Interpolating Scaling Vectors

Karsten Koch*
Philipps–Universität Marburg
FB 12 Mathematik und Informatik
Hans-Meerwein Str., Lahnberge
35032 Marburg
Germany

Abstract. In this paper we construct a one–parameter family of interpolating multigenerators which generalize the scalar generators constructed in [2] in a natural way. Furthermore we extend this approach to design a family of interpolating orthonormal multigenerators. The construction is based on the factorization techniques introduced in [21].

Key words: Interpolating scaling vector, multiwavelet, orthogonal bases.

AMS subject classification: 42C40, 65T50

1 Introduction

In recent years, wavelet analysis has become a very powerful tool in applied mathematics. Wavelet algorithms have been successfully applied in image/signal analysis/compression as well as in numerical analysis, geophysics, meteorology and in many other fields. Mostly, the interest has centered around the scalar wavelet case, i. e., the signal is analyzed by dilating and translating one mother wavelet. However, in recent studies, it has turned out that this setting very often does not provide enough flexibility in the sense that many desirable features such as smoothness, orthogonality and interpolation properties cannot be achieved at the same time. One possible way out could be to increase flexibility by using multiwavelets, i. e., the signal is analyzed by a family of wavelets. Because of this, multiwavelets have become more and more the center of attraction, see, e. g., [1, 8, 10, 20, 23, 24, 27].

Usually, these multiwavelets are constructed by means of a scaling vector, i. e., by a vector of functions in L_2 which satisfies a refinement equation of the form

$$\Phi(x) = \sum_{k\in\mathbb{Z}} A_k \Phi(2x-k). \tag{1}$$

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It can be shown that almost all important properties of the scaling vector carry over to the resulting multiwavelets, see Section 2 for details. Consequently, this paper is concerned with the systematic construction of smooth scaling vectors. We are especially interested in scaling vectors satisfying additional properties such as interpolation requirements

$$\Phi\left(\frac{n}{2}\right) = \begin{pmatrix} \delta_{0,n} \\ \delta_{1,n} \end{pmatrix}. \tag{2}$$

For several reasons, scaling vectors satisfying (2) have become of increasing interest. For instance, they provide a Shannon-like sampling theorem, see Section 2 for details. In many applications such as, e. g., image denoising also orthonormality is very useful. Therefore another important issue we are concerned with is the construction of orthonormal scaling vectors and multiwavelets, especially in the context of interpolation.

In [29], it was shown that the Haar function is the only compactly supported scalar generator which is as well orthonormal as interpolating. But these restrictions vanish if we switch over to the vector case. Therefore it turns out that the flexibility of multiwavelets can indeed be used to satisfy several remarkable properties at the same time.

A commonly used way to construct (scalar) scaling functions is to convolve a cardinal B–spline with a refinable distribution which enforces the desired additional properties such as orthogonality or interpolation. This is performed in the Fourier domain by multiplying the corresponding symbols, cf. [2, 3]. The generalization of this approach to the vector case is rather difficult due to the more complex algebraic structure. In this case the B–spline part of the symbol in the construction is replaced by the so–called Plonka factorization, see [21]. As shown in Section 3, applying the interpolation property to this factorization yields a one–parameter family of interpolating scaling vectors which appears as a natural generalization of (univariate versions of) the scalar functions in [2]. A similar approach has been derived in [4], where the Plonka factorization is used for the characterization of biorthogonal finite impulse response multifilter banks with a specific interpolation property adapted to signal processing.

By extending our construction in Section 4 we also obtain a whole family of interpolating orthonormal scaling vectors which possess some nice additional properties such as compact support and approximation power. This family also contains the scaling vectors constructed by Selesnick in [26], where the Plonka factorization is substituted by the notion of balancing. We show in Section 4.3 that interpolating scaling vectors which provide a certain approximation order are also balanced.

One of the main goals of this paper is to provide a very systematic approach which allows to interprete all the necessary building blocks such as balancing, approximation order, interpolation and so on in a unified framework and which is based on a sound mathematical foundation. This is especially important when it comes to generalizations to higher dimensions. As e. g., there already exist generalizations of the sum rules associated to the Plonka factorizations (cf. [14]), the extension of our approach seems to be very promising. This will be studied in a forthcoming paper.

2 Preliminaries

In this section we fix some notations and state some propositions needed throughout this paper.

An *r-scaling vector* $\Phi := (\phi_0, \ldots, \phi_{r-1})^T$, $r > 0$, is a vector of $L_2(\mathbb{R})$–functions which satisfies a *matrix refinement equation*

$$\Phi(x) = \sum_{k\in\mathbb{Z}} A_k \Phi(2x - k) \tag{3}$$

with the *mask* $\mathbf{A} := (A_k)_{k\in\mathbb{Z}}$ of real $r \times r$ matrices. The *length* of $\mathbf{A}$ is defined by

$$l(\mathbf{A}) := \min\Big\{|R - L| : \{k : A_k \neq \mathbf{0}\} \subset [L, R]\Big\}.$$

Applying the Fourier transform component–wise to equation (3) yields

$$\widehat{\Phi}(\omega) = \frac{1}{2}\mathbf{A}\left(e^{-i\frac{\omega}{2}}\right)\widehat{\Phi}\left(\frac{\omega}{2}\right) \tag{4}$$

with the *symbol*

$$\mathbf{A}(z) := \sum_{k\in\mathbb{Z}} A_k z^k, \quad z \in \mathbb{T},$$

where $\mathbb{T}$ denotes the complex unit circle

$$\mathbb{T} = \left\{\, z \in \mathbb{C} \;\middle|\; z = e^{-i\omega}, \omega \in \mathbb{R} \,\right\}.$$

In general, r–scaling vectors are used to construct r*–multiwavelets*, i. e., vectors $\Psi := (\psi_0, \ldots, \psi_{r-1})^T$ of $L_2(\mathbb{R})$–functions for which

$$\left\{\psi_0\left(2^j\cdot - k\right), \ldots, \psi_{r-1}\left(2^j\cdot - k\right) \;\middle|\; j, k \in \mathbb{Z}\right\}$$

forms a (Riesz) basis of $L_2(\mathbb{R})$. This is performed by means of a *multiresolution analysis*, i. e., we assume that the spaces

$$V_j = \overline{\operatorname{span}\{\phi_\rho(2^j\cdot - k),\, k \in \mathbb{Z},\, 0 \le \rho < r\}} \tag{5}$$

form a nested sequence $(V_j)_{j\in\mathbb{Z}}$ of shift–invariant closed subspaces of $L_2(\mathbb{R})$ with

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}) \quad \text{and} \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

Then we have to find Ψ such that the translates of dilated versions of the component functions $\psi_\rho, 0 \le \rho < r$, span an algebraic complement of V_j in V_{j+1}. It follows that Ψ can be represented as

$$\Psi(x) = \sum_{k\in\mathbb{Z}} B_k \Phi(2x - k)$$

with real $r \times r$ matrices B_k, see [3, 18] (scalar case) and [7] (vector case) for details. By applying the Fourier transform to this equation we obtain

$$\widehat{\Psi}(\omega) = \frac{1}{2}\mathbf{B}\left(e^{-i\frac{\omega}{2}}\right)\widehat{\Phi}\left(\frac{\omega}{2}\right)$$

with the symbol

$$\mathbf{B}(z) := \sum_{k\in\mathbb{Z}} B_k z^k, \quad z \in \mathbb{T}.$$

Many fundamental properties of the solution of the matrix refinement equation (3) can be characterized in terms of the symbol. For our purposes, especially the following conditions for compact support and L_2–stability, given in [24], are relevant.

Lemma 2.1. *The matrix refinement equation* (3) *has a compactly supported distributional solution* Φ *if and only if* $\mathbf{A}(1)$ *has an eigenvalue of the form* $2^n, n \in \mathbb{N} \setminus \{0\}$.

Lemma 2.2. *If* Φ *is a compactly supported,* L_2*–stable solution vector of* (3)*, then* $\mathbf{A}(1)$ *has a simple eigenvalue* 2 *and the moduli of all its other eigenvalues are less than* 2.

In this context Φ is said to be L_2*–stable* if there are constants $0 < A \leq B < \infty$, such that

$$A\sum_{\rho=0}^{r-1}\|c_\rho\|_{\ell_2}^2 \leq \|\sum_{\rho=0}^{r-1}\sum_{k\in\mathbb{Z}}\overline{c}_{\rho,k}\phi_\rho(\cdot-k)\|_{L_2}^2 \leq B\sum_{\rho=0}^{r-1}\|c_\rho\|_{\ell_2}^2$$

holds for any sequences $c_\rho = (c_{\rho,k})_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, $\rho = 0, \ldots, r-1$.

For the case $r = 2$ an interpolation property similar to the scalar case can be defined. A continuous 2–scaling vector Φ is called *interpolating* if for $\rho \in \{0, 1\}$ Φ satisfies

$$\phi_\rho\left(\frac{n}{2}\right) = \delta_{\rho,n} = \begin{cases} 0, & \text{if } n \in \mathbb{Z}\setminus\{\rho\} \\ 1, & \text{if } n = \rho. \end{cases} \tag{6}$$

Interpolating scaling vectors satisfy a Shannon–like sampling theorem, i. e., for any function $f \in \operatorname*{span}_{k,l}\{\phi_0(\cdot-k), \phi_1(\cdot-l)\}$ the equation

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\phi_0(x-k) + f(k+1/2)\phi_1(x-k)$$

holds. Note that for compactly supported Φ the interpolation property already implies *(algebraically) linearly independent translates*, i. e., for $(v_k)_{k\in\mathbb{Z}} := ((c_k, d_k))_{k\in\mathbb{Z}}$ with $c_k, d_k \in \mathbb{C}$ it holds that

$$\sum_{k\in\mathbb{Z}} v_k\Phi(x-k) = 0 \quad \Longrightarrow \quad v_k = \mathbf{0} \text{ for all } k \in \mathbb{Z}.$$

The simplest examples of interpolating 2–scaling vectors stem from interpolating scaling functions. Suppose ϕ is a scaling function with symbol $b(z)$, given by the Fourier transformed refinement equation

$$\widehat{\phi}(\omega) = \frac{1}{2}b\left(e^{\frac{-i\omega}{2}}\right)\widehat{\phi}\left(\frac{\omega}{2}\right).$$

Then $\Phi(x) := (\phi(2x), \phi(2x+1))^T$ is always a 2–scaling vector with symbol

$$\mathbf{A}(z) = \begin{pmatrix} b^0(z) & b^1(z) \\ zb^0(z) & zb^1(z) \end{pmatrix}. \tag{7}$$

Here b^0, b^1 denote the *subsymbols* of ϕ, defined by

$$\begin{aligned} b^0(z) &:= \sum_{k\in\mathbb{Z}} b_{2k} z^k, \\ b^1(z) &:= \sum_{k\in\mathbb{Z}} b_{2k+1} z^k, \end{aligned} \tag{8}$$

so that $b(z) = b^0(z^2) + zb^1(z^2)$ holds. Note that this reinterpretation of ϕ preserves interpolation properties.

Besides interpolation, one of the most desired properties of scaling functions is orthonormality. A scaling vector Φ is called *orthonormal*, if its integer translates are orthonormal, i. e.,

$$\langle \phi_\rho, \phi_\mu(\cdot - k)\rangle = \delta_{0,k}\delta_{\rho,\mu}, \quad 0 \le \rho, \mu < r, \quad k \in \mathbb{Z},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual L_2–inner product. For an orthonormal scaling vector the symbol has to satisfy

$$\mathbf{I}_r = \frac{1}{4}\left(\mathbf{A}(z)\overline{\mathbf{A}(z)}^T + \mathbf{A}(-z)\overline{\mathbf{A}(-z)}^T\right), \tag{9}$$

where $\mathbf{I}_r$ denotes the r-dimensional unit matrix, see [22] for details.

Another important property of a wavelet or multiwavelet basis is its approximation order m, i. e., for $f \in H^m$ one has Jackson's inequality

$$\inf_{g\in V_j} \{\|f - g\|_{L_2}\} \le C2^{-jm}\|f\|_{H^m}$$

with V_j as in (5). In the scalar case ($r = 1$) approximation order m is connected with a specific factorization of the symbol. Plonka has shown in [21] that this holds for the case $r > 1$ as well, see also [1]. Let D denote the differential operator with respect to ω in terms of $z = e^{-i\omega}$, i. e., $\mathrm{D}\mathbf{A}(e^{-i\omega}) := \left(\frac{d}{d\omega}\mathbf{A}(e^{-i\cdot})\right)(\omega)$.

Theorem 2.1 (Plonka). *Let Φ be a compactly supported r–scaling vector, and let $\{\phi_\rho(\cdot - n) : n \in \mathbb{Z}, \rho = 0, \ldots, r-1\}$ form a (algebraically) linearly independent basis of their closed linear span. Then Φ provides approximation order m if and only if the symbol $\mathbf{A}(z)$ of Φ satisfies the following conditions:*
The elements of $\mathbf{A}(z)$ are Laurent polynomials, and there are vectors $\mathbf{y}_k \in \mathbb{R}^r, \mathbf{y}_0 \neq \mathbf{0}, k = 0, \ldots, m-1$, such that for $n = 0, \ldots, m-1$

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (\mathbf{y}_k)^T (2i)^{k-n} \left(\mathrm{D}^{n-k}\mathbf{A}\right)(1) &= 2^{1-n}(\mathbf{y}_n)^T, \\ \sum_{k=0}^{n} \binom{n}{k} (\mathbf{y}_k)^T (2i)^{k-n} \left(\mathrm{D}^{n-k}\mathbf{A}\right)(-1) &= \mathbf{0}^T \end{aligned} \tag{10}$$

holds. Furthermore there exist matrices $\mathbf{C}_k, k = 0, \ldots, m-1$, such that $\mathbf{A}(z)$ factorizes like

$$\mathbf{A}(z) = \frac{1}{2^{m-1}}\mathbf{C}_0(z^2)\cdots\mathbf{C}_{m-1}(z^2)\mathbf{A}_m(z)\mathbf{C}_{m-1}^{-1}(z)\cdots\mathbf{C}_0^{-1}(z), \tag{11}$$

where $\mathbf{A}_m(z)$ is a suitable matrix with Laurent polynomials as entries.

The equations (10) are some of the main ingredients for our construction. They are called *sum rules of order m* and were also obtained by Heil et al. in [9].

3 Interpolating scaling vectors with arbitrary approximation order

The aim of this section is a systematical symbol–based construction of compactly supported interpolating 2–scaling vectors providing a high approximation order. Under the interpolation condition (6) the symbol $\mathbf{A}(z)$ of Φ has to fulfill some necessary conditions. Inserting the refinement equation (3) into (6) and using the coefficient matrices

$$A_k := \begin{pmatrix} a_k^{00} & a_k^{01} \\ a_k^{10} & a_k^{11} \end{pmatrix},$$

leads to

$$a_k^{\rho 0} = \delta_{\rho,k}.$$

Thus $\mathbf{A}(z)$ has to be of the form

$$\mathbf{A}(z) = \begin{pmatrix} 1 & a_0(z) \\ z & a_1(z) \end{pmatrix}, \quad z \in \mathbb{T}, \tag{12}$$

with entries

$$a_\rho(z) := \sum_{k\in\mathbb{Z}} a_k^{\rho 1} z^k, \quad \rho \in \{0,1\}.$$

This specific form of the symbol leads to remarkable simplifications concerning its Plonka–factorization and the corresponding sum rules.

Theorem 3.1. *Let Φ be a compactly supported interpolating 2–scaling vector with symbol $\mathbf{A}(z)$ as in (12). For $n \geq 0$ and $\rho \in \{0,1\}$ define the functions*

$$a_\rho^{(n)}(z) := \begin{cases} a_\rho(z) & \text{if } n = 0, \\ \sum_{k\neq 0} k^n a_k^{\rho 1} z^k & \text{else.} \end{cases}$$

Then Φ provides approximation order m if and only if $a_0(z)$ and $a_1(z)$ are Laurent polynomials and for $n = 0, \ldots, m-1$ the equations

$$\begin{aligned} 2^{1-n} &= (-1)^n a_0^{(n)}(1) + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_1^{(n-k)}(1) \\ 0 &= (-1)^n a_0^{(n)}(-1) + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_1^{(n-k)}(-1) \end{aligned} \tag{13}$$

hold.

Proof. As stated above, the interpolation property implies linearly independent translates, therefore the hypotheses of Theorem 2.1 are satisfied. In the following we show the equivalence of the sum rules (10) and the equations (13). For $n \geq 0$ the derivatives $\mathrm{D}^n\mathbf{A}(z)$ have the form

$$\mathrm{D}^n\mathbf{A}(z) = \begin{pmatrix} 0 & (-i)^n a_0^{(n)}(z) \\ (-i)^n z & (-i)^n a_1^{(n)}(z) \end{pmatrix}.$$

With $\mathbf{y}_n := (u_n, v_n)^T$ the first columns of the equations (10) are equivalent to

$$\begin{aligned} 2^{1-n}\, u_n &= u_n + \sum_{k=0}^{n} \binom{n}{k} (-2)^{k-n}\, v_k \\ 0 &= u_n - \sum_{k=0}^{n} \binom{n}{k} (-2)^{k-n}\, v_k. \end{aligned}$$

By induction we obtain

$$\mathbf{y}_n = u_0 \begin{pmatrix} \delta_{0,n} \\ \frac{1}{2^n} \end{pmatrix} \tag{14}$$

with some $u_0 \neq 0$. The second columns of the equations (10) are equivalent to

$$\begin{aligned} 2^{1-n}\, v_n &= \sum_{k=0}^{n} \binom{n}{k} (2i)^{k-n} \left((-i)^{n-k} u_k a_0^{(n-k)}(1) + (-i)^{n-k} v_k a_1^{(n-k)}(1)\right) \\ 0 &= \sum_{k=0}^{n} \binom{n}{k} (2i)^{k-n} \left((-i)^{n-k} u_k a_0^{(n-k)}(-1) + (-i)^{n-k} v_k a_1^{(n-k)}(-1)\right). \end{aligned}$$

Applying (14) we obtain (13). □

Remark. The corresponding factorization matrices $\mathbf{C}_n$ have the form

$$\mathbf{C}_n(z) = \left(\frac{1}{2u_0}\right)^{\delta_{0,n}} \begin{pmatrix} 2 & -2 \\ -2z & 2 \end{pmatrix}, \tag{15}$$

see [21] for details concerning their construction.

The eigenvalue properties of $\mathbf{A}(z)$ required for compact support and L_2–stability of Φ by Lemma 2.1 and Lemma 2.2 are likewise simplified by the specific structure of $\mathbf{A}(z)$. To provide a simple eigenvalue 2 for $z = 1$ the entries of $\mathbf{A}(z)$ have to satisfy

$$a_0(1) + a_1(1) = 2 \tag{16}$$

which corresponds to the first equation in (13) with $n = 0$. The second eigenvalue λ of $\mathbf{A}(1)$ is given by

$$a_0(1) = (\lambda - 1)(\lambda - a_1(1)). \tag{17}$$

Based on the fundamental relationships (12)–(13) we suggest the following construction. The examples obtained by this procedure are one–parameter families of interpolating 2-scaling vectors ${}_m\Phi_\alpha$ with approximation order $l(\mathbf{A}) - 1$. The corresponding symbols ${}_m\mathbf{A}_\alpha(z)$ are generated as follows:

- Start with a mask of length $m+1$ by defining general symbol entries of equal degree

$$a_\rho(z) := \sum_{k=\nu}^{\nu+m} a_k^{(\rho)} z^k, \quad \rho \in \{0,1\}, \nu \in \mathbb{Z},$$

  i. e.,there are $2(m+1)$ degrees of freedom. Centering the coefficients around $a_0^{(\rho)}$ seems to provide the highest regularity, therefore $\nu = -\lfloor \frac{m}{2} \rfloor$ is chosen.

- Apply the eigenvalue conditions (16) and (17) to the symbol. We chose $a_\rho(1) = 1, \rho = 0, 1$, which corresponds to the eigenvalues 2 and 0 of $\mathbf{A}(1)$. Now we are left with $2m$ degrees of freedom.

- For $n = 0, \ldots, m-1$ apply the factorization conditions (13). Finally one degree of freedom remains because of the redundancy produced by the conditions (16) and (13). This remaining parameter will be denoted by α.

Different methods of solving the above equations can lead to different scaling vectors ${}_m\Phi_\alpha$ for the same value of α, depending on the remaining free variable. The parameter α can be used to optimize the regularity of ${}_m\Phi_\alpha$.

4 Interpolating orthonormal scaling vectors

4.1 Construction

In this section we construct interpolating 2–scaling vectors which are also orthonormal by extending the construction method obtained in the previous section. So, first of all, we have to develop applicable orthonormality conditions in terms of the symbol entries $a_0(z)$ and $a_1(z)$.

Theorem 4.1. *Let Φ be a compactly supported interpolating 2–scaling vector with symbol*

$$\mathbf{A}(z) = \begin{pmatrix} 1 & a_0(z) \\ z & a_1(z) \end{pmatrix}, \quad a_0(z) \neq z^\nu,\ nu \in \mathbb{Z}.$$

If Φ is also orthonormal, then the symbol entries $a_0(z)$ and $a_1(z)$ have to satisfy

$$2 = |a_0(z)|^2 + |a_0(-z)|^2 \tag{18}$$

and

$$a_1(z) = \pm z^{2\kappa+1}\overline{a_0(-z)}, \quad \kappa \in \mathbb{Z}, \tag{19}$$

with

$$a_0(z) = \sum_{k=M}^{N} a_k z^k, \quad M, N \in \mathbb{Z}, \quad N - M \text{ odd}.$$

Proof. Inserting (12) into (9) leads to

$$\begin{aligned} 2 &= |a_0(z)|^2 + |a_0(-z)|^2 & (20)\\ 2 &= |a_1(z)|^2 + |a_1(-z)|^2 & (21)\\ 0 &= a_0(z)\overline{a_1(z)} + a_0(-z)\overline{a_1(-z)} & (22) \end{aligned}$$

for all $z \in \mathbb{T}$. Because of Φ being compactly supported and also orthonormal, the symbol entries $a_\rho(z)$, $\rho = 0, 1$, have to be Laurent polynomials, i. e.,

$$a_\rho(z) = \sum_{k=M_\rho}^{N_\rho} a_k^{(\rho)} z^k, \quad M_\rho, N_\rho \in \mathbb{Z}, \quad M_\rho \leq N_\rho,$$

with $a_{N_\rho}^{(\rho)} \neq 0$ and $a_{M_\rho}^{(\rho)} \neq 0$. Therefore there exist polynomials $\tilde{a}_\rho(z)$, $\tilde{\overline{a}}_\rho(z)$ with

$$a_\rho(z) = z^{M_\rho}\tilde{a}_\rho(z), \quad a_\rho(z^{-1}) = z^{-N_\rho}\tilde{\overline{a}}_\rho(z), \quad \rho = 0, 1.$$

The coefficients of $a_\rho(z)$ are real and z is contained in the unit circle, therefore $\overline{a_\rho(z)} = a_\rho(z^{-1})$. Thus (20) is equivalent to

$$\tilde{a}_0(z)\tilde{\overline{a}}_0(z) + (-1)^{M_0-N_0}\tilde{a}_0(-z)\tilde{\overline{a}}_0(-z) = 2z^{N_0-M_0}.$$

Consisting of polynomials only, this equation holds for all $z \in \mathbb{C}$ and implies for the greatest common divisor of $\tilde{a}_0(z)$ and $\tilde{a}_0(-z)$

$$\mathrm{GCD}(\tilde{a}_0(z), \tilde{a}_0(-z)) = 1.$$

Furthermore, $N_0 - M_0$ is odd, because

$$\tilde{a}_0(0) = a_{M_0}^{(0)} \neq 0.$$

With the above notation equation (22) is equivalent to

$$\tilde{a}_0(z)\tilde{\overline{a}}_1(z) = -(-1)^{M_0-N_1}\tilde{a}_0(-z)\tilde{\overline{a}}_1(-z),$$

which also holds for all $z \in \mathbb{C}$. Comparing the linear factors of the polynomials on both sides, we obtain that $\tilde{\overline{a}}_1(z)$ has to contain the linear factors which are contained in $\tilde{a}_0(-z)$ and which are not contained in $\tilde{a}_0(z)$. Therefore $\tilde{\overline{a}}_1(z)$ has to be of the form

$$\tilde{\overline{a}}_1(z) = \frac{\tilde{a}_0(-z)}{\mathrm{GCD}(\tilde{a}_0(z), \tilde{a}_0(-z))} p(z) = \tilde{a}_0(-z)p(z) \tag{23}$$

with a polynomial $p(z)$. Applying this to (22) yields

$$p(z) = -(-1)^{M_0-N_1}p(-z)$$

and by equation (21) there exists a $k \in \mathbb{Z}$ such that

$$p(z) = \pm z^{N_1-M_0+2k+1}.$$

But because of $\tilde{\overline{a}}_1(0) = a_{N_1}^{(1)} \neq 0$ equation (23) yields $p(z) \equiv \pm 1$ and $N_1 - M_0$ is odd. Therefore $a_1(z)$ has to be of the form

$$a_1(z) = \pm z^{N_1-M_0}\overline{a_0(-z)} =: \pm z^{2\kappa+1}\overline{a_0(-z)}.$$

□

Remark. If $a_0(z)$ is a (Laurent–) monomial, then due to the above equations the same holds for $a_1(z)$ with

$$a_0(z) = z^{\nu}, \quad a_1(z) = z^{\nu+2\kappa+1}, \quad \nu, \kappa \in \mathbb{Z}.$$

By Theorem 1.2 in [5] the corresponding scaling vector is of Haar–type, i. e., it is the characteristic function of a self–affine multi–tiling. Therefore this scaling vector can not even be continuous.

The next step is to find a more applicable version of condition (18). First of all, similar to the construction of orthonormal wavelets in [3], we represent $|a_0(z)|^2$, $z = e^{-i\omega}$, as a polynomial in $\sin^2 \frac{\omega}{2}$

$$|a_0(e^{-i\omega})|^2 =: P\left(\sin^2 \frac{\omega}{2}\right).$$

Then equation (18) transforms to

$$2 = P\left(\sin^2 \frac{\omega}{2}\right) + P\left(1 - \sin^2 \frac{\omega}{2}\right) =: \sum_{k=0}^{N-M} p_k \left(\sin^2 \frac{\omega}{2}\right)^k$$

and we obtain

$$p_k = 2\delta_{0,k}, \quad k = 0 \ldots N - M.$$

A simple computation yields the following corollary.

Corollary 4.1. *Let $a_0(z)$ be a Laurent polynomial defined on the unit circle*

$$a_0(z) = \sum_{k=M}^{N} a_k z^k, \quad z \in \mathbb{T}.$$

Then, with $K := N - M$, equation (18) is equivalent to

$$(a_M, \ldots, a_N)\mathbf{M}_k \begin{pmatrix} a_M \\ \vdots \\ a_N \end{pmatrix} = \delta_{0,k} \quad \textit{for} \quad k = 0, \ldots, \left\lfloor \frac{K}{2} \right\rfloor, \tag{24}$$

with $(K+1) \times (K+1)$ matrices

$$\mathbf{M}_k := \sum_{n=k}^{\lfloor K/2 \rfloor} c_{n,k} \begin{pmatrix} \mathbf{0} & \mathbf{I}_{K-2n+1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$c_{n,k} := 2^{\delta_{0,k}(1-\delta_{n,0})} \sum_{l=0}^{n-k} \sum_{j=l}^{n} \binom{2n}{2j} \binom{j}{j-l}.$$

As stated above, linearly independent translates of a scaling vector are implied by the interpolation property. Thus applying the orthonormality condition (19) to the sum rules (13) yields:

Corollary 4.2. *With the notations and conditions of Theorem* 4.1 *the* 2*–scaling vector* Φ *provides approximation order* m *iff for* $n = 0, \ldots, m-1$ *the mask coefficients of the symbol entry* $a_0(z)$ *satisfy*

$$\begin{aligned} 2^{-n} &= \sum_k (-2k)^n a_{2k} - \sum_k (2(k-\kappa)+1)^n a_{2k+1} \\ 2^{-n} &= \sum_k (-2k-1)^n a_{2k+1} + \sum_k (2(k-\kappa))^n a_{2k}. \end{aligned} \tag{25}$$

Note that the sum rules of order one together with the orthonormality condition (19) already imply that $\mathbf{A}(1)$ has the eigenvalues 2 and 0.

Based on these corollaries we suggest the following construction principle for the symbol $\mathbf{A}(z)$ of an orthonormal interpolating 2–scaling vector:

- Start with a first symbol entry $a_0(z)$ with an even number of coefficients

  $$a_0(z) := \sum_{k=-n+\nu}^{n+1+\nu} a_k z^k, \qquad \nu \in \mathbb{Z},$$

  by choosing the length $2n$ of $(a_k)_{k\in\mathbb{Z}}$. Again, centering the coefficients around a_0 seems to provide the highest regularity, therefore $\nu = 0$ is chosen.

- By equation (19) the second symbol entry $a_1(z)$ has to be of the form

  $$a_1(z) := \pm z^{2\kappa+1} a_0\left(-z^{-1}\right).$$

  It turns out that the choice $\kappa = 0$ and a positive sign provide the highest regularity and the shortest support. Now we have $2n$ degrees of freedom.

- Apply the orthonormality condition (24) to the coefficient sequence $(a_k)_{k\in\mathbb{Z}}$. Then we are left with n degrees of freedom.

- Finally, apply the sum rules (25) up to the highest possible order to the coefficient sequence $(a_k)_{k\in\mathbb{Z}}$.

The symbols being constructed in this way correspond to 2–scaling vectors supported on the interval $[-n, n+1]$. But, as the applied conditions are just necessary, we have to check whether the scaling vectors really possess the desired properties or not. As stated above, the interpolation property implies linearly independent translates. Jia and Micchelli have shown in [11] that for a compactly supported scaling vector linear independence implies L_p–stability. By Theorem 5 in [24] and Theorem 6 in [22] it follows that the above conditions combined with stability imply orthonormality. This was also proven in [27]. Therefore, to ensure that the constructed scaling vectors do possess the desired properties, we just have to check if they are interpolating.

4.2 Construction of multiwavelets

In the scalar case the construction of wavelets corresponding to orthonormal scaling functions is by now well–understood due to the pioneering work of [3]. All possible wavelets are related to a canonical wavelet, which is completely determined by the symbol of the scaling function, see, e. g., [3]. In the vector case the situation is more involved, therefore we have to go into some details. First of all we have to introduce some notations.

The *polyphase matrix* of an r–scaling vector Φ is the $r \times 2r$ matrix

$$\mathbf{P}(z) := \frac{1}{\sqrt{2}} \left(\mathbf{A}_0(z) \,\middle|\, \mathbf{A}_1(z) \right),$$

consisting of the *subsymbols* of Φ, defined by

$$\mathbf{A}_\rho(z) = \sum_{k\in\mathbb{Z}} A_{2k+\rho} z^k, \quad \rho = 0, 1,$$

where $(A_k)_{k\in\mathbb{Z}}$ denotes the mask of Φ, compare with (8). It was shown in [6, 7] that the construction of an r–multiwavelet Ψ corresponding to an orthonormal r–scaling vector Φ can be reduced to the problem of extending the polyphase matrix $\mathbf{P}(z)$ of Φ to an quadratic matrix $\mathbf{Q}(z)$, such that $\mathbf{Q}(z)$ is unitary for all $z \in \mathbb{T}$. Then the symbol $\mathbf{B}(z)$ of a corresponding r–multiwavelet Ψ is given by

$$\mathbf{B}(z) := \mathbf{B}_0\left(z^2\right) + z\mathbf{B}_1\left(z^2\right), \tag{26}$$

where $\mathbf{B}_0(z)$ and $\mathbf{B}_1(z)$ are submatrices of $\mathbf{Q}(z)$, i. e.,

$$\mathbf{Q}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{A}_0(z) & \mathbf{A}_1(z) \\ \mathbf{B}_0(z) & \mathbf{B}_1(z) \end{pmatrix}.$$

Since there might exist several unitary extensions, the construction of multiwavelets is by no means unique. For compactly supported r–scaling vectors an effective algorithm for the extension of $\mathbf{P}(z)$ has been given in [15]. Most of the examples of multiwavelets shown in this paper are obtained by an application of this algorithm.

If a 2–multiwavelet corresponding to a compactly supported interpolating and orthonormal 2–scaling vector Φ is desired to be also interpolating, i. e.,

$$\Psi\left(\frac{n}{2}\right) = \begin{pmatrix} \delta_{0,n} \\ \delta_{1,n} \end{pmatrix},$$

then our construction leaves us much less freedom in the sense that the symbol $\mathbf{B}(z)$ of Ψ is completely determined by the symbol $\mathbf{A}(z)$ of Φ. The following corollary of Theorem 4.1 has already been derived in [26] but without a proof. For the reader's convenience we sketch the proof in our setting.

Corollary 4.3. *The symbol $\mathbf{B}(z)$ of an interpolating multiwavelet corresponding to a compactly supported interpolating and orthonormal 2–scaling vector with symbol $\mathbf{A}(z)$ as in Theorem 4.1 has to satisfy*

$$\mathbf{B}(z) = \begin{pmatrix} 1 & -a_0(z) \\ z & -a_1(z) \end{pmatrix}, \qquad z \in \mathbb{T}.$$

Proof. Due to the interpolation property $\mathbf{B}(z)$ has to be of the form

$$\mathbf{B}(z) = \begin{pmatrix} 1 & b_0(z) \\ z & b_1(z) \end{pmatrix}, \qquad z \in \mathbb{T},$$

with Laurent series $b_0(z)$, $b_1(z)$. Applying (26) a direct computation shows that the unitarity of $\mathbf{Q}(z)$ is equivalent to the unitarity of the matrix

$$\frac{1}{2}\begin{pmatrix} \mathbf{A}(z) & \mathbf{A}(-z) \\ \mathbf{B}(z) & \mathbf{B}(-z) \end{pmatrix}.$$

This leads to

$$\begin{aligned} -2 &= a_\rho(z)\overline{b_\rho(z)} + a_\rho(-z)\overline{b_\rho(-z)} \\ 0 &= a_\rho(z)\overline{b_{1-\rho}(z)} + a_\rho(-z)\overline{b_{1-\rho}(-z)} \\ 2 &= |b_\rho(z)|^2 + |b_\rho(-z)|^2 \end{aligned}$$

for $\rho = 0, 1$. Following the lines of the proof of Theorem 4.1 we obtain

$$b_0(z) = -a_0(z) \quad \text{and} \quad b_1(z) = -a_1(z).$$

□

4.3 Balancing

For application purposes the reproduction of polynomials by scaling vectors and the vanishing moments of the corresponding multiwavelets play a crucial role. In the continuous setting these properties are closely related to approximation order. Unfortunately in the vector case, unlike the scalar case, these properties are not carried forward to the filter banks associated with scaling vectors. So the corresponding discrete multiwavelet transform lacks some important characteristics, see [16] and [25] for details. To bypass this problem Lebrun and Vetterli introduced the notion of *balancing*, cf. [16, 17].

Definition 4.1. An orthonormal 2–scaling vector is called *balanced of order m*, if the corresponding subdivision operator S_A, defined by

$$S_A v(\alpha) := \sum_{\beta \in \mathbb{Z}} v_\beta A_{\beta - 2\alpha} \qquad \text{for } v \in \ell^{1\times 2}(\mathbb{Z}), \quad \alpha \in \mathbb{Z},$$

preserves the sequence

$$u_n := \begin{pmatrix} (2k)^n \\ (2k+1)^n \end{pmatrix}_{k\in\mathbb{Z}},$$

for all $n = 0, \ldots, m-1$, i. e.,

$$S_A u_n^T = 2^{-n} u_n^T.$$

In the following we will show that the interpolation property already implies balancing of an order equal to the approximation order. Therefore we need the following theorem which was proven in [17].

Theorem 4.2. *An orthonormal* 2*–scaling vector is balanced of order* m *iff* $\mathbf{A}(z)$ *can be factorized as*

$$\mathbf{A}(z) = \frac{1}{2^{m-1}} \mathbf{T}^m(z^2)\mathbf{A}_m(z)\mathbf{T}^{-m}(z) \tag{27}$$

with

$$\mathbf{A}_m(z)\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \mathbf{T}(z) = \begin{pmatrix} 1 & -1 \\ -z & 1 \end{pmatrix}.$$

Now, if an compactly supported orthonormal 2–scaling vector provides approximation order m, then by Theorem 2.1 its symbol $\mathbf{A}(z)$ has already a specific factorization (11) with matrices $\mathbf{C}_k$. Equation (15) yields

$$\mathbf{C}_k(z) = 2\left(\frac{1}{2u_0}\right)^{\delta_{0,k}} \mathbf{T}(z),$$

therefore the factorization condition (27) holds true. Because of

$$\mathbf{A}(z)\mathbf{T}^m(z) = \frac{1}{2^{m-1}}\mathbf{T}^m(z^2)\mathbf{A}_m(z)$$

and $\ker \mathbf{T}(1) = \operatorname{span}\{(1,1)^T\}$ the vector $v := (1,1)^T$ has to be a right eigenvector of $\mathbf{A}_m(1)$. $\mathbf{T}(1)$ is symmetric, therefore v^T is also a left eigenvector of $\mathbf{T}(1)$ corresponding to the eigenvalue 0. Furthermore by the interpolation condition it holds that

$$v^T\mathbf{A}(1) = (2, a_0(1) + a_1(1))\,.$$

Due to (16) v^T is also a left eigenvector of $\mathbf{A}(1)$ corresponding to the eigenvalue 2. By iterating Lemma 3 in [28] we obtain that v is a right eigenvector of $\mathbf{A}_m(1)$ corresponding to the eigenvalue 1. So we have proven the following theorem.

Theorem 4.3. *Any compactly supported interpolating orthonormal* 2*–scaling vector providing approximation order* m *is balanced of order* m*.*

5 Examples

5.1 Interpolating scaling vectors

The interpolating scaling functions constructed in [2], interpreted as univariate refinable functions for dyadic scaling, provide approximation order m with symbol entries of a maximum degree m. According to equation (7) they can be reinterpreted as 2–scaling vectors. Therefore they belong to the corresponding sets $\{{}_m\Phi_\alpha : \alpha \in \mathbb{R}\}$. In the following examples α is chosen in such a way that ${}_m\Phi_0$ corresponds to these (scalar) scaling functions in [2].

The case $m = 2$:

For $m = 2$ the symbol ${}_2\mathbf{A}_\alpha(z)$ has the form

$$
{}_2\mathbf{A}_\alpha(z) = \begin{pmatrix} 1 & (\frac{1}{2} - \alpha)z^{-1} + \frac{1}{2} + \alpha z \\ z & \alpha z^{-1} + \frac{1}{2} + (\frac{1}{2} - \alpha)z \end{pmatrix}.
$$

Hence for $\alpha = 0$ the symbol of the symmetrized cardinal B–spline of order 2, interpreted as a 2–scaling vector, is obtained. In Figure 1 we have displayed the

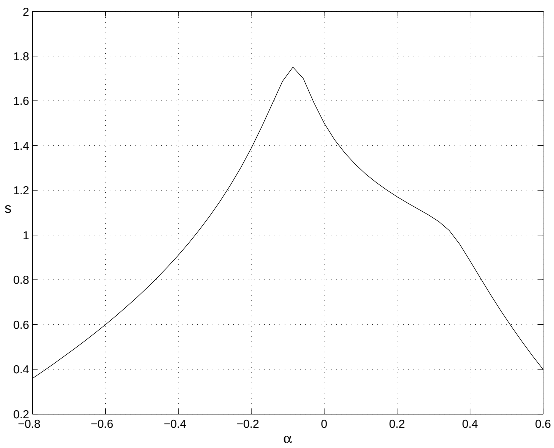


Figure 1: Sobolev exponent s of ${}_2\Phi_\alpha$.

Sobolev regularity of the scaling vectors ${}_2\Phi_\alpha$. We observe that the maximum is attained for $\alpha = -1/12$. The reader should remember that cardinal B-splines are the most regular refinable functions with respect to their supports. The results in [19] imply that the support of ${}_2\Phi_\alpha$ is contained in $[-1, 1]$, therefore a scaling vector with the same support but higher regularity compared to the B–spline is constructed. This emphasizes the benefit of matrix refinability. For $\alpha = -1/12$ the Sobolev exponent of ${}_2\Phi_\alpha$ is $s = 1.751$ in contrast to $s = 1.5$ for the B–spline. Thus the scaling vector shown in Figure 2 is one times continuously differentiable, in fact, by the Sobolev embedding theorem, its Hölder exponent is bounded from below by 1.251.

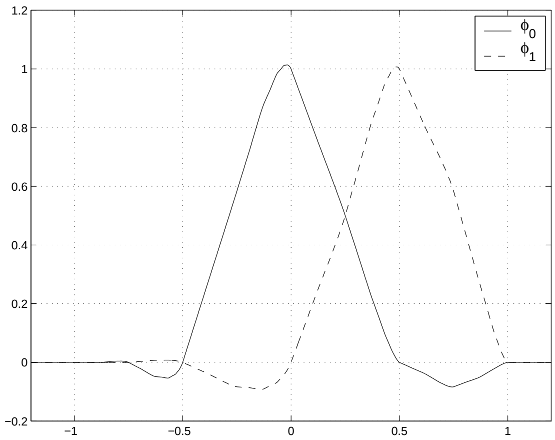


Figure 2: ϕ_0 and ϕ_1 of ${}_2\Phi_\alpha$ with $\alpha = -1/12$.

The case $m = 3$:

In the case $m = 3$, the above construction produces a symbol of the form

$$ {}_3\mathbf{A}_\alpha(z) = \begin{pmatrix} 1 & (\frac{3}{8} - 3\alpha)z^{-1} + \frac{3}{4} + 3\alpha - (\alpha + \frac{1}{8})z + \alpha z^2 \\ z & \alpha z^{-1} + \frac{3}{8} - \alpha + (\frac{3}{4} + 3\alpha)z - (\frac{1}{8} + 3\alpha)z^2 \end{pmatrix}. $$

Again, as shown in Figure 3, matrix refinability yields a gain of regularity compared to the scalar case ($\alpha = 0$). The scaling vector depicted in Figure 4 cor-

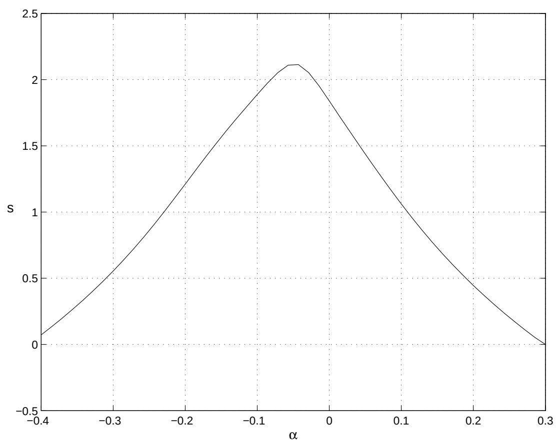


Figure 3: Sobolev exponent s of ${}_3\Phi_\alpha$.

responds to the maximum Sobolev exponent $s = 2.119$ of ${}_3\Phi_\alpha$ for $\alpha = -1/20$. The Sobolev exponent of the corresponding scalar scaling function ($\alpha = 0$) is $s = 1.839$.

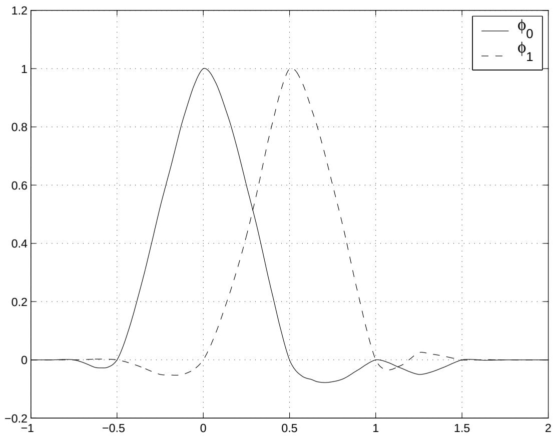


Figure 4: ϕ_0 and ϕ_1 of ${}_3\Phi_\alpha$ with $\alpha = -1/20$.

The case $m = 4$:

Starting with $m = 4$ leads to the symbol

$${}_4\mathbf{A}_\alpha(z) = \begin{pmatrix} 1 & -(\frac{1}{16} + 3\alpha)z^{-2} + \frac{9}{16}z^{-1} + \frac{9}{16} + 2\alpha - \frac{1}{16}z + \alpha z^2 \\ z & \alpha z^{-2} - \frac{1}{16}z^{-1} + \frac{9}{16} + 2\alpha + \frac{9}{16}z - (\frac{1}{16} + 3\alpha)z^2 \end{pmatrix}.$$

As shown in Figure 5 the Sobolev exponent of ${}_4\Phi_\alpha$ exceeds $s = 2.441$ corresponding to the scalar case. The choice of $\alpha = 1/50$ yields the scaling vector depicted in

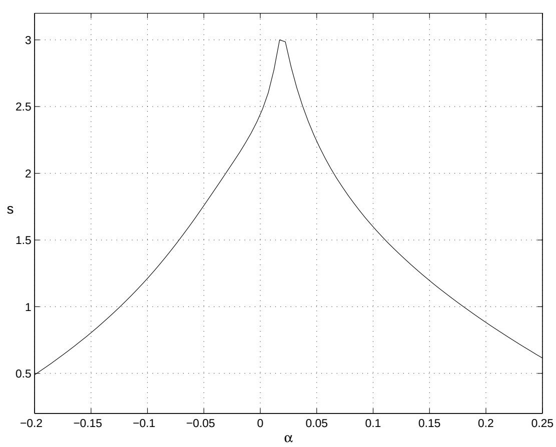


Figure 5: Sobolev exponent s of ${}_4\Phi_\alpha$.

Figure 6 with Sobolev exponent $s = 3.078$.

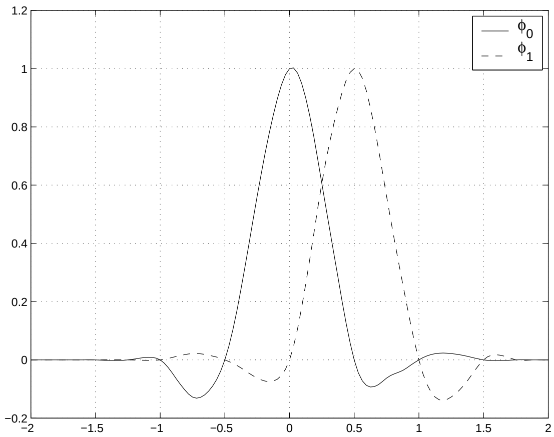


Figure 6: ϕ_0 and ϕ_1 of ${}_4\Phi_\alpha$ with $\alpha = 1/46$.

5.2 Interpolating orthonormal scaling vectors

Due to the quadratic conditions the solutions obtained by our construction are not unique. In the following examples we have chosen those solutions which possess the highest regularity.

The case $n = 1$:

Similar to the non-orthonormal case for $n = 1$ our construction leads to a one–parameter set of 2–scaling vectors Φ_α with symbols

$$\mathbf{A}_\alpha(z) = \begin{pmatrix} 1 & \sqrt{-\alpha(\alpha-1)}z^{-1} + \alpha - \sqrt{-\alpha(\alpha-1)}z + (1-\alpha)z^2 \\ z & (1-\alpha)z^{-1} + \sqrt{-\alpha(\alpha-1)} + \alpha z - \sqrt{-\alpha(\alpha-1)}z^2 \end{pmatrix}.$$

Figure 7 shows the Sobolev exponent s of Φ_α. For $\alpha \approx 0.9486$ we obtain an

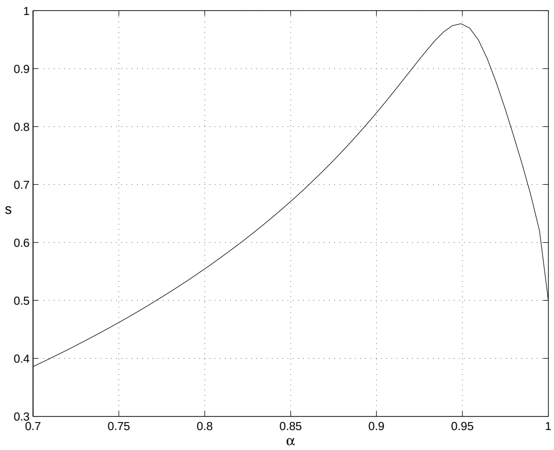
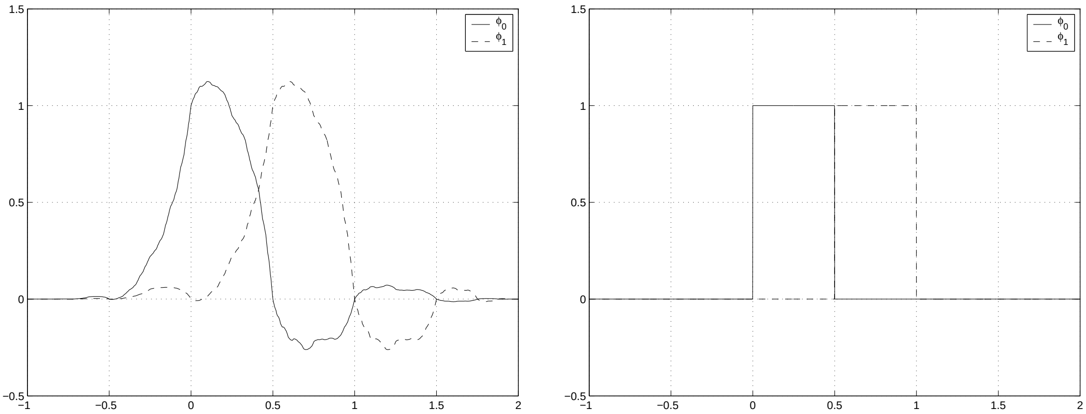


Figure 7: Sobolev exponent s of Φ_α

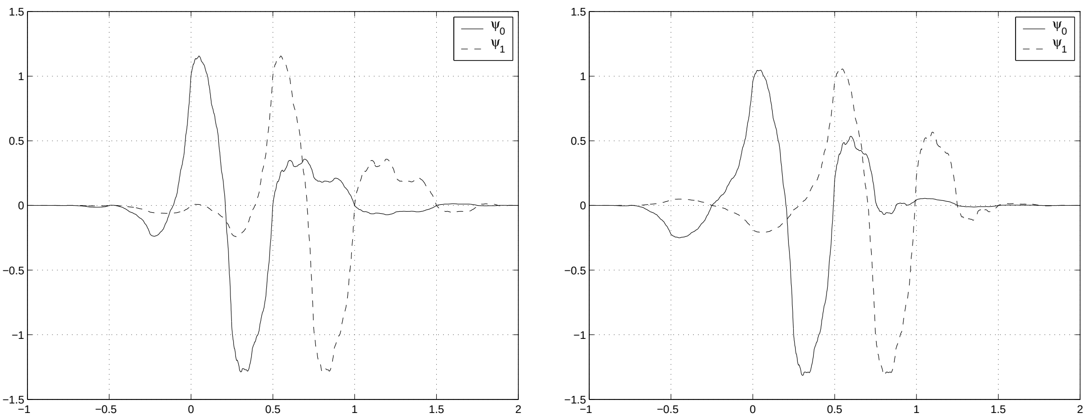
interpolating and orthonormal 2–scaling vector depicted in Figure 8(a) which provides approximation order 1 and is supported on $[-1, 2]$. The Sobolev exponent of $\Phi_{0.9486}$ is $s = 0.9777$. For $\alpha = 1$ we also obtain the Haar generator interpreted as a scaling vector as is shown in Figure 8(b).



(a) $\alpha = 0.9486$ (b) $\alpha = 1$

Figure 8: 2–Scaling vector Φ_α

In Figure 9 an interpolating and a non-interpolating 2–multiwavelet corresponding to $\Phi_{0.9486}$ are shown. Note that both multiwavelets possess the same support properties.



(a) interpolating (b) not interpolating

Figure 9: Multiwavelets corresponding to $\Phi_{0.9486}$

The case $n > 1$:

For $n > 1$ all degrees of freedom are consumed by our construction. So, for each n we obtain just a discrete set of solutions and not a parameter depending family

n	supp Φ	approximation order m	Sobolev exponent s
2	[-2,3]	2	1.50
3	[-3,4]	3	1.51
4	[-4,5]	3	1.74
5	[-5,6]	4	1.80
6	[-6,7]	4	2.01
7	[-7,8]	5	1.84
8	[-8,9]	5	2.04

Table 1: Sobolev regularity and approximation order of Φ_n

of solutions. In this example we concentrate on the family Φ_n of the most regular elements of these sets. All these Φ_n depicted in Figure 10 are very similar in shape. The main mass of the scaling vector is concentrated in the interval $[-1, 2]$ and with increasing n there is just some oscillation added outside this interval. The coefficient sequences $(a_k)_{k\in\mathbb{Z}}$ of the corresponding symbol entries $a_0(z)$ also reveal this similarity, see Table 3.

The non–interpolating and interpolating multiwavelets Ψ_n corresponding to the Φ_n are shown in Figure 11 and Figure 12 respectively. Both multiwavelet families possess rather similar support properties, but the non–interpolating multiwavelets reveal stronger oscillations. In general, the vanishing moments of a wavelet are strongly connected to its oscillation behaviour. As the matrix extension algorithm used to construct the non–interpolating multiwavelets still contains some freedom, this freedom could possibly be used to optimize the vanishing moments properties of Ψ_n. This will be studied in detail in a forthcoming paper.

As is shown in Table 1 similar to the non–orthonormal case the Sobolev regularity and the provided approximation order increase with the support length of Φ_n. By enlarging the parameter n by one either the Sobolev exponent of Φ_n or the provided approximation order is alternately increased. In our examples the approximation order equals $\lfloor\frac{n+1}{2}\rfloor+1$. Note that the examples for $n = 2$ and $n = 4$ were also obtained by Selesnick in [26]. For $n = 3$ our construction also yields the corresponding example of Selesnick and, in addition, a scaling vector that provides a higher Sobolev regularity. Therefore the more regular scaling vector is depicted in Figure 10.

For $1 \le n < 6$ we were able to compute the sequences $(a_k)_{k\in\mathbb{Z}}$ analytically. The corresponding entries of the symbol $a_0(z)$ are shown in Table 2 with the notation

$$a_0(z) := c\tilde{a}_0(z).$$

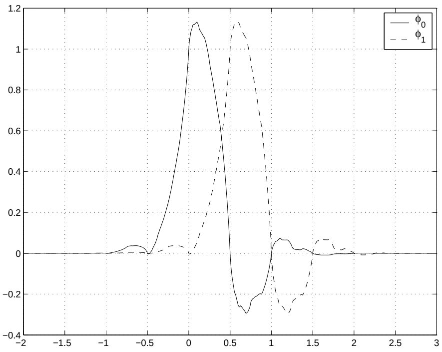
n	c	$\tilde{a}_0(z)$	roots
2	$\frac{1}{32}$	$z^{-2} + (4+r)z^{-1} + 30 - 2rz + z^2 + (r-4)z^3$	$r = \sqrt{15}$
3	$\frac{1}{2560}$	$(17-2r)z^{-3} + (74+r)z^{-2} + (589+6r)z^{-1} + (2418-3r)$ $- (589+6r)z + (62+3r)z^2 + (2r-17)z^3 + (6-r)z^4$	$r = \sqrt{31}$
4	$\frac{1}{7680}$	$-10z^{-4} + (48-2r)z^{-3} + (246+r)z^{-2} + (1771+6r)z^{-1} + 7242 -$ $3r - (1761+6r)z + (178+3r)z^2 - (63-2r)z^3 + (24-r)z^4 + 5z^5$	$r = \sqrt{1991}$
5	$\frac{1}{81920}$	$(2r_0 - 303 - 4r_1)z^{-5} - (66+r_0)z^{-4} + (875+12r_1-10r_0)z^{-3}$ $+ (5r_0+2690)z^{-2} + (20r_0+19370-8r_1)z^{-1} + 76940 - 10r_0$ $- (20r_0+8r_1+19370)z + (2020+10r_0)z^2 - (875-12r_1-10r_0)z^3$ $+ (470-5r_0)z^4 + (303-2r_0-4r_1)z^5 + (r_0-134)z^6$	$r_0 = \sqrt{17951}$ $r_1 = \sqrt{10795 - 80r_0}$

Table 2: Exact symbol entries $a_0(z)$

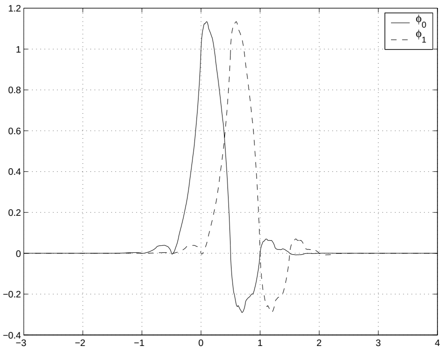
	$n=1$	$n=2$	$n=3$	$n=4$
a_{-4}				-0.0013020833333333
a_{-3}			0.002290809091539	-0.0053699539152725
a_{-2}		0.03125	0.03108115795423	0.037841226957636
a_{-1}	0.22081222792228	0.24603072956898	0.24312757272538	0.26545882007915
a_0	0.9486	0.9375	0.93800652613731	0.92553881912709
a_1	-0.22081222792228	-0.24206145913796	-0.24312757272538	-0.26415673674582
a_2	0.0514	0.03125	0.030743473862691	0.040607014206242
a_3		-0.0039692704310182	-0.002290809091539	0.0034168289152725
a_4			0.00016884204576952	-0.0026849769576362
a_5				0.00065104166666667

	$n=5$	$n=6$	$n=7$	$n=8$
a_{-8}				0.000302723900211519
a_{-7}			$8.46474488236856e{-05}$	0.000951764287700244
a_{-6}		$-5.9847542317558e-05$	$8.71705538516523e{-05}$	0.00151857269403843
a_{-5}	-0.00085475274252818	-0.00023304881297226	-0.000323778133068634	-0.000384845790624145
a_{-4}	-0.0024411784910232	-0.0030220063150712	-0.00308687194730986	-0.0149422995429387
a_{-3}	-0.0043928385490083	-0.0088609884112656	-0.00745963593193362	-0.0212438942302372
a_{-2}	0.041014486205116	0.044816338460119	0.0429318167079586	0.0708398092279952
a_{-1}	0.26830638162131	0.27973599714156	0.273567190946869	0.312834783122487
a_0	0.92285384008977	0.91465166015659	0.919901620035047	0.886903323447953
a_1	-0.27001458614461	-0.28348562819413	-0.273972002515595	-0.322106899626051
a_2	0.041013347410232	0.049514765055002	0.0435118223416151	0.0664820220838195
a_3	0.0069551453339558	0.014146518040279	0.00818829675564037	0.0373159544447045
a_4	-0.002440267455116	-0.0067807475909773	-0.00354085673595051	-0.0142459018557781
a_5	$6.5048087899631e-07$	-0.0010769051302435	$-8.10334356573321e{-05}$	-0.00756352180162508
a_6	$-2.2775897679288e-07$	0.00087983777665897	0.000191720573972034	0.00383539472598041
a_7		-0.00022594463322777	$-3.68513507849262e{-06}$	$-2.39652249074088e{-05}$
a_8			$3.57847081591257e{-06}$	-0.000693644681281331
a_9				0.000220624818552337

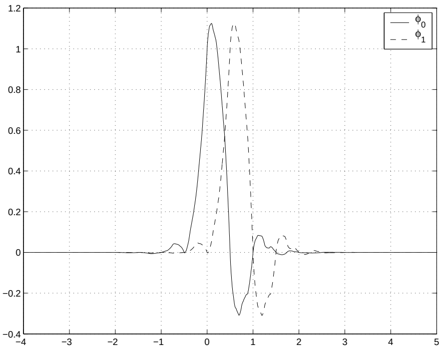
Table 3: Coefficient sequences $(a_k)_{k\in\mathbb{Z}}$



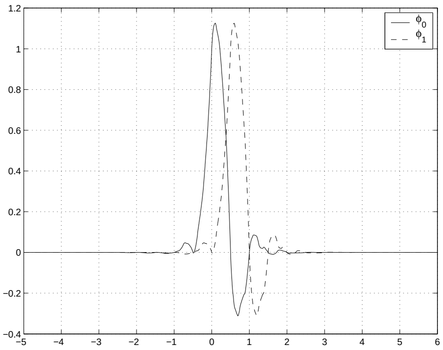
(a) $n = 2$



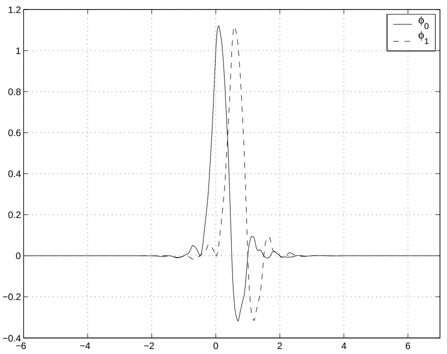
(b) $n = 3$



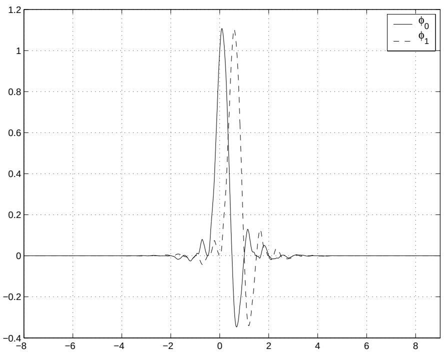
(c) $n = 4$



(d) $n = 5$

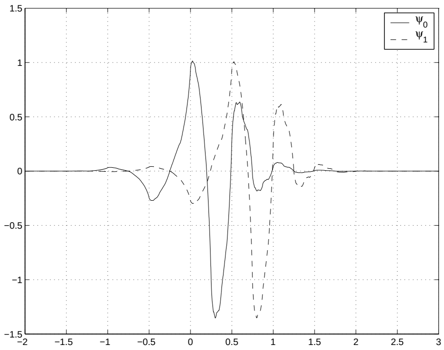


(e) $n = 6$

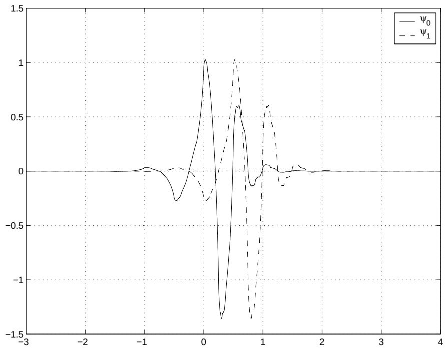


(f) $n = 8$

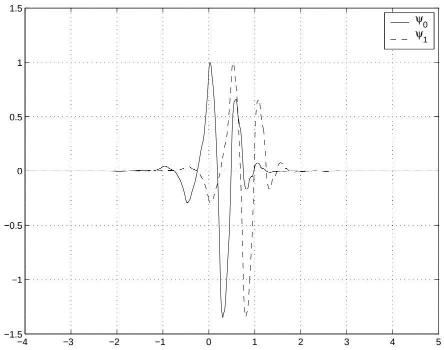
Figure 10: 2–Scaling vector Φ_n for $n > 1$



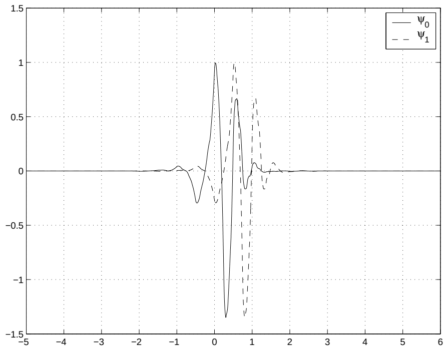
(a) $n = 2$



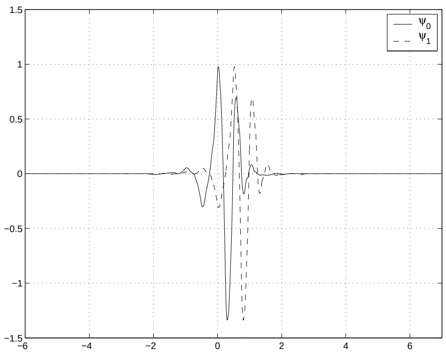
(b) $n = 3$



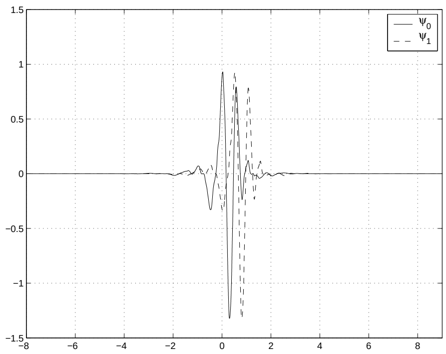
(c) $n = 4$



(d) $n = 5$

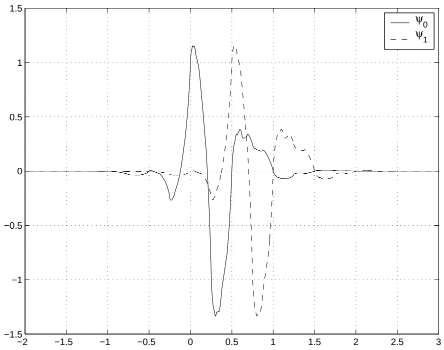


(e) $n = 6$

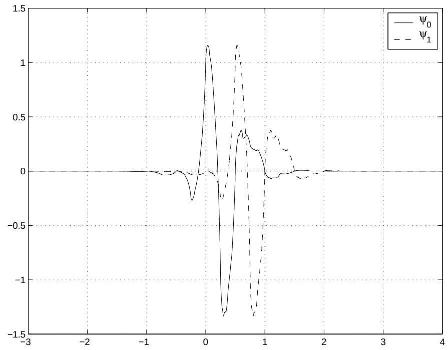


(f) $n = 8$

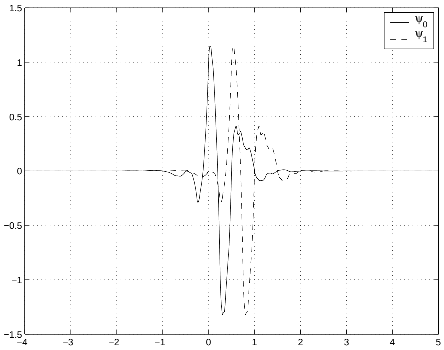
Figure 11: Non–interpolating 2–multiwavelet Ψ_n for $n > 1$



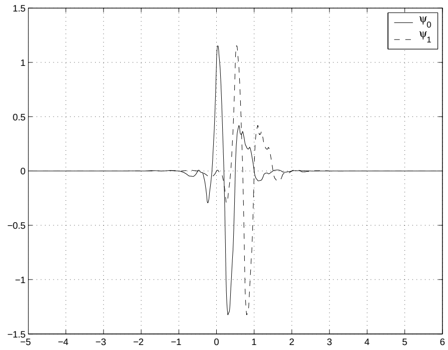
(a) $n = 2$



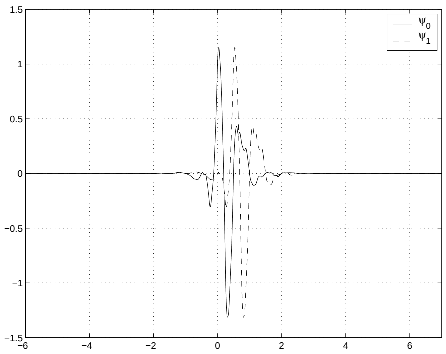
(b) $n = 3$



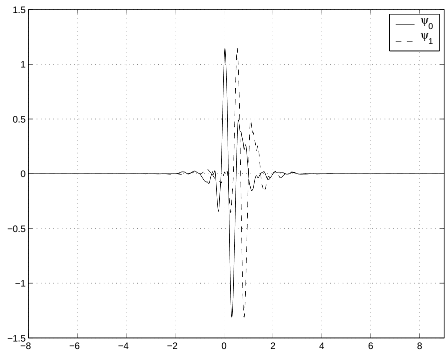
(c) $n = 4$



(d) $n = 5$



(e) $n = 6$



(f) $n = 8$

Figure 12: Interpolating 2–multiwavelet Ψ_n for $n > 1$

The masks $(B_k)_{k\in\mathbb{Z}}$ of the non–interpolating multiwavelets Ψ_n are shown in Table 4 with the notation

$$B_k := \begin{pmatrix} b_k^{00} & b_k^{01} \\ b_k^{10} & b_k^{11} \end{pmatrix}.$$

Again, the masks also reveal the similarity of the multiwavelets shown in Figure 11.

Remark. The regularity of the constructed scaling vectors was estimated with a MATLAB routine written by Q.T. Jiang [13], for the theoretical background see [10] and [12].

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k	b_k^{00}	b_k^{01}	b_k^{10}	b_k^{11}
$n=1$, $\alpha=0.9486$				
-1	-0.222428080109859	0	0.0438833587894344	0
0	0.955541632714646	-0.999424991504081	-0.188521054922329	-0.0339070251875302
1	0.188521054922329	0.0339070251875302	0.955541632714647	-0.999424991504081
2	0.0438833587894344	0	0.222428080109859	0
$n=2$				
-2	0.0311168812088972	0	-0.00490482533694517	0
-1	-0.244982687543562	0	0.0386156081938253	0
0	0.929817562119305	-0.999550051522028	-0.270072770146329	0.0299949079397124
1	0.270072770146329	-0.0299949079397124	0.929817562119305	-0.999550051522028
2	0.0386156081938253	0	0.244982687543562	0
3	0.00490482533694487	0	0.0311168812088972	0
$n=3$				
-3	-0.00229081095067021	0	0.000168816819622176	0
-2	0.0310811831784831	0	-0.00229046682911585	0
-1	-0.243127911254386	0	0.0307407965668045	0
0	0.938009203374459	-0.999999999939369	-0.243117243486203	$-1.10118897374539e{-05}$
1	0.243117243486203	$1.10118897373165e{-05}$	0.93800920337446	-0.99999999993937
2	0.0307407965668045	0	0.243127911254386	0
3	0.00229046682911585	0	0.0310811831784831	0
4	0.000168816819622152	0	0.00229081095067003	0
$n=4$				
-4	-0.00268328498886854	0	0.000657980382143995	0
-3	0.00536656997773707	0	-0.00131596076428805	0
-2	0.0406157113141723	0	0.00331184495802566	0
-1	-0.265360110344017	0	0.038527333851246	0
0	0.924852859252655	-0.999996658664916	-0.266548449651388	0.00258508394493862
1	0.266548449651388	-0.00258508394493878	0.924852859252655	-0.999996658664917
2	0.038527333851246	0	0.265360110344017	0
3	-0.00331184495802567	0	0.0406157113141723	0
4	-0.00131596076428804	0	-0.00536656997773707	0
5	-0.000657980382144195	0	-0.00268328498886852	0
$n=5$				
-5	$6.50480918150475e{-07}$	0	$-2.27758865018356e{-07}$	0
-4	-0.00244117863790441	0	0.000854752323034522	0
-3	0.00695514575329282	0	-0.00244026625993947	0
-2	0.0410144854502473	0	0.00439284559696321	0
-1	-0.270014593192369	0	0.041013301010756	0
0	0.922853886195691	-0.999999999999985	-0.26830622303753	$-1.7184062555909e-07$
1	0.26830622303753	$1.71840625647408e{-07}$	0.922853886195691	-0.999999999999985
2	0.041013301010756	0	0.270014593192369	0
3	-0.00439284559696321	0	0.0410144854502473	0
4	-0.00244026625993949	0	-0.00695514575329281	0
5	-0.000854752323034523	0	-0.00244117863790442	0
6	$-2.27758865005465e{-07}$	0	$-6.50480918022666e{-07}$	0

Table 4: Multiwavelet masks $(B_k)_{k\in\mathbb{Z}}$

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