On Multivariate Compactly Supported Bi–Framelets

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Abstract

In this paper, we apply the mixed oblique extension principle to construct compactly supported bi–framelets with an arbitrary dilation matrix. Based on a refinable function which generates a multiresolution analysis, our construction is explicit and leads to wavelets with a high number of vanishing moments. The refinable function is interpolating and the approximation order of the wavelet system reaches the approximation order of the multiresolution analysis. This is the highest possible. We give two classes of examples: the first one yields wavelets for the quincunx dilation matrix. The second class deals with the construction of wavelets from "interpolating versions" of box–spline refinable functions. All wavelets in the examples satisfy various symmetry conditions.

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1 Introduction

Wavelet analysis with its fast algorithms is used in many fields of applied mathematics, such as image and signal analysis.

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A wavelet basis consists of dilates and shifts of a finite number of functions, called wavelets, such that this system builds a basis in $L_2(\mathbb{R}^d)$. In general, the construction of compactly supported wavelets is based on a refinable function which generates a multiresolution analysis. Then, the wavelets are constructed by finite linear combinations of the refinable function, see Section 2.2.

Orthogonal wavelet bases in one dimension have successfully been constructed, see [10]. There, the Fejer–Riesz Lemma was applied, which does not hold in several dimensions. One has overcome this problem of the multivariate setting with the construction of biorthogonal wavelet bases. Moreover, symmetric wavelets could be achieved which is a convenient property in image and signal analysis, see [24]. Nevertheless, in general, the support of the biorthogonal wavelets rapidly grows with increasing smoothness, which seems to be problematic in applications. See for example [13, 18] for the construction of biorthogonal wavelets.

By skipping the geometrical orthogonality and biorthogonality conditions, wavelet frames allow redundancies in the wavelet expansion. This weakened concept, contrarily to unique expansions in the basis setting, offers much more freedom in the construction of wavelets. But, in contrast to the basis setting, smoothness no longer assures a high approximation order. In the frame setting, vanishing moments are a crucial additional ingredient: we say that a wavelet ψ has L **vanishing moments** if

$$\int_{\mathbb{R}^d} x^\alpha \psi(x) dx = 0\,, \quad \text{for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| < L\,.$$

Vanishing moments of the wavelet frame system together with smoothness lead to high approximation order and compression rates, see [11] for details.

Recently, there are many results available about the construction of wavelet frames. Most constructions are based on one of the following extension principles: the unitary (UEP) and the oblique extension principle (OEP) lead to tight wavelet frames. Applying the mixed (MEP) or the mixed oblique extension principle (MOEP) yields pairs of dual wavelet frames, shortly called bi–frames in literature. In one dimension, smooth symmetric tight wavelet frames with a high number of vanishing moments have successfully been constructed by applying these extension principles, see for example [6, 11, 27, 28, 30].

In the multivariate wavelet frame setting, most of the existing non–separable constructions of compactly supported wavelet frames lead to a lack of vanishing moments, see [15, 21, 29] where the UEP is applied. There, the approximation order of the wavelet system is at most two. Moreover, in [21], also the OEP is applied to obtain wavelets with a high number of vanishing moments. However, this is paid for with the loss of compact support. For the very important two–dimensional case of the quincunx dilation matrix, the approach in [17] leads to separable wavelets or to a high number of wavelets which might be problematic in applications. Symmetries are not treated there. In [5], the number of wavelets rapidly increases by growing smoothness. In [14], we could overcome all the problems mentioned above with the construction of bi–frames by applying the MEP. Moreover, the corresponding refinable function is interpolating, see Section 2.2 for details about interpolation and its advantages. Compactly supported smooth and symmetric wavelets are obtained which have a high number of vanishing moments. However, the number of wavelets is $m^2 - 1$ where m is the determinant of the dilation matrix. For $m = 2$, there is no problem at all but a larger m might yield problems in applications. A method to reduce the number of wavelets to $2m-2$ has also been given there but interpolation was lost.

In this paper, we apply the MOEP in the multivariate setting to an arbitrary dilation matrix. We obtain smooth compactly supported bi–framelets whose refinable function is interpolating. Moreover, our constructed wavelets have a high number of vanishing moments. More precisely, the wavelet system reaches the approximation order of the multiresolution analysis which is the highest possible. We need $3m-4$ primal and dual wavelets. Thus, as in [14], the number is independent of the smoothness and the number of vanishing moments. It only depends on the choice of the dilation matrix. In two dimensions with dyadic dilation, we obtain 8 wavelets. The interpolating approach in [14] leads to 15 wavelets. There, the reduced version yields 6 wavelets but, as mentioned above, it also results in the loss of interpolation. Thus, in this paper, we buy interpolation at the price of two wavelets. For dilation matrices with determinant 2, we obtain only 2 wavelets which is already the number of the reduced case in [14]. The bi–framelets in our examples satisfy various symmetry conditions.

As far as we know, this is the first explicit application of an oblique extension principle in several dimensions that leads to non–separable compactly supported wavelets.

The outline of the paper is as follows: in Section 2, we introduce wavelet frames and recall the concept of multiresolution analysis and the MOEP. In Section 3, our construction idea is presented with detailed construction steps. In Section 4, we apply our construction steps to two well–known classes of refinable functions in two dimensions: the first two examples deal with refinable functions for the quincunx dilation. Finally, we construct bi–framelets from "interpolating versions" of three–direction box–splines with equal multiplicities which were obtained in [26].

2 General Setting

In this section, we recall the concept of bi–frames and the well–known multiresolution analysis approach for the construction of wavelets. Finally, we state the mixed oblique extension principle which was obtained in [6, 11].

2.1 Wavelet Frames

We call an integer matrix M a **dilation matrix** if all its eigenvalues are larger than one in modulus. From here on, let M be a dilation matrix throughout the paper with $m := |\det(M)|$. Moreover, let Γ_M with $0 \in \Gamma_M$ and Γ_M^* with $0 \in \Gamma_M^*$ be a complete set of representatives of ${M^t}^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ and $\mathbb{Z}^d/M\mathbb{Z}^d$, respectively.

For a function $\psi \in L_2(\mathbb{R}^d)$ and $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, we define

$$\psi_{j,k}(\cdot) := m^{\frac{j}{2}}\psi(M^j \cdot -k) \; . \qquad (1)$$

We say that $\{\psi^1, \ldots, \psi^r\} \subset L_2(\mathbb{R}^d)$ generates a **wavelet frame** if there exist two positive constants A and B such that

$$A\|f\|_{L_2}^2 \leq \sum_{\mu,j,k} |\langle f|\psi_{j,k}^\mu\rangle_{L_2}|^2 \leq B\|f\|_{L_2}^2 , \quad \text{for all } f \in L_2(\mathbb{R}^d) .$$

The concept of wavelet frames is weaker than classical bases concepts. Thus, it offers more freedom in the construction methods.

To determine the coefficients in a wavelet frame expansion, the following concept is useful: we say that

$$\{(\psi^\mu, \widetilde{\psi}^\mu) \mid \mu = 1, \ldots, r\} \tag{2}$$

generates a **bi–frame** if both sets $\{\psi^\mu \mid \mu = 1, \ldots, r\}$ and $\{\widetilde{\psi}^\mu \mid \mu = 1, \ldots, r\}$ generate a wavelet frame in $L_2(\mathbb{R}^d)$ and the expansion

$$f = \sum_{\mu,j,k} \left\langle f \middle| \widetilde{\psi}_{j,k}^\mu \right\rangle_{L_2(\mathbb{R}^d)} \psi_{j,k}^\mu \tag{3}$$

is valid for every $f \in L_2(\mathbb{R}^d)$. We call $\{\psi^\mu \mid 1, \ldots, r\}$ and $\{\widetilde{\psi}^\mu \mid 1, \ldots, r\}$ the set of **primal** and **dual** wavelets, respectively. The elements are also called **bi–framelets** in literature.

2.2 Multiresolution Analysis and the MOEP

An increasing sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces in $L_2(\mathbb{R}^d)$ is called a **multiresolution analysis** if the following holds:

(i) $f(\cdot) \in V_j$ if and only if $f(M^{-j}\cdot) \in V_0$,

(ii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(iii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}^d)$.

(iv) There exists $\phi \in V_0$ such that V_0 is the closed linear span of its integer shifts.

If ϕ is given, then, by applying (i) and (iv), the sequence $(V_j)_{j\in\mathbb{Z}}$ is already determined. The starting point for our construction is a finite sequence $(a_k^0)_{k\in\mathbb{Z}^d}$ with $\sum_{k\in\mathbb{Z}^d} a_k^0 = m$. It is well–known that there exists a unique tempered distribution ϕ with compact support that is **refinable** with respect to $(a_k^0)_{k\in\mathbb{Z}^d}$ which means it satisfies the **refinement equation**

$$\phi(\cdot) = \sum_{k\in\mathbb{Z}^d} a_k^0 \phi(M\cdot - k) . \tag{4}$$

The Fourier transform of this unique solution ϕ can be obtained directly by an infinite product, see for example [10]. The following has been shown in [12]: if ϕ is refinable and not only a tempered distribution but contained in $L_2(\mathbb{R}^d)$, then it already generates a multiresolution analysis.

In classical wavelet bases constructions, based on a multiresolution analysis, the closed linear span of the wavelets is an algebraic complement of V_0 in V_1. Thus, the wavelets are contained in

V_1, which implies that they can be represented by linear combinations of $\phi(M\cdot -k)$, for $k \in \mathbb{Z}^d$. In the wavelet frame setting, we still use this idea: we start with two refinable functions ϕ and $\widetilde{\phi}$ contained in $L_2(\mathbb{R}^d)$ which are given implicitly by two finite sequences $(a_k^0)_{k\in\mathbb{Z}^d}$ and $(b_k^0)_{k\in\mathbb{Z}^d}$. For other finite sequences $(a_k^1)_{k\in\mathbb{Z}^d}$, $(b_k^1)_{k\in\mathbb{Z}^d}$, ..., $(a_k^r)_{k\in\mathbb{Z}^d}$, $(b_k^r)_{k\in\mathbb{Z}^d}$, the wavelets are constructed by

$$\psi^\mu(\cdot) := \sum_{k\in\mathbb{Z}^d} a_k^\mu \phi(M\cdot -k) \quad \text{and} \quad \widetilde{\psi}^\mu(\cdot) := \sum_{k\in\mathbb{Z}^d} b_k^\mu \widetilde{\phi}(M\cdot -k)\ . \tag{5}$$

Thus, $\psi^\mu, \widetilde{\psi}^\mu \in V_1$ for all $\mu = 1, \ldots, r$. Then, one needs conditions on those finite sequences such that the system

$$\{(\psi^\mu, \widetilde{\psi}^\mu) \mid \mu = 1, \ldots, r\} \tag{6}$$

generates a bi–frame. For this, some preparations are necessary: we define the Fourier transform of a function $f \in L_1(\mathbb{R}^d)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e_{-\xi}(x)\, dx\,, \quad \text{for } \xi \in \mathbb{R}^d\,,$$

where

$$e_\omega(x) := e^{2\pi i\, \omega\cdot x}\,, \quad \text{for } \omega, x \in \mathbb{R}^d.$$

For a finite sequence $(a_k)_{k\in\mathbb{Z}^d}$, we call the trigonometric polynomial

$$a := \frac{1}{m} \sum_{k\in\mathbb{Z}^d} a_k e_{-k}$$

a **symbol** and the finite sequence $(a_k)_{k\in\mathbb{Z}^d}$ the **mask** or the **filter**.

Let θ be a trigonometric polynomial with $\theta(0) = 1$. We say that the symbol family $\{(a^\mu, b^\mu) \mid \mu = 0, \ldots, r\}$ satisfies **condition (I)** with respect to θ if the following conditions hold:

(a) $a^0(0) = b^0(0) = 1$.

(b) $a^\mu(0) = b^\mu(0) = 0$, for all $\mu = 1, \ldots, r$.

(c) For all $\gamma \in \Gamma_M$,

$$a^0(\cdot)\overline{b^0(\cdot+\gamma)}\theta(M^t\cdot) + \sum_{\mu=1}^{r} a^\mu \overline{b^\mu(\cdot+\gamma)} = \theta(\cdot)\delta_{0,\gamma}\ . \tag{7}$$

Our bi–frame construction is based on the following theorem that has been shown in [6] and [11]. Implicitly, it goes back to the fundamental work in [27]. We state it here with slightly changed assumptions, see [2] and [17] for their validity.

Theorem 2.1 (MOEP). *Let θ be a symbol with $\theta(0) = 1$ and let the symbol family $\{(a^\mu, b^\mu) \mid \mu = 0, \ldots, r\}$ satisfy condition* (I) *with respect to θ. Furthermore, let ϕ and $\widetilde{\phi}$, contained in $L_2(\mathbb{R}^d)$, be refinable with respect to a^0 and b^0, respectively. For $\mu = 1, \ldots, r$, we define*

$$\psi^\mu(\cdot) := \sum_{k\in\mathbb{Z}^d} a_k^\mu \phi(M\cdot -k) \quad \textit{and} \quad \widetilde{\psi}^\mu(\cdot) := \sum_{k\in\mathbb{Z}^d} b_k^\mu \widetilde{\phi}(M\cdot -k)\ . \tag{8}$$

Then, the set

$$\{(\psi^\mu, \widetilde{\psi}^\mu) \mid \mu = 1, \ldots, r\}$$

generates a bi–frame.

For $\theta = 1$, Theorem 2.1 is called the MEP, see [27].

Our construction of bi–frames will extremely be simplified by dealing with **interpolating** refinable functions ϕ, which means that ϕ is continuous and

$$\phi(k) = \delta_{0,k} \quad \text{for all } k \in \mathbb{Z}^d \ .$$

Beside the convenience for our construction, interpolation itself is a desired property: let a function f be a linear combination of the integer shifts of ϕ, for example $f \in V_0$,

$$f(\cdot) = \sum_{k \in \mathbb{Z}^d} \lambda_k \phi(\cdot + k) \ .$$

Then, the coefficients are already given by the sample values $\lambda_k = f(k)$. For example, this extremely simplifies preprocessing in filter bank applications, see [24].

The key ingredient for our construction is that the symbol $a = a^0$ satisfies the **interpolation condition** which means

$$\sum_{\gamma \in \Gamma_M} a(\cdot + \gamma) = 1 \ . \tag{9}$$

The relation of symbol and refinable function with respect to interpolation is given by the following theorem in [22].

Theorem 2.2. *A continuous refinable function ϕ corresponding to a symbol a with $a(0) = 1$ is interpolating if and only if ϕ has stable integer shifts and a satisfies the interpolation condition* (9)*.*

To obtain wavelets with a high number of vanishing moments, the following preparations are useful: first, we rewrite (8) in terms of its Fourier transform. We obtain

$$\widehat{\psi^\mu}(\cdot) = a^\mu(M^{t^{-1}} \cdot)\, \widehat{\phi}(M^{t^{-1}} \cdot) \quad \text{and} \quad \widehat{\widetilde{\psi}^\mu}(\cdot) = b^\mu(M^{t^{-1}} \cdot)\, \widehat{\widetilde{\phi}}(M^{t^{-1}} \cdot) \ . \tag{10}$$

We say that a symbol a satisfies the **Strang Fix conditions** of order L if

$$\partial^\alpha a(\gamma) = 0 \,, \quad \text{for all } \gamma \in \Gamma_M \backslash \{0\} \text{ and } \alpha \in \mathbb{N}^d, |\alpha| < L \ .$$

Moreover, we say that a symbol a has L **vanishing moments** if

$$\partial^\alpha a(0) = 0 \,, \quad \text{for all } \alpha \in \mathbb{N}^d, |\alpha| < L \ .$$

In the following sections, we will see that Strang Fix conditions are a crucial property for the symbols of refinable functions. In our construction, it will induce vanishing moments of wavelet symbols which imply vanishing moments of the corresponding wavelets by (10).

3 Construction of Oblique Wavelet Frames

In this section, we show how to use the interpolation condition of a symbol for the construction of bi–frames with a high number of vanishing moments. The general idea is the following: from a given starting symbol corresponding to a refinable function, we construct wavelet symbols. Under certain assumptions on the starting symbol, Strang Fix conditions transform into vanishing moments of the wavelet symbols. In our construction, the key ingredient that makes this transformation work is the interpolation condition. Some wavelet symbols are constructed by a polyphase approach where the interpolation condition provides the existence of a certain matrix extension. The rest of the wavelet symbols is constructed by splitting a trigonometric polynomial into a sum of products. There, the interpolation condition yields vanishing moments of this sum.

3.1 Oblique Polyphase Conditions

In this section, we express Equation (7) in polyphase terms which we will apply in the further construction. Therefore, we need some notations and auxiliary results.

To simplify notation in the following, we denote $\Gamma_M = \{\gamma_0, \dots, \gamma_{m-1}\}$ as well as $\Gamma_M^* = \{\gamma_0^*, \dots, \gamma_{m-1}^*\}$ with $\gamma_0 = \gamma_0^* = 0$. Then, for the symbols a^μ and b^μ for $\mu = 0, \dots, r$, we define the **modulation matrices** by

$$\mathbf{a} := \left(a^\mu(\cdot + \gamma_\nu) \right)_{\substack{\nu=0,\dots,m-1 \\ \mu=0,\dots,r}} \quad \text{and} \quad \mathbf{b} := \left(b^\mu(\cdot + \gamma_\nu) \right)_{\substack{\nu=0,\dots,m-1 \\ \mu=0,\dots,r}} .$$

In the following, we split a symbol into smaller symbols depending on the dilation matrix M. For a given symbol a and $\gamma^* \in \Gamma_M^*$, we call

$$A_{\gamma^*} := \sum_{k \in \mathbb{Z}^d} a_{Mk+\gamma^*} e_{-k}$$

the (γ^*–)**subsymbol** of a. For $k \in \mathbb{Z}^d$, a well–known lemma about character sums states that

$$\sum_{\gamma \in \Gamma_M} e_k(\gamma) = \begin{cases} m\,, & \text{if} \quad k \in M\mathbb{Z}^d\,, \\ 0\,, & \text{otherwise} \end{cases} . \tag{11}$$

The following relations can easily be obtained by applying (11):

$$A_{\gamma^*}(M^t \cdot) = \sum_{\gamma \in \Gamma_M} e_{\gamma^*}(\cdot + \gamma)\, a(\cdot + \gamma) \tag{12}$$

and

$$a(\cdot) = \frac{1}{m} \sum_{\gamma^* \in \Gamma_M^*} A_{\gamma^*}(M^t \cdot)\, e_{-\gamma^*}(\cdot)\,. \tag{13}$$

For two given symbol families a^μ and b^μ, for $\mu = 0, \dots, r$, we denote the corresponding subsymbols by $A^\mu_{\gamma^*_\nu}$ and $B^\mu_{\gamma^*_\nu}$, respectively. Then, the two matrices

$$\mathbf{A} = \left(A^\mu_{\gamma^*_\nu} \right)_{\substack{\nu=0,\dots,m-1 \\ \mu=0,\dots,r}} \quad \text{and} \quad \mathbf{B} = \left(B^\mu_{\gamma^*_\nu} \right)_{\substack{\nu=0,\dots,m-1 \\ \mu=0,\dots,r}}$$

are called the primal and dual, respectively, **polyphase** matrices corresponding to this symbol families.

For $\theta = 1$, the following theorem is already well–known in the setting of biorthogonal wavelet constructions, see for example [19]. We generalize the equivalence between modulation and polyphase matrices to arbitrary symbols θ. The subsymbols of θ are denoted by Θ_{γ^*}:

Theorem 3.1. *Given a symbol family $\{(a^\mu, b^\mu) \mid \mu = 0, \dots, r\}$ and a symbol θ, then*

$$\mathbf{a}(\cdot)\overline{\mathbf{b}}^t(\cdot) = \mathrm{diag}(\theta(\cdot), \dots, \theta(\cdot + \gamma_{m-1})) \qquad \text{iff} \qquad \mathbf{A}\overline{\mathbf{B}}^t = \left(\Theta_{\gamma^*_\nu - \gamma^*_\mu}\right)_{\substack{\nu=0,\dots,m-1\\ \mu=0,\dots,m-1}} . \tag{14}$$

Proof. With the matrix

$$U := \left(e_{-\gamma^*_\mu}(\cdot + \gamma_\nu) \right)_{\substack{\nu=0,\dots,m-1\\ \mu=0,\dots,m-1}} \tag{15}$$

and by applying (13), we have

$$\mathbf{a}(\cdot) = \frac{1}{m} U(\cdot)\mathbf{A}(M^t\cdot) \qquad \text{and} \qquad \mathbf{b}(\cdot) = \frac{1}{m} U(\cdot)\mathbf{B}(M^t\cdot) . \tag{16}$$

This leads to

$$\mathbf{a}(\cdot)\overline{\mathbf{b}}^t(\cdot) = \frac{1}{m^2} U(\cdot)\mathbf{A}(M^t\cdot)\overline{\mathbf{B}}^t(M^t\cdot)\overline{U}^t(\cdot) .$$

Thus, the left hand side of (14) is equivalent to

$$U(\cdot)\mathbf{A}(M^t\cdot)\overline{\mathbf{B}}^t(M^t\cdot)\overline{U}^t(\cdot) = m^2\, \mathrm{diag}(\theta(\cdot), \dots, \theta(\cdot + \gamma_{m-1})) .$$

By applying $U\overline{U}^t = m\, I_m$, this is equivalent to

$$\begin{aligned} \mathbf{A}(M^t\cdot)\overline{\mathbf{B}}^t(M^t\cdot) &= \overline{U}^t(\cdot)\, \mathrm{diag}(\theta(\cdot), \dots, \theta(\cdot + \gamma_{m-1}))U(\cdot) \\ &= \left(\sum_{\gamma \in \Gamma_M} e_{\gamma^*_\nu - \gamma^*_\mu}(\cdot + \gamma)\theta(\cdot + \gamma) \right)_{\substack{\nu=0,\dots,m-1\\ \mu=0,\dots,m-1}} \\ &= \left(\Theta_{\gamma^*_\nu - \gamma^*_\mu}(M^t\cdot)\right)_{\substack{\nu=0,\dots,m-1\\ \mu=0,\dots,m-1}} , \end{aligned}$$

where we have applied (12) to obtain the last equation. Now, the invertibility of M^t leads to the equivalence in (14). $\square$

Applying the group structure of Γ_M, it can be verified that (7) is equivalent to

$$\mathbf{a}\overline{\mathbf{b}}^t = \mathrm{diag}(\theta(\cdot), \dots, \theta(\cdot + \gamma_{m-1}))$$

where we have replaced b^0 by $b^0(\cdot)\theta(M^t\cdot)$ in $\overline{\mathbf{b}}^t$.

Remark 3.2. By a straightforward computation, one obtains that the subsymbols of $b^0(\cdot)\theta(M^t\cdot)$ are given by $B^0_{\gamma^*}\theta$, for $\gamma^* \in \Gamma^*_M$. Thus, by applying Theorem 3.1, we obtain an equivalent version of (7) in terms of subsymbols:

$$(7) \quad \text{iff} \quad A^0_{\gamma^*_\nu}\overline{B^0_{\gamma^*_\mu}}\theta + \sum_{j=1}^{r} A^j_{\gamma^*_\nu}\overline{B^j_{\gamma^*_\mu}} = \Theta_{\gamma^*_\nu - \gamma^*_\mu} \quad \text{for } \nu, \mu = 0, \dots, m-1. \tag{17}$$

As far as we know, the Equivalence (17) has not yet been stated elsewhere.

It does not seem to be of advantage to change into the polyphase setting because the right hand side of (14) seems to be quite complicated. Nevertheless, the following corollary deals with the natural restriction of the function θ to obtain a simplified version.

Corollary 3.3. *Let $\{(a^\mu, b^\mu) \mid \mu = 0, \dots, r\}$ be a symbol family and let θ be Γ_M–periodical, then* (14) *simplifies to*

$$\mathbf{a}\overline{\mathbf{b}}^t = \theta \cdot I_m \qquad \text{iff} \qquad \mathbf{A}\overline{\mathbf{B}}^t = \Theta_0 \cdot I_m \ . \tag{18}$$

Proof. The set Γ_M^* is a complete set of representatives of $\mathbb{Z}^d / M\mathbb{Z}^d$. Thus,

$$\gamma_\nu^* - \gamma_\mu^* \in M\mathbb{Z}^d \quad \text{iff} \quad \nu = \mu \ . \tag{19}$$

By (12), the Γ_M–periodicy of θ and (11), we obtain

$$\begin{aligned} \Theta_{\gamma_\nu^* - \gamma_\mu^*}(M^t \cdot) &= \sum_{\gamma \in \Gamma_M} e_{\gamma_\nu^* - \gamma_\mu^*}(\cdot + \gamma) \theta(\cdot + \gamma) \\ &= \theta(\cdot) e_{\gamma_\nu^* - \gamma_\mu^*}(\cdot) \sum_{\gamma \in \Gamma_M} e_{\gamma_\nu^* - \gamma_\mu^*}(\gamma) \\ &= m\theta(\cdot)\delta_{\nu,\mu} \ . \end{aligned} \tag{20}$$

Thus, Theorem 3.1 implies (18). □

3.2 Interpolation Property and the Oblique Construction

In this section, we present our construction principle which is based on a multivariate starting symbol satisfying the interpolation condition. We use the right hand side of (18) for the construction of wavelet symbols. At first, we explain the general construction idea. It is an analogy of the one–dimensional construction in [6] and [11], where it is applied to reduce the number of wavelets. See also Corollary 14.8.3 in [4].

Theorem 3.4. *Let $\{(a^\mu, b^\mu) \mid \mu = 0, \dots, m-1\}$ be a symbol family with $a^0(0) = b^0(0) = 1$ and let us denote*

$$\theta(\cdot) := \sum_{\gamma \in \Gamma_M} \left(a^0 \overline{b^0} \right)(\cdot + \gamma) \qquad \text{and} \qquad \eta := 1 - \theta \ . \tag{21}$$

We assume that

(1) $\mathbf{a}\overline{\mathbf{b}}^t = \theta I_m$ *(notify that* $\mathbf{a}$ *and* $\mathbf{b}$ *are square),*

(1*) $a^1, b^1, \dots, a^{m-1}, b^{m-1}$ *have* $L_1 \geq 0$ *vanishing moments,*

(2) *there exist* $\eta_1, \widetilde{\eta}_1, \dots, \eta_n, \widetilde{\eta}_n$ *such that* $\eta = \sum_{\nu=1}^n \eta_\nu \overline{\widetilde{\eta}_\nu}$,

(2*) $\eta_1, \widetilde{\eta}_1, \dots, \eta_n, \widetilde{\eta}_n$ *have* $L_2 \geq 0$ *vanishing moments.*

By defining

$$a^{m-1+\nu}(\cdot) := \eta_\nu(M^t\cdot)a^0(\cdot) \qquad \textit{and} \qquad b^{m-1+\nu}(\cdot) := \widetilde{\eta}_\nu(M^t\cdot)b^0(\cdot)\ , \tag{22}$$

for $\nu = 1, \ldots, n$*, the symbol family* $\{(a^\mu, b^\mu) \mid \mu = 0, \ldots, m-1+n\}$ *satisfies condition* (I) *with respect to* θ*. All wavelet symbols have at least* $\min\{L_1, L_2\}$ *vanishing moments.*

The choice of θ and η in (21) is only a special case of the idea in [4, 6, 11]. But there, only the case $M = 2$, which implies $m = 2$, as well as $n = 1$ in (1) and (2) of Theorem 3.4 has been treated. We have generalized the construction idea to arbitrary dilation matrices M, which leads to $m \geq 2$. Instead of assuming a direct factorization, we allow the more general case of a sum of products in (2).

Proof. By applying $\eta = 1 - \theta$, we obtain

$$\begin{aligned}
&a^0(\cdot)\overline{b^0(\cdot+\gamma)}\theta(M^t\cdot) + \sum_{\mu=1}^{m-1+n} a^\mu(\cdot)\overline{b^\mu(\cdot+\gamma)} \\
&\quad= a^0(\cdot)\overline{b^0(\cdot+\gamma)}\theta(M^t\cdot) + \sum_{\mu=1}^{m-1} a^\mu(\cdot)\overline{b^\mu(\cdot+\gamma)} + \sum_{\nu=1}^{n} a^0(\cdot)\overline{b^0(\cdot+\gamma)}\eta_\nu(M^t\cdot)\overline{\widetilde{\eta}_\nu(M^t\cdot)} \\
&\quad= a^0(\cdot)\overline{b^0(\cdot+\gamma)}\theta(M^t\cdot) + \sum_{\mu=1}^{m-1} a^\mu(\cdot)\overline{b^\mu(\cdot+\gamma)} + a^0(\cdot)\overline{b^0(\cdot+\gamma)}\eta(M^t\cdot) \\
&\quad= a^0(\cdot)\overline{b^0(\cdot+\gamma)} + \sum_{\mu=1}^{m-1} a^\mu(\cdot)\overline{b^\mu(\cdot+\gamma)} \\
&\quad= \theta(\cdot)\delta_{0,\gamma}\ .
\end{aligned}$$

The number of vanishing moments are the direct consequence of the assumptions (1*) and (2*). ☐

To construct bi–frames, we would like to apply Theorem 3.4. The idea is that we only need the symbol a^0 with nice properties. Then, we are able to construct the rest of the symbols by using this starting symbol a^0. It is worth mentioning that the stepwise restrictions we make in the following still leave us a lot of freedom. To show the dependencies directly, we will proceed in the restriction process step by step. First, to use the right hand side of (18), we need a representation of Θ_0 in terms of subsymbols of $a = a^0$ and $b = b^0$.

Lemma 3.5. *Let* θ *be given by* (21)*. Then,*

$$\Theta_0 = \sum_{\gamma^* \in \Gamma_M^*} A_{\gamma^*}\overline{B_{\gamma^*}}\ .$$

Proof. Applying (20) for $\mu = \nu$ and (13), we obtain

$$\begin{aligned}
\Theta_0(M^t\cdot) &= m\theta(\cdot) \\
&= m \sum_{\gamma\in\Gamma_M} a(\cdot+\gamma)\overline{b(\cdot+\gamma)} \\
&= m \sum_{\gamma\in\Gamma_M} \frac{1}{m} \sum_{\gamma^*\in\Gamma_M^*} A_{\gamma^*}(M^t\cdot)e_{-\gamma^*}(\cdot+\gamma)\frac{1}{m} \sum_{\widetilde{\gamma}^*\in\Gamma_M^*} \overline{B_{\widetilde{\gamma}^*}(M^t\cdot)} e_{\widetilde{\gamma}^*}(\cdot+\gamma) \\
&= \frac{1}{m} \sum_{\gamma^*,\widetilde{\gamma}^*} A_{\gamma^*}(M^t\cdot)\overline{B_{\widetilde{\gamma}^*}(M^t\cdot)} \sum_{\gamma\in\Gamma_M} e_{\widetilde{\gamma}^*-\gamma^*}(\cdot+\gamma)\ .
\end{aligned}$$

Applying (11) yields

$$\begin{aligned}
\Theta_0(M^t\cdot) &= \frac{1}{m} \sum_{\gamma^*,\widetilde{\gamma}^*} A_{\gamma^*}(M^t\cdot)\overline{B_{\widetilde{\gamma}^*}(M^t\cdot)}\, m\, \delta_{\widetilde{\gamma}^*,\gamma^*} \\
&= \sum_{\gamma^*\in\Gamma_M^*} A_{\gamma^*}(M^t\cdot)\overline{B_{\gamma^*}(M^t\cdot)}\ .
\end{aligned}$$

□

The constructed symbols in the following lemma are a generalized result of the wavelet construction algorithm given in [19] for interpolating refinable functions. There, from a given interpolating refinable function and its dual, which means $\theta = 1$ in (21), a biorthogonal wavelet basis has been constructed. See also [14] for a summary. We generalize this idea to θ not necessarily identical to 1. Vanishing moments are assured by applying a result in [3].

Lemma 3.6 (Step 1). *Under the notation of Theorem* 3.4*, let the symbols a^0 and b^0 satisfy $a^0(0) = b^0(0) = 1$ as well as the Strang Fix conditions of order $2L \geq 0$. Moreover, let a^0 satisfy the interpolation condition. For $\mu = 1, \dots, m-1$, we define*

$$a^\mu(\cdot) \;=\; e_{-\gamma_\mu^*}(\cdot) \sum_{\gamma\in\Gamma_M\setminus\{0\}} \left(a^0(\cdot+\gamma) - a^0(\cdot)e_{-\gamma_\mu^*}(\gamma)\right) \overline{b^0(\cdot+\gamma)}\ , \tag{23}$$

$$b^\mu(\cdot) \;=\; \frac{1}{m} e_{-\gamma_\mu^*}(\cdot) \sum_{\gamma\in\Gamma_M\setminus\{0\}} \left(1 - e_{-\gamma_\mu^*}(\gamma)\right) \overline{a^0(\cdot+\gamma)}\ . \tag{24}$$

Then, the corresponding modulation matrices **a** *and* **b** *satisfy* (1) *in Theorem* 3.4 *and* (1*) *for $L_1 = 2L$.*

Proof. Let us define the following polyphase matrices,

$$\mathbf{A} := \begin{pmatrix} A_0 & -\overline{B_{\gamma_1^*}} & & \dots & & -\overline{B_{\gamma_{m-1}^*}} \\ \vdots & & & & & \\ \vdots & & & \left(\Theta_0 I_{m-1} - (A_{\gamma_\nu^*}\overline{B_{\gamma_\mu^*}})_{\substack{\nu=1,\dots,m-1\\ \mu=1,\dots,m-1}}\right) & & \\ \vdots & & & & & \\ A_{\gamma_{m-1}^*} & & & & & \end{pmatrix}$$

and

$$\overline{\mathbf{B}}^t := \begin{pmatrix} \overline{B_0} & \overline{B_{\gamma_1^*}} & \dots & \overline{B_{\gamma_{m-1}^*}} \\ -A_{\gamma_1^*} & & & \\ \vdots & & I_{m-1} & \\ -A_{\gamma_{m-1}^*} & & & \end{pmatrix} .$$

By these polyphase matrices, we implicitly define the symbols $a^1, \dots, a^{m-1}$ as well as $b^1, \dots, b^{m-1}$. Applying (12) and (13), a simple computation leads to

$$\begin{aligned} a^\mu(\cdot) &= e_{-\gamma_\mu^*}(\cdot)\theta(\cdot) - a^0(\cdot) \sum_{\gamma\in\Gamma_M} \left(e_{-\gamma_\mu^*}\overline{b^0}\right)(\cdot+\gamma) , \\ b^\mu(\cdot) &= \frac{1}{m}\left(e_{-\gamma_\mu^*}(\cdot) - \sum_{\gamma\in\Gamma_M} \left(e_{-\gamma_\mu^*}\overline{a^0}\right)(\cdot+\gamma)\right) . \end{aligned}$$

It can easily be verified that the definition of θ in (21) leads to the representation (23) as well as the interpolation condition for $\overline{a^0}$ yields (24).
By Lemma 3.5, we know that

$$\Theta_0 = \sum_{\gamma^*\in\Gamma_M^*} A_{\gamma^*}\overline{B_{\gamma^*}} .$$

The symbol a^0 satisfies the interpolation condition. It is well–known that this implies $A_0 = 1$, see for example [9]. Thus, a straightforward computation yields

$$\mathbf{A}\overline{\mathbf{B}}^t = \Theta_0 \cdot I_m . \tag{25}$$

Obviously, θ given by (21) is Γ_M–periodical. Thus, applying Corollary 3.3 yields

$$\mathbf{a}\overline{\mathbf{b}}^t = \theta \cdot I_m \tag{26}$$

for the corresponding modulation matrices **a** and **b**. We obtain vanishing moments directly by applying the Strang Fix conditions to (23) and (24). The following approach is independent of the concret representations: both modulation matrices **a** and **b** are square. By applying the Strang Fix conditions to (21), we can conclude $\theta(0) = 1$. Thus, θ is not identical to zero, which implies

$$\mathbf{a}^t\overline{\mathbf{b}} = \theta \cdot I_m .$$

Then, by a result in [3], the wavelet symbols $a^1, \dots, a^{m-1}$, and $b^1, \dots, b^{m-1}$ have $2L$ vanishing moments. □

The following lemma shows that the interpolation condition yields the potential of vanishing moments in our construction.

Lemma 3.7. *Under the assumptions of Lemma* 3.6*, let also b^0 satisfy the interpolation condition. Then, η has $2L$ vanishing moments.*

Proof. Applying the interpolation condition, we obtain

$$\begin{aligned} 1 &= \sum_{\gamma\in\Gamma_M} a^0(\cdot+\gamma) \sum_{\tilde\gamma\in\Gamma_M} \overline{b^0(\cdot+\tilde\gamma)} \\ &= \sum_{\gamma\in\Gamma_M} \left(a^0\overline{b^0}\right)(\cdot+\gamma) + \sum_{\gamma\neq\tilde\gamma} a^0(\cdot+\gamma)\overline{b^0(\cdot+\tilde\gamma)} \, . \end{aligned}$$

Thus,

$$\eta = \sum_{\gamma\neq\tilde\gamma} a^0(\cdot+\gamma)\overline{b^0(\cdot+\tilde\gamma)} \, . \tag{27}$$

Applying the Strang Fix conditions, we obtain that η has $2L$ vanishing moments. □

By (27), we already have a sum of products but it does not lead to vanishing moments because $a^0(0) = 1 = b^0(0)$. To obtain η_ν and $\widetilde{\eta}_\nu$ with a high number of vanishing moments, we need that a^0 can be factorized:

Lemma 3.8 (Step 2). *Under the assumptions of Lemma* 3.6*, let* $b^0 = \overline{a^0}$*. Moreover, let there exist symbols* c *and* d *satisfying the Strang Fix conditions of order* L *such that* $a^0 = c\overline{d}$*. For* $\mu = 1,\dots,m-1$ *and* $\nu = 1,\dots,m-2$*, we define*

$$\begin{aligned} \eta_\mu &:= c(\cdot)\overline{d(\cdot+\gamma_\mu)} \, , & \widetilde{\eta}_\mu &:= 2d(\cdot)\overline{c(\cdot+\gamma_\mu)} \, , \\ \eta_{m-1+\nu} &:= a^0(\cdot+\gamma_\nu) \, , & \widetilde{\eta}_{m-1+\nu} &:= 2\sum_{j=\nu+1}^{m-1} \overline{a^0(\cdot+\gamma_j)} \, . \end{aligned}$$

Then, (2) *and* (2*) *in Theorem* 3.4 *are satisfied with* $n = 2m-3$ *and* $L_2 = L$*.*

Proof. By $b^0 = \overline{a^0}$, the assumptions of Lemma 3.7 are satisfied. Thus, we can apply (27), which leads to the following sum of products:

$$\begin{aligned} \eta &= \sum_{\gamma\neq\tilde\gamma} (c\overline{d})(\cdot+\gamma)(\overline{d}c)(\cdot+\tilde\gamma) \\ &= 2 \sum_{0\le i<j\le m-1} c(\cdot+\gamma_i)\overline{d(\cdot+\gamma_j)d(\cdot+\gamma_i)}c(\cdot+\gamma_j) \\ &= \sum_{\mu=1}^{m-1} c(\cdot)\overline{d(\cdot+\gamma_\mu)}2\overline{d(\cdot)}c(\cdot+\gamma_\mu) + \sum_{\nu=1}^{m-2} c(\cdot+\gamma_\nu)\overline{d(\cdot+\gamma_\nu)}2\sum_{j=\nu+1}^{m-1} c(\cdot+\gamma_j)\overline{d(\cdot+\gamma_j)} \, . \end{aligned}$$

□

Step 1 and step 2 lead us to all ingredients of Theorem 3.4 that yields $3m-4$ wavelets which have at least L vanishing moments.

Remark 3.9. The assumption about the factorization of the symbol a^0 is not as restrictive as it seems. Most of the well–known symbols that satisfy the interpolation condition are obtained by some iteration process. Thus, the symbols c and d are already known.

3.3 Approximation Order

In this section, we recall the concept of approximation order, as discussed in [11]. The following definitions can only be applied to isotropic dilation matrices. Thus, we assume from here on that the dilation matrix M is isotropic, which is also valid in our examples of the last section.

Let λ be the modulus of the eigenvalues of M. We say that a multiresolution analysis $(V_j)_{j\in\mathbb{Z}}$ provides **approximation order** L if, for every f in the Sobolev space $W^L(L_2(\mathbb{R}^d))$,

$$\operatorname{dist}(f, V_n) := \min\{\|f - g\|_{L_2(\mathbb{R}^d)} \mid g \in V_n\} = O(\lambda^{-nL}) .$$

We say that the bi–frame $\{(\psi^\mu, \widetilde{\psi}^\mu) \mid \mu = 1, \ldots, r\}$ provides **approximation order** L if, for all $f \in W^L(L_2(\mathbb{R}^d))$,

$$\|f - Q_n(f)\|_{L_2(\mathbb{R}^d)} = O(\lambda^{-nL}) ,$$

where the truncated representation Q_n is given by

$$Q_n(f) := \sum_{\substack{1\le\mu\le r \\ k\in\mathbb{Z}^d,\ j<n}} \left\langle f \middle| \widetilde{\psi}^\mu_{j,k} \right\rangle \psi^\mu_{j,k}\,, \quad \text{for all } f \in L_2(\mathbb{R}^d).$$

We summarize the following results from [11, 20, 23], see also [14] for a more detailed summary.

Theorem 3.10. *Let the dilation matrix M be isotropic and let the symbol family $\{(a^\mu, b^\mu) \mid \mu = 0, \ldots, r\}$ lead to a bi–frame by applying the* MOEP*. Let a^0 satisfy the Strang Fix conditions of order L_0. We assume further that, for $\mu = 1, \ldots, r$, all products $a^\mu b^\mu$ have $2L_1$ vanishing moments. Then*

(a) *the multiresolution analysis generated by ϕ provides approximation order L_0,*

(b) *the approximation order L of the bi–frame satisfies*

$$\min\{L_0, 2L_1\} \le L \le L_0 . \tag{28}$$

In all our examples of the following section, both kinds of approximation order coincide. Thus, the approximation order of the bi–frame reaches that of the multiresolution analysis. Due to (28), this is the highest possible.

4 Examples of Interpolating Bi–Frames

In the following, we give two concrete examples for the construction of bi–frames by applying step 1 and step 2. To simplify notation, we introduce the well–known z–notation: let $z_1 := e_{-1}(\xi_1)$ and $z_2 := e_{-1}(\xi_2)$. The smoothness estimates of the quincunx examples were done by an implementation of the Villemoes algorithm, see [1, 31].

Example 4.1 (Quincunx A). For the quincunx dilation matrix

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} ,$$

the Laplace symbol

$$c := \frac{1}{2}\left(1 + \frac{1}{4}z_1 + \frac{1}{4}\frac{1}{z_1} + \frac{1}{4}z_2 + \frac{1}{4}\frac{1}{z_2}\right)$$

satisfies the Strang Fix conditions of order 2 as well as the interpolation condition, see [7]. Let d be the symbol given by the equation

$$d = c(3 - 2c) .$$

Then, c as well as d is real–valued, and it is well–known that $a^0 = cd$ satisfies the interpolation condition, see [10, 13, 14]. Hence, according to step 2, we define $b^0 = a^0$. Applying $\gamma = (\frac{1}{2}, \frac{1}{2})^t$ and $\gamma^* = (1, 0)^t$, it can easily be verified that step 1 leads to

$$a^1(\cdot) = e_{-\gamma^*}(\cdot)\, a^0(\cdot + \gamma) \quad \text{and} \quad b^1 = a^1 .$$

Furthermore, by applying step 2, we obtain

$$\eta_1(\cdot) = c(\cdot)d(\cdot + \gamma) \quad \text{and} \quad \widetilde{\eta}_1(\cdot) = 2d(\cdot)c(\cdot + \gamma) .$$

This leads to

$$a^2(\cdot) = c(M^t\cdot)d(M^t \cdot + \gamma)a^0(\cdot) \quad \text{and} \quad b^2(\cdot) = 2d(M^t\cdot)c(M^t \cdot + \gamma)a^0(\cdot) .$$

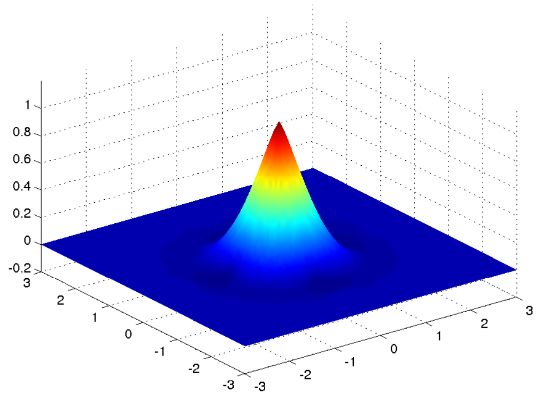
Primal and dual functions are contained in the Hölder class 1.3 and all wavelets have at least 2 vanishing moments. The approximation order of the bi–frame is 4. We define

$$\Diamond_s := \{k \in \mathbb{Z}^2 \mid |k_1| + |k_2| \leq s\} ,$$

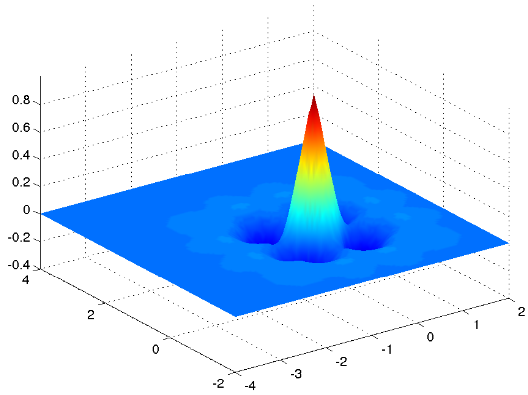
for $s \geq 0$. Then, the mask of a^0 is contained in $\Diamond_3$ and the mask of a^1 in $(1, 0)^t + \Diamond_3$. Both masks of a^2 and b^2 are contained in $\Diamond_9$. Due to the symmetry properties of the masks, the refinable function as well as all wavelets are symmetric modulo translation to the four lines $x_1 = 0$, $x_2 = 0$, $x_1 = x_2$, and $x_1 = -x_2$, see [16] for details. See Figure 1 for the plotted functions.

For the quincunx dilation matrix, another interpolating refinable function has been obtained in [8]:

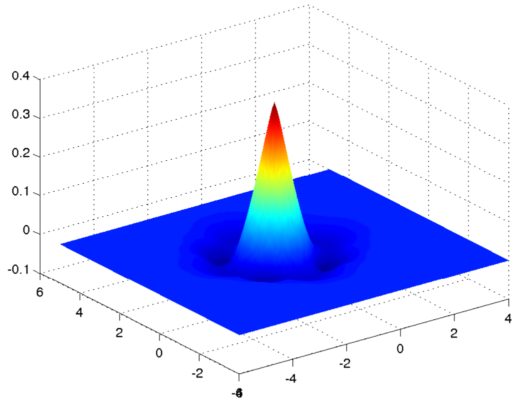
Example 4.2 (Quincunx B). Instead of the Laplace symbol, we apply the symbol c given by the mask in Figure 2. It satisfies the interpolation condition and the Strang Fix conditions of order 4. By following the lines in Example 4.1, we obtain two primal and two dual wavelets that are contained in the Hölder class 2.9 and they inherit the same symmetry properties as those in Example 4.1. All wavelets have at least 4 vanishing moments and the approximation order of the bi–frame is 8.



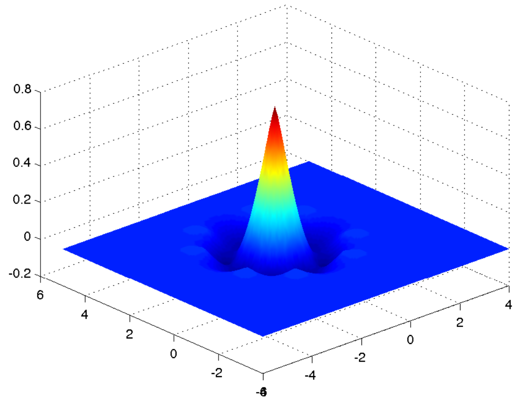
(a) $\phi = \widetilde{\phi}$



(b) $\psi^1 = \widetilde{\psi}^1$



(c) ψ^2



(d) $\widetilde{\psi}^2$

Figure 1: Functions in Example 4.1

$$
\begin{array}{ccccccc}
 & & & \frac{1}{256} & & & \\
 & & -\frac{9}{256} & 0 & -\frac{9}{256} & & \\
 & -\frac{9}{256} & 0 & \frac{81}{256} & 0 & -\frac{9}{256} & \\
\frac{1}{256} & 0 & \frac{81}{256} & 1 & \frac{81}{256} & 0 & \frac{1}{256} \\
 & -\frac{9}{256} & 0 & \frac{81}{256} & 0 & -\frac{9}{256} & \\
 & & -\frac{9}{256} & 0 & -\frac{9}{256} & & \\
 & & & \frac{1}{256} & & &
\end{array}
$$

Figure 2: The mask of c in Example 4.2

The following example deals with "interpolating versions" of box–splines obtained in [26].

Example 4.3 (Box–Splines). Let $M = 2I_2$ and, for $N = 2, 4, 6$, and 8, let

$$\widetilde{c}(z) := \left(\frac{1+z_1}{2}\right)^N \left(\frac{1+z_2}{2}\right)^N \left(\frac{1+z_1 z_2}{2}\right)^N \tag{29}$$

be the symbol of the three–direction box–spline with equal multiplicities N. In [26], a symbol $\widetilde{d}$ has been constructed such that $\widetilde{c}\widetilde{d}$ is non–negative and satisfies the interpolation condition. We define

$$a(z) := \left(\frac{1+z_1}{2}\right)^N \left(\frac{1+z_2}{2}\right)^N \left(\frac{1+z_1 z_2}{2}\right)^N \widetilde{d}(z) \ .$$

It can be checked that the mask of a is symmetric with respect to the symmetry group

$$S := \left\{ \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}, \pm\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix} \right\} ,$$

which means

$$a_{Ek} = a_k \ , \qquad \text{for all } k \in \mathbb{Z}^2 \text{ and } E \in S,$$

see [26] for details. By applying $M = 2I_2$ to a result in [16], one obtains that the refinable function ϕ corresponding to a is symmetric with respect to S, i.e. that is

$$\phi(E\cdot) = \phi(\cdot) \ , \qquad \text{for all } E \in S.$$

Thus, ϕ has still the full symmetries of the three–direction box–spline.

Due to [25], we have the opportunity to change step 1 in our construction process: we define the wavelet symbols by

$$\begin{aligned}
a^1(z_1, z_2) &:= \frac{z_1}{z_2} a^0(-z_1, z_2)\,,\\
a^2(z_1, z_2) &:= z_2 a^0(z_1, -z_2)\,,\\
a^3(z_1, z_2) &:= \frac{1}{z_1} a^0(-z_1, -z_2)\,.
\end{aligned}$$

Then, it can be verified that the corresponding modulation matrix $\mathbf{a}$ satisfies

$$\mathbf{a}\overline{\mathbf{a}}^t = \theta \cdot I_2 \ .$$

Thus, with $\mathbf{b} := \mathbf{a}$, our first 3 primal and dual wavelets coincide. Nevertheless, we do not obtain a tight frame because, in step 2, we have to add 5 primal and dual wavelets that do not coincide. We define

$$c(z) := \left(\frac{1+z_1}{2}\right)^{\frac{N}{2}} \left(\frac{1+z_2}{2}\right)^{\frac{N}{2}} \left(\frac{1+z_1 z_2}{2}\right)^{\frac{N}{2}}$$

as well as

$$\overline{d(z)} := \left(\frac{1+z_1}{2}\right)^{\frac{N}{2}} \left(\frac{1+z_2}{2}\right)^{\frac{N}{2}} \left(\frac{1+z_1 z_2}{2}\right)^{\frac{N}{2}} \widetilde{d}(z) \ .$$

This yields $a = c\overline{d}$ and we can apply step 2. In the end, we obtain 8 wavelets that have at least N vanishing moments. Thus, the approximation order of the bi–frame is $2N$ which is also the approximation order of the corresponding multiresolution analysis. See Table 1 for the smoothness and the mask sizes of the symbols. We quoted the cubes the masks are contained in. The largest wavelet mask, up to shifts, is contained in a cube that is three times the size of the cube of the refinable function. The smoothness estimates are borrowed from [26].

N	mask size ϕ	smoothness	van. mom.	appr. order
2	$[-3,3]^2$	$\mathcal{C}^{2-\epsilon}(\mathbb{R}^2)$	2	4
4	$[-7,7]^2$	$\mathcal{C}^{3.55}(\mathbb{R}^2)$	4	8
6	$[-11,11]^2$	$\mathcal{C}^{4.77}(\mathbb{R}^2)$	6	12
8	$[-15,15]^2$	$\mathcal{C}^{5.82}(\mathbb{R}^2)$	8	16

Table 1: Box–Spline Wavelet Frames in Example 4.3

The examples above show that we are able to construct symmetric bi–framelets whose refinable function is interpolating while the number of the wavelets is under control. The wavelets are very smooth and have a high number of vanishing moments. The approximation order of the bi–frame reaches the approximation order of the corresponding multiresolution analysis. This is the highest possible. Moreover, the bi–framelets in the examples above satisfy various symmetry conditions.

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