive constant which is independent of the main parameters, but its value may change from line to line. The expression $A \lesssim B$ means that $A \leq c\,B$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.

Given two quasi-Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded.

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space and $\mathcal{S}'(\mathbb{R}^d)$ its topological dual. Moreover, $\mathcal{F}$ stands for the Fourier-transform on $\mathcal{S}'(\mathbb{R}^d)$ with inverse $\mathcal{F}^{-1}$.
A domain $\Omega$ is an open bounded set in $\mathbb{R}^d$. The test functions on $\Omega$ are denoted by $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ stands for the set of distributions on $\Omega$. Let $L_p(\Omega)$, $1 \leq p \leq \infty$, be the Lebesque spaces on $\Omega$ as usual. We denote by $C(\Omega)$ the space of all bounded continuous functions $f : \Omega \to \mathbb{R}$ and $C^k(\Omega)$, $k \in \mathbb{N}_0$, is the space of all functions $f \in C(\Omega)$ such that $\partial^\alpha f \in C(\Omega)$ for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \leq k$, endowed with the norm $\sum_{|\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha f(x)|$.
Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then $W_p^m(\Omega)$ denotes the standard Sobolev space on the domain $\Omega$, equipped with the norm

$$\| u \,|W_p^m(\Omega)\| := \Big( \sum_{|\alpha| \leq m} \int_\Omega |\partial^\alpha u(x)|^p \, dx \Big)^{1/p}$$

(with the usual modification if $p = \infty$). If $p = 2$ we shall also write $H^m(\Omega)$ instead of $W_2^m(\Omega)$. By $H_0^m(\Omega)$ we denote the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$. The dual space $(H_0^m(\Omega))'$ of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$.

Moreover, for $s \in \mathbb{R}$ we define fractional Sobolev spaces $H_p^s(\mathbb{R}^d)$, which contain all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|H_p^s(\mathbb{R}^d)\| := \|\mathcal{F}^{-1}((1 + |\xi|^2)^{s/2} \mathcal{F} f)|L_p(\mathbb{R}^d)\| < \infty.$$

These spaces partially coincide with the classical Sobolev spaces, i.e., we have $H^m_p(\mathbb{R}^d) = W^m_p(\mathbb{R}^d)$ for $m \in \mathbb{N}_0$ and $1 < p < \infty$.

# 2 Kondratiev spaces

In order to study the regularity of solutions to elliptic PDEs we now introduce scales of weighted Sobolev spaces. These spaces have their origin in the midsixties in the pioneering work of Kondratiev [37, 38], see also the survey of Kondratiev and Oleinik [39]. Later these kind of spaces, partly more general, have been considered by Kufner, Sändig [43], Babuska, Guo [2], Maz'ya, Rossmann [40, 46], Nistor, Mazzucato [47], and Costabel, Dauge, Nicaise [11], to mention at least a few.
Whereas in the mentioned references the weight was always chosen to be a power of the distance to the singular set of the boundary, there are also publications dealing with the weight being a power of the distance to the whole boundary. We refer e.g. to Kufner, Sändig [43], Triebel [52, 3.2.3] and Lototsky [44].

## 2.1 Definition and basic properties

**Definition 1.** *Let $\Omega$ be a domain in $\mathbb{R}^d$ and let $M$ be a nontrivial closed subset of its boundary $\partial\Omega$. Furthermore, let $1 \le p \le \infty$, $m \in \mathbb{N}_0$, and $a \in \mathbb{R}$. We define the space $\mathcal{K}^m_{a,p}(\Omega, M)$ as the collection of all measurable functions, which admit $m$ weak derivatives in $\Omega$ satisfying*

$$\|u|\mathcal{K}^m_{a,p}(\Omega, M)\| := \Big( \sum_{|\alpha|\le m} \int_\Omega |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx \Big)^{1/p} < \infty$$

*if $p < \infty$, modified by*

$$\|u|\mathcal{K}^m_{a,\infty}(\Omega, M)\| := \sum_{|\alpha|\le m} \sup_{x\in\Omega} |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)| < \infty$$

*if $p = \infty$. Therein, the weight function $\rho$ is defined by*

$$\rho(x) := \min\{1, \operatorname{dist}(x, M)\}, \qquad x \in \Omega .$$

*Finally, for $1 < p < \infty$ and $m \in \mathbb{N}$ we also define $\mathcal{K}^{-m}_{a,p}(\Omega, M) = \big(\mathcal{K}^m_{-a,p'}(\Omega, M)\big)'$, the dual space equipped with its usual norm, where $\frac{1}{p'} = 1 - \frac{1}{p}$.*

**Remark 1.** In our setting the set $M$ will very often be the *singularity set* $S$ of the domain $\Omega$, i.e., the set of all points $x \in \partial\Omega$ for which for any $\varepsilon > 0$ the set $\partial\Omega \cap B_\varepsilon(x)$ is not smooth. In this case, we simply abbreviate

$$\mathcal{K}^m_{a,p}(\Omega) := \mathcal{K}^m_{a,p}(\Omega, S).$$

The relevance of these spaces stems from the fact that within this scale of spaces one can prove shift theorems also on nonsmooth domains analogously to those in the usual Sobolev scale on smooth domains. As a generic example we cite the following shift theorem from [3, 47]. This fundamental result will form the basis for the investigations presented in this paper.

**Proposition 1.** *Let $D$ be some bounded polyhedral domain without cracks in $\mathbb{R}^d$, $d = 2, 3$. Consider the problem*

$$-\nabla\big(A(x) \cdot \nabla u(x)\big) = f \quad \text{in} \quad D\,, \qquad u|_{\partial D} = 0\,, \tag{5}$$

*where $A = (a_{i,j})_{i,j=1}^d$ is symmetric and*

$$a_{i,j} \in \mathcal{K}^m_{0,\infty}(D) = \big\{v : D \longrightarrow \mathbb{R} : \rho^{|\alpha|}\partial^\alpha v \in L_\infty(D)\,, |\alpha| \le m\big\}\,, \qquad 1 \le i, j \le d\,.$$

*Let the associated bilinear form*

$$B(v, w) = \int_D \sum_{i,j} a_{i,j}(x)\partial_i v(x)\partial_j w(x)dx$$

*satisfy*

$$|B(v, w)| \le R\|v|H^1(D)\| \cdot \|w|H^1(D)\| \qquad \text{and} \qquad r\|v|H^1(D)\|^2 \le B(v, v)$$

*for all $v, w \in H^1_0(D)$ and some constants $0 < r \le R < \infty$. Then there exists some $\overline{a} > 0$ such that for any $m \in \mathbb{N}_0$, any $|a| < \overline{a}$, and any $f \in \mathcal{K}^{m-1}_{a-1,2}(D)$ the problem (5) admits a uniquely determined solution $u \in \mathcal{K}^{m+1}_{a+1,2}(D)$, and it holds*

$$\|u|\mathcal{K}^{m+1}_{a+1,2}(D)\| \le C\,\|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$$

*for some constant $C > 0$ independent of $f$.*

**Remark 2.** (i) The conditions of Proposition 1 clearly imply that the bilinear form $B$ is continuous and coercive in $H_0^1(D)$. Hence, assuming $f \in H^{-1}(D)$, there exists a unique weak solution $u \in H_0^1(D)$ to problem (5), see, e.g. [30, Ch. 6.5] for details. Proposition 1 implies that under certain conditions on the coefficients $a_{i,j}, 1 \leq i, j \leq d$, and the right–hand side $f$ this weak solution possesses additional regularity in the scale of Kondratiev spaces.

(ii) For a polygon $\Omega \subset \mathbb{R}^2$ without cracks we have $\overline{a} = \frac{\pi}{\alpha_{\max}}$ in Proposition 1, where $\alpha_{\max}$ is the largest angle of $\Omega$, cf. [4, Sect. 2.1].

(iii) In the literature there are further results of this type, either treating different boundary conditions, or using slightly different scales of function spaces. We particularly refer to [40, Ch. 6] and [46, Part 1, Ch. 4].

**Remark 3.** Let us summarize some of the basic properties of Kondratiev spaces that will be used in the sequel. For further information, we refer e.g. to [16].

- $\mathcal{K}^m_{a,p}(\Omega, M)$ is a Banach space, see [41, 42].
- The scale of Kondratiev spaces is monotone in $m$ and $a$, i.e.,

$$\mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow \mathcal{K}^{m'}_{a,p}(\Omega, M) \qquad \text{and} \qquad \mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow \mathcal{K}^m_{a',p}(\Omega, M) \tag{6}$$

  if $m' < m$ and $a' < a$.
- Let $b \in \mathbb{R}$. The mapping $T_b :\ u \mapsto \rho^b\, u$ yields an isomorphism of $\mathcal{K}^m_{a,p}(\Omega, M)$ onto $\mathcal{K}^m_{a+b,p}(\Omega, M)$. To see this one may use [50, Thm. VI.2.2].
- Let $m \in \mathbb{N}$ and $|\alpha| \leq m$. Then $u \in \mathcal{K}^m_{a,p}(\Omega, M)$ implies $\partial^\alpha u \in \mathcal{K}^{m-|\alpha|}_{a-|\alpha|,p}(\Omega, M)$.
- Let $a \geq 0$. Then $\mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow L_p(\Omega)$.
- A function $\psi \in C^m(\Omega)$ is a pointwise multiplier for $\mathcal{K}^m_{a,p}(\Omega, M)$, i.e., $\psi\, u \in \mathcal{K}^m_{a,p}(\Omega, M)$ for all $u \in \mathcal{K}^m_{a,p}(\Omega, M)$.

## 2.2 Domains of polyhedral type

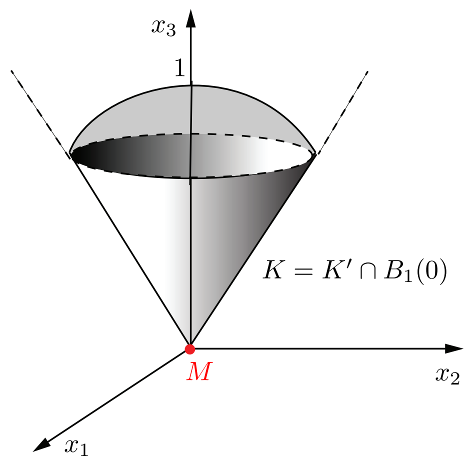
The analysis presented in this paper is based on the fundamental Proposition 1. Therefore, in the sequel we will always assume that the underlying domain $\Omega$ satisfies the conditions stated there. Some additional restrictions are also necessary. It is well-known that not all polyhedral domains are of Lipschitz type, see, e.g. [22, Ex. 6.5]. However, some parts of the analysis presented below rely on Lipschitz properties, e.g. the generalization of Stein's extension operator to Kondratiev spaces. Therefore, in the course of this paper we will assume that the underlying domain is of Lipschitz type. Moreover, we will heavily use several properties of Kondratiev spaces that have been proved in [16]. In this paper, specific polyhedral domains for which the analysis of Kondratiev spaces can be reduced to four basic cases have been considered, resulting in the fundamental Definition 2 below. Therefore, in the sequel, we will additionally assume that $\Omega$ is of polyhedral type according to Definition 2. Let us now briefly recall the setting of [16].

As usual, an infinite smooth cone with vertex at the origin is the set

$$K := \{x \in \mathbb{R}^d : \ 0 < |x| < \infty\,,\ x/|x| \in \Omega\}\,,$$

where $\Omega$ is a subdomain of the unit sphere $S^{d-1}$ with $C^\infty$ boundary.

**Case I:** *Kondratiev spaces on smooth cones.* Let $K'$ be an infinite smooth cone contained in $\mathbb{R}^d$ with vertex at the origin which is rotationally invariant with respect to the axis $\{(0,\dots,0,x_d) : \ x_d \in \mathbb{R}\}$. Then we define the truncated cone $K$ by $K := K' \cap B_1(0)$. In this case we choose $M := \{0\}$, i.e.,

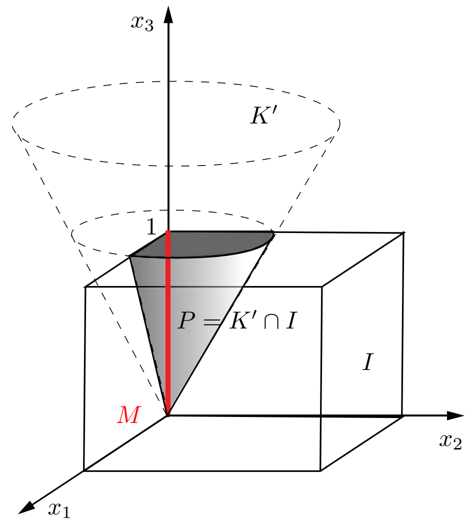



$$\|u|\mathcal{K}^m_{p,a}(K,M)\|^p = \|u|\mathcal{K}^m_{p,a}(K,\{0\})\|^p = \sum_{|\alpha|\le m} \int_K |\,|x|^{|\alpha|-a}\partial^\alpha u(x)|^p\,dx \qquad (7)$$

There is still one degree of freedom in the choice of the smooth cone, namely the opening angle $\gamma \in (0,\pi)$ of the cone. Since this will be unimportant in what follows we will not indicate this in the notation.

**Case II:** *Kondratiev spaces on specific nonsmooth cones.* Let again $K'$ denote a rotationally symmetric cone as described in **Case I** with opening angle $\gamma \in (0, \pi)$. Then we define the specific poyhedral cone $P$ by $P = K' \cap I$, where $I$ denotes the unit cube

$$I := \{x \in \mathbb{R}^d \, : \, 0 < x_i < 1, \; i = 1, \ldots, d\}. \tag{8}$$

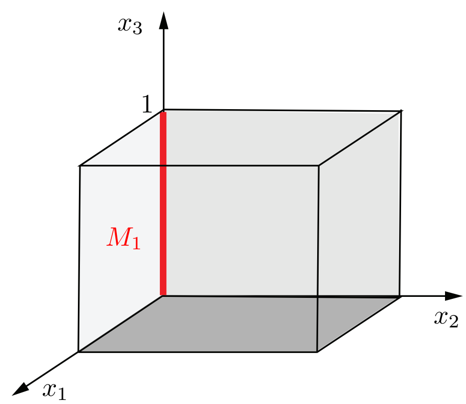



In this case, we choose $M = \Gamma := \{x \in \mathbb{R}^d, x = (0, \ldots, 0, x_d), \; 0 \leq x_d \leq 1\}$ and see that

$$\|u|\mathcal{K}^m_{a,p}(P, \Gamma)\|^p = \sum_{|\alpha| \leq m} \int_P |\, \rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx, \tag{9}$$

where $\rho(x)$ denotes the distance of $x$ to $\Gamma$, i.e., $\rho(x) = |(x_1, \ldots, x_{d-1})|$. Again the opening angle $\gamma$ of the cone $K'$ will be of no importance.

**Case III:** *Kondratiev spaces on specific dihedral domains.* Let $1 \leq l < d$ and let $I$ be the unit cube defined in (8). For $x \in \mathbb{R}^d$ we write $x = (x', x'') \in \mathbb{R}^{d-l} \times \mathbb{R}^l$, where $x' := (x_1, \ldots, x_{d-l})$ and $x'' := (x_{d-l+1}, \ldots, x_d)$. Hence $I = I' \times I''$ with the obvious interpretation.




Then we choose

$$M_l := \{x \in I : \; x_1 = \ldots = x_{d-l} = 0 \,, \; 0 \leq x_i \leq 1, \; i = d-l+1, \ldots, d\} \tag{10}$$

and define

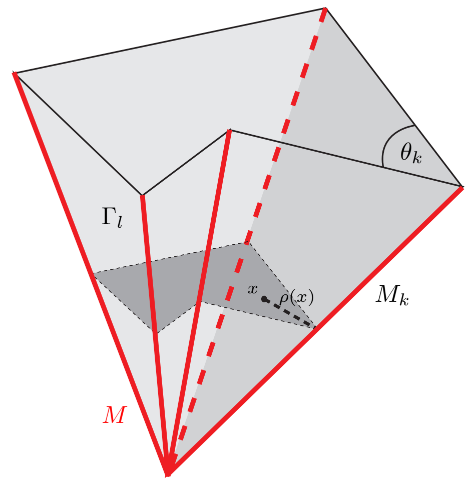
$$\|u|\mathcal{K}^m_{a,p}(I, M_l)\|^p = \sum_{|\alpha| \leq m} \int_I |\, |x'|^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx. \tag{11}$$

This time the set $M_l$ is a subset of the singularity set of $I$ if, and only if, $l \leq d-2$.

**Case IV:** *Kondratiev spaces on polyhedral cones.* Let

$K' := \{x \in \mathbb{R}^3 : 0 < |x| < \infty\,, x/|x| \in \Omega\}\,,$

be an infinite cone in $\mathbb{R}^3$. We suppose that the boundary $\partial K'$ consists of the vertex $x = 0$, the edges (half lines) $M_1, \ldots, M_n$, and smooth faces $\Gamma_1, \ldots \Gamma_n$. This means $\Omega := K' \cap S^2$ is a domain of polygonal type on the unit sphere with sides $\Gamma_k \cap S^2$. Without loss of generality we may assume that the positive part of the $x_3$-axis belongs to $K'$. We further assume that the angles between the edges $M_j$ and the positive part of the $x_3$-axis all are smaller than $\frac{\pi}{2}$. Then we put




$$Q := K' \cap \{x \in \mathbb{R}^3 :\ 0 < x_3 < 1\}\,.$$

In this case, we choose $M := M_1 \cup \ldots \cup M_n$ and define

$$\|u|\mathcal{K}^m_{a,p}(Q, M)\|^p = \sum_{|\alpha|\le m} \int_Q |\,\rho(x)^{|\alpha|-a}\partial^\alpha u(x)|^p\, dx, \qquad (12)$$

where $\rho(x)$ denotes the distance of $x$ to $M$.

Based on these four cases, we define the specific domains we will be concerned with in this paper.

**Definition 2.** *Let $D$ be a domain in $\mathbb{R}^d$ with singularity set $S$. Then $D$ is of* polyhedral type*, if there exists a finite covering $(U_i)_i$ of bounded open sets such that*

$$\overline{D} \subset \Big(\bigcup_{i\in\Lambda_1} U_i\Big) \cup \Big(\bigcup_{j\in\Lambda_2} U_j\Big) \cup \Big(\bigcup_{k\in\Lambda_3} U_k\Big) \cup \Big(\bigcup_{l\in\Lambda_4} U_l\Big)\,,$$

*where*

*i) $i \in \Lambda_1$ if $U_i$ is a ball and $\overline{U_i} \cap S = \emptyset$.*

*ii)* $j \in \Lambda_2$ *if there is a* $C^\infty$*-diffeomorphism* $\eta_j : \overline{U_j} \longrightarrow \eta_j(\overline{U_j}) \subset \mathbb{R}^d$ *such that* $\eta_j(U_j \cap D)$ *is the smooth cone* $K$ *as described in* **Case I**. *Moreover, we assume that for all* $x \in U_j \cap D$ *the distance to the singularity set* $S$ *of* $D$ *is equivalent to the distance to the point* $x^j := \eta_j^{-1}(0)$.

*iii)* $k \in \Lambda_3$ *if there exists a* $C^\infty$*-diffeomorphism* $\eta_k : \overline{U_k} \longrightarrow \eta_k(\overline{U_k}) \subset \mathbb{R}^d$ $(d \geq 3)$ *such that* $\eta_k(U_k \cap D)$ *is the nonsmooth cone* $P$ *as described in* **Case II**. *Moreover, we assume that for all* $x \in U_k \cap D$ *the distance to* $S$ *is equivalent to the distance to the set* $\Gamma^k := \eta_k^{-1}(\Gamma)$.

*iv)* $l \in \Lambda_4$ *if there exists a* $C^\infty$*-diffeomorphism* $\eta_l : \overline{U_l} \longrightarrow \eta_l(\overline{U_l}) \subset \mathbb{R}^d$ $(d \geq 3)$ *such that* $\eta_l(U_l \cap D)$ *is a specific dihedral domain as described in* **Case III**. *Moreover, we assume that for all* $x \in U_l \cap D$ *the distance to* $S$ *is equivalent to the distance to the set* $M^l := \eta_l^{-1}(M_n)$ *for some* $n \in \{1, \ldots, d-1\}$.

*Particularly in* $d = 3$ *we permit another type of subdomain: Here*

$$\overline{D} \subset \Big(\bigcup_{i\in\Lambda_1} U_i\Big) \cup \Big(\bigcup_{j\in\Lambda_2} U_j\Big) \cup \Big(\bigcup_{k\in\Lambda_3} U_k\Big) \cup \Big(\bigcup_{l\in\Lambda_4} U_l\Big) \cup \Big(\bigcup_{m\in\Lambda_5} U_m\Big),$$

*where*

*v)* $m \in \Lambda_5$ *if there exists a* $C^\infty$*-diffeomorphism* $\eta_m : \overline{U_m} \longrightarrow \eta_m(\overline{U_m}) \subset \mathbb{R}^d$ *such that* $\eta_m(U_m \cap D)$ *is a polyhedral cone as described in* **Case IV**. *Moreover, we assume that for all* $x \in U_m \cap D$ *the distance to* $S$ *is equivalent to the distance to the set* $M'_m := \eta_m^{-1}(M)$.

In summary, unless otherwise stated, throughout this paper we will always assume that the domain $D$ under consideration satisfies the

**Assumption 1.** *Let* $D \subset \mathbb{R}^d$, $d = 2, 3$, *be a bounded Lipschitz domain of polyhedral type according to Definition 2.*

**Remark 4.** The domains $D$ satisfying Assumption 1 are special polyhedral domains. Moreover, they are also special cases of the domains considered in Proposition 1.

**Remark 5.** As already mentioned in Remark 1, the set $M$ according to Definition 1 will not be mentioned explicitly in case that it coincides with the singularity set

$S$. In the sequel, we will use the same convention if $M$ is one of the specific sets introduced in **Case I - Case IV**, i.e., we simply write

$$\mathcal{K}^m_{p,a}(D) := \mathcal{K}^m_{p,a}(D, M).$$

Since for the specific domains in **Case I - Case IV** the set $M$ does not coincide with the singularity set $S$, this clearly causes some ambiguities. However, throughout the paper this is not a serious problem since for the specific domains from **Case I - Case IV** Kondratiev spaces with respect to the whole singularity set are not considered at all in what follows.

## 2.3 Embeddings and pointwise multiplication

Throughout this subsection let $D \subset \mathbb{R}^d$ be as in Definition 2. Embeddings of Kondratiev spaces have been discussed in [16], but see also Maz'ya and Rossmann [46] (Lemma 1.2.2, Lemma 1.2.3 (smooth cones), Lemma 2.1.1 (dihedron), Lemma 3.1.3, Lemma 3.1.4 (cones with edges), Lemma 4.1.2 (domains of polyhedral type)).

**Theorem 1.** *Let $1 \le p \le q \le \infty$, $a \in \mathbb{R}$, and $m \in \mathbb{N}$.*
(i) *Let $q < \infty$. Then $\mathcal{K}^m_{a,p}(D)$ is embedded into $\mathcal{K}^{m'}_{a',q}(D)$ if, and only if,*

$$m - \frac{d}{p} \ge m' - \frac{d}{q} \qquad \text{and} \qquad a - \frac{d}{p} \ge a' - \frac{d}{q}\,.$$

(ii) *Let $1 < p < \infty$ and $q = \infty$. Then $\mathcal{K}^m_{a,p}(D)$ is embedded into $\mathcal{K}^{m'}_{a',\infty}(D)$ if, and only if,*

$$m - \frac{d}{p} > m' \qquad \text{and} \qquad a - \frac{d}{p} \ge a'\,.$$

(iii) *Let $p = 1$ and $q = \infty$. Then $\mathcal{K}^m_{a,1}(D)$ is embedded into $\mathcal{K}^{m'}_{a',\infty}(D)$ if, and only if,*

$$m - d \ge m' \qquad \text{and} \qquad a - d \ge a'\,.$$

Concerning pointwise multiplication we recall a result taken from [16].

**Theorem 2.** *Let $d/2 < p < \infty$, $m \in \mathbb{N}$, and $a \ge \frac{d}{p} - 1$. Then there exists a constant $c > 0$ such that*

$$\| \, u\, v \,|\mathcal{K}^{m-1}_{a-1,p}(D)\| \le c \, \| \, u \,|\mathcal{K}^{m+1}_{a+1,p}(D)\| \, \| \, v \,|\mathcal{K}^{m-1}_{a-1,p}(D)\|$$

*holds for all $u \in \mathcal{K}^{m+1}_{a+1,p}(D)$ and $v \in \mathcal{K}^{m-1}_{a-1,p}(D)$.*

## 2.4 Composition operators in Kondratiev spaces

The goal of this paper is to establish regularity estimates for the solution to semi-linear problems of the form (1). Let $L$ be the solution operator associated to (5), i.e., $Lf = u$. Then, by Proposition 1, we know that $L$ is well-defined on $\mathcal{K}^{m-1}_{a-1,2}(D)$ with values in the set

$$\mathcal{K}^{m+1}_{a+1,2,0}(D) := \{u \in \mathcal{K}^{m+1}_{a+1,2}(D) : \ u_{|\partial D} = 0\} . \tag{13}$$

Vice versa, to each $u \in \mathcal{K}^{m+1}_{a+1,2,0}(D)$ there exists an $f \in \mathcal{K}^{m-1}_{a-1,2}(D)$ such that (5) is satisfied, see Remark 3, and this operation is bounded as well. Hence, $L$ is an isomorphism.

By defining the linear operator

$$L^{-1}u(x) := -\nabla\big(A(x)\cdot\nabla u(x)\big), \qquad x \in D, \tag{14}$$

and the nonlinear map

$$N(u)(x) := f(x) - g(x, u(x)), \qquad x \in D, \tag{15}$$

we see that equation (1) can be written as

$$u = (L \circ N)u,$$

which will enable us to apply suitable fixed point theorems in the regularity spaces we are interested in. Therefore, we have to study the mapping properties of $L \circ N$ and in this context derive bounds for the nonlinear map $N$ in (15). For this purpose, we will directly estimate the weighted $L_p$-norms of partial derivatives of the function $g$, which requires imposing certain growth conditions on the classical partial derivatives of the function. Our result can be formulated as follows:

**Theorem 3.** *Let $D$ be as in Definition 2 and denote by $S$ the singularity set of $D$. Let $1 < p < \infty$, $a \geq \frac{d}{p} - 1$, $m \in \mathbb{N}$, and $\delta \geq \max(1, m-1)$ such that the continuous function $g : \ D \times \mathbb{R} \to \mathbb{R}$ and its continuous classical derivatives fulfill the growth conditions*

$$|\partial^l_\xi \partial^\alpha_x g(x,\xi)| \leq c_{\alpha,l}|\xi|^{\delta - l}\,, \qquad l \in \mathbb{N}_0, \quad \alpha \in \mathbb{N}^d_0, \quad l + |\alpha| \leq m-1\,. \tag{16}$$

*Moreover, let either*

$$\min(m+1,3) - \frac{d}{p} > 0 \tag{17}$$

*or*

$$0 > m+1-\frac{d}{p} \geq -\frac{2}{\delta-1}$$

*(no lower restriction in case $\delta = 1$).*

*Then the nonlinear operator $T_G:\ u \mapsto G(u)(x) = g(x,u(x))$ maps functions from $\mathcal{K}^{m+1}_{a+1,p}(D)$ to functions in $\mathcal{K}^{m-1}_{a-1,p}(D)$. Moreover, it holds*

$$\|G(u)|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq C\,\|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^{\delta}$$

*with some constant $C > 0$ independent of $u$.*

*Proof. Step 1.* Preparations. As a first step we shall have a look at partial derivatives of the composed function $g(x,u(x))$. We first observe that, except for $\partial^\alpha_x g(x,\xi)|_{\xi=u(x)}$, all occurring terms are of the form

$$\partial^l_\xi \partial^{\alpha'}_x g(x,\xi)\big|_{\xi=u(x)} \partial^{\alpha-\alpha'}_x\big(u^l\big) \qquad \alpha' \leq \alpha\,, \quad 1 \leq l \leq |\alpha-\alpha'|\,. \tag{18}$$

This can be seen by induction. Here is the induction step. A further partial derivative $\partial^{e_j}$ ($e_j$ having entry 1 at position $j$, all the other entries being 0) gives

$$\begin{aligned}\partial^l_\xi \partial^{\alpha'+e_j}_x g(x,\xi)\big|_{\xi=u(x)} \partial^{\alpha-\alpha'}_x\big(u^l\big) + \partial^{l+1}_\xi \partial^{\alpha'}_x g(x,\xi)\big|_{\xi=u(x)} \partial^{\alpha-\alpha'}_x\big(u^l\big)\partial^{e_j}_x u \\ + \partial^l_\xi \partial^{\alpha'}_x g(x,\xi)\big|_{\xi=u(x)} \partial^{\alpha-\alpha'+e_j}_x\big(u^l\big)\,,\end{aligned}$$

all three terms clearly being covered by (18) (with $\alpha$ being replaced by $\alpha+e_j$). Further, applying the Leibniz formula to the second factor, (18) results in pointwise estimates

$$\begin{aligned}\big|\partial^\alpha\big(g(x,u(x))\big)\big| &\lesssim \big|\partial^\alpha_x g(x,\xi)|_{\xi=u(x)}\big| \\ &\quad + \sum_{\alpha'\leq\alpha}\sum_{l=1}^{|\alpha-\alpha'|}\Big|\partial^l_\xi\partial^{\alpha'}_x g(x,\xi)\big|_{\xi=u(x)}\Big| \sum_{\beta_1+\cdots+\beta_l=\alpha-\alpha'}\Big|\partial^{\beta_1}_x u(x)\cdots\partial^{\beta_l}_x u(x)\Big|\,.\end{aligned} \tag{19}$$

A comparison with the general Faa di Bruno formula further yields that we can restrict the last sum to multiindices $|\beta_j| \geq 1$, $j = 1,\ldots,l$ (recall that we originally started with the chain rule in (18) therefore no terms of $u(x)$ without derivatives appear).

Secondly, let use mention that

$$\|\rho^{-a}\, u|L_p(D)\| + \sum_{|\alpha|=m} \|\rho^{m-a}\, \partial^\alpha u|L_p(D)\|$$

generates an equivalent norm for the Kondratiev space $\mathcal{K}^m_{a,p}(D)$, see [16]. Below we shall work with this norm without further reference.

*Step 2.* Now assume first $m+1-\frac{d}{p}<0$. Then for a typical term in (19) we can estimate

$$\begin{aligned}
&\int_D \left(\rho^{|\alpha|-a+1}(x)\Big|\partial^l_\xi \partial^{\alpha'}_x g(x,\xi)\big|_{\xi=u(x)}\Big|\,\Big|\partial^{\beta_1}_x u(x)\cdots\partial^{\beta_l}_x u(x)\Big|\right)^p dx \\
&\quad\lesssim \int_D \left(\rho^{|\alpha|-a+1}(x)|u(x)|^{\delta-l}\,\Big|\partial^{\beta_1}_x u(x)\cdots\partial^{\beta_l}_x u(x)\Big|\right)^p dx \\
&\quad\lesssim \left(\int_D \Big(\rho^{\gamma_0}(x)|u(x)|\Big)^{q_0} dx\right)^{\frac{p(\delta-l)}{q_0}} \prod_{j=1}^{l}\left(\int_D \Big(\rho^{|\beta_j|+\gamma_j}(x)|\partial^{\beta_j}u(x)\Big)^{q_j} dx\right)^{\frac{p}{q_j}} \qquad (20)\\
&\quad\le \|u|\mathcal{K}^0_{-\gamma_0,q_0}(D)\|^{p(\delta-l)} \prod_{j=1}^{l}\|u|\mathcal{K}^{|\beta_j|}_{-\gamma_j,q_j}(D)\|^p,
\end{aligned}$$

where we first used the growth condition (16), and then Hölder's inequality assuming

$$\frac{\delta-l}{q_0}+\frac{1}{q_1}+\cdots+\frac{1}{q_l}\le\frac{1}{p}$$

(since $D$ is a bounded domain), as well as

$$\gamma_0(\delta-l)+\gamma_1+\cdots+\gamma_l+|\alpha-\alpha'|\le|\alpha|-a+1.$$

Note that this step also required the condition $\delta-l\ge 0$ for all $l$, hence $\delta\ge m-1$. To satisfy these two conditions, we choose

$$\frac{d}{q_0}=\frac{d}{p}-m-1>0 \qquad\text{and}\qquad \gamma_0=-\frac{d}{q_0}+\frac{d}{p}-a-1$$

as well as

$$\frac{d}{q_j}=|\beta_j|+\frac{d}{p}-m-1>0 \qquad\text{and}\qquad \gamma_j=-\frac{d}{q_j}+\frac{d}{p}-a-1.$$

This choice implies

$$\frac{d(\delta-l)}{q_0}+\frac{d}{q_1}+\cdots+\frac{d}{q_l}=\delta\frac{d}{p}-\delta(m+1)+|\alpha-\alpha'|\le\delta\frac{d}{p}-\delta(m+1)+m-1,$$

which is bounded by $\frac{d}{p}$ if, and only if, $\frac{d}{p} - m - 1 \leq \frac{2}{\delta-1}$ (in case $\delta = 1$ there is no extra condition). This reasoning particularly ensures $p < q_j < \infty$ for $j = 0, \dots, l$. Similarly, we find

$$\begin{aligned}
&\gamma_0(\delta - l) + \gamma_1 + \cdots + \gamma_l + |\alpha - \alpha'| \\
&\qquad = \delta\Big(\frac{d}{p} - a - 1\Big) - \frac{d(\delta - l)}{q_0} - \frac{d}{q_1} - \cdots - \frac{d}{q_l} + |\alpha - \alpha'| \\
&\qquad = \delta\Big(\frac{d}{p} - a - 1\Big) - \delta\Big(\frac{d}{p} - m - 1\Big) \\
&\qquad = \delta\big(m - a\big) \overset{!}{\leq} |\alpha| - a + 1
\end{aligned}$$

which is fulfilled with $|\alpha| = m - 1$ (sufficient by Step 1) due to $a \geq m$, which follows from our assumptions in view of $a \geq \frac{d}{p} - 1 = \frac{d}{p} - 1 - m + m \geq m$ (again for $\delta = 1$ no assumption on the parameters is required).

With this choice of parameters $\gamma_j$ and $q_j$ we now can further argue using the Sobolev-embedding from Theorem 1, which yields

$$\begin{aligned}
&\int_D \bigg(\rho^{|\alpha|-a+1}(x)\Big|\partial_\xi^l \partial_x^{\alpha'} g(x,\xi)\big|_{\xi=u(x)}\Big| \, \Big|\partial_x^{\beta_1} u(x) \cdots \partial_x^{\beta_l} u(x)\Big|\bigg)^p dx \\
&\qquad \leq \|u|\mathcal{K}^0_{-\gamma_0,q_0}(D)\|^{p(\delta-l)} \prod_{j=1}^{l} \|u|\mathcal{K}^{|\beta_j|}_{-\gamma_j,q_j}(D)\|^p \\
&\qquad \leq \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^{p(\delta-l)} \prod_{j=1}^{l} \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^p = \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^{p\delta} .
\end{aligned}$$

This proves the claim in case $\frac{d}{p} - m - 1 > 0$.

*Step 3.* Now assume $m + 1 - \frac{d}{p} > 0$.

*Substep 3.1.* For the first term in (19), corresponding to $l = 0$, we use (16) and see that $|\partial_x^\alpha g(x, u(x))| \leq c_{\alpha,0}|u(x)|^\delta$. This yields

$$\begin{aligned}
&\max_{|\alpha|\leq m-1} \int_D |\rho(x)^{|\alpha|-a+1} \left(\partial_x^\alpha g\right)(x, u(x))|^p dx \\
&\qquad \lesssim \int_D (\rho(x)^{-a+1} \, |u(x)|^\delta)^p dx \\
&\qquad \leq \left(\sup_{x\in D} \rho(x)^{(\delta-1)\gamma} |u(x)|^{\delta-1}\right)^p \int_D \left(\rho(x)^{-a-1}|u(x)|\right)^p \mathrm{d}x \\
&\qquad \leq \|u|\mathcal{K}^0_{-\gamma,\infty}(D)\|^{(\delta-1)p} \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^p, \qquad (21)
\end{aligned}$$

where the second but last step holds if $-a+1 \geq (\delta-1)\gamma - a - 1$, i.e., we may choose $\gamma := \frac{2}{\delta-1}$ if $\delta > 1$. Furthermore, by Theorem 1 we see that

$$\mathcal{K}^{m+1}_{a+1,p}(D) \hookrightarrow \mathcal{K}^0_{-\gamma,\infty}(D) \tag{22}$$

if

$$m+1-\frac{d}{p} > 0 \qquad \text{and} \qquad a+1-\frac{d}{p} \geq -\gamma,$$

which is satisfied by our assumptions. Now (21) and (22) give the desired estimate for the first term in (19). If $\delta = 1$ the same result follows from a slight modification of (21). In this case we have

$$\begin{aligned} &\max_{|\alpha|\leq m-1} \int_D |\rho(x)^{|\alpha|-a+1} (\partial_x^\alpha g)(x, u(x))|^p dx \\ &\qquad \leq \int_D (\rho(x)^{-a+1} |u(x)|)^p dx = \|u|\mathcal{K}^0_{a-1,p}(D)\|^p \leq \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^p, \end{aligned}$$

where the last step is a consequence of the elementary embeddings for Kondratiev spaces, cf. Remark 3.

*Substep 3.2.* Next we shall deal with the terms in (19) with $l = 1$, i.e., the terms $\left|\partial_\xi \partial_x^{\alpha'} g(x,\xi)\big|_{\xi=u(x)}\right| \left|\partial_x^{\alpha-\alpha'} u(x)\right|$. Note that this step is only relevant for $m \geq 2$ since we consider derivatives up to order $m-1 \geq |\alpha| = |\alpha - \alpha'| + |\alpha'| \geq l$. Using the growth condition (16) we find

$$\begin{aligned} &\int_D \left(\rho^{|\alpha|-a+1}(x) \left|\partial_\xi \partial_x^{\alpha'} g(x,\xi)\big|_{\xi=u(x)}\right| \left|\partial_x^{\alpha-\alpha'} u(x)\right|\right)^p dx \\ &\quad \lesssim \int_D \left(\rho^{|\alpha|-a+1}(x) |u(x)|^{\delta-1} \left|\partial_x^{\alpha-\alpha'} u(x)\right|\right)^p dx \\ &\quad \lesssim \left(\sup_{x\in D} \rho^{\gamma_0}(x)|u(x)|\right)^{(\delta-1)p} \int_D \left(\rho^{|\alpha-\alpha'|+\gamma_1}(x) |\partial^{\alpha-\alpha'} u(x)|\right)^p dx \\ &\quad \leq \|u|\mathcal{K}^0_{-\gamma_0,\infty}(D)\|^{p(\delta-1)} \|u|\mathcal{K}^{|\alpha-\alpha'|}_{-\gamma_1,p}(D)\|^p. \end{aligned}$$

For this it needs to hold $\gamma_0(\delta-1) + |\alpha - \alpha'| + \gamma_1 \leq |\alpha| - a + 1$. In addition we want the embedding $\mathcal{K}^{m+1}_{a+1,p}(D) \hookrightarrow \mathcal{K}^0_{-\gamma_0,\infty}(D)$ to be valid, which requires

$$m+1-\frac{d}{p} > 0 \qquad \text{and} \qquad a+1-\frac{d}{p} \geq -\gamma_0.$$

In view of our assumption $a \geq \frac{d}{p} - 1$ this is fulfilled for arbitrary $\gamma_0 \geq 0$. Hence choosing $-\gamma_1 = a+1$ and $\gamma_0(\delta-1) = |\alpha'| + 2$ (i.e., $\gamma_0 \geq 0$ is arbitrary for $\delta = 1$), the mentioned condition is satisfied and we conclude

$$\begin{aligned}
&\int_D \left(\rho^{|\alpha|-a+1}(x)\Big|\partial_\xi \partial_x^{\alpha'} g(x,\xi)\big|_{\xi=u(x)}\Big|\,\Big|\partial_x^{\alpha-\alpha'} u(x)\Big|\right)^p dx \\
&\quad \lesssim \|u|\mathcal{K}^0_{-\gamma_0,\infty}(D)\|^{p(\delta-1)} \|u|\mathcal{K}^{|\alpha-\alpha'|}_{a+1,p}(D)\|^p \\
&\quad \lesssim \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^{p(\delta-1)} \|u|\mathcal{K}^{|\alpha-\alpha'|}_{a+1,p}(D)\|^p \le \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^{p\delta} . \qquad (23)
\end{aligned}$$

*Substep 3.3.* Now consider the terms in (19) with $l \ge 2$. Note that this step is only relevant for $m \ge 3$ since as before $l \le m-1$. Once again we shall use the growth condition (16). We obtain this time

$$\begin{aligned}
&\max_{|\alpha|\le m-1} \int_D \Big|\rho(x)^{|\alpha|-a+1}\, \partial_\xi^l \partial_x^{\alpha'} g(x,u(x))\, \partial_x^{\beta_1} u(x) \cdots \partial_x^{\beta_l} u(x)\Big|^p dx \\
&\quad \lesssim \int_D \Big(|\rho^{|\alpha|-a+1}|u(x)|^{\delta-l} \prod_{j=1}^{l} |\partial^{\beta_j} u(x)|\Big)^p dx \\
&\quad \lesssim \left(\sup_{x\in D} \rho(x)^{\gamma(\delta-l)} |u(x)|^{\delta-l}\right)^p \prod_{j=2}^{l} \left(\sup_{x\in D} \rho(x)^{|\beta_j|+\gamma} |\partial^{\beta_j} u(x)|\right)^p \cdot \\
&\qquad \cdot \left(\int_D \rho(x)^{p(|\beta_1|-a-1)} |\partial^{\beta_1} u(x)|^p dx\right) . \qquad (24)
\end{aligned}$$

Here, in the last step, we have used that $\rho(x) \le 1$ and thus, in order to obtain an estimate from above, the exponents of $\rho$ have to satisfy

$$\begin{aligned}
|\alpha| - a + 1 &\overset{!}{\ge} \gamma(\delta - l) + \sum_{j=2}^{l} (|\beta_j| + \gamma) + |\beta_1| - a - 1 \\
&= \gamma(\delta-1) + |\alpha| - |\alpha'| - a - 1 ,
\end{aligned}$$

which leads to $\gamma \le \frac{2+|\alpha'|}{\delta-1}$ in case $\delta \ne 1$ and $\gamma$ arbitrary in case $\delta = 1$. Therefore we may choose $\gamma = \frac{2}{\delta-1}$ if $\delta > 1$ and $\gamma = 0$ if $\delta = 1$. In addition, without loss of generality, we assume $|\beta_1| \ge |\beta_j|$ for all $2 \le j \le l$.
Clearly,

$$\begin{aligned}
\left(\sup_{x\in D} \rho(x)^{\gamma}\, |u(x)|\right)^{\delta-l} &= \|u|\mathcal{K}^0_{-\gamma,\infty}(D)\|^{\delta-l}, \\
\left(\sup_{x\in D} \rho(x)^{|\beta_j|+\gamma} |\partial^{\beta_j} u(x)|\right) &\le \|u|\mathcal{K}^{|\beta_j|}_{-\gamma,\infty}(D)\|, \\
\left(\int_D \rho(x)^{p(|\beta_1|-a-1)} |\partial^{\beta_1} u(x)|^p dx\right)^{1/p} &\le \|u|\mathcal{K}^{|\beta_1|}_{a+1,p}(D)\| .
\end{aligned}$$

From Theorem 1 we conclude

$$\mathcal{K}^{m+1}_{a+1,p}(D) \hookrightarrow \mathcal{K}^{0}_{-\gamma,\infty}(D) \qquad \text{if} \qquad m+1-\frac{d}{p}>0\,, \quad a+1-\frac{d}{p}\geq -\gamma\,.$$

Furthermore, we have

$$\mathcal{K}^{m+1}_{a+1,p}(D) \hookrightarrow \mathcal{K}^{m-2}_{-\gamma,\infty}(D) \qquad \text{if} \qquad 3-\frac{d}{p}>0\,, \quad a+1-\frac{d}{p}\geq -\gamma\,.$$

Observe that $|\beta_1|+|\beta_j| \leq |\alpha|-|\alpha'| \leq m-1$. Hence, by (21), (23) and (24) we get

$$\|G(u)|\mathcal{K}^{m-1}_{a-1,p}(D)\| \lesssim \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^{\delta}$$

as claimed. □

**Remark 6.** Some more remarks concerning Theorem 3 seem to be in order. Estimates for Nemytskij operators $T_G: \ u \mapsto G(u)(x) = g(x,u(x))$ in Sobolev spaces are a delicate topic. Even in the more simple case of composition operators $u \mapsto F(u)(x) := f(u(x))$ there are many open questions, we refer, e.g., to [6], [9], [45] or the survey [8]. The naive conjecture that $F$ maps a Sobolev space into itself if $f$ is sufficiently smooth is known to be true if $W^m_p(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$. On the other hand, if the Sobolev space contains unbounded functions, such a statement is not true. One may think on composition operators related to $f(t) := t^n$ for some $n \geq 2$. Then, under the mapping $F$, the unboundedness is enhanced. Since we allow a shift in the smoothness from $m+1$ to $m-1$ (and from $a+1$ to $a-1$) we can deal also with Kondratiev spaces containing unbounded functions as long as the unboundedness is small enough. This is expressed by the restriction $0 > m+1-\frac{d}{p} \geq -\frac{2}{\delta-1}$.

We finish this subsection by taking a closer look on the mapping $u \mapsto u^n$, $n \geq 2$.

**Corollary 1.** *Let $D$ be as in Definition 2 and denote by $S$ the singularity set of $D$. Let $1<p<\infty$, $a \geq \frac{d}{p}-1$, $m \in \mathbb{N}$, and $n \in \mathbb{N}$, $n \geq 2$. Moreover, assume either*

$$\min(m+1,3)-\frac{d}{p}>0 \tag{25}$$

*or*

$$0>m+1-\frac{d}{p}\geq -\frac{2}{n-1}. \tag{26}$$

*Then the nonlinear operator $T_n: \ u \mapsto u^n$ maps functions from $\mathcal{K}^{m+1}_{a+1,p}(D)$ to functions in $\mathcal{K}^{m-1}_{a-1,p}(D)$. Moreover, it holds*

$$\|u^n|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq C\, \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^n \tag{27}$$

*with some constant $C>0$ independent of $u$.*

*Proof.* It will be enough to give a few comments.

Clearly, monomials $g(x,\xi) = \xi^n$, where $n \in \mathbb{N}$ and $n \geq 2$, satisfy the growth condition (16) for $\delta = n$. Hence, we may follow the proof of Theorem 3 step by step. Observe that in formula (19) the summation with respect to $l$ is limited by $\min(|\alpha|, n)$ in this case. If $l = n \leq |\alpha|$, then we drop the first factor in (20). All other arguments can be repeated which proves the claim in case $0 > m + 1 - \frac{d}{p} \geq -\frac{2}{n-1}$. Also the proof of (i) under the restriction (25) follows along the lines of the previous arguments in Step 3, where again $\delta = n$. □

**Remark 7.** Under the stronger assumption $d/p < 2$ instead of (25), an alternative proof can be given applying Theorem 2 together with an obvious induction argument.

# 3 Fixed points of nonlinear operators in Banach spaces

As already outlined above, our regularity results will be established by means of suitable fixed point operators. Here, we will use the fixed point theorem as stated in Proposition 2 below that works for admissible spaces.

**Definition 3.** *A quasi-normed space $A$ is said to be* admissible*, if for every compact subset $K \subset A$ and for every $\varepsilon > 0$ there exists a continuous map $\tilde{T} : K \to A$ such that $\tilde{T}(K)$ is contained in a finite-dimensional subset of $A$ and $x \in K$ implies*

$$\|\tilde{T}x - x|A\| \leq \varepsilon.$$

Let $X$ and $Y$ be admissible quasi-Banach spaces. Furthermore, we assume that $\tilde{L} : Y \to X$ is a linear and continuous operator and $\tilde{N} : X \to Y$ is (in general) a nonlinear map. We are looking for a fixed point of the problem

$$u = (\tilde{L} \circ \tilde{N})u. \tag{28}$$

For this we make use of the following result.

**Proposition 2.** *Let $X$, $Y$, $\tilde{L}$, and $\tilde{N}$ be as above. Suppose that there exist $\eta \geq 0$, $\vartheta \geq 0$, and $\delta \geq 0$ such that*

$$\| \tilde{N}u|Y\| \leq \eta + \vartheta \ \|u|X\|^{\delta} \tag{29}$$

*holds for all $u \in X$. Furthermore, we assume that the mapping $\tilde{L} \circ \tilde{N} : X \to X$ is completely continuous. Then there exists at least one solution $u \in X$ of* (28) *provided one of the following conditions is satisfied:*

$$
\begin{aligned}
&\text{(a)} \qquad \delta \in [0,1), \\
&\text{(b)} \qquad \delta = 1,\ \vartheta < \| \tilde{L} \|^{-1}, \\
&\text{(c)} \qquad \delta > 1 \ \ \textit{and} \ \ \eta \, \| \tilde{L} \| < \left[ \frac{1}{\vartheta \, \| \tilde{L} \|} \right]^{\frac{1}{\delta-1}} \left[ \left(\frac{1}{\delta}\right)^{\frac{1}{\delta-1}} - \left(\frac{1}{\delta}\right)^{\frac{\delta}{\delta-1}} \right].
\end{aligned}
\tag{30}
$$

Clearly, we will apply Proposition 2 to the case when $\tilde{L} = L$ and $\tilde{N} = N$ as defined in (14) and (15), respectively.

**Remark 8.** A proof of this proposition, which is based on the Schauder fixed point theorem, can be found in [25]. Later on, in [48], by means of the Leray-Schauder principle, these results have also been generalized to admissible quasi-Banach spaces.

**Remark 9.** In [19] semilinear problems associated with the Poisson equation have been studied. There, the authors considered nonlinear terms that lead to bounds with $\delta \leq 1$. For this reason in this paper we particularly discuss the case $\delta > 1$.

# 4 Regularity of semilinear elliptic problems in Kondratiev spaces

We want to establish our regularity results by applying Proposition 2 to Kondratiev spaces. Therefore, we have to clarify that all the necessary assumptions are satisfied in this case.

## 4.1 Admissibility of Kondratiev spaces

Before we come to the existence of solutions of problem (1), we need to discuss the admissibility of Kondratiev spaces, this being one of the requirements of our main tool Proposition 2.

**Proposition 3.** *Let the domain $D$ satisfy Assumption 1. Then the spaces $\mathcal{K}^m_{a,p}(D)$ and $\mathcal{K}^{m+1}_{a+1,2,0}(D)$ are admissible for all $1 < p < \infty$, $m \in \mathbb{N}$, and $a \in \mathbb{R}$.*

*Proof.* The admissibility of the spaces $\mathcal{K}^m_{a,p,0}(D)$ is an immediate consequence of the admissibility of the Kondratiev spaces $\mathcal{K}^m_{a,p}(D)$ itself.
The admissibility of the Kondratiev spaces $\mathcal{K}^m_{a,p}(D)$ can be traced back to the one of the so-called refined localization spaces $F^{s,\mathrm{rloc}}_{p,q}(D)$ and their relation to the spaces $\mathcal{K}^m_{m,p}(D)$. In turn, the admissibility of the spaces $F^{s,\mathrm{rloc}}_{p,q}(D)$ follows from the existence of wavelet bases.

**Step 1:** Concerning the definition and further properties of refined localization spaces $F^{s,\mathrm{rloc}}_{p,q}(D)$ we refer to [53, Ch. 4], [54, Ch. 2], and [33]. In particular, the latter reference provides the following equivalent characterization of their norm: With $\delta$ being the distance to the boundary, $\delta(x) = \mathrm{dist}\,(x, \partial D)$, we have

$$\|u|F^{s,\mathrm{rloc}}_{p,q}(D)\| \sim \|u|F^s_{p,q}(D)\| + \|\delta^{-s}u|L_p(D)\|.$$

This equivalence holds for $s > \sigma_{p,q} := d\left(\frac{1}{\min(1,p,q)} - 1\right)$, $0 < p < \infty$, and $0 < q \leq \infty$. One of the key properties of these refined localization spaces is their characterization by suitable wavelet systems. Moreover, these wavelet systems then form a basis in case $q < \infty$. From the existence of such a basis, the admissibility now follows by standard arguments: expanding $x \in K$ as a series $x = \sum_{j=1}^{\infty} \lambda_j(x) w_j$, where $(w_j)_{j=1}^{\infty}$ is the wavelet system and $\lambda_j \in (F^{s,\mathrm{rloc}}_{p,q}(D))'$, we can find $j_0(x)$ such that $\| \sum_{j=1}^{j_0(x,\varepsilon)} \lambda_j(x) w_j - x|F^{s,\mathrm{rloc}}_{p,q}(D)\| \leq \varepsilon$. Now a standard compactness argument ensures that we can choose $j_0(x,\varepsilon)$ independent of $x \in K$, so that defining $T_\varepsilon x = \sum_{j=1}^{j_0(\varepsilon)} \lambda_j(x) w_j$ satisfies the requirements of the definition for admissibility of $F^{s,\mathrm{rloc}}_{p,q}(D)$.

**Step 2:** In [31] it was shown that $\mathcal{K}^m_{m,p}(\mathbb{R}^d \setminus E) = F^{m,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus E)$, where $E$ is an arbitrary closed set with Lebesgue measure $|E| = 0$. This particularly applies to the case where $E$ is the singular set $S$ of a bounded Lipschitz domain $D$ of polyhedral type. Moreover, the spaces $\mathcal{K}^m_{m,p}(D)$ and $\mathcal{K}^m_{m,p}(\mathbb{R}^d \setminus S)$ are related via the boundedness of Stein's extension operator, $\mathcal{E} : \mathcal{K}^m_{m,p}(D) \to \mathcal{K}^m_{m,p}(\mathbb{R}^d \setminus S)$, which is proven in [32, Lem. 5.1]. For general $a$, the admissibility of $\mathcal{K}^m_{a,p}(D)$ now follows from the admissibility of $\mathcal{K}^m_{m,p}(D)$ since both spaces are isomorphic, cf. the listed properties of Kondratiev spaces on page 9. □

## 4.2 Complete continuity of $L \circ N$

In this subsection we shall prove another technical aspect for the application of the fixed point result (Proposition 2), namely the complete continuity of the composed mapping $L \circ N$. For an operator $T$ to be completely continous it is sufficient to know that it is continuous and compact. Hence, we shall prove compactness of the linear solution map $L$ for appropriate pairs of spaces, together with continuity of $N$, which will ultimately yield complete continuity of the composed map.

A natural idea to prove compactness of $L$ would be to use compact embeddings of Kondratiev spaces. For classical non-weighted Sobolev spaces, this strategy has successfully been worked out in [19]. Moreover, in [46, Lemma 4.1.4] and [16, Thm. 4] compact embeddings of Kondratiev spaces have already been established.

If $p \leq q$, then $\mathcal{K}^m_{a,p}(D,M)$ is compactly embedded into $\mathcal{K}^{m'}_{a',q}(D,M)$ if, and only if,

$$m - \frac{d}{p} > m' - \frac{d}{q} \qquad \text{and} \qquad a - \frac{d}{p} > a' - \frac{d}{q}\,.$$

However, by using this result with $p = q$ directly, we would loose at least one order in the regularity in the end. To avoid this problem, our strategy is now to define fractional families of Kondratiev spaces (by means of complex interpolation of the usual Kondratiev spaces) with the hope that we only loose an arbitrarily small $\varepsilon$ of smoothness. We refer to [33] in this context, where these spaces are studied in detail. Concerning the basics in complex interpolation theory we furthermore refer to [5, 52].

**Definition 4.** *Let $D$ be a domain which satisfies Assumption 1, $s \in \mathbb{R}_+$, $1 \leq p \leq \infty$, and $a \in \mathbb{R}$. For $s \in \mathbb{N}$, put $\mathfrak{K}^s_{a,p}(D) = \mathcal{K}^s_{a,p}(D)$. Otherwise, for $s \geq 0$ with $s \notin \mathbb{N}$, let $m = [s]$ denote its integer part and $\theta = \{s\} := s - [s]$. Then we define*

$$\mathfrak{K}^s_{a,p}(D) := \left[\mathcal{K}^m_{a,p}(D), \mathcal{K}^{m+1}_{a,p}(D)\right]_\theta.$$

**Remark 10.** We collect some further properties of fractional Kondratiev spaces which will be important for our later considerations.

(i) The spaces $\mathfrak{K}^s_{a,p}(D)$ are admissible for all $1 < p < \infty$, $s \geq 0$, and $a \in \mathbb{R}$. We sketch the proof which follows from the observations in Proposition 3 together with the results presented in [33]. There, for $s \geq 0$ and $1 < p < \infty$, alternative fractional Kondratiev spaces

$$\tilde{\mathfrak{K}}^s_{s,p}(\mathbb{R}^d \setminus S) = F^{s,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S)$$

are introduced, where $S$ is the singular set of a bounded Lipschitz domain of polyhedral type. Moreover, for $a \in \mathbb{R}$ put

$$\tilde{\mathfrak{K}}^s_{a,p}(\mathbb{R}^d \setminus S) := T_{s-a}\tilde{\mathfrak{K}}^s_{s,p}(\mathbb{R}^d \setminus S),$$

where $T_{s-a}(u) = \rho^{s-a}u$. Since refined localization spaces are admissible and $T_{s-a}$ is an isomorphism we conclude that the spaces $\tilde{\mathfrak{K}}^s_{a,p}(\mathbb{R}^d \setminus S)$ are admissible as well. Furthermore, it is shown in [33] that $\tilde{\mathfrak{K}}^s_{a,p}(D)$ and $\tilde{\mathfrak{K}}^s_{a,p}(\mathbb{R}^d \setminus S)$ are related via the boundedness of Stein's extension operator. This implies admissibility of $\tilde{\mathfrak{K}}^s_{a,p}(D)$. Finally, in [33] it is proven that for $1 < p < \infty$, $s \geq 0$, and $a \in \mathbb{R}$, we have the coincidence

$$\mathfrak{K}^s_{a,p}(D) = \tilde{\mathfrak{K}}^s_{a,p}(D), \tag{31}$$

which shows admissibility of the spaces $\mathfrak{K}^s_{a,p}(D)$.

(ii) From the coincidence (31) and the interpolation results for the spaces $\tilde{\mathfrak{K}}^s_{a,p}(D)$ established in [33], we can conclude a generalized interpolation result for arbitrary pairs of Kondratiev spaces within the full scale of parameters $s$ and $a$. Let $1 < p < \infty$, $s_0, s_1 \geq 0$, $a_0, a_1 \in \mathbb{R}$, and $0 < \theta < 1$. Then

$$\mathfrak{K}^s_{a,p}(D) = \left[\mathfrak{K}^{s_0}_{a_0,p}(D), \mathfrak{K}^{s_1}_{a_1,p}(D)\right]_\theta, \tag{32}$$

where $s = (1-\theta)s_0 + \theta s_1$ and $a = (1-\theta)a_0 + \theta a_1$.

In terms of compact embeddings the following result is proven in [33].

**Proposition 4.** *Let $D$ be a domain which satisfies Assumption 1. Moreover, let $m \in \mathbb{N}$, $a \in \mathbb{R}$, and $1 < p < \infty$. Then the embedding*

$$\mathcal{K}^m_{a,p}(D) \hookrightarrow \mathfrak{K}^{m-\varepsilon}_{a-\varepsilon,p}(D) \tag{33}$$

*is compact for arbitrary $0 < \varepsilon < 1$.*

**Remark 11.** From (33) together with [12, Thm. 10] we obtain that the embedding

$$\mathfrak{K}^{m-\varepsilon'}_{a-\varepsilon',p}(D) = \left[\mathcal{K}^{m-1}_{a-1,p}(D), \mathcal{K}^m_{a,p}(D)\right]_{1-\varepsilon'} \hookrightarrow \left[\mathfrak{K}^{m-1-\varepsilon}_{a-1-\varepsilon,p}(D), \mathfrak{K}^{m-\varepsilon}_{a-\varepsilon,p}(D)\right]_{1-\varepsilon'} = \mathfrak{K}^{m-(\varepsilon+\varepsilon')}_{a-(\varepsilon+\varepsilon'),p}(D)$$

is also compact for arbitrary $0 < \varepsilon, \varepsilon' < 1$.

In order to deal with the complete continuity of $L \circ N$, we first show that $N : \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \to \mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D)$ is continuous for sufficiently small $\varepsilon > \varepsilon' > 0$. In particular, this requires a slight strengthening of Theorem 3.

**Proposition 5.** *Let $D$ be a domain which satisfies Assumption 1, $d/2 < p < \infty$, $a \geq \frac{d}{p} - 1$, and $m \in \mathbb{N}$ with $m > \frac{d}{p}$. Moreover, assume $\delta \geq \max(1, m-1)$ and let the function $g$ satisfy the growth-condition (16). Then for sufficiently small $\varepsilon > 0$ the nonlinear operator $G(u)(x) = g(x, u(x))$ maps functions from $\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)$ to functions in $\mathcal{K}^{m-1}_{a-1,p}(D)$ and it holds*

$$\|G(u)|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq C \, \|u|\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\|^{\delta} \,,$$

*for some constant $C > 0$ independent of $u$.*

*Proof. Step 1.* For now let $0 < \varepsilon < 1$. In what follows we will have to choose $\varepsilon$ small enough to suit our needs (from the proof $\varepsilon < \frac{1}{\delta}$ will turn out to be sufficient, but below we won't further comment on the specific choice). We reuse the estimate

$$\begin{aligned} \left|\partial^{\alpha}\big(g(x,u(x))\big)\right| &\lesssim \left|\partial^{\alpha}_x g(x,\xi)|_{\xi=u(x)}\right| \\ &\quad + \sum_{\alpha' \leq \alpha} \sum_{l=1}^{|\alpha-\alpha'|} \left|\partial^l_{\xi}\partial^{\alpha'}_x g(x,\xi)\big|_{\xi=u(x)}\right| \sum_{\substack{\beta_1+\cdots+\beta_l=\alpha-\alpha',\\ |\beta_1|,\ldots,|\beta_l|\geq 1}} \left|\partial^{\beta_1}_x u(x)\cdots\partial^{\beta_l}_x u(x)\right|, \end{aligned} \tag{34}$$

obtained in Step 1 of the proof of Theorem 3. Again we start dealing with the first term in (34), corresponding to $l = 0$. From (16) it follows that $|\partial^{\alpha}_x g(x, u(x))| \leq c_{\alpha,0}|u(x)|^{\delta}$, which now yields

$$\begin{aligned} \max_{|\alpha|\leq m-1} &\int_D \left|\rho(x)^{|\alpha|-a+1}\,(\partial^{\alpha}_x g)(x,u(x))\right|^p dx \\ &\lesssim \int_D \left(\rho(x)^{-a+1}\,|u(x)|^{\delta}\right)^p dx \\ &\leq \left(\sup_{x\in D}\rho(x)^{\gamma(\delta-1)}u(x)^{\delta-1}\right)^p \left(\int_D \left(\rho(x)^{-a-1+\varepsilon}|u(x)|\right)^p \mathrm{d}x\right) \\ &\lesssim \|u|\mathcal{K}^0_{-\gamma,\infty}\|^{(\delta-1)p}\|u|\mathcal{K}^0_{a+1-\varepsilon,p}\|^p \\ &\lesssim \|u|\mathcal{K}^m_{a+1-\varepsilon,p}\|^{(\delta-1)p}\|u|\mathcal{K}^m_{a+1-\varepsilon,p}\|^p, \end{aligned} \tag{35}$$

where the second step holds if

$$-a+1 \geq \gamma(\delta-1) - a - 1 + \varepsilon,$$

i.e., we choose $\gamma := \frac{2-\varepsilon}{\delta-1}$ if $\delta > 1$ and the 4th step is a consequence of the elementary embeddings for Kondratiev spaces, cf. Remark 3 and Theorem 1(ii), which holds if

$$m - \frac{d}{p} > 0 \qquad \text{and} \qquad a + 1 - \varepsilon - \frac{d}{p} \geq -\gamma,$$

and is satisfied by our assumptions upon choosing $\varepsilon$ small enough. Note that for $\delta = 1$ (35) is just a consequence of the elementary embeddings of Kondratiev spaces, i.e., we have

$$\begin{aligned} \max_{|\alpha| \leq m-1} & \int_D \left| \rho(x)^{|\alpha|-a+1} \left(\partial_x^\alpha g\right)(x, u(x)) \right|^p dx \\ & \lesssim \int_D \left( \rho(x)^{-a+1} \left| u(x) \right| \right)^p dx = \| u | \mathcal{K}^0_{a-1,p}(D) \|^p \leq \| u | \mathcal{K}^m_{a+1-\varepsilon,p}(D) \|^p. \end{aligned} \tag{36}$$

By definition of fractional Kondratiev spaces and the properties of complex interpolation it follows that

$$\begin{aligned} \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) &= \left[ \mathcal{K}^m_{a+1-\varepsilon,p}(D), \mathcal{K}^{m+1}_{a+1-\varepsilon,p}(D) \right]_{1-\varepsilon} \\ & \hookrightarrow \left[ \mathcal{K}^m_{a+1-\varepsilon,p}(D), \mathcal{K}^m_{a+1-\varepsilon,p}(D) \right]_{1-\varepsilon} = \mathcal{K}^m_{a+1-\varepsilon,p}(D). \end{aligned} \tag{37}$$

This together with (35) and (36) yields for $\delta \geq 1$,

$$\max_{|\alpha| \leq m-1} \int_D \left| \rho(x)^{|\alpha|-a+1} \left(\partial_x^\alpha g\right)(x, u(x)) \right|^p dx \lesssim \| u | \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \|^{\delta p}. \tag{38}$$

*Step 2.* Next we shall deal with the terms in (19) with $l = 1$, i.e., the terms $\left| \partial_\xi \partial_x^{\alpha'} g(x,\xi) \big|_{\xi = u(x)} \right| \left| \partial_x^{\alpha - \alpha'} u(x) \right|$. Note that this step is only relevant for $m \geq 2$ since we consider derivatives up to order $m - 1 \geq |\alpha| = |\alpha - \alpha'| + |\alpha'| \geq l$. Using the growth condition (16) we find

$$\begin{aligned} & \int_D \left( \rho^{|\alpha|-a+1}(x) \left| \partial_\xi \partial_x^{\alpha'} g(x,\xi) \big|_{\xi=u(x)} \right| \left| \partial_x^{\alpha-\alpha'} u(x) \right| \right)^p dx \\ & \quad \lesssim \int_D \left( \rho^{|\alpha|-a+1}(x) |u(x)|^{\delta-1} \left| \partial_x^{\alpha-\alpha'} u(x) \right| \right)^p dx \\ & \quad \lesssim \left( \sup_{x \in D} \rho^{\gamma_0}(x) |u(x)| \right)^{(\delta-1)p} \int_D \left( \rho^{|\alpha-\alpha'|+\gamma_1}(x) |\partial^{\alpha-\alpha'} u(x)| \right)^p dx \\ & \quad \leq \| u | \mathcal{K}^0_{-\gamma_0,\infty}(D) \|^{p(\delta-1)} \| u | \mathcal{K}^{|\alpha-\alpha'|}_{-\gamma_1,p}(D) \|^p. \end{aligned}$$

For this it needs to hold $\gamma_0(\delta - 1) + |\alpha - \alpha'| + \gamma_1 \leq |\alpha| - a + 1$. In addition we want the embedding $\mathcal{K}^m_{a+1-\varepsilon,p}(D) \hookrightarrow \mathcal{K}^0_{-\gamma_0,\infty}(D)$ to be valid, which requires

$$m - \frac{d}{p} > 0 \qquad \text{and} \qquad a + 1 - \varepsilon - \frac{d}{p} \geq -\gamma_0.$$

In view of our assumption $a \geq \frac{d}{p} - 1$ this is fulfilled for $\varepsilon$ small (i.e., $0 < \varepsilon < \gamma_0$). Hence choosing $-\gamma_1 = a+1-\varepsilon$ and $\gamma_0(\delta-1) = |\alpha'|+2-\varepsilon$ (i.e., $\gamma_0 > 0$ is arbitrary for $\delta = 1$), the mentioned condition is satisfied and we conclude

$$\begin{aligned}
&\int_D \left( \rho^{|\alpha|-a+1}(x) \left| \partial_\xi \partial_x^{\alpha'} g(x,\xi) \big|_{\xi=u(x)} \right| \left| \partial_x^{\alpha-\alpha'} u(x) \right| \right)^p dx \\
&\quad \lesssim \|u|\mathcal{K}^0_{-\gamma_0,\infty}(D)\|^{p(\delta-1)} \|u|\mathcal{K}^{|\alpha-\alpha'|}_{a+1-\varepsilon,p}(D)\|^p \\
&\quad \lesssim \|u|\mathcal{K}^m_{a+1-\varepsilon,p}(D)\|^{p(\delta-1)} \|u|\mathcal{K}^{|\alpha-\alpha'|}_{a+1-\varepsilon,p}(D)\|^p \\
&\quad \leq \|u|\mathcal{K}^m_{a+1-\varepsilon,p}(D)\|^{p\delta} \lesssim \|u|\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\|^{p\delta},
\end{aligned} \tag{39}$$

where the last embedding is a consequence of (37).

*Step 3.* Estimate of the terms in (34) with $l \geq 2$ (this case only occurs for $m \geq 3$). We use the inequality (16) in case $2 \leq l \leq |\alpha| - |\alpha'|$ and obtain

$$\begin{aligned}
&\max_{|\alpha|\leq m-1} \int_D \left| \rho(x)^{|\alpha|-a+1}\, \partial_\xi^l \partial_x^{\alpha'} g(x,u(x))\, \partial_x^{\beta_1} u(x) \cdots \partial_x^{\beta_j} u(x) \right|^p dx \\
&\quad \lesssim \int_D \left| \rho(x)^{|\alpha|-a+1} |u(x)|^{\delta-l} \prod_{j=1}^l |\partial^{\beta_j} u(x)| \right|^p dx \\
&\quad \lesssim \left( \sup_{x\in D} \rho(x)^{\gamma(\delta-l)} |u(x)|^{\delta-l} \right)^p \prod_{j=2}^l \left( \sup_{x\in D} \rho(x)^{|\beta_j|+\gamma-\varepsilon} |\partial^{\beta_r} u(x)| \right)^p \\
&\qquad \cdot \left( \int_D \rho(x)^{p(|\beta_1|-a-1+\varepsilon)} |\partial^{\beta_1} u(x)|^p dx \right).
\end{aligned} \tag{40}$$

Here, in the last step, we have used that $\rho(x) \leq 1$ and thus, in order to obtain an estimate from above, the exponents of $\rho$ have to satisfy

$$\begin{aligned}
|\alpha| - a + 1 \quad &\overset{!}{\geq} \quad \gamma(\delta - l) + \sum_{j=2}^l (|\beta_j| + \gamma - \varepsilon) + |\beta_1| - a - 1 + \varepsilon \\
&= \quad \gamma(\delta-1) + |\alpha| - |\alpha'| - a - 1 - \varepsilon(l-1) + \varepsilon\,,
\end{aligned}$$

which leads to $\gamma \leq \frac{(2-\varepsilon)+|\alpha'|+\varepsilon(l-1)}{\delta-1}$ in case $\delta \neq 1$ and $\gamma$ arbitrary in case $\delta = 1$. Therefore, we may choose $\gamma = \frac{1}{\delta-1}$ if $\delta > 1$ and $\gamma = 1$ if $\delta = 1$. In addition, without loss of generality, we assume $|\beta_1| \geq |\beta_j|$ for all $2 \leq j \leq l$.

Clearly,

$$\begin{aligned} \left(\sup_{x\in D}\rho(x)^{\gamma}\,|u(x)|\right)^{(\delta-l)} &\leq \|u|\mathcal{K}^{0}_{-\gamma,\infty}(D)\|^{\delta-l}, \\ \left(\sup_{x\in D}\rho(x)^{|\beta_j|+\gamma-\varepsilon}|\partial^{\beta_r}u(x)|\right) &\leq \|u|\mathcal{K}^{|\beta_j|}_{-\gamma+\varepsilon,\infty}(D)\|, \\ \left(\int_D \rho(x)^{p(|\beta_1|-a-1+\varepsilon)}|\partial^{\beta_1}u(x)|^p dx\right)^{1/p} &\leq \|u|\mathcal{K}^{|\beta_1|}_{a+1-\varepsilon,p}(D)\|\,. \end{aligned}$$

From (37) and Theorem 1 we conclude

$$\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\hookrightarrow \mathcal{K}^{m}_{a+1-\varepsilon,p}(D)\hookrightarrow \mathcal{K}^{0}_{-\gamma,\infty}(D)$$

if $m-\frac{d}{p}>0$ and $a+1-\varepsilon-\frac{d}{p}\geq -\gamma$ . Both inequalities are guaranteed by our assumptions on $m$ and $a$ (if $\varepsilon$ is chosen small enough such that $\varepsilon<\gamma$).
Furthermore, we have

$$\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\hookrightarrow \mathcal{K}^{m}_{a+1-\varepsilon,p}(D)\hookrightarrow \mathcal{K}^{m-2}_{-\gamma+\varepsilon,\infty}(D)$$

if $m-\frac{d}{p}>m-2$ and $a+1-\varepsilon-\frac{d}{p}\geq -\gamma+\varepsilon$. Both inequalities are satisfied by our assumptions $d/2<p$ and $a\geq \frac{d}{p}-1$ (if $\varepsilon$ is chosen small enough such that $\varepsilon<\gamma/2$).
Observe that $|\beta_1|+|\beta_j|\leq|\alpha|-|\alpha'|\leq m-1$. Hence, by (38), (39), and (40) we get

$$\begin{aligned} \|G(u)|\mathcal{K}^{m-1}_{a-1,p}(D)\| &\lesssim \|u|\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\|^{\delta} \\ &\quad + \|u|\mathcal{K}^{0}_{-\gamma,\infty}(D)\|^{\delta-l}\|u|\mathcal{K}^{m-2}_{-\gamma+\varepsilon,\infty}(D)\|^{l-1}\|u|\mathcal{K}^{m-1}_{a+1-\varepsilon,p}(D)\| \\ &\lesssim \|u|\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\|^{\delta} \end{aligned}$$

as claimed. □

**Remark 12.** Since we assume $p>\frac{d}{2}$ in Proposition 5 the condition $m>\frac{d}{p}$ is always satisfied if $m\geq 2$.
Moreover, using the fact that $\mathcal{K}^{m-1}_{a-1,p}(D)\hookrightarrow \mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D)$ for some $\varepsilon'>0$, we see from Proposition 5 that

$$G:\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\hookrightarrow \mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D) \tag{41}$$

is bounded as well.

So far from Proposition 5 we derive boundedness of $N : \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \to \mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D)$. However, since $N$ is nonlinear, continuity does not follow automatically from this. The fact that $N$ is continuous nevertheless is now proven in Lemma 1 below.

**Lemma 1.** *Let $D$ be a domain which satisfies Assumption 1, $d/2 < p < \infty$, $a \geq \frac{d}{p} - 1$, and $m \in \mathbb{N}$ with $m > \frac{d}{p}$. Moreover, assume $\delta \geq \max(1, m-1)$ and let the function $g$ satisfy the growth-condition (16). Then for sufficiently small $\varepsilon > \varepsilon' > 0$ the nonlinear mapping*

$$N : \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \to \mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D)$$

*is continuous.*

*Proof.* Since $N(u)(x) = f(x) - g(x, u(x))$ Proposition 5 implies

$$\|N(u)|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq \|f|\mathcal{K}^{m-1}_{a-1,p}(D)\| + c\|u|\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\|^{\delta}. \tag{42}$$

Making use of (16), we have $|G(u)(x)| = |g(x, u(x))| \leq c\,|u(x)|^{\delta}$ for some $\delta \geq 1$. From this we see that

$$\|Gu|L_p(D)\| \leq c \left( \int_D |u(x)|^{\delta p} \mathrm{d}x \right)^{1/p} = c\|u|L_{\delta p}\|^{\delta},$$

therefore, $G : L_{\delta p}(D) \to L_p(D)$ is bounded. Continuity of $G$ follows from [1, Thm. 3.7] since $g$ is at least continuous in both variables. Hence,

$$N : L_{\delta p}(D) \to L_p(D) \tag{43}$$

is continuous as well. By (37) and Theorem 1 we have

$$\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \hookrightarrow \mathcal{K}^{m}_{a+1-\varepsilon,p}(D) \hookrightarrow L_{\delta p}(D) = \mathcal{K}^{0}_{0,\delta p}(D), \tag{44}$$

if, and only if,

$$m \geq \frac{d}{p} - \frac{d}{\delta p} \quad \text{and} \quad a \geq \frac{d}{p} - 1 - \frac{d}{\delta p} + \varepsilon.$$

This is satisfied by our assumptions $m > \frac{d}{p}$ and $a \geq \frac{d}{p} - 1$ if $\varepsilon > 0$ is chosen small enough. We now make use of the interpolation result

$$\left[L_p(D), \mathcal{K}^{m-1}_{a-1,p}(D)\right]_{\theta} = \left[\mathcal{K}^{0}_{0,p}(D), \mathcal{K}^{m-1}_{a-1,p}(D)\right]_{\theta} = \mathfrak{K}^{\theta(m-1)}_{\theta(a-1),p}(D), \qquad 0 < \theta < 1,$$

which follows from formula (32). By employing the associated interpolation inequality we have for all $u \in \mathcal{K}^{m-1}_{a-1,p}(D) \cap L_p(D)$,

$$\|u|\mathfrak{K}^{\theta(m-1)}_{\theta(a-1),p}(D)\| \leq \|u|\mathcal{K}^{m-1}_{a-1,p}(D)\|^{\theta}\|u|L_p(D)\|^{1-\theta}, \tag{45}$$

cf. [5, Thm. 4.1.4]. Replacing $u$ by $Nu_1 - Nu_2 = Gu_1 - Gu_2$ and using (42) - (45) we find

$$\begin{aligned}
&\|Nu_1 - Nu_2|\mathfrak{K}^{\theta(m-1)}_{\theta(a-1),p}(D)\| \\
&\quad \leq \|Gu_1 - Gu_2|\mathcal{K}^{m-1}_{a-1,p}(D)\|^{\theta}\|Nu_1 - Nu_2|L_p(D)\|^{1-\theta} \\
&\quad \leq c^{\theta}\left(\|u_1|\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\|^{\delta} + \|u_2|\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\|^{\delta}\right)^{\theta}\ \|Nu_1 - Nu_2|L_p(D)\|^{1-\theta}.
\end{aligned}$$

Observe that $\varepsilon$ and $\theta$ can be chosen independent from each other. Since we have $\|Nu_1 - Nu_2|L_p(D)\| \to 0$ if $u_1 \to u_2$ in $\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \hookrightarrow L_{\delta p}(D)$ as well as $\mathfrak{K}^{\theta(m-1)}_{\theta(a-1),p}(D) \hookrightarrow \mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D)$ for some $\varepsilon' < \varepsilon$ (choosing $\theta$ close enough to 1), the above calculations show that

$$N : \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \to \mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D)$$

is continuous. □

**Remark 13.** Continuity of Nemytskij operators (composition operators) is even more delicate than boundedness. In the framework of Sobolev spaces we refer to [1], [7], [9] and [45].

We are now finally in a position to prove complete continuity of $L \circ N$ on the fractional Kondratiev space $\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)$. Recall, $\overline{a}$ has been defined in Proposition 1.

**Theorem 4.** *Let $D$ be a domain which satisfies Assumption 1, $d/2 < p < \infty$, $\frac{d}{p} - 1 \leq a < \overline{a}$, and $m \in \mathbb{N}$ with $m > \frac{d}{p}$. Moreover, assume $\delta \geq \max(1, m-1)$ and let the function $g$ satisfy the growth-condition* (16)*. Then for sufficiently small $\varepsilon > 0$ it follows that the operator*

$$(L \circ N) : \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \hookrightarrow \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)$$

*is completely continuous.*

*Proof.* From Proposition 1 we know that the linear operator $L$ maps $\mathcal{K}^{m-1}_{a-1,p}(D)$ into $\mathcal{K}^{m+1}_{a+1,p}(D)$ if $m \in \mathbb{N}_0$ and $|a| < \overline{a}$. Using complex interpolation and $m \geq 1$ we obtain

$$L : \underbrace{\left[\mathcal{K}^{m-2}_{a-1-\varepsilon',p}(D), \mathcal{K}^{m-1}_{a-1-\varepsilon',p}(D)\right]_{1-\varepsilon'}}_{=\mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D)} \to \underbrace{\left[\mathcal{K}^{m}_{a+1-\varepsilon',p}(D), \mathcal{K}^{m+1}_{a+1-\varepsilon',p}(D)\right]_{1-\varepsilon'}}_{=\mathfrak{K}^{m+1-\varepsilon'}_{a+1-\varepsilon',p}(D)},$$

which together with the compact embedding $\mathfrak{K}^{m+1-\varepsilon'}_{a+1-\varepsilon',p}(D) \hookrightarrow \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)$ for $0 < \varepsilon' < \varepsilon$ (cf. Remark 11) shows that

$$L : \mathfrak{K}^{m-1-\varepsilon'}_{a-1-\varepsilon',p}(D) \to \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D), \tag{46}$$

is a compact operator. Now using (46) together with Lemma 1 yields complete continuity of

$$(L \circ N) : \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D) \hookrightarrow \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D).$$

□

### 4.3 Existence of solutions in Kondratiev spaces

We are now in a position to formulate existence results for solutions to problem (1) within Kondratiev spaces.

**Proposition 6.** *Let $D$ be as in Definition 2 with singularity set $S$. Consider the mapping $N(u)(x) = f(x) - g(x, u(x))$ and let $f \in \mathcal{K}^{m-1}_{a-1,p}(D)$ and $u \in \mathcal{K}^{m+1}_{a+1,p}(D)$. Further, let us assume $a \geq \frac{d}{p} - 1$ and $d/2 < p < \infty$.*

*(i) Let $g(x, \xi) = \xi^n$ for some natural number $n > 1$. Then $N(u)$ satisfies the estimate*

$$\|N(u)|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq \|f|\mathcal{K}^{m-1}_{a-1,p}(D)\| + c^n \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^n,$$

*where $c$ denotes the constant in Theorem 2.*

*(ii) Now let $g(x, \xi)$ satisfy the growth condition* (16)*. Then it holds*

$$\|N(u)|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq \|f|\mathcal{K}^{m-1}_{a-1,p}(D)\| + C(m-1, g) \|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^\delta.$$

*Proof.* Part (i) follows simply by repeatedly applying Theorem 2. This results in an estimate

$$\|f - u^n|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq \|f|\mathcal{K}^{m-1}_{a-1,p}(D)\| + c^{n-1}\|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|^{n-1}\|u|\mathcal{K}^{m-1}_{a-1,p}(D)\|,$$

and in view of the embedding properties of Kondratiev spaces as stated in (6) we have $\|u|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq c\|u|\mathcal{K}^{m+1}_{a+1,p}(D)\|$.

Part (ii) follows immediately from Theorem 3. □

**Remark 14.** Proposition 6 can further be strengthened with the help of our results from Proposition 5. The result then reads as follows. Let $D$ be a domain which satisfies Assumption 1, $d/2 < p < \infty$, $a \geq \frac{d}{p} - 1$, and $m \in \mathbb{N}$ with $m > \frac{d}{p}$. Moreover, assume $\delta \geq \max(1, m-1)$ and let the function $g$ satisfy the growth-condition (16). Then $N(u)$ satisfies the estimate

$$\|N(u)|\mathcal{K}^{m-1}_{a-1,p}(D)\| \leq \|f|\mathcal{K}^{m-1}_{a-1,p}(D)\| + C\,\|u|\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,p}(D)\|^{\delta} \tag{47}$$

for $f \in \mathcal{K}^{m-1}_{a-1,p}(D)$, $u \in \mathcal{K}^{m+1}_{a+1,p}(D)$, and sufficiently small $\varepsilon > 0$. In the particular case of $g(x,\xi) = \xi^n$ for $n \in \mathbb{N}$, $n > 1$, in view of Corollary 1, we have $\delta = n$ in (47).

With this we now obtain the following existence and regularity result for problem (1). Since we rely on Proposition 1 in the sequel, we restrict our considerations to the case when $p = 2$.

**Theorem 5.** *Let $D$ be a domain which satisfies Assumption 1. Assume $m \in \mathbb{N}$ with $m > \frac{d}{2}$ and $\frac{d}{2} - 1 \leq a < \overline{a}$, where $\overline{a}$ is the constant from Proposition 1. Let the function $g$ satisfy the growth condition* (16) *and assume that the condition* (30)*(c) is satisfied for $\eta = \|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$ and $\vartheta = C$ from* (47)*. Then there exists a solution $u \in \mathcal{K}^{m+1}_{a+1,2}(D)$ of problem* (1)*.*

*Proof.* This is almost an immediate consequence of Proposition 2 and (47). The admissibility of the fractional Kondratiev space $\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,2}(D)$ follows from the explanations given in Remark 10. Furthermore, the complete continuity of the composition operator $L \circ N$ in $\mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,2}(D)$ was proven in Theorem 4. Hence we may apply Proposition 2 with $X := \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,2}(D)$ and $Y := \mathcal{K}^{m-1}_{a-1,2}(D)$ for sufficiently small $\varepsilon > 0$. This yields the existence of a function $u \in \mathfrak{K}^{m+1-\varepsilon}_{a+1-\varepsilon,2}(D)$ which solves the partial differential equation but does not necessarily fulfil the boundary condition. To see that $u_{|\partial D} = 0$ holds we argue as follows. Because $u$ is a fixed point, i.e., $u = (L \circ N)u$, and $N(u) \in \mathcal{K}^{m-1}_{a-1,2}(D)$, see Proposition 5, we conlude that $(L \circ N)u \in \mathcal{K}^{m+1}_{a+1,2,0}(D)$ by using Proposition 1. □

**Remark 15.** Clearly, $C$ from (47) depends on the nonlinearity and the Kondratiev space. By choosing $\eta = \|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$ small enough we always can apply Theorem 5.

**Remark 16.** The inclined reader might wonder why the results in this section can be established without any essential restriction on the power $\delta$ in the nonlinear term. The reason is that so far we have studied distributional solutions to (1). Our solutions are regular distributions contained in Kondratiev spaces, but so far we have not claimed that they are classical weak solutions contained in $H^1_0(D)$. We will come back to these problems in the following sections.

## 4.4 Uniqueness of solutions

Unfortunately, Proposition 2 doesn't make any assertions on uniqueness of the fixed point. In the simplified setting, where the nonlinearity is just of monomial type, it turns out that we may also use Banach's fixed point theorem instead of Proposition 2. Then we additionally obtain (under the same assumptions) uniqueness of the solution to the semilinear problem in a (sufficiently small) ball around the solution of the corresponding linear problem (this time in $\mathcal{K}^{m+1}_{a+1,2}(D)$). The precise result reads as follows.

**Theorem 6.** *Let $D$ be a domain which satisfies Assumption 1, $m \in \mathbb{N}$ and $\frac{d}{2} - 1 \leq a < \overline{a}$, where $\overline{a}$ is the constant from Proposition 1. Furthermore, let $g(x, \xi) = \xi^n$ for some natural number $n > 1$ and let $f \in \mathcal{K}^{m-1}_{a-1,2}(D)$. Assume the condition* (30)*(c) to be satisfied for $\delta = n$, $\eta = \|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$, and $\vartheta = c^n$ (c taken from Theorem 2). Then there exists a unique solution $u \in K_0 \subset \mathcal{K}^{m+1}_{a+1,2}(D)$ of problem* (1)*, where $K_0$ denotes a small closed ball in $\mathcal{K}^{m+1}_{a+1,2}(D)$ with center $Lf$ (the solution of the corresponding linear problem) and radius $r := \frac{\|L\|\eta}{n-1}$.*

*Proof.* We wish to apply Banach's fixed point theorem, which guarantees uniqueness of the solution if we can show that $T := (L \circ N) : K_0 \to K_0$ is a contraction mapping, i.e.,

$$\|T(u) - T(v)|\mathcal{K}^{m+1}_{a+1,2}(D)\| \leq q\|u - v|\mathcal{K}^{m+1}_{a+1,2}(D)\| \quad \text{for all} \quad u, v \in K_0, \quad q \in [0, 1),$$

where $K_0$ is a closed ball in $\mathcal{K}^{m+1}_{a+1,2}(D)$ with center $Lf$ and suitable radius $r$. Observe that

$$(L \circ N)(u) - (L \circ N)(v) = L(f - G(u)) - L(f - G(v)) = (L \circ G)(v) - (L \circ G)(u),$$

thus, $L \circ N$ is a contraction if, and only if, $L \circ G$ is. Let us analyze the resulting scaling condition in the monomial case $g(x, \xi) = \xi^n$. Since we have $G(u) - G(v) = u^n - v^n = (u - v) \sum_{j=0}^{n-1} u^j v^{n-1-j}$, we can apply Proposition 1 and Theorem 2 to obtain the estimate

$$\begin{aligned}
&\|(L \circ G)(u) - (L \circ G)(v)|\mathcal{K}^{m+1}_{a+1,2}(D)\| \\
&\qquad \le \|L\| \, \|G(u) - G(v)|\mathcal{K}^{m-1}_{a-1,2}(D)\| \\
&\qquad = \|L\| \, \|u^n - v^n|\mathcal{K}^{m-1}_{a-1,2}(D)\| \\
&\qquad \le \|L\| \, c \, \|u - v|\mathcal{K}^{m+1}_{a+1,2}(D)\| \sum_{j=0}^{n-1} c^{n-1} \, \|u|\mathcal{K}^{m+1}_{a+1,2}(D)\|^j \, \|v|\mathcal{K}^{m+1}_{a+1,2}(D)\|^{n-1-j} \\
&\qquad \le \|L\| \, n \, c^n \, (r + \|L\|\eta)^{n-1} \, \|u - v|\mathcal{K}^{m+1}_{a+1,2}(D)\| \qquad (48)
\end{aligned}$$

for all $u, v \in K_0$, the closed ball in $\mathcal{K}^{m+1}_{a+1,2}(D)$ with center $Lf$ and radius $r$. With $r = \frac{\|L\|\eta}{n-1}$ we conlude that $L \circ G$ is a contraction if

$$n \left(1 + \frac{1}{n-1}\right)^{n-1} c^n \, \eta^{n-1} \, \|L\|^n < 1 \, . \qquad (49)$$

Elementary calculations yield that this equivalent to (30)(c).

Moreover, we need $(L \circ N)(B_r(Lf)) \subset B_r(Lf)$. Since $(L \circ N)(0) = Lf$, due to $G(0) = 0$, we see that

$$\begin{aligned}
&\|(L \circ N)(u) - Lf|\mathcal{K}^{m+1}_{a+1,2}(D)\| \\
&\qquad = \|(L \circ N)(u) - (L \circ N)(0)|\mathcal{K}^{m+1}_{a+1,2}(D)\| \\
&\qquad \le \|L\| \, \|u^n - 0|\mathcal{K}^{m-1}_{a-1,2}(D)\| \\
&\qquad \le \|L\| \, c^n \, (r + \|L\|\eta)^n \overset{!}{\le} r = \frac{\|L\|\eta}{n-1} \qquad (50)
\end{aligned}$$

with $u \in K_0$. The claimed inequality is equivalent to

$$\left(1 + \frac{1}{n-1}\right)^n c^n \, \eta^{n-1} \, \|L\|^n \le \frac{1}{n-1} \, .$$

Using (49) we conclude

$$\left(1 + \frac{1}{n-1}\right)^n c^n \, \eta^{n-1} \, \|L\|^n < \left(1 + \frac{1}{n-1}\right)^n \frac{1}{n(1 + \frac{1}{n-1})^{n-1}} = \frac{1}{n-1} \, . \qquad \square$$

**Remark 17.** The main restriction in Theorem 6 stems from condition (30)(c), which upon inserting $\eta = \|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$ turns into a scaling condition for the right-hand side. In other words: A unique solution can only exist in case of "sufficiently small" right-hand sides.

This observation is in accordance to what is known from the classical theory for semilinear elliptic problems in Sobolev spaces, as it can be found, e.g., in [51]. Particularly for semilinear problems with monomial nonlinearities $\pm|u|^{p-2}u$, much is known about existence and (non)uniqueness of solutions. The delicate dependence on the sign of the nonlinearity is eliminated in our setting by the usage of the simple growth-condition (16).

More precisely, for the problem

$$-\Delta u = u|u|^{p-2} + f, \quad u|_{\partial D} = 0$$

it is known that, for a certain range of parameters $p > 2$, for arbitrary $f \in L_2(D)$ we have an unbounded sequence of solutions in $H^1_0(D)$; we refer to [51, Theorem 7.2, Remark 7.3]. Thus to nevertheless obtain any notion of uniqueness, additional restriction, usually taking the form of scaling conditions, become necessary.

In order to obtain classical weak solutions contained in $H^1_0$ we can strengthen Theorem 6 in the following way.

**Corollary 2.** *Let $D$ be a domain which satisfies Assumption 1, $m \in \mathbb{N}$, and $\frac{d}{2} - 1 \leq a < \overline{a}$, where $\overline{a}$ is the constant from Proposition 1. Furthermore, let $g(x,\xi) = \xi^n$ where $n$ is an arbitrary natural number $\geq 2$ if $d = 2$ and $n \in \{2, \ldots, 5\}$ if $d = 3$. Let $f \in \mathcal{K}^{m-1}_{a-1,2}(D) \cap H^{-1}(D)$ and assume the condition* (30)*(c) to be satisfied for $\delta = n$,*

$$\eta := \|f|\mathcal{K}^{m-1}_{a-1,2}(D) \cap H^{-1}(D)\| := \max\left(\|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|, \|f|H^{-1}(D)\|\right),$$

$$\|L\| := \max(\|L|\mathcal{L}(\mathcal{K}^{m-1}_{a-1,2}(D), \mathcal{K}^{m+1}_{a+1,2}(D))\|, \|L|\mathcal{L}(H^{-1}(D), H^1_0(D))\|),$$

*and*

$$\vartheta := \max(c^n, \|I_D\|^{n+1}_{n+1} 3^{n-1}) \qquad \text{(c taken from Theorem 2).}$$

*Then there exists a unique solution $u \in \tilde{K}_0 \subset \mathcal{K}^{m+1}_{a+1,2}(D) \cap H^1_0(D)$ of problem* (1)*, where $\tilde{K}_0$ denotes the small closed ball in this space with center $Lf$ (the solution of the corresponding linear problem) and radius $r = \frac{\|L\|\eta}{n-1}$.*

*Proof.* *Step 1.* In this step we show that

$$L \circ N : \mathcal{K}^{m+1}_{a+1,2}(D) \cap H^1_0(D) \to \mathcal{K}^{m+1}_{a+1,2}(D) \cap H^1_0(D)$$

is a bounded and continuous operator. For this we first show that already $N : H^1_0(D) \to H^{-1}(D)$ is a bounded and continuous operator, where $N(u) = f - u^n$. Since $f \in H^{-1}(D)$ this is the case if $G : H^1_0(D) \to H^{-1}(D)$, $G(u) = u^n$, is bounded and continuous. Let $2 < q < \infty$. By Sobolev's embedding theorem we have

$$H^1(D) \hookrightarrow L_q(D) \quad \text{if} \quad 1 - \frac{d}{2} \geq -\frac{d}{q}, \tag{51}$$

i.e., $q \leq \frac{2d}{d-2}$ if $d = 3$ and no extra condition for $d = 2$. The operator norm $\|I_D|H^1(d) \to L_q(D)\|$ of the embedding operator $I_D$ will be abbreviated by $\|I_D\|_q$. Because of (51) and

$$\Big| \int_D |u|^n(x)\, \varphi(x)\, dx \Big| \leq \|\varphi|L_q(D)\| \cdot \|u^n|L_{q'}(D)\| < \infty \quad \text{for all} \quad \varphi \in H^1_0(D)\,, \tag{52}$$

we conclude $u^n \in H^{-1}(D)$ if $u^n \in L_{q'}(D)$, where $1/q' := 1 - 1/q$. This holds if $nq' \leq q$, i.e., $n \leq \frac{q}{q'} = q - 1 \leq \frac{d+2}{d-2}$, which for $d = 3$ yields $n \leq 5$. Thus, $G : H^1_0(D) \to H^{-1}(D)$, $G(u) = u^n$, is a bounded operator for $n \leq 5$ if $d = 3$ and for all $n$ if $d = 2$. Concerning continuity we make use of the formula $u^n - v^n = (u - v) \sum_{j=0}^{n-1} u^j v^{n-1-j}$. Since $q > 2$ applying Hölder's inequality twice with $r := \frac{q}{2}$, $r' = \frac{q}{q-2}$, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$, and afterwards with $r = r' = 2$, we obtain for $\varphi \in H^1_0(D)$,

$$\begin{aligned}
&\Big| \int_D (u^n(x) - v^n(x))\, \varphi(x)\, dx \Big| \qquad (53)\\
&\leq \sum_{j=0}^{n-1} \int_D |u(x) - v(x)|\, |u^j(x) v^{n-1-j}(x)|\, |\varphi(x)|\, dx \\
&\leq \sum_{j=0}^{n-1} \left( \int_D |(u(x) - v(x))\varphi(x)|^{\frac{q}{2}}\, dx \right)^{\frac{2}{q}} \left( \int_D |u^j(x) v^{n-1-j}(x)|^{\frac{q}{q-2}}\, dx \right)^{\frac{q-2}{q}} \\
&\leq \sum_{j=0}^{n-1} \left( \int_D |u(x) - v(x)|^q\, dx \right)^{\frac{1}{q}} \left( \int_D |\varphi(x)|^q\, dx \right)^{\frac{1}{q}} \left( \int_D |u^j(x) v^{n-1-j}(x)|^{\frac{q}{q-2}}\, dx \right)^{\frac{q-2}{q}} \\
&\leq n \cdot \|I_D\|_q^2 \cdot \|u - v|H^1(D)\| \cdot \|\varphi|H^1(D)\| \cdot \left( \int_D \left( \max(|u(x)|, |v(x)|)^{n-1} \right)^{\frac{q}{q-2}}\, dx \right)^{\frac{q-2}{q}}.
\end{aligned}$$

If $u, v \in H^1(D)$ it follows from [27, Lem. 7.6, p. 152] that $u^+, |u| \in H^1(D)$, where $u^+ := \max(u, 0)$, which together with the formula

$$\max(|u|, |v|) = \frac{1}{2} \left( (|u| - |v|)^+ + (|v| - |u|)^+ + |u| + |v| \right)$$

shows that $\max(|u|,|v|) \in H^1(D)$. Hence, in order for the integral in the last line of (53) to be bounded, we require from Sobolev's embedding theorem that $(n-1)\frac{q}{q-2} \overset{!}{\leq} q$, i.e., $n \leq q-1 \leq \frac{d+2}{d-2}$, which for $d=3$ gives $n \leq 5$ and for $d=2$ no extra condition. Under the given restrictions, (53) yields

$$\|u^n - v^n|H^{-1}(D)\| \lesssim \|u-v|H^1(D)\|$$

where the suppressed constant depends on $n$ and $R$ for $u,v \in B_R(0) \subset H^1_0(D)$. Hence, $G: H^1_0(D) \to H^{-1}(D)$ is locally Lipschitz continuous. This, together with Remark 2(i), shows that

$$L \circ N : H^1_0(D) \to H^1_0(D)$$

is a bounded and continuous operator. From the proof of Theorem 6 (in particular the calculations in (48)) we already know that

$$L \circ N : \mathcal{K}^{m+1}_{a+1,2}(D) \to \mathcal{K}^{m+1}_{a+1,2}(D)$$

is a bounded and continuous operator. Thus,

$$L \circ N : \mathcal{K}^{m+1}_{a+1,2}(D) \cap H^1_0 \to \mathcal{K}^{m+1}_{a+1,2}(D) \cap H^1_0$$

is bounded and continuous as well.

*Step 2.* Now we wish to apply Banach's fixed point theorem. Applying again [27, Lem. 7.6, p. 152] we see that

$$\max(\|u^+|H^1(D)\|, \| \, |u| \, |H^1(D)\|) \leq \|u|H^1(D)\|\,, \qquad u \in H^1(D)\,.$$

Let $u,v \in \tilde{K}_0$. A close inspection of (53) gives that

$$\begin{aligned}
\|u^n - v^n|H^{-1}(D)\| &\leq n\,\|I_D\|_q^{n+1}\left(\frac{3}{2}\,\|u|H^1(D)\| + \frac{3}{2}\,\|v|H^1(D)\|\right)^{n-1}\|u-v|H^1(D)\| \\
&\leq n\,\|I_D\|_q^{n+1}\,3^{n-1}\,(r+\eta\|L\|)^{n-1}\,\|u-v|H^1(D)\| \\
&\leq n\,\|I_D\|_q^{n+1}\,3^{n-1}\,\|L\|^{n-1}\,\eta^{n-1}\left(1+\frac{1}{n-1}\right)^{n-1}\|u-v|H^1(D)\|.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|(L\circ N)(u) - (L\circ N)(v)|H^1_0(D)\| \\
&\quad= \|(L\circ G)(u) - (L\circ G)(v)|H^1_0(D)\| \\
&\quad\leq \|L\|\|G(u)-G(v)|H^{-1}(D)\| \\
&\quad\leq n\,\|I_D\|_q^{n+1}\,3^{n-1}\,\|L\|^n\,\eta^{n-1}\left(1+\frac{1}{n-1}\right)^{n-1}\|u-v|H^1(D)\|. \qquad (54)
\end{aligned}$$

The structure of this estimate is exactly as in (48) except that $c^n$ is replaced by $\|I_D\|_q^{n+1}\, 3^{n-1}$. Hence, by our assumptions,

$$n\,\|I_D\|_q^{n+1}\, 3^{n-1}\,\|L\|^n\,\eta^{n-1}\left(1+\frac{1}{n-1}\right)^{n-1} < 1 \tag{55}$$

and $L \circ N$ is a contraction. We also need $(L\circ N)(B_r(Lf)) \subset B_r(Lf)$. Therefore we apply (52) and find

$$\begin{aligned}\left|\int_D |u|^n(x)\,\varphi(x)\,dx\right| &\leq \|\varphi|L_q(D)\|\cdot\|u|L_{q'n}(D)\|^n \\ &\leq \|I_D\|_q^{n+1}\,\|\varphi|H^1(D)\|\cdot(r+\|L\|\eta)^n\end{aligned}$$

Hence

$$\begin{aligned}&\|(L\circ N)(u) - Lf|H_0^1(D)\| = \|(L\circ G)(u)-(L\circ G)(0)|H_0^1(D)\| \\ &\quad\leq \|L\|\|G(u)|H^{-1}(D)\| \\ &\quad\leq \|I_D\|_q^{n+1}\,\|L\|^{n+1}\,\eta^n\left(1+\frac{1}{n-1}\right)^n.\end{aligned} \tag{56}$$

Because of (55) it follows

$$\|I_D\|_q^{n+1}\,\|L\|^{n+1}\,\eta^n\left(1+\frac{1}{n-1}\right)^n \leq r = \frac{\|L\|\,\eta}{n-1}.$$

Combining these arguments with those used in the proof of Theorem 6 and taking into account that the smallest possible $q$ is given by $q = n+1$, see Step 1, the claim follows. □

**Remark 18.** If $m \geq 1$ and $a \geq 1$ we see from the embedding

$$\mathcal{K}^{m-1}_{a-1,2}(D) \hookrightarrow \mathcal{K}^0_{0,2}(D) = L_2(D) \hookrightarrow H^{-1}(D)$$

that no additional restrictions on the right hand side $f$ are needed in Corollary 2 compared to Theorem 6.

# 5 Besov regularity of semilinear elliptic problems

The results of the last subsection are the basis for assertions on Besov regularity of solutions to (1), more precisely, statements that these solutions belong to spaces $F^{m+1}_{\tau,\infty}(D) \hookrightarrow L_2(D)$ for a suitable parameter $0 < \tau < 2$, which in turn are closely related to approximation spaces for $n$-term wavelet approximation and adaptive finite element approximation.

We first cite the following embedding result from [34, Thm. 4.9].

**Proposition 7.** *Let $D \subset \mathbb{R}^d$ be some bounded Lipschitz domain of polyhedral type with singularity set $S$ of dimension $l$ and let $1 < p < \infty$, $0 < \tau < p$, $m \in \mathbb{N}$, and $a > 0$. Further assume*

$$m - a < (d - l)\Big(\frac{1}{\tau} - \frac{1}{p}\Big).$$

*Then it holds*

$$\mathcal{K}^m_{a,p}(D) \hookrightarrow F^m_{\tau,2}(D). \tag{57}$$

**Remark 19.** Proposition 7 in [34] is there stated for bounded Lipschitz domains with piecewise smooth boundary, which covers our bounded Lipschitz domains of polyhedral type.
The result extends to $m = 0$ since for $a > 0$ and $\tau < p$ we have

$$\mathcal{K}^0_{a,p}(D) \hookrightarrow L_p(D) = F^0_{p,2}(D) \hookrightarrow F^0_{\tau,2}(D),$$

where the identity follows from (65) in Appendix A. Moreover, using interpolation the embedding (57) also holds for the fractional Kondratiev spaces. In particular, under the same assumptions as in Proposition 7 with $m \in \mathbb{N}$ replaced by $s \geq 0$, now assuming that

$$s - a < (d - l)\left(\frac{1}{\tau} - \frac{1}{p}\right),$$

we have

$$\mathfrak{K}^s_{a,p}(D) \hookrightarrow F^s_{\tau,2}(D). \tag{58}$$

We sketch the proof. Using complex interpolation and Proposition 7 we see that for $s \notin \mathbb{N}$, $m_0 = [s]$, $m_1 = [s] + 1$, $\theta = s - [s]$ and $a_i = m_i - s + a$, $i = 0, 1$, we have

$$s = (1 - \theta)m_0 + \theta m_1 \quad \text{and} \quad a = (1 - \theta)a_0 + \theta a_1$$

as well as

$$m_i - a_i = s - a < (d - l)\left(\frac{1}{\tau} - \frac{1}{p}\right), \qquad i = 0, 1,$$

hence we conclude

$$\mathfrak{K}^s_{a,p}(D) = \big[\mathcal{K}^{m_0}_{a_0,p}(D), \mathcal{K}^{m_1}_{a_1,p}(D)\big]_\theta \hookrightarrow \big[F^{m_0}_{\tau,2}(D), F^{m_1}_{\tau,2}(D)\big]_\theta = F^s_{\tau,2}(D).$$

In particular, for the interpolation of Kondratiev spaces we used formula (32) whereas the interpolation formula for Triebel-Lizorkin spaces may be found in [36].

We further specialize to the case $p = 2$. Combining the embedding (58) with the existence result from Theorem 5 we can now state our main result.

**Theorem 7.** *Let $D \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain of polyhedral type with singularity set $S$ of dimension $l$. Let $\overline{a}, m, g, f, \eta, C$ be as in Theorem 5. Let $a$ and $0 < \tau < 2$ be such that*

$$\frac{d}{2} - 1 \leq a < \overline{a} \qquad \text{and} \qquad \frac{m-a}{d-l} + \frac{1}{2} < \frac{1}{\tau} \leq \frac{2m+d}{2d}.$$

*Then there exists a solution $u \in F^{m+1}_{\tau,2}(D) \hookrightarrow H^1(D)$ of problem (1).*

*Proof.* The claim is an immediate consequence of Theorem 5 and (58). Let us mention that (63) in Appendix A yields

$$F^{m+1}_{\tau,2}(D) \hookrightarrow F^1_{2,2}(D) = H^1(D),$$

if $m - \frac{d}{\tau} \geq -\frac{d}{2}$, i.e., if $\tau \geq \frac{2d}{2m+d}$. □

# 6 Applications to adaptive approximation schemes

We shall apply our regularity results to obtain convergence rates for adaptive algorithms, in particular, either adaptive wavelet algorithms or adaptive finite element algorithms. For both approaches algorithms are known which (provably) perform at the optimal convergence rate in the following sense: If the solution $u$ belongs to a related Approximation class $\mathcal{A}^\alpha$ (to be specified below), i.e., the error for the optimal approximation is proportional to $N^{-\alpha}$, then the algorithm indeed produces an approximation with error proportional to $N^{-\alpha}$. Therein $N$ corresponds to the number of degrees of freedom used in the construction of the approximation, and it also corresponds to the computational cost of the algorithm. Thus, in order to analyze the potential performance of adaptive solvers of the semilinear problem (1), we have to study relations between the Approximation classes and regularity classes in which solutions exist.

## 6.1 $n$-term approximation and adaptive wavelet algorithms

It is nowadays well-known that certain Besov spaces are closely related to approximation spaces for $N$-term wavelet approximation. To describe related results, let $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ be a wavelet system with sufficiently high differentiability and vanishing moments, such that all relevant (unweighted) Sobolev and Besov spaces can be characterized in terms of expansion coefficients w.r.t. $\Psi$.

Let $X$ be some Banach space. The (error of the) best $N$-term approximation is defined as

$$\sigma_N\big(u;X\big) = \inf_{\Gamma\subset\Lambda:\#\Gamma\le N}\inf_{c_\lambda}\Big\|u - \sum_{\lambda\in\Gamma} c_\lambda\psi_\lambda\big|X\Big\|,$$

i.e., as the name suggests we consider the best approximation by linear combinations of the basis functions consisting of at most $N$ terms. We further introduce *approximation classes* $\mathcal{A}_q^\alpha(X)$, $\alpha > 0$, $0 < q \le \infty$ by requiring

$$\|u|\mathcal{A}_q^\alpha(X)\| = \Bigg(\sum_{N=0}^{\infty}\Big((N+1)^\alpha\sigma_N\big(u;X\big)\Big)^q\frac{1}{N+1}\Bigg)^{1/q} < \infty\,, \tag{59}$$

if $0 < q < \infty$ as well as

$$\|u|\mathcal{A}_\infty^\alpha(X)\| = \sup_{N\ge 0}(N+1)^\alpha\sigma_N\big(u;X\big) < \infty\,.$$

A famous result of DeVore, Jawerth and Popov then may be formulated as

$$\mathcal{A}_\tau^{m/d}(L_p(\mathbb{R}^d)) = B^m_{\tau,\tau}(\mathbb{R}^d)\,, \qquad \frac{1}{\tau} = \frac{m}{d} + \frac{1}{p}\,.$$

For our purposes we shall consider a result from [17, Thm. 11, p. 586], which reads as

$$B^{m+s}_{\tau,q}(D) \hookrightarrow \mathcal{A}_\infty^{m/d}(H^s_p(D))\,, \qquad \frac{1}{\tau} < \frac{m}{d} + \frac{1}{p}\,, \tag{60}$$

where $s \in \mathbb{R}$, $\tau < p$ and $m > 0$, independent of the parameter $0 < q \le \infty$. Now Theorem 7, together with (64) from Appendix A and (60), applied with $s = 1$ and $p = 2$, gives the following result.

**Theorem 8 (Approximation in $H^1$).** *Let $D \subset \mathbb{R}^d$, $d = 2,3$, be a bounded Lipschitz domain of polyhedral type with singularity set $S$ of dimension $l$. Let $\overline{a}, m, g, f, \eta, C$ be as in Theorem 5. Let $a$ satisfy*

$$\frac{d}{2} - 1 \le a < \overline{a} \qquad \text{and} \qquad \frac{ml}{d} < a\,.$$

*Then there exists a solution* $u \in H^1(D)$ *of problem* (1) *which belongs to the approximation class* $\mathcal{A}^{m/d}_\infty(H^1(D))$, *i.e., it satisfies the estimate*

$$\sigma_N(u; H^1(D)) \lesssim N^{-m/d} \|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$$

*for* $N$*-term wavelet approximation.*

*Proof.* Observe that our assumption $\frac{ml}{d} < a$ is equivalent to

$$\frac{m-a}{d-l} + \frac{1}{2} < \frac{m}{d} + \frac{1}{2} = \frac{2m+d}{2d},$$

hence there exist parameters $\tau$ fulfilling (60) as well as the assumptions of Theorem 7. We conclude that we have a solution

$$u \in F^{m+1}_{\tau,2}(D) \hookrightarrow B^{m+1}_{\tau,\infty}(D) \hookrightarrow \mathcal{A}^{m/d}_\infty(H^1(D))$$

of problem (1). A reformulation of this inclusion gives the claimed approximation result. □

**Remark 20.** (i) The reader should observe that in the case $d = 2$ the lower bound for $a$ reads as $a > 0$ (since $l = 0$). Therefore, Theorem 8 implies that, by increasing the Kondratiev regularity $m$ of the right–hand side $f$ and of the coefficients $a_{i,j}, 1 \leq i, j \leq 2$, solutions with arbitrarily high Kondratiev regularity exist. This means that, in principle, these solutions can be approximated by best $N$-term wavelet approximation up to any order! But here we have to mention that the condition (30)(c), which has to be satisfied for $\eta = \|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$ and $\vartheta = C$ from (47) implies that $\|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$ has to be sufficiently small which is a serious restriction. On the other hand, high orders of best $N$-term wavelet approximation are hard to realize in practice since one has to work with wavelet bases that characterize the corresponding approximation classes, e.g. the Besov spaces.

(ii) Of course nonlinearities of the form $g(x, \xi) = \xi^n$, $n \in \mathbb{N}$, $n \geq 2$, are admissible in Theorem 8. In such a situation we may replace Theorem 5 by Corollary 2. As a consequence, we may change the phrase *Then there exists a solution* $u \in H^1(D)$ *of problem* (1) into *Then there is a unique solution* $u \in H^1(D)$ *of problem* (1) *in the small closed ball* $K_0$. Also from the practical point of view we consider Theorem 8 as important. Indeed, semilinear problems with nonlinear polynomial part are the standard test cases for adaptive algorithms, see, e.g. [10].

(iii) It is of course not surprising that for $d = 3$ our results are more restrictive. Due to the upper bound $\overline{a}$, we cannot choose $m$ arbitrarily high except the case that $l = 0$ (and $\overline{a} > 1/2$). One particular case, namely problems on smooth cones, has been studied before. The 3d-results of Theorem 8 can be improved in this situation. Indeed, it has been shown in [40, Thm. 6.1.1], that in this case the upper bound $\overline{a}$ can be avoided. Instead, there is a countable set of parameters $a$ which are excluded.

Concerning approximation in $L_2(D)$ we obtain the following result from (60).

**Theorem 9 (Approximation in $L_2$).** *Let $D \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain of polyhedral type with singularity set $S$ of dimension $l$. Let $\overline{a}, m, g, f, \eta, C$ be as in Theorem 5. Let $a$ satisfy*

$$\frac{d}{2} - 1 \le a < \overline{a} \qquad \text{and} \qquad \frac{(m+1)l}{d} - 1 < a\,.$$

*Then there exists a solution $u \in L_2(D)$ of problem (1) which belongs to the approximation class $\mathcal{A}_\infty^{(m+1)/d}(L_2(D))$, i.e., it satisfies the estimate*

$$\sigma_N(u; H^1(D)) \lesssim N^{-(m+1)/d} \| f | \mathcal{K}^{m-1}_{a-1,2}(D) \|$$

*for $N$-term wavelet approximation.*

*Proof.* By Theorem 7 we have a solution $u \in F^{m+1}_{\tau,2}(D)$ of problem (1) for some $0 < \tau < 2$. Observe that this time the interval

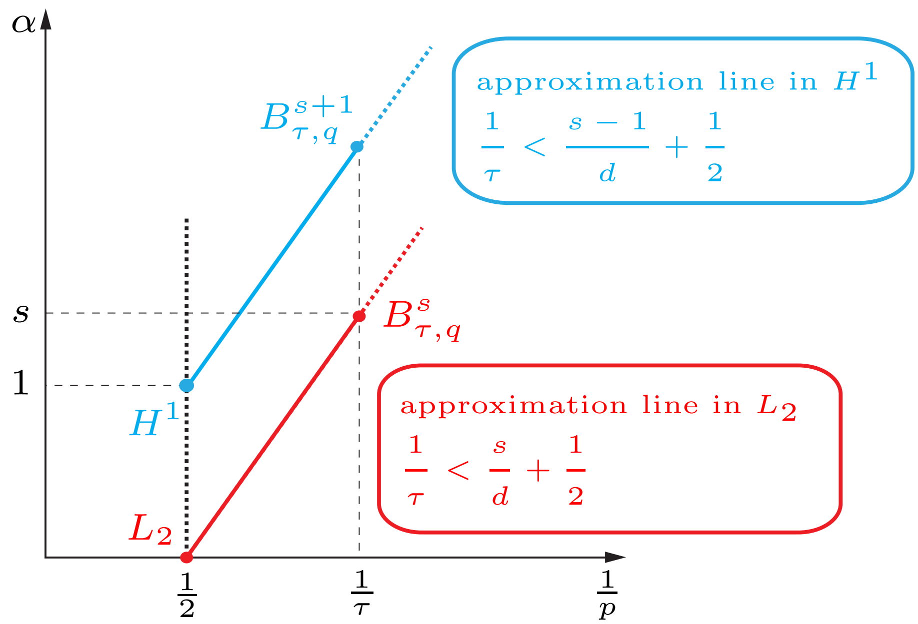
$$\frac{m-a}{d-l} + \frac{1}{2} < \frac{1}{\tau} \le \frac{2m+2+d}{2d}\,. \tag{61}$$

gives the admissible $\tau$ there. This need to be combined with $1/\tau < (m+1)/d + 1/2$, see (60). We choose $\tau$ such that $1/\tau$ is close to $(m+1)/d + 1/2$. Then the left-hand side in (61) turns into

$$\frac{m-a}{d-l} < \frac{m+1}{d} \qquad \Longleftrightarrow \qquad \frac{(m+1)l}{d} - 1 < a\,.$$

□

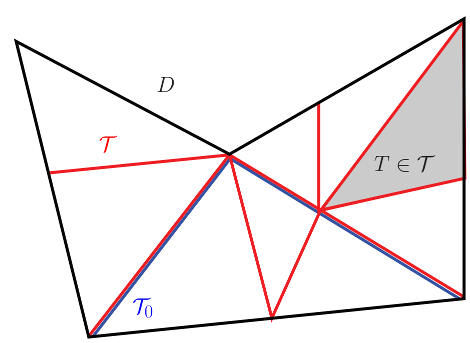
The diagram below illustrates the approximation lines in $H^1$ and $L_2$ as provided by (60).




## 6.2 Finite element approximation

Recent results by Gaspoz and Morin [26] show that very similar statements are true for finite element approximations with adaptive $h$-refinement.

The starting point is an initial triangulation $\mathcal{T}_0$ of the polyhedral domain $D$. Furthermore, $\mathbb{T}$ denotes the family of all conforming, shape-regular partitions $\mathcal{T}$ of $D$ obtained from $\mathcal{T}_0$ by refinement using bisection rules. Moreover, $V_{\mathcal{T}}$ denotes the finite element space of continuous piecewise polynomials of degree at most $r$, i.e.,




$$V_{\mathcal{T}} = \left\{ v \in C(\overline{D}) : v|_T \in \mathcal{P}_r \text{ for all } T \in \mathcal{T} \right\}.$$

In this setting the counterpart to the quantity $\sigma_N(u; X)$ is given by

$$\sigma_N^{FE}(u; X) = \min_{\substack{\mathcal{T} \in \mathbb{T}: \\ \#\mathcal{T} - \#\mathcal{T}_0 \leq N}} \inf_{v \in V_{\mathcal{T}}} \|u - v|X\|.$$

Then [26, Theorem 2.2] gives direct estimates,

$$\sigma_N^{FE}\big(u; L_p(D)\big) \leq C\, N^{-s/d} \|f|B^s_{\tau,\tau}(D)\|\,,$$

as well as

$$\sigma_N^{FE}\big(u; B^\alpha_{p,p}(D)\big) \leq C\, N^{-s/d} \|f|B^{s+\alpha}_{\tau,\tau}(D)\|\,,$$

where $1 < p < \infty$, $0 < \alpha < r+1$, $0 < s+\alpha \leq r + \frac{1}{\tau_*}$, $\tau_* = \min(1,\tau)$, and $\frac{1}{\tau} < \frac{s}{d} + \frac{1}{p}$. In [31] it was shown that this extends to embeddings

$$B^s_{\tau,\infty}(D) \hookrightarrow \mathcal{A}^{s/d}_{\infty,FE}(L_p(D)) \quad \text{and} \quad B^{s+\alpha}_{\tau,\infty}(D) \hookrightarrow \mathcal{A}^{s/d}_{\infty,FE}(B^\alpha_{p,p}(D))\,, \tag{62}$$

where the approximation class $\mathcal{A}^{s/d}_{\infty,FE}(X)$ is defined as in (59) with $\sigma_N$ being replaced by $\sigma_N^{FE}$. The embeddings (62) are the immediate counterparts of (60). With this, from (64) in Appendix A and the fact that Besov spaces satisfy the elementary embedding $B^s_{p,q} \hookrightarrow B^s_{p,\infty}$ for all $0 < q \leq \infty$, we obtain the following result.

**Theorem 10 (Finite element approximation in $H^1$).** *In the setting of Theorem 8 the function $u$ belongs to the approximation space $\mathcal{A}^{m/d}_{\infty,FE}(H^1(D))$, i.e., we also have the estimate*

$$\sigma_N^{FE}(u; H^1(D)) \lesssim N^{-m/d} \|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$$

*for finite element approximation on shape-regular conforming triangulations with adaptive $h$-refinement.*

*Proof.* For the solution $u \in B^{m+1}_{\tau,2}(D)$ from Theorem 8 we see from the elementary embedding that $u \in B^{m+1}_{\tau,\infty}(D)$, which together with the second embedding in (62) finishes the proof. □

# A Besov and Triebel-Lizorkin spaces

Besov and Triebel-Lizorkin spaces can be defined in a number of ways, including definitions in terms of finite differences, Littlewood-Paley decompositions or via their wavelet characterizations. Here we shall provide the Fourier-analytical version in terms of dyadic Littlewood-Paley decompositions. For further information on these function spaces we refer to [53] and the references therein.

We start with a function $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ with $\varphi_0 = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq \frac{3}{2}$. Define $\varphi_1(x) = \varphi_0(2x) - \varphi_0(x)$, and put $\varphi_j(x) = \varphi_1(2^{-j+1}x)$. Then $\{\varphi_j\}_{j\in\mathbb{N}_0}$

forms a so-called dyadic resolution of unity; in particular, we have $\sum_{j\geq 0}\varphi_j(x)=1$ for every $x\in\mathbb{R}^d$. Based on such resolutions of unity, we can decompose every tempered distribution $f\in\mathcal{S}'(\mathbb{R}^d)$ into a series of entire analytical functions,

$$f=\sum_{j=0}^{\infty}\mathcal{F}^{-1}(\varphi_j\mathcal{F}f),$$

converging in $\mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ denotes its inverse. Then, for $s\in\mathbb{R}$ and $0<p,q\leq\infty$ ($p<\infty$ in case of Triebel-Lizorkin spaces), the Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ are defined as the collection of all distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|B^s_{p,q}(\mathbb{R}^d)\|:=\left(\sum_{j=0}^{\infty}2^{jsq}\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)|L_p(\mathbb{R}^d)\|^q\right)^{1/q}<\infty,$$

with a supremum instead of a sum if $q=\infty$. Moreover, the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ are defined in a similar way by interchanging the order in which the norms are taken. In particular, they contain all distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|F^s_{p,q}(\mathbb{R}^d)\|:=\left\|\left(\sum_{j=0}^{\infty}2^{jsq}|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(\cdot)|^q\right)^{1/q}\Big|L_p(\mathbb{R}^d)\right\|<\infty.$$

Now the most direct way to introduce spaces on domains $\Omega\subset\mathbb{R}^d$ is via restriction. Let $A\in\{B,F\}$. Then we define

$$A^s_{p,q}(\Omega):=\{f\in\mathcal{D}'(\Omega):\ \exists g\in A^s_{p,q}(\mathbb{R}^d),\ g\big|_{\Omega}=f\},$$

normed by

$$\|f|A^s_{p,q}(\Omega)\|:=\inf_{g|_\Omega=f}\|g|A^s_{p,q}(\mathbb{R}^d)\|.$$

Within the scales we have Sobolev-type embeddings, i.e., for $\sigma<s$ and $p<\tau$ it holds

$$A^s_{p,q}(\Omega)\hookrightarrow A^{\sigma}_{\tau,r}(\Omega)\qquad\text{if}\qquad s-\frac{d}{p}\geq\sigma-\frac{d}{\tau},\tag{63}$$

where $0<r\leq\infty$ and, additionally, $q\leq r$ if $A=B$. Furthermore, the two scales of function spaces are linked via

$$B^s_{p,\min(p,q)}(\Omega)\hookrightarrow F^s_{p,q}(\Omega)\hookrightarrow B^s_{p,\max(p,q)}(\Omega)\tag{64}$$

and they coincide for $p = q$, i.e., we have $F^s_{p,p}(\Omega) = B^s_{p,p}(\Omega)$. A final important aspect of Triebel-Lizorkin spaces is their close relation to many classical function spaces. For our purposes, we especially mention the identities

$$F^s_{p,2}(\mathbb{R}^d) = H^s_p(\mathbb{R}^d) \quad \text{and} \quad F^m_{p,2}(\mathbb{R}^d) = H^m_p(\mathbb{R}^d) = W^m_p(\mathbb{R}^d), \tag{65}$$

where $m \in \mathbb{N}_0$, $s \in \mathbb{R}$, and $1 < p < \infty$.

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