THE UNCERTAINTY PRINCIPLE ASSOCIATED WITH THE CONTINUOUS SHEARLET TRANSFORM

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Abstract. Finding optimal representations of signals in higher dimensions, in particular directional representations, is currently the subject of intensive research. Since it might be difficult to obtain directional information by means of wavelets, several new representation systems were proposed in the past, including ridgelets, curvelets and, more recently, shearlets. In this paper we study and visualize the *continuous Shearlet transform*. Moreover, we aim at deriving mother shearlet functions which ensure optimal accuracy of the parameters of the associated transform. For this, we first show that this transform is associated with a unitary group representation coming from the so-called *Shearlet group* and compute the associated admissibility condition. This enables us to employ the general uncertainty principle in order to derive mother shearlet functions that minimize the uncertainty relations derived for the infinitesimal generators of the Shearlet group: scaling, shear and translations. We further discuss methods to ensure square-integrability of the derived minimizers by considering weighted L^2-spaces. Moreover, we study whether the minimizers satisfy the admissibility condition, thereby proposing a method to balance between the minimizing and the admissibility property.

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1. Introduction

Optimal representation of functions is an important topic in signal and image processing. Usually, a family of functions, that is a frame or a Riesz basis, is utilized for the sake of representation of signals. One way is to create this family of functions by applying certain group operations on a mother function. The classical examples are the Gabor functions, which account for time-frequency representations of signals and are associated with the Weyl-Heisenberg group, or wavelets, which account for time-scale representations and are associated with the affine group. The ultimate goal is to faithfully describe functions using as few basic elements as possible, and to obtain a compact representation that can be useful in various applications, e.g., in compression.

The representation of signals in both, the time and frequency domain, has been given enormous attention in the last century. It was already studied more than one hundred years ago in the context of coherent states of quantum systems. In the seminal work of Gabor in 1946 the information uncertainty principle was derived [13]. Since then the theoretical and applicative aspects of time-frequency representations were intensively studied. The time representation provides accurate information regarding the value of the signal at each instant in time, while the Fourier representation provides accurate information regarding the frequency contents of the signal. Two interesting questions have risen at this point: the first involves a search for an optimal representation in both, the time and the frequency domain. The second aims at finding other possible representations of functions, accounting for other attributes they may have.

The first question was addressed and answered by Gabor [13]. He introduced the Gaussian modulated sine-wave functions that later on were named after him. The Gabor transform is a convolution of a function with Gaussian-modulated complex exponentials, thus a local Fourier transform with a Gaussian window. This approach proved to have minimal inaccuracy with respect to both the time and frequency attributes of the function. In quantum mechanics, these Gaussian window functions, for which the minimal combined uncertainty is obtained, are a family of canonical coherent states generated by the Weyl-Heisenberg group.

The second question led to a vast research yielding numerous transforms such as the wavelet transform, and more recently the directional wavelet [3], complex wavelet [18], ridgelet [5], curvelet [6], contourlet [11] and shearlet [14] transform. Each transform is unique in emphasizing different attributes of functions: frequency, scale, orientation etc. Many of these transforms can be considered as the integral transform related to a unitary representation of some group: the windowed Fourier transform is the integral transform related to the Weyl-Heisenberg group and the wavelet transform is related to the affine group.

Given such an integral transform, then, similar to the classical Gabor case, a very natural question arises: do there exist representations that are optimal with respect to this transform? That is, can we find nontrivial analyzing functions that minimize the uncertainty principles related to the transform? Previous studies have already considered this issue for the affine group in one dimension and the similitude as well as the affine group in two dimensions [1, 3, 7, 22]. For the one dimensional affine group it was possible to find an analytical solution of the form:

$$\psi(x) = c(x-\eta)^{-\frac{1}{2}-i\eta\mu_2+i\mu_1},$$

where c is some constant, η is purely imaginary and $\mu_1, \mu_2 \in \mathbb{R}$. This function, when used as the basic window function, provides the minimal combined uncertainties with respect to time and scale. For the two dimensional $SIM(2)$ group, it was not possible to find solutions which simultaneously minimize the combined uncertainty with respect to all the parameters involved: position, scale and orientation, and therefore solutions that accounted for various subgroups were employed [1, 7].

In this study, we apply the analysis of finding uncertainty minimizers to the Shearlet transform, and the group associated with it, the Shearlet group. This transform was introduced in [14]. The basic elements of this transform are dilations, shear and translations. Thus shearlets are given by

$$\psi_{ast}(x) = a^{-\frac{3}{4}}\psi\left(A_a^{-1}S_s^{-1}(x-t)\right), \tag{1}$$

where

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \tag{2}$$

and

$$S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}. \tag{3}$$

The Shearlet transform is advantageous over the classical wavelet transform as it provides information about the directionality within the image [19]. In previous studies it was shown that the continuous Shearlet transform precisely resolves the wavefront set of a distribution [19]. Moreover, the associated discrete shearlet system provides sparse representation for $2D$ functions which are smooth away from discontinuities along curves [14, 15], and a multi-resolution analysis, similar to the one offered for classical wavelets, was offered [20].

So far, the known machinery of finding uncertainty minimizers is based on unitary group representations. Therefore, if we want to apply these concepts to the shearlet transform, then it is necessary to verify that this transform indeed stems from a suitable group representation. A priori, this is by no means clear, and many other directional transforms do *not* have this property. Nevertheless, in this paper we show that the continuous Shearlet transform can indeed be treated within group theory framework. The group associated with this transform is a subgroup of the affine group in $2D$ and will be referred to in the following as the *Shearlet group*. We use the unitary representation of the Shearlet group to derive its infinitesimal generators, and calculate the uncertainty equations related to each couple of them.

This paper is organized as follows. In Section 2 we first show that the continuous Shearlet transform is associated with a group representation, and then determine the admissibility condition associated with it. In Section 3 we illustrate directional properties of the continuous shearlet transform with some numerical examples. In Section 4 we very briefly review the uncertainty principle theorem. This framework is then applied to calculate the uncertainty minimizers to the Shearlet group in Section 5, where we provide possible solutions. Section 6 deals with ensuring square-integrability of the derived minimizers by considering weighted L^2-spaces and discussing methods to provide "almost" minimizers which satisfy the admissibility condition. We finish with stating some conclusions in Section 7.

2. The Shearlet Transform along with the Group Associated with it

Set $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ and $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. Let $\psi \in L^2(\mathbb{R}^2)$, and for each $a \in \mathbb{R}^+$, $s \in \mathbb{R}$, and $t \in \mathbb{R}^2$ define $\psi_{ast} \in L^2(\mathbb{R}^2)$ by

$$\psi_{ast}(x) = a^{-\frac{3}{4}} \psi(A_a^{-1} S_s^{-1}(x - t)).$$

Then the *shearlet system* generated by ψ is defined by

$$\{\psi_{ast} : a \in \mathbb{R}^+,\ s \in \mathbb{R},\ t \in \mathbb{R}^2\}.$$

The associated *continuous Shearlet transform* of some $f \in L^2(\mathbb{R}^2)$ is given by

$$\mathcal{SH}_f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}, \quad \mathcal{SH}_f(a, s, t) = \langle f, \psi_{ast} \rangle.$$

These definitions are taken from [14] and [19].

In this section we show that the elements of a shearlet system can be generated by using a representation of a special group that we will refer to as the Shearlet group. Moreover, we calculate the left and right Haar measures for the Shearlet group, and obtain the admissibility condition associated with the considered unitary representation of this group.

2.1. **The Group Structure of the Shearlet Group.** We consider a special multiplication on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$. In the following lemma we show that this is indeed a group multiplication.

Lemma 2.1. *The set* $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ *equipped with multiplication given by*

$$(a, s, t) \cdot (a', s', t') = (aa', s + s'\sqrt{a}, t + S_s A_a t')$$

forms a group.

Proof. It can be easily checked that $(1, 0, 0)$ is the neutral element. The inverse of some $(a, s, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ is given by

$$(a, s, t)^{-1} = (\tfrac{1}{a}, -\tfrac{s}{\sqrt{a}}, -A_a^{-1} S_s^{-1} t),$$

since

$$(a, s, t) \cdot (\tfrac{1}{a}, -\tfrac{s}{\sqrt{a}}, -A_a^{-1} S_s^{-1} t) = (a\tfrac{1}{a}, s - \tfrac{s}{\sqrt{a}}\sqrt{a}, t - S_s A_a A_a^{-1} S_s^{-1} t) = (1, 0, 0)$$

and

$$(\tfrac{1}{a}, -\tfrac{s}{\sqrt{a}}, -A_a^{-1} S_s^{-1} t) \cdot (a, s, t) = (\tfrac{1}{a}a, -\tfrac{s}{\sqrt{a}} + s\tfrac{1}{\sqrt{a}}, -A_a^{-1} S_s^{-1} t + S_{-\frac{s}{\sqrt{a}}} A_{\frac{1}{a}} t) = (1, 0, 0)$$

by using the fact that

$$S_{-\frac{s}{\sqrt{a}}} A_{\frac{1}{a}} = \begin{pmatrix} 1 & -\frac{s}{\sqrt{a}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{s}{a} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} = A_a^{-1} S_s^{-1}.$$

The multiplication is also transitive as the following computation shows:

$$\begin{aligned} &((a, s, t) \cdot (a', s', t')) \cdot (a'', s'', t'') \\ = \; & (aa', s + s'\sqrt{a}, t + S_s A_a t') \cdot (a'', s'', t'') \\ = \; & (aa'a'', s + s'\sqrt{a} + s''\sqrt{aa'}, t + S_s A_a t' + S_{s+s'\sqrt{a}} A_{aa'} t'') \\ = \; & (a(a'a''), s + (s' + s''\sqrt{a'})\sqrt{a}, t + S_s A_a (t' + S_{s'} A_{a'} t'')) \\ = \; & (a, s, t) \cdot (a'a'', s' + s''\sqrt{a'}, t' + S_{s'} A_{a'} t'') \\ = \; & (a, s, t) \cdot ((a', s', t') \cdot (a'', s'', t'')). \end{aligned}$$

For the third equality observe that

$$\begin{aligned} S_s A_a S_{s'} A_{a'} &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & \sqrt{a'} \end{pmatrix} \\ &= \begin{pmatrix} a & s\sqrt{a} \\ 0 & \sqrt{a} \end{pmatrix} \begin{pmatrix} a' & s'\sqrt{a'} \\ 0 & \sqrt{a'} \end{pmatrix} \\ &= \begin{pmatrix} aa' & s'a\sqrt{a'} + s\sqrt{aa'} \\ 0 & \sqrt{aa'} \end{pmatrix} \\ &= \begin{pmatrix} 1 & s + s'\sqrt{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} aa' & 0 \\ 0 & \sqrt{aa'} \end{pmatrix} \\ &= S_{s+s'\sqrt{a}} A_{aa'}. \quad \square \end{aligned}$$

Using this lemma, we can now define the Shearlet group.

Definition 2.2. Let the *Shearlet group* $\mathbb{S}$ be defined to be the set $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ along with the multiplication law given by

$$(a, s, t) \cdot (a', s', t') = (aa', s + s'\sqrt{a}, t + S_s A_a t').$$

Remark 2.3. Notice that the Shearlet group is isomorphic to the locally compact group $G \ltimes \mathbb{R}^2$, where

$$G = \{S_s A_a : a \in \mathbb{R}^+,\ s \in \mathbb{R}\}.$$

Thus it is a subgroup of the full group of motions $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ with multiplication defined by $(M, t) \cdot (M', t') = (MM', t + Mt)$. We also refer to [4] for wavelet-related results for those groups.

In the next lemma we define the mapping $\sigma : \mathbb{S} \to \mathcal{U}(L^2(\mathbb{R}^2))$, where $\mathcal{U}(L^2(\mathbb{R}^2))$ denotes the group of unitary operators on $L^2(\mathbb{R}^2)$, and prove that it is indeed a unitary representation of the Shearlet group. This representation can be related to the continuous Shearlet transform in the following way:

$$\mathcal{SH}_f(a, s, t) = \langle f, \psi_{ast} \rangle = \langle f, \sigma(a, s, t)\psi \rangle \quad \text{for all } f \in L^2(\mathbb{R}^2).$$

This observation will turn out to become essential for deriving an admissibility condition associated with the continuous Shearlet transform in the next subsection and for studying uncertainty relations in Section 5.

Lemma 2.4. *Define $\sigma : \mathbb{S} \to \mathcal{U}(L^2(\mathbb{R}^2))$ by*

$$\sigma(a, s, t)\psi(x) = \psi_{ast}(x) = a^{-\frac{3}{4}} \psi(A_a^{-1} S_s^{-1}(x - t)). \tag{4}$$

Then, for all $(a, s, t), (a', s', t') \in \mathbb{S}$, we have

$$\sigma(a, s, t)\sigma(a', s', t') = \sigma((a, s, t) \cdot (a', s', t')),$$

and, moreover, for all $(a, s, t) \in \mathbb{S}$, the operator $\sigma(a, s, t)$ belongs indeed to $\mathcal{U}(L^2(\mathbb{R}^2))$.

Proof. Let $\psi \in L^2(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. Then

$$\begin{aligned}
\sigma(a, s, t)(\sigma(a', s', t')\psi)(x) &= a^{-\frac{3}{4}} \sigma(a', s', t')\psi(A_a^{-1} S_s^{-1}(x - t)) \\
&= (aa')^{-\frac{3}{4}} \psi(A_{a'}^{-1} S_{s'}^{-1}(A_a^{-1} S_s^{-1}(x - t) - t')) \\
&= (aa')^{-\frac{3}{4}} \psi(A_{a'}^{-1} S_{s'}^{-1} A_a^{-1} S_s^{-1}(x - (t + S_s A_a t'))) \\
&= (aa')^{-\frac{3}{4}} \psi(A_{aa'}^{-1} S_{s+s'\sqrt{a}}^{-1}(x - (t + S_s A_a t'))),
\end{aligned}$$

since

$$\begin{aligned}
A_{a'}^{-1} S_{s'}^{-1} A_a^{-1} S_s^{-1} &= \begin{pmatrix} \frac{1}{a'} & 0 \\ 0 & \frac{1}{\sqrt{a'}} \end{pmatrix} \begin{pmatrix} 1 & -s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{a'} & -\frac{s'}{a'} \\ 0 & \frac{1}{\sqrt{a'}} \end{pmatrix} \begin{pmatrix} \frac{1}{a} & -\frac{s}{a} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{aa'} & -\frac{s}{aa'} - \frac{s'}{a'\sqrt{a}} \\ 0 & \frac{1}{\sqrt{aa'}} \end{pmatrix}
\end{aligned}$$

and

$$A_{aa'}^{-1}S_{s+s'\sqrt{a}}^{-1} = \begin{pmatrix} \frac{1}{aa'} & 0 \\ 0 & \frac{1}{\sqrt{aa'}} \end{pmatrix} \begin{pmatrix} 1 & -s - s'\sqrt{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{aa'} & -\frac{s}{aa'} - \frac{s'}{a'\sqrt{a}} \\ 0 & \frac{1}{\sqrt{aa'}} \end{pmatrix}.$$

The second assertion follows immediately by that fact that $Id = \sigma(a,s,t)^*\sigma(a,s,t)$ which in turn yields $\sigma(a,s,t)^* = \sigma((a,s,t)^{-1})$. □

In the sequel let the unitary representation $\sigma : \mathbb{S} \to \mathcal{U}(L^2(\mathbb{R}^2))$ be always defined as in Lemma 2.4.

2.2. **The Admissibility Condition associated with the Shearlet Group.** Having established the group structure of the Shearlet group and determined the unitary representation of $\mathbb{S}$ which gives rise to the continuous Shearlet transform, we are ready to extract the associated admissibility condition. As a first step we will obtain the left and right Haar measures for this group, and then formulate the admissibility condition.

2.2.1. *The Left and Right Haar Measures for the Shearlet Group.* It is already known that for locally compact groups, there always exist left and right invariant Haar integrals [16]. For the Shearlet group, this invariance implies

$$\int_{\mathbb{S}} f(a,s,t)d\mu_l = \int_{\mathbb{S}} f(a'a, s' + s\sqrt{a'}, t' + S'_s A'_a t)d\mu_l$$

where $d\mu_l = \nu(a,s,t)dadsdt$ and ν is a function of the parameters of the Shearlet transform. The calculation of ν can be done by calculating the Jacobian of the change of variables:

$$a'' = a'a, s'' = s' + s\sqrt{a'}, t'' = t' + S'_s A'_a t$$

which turns out to be $\frac{1}{a^3}$. Thus, the left Haar measure is $\frac{dadsdt}{a^3}$. This result can also be found in [19]. In order to calculate the right Haar measure, we apply the same procedure, but this time with the right-operation of the group. This yields the right Haar measure $\frac{dadsdt}{a}$.

2.2.2. *The Admissibility Condition.* Next we intend to derive the admissibility condition associated with the representation σ of the Shearlet group $\mathbb{S}$ given by (4). In general, given a unitary representation U of a locally compact group G on a Hilbert space $\mathcal{H}$, a function $\psi \in \mathcal{H}$ is called *admissible*, if

$$\int_G |\langle \psi, U(g)\psi \rangle|^2 d\mu_l(g) < \infty.$$

The admissibility condition is important, since it usually yields to a resolution of identity that allows the reconstruction of signals $f \in \mathcal{H}$ from the representation coefficients $(\langle f, U(g)\psi \rangle)_{g \in G}$.

Let us now turn to the computation of the admissibility condition for the representation σ of the Shearlet group $\mathbb{S}$. We wish to mention that a similar approach can be found in [10] for the affine group and the wavelet transform.

Theorem 2.5. *If $f, \psi \in L^2(\mathbb{R}^2)$, then*

$$\begin{aligned}\int_{\mathbb{S}} |\langle f, \psi_{ast}\rangle|^2 \frac{da\, ds\, dt}{a^3} \;&=\; \int_{\mathbb{R}}\int_0^{\infty} |\hat{f}(\omega)|^2\, d\omega_x d\omega_y \int_0^{\infty}\int_{\mathbb{R}} \frac{|\hat{\psi}(\nu_x,\nu_y)|^2}{\nu_x^2}\, d\nu_y\, d\nu_x \\ &\quad + \int_{\mathbb{R}}\int_{-\infty}^{0} |\hat{f}(\omega)|^2\, d\omega_x d\omega_y \int_{-\infty}^{0}\int_{\mathbb{R}} \frac{|\hat{\psi}(\nu_x,\nu_y)|^2}{\nu_x^2}\, d\nu_y\, d\nu_x.\end{aligned} \tag{5}$$

Proof. We first observe that the Shearlet transform of some function $f \in L^2(\mathbb{R}^2)$ can be regarded as a convolution. In fact, we have

$$\mathcal{SH}_f(a,s,t) = \langle f, a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(\cdot - t))\rangle = f * \psi_{as0}^*(t), \tag{6}$$

where $\psi_{as0}^*(x) = \overline{\psi_{as0}(-x)}$ for all $x \in \mathbb{R}^2$. Furthermore, we will need the Fourier transform of the shearlets, which can be easily computed to be

$$\widehat{\psi_{ast}}(\omega) = a^{\frac{3}{4}} e^{-2\pi i t\omega}\hat{\psi}(A_a^T S_s^T \omega) = a^{\frac{3}{4}} e^{-2\pi i t\omega}\hat{\psi}(a\omega_x, \sqrt{a}(\omega_y + s\omega_x)), \tag{7}$$

where M^T denotes the transpose of a matrix M. Employing (6), the Plancherel theorem, and (7), we obtain

$$\begin{aligned}\int_{\mathbb{S}} |\langle f, \psi_{ast}\rangle|^2 \frac{da\, ds\, dt}{a^3} &= \int_{\mathbb{S}} |f * \psi_{as0}^*(t)|^2\, dt\, ds\, \frac{da}{a^3} \\ &= \int_0^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}^2} |\hat{f}(\omega)|^2 |\widehat{\psi_{as0}^*}(\omega)|^2\, d\omega\, ds\, \frac{da}{a^3} \\ &= \int_0^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}^2} |\hat{f}(\omega)|^2 a^{\frac{3}{2}} |\hat{\psi}(A_a^T S_s^T\omega)|^2\, d\omega\, ds\, \frac{da}{a^3} \\ &= \int_0^{\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}} |\hat{f}(\omega)|^2 a^{-\frac{3}{2}} |\hat{\psi}(a\omega_x, \sqrt{a}(\omega_y + s\omega_x))|^2\, ds\, d\omega\, da \\ &= \int_{\mathbb{R}}\int_0^{\infty}\int_0^{\infty}\int_{\mathbb{R}} |\hat{f}(\omega)|^2 a^{-2}\omega_x^{-1} |\hat{\psi}(a\omega_x, \nu_y)|^2\, d\nu_y\, da\, d\omega_x d\omega_y \\ &\quad - \int_{\mathbb{R}}\int_{-\infty}^{0}\int_0^{\infty}\int_{\mathbb{R}} |\hat{f}(\omega)|^2 a^{-2}\omega_x^{-1} |\hat{\psi}(a\omega_x, \nu_y)|^2\, d\nu_y\, da\, d\omega_x d\omega_y \\ &= \int_{\mathbb{R}}\int_0^{\infty} |\hat{f}(\omega)|^2\, d\omega_x d\omega_y \int_0^{\infty}\int_{\mathbb{R}} \frac{|\hat{\psi}(\nu_x,\nu_y)|^2}{\nu_x^2}\, d\nu_y\, d\nu_x \\ &\quad + \int_{\mathbb{R}}\int_{-\infty}^{0} |\hat{f}(\omega)|^2\, d\omega_x d\omega_y \int_{-\infty}^{0}\int_{\mathbb{R}} \frac{|\hat{\psi}(\nu_x,\nu_y)|^2}{\nu_x^2}\, d\nu_y\, d\nu_x.\end{aligned}$$

□

The following two corollaries follow immediately from Theorem 5.

Corollary 2.6. *The representation σ of the shearlet group $\mathbb{S}$ is square integrable, and every $\psi \in L^2(\mathbb{R}^2)$ that satisfies*

$$\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\nu_x,\nu_y)|^2}{\nu_x^2}\, d\nu_y\, d\nu_x < \infty. \tag{8}$$

is admissible.

Corollary 2.7. *Given an admissible $\psi \in L^2(\mathbb{R}^2)$, define*

$$c_\psi^+ = \int_0^\infty \int_{\mathbb{R}} \frac{|\hat{\psi}(\nu_x,\nu_y)|^2}{\nu_x^2}\, d\nu_y\, d\nu_x, \qquad c_\psi^- = \int_{-\infty}^0 \int_{\mathbb{R}} \frac{|\hat{\psi}(\nu_x,\nu_y)|^2}{\nu_x^2}\, d\nu_y\, d\nu_x.$$

If $c_\psi^- = c_\psi^+ = c_\psi$, then the shearlet transform is a c_ψ-multiple of an isometry.

Definition 2.8. A function $\psi \in L^2(\mathbb{R}^2)$ is called a *continuous shearlet*, if it satisfies the admissibility condition (8).

Now we would like to obtain an inversion formula for the shearlet transform. The following result is again similar to the wavelet setting.

Theorem 2.9. *Suppose $\psi \in L^2(\mathbb{R}^2)$ is admissible with $c_\psi^+ = c_\psi^- = 1$. Let $\{\rho_n\}_{n=1}^\infty$ be an approximate identity such that $\rho_n \in L^2(\mathbb{R}^2)$ and $\rho_n(x) = \rho_n(-x)$ for all x. Then $\lim_{n\to\infty} \|f - f_n\|_2 = 0$ for all $f \in L^2(\mathbb{R}^2)$, where*

$$f_n(x) = \int_{\mathbb{S}} \mathcal{SH}_f(a,s,t)(\rho_n * \psi_{a,s,t})(x)\, dt\, ds\, \frac{da}{a^3}.$$

Proof. Let T_x, $x \in \mathbb{R}^2$ denote the translation operator $T_x f(y) = f(x-y)$. Since ρ_n is even and the Shearlet transform is an isometry, we obtain

$$\begin{aligned}
(f * \rho_n)(x) &= \int_{\mathbb{R}^2} f(y)\rho_n(x-y)dy \\
&= \langle f, \overline{T_x\rho_n}\rangle \\
&= \langle \mathcal{SH}_f, \mathcal{SH}_{\overline{T_x\rho_n}}\rangle \\
&= \int_{\mathbb{S}} \mathcal{SH}_f(a,s,t)\overline{\langle \overline{\rho_n(\cdot - x)}, \psi_{a,s,t}(\cdot)\rangle} dt\, ds\, \frac{da}{a^3} \\
&= \int_{\mathbb{S}} \mathcal{SH}_f(a,s,t)(\rho_n * \psi_{a,s,t})(x) dt\, ds\, \frac{da}{a^3}.
\end{aligned}$$

But $\{\rho_n\}_{n=1}^\infty$ is an approximate identity, so $\lim_{n\to\infty} \|f - f_n\|_2 = 0$. □

Remark 2.10. Consider the function $\psi \in L^2(\mathbb{R}^2)$ defined by

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1,\xi_2) = \hat{\psi}_1(\xi_1)\, \hat{\psi}_2(\tfrac{\xi_2}{\xi_1}),$$

where ψ_1 is a continuous wavelet, $\hat{\psi}_1 \in C^\infty(\mathbb{R})$, and supp $\hat{\psi}_1 \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, and where ψ_2 is such that $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ and supp $\hat{\psi}_2 \subseteq [-1, 1]$. This generating function was employed in [19] to show that the continuous Shearlet transform precisely resolves the wavefront set. It is an easy computation to show that this function is indeed a continuous shearlet as defined above.

In Section 3 we will underpin the directional properties of the transform $f \mapsto \langle f, \psi_{ast}\rangle$ by showing and discussing some numerical examples. After that, equipped with the group structure of the Shearlet transform and the relevant associated unitary representation, in Section 5 we can use the framework reviewed in Section 4, which provides those functions that are the minimizers of uncertainty relations.

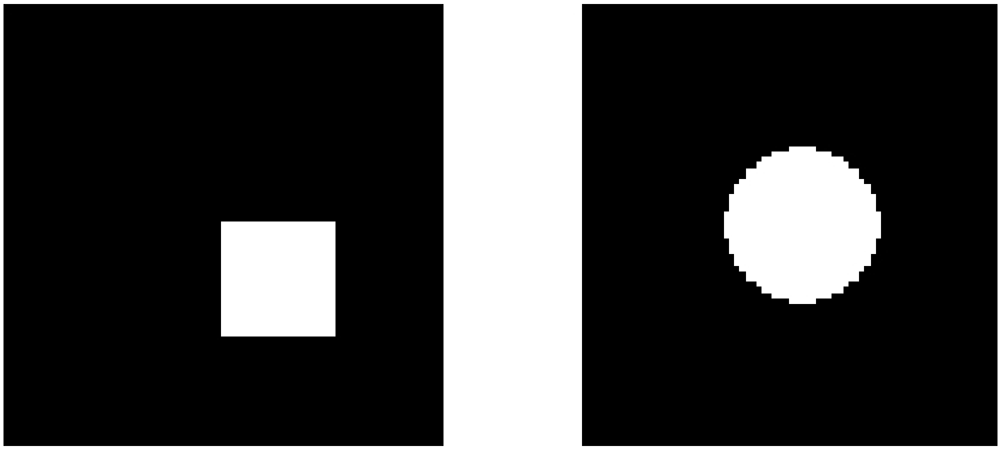

FIGURE 1. Originals: Square (left) and circle(right).

3. Visualization of the Shearlet Coefficients

The purpose of this section is, to visualize the behavior of the *shearlet coefficients* $\langle f, \psi_{ast}\rangle$, i.e., of the values of the Shearlet transform $\mathcal{SH}_f(a,s,t)$. Given the original f we shall select specific values for a and s, respectively, and display $\langle f, \psi_{ast}\rangle$ as a function of $t \in \mathbb{R}^2$, i.e., as an image. For simplicity this image is denoted with $\mathbb{T}_{(\psi,a,s)}$.

The admissibility condition is needed to ensure invertibility of the map $f \mapsto \langle \psi_{ast}, f\rangle$. Having possible applications in texture *analysis* [22] in mind, e.g., tunable detection of orientations in images, it seems to be justified to investigate $\langle \psi_{ast}, f\rangle$ even if ψ does *not* satisfy the admissibility condition.

In this section we study

$$\psi_1(x,y) = \psi_H(x)\chi_{\left(-\frac{1}{2},\frac{1}{2}\right)}(y) \text{ with } \psi_H(x) = \begin{cases} 1 & : \ -\frac{1}{2} \le x < 0 \\ -1 & : \ 0 \le x \le \frac{1}{2} \\ 0 & : \ \text{else} \end{cases}$$

and $\chi_{\left(-\frac{1}{2},\frac{1}{2}\right)}$ denoting the characteristic function of the open interval $\left(-\frac{1}{2},\frac{1}{2}\right)$. Clearly, ψ_H is a shifted version of the well-known one-dimensional Haar-wavelet. Moreover, we consider

$$\psi_2(x,y) = e^{-4r^2}(32r^2 - 4) \text{ with } r^2 = x^2 + y^2,$$

i.e., the second radial derivative of a Gaussian. Both ψ_1 and ψ_2 have zero mean, thus "reacting on" edges in f. Obviously ψ_1 will respond on vertical edges, whereas ψ_2 is isotropic.

Note that in all plots discussed below the gray-level-coding is *scaled*, i.e., the smallest value is displayed black, the largest white. Since we want to focus on shear, in all plots $a = 1$ was chosen.

Fig. 1 shows the respective originals f, to be analyzed. In order to investigate, how $\mathbb{T}_{(\psi,a,s)}$ responds on shear, we compute and display $\mathbb{T}_{(\psi,a,s)}$ for sheared versions of f, i.e., for $f_{a's't'}$. We restrict ourselves to $a' = 1$ and, in order to keep notation simple, regard $f_{a's't'}$ ($t' \in \mathbb{R}^2$) as an image, which is denoted with $f_{s'}$.

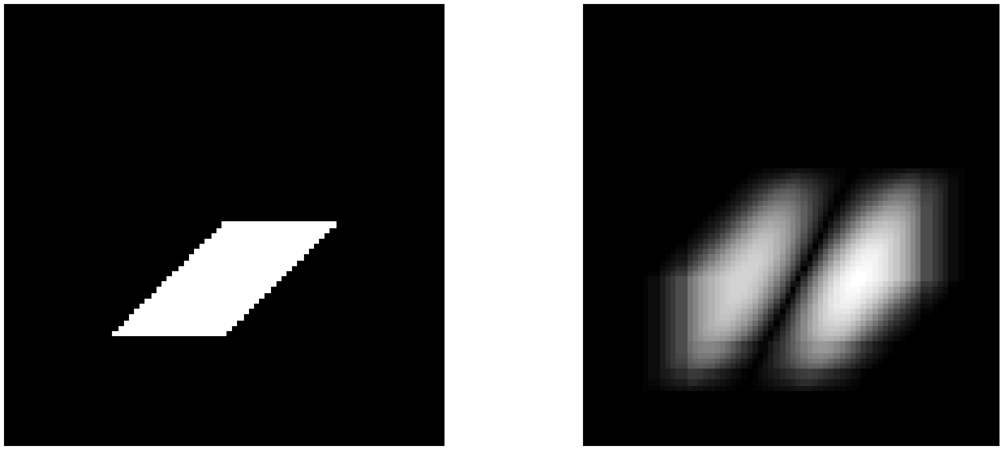

FIGURE 2. Sheared square f_1 (left) and $|\mathbb{T}_{(\psi_1,1,0)}|$ (right, "wrong" shear factor).

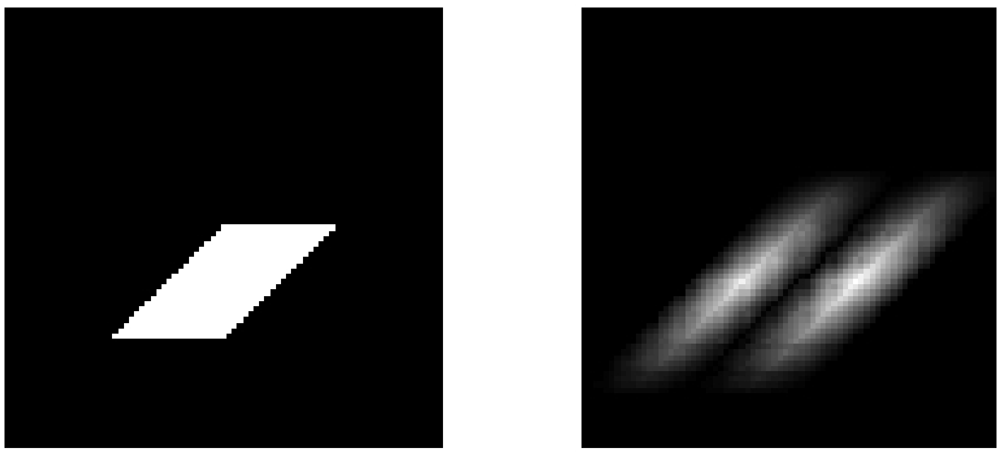

FIGURE 3. Sheared square f_1 (left) and $|\mathbb{T}_{(\psi_1,1,1)}|$ (right, "matching" shear factor).

Figs. 2-9 demonstrate a reasonable behavior, both with respect to concentration properties at matching shear factors and with respect to directional properties of the underlying ψ. Thus, e.g., Figs. 5 and 9 clearly illustrate the isotropy of ψ_2, whereas Figs. 2 and 3 exhibit the above mentioned concentration effect.

4. Uncertainty Relations and Uncertainty Minimizers

In this section we briefly review the basics of the uncertainty relations in terms of group theory and harmonic analysis. A general theorem which is well known in quantum mechanics and harmonic analysis [12] relates an uncertainty principle to any two self-adjoint operators and provides a mechanism for deriving a minimizing function for the uncertainty equation. We first recall some basic definitions and fix some notations.

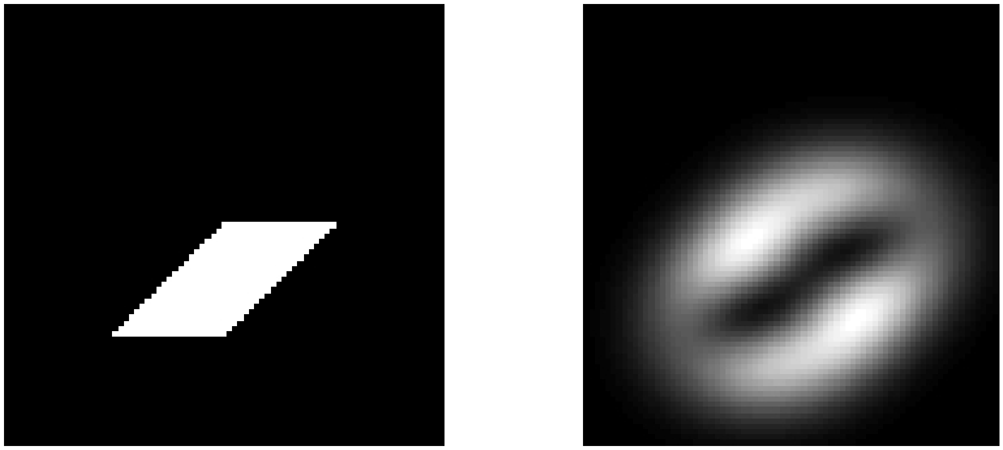

FIGURE 4. Sheared square f_1 (left) and $|\mathbb{T}_{(\psi_2,1,0)}|$ (right, "wrong" shear factor).

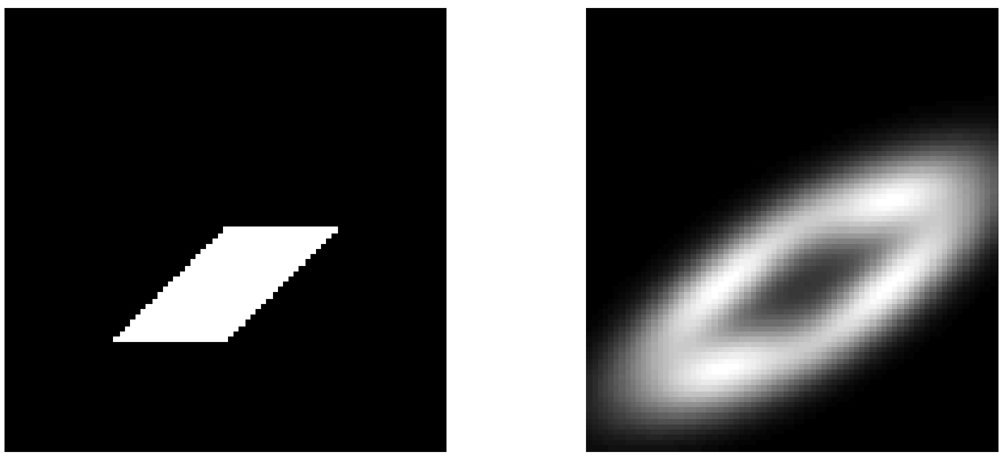

FIGURE 5. Sheared square f_1 (left) and $|\mathbb{T}_{(\psi_2,1,1)}|$ (right, "matching" shear factor).

Definition/Notation 4.1. Let P be an operator on a Hilbert space $\mathcal{H}$ and let ψ be an element of $\mathcal{H}$. The *mean* of the action of P on ψ is defined by $\mu_P(\psi) = \langle P \rangle$, where the triangular parenthesis mean an average over the signal, i.e., $\langle X \rangle = \int \psi^* X \psi$. The *variance* of P with respect to ψ is defined by

$$\Delta P_\psi := \langle (P - \mu_P(\psi))^2 \psi, \psi \rangle. \tag{9}$$

Let Q be also an operator on $\mathcal{H}$. Then the *commutator* $[P, Q]$ is given by

$$[P, Q] := PQ - QP.$$

Now we can state the general uncertainty principle, which can for instance be found in the book [12].

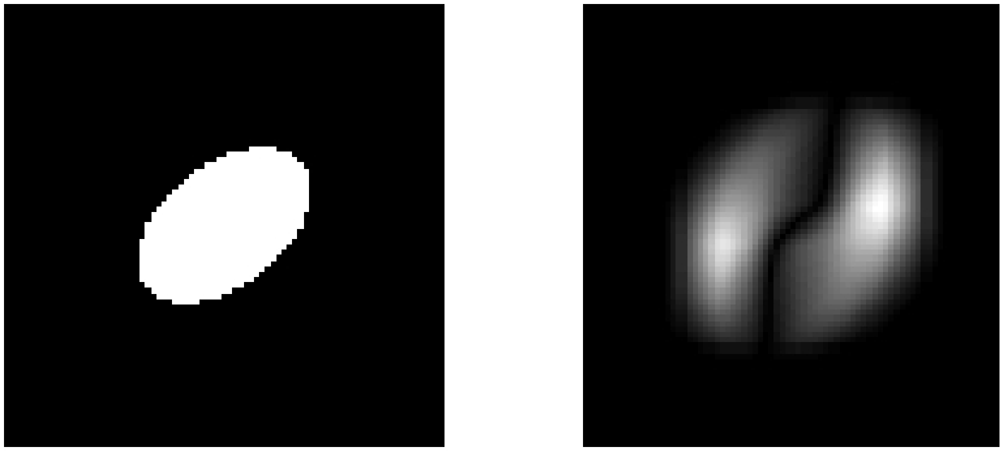

FIGURE 6. Sheared circle $f_{0.5}$ (left) and $|\mathbb{T}_{(\psi_1,1,0)}|$ (right, "wrong" shear factor).

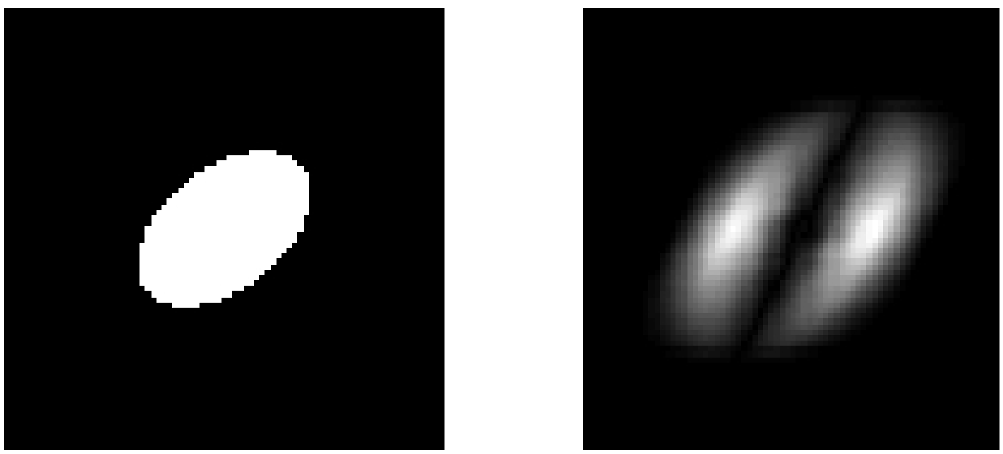

FIGURE 7. Sheared circle $f_{0.5}$ (left) and $|\mathbb{T}_{(\psi_1,1,0.5)}|$ (right, "matching" shear factor).

Theorem 4.2. *Two self-adjoint operators A and B on a Hilbert space $\mathcal{H}$ obey the uncertainty relation*

$$\Delta A_\psi \Delta B_\psi \ \geq \frac{1}{2}|\langle [A,B] \rangle| \quad \textit{for all } \psi \in \mathcal{H}. \tag{10}$$

A function ψ is said to have minimal uncertainty if the inequality (10) turns into an equality. This happens if and only if there exists an $\eta \in i\mathbb{R}$ such that

$$(A - \mu_A(\psi))\psi = \eta(B - \mu_B(\psi))\psi. \tag{11}$$

In many cases, this last relation yields a partial differential equation for each non-commuting couple of operators.

In this study, we consider a bank of filters that is generated by the application of the operations of the Shearlet group to some mother function. The operation of the group close to the identity element can be described using the infinitesimal generators of the group.

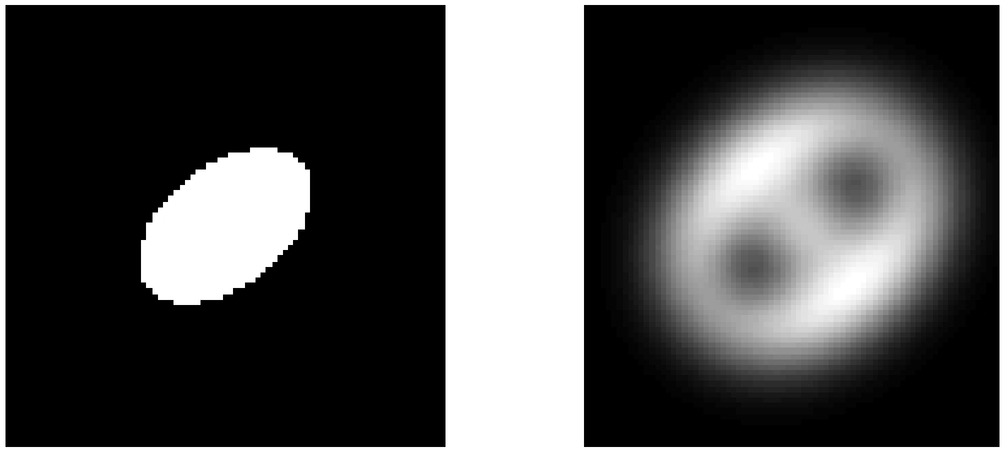

FIGURE 8. Sheared circle $f_{0.5}$ (left) and $|\mathbb{T}_{(\psi_2,1,0)}|$ (right, "wrong" shear factor).

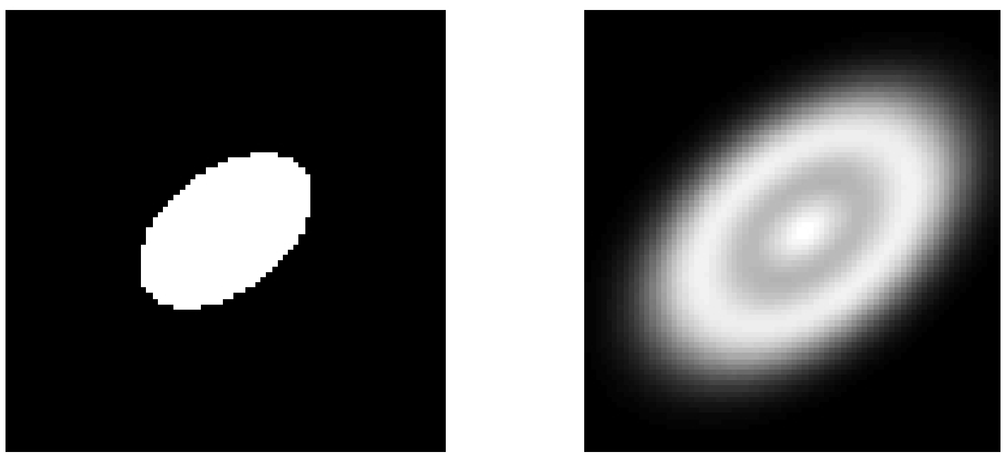

FIGURE 9. Sheared circle $f_{0.5}$ (left) and $|\mathbb{T}_{(\psi_2,1,0.5)}|$ (right, "matching" shear factor).

The group representation is unitary and thus the infinitesimal generators can be made self-adjoint operators. Therefore, the general uncertainty theorem stated above provides a tool for obtaining uncertainty principles using these infinitesimal generators.

This study aims at providing the minimizers of the uncertainty relations associated with the Shearlet group, thereby providing mother shearlets for optimal accuracy of the parameters of the continuous Shearlet transform. Before doing so, we shall make an important comment about the minimizers, determined using Theorem 4.2.

Minimizing the combined uncertainties $\Delta A_\psi \Delta B_\psi$ *globally* means to determine the set

$$\mathcal{M} := \{\psi \in \mathcal{H} \mid \Delta A_\psi \Delta B_\psi \leq \Delta A_{\psi'} \Delta B_{\psi'} \,\forall\, \psi' \in \mathcal{H}\}.$$

Minimizers corresponding to Theorem 4.2 belong to the set

$$\mathcal{M}' := \{\psi \in \mathcal{H} \mid \Delta A_\psi \Delta B_\psi = \tfrac{1}{2}|\langle \psi, [A,B]\psi \rangle|\}.$$

Thus, the combined uncertainties of those minimizers reach their *individual* lower bound, which not necessarily means, that they are global minimizers in the sense of $\mathcal{M}$.

Note that $\mathcal{M} = \mathcal{M}'$ if $\frac{1}{2}|\langle [A,B]\rangle|$ does *not* depend on ψ. This will be the case if $[A,B]$ is a multiple of the identity, which is true for the Weyl-Heisenberg group in the classic work of Gabor [13]. If $[A,B]$ does not have this property, we may not necessarily expect, that minimizers derived according to Theorem 4.2 are global minimizers in the sense, discussed above. This applies to the shearlet group, discussed in this paper, as well as to earlier work [7], [22].

5. The Minimizers of the Uncertainty Principle Related to the Shearlet Transform

In this section we will formulate the uncertainty principles that are related to the Shearlet group and the Shearlet transform and derive functions that attain minimal uncertainty.

Theorem 5.1. (i) *The infinitesimal generators of the Shearlet group* $\mathbb{S}$, *with respect to the scaling* $a \in \mathbb{R}^+$, *the shear parameter* $s \in \mathbb{R}$ *and the translation parameters* $t_1, t_2 \in \mathbb{R}$, *are given by*

$$\begin{aligned}
(T_a\psi)(x,y) &= -i(\frac{3}{4}\psi + x\frac{\partial}{\partial x}\psi + \frac{y}{2}\frac{\partial}{\partial y}\psi), \\
(T_s\psi)(x,y) &= -iy\frac{\partial}{\partial x}\psi, \\
(T_{t_1}\psi)(x,y) &= -i\frac{\partial}{\partial x}\psi, \\
(T_{t_2}\psi)(x,y) &= -i\frac{\partial}{\partial y}\psi.
\end{aligned}$$

(ii) *We have*

$$[T_s, T_{t_1}] = [T_{t_1}, T_{t_2}] = 0.$$

Moreover, the partial differential equations that result from the non-vanishing commutation relations are

(a) *for the scale and shear operators,*

$$-\frac{3i}{4}\psi - ix\psi_x - -i\frac{y}{2}\psi_y - \mu_a\psi = \lambda_1(-iy\psi_x - \mu_s\psi), \tag{12}$$

(b) *for the scale and* x*-translation operator,*

$$-\frac{3i}{4}\psi - ix\psi_x - -i\frac{y}{2}\psi_y - \mu_a\psi = \lambda_2(-i\psi_x - \mu_{t_1}\psi), \tag{13}$$

(c) *for the scale and* y*-translation operator,*

$$-\frac{3i}{4}\psi - ix\psi_x - -i\frac{y}{2}\psi_y - \mu_a\psi = \lambda_3(-i\psi_y - \mu_{t_2}\psi), \tag{14}$$

(d) *and for the shear and translation operator in the* y*-direction,*

$$-iy\psi_x - \mu_s\psi = \lambda_4(-i\psi_y - \mu_{t_2}\psi). \tag{15}$$

(iii) *The partial differential equations* (12)–(15) *do not possess a simultaneous solution.*

(iv) *There exists a solution for the differential equations* (12)–(14)*, hence a minimizer for the scale-shear and scale-translations uncertainty, which is obtained on the characteristic lines $x = cy^2$ and is given by*

$$\psi(x,y) = \psi(cy^2, y) = \tau(\frac{x}{y^2})y^{-\frac{3}{2}+2i\mu_a}, \quad \textit{where } \tau \in L^2(\mathbb{R}^2).$$

(v) *There exists a solution for the differential equation* (15)*, hence a minimizer for the scale and y-translation uncertainty, which is given by*

$$\psi(x,y) = ce^{i\mu_s \frac{x}{y}}.$$

Proof. (i) The derivation of the infinitesimal generators is straightforward and is done by calculating the appropriate derivatives of the unitary representation of the Shearlet group with respect to the parameters of $\mathbb{S}$. We obtain

$$\begin{aligned}
(T_a\psi)(x,y) &:= i\frac{\partial}{\partial a}[\sigma(a,s,t_1,t_2)\psi](x,y)|_{a=1,s=0,t_1=0,t_2=0} \\
&= -i(\frac{3}{4}\psi + x\frac{\partial}{\partial x}\psi + \frac{y}{2}\frac{\partial}{\partial y}\psi) \\
(T_s\psi)(x,y) &:= i\frac{\partial}{\partial s}[\sigma(a,s,t_1,t_2)\psi](x,y)|_{a=1,s=0,t_1=0,t_2=0} = -iy\frac{\partial}{\partial x}\psi \\
(T_{t_1}\psi)(x,y) &:= i\frac{\partial}{\partial t_1}[\sigma(a,s,t_1,t_2)\psi](x,y)|_{a=1,s=0,t_1=0,t_2=0} = -i\frac{\partial}{\partial x}\psi \\
(T_{t_2}\psi)(x,y) &:= i\frac{\partial}{\partial t_2}[\sigma(a,s,t_1,t_2)\psi](x,y)|_{a=1,s=0,t_1=0,t_2=0} = -i\frac{\partial}{\partial y}\psi
\end{aligned}$$

(ii) To prove the first claim we compute

$$[T_s, T_{t_1}] = (-i)^2 y\frac{\partial}{\partial x}\frac{\partial}{\partial x} - (-i)^2\frac{\partial}{\partial x}y\frac{\partial}{\partial x} = 0.$$

A similar computation shows that $[T_{t_1}, T_{t_2}] = 0$. On the other hand, it is easy to see that the operator T_a does not commute with T_s, T_{t_1} and T_{t_2}, and T_s does not commute with T_{t_2}. By using (11), we obtain the set of four differential equations for those pairs of operators.

(iii) Since τ cannot be chosen such that the solution for (12)–(14) in (iv) does coincide with the solution for (15) in (v), there is no simultaneous solution for the full set of differential equations.

(iv) Our analysis involves solving each differential equation independently, thus to minimize the uncertainties between pairs of operators. First, we find a solution that minimizes the combined uncertainties of the scale and shear operators (12), which is given by

$$\psi(cy^2 + 2\lambda_1 y, y) = \tau(\frac{2\lambda_1 y + cy^2}{y^2})y^{-\frac{3}{2}+2i\mu_a-2i\lambda_1\mu_s},$$

where τ is a L^2 function. This solution is given along the characteristic line $x = 2\lambda_1 y + cy^2$. As x and y are both real valued variables, we are forced to constraint $\lambda_1 = 0$ to obtain a valid expression: $x = cy^2$. Next, we consider the differential

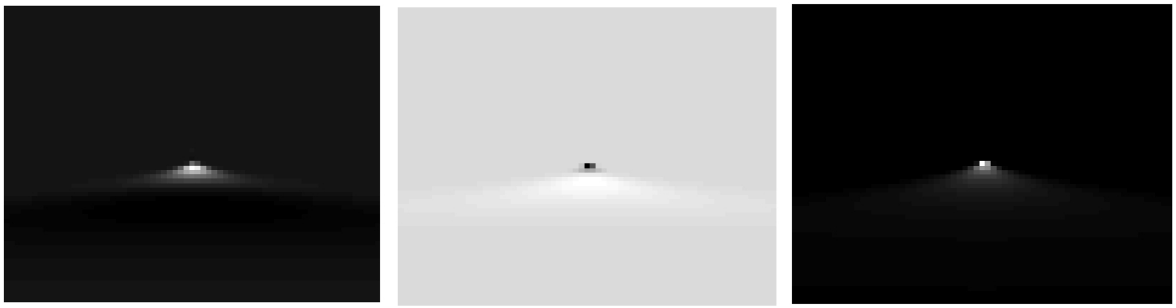

FIGURE 10. A possible solution for minimal uncertainty for the scale-shear and scale-translations uncertainty. The minimizer shown here is given by: $\psi(x,y) = e^{-\frac{1}{2}(\frac{x}{y^2})^2} y^{-\frac{3}{2}+2i\mu_a}$. On the left we see the real value of the function, at the middle its imaginary value and on the right its absolute value.

equation (13) that arises for the scale and x-translation operators. The function that minimizes their mutual uncertainty is calculated to be

$$\psi(cy^2+\lambda_2, y) = \tau(\frac{\lambda_2+cy^2}{y^2})y^{-\frac{3}{2}+2i\mu_a-2i\lambda_2\mu_{t_1}}. \tag{16}$$

This solution is given for values of x that satisfy: $x = \lambda_2 + cy^2$. Again, we are forced to set $\lambda_2 = 0$ to obtain real values for the coordinates. This solution then coincides with the one obtained for the scale and shear uncertainties. The function that minimizes the combined uncertainty of the scale and y-translations operators (14) is given by

$$\psi(c(y-2\lambda_3)^2, y) = \tau(\frac{(y-2\lambda_3)^2}{y^2})(y-2\lambda_3)^{-\frac{3}{2}+2i\mu_a-2i\lambda_3\mu_{t_2}}.$$

As can be seen, also here $\lambda_3 = 0$ and the three solutions coincide for the setting $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In order for this solution to be square integrable the function $\tau(\frac{x}{y^2})$ should be square integrable.

(v) We set, in the original equation, $\lambda_4 = 0$, and obtain $T_s\psi = 0$ namely

$$-iy\psi_x - \mu_s\psi = 0$$

which has the solution

$$\psi(x,y) = ce^{i\mu_s\frac{x}{y}}.$$

□

A sketch of the solution in (iv) can be seen in Figure 10 for which τ is selected to be $\tau\left(\frac{x}{y^2}\right) = e^{-\frac{1}{2}(\frac{x}{y^2})^2}$.

Part (ii) of Theorem 5.1 shows that the exact shear attribute can be simultaneously known with the position in the x direction and that the exact positions in the x- and y-direction can also be simultaneously known.

As an adding to part (iii) we mention that considering a general solution of the type of the rotation invariant solution that is offered in [7] for the $SIM(2)$ group, we may get "shear-invariant" solutions. These, however, are simply constant along one direction, and are therefore not very interesting.

We further remark that the minimizer derived in (v) gives us a shear-invariant solution, that minimizes the uncertainties between the shear and the y-translations as well as the shear and the scaling. Any function that is constant along its x-direction (solutions with $\mu_s = 0$) or that depends only on $\frac{x}{y}$ can be valid solutions.

6. Square-Integrability and Admissibility of the Minimizers

In the previous section it was proven that the minimal uncertainty for the scale-shear and scale-translation relations is attained by choosing a function of the form

$$\psi(x, y) = \tau(\frac{x}{y^2})y^{-\frac{3}{2}+2i\mu_a}. \tag{17}$$

The first expression of the RHS ($\tau(\frac{x}{y^2})$) is defined along the characteristic lines $x = cy^2$. It is easy to observe that this function can never be contained in L^2 unless it coincides with the zero function. Moreover, in Section 2.2 we have formulated the admissibility condition related to the unitary representation σ of the Shearlet group (see (8)). Naturally, we would like the minimizer of the uncertainty related to the Shearlet group to be an admissible function. In the next subsections, we first address the issue of square integrability, and choose an appropriate weight function to obtain this quality. Then, we reformulate the admissibility condition as a variational problem in which we search for a function that is the closest to our minimizer in the L^2-sense, and in the same time satisfies the admissibility equation.

6.1. **Square-Integrability of the Minimizers.** It is possible to choose an appropriate weight function on $\mathbb{R}^2$ in order to obtain square integrability. The idea will be to choose a parameterization of $\mathbb{R}^2\backslash((\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}))$ in such a way that one parameter indicates to which characteristic line a point belongs to, whereas the second parameter parameterizes the particular characteristic line itself. Notice that the cross $(\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$ needs to be excluded due to the fact that no characteristic line intersects it except for the origin. The weight function will then be chosen accordingly in the sense that it does not act on the first parameter, but provides sufficient decay with respect to the second one.

The parameterization of

$$R := \mathbb{R}^2\backslash((\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})) \tag{18}$$

that we employ here is given by

$$\varphi : R \to R, \quad \varphi : (c, t) \mapsto (ct^2, t) = (x, y). \tag{19}$$

It is easy to see that this map is indeed bijective with the inverse given by $\varphi^{-1}(x, y) = (\frac{x}{y^2}, y)$. The Jacobian determinant of φ is t^2. We also wish to mention that this is by far not the only possible choice. We chose this parameterization because the form of the characteristic lines $c = \frac{x}{y^2}$ seems to mark it as the most natural one.

The following proposition makes these considerations precise.

Proposition 6.1. *Let $\tau \in L^2(\mathbb{R})$, let $w_2 : \mathbb{R} \to [0, \infty)$ be a weight function which satisfies $\int_{\mathbb{R}} |t|^{-1} w_2(t)\, dt < \infty$, and let R and φ be defined by* (18) *and* (19)*, respectively. Define*

$w : R \to \mathbb{R}^+$ by $w(x,y) = (1 \otimes w_2)(\varphi^{-1}(x,y))$, where $1(t) = 1$ for all $t \in \mathbb{R}$. Then the minimizer ψ defined by (17) satisfies

$$\psi \in L^2_w(\mathbb{R}^2).$$

Proof. Employing the definition of the parameterization φ, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} |\psi(x,y)|^2 w(x,y)\, d(x,y) &= \int_{\mathbb{R}^2} |\psi(ct^2,t)|^2 w(ct^2,t)\, t^2\, d(c,t) \\
&= \int_{\mathbb{R}^2} |\tau(c)|^2 |t|^{-3} t^2 (1 \otimes w_2)(c,t)\, d(c,t) \\
&= \|\tau\|_2^2 \int_{\mathbb{R}} |t|^{-1} w_2(t)\, dt \\
&< \infty,
\end{aligned}$$

where the last step follows from the choice of the weight w_2. □

In the following we provide an example of a family of appropriate weight functions w_2.

Example 6.2. Let $w_2 : \mathbb{R} \to \mathbb{R}^+$ be defined by

$$w_2(t) = |t|^{\alpha+1}(1+t^2)^{-\frac{\alpha+\beta}{2}}.$$

Provided that $\alpha > 1$ and $\beta > 1$, it is straightforward to show that w_2 then satisfies

$$\int_{\mathbb{R}} |t|^{-1} w_2(t)\, dt < \infty,$$

thereby serving as a possible choice for the weight function in Proposition 6.1.

6.2. **Admissibility of the Minimizers.** In Section 2.2 we have formulated the admissibility condition associated with the Shearlet group,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega_x,\omega_y)|^2}{\omega_x^2}\, d\omega_y\, d\omega_x < +\infty.$$

In Section 5 we have obtained the minimizer with respect to the scale-shear and scale-translation uncertainty relations,

$$\tilde{\psi}(x,y) = \tau(\frac{x}{y^2}) y^{-\frac{3}{2}+2i\mu_a}.$$

In this section, we would like to obtain the function that is closest to this minimizer in the L^2-sense, and that is minimizing the admissibility condition integral. We formulate this problem in a variational setting and obtain the solution as the minimizer of a functional.

Proposition 6.3. *Let us define the following functional*

$$F(\hat{\psi}) = \alpha \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega_x,\omega_y)|^2}{|\omega_x|^2}\, d\omega_y\, d\omega_x + \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{\psi}(\omega_x,\omega_y) - \hat{\tilde{\psi}}(\omega_x,\omega_y)|^2\, d\omega_y\, d\omega_x \qquad (20)$$

where $\tilde{\psi}$ is the above mentioned minimizer obtained for the scale-shear and scale-translation uncertainty relation. Then, the minimizer for this functional is given by

$$\hat{\psi}(\omega_x, \omega_y) = \frac{|\omega_x|^2}{\alpha + |\omega_x|^2} \hat{\tilde{\psi}}(\omega_x, \omega_y). \tag{21}$$

Proof. Let us consider the functional $\Phi(t) = F(\hat{\psi} + t\hat{h})$ and look for the minimizer of this functional by calculating its derivative with respect to t and evaluating it for $t = 0$. This gives the following equation

$$\begin{aligned}
&\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\overline{\hat{\psi}(\omega_x, \omega_y)} \hat{h}(\omega_x, \omega_y) + \hat{\psi}(\omega_x, \omega_y) \overline{\hat{h}(\omega_x, \omega_y)}}{|\omega_x|^2} \right) d\omega_y \, d\omega_x \\
&+ \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\overline{(\hat{\psi}(\omega_x, \omega_y) - \hat{\tilde{\psi}}(\omega_x, \omega_y))} \hat{h}(\omega_x, \omega_y) + (\hat{\psi}(\omega_x, \omega_y) - \hat{\tilde{\psi}}(\omega_x, \omega_y)) \overline{\hat{h}(\omega_x, \omega_y)} \right) d\omega_y \, d\omega_x \\
&= 0,
\end{aligned}$$

that may be rearranged as

$$\alpha \hat{\psi}(\omega_x, \omega_y) + |\omega_x|^2 \hat{\psi}(\omega_x, \omega_y) = \hat{\tilde{\psi}}(\omega_x, \omega_y) |\omega_x|^2 \tag{22}$$

to finally obtain the result

$$\hat{\psi}(\omega_x, \omega_y) = \frac{|\omega_x|^2}{\alpha + |\omega_x|^2} \hat{\tilde{\psi}}(\omega_x, \omega_y). \tag{23}$$

□

Thus, we obtain an approximation to the minimizer with respect to the scale-shear and scale-translation uncertainties that is admissible and has a minimal distance to the minimizer in the L^2-sense. The real value of a possible minimizer is presented in Figure 11 along with its various versions with respect to different values of α (Figure 12).

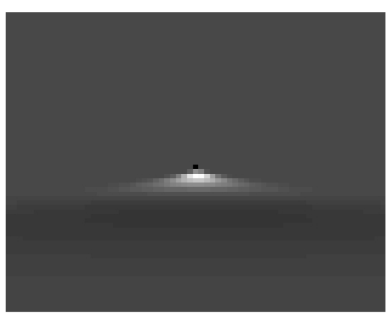

FIGURE 11. Original minimizer

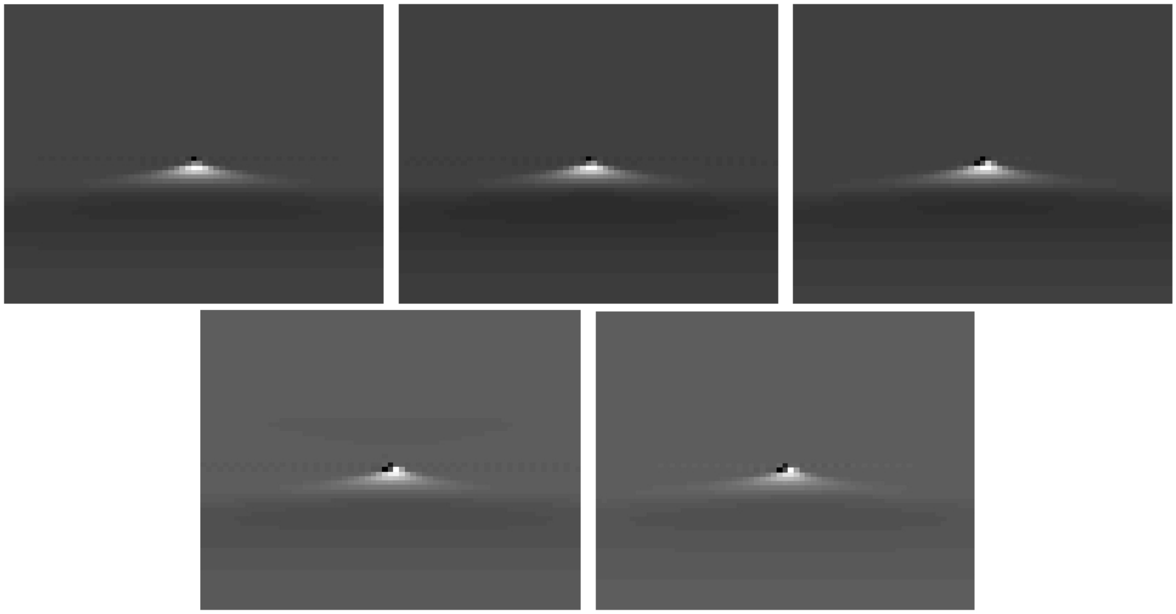

Figure 12. A possible approximation for the solution of minimal uncertainty for the scale-shear and scale-translations uncertainty that is admissible in the sense that it minimizes the admissibility condition integral. We provide this function for several values of the weight factor α: $\alpha = 0.1$ (top left), $\alpha = 1$ (top middle), $\alpha = 10$ (top right), $\alpha = 100$ (bottom left) and $\alpha = 10^6$ (bottom right).

7. Discussion

Wavelets have proven to be a useful tool in several signal and image processing applications. Still, there is a growing interest in recent years in oriented wavelets, as the isotropic ones do not capture singularities along curves in images. This interest has led to the introduction of ridgelets, curvelets and shearlets to name a few.

Usually, the generation of $2D$ wavelets is done by applying the operations of the $SIM(2)$ group to some mother function. The full span of linear transforms can naturally be given by the full affine group. Looking at subsets of the full affine group, we may obtain the $SIM(2)$ group that accounts for rotations and a single scaling that generate the usual $2D$ wavelet transform, or alternatively, the Shearlet group that accounts for some coupled scaling and shear, with no rotations. It seems that the shear and rotation operations play similar roles in terms of the tiling of the frequency plane. The shear operation can be seen as a partial rotation in terms of the infinitesimal generators of the associated group.

Once we select some representation, we also have to determine a mother function. Then, a whole bank of filters is generated by applying the actions of the associated group on this mother function. The selection of this mother function can be done according to the application and numerical simplicity. It can also be the function that provides the minimal uncertainty regarding the localization of information in the feature space (where features can be location in space, frequency, scale, shear etc.).

In this study, we have shown that the Shearlet transform can be related to a group, and provided the representation of this group. Then, we have applied the mechanism of calculating the uncertainty relations between the infinitesimal operators that generate this Shearlet group: scaling, shear and translations. Two commutation relations have vanished: those between the translations in the x and the y directions, and those between the shear and the translations in the x direction. This means that knowing exactly our location in both the x and y directions is possible. Moreover, we may exactly know our location in the x direction, as well as the shear value. For those commutation relations that did not vanish, we have learnt that finding a solution with respect to all of them is not possible. Nevertheless, we obtain a solution that allows good localization with respect to scale and shear, and scale and translations. This solution, however, does not allow good localization (in terms of minimal uncertainty) for shear and translations in the y-direction.

To conclude, even in cases where the efforts to obtain a global minimizer with respect to all the pairs of non-commuting operators are futile, there is a possibility to find solutions that minimize some sub-set of these uncertainties. Determining which parameters are needed with greater accuracy and which are not, probably depends on the application at hand.

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