On Finite Element Methods for Fully Nonlinear Elliptic Equations and Systems

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Dedicated to My Friend and Co Author of Many Papers,
Prof. Dr. E. L. Allgower, on the Occasion of His 70-th Birthday

Abstract

For the first time, we present for the general case of fully nonlinear elliptic differential equations and systems of order 2 and $2m$ in $\mathbf{R}^n$ on C^{2m} domains, a stability, consistency and convergence proof. A nonstandard nonconforming C^{2m-1} finite element method with variational crimes is used. The necessary quadrature approximations are included as well. The classical theory of discretization methods is applied to an operator with two components, the differential and the boundary operator. The violated boundary conditions on curved domains have to be estimated and the stability has to be proved, both in an unusual way. This is the hard core of the paper. Essential tools are linearization, a compactness argument, the interplay between the weak and strong form of the linearized operator and a new regularity result for solutions of finite element equations. An essential basis for computations and our proofs are Davydov's C^1 finite elements on polygonal domains. These results are fully available in $\mathbf{R}^2$ of local degree 5 and soon in $\mathbf{R}^n$ yielding second order convergence. A better convergence is to be expected from his forthcoming results on curved domains. Our proofs work for both cases. The method applies to nondivergent quasilinear elliptic problems as well.

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Key words: finite element, fully nonlinear elliptic differential equations and systems, order 2 and $2m$, in $\mathbf{R}^n$, C^{2m-1} finite elements, non divergent quasilinear elliptic forms, stability and convergence, quadrature and cubature approximation, regularity for finite element solutions, algorithm, discrete Newton method, locally quadratic convergence

1 Introduction

We present, to the authors knowledge for the first time, a finite element method (FEM) for the general case of fully nonlinear elliptic differential equations and systems of orders

2 and $2m$ in $\mathbf{R}^n$ including quadrature approximations. Our FEM is a natural translation of the nonstandard situation for fully nonlinear elliptic differential equations and systems, including nondivergent quasilinear problems. Linear and divergent quasilinear problems usually are discussed in the weak form with derivatives up to the order m, see e.g., (3.3). This is not possible for fully nonlinear and nondivergent quasilinear elliptic problems. Here the solution has to be determined from the original strong form with derivatives up to the order $2m$. This situation is directly translated into our strong form of a FEM, see Section 2 for more details. This FEM, discussed as an example for the classical discretization theory, strongly differs from the earlier approaches.

Since the standard approaches for FEMs did not work, see below, our FEM had to incorporate several new ideas: The FEMs for general fully nonlinear elliptic problems, based upon the strong form, yield an unusual variational characterization of the discrete solution. This is studied via the classical theory of discretization methods, applied to an operator with two components, the differential and the boundary operator. We need a domain $\Omega \in C^{2m}$ and have to choose $C^{2m-1}-$FEs with local support, see Section 6, exactly vanishing on an approximate boundary, $\partial\Omega_c^h$, but violating the original (trivial) Dirichlet boundary conditions on $\partial\Omega \in C^{2m}$. We handle this variational crime by adding the geometric part of the error, due to $\partial\Omega \neq \partial\Omega_c^h$, and the FE error for a conforming FEM on Ω_c^h. For FEs in C^{2m-1}, there are still many approximation theoretic gaps for our approach. As indicated, the results are fully available for C^1-FEs in $\mathbf{R}^2$ of degree 5 and soon for higher degrees in $\mathbf{R}^n$ yielding second order convergence. A better convergence is to be expected from Davydov's forthcoming results on curved domains in $\mathbf{R}^2$ and $\mathbf{R}^3$. It is not clear when and if the corresponding results for $C^{2m-1}-$FEs will be available. The proofs in this paper are valid for all these cases.

For the stability proof, we combine linearization, coinciding strong and weak bilinear forms for the linearized differential operator and its FE approximation, with compact perturbation arguments and a new regularity result for FE solutions of weak linear FE equations. For computations mainly the weak linear forms, and thus the many known powerful numerical methods apply to our FEM as well. However the standard C^0-FEs have to be replaced by C^1-FEs or even $C^{2m-1}-$FEs. Different strategies for solving the highly nonlinear FE equations are discussed in Section 2 and in [15], for the case of a second order differential equation. The main tool is the mesh independence principle guaranteeing a quadratically converging Newton's method.

For comparing our FEM with the other available results in the literature, we shortly summarize them. There are only a few papers on finite element methods for special cases of fully nonlinear elliptic differential equations, the Monge-Ampere equations in $\mathbf{R}^2$ or modifications, [42, 31, 29]. Oliker and Prussner, [42], use convexity arguments and construct sequences monotonically converging to the solution. Fulton, [31], introduces, without a proof, a multi grid solution of generalizations of the Monge-Ampere equation as models of non-linear balance equations in meteorology. Finally, Dean and Glowinski, [29], reformulate it as a problem of Calculus of Variations involving the biharmonic (or a closely related) operator. A reinterpretation as saddle-point problem for a well-chosen augmented Lagrangian functional leads to iterative methods such as Uzawa-Douglas-Rachford. This methodology can be applied to a related problem, the Pucci equation.

These FE approaches are based upon either combinations of the special properties of a specific problem with the proposed numerical method or upon maximum and/or monotonicity arguments. But none of them seems to be applicable nor has been applied to the general fully nonlinear elliptic differential equations and systems as treated in this paper. This seems to indicate the need for new ideas.

2 Main Ideas and Results for Our FEM, an Extended Summary

We propose an approach totally different from those in the above papers. As mentioned, a major part of the analysis for fully nonlinear differential equations is based upon the strong form of these equations, see, e.g., Gilbarg, Trudinger, [32], Showalter, [45], and Taylor, [51, 52, 53]. This motivates a FEM version, based upon the strong form of the differential equation. For the analysis, e.g. monotonicity, maximum principles and the Schauder fixed point theorem are needed to prove existence, uniqueness and related results for the exact solution of the original equations. We employ, as for many other numerical approaches, the existence, uniqueness and regularity of exact solutions as the basis for our new FEM, essentially independent of these concepts.

In this introductory Section we formulate the main ideas and results. However, avoiding the technicalities for higher order equations and systems, we restrict the FEM and its results to one second order equation, see (2.1)- (2.9). We prove convergence for the FE in the framework of general discretization theory, based upon consistency and stability, see e.g. Stetter, [48]. We need $\Omega \in C^{2m}$ for the regularity of the solutions for the original and the FE equations.

The first major step is the proof of consistency. For the strong form we require FEs in C^{2m-1}, allowing an L^2 evaluation of the differential equation or system and avoiding variational crimes by interior jumps of derivatives along interior edges of the triangulation. So we only have to care for the violated boundary conditions due to the unavoidable $\partial\Omega_c^h \neq \partial\Omega \in C^{2m}$. The earlier approaches, see e.g. Lenoir, [40], and Brenner, Scott, [17], for linear problems, do not seem to be applicable to the fully nonlinear problems. They are based upon the interplay of the strong and the weak form, the latter unavailable for fully nonlinear problems. So other nonstandard, not surprisingly, similarly complicated considerations are necessary. The clue is the idea of an operator, defined by the differential and the boundary operator as two components. This requires a more complicated version of the general discretization theory. Specifically, we have to very carefully introduce the necessary projectors and spaces for including the unusual situation of violated boundary conditions into the "classical discretization theory". This avoids the contrast between "inner" and "external discretization methods", see e.g. Zeidler, [62], but implies several technicalities. Then the "classical consistency error" for the differential and the boundary operator can be estimated in a simple proof, see (7.22) and Theorem 8.1 below. Essentially we split this error into the FE error for a conforming FEM on Ω_c^h and its complementary geometric error, due to $\partial\Omega \neq \partial\Omega_c^h$, according to a new idea of Tiihonen, [54].

The most difficult part is the proof of stability. We did not see a direct possibility for the strong form. So we combine linearization with regularity of FE solutions and variational crimes. This is the essential reason for the difference of our FEM from the standard methods and those in the above papers. Stability is first proved for the linearized weak form. Unfortunately, we loose the powerful machinery of weak bilinear forms and their FEMs, *unless* we find a kind of *connection* between the two levels. This is achieved by Proposition 9.1 and Lemma 9.1. The first shows equal weak and strong FE bilinear forms in our case. The latter new result states, similarly to the exact solution of a "smooth" linear elliptic problem, a regularity and stability result for our FE solutions. This, and the available existence, uniqueness and regularity results, require $\Omega \in C^{2m}$, implying the above violated boundary conditions for our FEM. Generalizing our earlier compact perturbation techniques, see Böhmer, Sassmannshausen, [16, 13, 12], to violated boundary conditions allows lifting the stability from coercive to general elliptic bilinear forms. A combination of these results transforms the weak back to the strong stability and allows the proof of convergence.

We have to discuss the highly unpleasant required smoothness, $\partial\Omega \in C^{2m}$ and FEs in C^{2m-1}. These FEs are chosen for avoiding interior variational crimes. They are available, e.g., in Zenísek, e.g. [63, 64], but require a pretty complicated handling. But we need additionally a stable splitting of these spaces into subspaces of FEs vanishing along $\partial\Omega_c^h$ and their complement. Presently this is available only for Davydov's, [25], C^1-FEs. He has extended some known results and proved the new results, necessary for our approach, in his Theorems 1, 2, 3 and his discussion of the Argyris element. Presently, these necessary "error estimates", see (6.10), the "inverse estimates", see (6.4), and the necessary boundary splitting are available on polygonal Ω^h in $\mathbf{R}^2$ and for a modified Argyris FE. This restricts the convergence of our FEM to the order 2. Davydov exspects generalizing his results to FEs in C^1, piecewise of degree $d \geq 5$ in $\mathbf{R}^n$, formulated as conjecture in the last remark in [25], still of order 2 or to approximate curved $\partial\Omega_c^h$, yielding convergence of an order $p > 2$, available soon. It is a matter of speculation whether the results in [25] can be extended to $C^{2m-1}-$FEs. If this should not be possible, elliptic problems of order $2m$ can be reduced to systems of order 2. This might be worthwhile anyway, since the compilation of $C^{2m-1}-$FEs is much more complicated than that of C^1- or even C^0-FEs. In this paper we generalize Davydov's results to $C^{2m-1}-$FEs to all these cases, indicated by using $d \geq 5, \Omega_c^h, \mathbf{R}^n$ and $C^{2m-1}-$FEs. The proofs in the present paper admit all these cases.

We extend our results to the necessary quadrature and cubature approximations, abbreviated as *quadrature approximations,* see Section 10: The test integrals for nonlinear operators, see (2.4), can be exactly computed only for special cases. So usually they have to be approximated. We prove the corresponding convergence results for quadrature approximated FEMs.

We only indicate another important aspect, published in [15]. The highly nonlinear FE equations have to be calculated and solved. The solution is obtained by combining continuation methods with a quadratically converging discrete Newton method, see (2.9), based upon the mesh independence principle, see Allgower, Böhmer, [2, 3], and with Potra, Rheinboldt, [4]. Again linear problems turn out to be essential for solving these equations. As a welcome consequence of our coinciding weak and strong FE bilinear forms, the standard solution techniques for weak linear problems are applicable to the fully nonlinear equations as well. However, the standard C^0-FEs have to be replaced by $C^{2m-1}-$FEs.

With the notation introduced below let G be a uniformly elliptic operator on Ω, see (5.2), (4.9). Violated boundary conditions for the FEM below will play a dominant role. Thus we do not include the homogeneous Dirichlet conditions into the space of solution functions, but impose two independent conditions. The exact solution, u_0, is determined by the differential equation and the boundary condition in the trace sense (mostly we omit Ω)

$$\begin{aligned} u_0 \in \mathcal{U} := H^2(\Omega): \ G(u_0)(\cdot) := G^w(\cdot, u_0(\cdot), Du_0(\cdot), D^2u_0(\cdot)) = 0 \ \in L^2 \\ \iff (G(u_0), v)_{L^2(\Omega)} = 0 \ \forall v \in L^2 = (L^2)', \\ \text{and } u_0|_{\partial\Omega} = 0 \iff (u_0|_{\partial\Omega}, v_b)_{L^2(\partial\Omega)} = 0 \ \forall v_b \in L^2(\partial\Omega), \end{aligned} \tag{2.1}$$

with $G^w : \Omega' \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n^2} \to \mathbf{R}, \overline{\Omega} \subset \Omega$, see (5.1). Since $L^2(\partial\Omega)$ is dense in $H^{3/2}(\partial\Omega)$ the testing in the last line with only $\forall v_b \in L^2(\partial\Omega)$ is correct. To allow a systematic study of differential and boundary operator, we introduce an equivalent form determining

$$\begin{aligned} u_0 \in \mathcal{U}: \quad 0 = F(u_0) := (G(u_0)|_\Omega = 0, u_0|_{\partial\Omega} = 0) \iff \\ (G(u_0), v)_{L^2(\Omega)} + (u_0|_{\partial\Omega}, v_b)_{L^2(\partial\Omega)} = 0 \ \forall v \in L^2(\Omega), v_b \in L^2(\partial\Omega), \end{aligned} \tag{2.2}$$

The difference between quasilinear and fully nonlinear problems for a second order equation is discussed in Section 3. This motivated our FEMs for fully nonlinear elliptic equations and systems of second and higher order. The standard ellipticity condition for nonlinear via the linear systems has to be modified for equations and systems of higher order, implying the coercivity of the linearized main part, see Section 4. This allows formulating some existence, uniqueness and regularity results in Section 5 and the corresponding FEM for fully nonlinear equations and systems of orders 2 and $2m$ in Section 7.

We choose FE subspaces of C^1-, later $C^{2m-1}-$FE spaces, of local degree $d \geq 5$,

$$\mathcal{U}^h := S^h \subset \mathcal{S}_d^1(\mathcal{T}_c^h) \text{ and } \mathcal{V}^h := \{u^h \in S^h : u^h|_{\partial\Omega_c^h} = 0\} \subset \mathcal{U}^h \neq \mathcal{V}^h, \tag{2.3}$$

defined on a triangulation $\mathcal{T}_c^h$ of a polyhedral Ω^h or curved $\Omega_c^h \approx \Omega$, with the same notation $\mathcal{T}_c^h$ and Ω_c^h for both cases. The $S^h \subset \mathcal{S}_d^1(\mathcal{T}_c^h)$ will be introduced in Section 6. Then the FE solution, $u_0^h \in \mathcal{U}^h$, is determined by testing the differential equation w.r.t. $v^h \in \mathcal{V}^h$, not in the standard H^1, but in the L^2 sense (note that $G(u_0^h)$ is defined on Ω_c^h):

$$u_0^h \in \mathcal{U}^h \text{ s.t. } (G(u_0^h), v^h)_{L^2(\Omega_c^h)} = 0 \ \forall v^h \in \mathcal{V}^h, \text{and } u_0^h|_{\partial\Omega_c^h} = 0 \Leftrightarrow u_0^h \in \mathcal{V}^h \tag{2.4}$$

see Theorem 9.1. The discrete counterpart to (2.2) is

$$\begin{aligned} u_0^h \in \mathcal{U}^h : 0 = F^h(u_0^h) := (G^h(u_0^h), u_0^h|_{\partial\Omega_c^h}) = 0 \iff (G(u_0^h), v^h)_{L^2(\Omega_c^h)} + \\ +(u_0^h|_{\partial\Omega_c^h}, v_b^h)_{L^2(\partial\Omega_c^h)} = 0 \ \forall v^h \in \mathcal{V}^h, v_b^h \in \mathcal{V}_b^h := \{u^h|_{\partial\Omega_c^h} : u^h \in \mathcal{U}^h\}. \end{aligned} \tag{2.5}$$

Comparing u_0^h and u_0 requires extension operators,$E_c : H^2(\Omega) \to H^2(\Omega_c^h)$.

Our FEM, transforming $F(u_0) = 0$ in (2.2) into $F^h(u_0^h) = 0$ in (2.5), and its generalizations in Section 7, is a classical discretization method in the sense of, e.g., Stetter, [48]. Related approaches for general discretization methods are discussed in Stummel, [49, 50], Vainikko, [55], Keller, [39], Zeidler, [60, 61, 62], Böhmer, [10, 13, 12]. Here Stetter's, [48], Theorems 1.2.3 - 1.2.5 provide the tools for showing convergence, if consistency and stability can be proved. Thus Sections 6 - 9 are the core of the paper.

Since this method is very unusual, we summarize the main ideas for the new FEM, referring to the corresponding Sections and Subsections. In the general case, we will update the H^1, H^2, H_0^1 and C^1-FEs into $H^m(\Omega, \mathbf{R}^q)$, H^{2m}, H_0^m and $C^{2m-1}(\Omega_c^h, \mathbf{R}^q)-$FEs, where $2m$ indicates the order and q the number of equations of the operators. Accordingly, (2.2) and (2.5) have to be generalized. Finally, the boundary condition $u|_{\partial\Omega} = 0$ has to be replaced by $(\partial^k u)/(\partial\nu)^k|_{\partial\Omega} = 0,\ k = 0, \dots, m-1$.

1. *The FEM, transforming (2.2) into (2.5), is a general discretization method:*

- The existence, uniqueness and regularity results for the original problems in Section 5 are the basic results for a general discretization method.
- The fully nonlinear elliptic problem (2.1), (2.2) is approximated by the *strong* form (2.4), (2.5), requiring FEs in C^1 as ansatz, their restriction to the boundary and $C^1 \cap H_0^1$ as test functions for the boundary and differential operators, resp., see (2.3) and see Section 6 for the relevant results and extension theorems.
- The FEM is a general linear discretization method, Section 7.
- In Subsections 7.1 - 7.3 we reformulate the FEM, introduce the necessary projectors and verify the required conditions: The nonconforming FEM is convergent if it is stable and consistent, see Theorem 7.1.
- As usual, for a "smooth" problem our nonconforming method is consistent, see Section 8, Theorem 8.1.

2. *Stability for the nonlinear* strong *equation, (2.5), is the hard problem, see Section 9:*

- The nonlinear stability is a consequence of the stability for the *linearized strong* problem, see (2.6) and Theorem 9.1. For given u_0^h and f, ϕ, determine the FE solution u_1^h from

$$u_1^h \in \mathcal{U}^h \text{ s.t. } (F^h)'(u_0^h)u_1^h = \big((G^h)'(u_0^h)u_1^h, u_1^h|_{\partial\Omega_c^h}\big) = (f, \phi). \tag{2.6}$$

  For this linearized problem, two FEMs complement each other:

- The first component of the linearized *strong* and *weak* forms coincide:

$$(G'(u_0^h)u^h,\ v^h)_{L^2(\Omega_c^h)} = \langle G'(u_0^h)u^h,\ v^h\rangle \ \ \forall\ u^h \in \mathcal{U}^h, v^h \in \mathcal{V}^h, \tag{2.7}$$

  with $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-1}(\Omega_c^h)\times H^1(\Omega_c^h)}$, see Proposition 9.1.

- For $(f,\phi) \in H^{-1}(\Omega_c^h) \times H^{1/2}(\partial\Omega_c^h)$, the linearized *weak* problem

$$(F^h)'(u_0^h)u_1^h = (f,\phi) \Leftrightarrow \langle G'(u_0^h)u_1^h - f, v^h\rangle + ((u_1^h - \phi)|_{\partial\Omega_c^h}, v_b^h)_{L^2(\partial\Omega_c^h)} = 0$$

  $\forall v^h \in \mathcal{V}^h, v_b^h \in \mathcal{V}_b^h$, is stable, essentially if u_0 is an isolated solution of $F(u_0) = 0$ with boundedly invertible $F'(u_0)$. This is a consequence of the discussion at the end of Section 5 and the compactness arguments in Theorem 9.3.

- The novel Lemma 9.1 transmits the regularity results for the exact solution of a linear elliptic problem to the corresponding FE solutions. In the *weak linearized* problem, $(F^h)'(u_0^h)u_1^h = (f,\phi)$, we determine $u_1^h \in \mathcal{U}^h$ for the smoother $(f,\phi) \in L^2(\Omega_c^h) \times H^{3/2}(\partial\Omega_c^h)$, see (2.5), (2.7). The stability of this *weak* problem implies the stability of the *strong* problem

$$(G'(u_0^h)u_1^h - f, v^h)_{L^2(\Omega_c^h)} + ((u_0^h - \phi)|_{\partial\Omega_c^h}, v_b^h)_{L^2(\partial\Omega_c^h)} = 0 \; \forall v^h \in \mathcal{V}^h, v_b^h \in \mathcal{V}_b^h.$$

  This allows estimates for discrete solutions w.r.t. a piecewise H^2-norm instead of the usual H^1-norm for a right hand side in $L^2(\Omega_c^h) \times H^{3/2}(\partial\Omega_c^h)$ instead of the usual $H^{-1}(\Omega_c^h) \times H^{1/2}(\partial\Omega_c^h)$, but it requires smooth coefficients and a domain $\Omega \in C^{2m}$, necessary for good convergence anyway.

3. *Summary: The FE solutions u_0^h of (2.4), (2.5), uniquely exist, for small enough h, and converge, see (2.8) and Theorem 9.5.*
4. *Quadrature approximations for (2.4), (2.6), see Section 10:*

- The exact equations (2.4), (2.6) usually have to be approximated by good enough quadrature to maintain convergence, see Theorem 10.2.

Above we have referred to the convergence of the FE and the discrete Newton method: Let (2.1), for smooth enough G and Ω, have a smooth isolated solution $u_0 \in H^s$ and choose FEs of local degree d in C^1. Then the discrete solutions u_0^h of (2.4), extended to Ω_c^h as $E_c u_0$, converge as, see Theorem 9.5,

$$\|E_c u_0 - u_0^h\|_{H^2(\Omega_c^h)}^h \leq Ch^{\min\{\ell-2,p\}}\|u_0\|_{H^\ell(\Omega)} \text{ for large enough } \ell,\ d, \tag{2.8}$$

with $p = 2$ and $p > 2$ for polyhedral and curved Ω^h and Ω_c^h. The Newton method for the discrete problem (2.5) has the form (2.9) and converges quadratically, see Theorem **??**: For a good enough approximation $u_1^h \approx u_0^h$ and $\nu = 1, \ldots$ compute, see Böhmer, [15],

$$\begin{aligned} u_{\nu+1}^h \in \mathcal{U}^h :\ (F^h)'(u_\nu^h)(u_{\nu+1}^h - u_\nu^h) = -F^h(u_\nu^h), \\ \Longrightarrow \|u_{\nu+1}^h - u_\nu^h\|_{H^2(\Omega_c^h)}^h \leq C(\|u_\nu^h - u_{\nu-1}^h\|_{H^2(\Omega_c^h)}^h)^2. \end{aligned} \tag{2.9}$$

3 FEMs for Divergent Quasilinear Equations

With the standard notations for derivatives, see (4.1), Sobolev spaces, see (4.5), and

$$\partial^0 u := u, \text{ and } (-1)_{j>0} := 1 \text{ for } j = 0 \text{ and } := -1 \text{ for } j \geq 1. \tag{3.1}$$

the strong and weak form of a divergent quasilinear elliptic equation $G : \mathcal{D}(G) \subset \mathcal{U} = H^2(\Omega) \to \mathcal{V} = L^2(\Omega)$ and $G : \mathcal{D}(G) \subset \mathcal{W} = H^1(\Omega) \to \mathcal{W}' = H^{-1}(\Omega)$, resp., with $\mathcal{U}_0 = \mathcal{U} \cap H_0^1(\Omega), \mathcal{W}_0 = H_0^1(\Omega)$, is solved by

$$u_0 \in \mathcal{U}_0 : \; Gu_0 - f = \sum_{j=0}^{n} (-1)_{j>0} \partial^j (a_j(x, u_0, Du_0)) - f = 0 \in \mathcal{V}, \tag{3.2}$$

$$\begin{aligned} u_0 \in \mathcal{W}_0 : \langle Gu_0, v \rangle_{\mathcal{W}' \times \mathcal{W}} &:= a(u_0, v) := \int_\Omega \sum_{j=0}^{n} a_j(x, u_0, Du_0) \partial^j v dx \\ = \langle f, v \rangle_{\mathcal{W}' \times \mathcal{W}}, \quad &\forall v \in \mathcal{W}_0, \text{ with nonlinear } a_j : \Omega \times \mathbf{R}^{n+1} \to \mathbf{R}, \end{aligned} \tag{3.3}$$

see Skrypnik, [46], Zeidler, [62], Böhmer, [14]. These a_j are "Nemyckii operators".

For the weak form (3.3) on a polygonal $\Omega = \Omega^h$, it is straight forward to formulate a conforming FEM. As test and approximation spaces $\mathcal{W}_0^h = \mathcal{V}^h \subset \mathcal{W}_0$ we use the standard continuous FEs vanishing along $\partial\Omega$. Thus all the following terms are well defined. We determine $u_0^h \in \mathcal{W}_0^h$ s.t.

$$\begin{aligned} a(u_0^h, v^h) = \langle Gu_0^h, v^h \rangle_{\mathcal{W}' \times \mathcal{W}} &:= \int_\Omega \sum_{j=0}^{n} a_j(x, u_0^h, Du_0^h) \partial^j v^h dx = \\ \langle f, v^h \rangle_{\mathcal{W}' \times \mathcal{W}} \text{ e.g., } &= \int_\Omega \sum_{j=0}^{n} f_j \partial^j v^h dx \;\; \forall\, v^h \in \mathcal{V}^h \subset \mathcal{W}_0. \end{aligned} \tag{3.4}$$

For a nondivergent quasilinear or a fully nonlinear G a weak form is impossible. We have to determine

$$u_0 \in \mathcal{U}_0 : \qquad Gu_0 - f = \sum_{j=0}^{n} a_{i,j}(\cdot, u_0, Du_0) \partial^i \partial^j u_0 - f(\cdot) = 0 \in \mathcal{V}, \text{ or} \tag{3.5}$$

$$u_0 \in \mathcal{U}_0 : \qquad G(u_0) = G(\cdot, u_0(\cdot), Du_0(\cdot), D^2 u_0(\cdot)) = 0 \in \mathcal{V}. \tag{3.6}$$

Consequently, a weak form of a FEM as in (3.4) is not possible for (3.5), (3.6). This has motivated our nonstandard FEM, based upon the strong form (3.6).

4 Linear Elliptic Equations and Systems

The discussion of the nonlinear problems is based upon an appropriate definition of ellipticity for the linearized operator for the different cases. Compared to the familiar situation of one equation of second order, there is a difference which is important in our context. For second order linearized equations the conditions of ellipticity (4.9) and of the $H_0^1(\Omega)$-coercivity of the principle part are "nearly equivalent". This simple transition is no longer possible for elliptic systems and equations of higher order, so $2m \geq 2$, and q denotes the number of equations in the system, see (4.11) and Theorem 4.2. For the different cases of nonlinear elliptic equations and systems, we will have to require appropriate conditions for the three combinations for the cases $m = q = 1$, $m > q = 1$, $m = 1 < q$ and $m, q > 1$, see below.

Furthermore, for $2m, q \geq 2$ the $H_0^m(\Omega, \mathbf{R}^q)$-coercivity of the principle part does not seem to be available in the appropriate generality. To formulate the necessary conditions and to prove the $H_0^m(\Omega, \mathbf{R}^q)$-coercivity is the main purpose of this Section. We define the *(partial) derivatives* and the corresponding vectors, reals and integers as

$$\begin{aligned}
&\partial^i u = \frac{\partial u}{\partial x_i}, \text{ and } \partial^\alpha u = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial {x_n}^{\alpha_n}} u, \ D^0 u = u, \\
&D^k u = (\partial^\alpha u)_{|\alpha|=k} = Du = D^1 u = (\partial^\alpha u)_{|\alpha|=1} = (\partial^1 u, \dots, \partial^n u), \\
&\vartheta = (\vartheta^1, \dots, \vartheta^n) \in \mathbf{R}^n, \vartheta^\alpha = (\vartheta^1)^{\alpha_1} \cdots (\vartheta^n)^{\alpha_n}, \vartheta^i, \Theta^0 \in \mathbf{R}, i \geq 0, \text{choose} \\
&\quad n_k, N_k, \text{ s.t. } \Theta^k = (\vartheta^\alpha)_{|\alpha|=k} \in \mathbf{R}^{n^k}, (\vartheta^\alpha)_{|\alpha| \leq k} \in \mathbf{R}^{N_k}, \Theta = \Theta^1, \text{ with} \\
&\partial^i u(x), \partial^\alpha u(x) \in \mathbf{R}, Du(x) \in \mathbf{R}^n, D^k u(x) \in \mathbf{R}^{n^k}, |\alpha| = 0 : \partial^\alpha u = u, \vartheta^\alpha = 1.
\end{aligned} \tag{4.1}$$

For systems we have to modify the above notations: We have to apply the partials $\partial^l, l = 1, \cdots, n$, to q components of vector functions, $u_j, v_i, i, j = 1, \cdots, q$, and modify (4.1) from reals $\vartheta^l, \vartheta^\alpha$, $\partial^l u(x), \partial^\alpha u(x) \in \mathbf{R}$, to vectors $\vec{\vartheta}^l, \vec{\vartheta}^\alpha$, $\partial^l \vec{u}(x), \partial^\alpha \vec{u}(x) \in \mathbf{R}^q$, and from $n-$vectors $\vartheta = (\vartheta^1, \dots, \vartheta^n), Du(x) \in \mathbf{R}^n$ to $n \times q-$matrices $\vec{\Theta}, D\vec{u}(x) \in \mathbf{R}^{n \times q}$. In addition to the notations in (4.1) we need – mind the difference of Θ in (4.1) and $\vec{\Theta}$ here: [1]

$$\begin{aligned}
&\vec{u} = (u_1, \cdots, u_q), \partial^\alpha \vec{u} = (\partial^\alpha u_1, \cdots, \partial^\alpha u_q), \vec{\vartheta}^l = (\vartheta_1^l, \cdots, \vartheta_q^l), \\
&l = 1, \dots, n, \ \vec{\vartheta}^\alpha = (\vartheta_1^\alpha, \cdots, \vartheta_q^\alpha), \text{with } \partial^\alpha \vec{u} = \vec{u} \text{ for } |\alpha| = 0 \text{ and} \\
&\vec{u}(x), \ \partial^\alpha \vec{u}(x), \vec{\vartheta}^l, \vec{\vartheta}^\alpha \in \mathbf{R}^q, \ \vec{\Theta} = (\vec{\vartheta}^1, \cdots, \vec{\vartheta}^n) \in \mathbf{R}^{n \times q}, |\vec{\Theta}| = |\vec{\Theta}|_{nq} \in \mathbf{R}, \\
&D\vec{u}(x) = (\partial^1 \vec{u}, \cdots, \partial^n \vec{u})(x) = (Du_1, \cdots, Du_q)(x) \in \mathbf{R}^{n \times q}.
\end{aligned} \tag{4.2}$$

We formulate the bilinear form and the linear operator for the case $m, q \geq 1$ and have to distinguish for $m + q \geq 3$ the different ellipticity conditions. To avoid too many technicalities we restrict the discussion to *Dirichlet boundary conditions*

$$B_D \vec{u} := (\frac{\partial^j \vec{u}}{\partial \nu^j}|_{\partial\Omega}, j = 0, \dots, m-1) = \vec{0} \ \Leftrightarrow \partial^\alpha \vec{u}|_{\partial\Omega} = 0 \ \forall |\alpha| \leq m-1, \tag{4.3}$$

in the trace sense. We formulate the conditions for the domain and introduce four Hilbert spaces with $\mathcal{W}_0$ for trivial Dirichlet boundary conditions:

$$\Omega \in C_L \text{ is an (open) bounded domain in } \mathbf{R}^n \text{ and} \tag{4.4}$$

$$\begin{aligned}
\mathcal{W} := H^m(\Omega, \mathbf{R}^q), \mathcal{U} := H^{2m}(\Omega, \mathbf{R}^q) \subset \mathcal{W} \subset \mathcal{V} := L^2(\Omega, \mathbf{R}^q) = \mathcal{V}', \\
\mathcal{W}_0 := H_0^m(\Omega, \mathbf{R}^q) = \{\vec{u} \in \mathcal{W} : D^j \vec{u}(x)|_{\partial\Omega} = 0 \forall j \leq m-1\}, \\
\mathcal{U}_0 := \mathcal{W}_0 \cap \mathcal{U}, \ \mathcal{W}' := H^{-m}(\Omega), \ \mathcal{V}_b := L^2(\partial\Omega, \mathbf{R}^q) = \mathcal{V}_b', \\
\mathcal{V}_D := H^{2m-1/2}(\partial\Omega, \mathbf{R}^q) \times \dots \times H^{m+1/2}(\partial\Omega, \mathbf{R}^q) = B_D \mathcal{U}, \\
\mathcal{W}_D := H^{m-1/2}(\partial\Omega, \mathbf{R}^q) \times \dots \times H^{1/2}(\partial\Omega, \mathbf{R}^q) = B_D \mathcal{W}, \ \vec{\Phi} \in \mathcal{V}_D \text{ or } \mathcal{W}_D,
\end{aligned} \tag{4.5}$$

[1] the usual notations for θ and $\vec{u}$ motivate the certainly not optimal notation $\vec{\vartheta}^l$, $\vec{\Theta}$, which turns out to be appropriate below

with the standard scalar products, norms, seminorms for $\mathcal{U}, \ldots$, and the Cartesian $(\cdot,\cdot)_q$ in $\mathbf{R}^q$ or $\mathbf{C}^q$. For Dirichlet conditions, we determine a strong and weak solution $\vec{u}^0 = (u_1^0, \cdots, u_q^0) \in \mathcal{U}_0$ for $\vec{f} \in \mathcal{V}$ and $\vec{u}^0 \in \mathcal{W}_0$ for $\vec{f} \in \mathcal{W}'$ for (4.6) and (4.7) from

$$A_s \vec{u}^0 := \sum_{|\alpha|,|\beta|\le m} (-1)^{|\alpha|} \partial^\alpha (A_{\alpha\beta} \partial^\beta \vec{u}^0) = \vec{f}, A_{\alpha\beta}(x) \in W^{|\alpha|,\infty}(\Omega, \mathbf{R}^{q\times q}), \tag{4.6}$$

$$< A\vec{u}^0, \vec{v} >_{\mathcal{W}'\times\mathcal{W}} := a(\vec{u}^0, \vec{v}) := \int_\Omega \sum_{|\alpha|,|\beta|\le m} (A_{\alpha\beta}\partial^\beta \vec{u}^0, \partial^\alpha \vec{v})_q dx = \tag{4.7}$$

$$=< \vec{f}, \vec{v} >_{\mathcal{W}'\times\mathcal{W}} \ \forall\ \vec{v} \in \mathcal{W}, A_{\alpha\beta}(x) \in L^\infty(\Omega, \mathbf{R}^{q\times q}),\ \ \vec{f} \in \mathcal{W}',$$
$$A_s : H^{2m}(\Omega, \mathbf{R}^q) \to L^2(\Omega, \mathbf{R}^q), A : H^m(\Omega, \mathbf{R}^q) \to H^{-m}(\Omega, \mathbf{R}^q).$$

Obviously, all these linear operators are well defined. As usual we transform the *strong into the weak form* by taking the standard scalar product of (4.6) with $\vec{v} = (v_1, \cdots, v_q) \in \mathcal{W}$ and integrating by parts.

We formulate the *strong Legendre* and the *strong Legendre-Hadamard condition* for the orders 2 and $2m$, resp., and allow variable coefficients $A_{\alpha\beta}$ for $m > 1$, see Remark 4.1.

Definition 4.1 Linear elliptic equations and systems of order $2m, m \ge 1 :$ *The* principle part, A_p, *in (4.8) satisfies the* strong Legendre *(4.9) and the* strong Legendre-Hadamard condition *(4.11), if* $0 < \lambda$ *exists s.t.:*

$$a_p(\vec{u}, \vec{v}) := \int_\Omega A_p \vec{u}\vec{v} dx := \int_\Omega \sum_{|\alpha|=|\beta|=m} (A_{\alpha\beta}\partial^\beta \vec{u}, \partial^\alpha \vec{v})_q dx,\ \ \forall\ \vec{u}, \vec{v} \in \mathcal{W}_0, \tag{4.8}$$

$$\forall x \in \Omega, \vec{\Theta} \in \mathbf{R}^{n\times q} : \sum_{|\alpha|=|\beta|=m} (A_{\alpha\beta}(x)\vec{\vartheta}^\beta)\vec{\vartheta}^\alpha \ge \lambda |\vec{\Theta}|^{2m},\ \textit{for the Legendre, and} \tag{4.9}$$

$$\tag{4.10}$$

$$\forall \vartheta \in \mathbf{R}^n, \eta \in \mathbf{C}^q : \overline{\eta}^T A_p(x, \vartheta)\eta = \overline{\eta}^T \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\vartheta^\beta \vartheta^\alpha \eta \ge \lambda |\vartheta|^{2m} |\eta|^2. \tag{4.11}$$

Upper bounds a.e. follow from the above $A_{\alpha\beta}(x) \in L^\infty(\Omega, \mathbf{R}^{q\times q})$.

Existence and uniqueness results for (4.7) follow via the Fredholm alternative. It is satisfied by the bilinear form $a(\vec{u}, \vec{v}) =: a_p(\vec{u}, \vec{v}) + b(\vec{u}, \vec{v})$. In fact, $a(\cdot,\cdot), a_p(\cdot,\cdot) : \mathcal{W}\times\mathcal{W} \to \mathbf{R}$ and $\vec{f} \in \mathcal{W}' \to \mathbf{R}$ are well defined *bounded* forms and $a_p(\cdot,\cdot)$ is shown to be coercive

Theorem 4.1 *For a system of the form (4.7), the bilinear and linear forms* $a(\vec{u}, \vec{v})$ *and* $< \vec{f}, \vec{v} >_{\mathcal{W}'\times\mathcal{W}}$ *, and the principle part* $a_p(\vec{u}, \vec{v})$ *as in (4.8) are continuous in* $\mathcal{W}$. *For* $A_{\alpha\beta} \in W^{|\alpha|,\infty}(\Omega, \mathbf{R}^{q\times q})$, *any solution* $\vec{u}^0 \in \mathcal{U}_0$ *of (4.6) necessarily satisfies the weak equation*

$$\vec{u}^0 \in \mathcal{W}_0 :\ \ a(\vec{u}^0, \vec{v}) =< \vec{f}, \vec{v} >_{\mathcal{W}'\times\mathcal{W}} \forall \vec{v} \in \mathcal{W}_0. \tag{4.12}$$

Vice versa, every smooth solution $\vec{u}^0 \in \mathcal{U}_0$ *of (4.7) satisfies (4.6).*

We summarize and prove the $\mathcal{W}_0-$coercivity of the principle part.

Theorem 4.2 Coercivity: *Let in (4.7) the coefficients satisfy*

$$A_{\alpha\beta} \in L^\infty(\Omega, \mathbf{R}^{q\times q}) \text{ for } |\alpha|, |\beta| \le m \text{ and } A_{\alpha\beta} \in C(\overline{\Omega}) \text{ for } |\alpha| = |\beta| = m > 1, \quad (4.13)$$

and with $A_{\alpha\beta} = a_{\alpha\beta}$ for $q = 1$ and $a_{\alpha\beta} \in C(\overline{\Omega})$ for $|\alpha| = |\beta| = m > 1$ and A_{ij} or a_{ij} for $m = 1, q > 1$ or $m = 1 = q$. Furthermore assume (4.9), the strong Legendre condition for $m \ge 1, q = 1$ and $m = 1, q \ge 1$, and (4.11), the strong Legendre-Hadamard condition, for $m > 1, q > 1$. Then the principle part $a_p(\vec{u}, \vec{v})$ is $\mathcal{W}_0-$coercive. [2]

Remark 4.1 *This coercivity, hence $a_p(\vec{u}, \vec{u}) \ge \alpha\|\vec{u}\|^2_{\mathcal{W}_0},\ \alpha > 0$, implies the existence of a unique solution $\vec{u}^0 \in \mathcal{W}_0$ for (4.7) with $a(\vec{u}, \vec{v})$ replaced by $a_p(\vec{u}, \vec{v})$. For any $\mathcal{W}_0-$coercive principle part $a_p(\vec{u}, \vec{v})$, the above $a(\vec{u}, \vec{v})$ is a $\mathcal{W}-$elliptic bilinear form. So the Fredholm alternative applies and the index A in $\mathcal{W}_0$ vanishes. This implies for (4.7) either none or infinitely many solutions if additional conditions are violated or satisfied or otherwise exactly one solution and $\|A^{-1}\|_{\mathcal{W}\hookleftarrow\mathcal{W}'} < \infty$.*

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[2] It is defined, with the imaginary unit ι with $\iota^2 = -1$, for the components $u = u_j \in C_0^\infty(\mathbf{R}^n)$:

$$\begin{aligned} F : C_0^\infty(\mathbf{R}^n) \to L^2(\mathbf{R}^n), &\quad \text{with } <\xi, x>_n = \sum_{k=1}^n \xi_k x_k \ \forall\ \xi, x \in \mathbf{R}^n, \text{ as} \\ \hat{u}(\xi) := (Fu)(\xi) &:= (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-\iota<\xi,x>_n} u(x)dx \\ &:= \lim_{R\to\infty} (2\pi)^{-n/2} \int_{|\xi|_\infty \le R} e^{-\iota<\xi,x>_n} u(x)dx. \end{aligned} \quad (4.14)$$

Since $C_0^\infty(\mathbf{R}^n)$ is dense in $L^2(\mathbf{R}^n)$, this F can be extended to $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ and inverted on $C_0^\infty(\mathbf{R}^n)$, dense in $L^2(\mathbf{R}^n)$, and again extended to $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ as

$$(F^{-1}\hat{u})(x) := (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{\iota<\xi,x>_n} \hat{u}(\xi)d\xi \ \forall\ u \in C_0^\infty(\mathbf{R}^n), \text{ extended to } L^2(\mathbf{R}^n).$$

Its characteristics are summarized in

Theorem 4.3 *The Fourier transform and its inverse are linear isometries:*

$$\begin{aligned} F, F^{-1} \quad : \quad & L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n), \text{ with} \\ & \|F\|_{L^2(\mathbf{R}^n)\hookleftarrow L^2(\mathbf{R}^n)} = \|F^{-1}\|_{L^2(\mathbf{R}^n)\hookleftarrow L^2(\mathbf{R}^n)} = 1, \\ & (u, v)_{L^2(\mathbf{R}^n)} = (\hat{u}, \hat{v})_{L^2(\mathbf{R}^n)} \ \forall\ u, v \in L^2(\mathbf{R}^n), \quad (4.15) \\ & \forall\ u \in H^k(\mathbf{R}^n), |\alpha| \le k \quad F(\partial^\alpha u)(\xi) = \iota^{|\alpha|}\xi^\alpha \hat{u}(\xi), \text{ and} \\ & \|u\|_{H^k(\mathbf{R}^n)} = \Big\|\sqrt{\sum_{|\alpha|\le k} |\xi^\alpha|^2}\hat{u}(\xi)\Big\|_{L^2(\mathbf{R}^n)} \text{ and} \\ & \text{the semi norm } |u|_{H^k(\mathbf{R}^n)} = \Big\|\sqrt{\sum_{|\alpha|=k} |\xi^\alpha|^2}\hat{u}(\xi)\Big\|_{L^2(\mathbf{R}^n)}. \quad (4.16) \end{aligned}$$

By applying F componentwise, this F can be extended to $F : L^2(\mathbf{R}^n, \mathbf{R}^q) \to L^2(\mathbf{R}^n, \mathbf{R}^q)$, mind that then $(u_i, v_j)_{L^2(\mathbf{R}^n)} = (\hat{u}_i, \hat{v}_j)_{L^2(\mathbf{R}^n)}\ \forall\ i, j = 1, \cdots, q$.

Proof: We only prove the result for $m > 1, q > 1$. All other cases are easily available in the Literature, see e.g., Hackbusch, [34], Chen, Wu [19]. We use the standard Fourier transforms, see, e.g., Hackbusch [34], Theorem 7.2.7.

We start the proof, assuming that the highest coefficients are constant.

$$a_p(\vec{u}, \vec{u}) = \int_\Omega \sum_{|\alpha|=|\beta|=m} \left(A_{\alpha\beta}\partial^\beta \vec{u}, \partial^\alpha \vec{v}\right)_q dx = \sum_{j,k=1}^{q} a_{\alpha\beta}^{jk} \partial^\beta u_k \partial^\alpha v_j \tag{4.17}$$

We extend the $\vec{u} \in \mathcal{W}_0$ to $\vec{u}_e \in H_0^m(\mathbf{R}^n, \mathbf{R}^q)$ by one of the many extension theorems, s.t. $\vec{u}$ has compact support $\overline{\Omega} \subset \operatorname{supp} \vec{u}_e$. Thus the *Fourier transform,* see, e.g., [34], Subsection 6.2.3, is well defined for this $\vec{u}$. For $a_p(\vec{u}, \vec{u})$ we obtain for constant coefficients

$$a_p(\vec{u}, \vec{u}) = \sum_{j,k=1}^{q} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^{jk} \int_{\mathbf{R}^n} \left(\partial^\beta u_k \cdot \partial^\alpha u_j\right) dx \ \forall\ \vec{u} \in \mathcal{W}_0$$

We apply the Fourier transform to each term $\partial^\beta u_k$ and $\partial^\alpha u_j$. By the invariance of the scalar products in (4.15), the strong Legendre-Hadamard condition (4.11) and with $\eta_k = \hat{u}_k(\xi)$ we get

$$\begin{aligned} a_p(\vec{u}, \vec{u}) &= \sum_{j,k=1}^{q} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^{jk} \int_{\mathbf{R}^n} \left([\iota^{|\beta|} \xi^\beta \hat{u}_k(\xi)] \overline{[\iota^{|\alpha|} \xi^\alpha \hat{u}_j(\xi)]}\right) d\xi \\ &= \int_{\mathbf{R}^n} \sum_{j,k=1}^{q} \sum_{|\alpha|=|\beta|=m} \left(a_{\alpha\beta}^{jk} \xi^\beta \hat{u}_k(\xi) \xi^\alpha \overline{[\hat{u}_j(\xi)]}\right) d\xi \\ &\geq \int_{\mathbf{R}^n} \lambda \Big(\sum_{\ell=1}^{n} (\xi_\ell)^2\Big)^m \sum_{i=1}^{q} |\hat{u}_i(\xi)|^2 d\xi \end{aligned} \tag{4.18}$$

For a fixed $m > 1$ there exists an $\epsilon > 0$, s.t. $\left(\sum_{\ell=1}^n (\xi_\ell)^2\right)^m > \epsilon \sum_{|\alpha|=m} (\xi^\alpha)^2$. So we find with (4.16)

$$a_p(\vec{u}, \vec{u}) > \lambda\epsilon \sum_{i=1}^{q} \int_{\mathbf{R}^n} \Big(\sum_{|\alpha|=m} (\xi^\alpha)^2\Big) |\hat{u}_i(\xi)|^2 d\xi = \lambda\epsilon |\vec{u}|^2_{H^m(\mathbf{R}^n, \mathbf{R}^q)}. \geq \lambda\epsilon |\vec{u}|^2_{H^m(\Omega, \mathbf{R}^q)}.$$

Since $|\vec{u}|_{H^m(\Omega,\mathbf{R}^q)}$ and $\|\vec{u}\|_{H^m(\Omega,\mathbf{R}^q)}$ are equivalent norms on $\mathcal{W}_0$, the $\mathcal{W}_0-$coer-civity of $a_p(\vec{u}, \vec{u})$ is proved.

We turn to the case of continuous coefficients. Not all the details are given, but only the main idea, employing the technique of frozen coefficients. For details see Wloka, [58], p 282 ff. or Hörmander, [36]. By the continuity of the $A_{\alpha\beta}$ in $\overline{\Omega}$ we can choose, for any $\epsilon > 0$, a finite number of points $x_i \in \Omega, i = 1, \ldots, N$, and their neighborhoods $\mathcal{U}_{x_i}$ s.t. $\forall |\alpha| + |\beta| = 2m$

$$\overline{\Omega} \subset \cup_{i=1}^N \mathcal{U}_{x_i} \text{s.t.} A_{\alpha\beta}^{c,i} := A_{\alpha\beta}(x_i) : \|A_{\alpha\beta}^{c,i} - A_{\alpha\beta}\|_{C(\mathcal{U}_{x_i})} < \epsilon \ \forall i = 1, \ldots, N. \tag{4.19}$$

We choose a first generation of the i s.t., after an appropriate renumbering, $\mathcal{U}_{x_i} \cap \partial\Omega \neq \emptyset, i = 1, \ldots, N_1 \leq N$. Then the standard technique allows us to prove, for the strongly-elliptic principle part, see (4.11), and for $u \in H_0^m(\Omega)$, the coercivity of a_p on each of the $\mathcal{U}_{x_i}, i = 1, \ldots, N_1$. Now choose the next inner generation of $\mathcal{U}_{x_i}$ intersecting the union of the first generation, so

$$\mathcal{U}_{x_i} \cap \left(\cup_{i=1}^{N_1} \mathcal{U}_{x_i} \right) \neq \emptyset \quad \forall i = N_1 + 1, \ldots, N_2.$$

By the Trace Theorem , the normal derivatives $\|\partial^k u/\partial\nu^k\|_{H^{m-k-1/2}(\partial\mathcal{U}_{x_i})}, i = 1, \ldots, N_1, k = 0, \ldots, m-1$ can be estimated by the $\|u\|_{H^m(\mathcal{U}_{x_i})}$. Each of these $\partial^k u/\partial\nu^k|_{H^{m-k-1/2}(\partial\mathcal{U}_{x_i})}$, can be employed to prove, for each of the $\mathcal{U}_{x_i}, i = N_1, \ldots, N_2$ in the next generation, the coercivity of a_p now for the above nontrivial boundary values $\|\partial^k u/\partial\nu^k\|_{H^{m-k-1/2}(\partial\mathcal{U}_{x_i})}, k = 0, \ldots, m-1$, a.s.o. $\Box$

5 Fully Nonlinear Elliptic Equations

The general theory of discretization methods used in this paper can be applied if the original problem has two properties: It admits a locally unique solution u_0, strongly related to the stability, and u_0 is smooth enough, implying the consistency and good enough convergence of our method. So we have to list some existence and regularity results for fully nonlinear elliptic equations, based upon rather technical conditions. We start with the most important case of equations of order 2, with $m = 1$ in (4.4), and then extend it to systems and to order $2m$. We summarize results from some textbooks, e.g., Gilbarg, Trudinger, [32], Chen, Wu, [19], and Skrypnik, [46]. We assume (4.4) for $\Omega \subset \mathbf{R}^n$, a slightly larger open $\Omega' \subset \mathbf{R}^n$ with $\overline{\Omega} \subset \Omega'$, $\overline{\Omega_c^h} \subset \Omega'$, where Ω_c^h indicates later approximations for Ω for the $u^h \in \mathcal{U}^h$. A real valued function G^w is defined, s.t.

$$G^w : \mathcal{D}(G^w) \to \mathbf{R}, w = (x, z, p, r) \in \mathcal{D}(G^w) = \Omega' \times \Gamma, \Gamma \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n^2}, \Omega', \Gamma \text{ open}, \tag{5.1}$$

with r restricted to the symmetric real valued $n \times n$ matrices. We consider the strong form of a fully nonlinear elliptic BVP, compare (2.1), (3.6),

$$u_0 \in \mathcal{D}(G) : G(u_0) := G^w(\cdot, u_0(\cdot), Du_0(\cdot), D^2u_0(\cdot))|_\Omega = 0, (u_0 - \phi)|_{\partial\Omega} \tag{5.2}$$
$$\text{often } \phi = 0, \mathcal{D}(G) := \{u \in \mathcal{U} : w_u(x) \in \mathcal{D}(G^w) \forall x \in \Omega \text{ and } G^w(u) \in \mathcal{V}\},$$
$$G : \mathcal{D}(G) \subset \mathcal{U} \to \mathcal{V}, u \to G(u), \ w(\cdot) := w_u(\cdot) := (\cdot, u(\cdot), Du(\cdot), D^2u(\cdot)). \tag{5.3}$$

We need the linearization $G'(u_0)$ in $w_0 := w_{u_0}$ applied to u, $u, u_0 \in \mathcal{D}(G)$:

$$G'(u_0)u = \frac{\partial G^w}{\partial z}(w_0)u + \sum_{i=1}^n \frac{\partial G^w}{\partial p_i}(w_0)\partial^i u + \sum_{i,j=1}^n \frac{\partial G^w}{\partial r_{ij}}(w_0)\partial^i \partial^j u. \tag{5.4}$$

The operator G is called *uniformly elliptic* in $u_0 \in \mathcal{D}(G)$ and all of $\mathcal{D}(G)$ if its linearization $G'(u)$ is uniformly elliptic for $u = u_0$ and $\forall u \in \mathcal{D}(G)$, resp., see (4.9) with $m = q = 1$.

We formulate some of the most important examples.

Example 5.1 *Monge-Ampere-Equation: This is, in particular for $n = 2$,*

$$0 = G(u) := u_{xx}u_{yy} - u_{xy}^2 - f(x, u, Du) = \det D^2u - f(x, u, Du). \tag{5.5}$$

This fully nonlinear $G(u)$ is uniformly elliptic in u_0 for a symmetric strictly positive matrix $r = D^2u_0$ in Ω, hence for a function $u_0 \in C^2(\Omega)$, strictly convex $\forall x \in \Omega$ on a convex Ω. With $\det(D^2u) > 0$, uniformly convex solutions are only possible for positive f, [32], pp 441 ff. For partially nonelliptic generalizations, see [32], p 467ff and Courant, Hilbert, [23], p 324. According to Haltiner, Kasahara and Fulton the leading term in the balance equation in dynamical Meteorology has the form (5.5), [35, 38, 31]. In Westcott, [56] it is related to geometric optics.

Among the many applications in Geometry we mention the following equation:

Equation for a Surface with prescribed Gauss Curvature $K = K(x)$ at x. It is closely related to the Monge-Ampere-Equation. (5.5) has to be modified as

$$G(u) := \det D^2u - K(1 + |Du|^2)^{(n+2)/2} = 0, \ K(x) > 0. \tag{5.6}$$

The existence of surfaces with prescribed Gauss curvature is known for appropriate K. Specific properties would become more flexibly available via our FEMs than by possible finite difference methods, e.g., by Crandall, Lions, [24] ff.: Our FEM allows C^1-FEs on quasi uniform grids and yields, for the forthcoming better boundary approximations, higher orders of convergence than the difference methods in [24], which are only defined on equidistant grids. □

To guarantee existence and uniqueness results we have to impose *growth conditions.* We distinguish the cases $n = 2$ and $n \geq 2$ and evaluate the following derivatives in $w = (x, z, p, r) \in \mathcal{D}(G^w), x \in \partial\Omega$. We require, for $G_x^w, G_z^w, G_p^w, G_r^w$ see (5.4), (4.9),

$$\text{for } n = 2: \quad |G_z^w(w)|, |G_p^w(w)| \leq \lambda\mu$$
$$|G_x^w(w)| \leq \mu\lambda(1 + |p| + |r|) \text{ and} \tag{5.7}$$
$$\text{for } n \geq 2: \quad |G_z^w(w)|, |G_p^w(w)|, |G_{rx}^w(w)|, |G_{px}^w(w)|, |G_{zx}^w(w)| \leq \lambda\mu$$
$$|G_x^w(w)|, |G_{xx}^w(w)| \leq \mu\lambda(1 + |p| + |r|). \tag{5.8}$$

The nonnegative $\lambda = \lambda(\cdot)$ and $\Lambda = \Lambda(\cdot), \mu = \mu(\cdot)$ in (4.9), (5.7) - (5.8) depend upon $|z|$ and have to be nonincreasing and nondecreasing in $z \in [0, \infty)$, resp.

Theorem 5.1 *([32], Theorem 17.12) Assume $n = 2$, $\partial\Omega \in C^3$, $\phi \in C^3(\overline{\Omega})$, see (5.2). Let G satisfy the conditions (4.9), (5.7), and $G_z^w \leq 0$ in Γ, see (5.1). Then the classical Dirichlet problem (5.2) has a unique solution $u_0 \in C^{2,\alpha}(\overline{\Omega})$ for all* $0 < \alpha < 1$.

Theorem 5.2 *([32], Theorem 17.17) Assume $n \geq 2$, $\phi \in C(\partial\Omega)$. Let Ω satisfy an exterior sphere condition for every $\xi \in \partial\Omega$ and G satisfies the conditions (4.9), (5.8), and $G^w \in C^2(\Gamma)$ is concave (or convex) w.r.t. (z, p, r), nonincreasing w.r.t. z. Then the classical Dirichlet problem (5.2) has a unique solution $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Remark 5.1 Ω *satisfies an exterior sphere condition, for every* $\xi \in \partial\Omega$*, if there exist a ball* $B = B_R(y) \subset \mathbf{R}^n$ *s.t.* $\overline{B} \cap \overline{\Omega} = \{\xi\}$*. Mind the difference in these Theorems. For* $n = 2$ *a* $\partial\Omega \in C^3$ *implies the global* $u_0 \in C^{2,\alpha}(\overline{\Omega})$ *for all* $0 < \alpha < 1$*, in contrast to the interior estimate* $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$ *for* $n \geq 2$ *and no condition for* $\partial\Omega$*.*

More detailed results for the above special cases in Example 5.1 are presented in [32] and Skrypnik, [46]. E.g., [46] considers the Monge-Ampere equation (5.5) on the two-dimensional ball $\Omega = B_R(0)$ with center 0. He assumes the existence of functions and combines them with the boundary condition s.t.

$$\begin{aligned}
&(u_0 - \phi)|_{\partial B_R(0)} = 0, \psi : B_R(0) \to \mathbf{R}_+,\ f_0 : \mathbf{R}^2 \to \mathbf{R}_+, m := \max_{x \in B_R(0)} \phi(x) \\
&f(x,u,p) \leq \psi(\|x\|^2) f_0(\|p\|^2) \text{ for } x,p,s \in \mathbf{R}^2,\ \mathbf{R} \ni u \leq m \text{ and let} \qquad (5.9)\\
&\int_{B_R(0)} \psi(\|x\|^2)dx \leq \int_{\mathbf{R}^2} \inf_{\|s-p\|^2 < M_H} 1/f_0(\|s\|^2)ds,
\end{aligned}$$

where M_H is the lower winding of the curve defined by $(u_0 - \phi)|_{\partial B_R(0)} = 0$, see e.g. Bakelman, [5]. Finally, let

$$H^4_+(\Omega) := \{u \in H^4(\Omega) : \det D^2 u(x) > 0, \partial^{20} u(x) > 0\ \forall x \in \overline{B_R(0)}\}.$$

Theorem 5.3 Regular solutions for the Monge-Ampere equation, Skrypnik, [46], Theorem 3.2.3. *Let (5.9) be satisfied and* $f \in C^{2,\gamma}(\Omega), \phi \in C^{4,\gamma}(\partial\Omega)$*. Then (5.5) has at least one solution in* $H^4_+(\Omega)$*.*

Regularity results for $u \in C^{k,\gamma}(\Omega), k \geq 2$ can be deduced from Chen, Wu, [19], Chapter 7, and are extended in Böhmer, [11].

We turn to an existence and regularity result for fully nonlinear equations of order $2m$. Skrypnik, [46], combines monotone operators and degree theory to get his results for fully nonlinear elliptic equations. He discusses some of these problems on narrow strips and in perforated domains. To give a flavor of his existence and regularity results we cite one of his Theorems, based upon continuation techniques. He determines solutions u_0 in *Hilbert spaces* for nonlinear Dirichlet problems. Determine u_0 from

$$\begin{aligned}
&G(u_0(\cdot)) = G^w(t=0,\cdot,u_0,\cdots,D^{2m}u_0)(\cdot) = 0, u_0 \in \mathcal{X}_0 := \mathcal{W}_0 \cap H^{2m+l}(\Omega) \\
&\quad \text{for } q = 1, \text{ with } G : \mathcal{D}(G) \subseteq \mathcal{W}_0 \cap \mathcal{U} \to \mathcal{V}, \text{ s.t. for } G^w_0 := G^w(t=0,\ldots) \qquad (5.10)\\
&G^w \in C(\mathcal{D}(G^w)),\ \mathcal{D}(G^w) := [0,1] \times \mathcal{D}(G^w_0) = [0,1] \times \Gamma' \subset [0,1] \times \bar{\Omega} \times \mathbf{R}^{N_{2m}}, \\
&\quad \text{with appropriate } N_{2m}, \text{ let } G^w(0,x,-\Theta^{N_{2m}}) = -G^w(0,x,\Theta^{N_{2m}}), \text{ and} \\
&\forall t \in [0,1]:\ G^w(t,\cdots) \in C^{l+1}(\mathcal{D}(G^w_0)),\ l \geq [n/2]+1, \quad \text{where } u_1 \in \mathcal{D}(G) \\
&\quad \text{if, e.g., } w_1(x) := (x, \partial^\gamma u_1(x), |\gamma| \leq 2m)\ \in \mathcal{D}(G^w_0) \forall x \in \Omega. \qquad (5.11)
\end{aligned}$$

He assumes the existence of constants $0 < k, 0 < \mu < 1, 0 < \Lambda < \infty$ by (5.10), and modifies the *ellipticity condition,* compare (4.9), s.t.

$$\begin{aligned}
&\exists k \in \mathbf{R}_+ :\ \forall t \in [0,1], v \in \mathcal{X}_0 :\ G(t,v) = 0 \text{ implies } \|v\|_{C^{2m,\mu}(\bar{\Omega})} \leq k, \text{ and} \qquad (5.12)\\
&\forall t \in [0,1], x \in \Omega :\ k^{-1}|\partial|^{2m} \leq \sum_{|\gamma|=2m} \left[\partial^\gamma\big(G^w(t,\cdot,v,\cdots,D^{2m}v)\big)(x)\right]\partial^\gamma \leq \Lambda|\partial|^{2m}.
\end{aligned}$$

Theorem 5.4 Regular solutions, Skrypnik, [46], Theorem 3.2.2. *Let G satisfy (5.10) – (5.12) and for Ω in (4.4) let $\partial\Omega \in C^\infty$. Then (5.11) has, for $t = 1$, at least one solution $u_0 \in \mathcal{X}_0$.*

For systems of q fully nonlinear equations we do not formulate any existence and regularity results. For systems of order 2 there exist some results, but we only refer to Yang, Fanciullo, Belopolskaya, Chen, Wu, Li, Liu, [59, 30, 6, 7, 19, 41]. Higher order nonlinear systems do not seem to be studied. This seems to be correct even for the important example of biharmonic fully nonlinear equations. But now, a combination of our FEMs below and continuation techniques allows studying this class of problems. Fully nonlinear systems of order $2m$ have the form (5.14) below.

The existence and uniqueness results for nonlinear elliptic boundary value problems are obtained by methods, usually independent of linearization. However, preparing the next Sections, we have to generalize the operator F in (2.2) to the fully nonlinear systems and boundary conditions. We slightly extend the notations in (4.1), (4.2) as

$$D^k\vec{u} = (\partial^\alpha\vec{u})_{|\alpha|=k}, n_k, N_k, \text{ s.t. } \vec{\Theta}^k = (\vec{\vartheta}^\alpha)_{|\alpha|=k} \in \mathbf{R}^{n^k\times q}, (\vec{\vartheta}^\alpha)_{|\alpha|\le k} \in \mathbf{R}^{N_k\times q}. \quad (5.13)$$

The differential operator has the form

$$\begin{aligned} &G(\vec{u}(\cdot)) := G^w(\cdot, \vec{u}, D\vec{u}, \dots, D^{2m}\vec{u})(\cdot), \quad \text{with } G : \mathcal{D}(G) \subseteq \mathcal{U} \to \mathcal{V}, \text{ where} \qquad (5.14)\\ &G^w : \mathcal{D}(G^w) \to \mathbf{R}^q, \mathcal{D}(G^w) = \Omega' \times \mathcal{D}(G_1^w) \subset \Omega' \times \mathbf{R}^{N_{2m}\times q}, \Omega'\mathcal{D}(G_1^w) \text{ open and}\\ &\mathcal{D}(G) := \{\vec{u} \in \mathcal{U} : \vec{w}(x) := (x, \partial^\gamma\vec{u}(x), |\gamma| \le 2m) \in \mathcal{D}(G^w)\forall x \in \Omega, G(\vec{u}) \in \mathcal{V}\}. \end{aligned}$$

The boundary operator for *Dirichlet conditions* is

$$B_D : \mathcal{U} \to \mathcal{V}_D \text{ or } B_D : \mathcal{W} \to \mathcal{W}_D \text{ and } B_D\vec{u}(x) := \big(\frac{\partial^j\vec{u}(x)}{\partial\nu^j}|_{\partial\Omega} \ \forall 0 \le j \le m-1\big) \quad (5.15)$$

in the trace sense. We assume a locally unique solution, $\vec{u}_0$, for the generalized operator

$$\begin{aligned} &F : \mathcal{D}(G) \subset \mathcal{U} \to \mathcal{V} \times \mathcal{V}_D, \vec{u} \to F(\vec{u}) := (G(\vec{u}), B_D\vec{u}) \text{ and locally } \exists_1\vec{u}_0 \text{ for}\\ &\qquad F(\vec{u}_0) = (\vec{0}, \vec{\Phi}), \text{ s.t. } F'(\vec{u}_0) \in \mathcal{L}(\mathcal{U}, \mathcal{V} \times \mathcal{V}_D) \text{ is boundedly invertible.} \qquad (5.16) \end{aligned}$$

We do have to defend (5.16), different from the above conditions: The arguments all are closely related to the numerically necessary condition of a (locally) well conditioned problem $F(\vec{u}) = 0$. This is often guaranteed by monotonicity arguments. For an existing locally unique solution the Frechet differentiability is satisfied for many practically relevant problems. As a consequence of the Taylor formula for operators, the bounded invertability of $F'(\vec{u}_0) \in \mathcal{L}(\mathcal{U}, \mathcal{V}\times\mathcal{V}_D)$ is very probably, but not necessarily, satisfied as well. Otherwise the second derivative has to be positive or negative definite or other combinations with higher derivatives have to be satisfied to fit to the above monotonicity arguments. Thus there will be (exceptional) operators with locally unique solutions violating (5.16). For our numerical approach we luckily are able situation, to monitor this danger. If this *exceptional case* is detected, the following numerical methods simply *do not work for this problem.* In fact, except monotonicity and small dimensional systems, there is hardly any numerical method for solving the discrete equation $F^h(\vec{u}_0^h) = 0$ without requiring (5.16) in one or the other form. The latter, in particular, has to be correct for the discrete Newton method, considered in Böhmer, [15]. Unless $F'(\vec{u}_0) \in \mathcal{L}(\mathcal{U}, \mathcal{V} \times \mathcal{V}_D)$ is nearly singular, the corresponding discrete $(F^h)'(\vec{u}_0^h) \in \mathcal{L}(\mathcal{U}^h, \mathcal{V}^{h'} \times \mathcal{V}_D^{h'})$ is stable, see Theorem 9.4, hence boundedly invertible. There is another important reason for assuming (5.16), strongly related to the Fredholm alternative for the linearized elliptic operator $F'(\vec{u}_0)$. If (5.16) would be violated, small perturbations of the original problem, e.g. corresponding to round off errors, could be embedded into a parameter dependent problem $F(\vec{u}, \lambda) = 0$ with $F(\vec{u}, \lambda = 0) = F(u)$. Then $F(\vec{u}, \lambda) = 0$ would usually have a bifurcation point or another singularity in $\vec{u}_0, \lambda = 0$. The usual numerical methods would have to be replaced by numerical methods for bifurcation, see e.g., Böhmer, [10, 13, 12], Cliff, Spence, Tavener, [22]. So there are enough good and numerically necessary reasons for (5.16).

6 Smooth FEs on Curved Domains and Extensions

We need FEs defined on curved approximating domains $\Omega_c^h \approx \Omega$. This causes violated boundary conditions. They are handled by splitting the FE spaces into interior and boundary parts. For comparing the original functions and its interpolating FEs, we need extension operators from Ω and Ω_c^h to $\Omega' \supset \Omega_c^h \cup \Omega$.

6.1 Smooth FEs on curved domains

The FEs $\vec{u}^h$ fit to our discrete equations (2.4), (2.5), their linearizations and generalizations to equations and systems of orders 2 and $2m$, if $G(\vec{u}^h) \in \mathcal{V}$, see (5.14), so for $\vec{u}^h \in C^{2m-1}(\Omega, \mathbf{R}^q)$. More precisely, we choose as ansatz and test functions FE spaces, $S_d(\mathcal{T}_c^h, \mathbf{R}^q)$, and $S_d^{2m-1}(\mathcal{T}_c^h, \mathbf{R}^q)$(splines), the spaces of all piecewise $n-$variate polynomial functions, $\Pi_d^n(\mathbf{R}^q)$, with q components of fixed total degree $\leq d, d \geq 0$ w.r.t. $\mathcal{T}_c^h$,

$$\vec{s} \in S_d(\mathcal{T}_c^h) := S_d(\mathcal{T}_c^h, \mathbf{R}^q) \iff \vec{s}|_T \in \Pi_d^n(\mathbf{R}^q)\ \forall T \in \mathcal{T}_c^h,\ h := \max_{T \in \mathcal{T}_c^h}\{\text{diam } T\} \text{ and}$$
$$S_d^{2m-1}(\mathcal{T}_c^h) := S_d(\mathcal{T}_c^h) \cap C^{2m-1}(\Omega^h),\ \text{necessarily with } d \geq (2m-1)2^n + 1, n \geq 2. \quad (6.1)$$

Here $\mathcal{T}_c^h$ indicates[3] a simplicial subdivision, often called triangulation, of a a polyhedral, Ω^h, or curved approximation, $\Omega_c^h \approx \Omega$, with the maximum diameter, h, of all simplices. We use the notation Ω_c^h for both cases, unless we emphasize the polyhedral Ω^h.

Why don't we use the standard FEs $\vec{u}^h \in C(\Omega, \mathbf{R}^q)$, automatically with $\vec{u}^h \in C^{2m-1}(\mathcal{T}_c^h)$, see (6.4), allowing $G(\vec{u}^h) \in \mathcal{V}$? Then we would have to study interior variational crimes caused by the nondifferentiability of these $\vec{u}^h \in C^0(\mathcal{T}_c^h)$ between neighboring $T \in \mathcal{T}_c^h$. The approved techniques use an interplay between the strong and weak forms of the corresponding operators, see the proof of Proposition 9.1 for a linear case. We have seen above that the weak form for fully nonlinear operators does not make sense. Thus we stay with $\vec{u}^h \in C^{2m-1}(\Omega, \mathbf{R}^q)$.

[3]we do not want to over-formalize the notation and have chosen $h \in H \subset \mathbf{R}, \inf_{h \in H} h = 0$

Before already, we emphasized that the proof of the "strong stability" will require

$$\partial\Omega \in C^{2m}, \text{ or even } \partial\Omega \in C^{\infty}, \text{ and a polyhedral or curved approximation } \Omega_c^h \approx \Omega. \quad (6.2)$$

Thus, by the necessary $\Omega_c^h \approx \Omega$, violated boundary conditions are unavoidable.

In the literature, different approaches for smooth FEs on curved domains are elaborated. We give a short survey. Most results are available for *two variate FEs.* C^m FEs only have been studied by Zenísek, e.g. [63, 64], C^1 FEs by Bernadou, e.g. [8, 9] for the "curved" Argyris and Bellman FEs, hence $d = 5$. The results in [63, 64, 8, 9] and the compilations only apply to $\mathbf{R}^2$. We need inverse and error estimates. To obtain these from the previous papers would require rewriting major parts.

The necessary results are proved in Davydov's [25], at least for the most important special case, some of them more generally. He extended his results in [26] to inverse and error estimates for polyhedral Ω^h, approximating Ω, and C^{2m-1} FEs with $n \geq 2, d \geq 5$. Finally, a special treatment of the boundary conditions, the splitting property as in (**??**), is most important for our proofs. This has not been considered in previous papers. In [25] it is proved for modified Argyris FEs in $\mathcal{S}_5^1(\mathcal{T}^h)$. Generalizations to $n \geq 2, d \geq 5$ are formulated as conjecture in the final Remark in [25]. There will certainly be specific conditions, relating the chosen degree with smoothness properties of $\partial\Omega_c^h$ and the modifications of the interpolation conditions at the boundary $\partial\Omega_c^h$, allowing a stable splitting. This is documented by the subtle discussion for the modification of the Argyris FE in [25]. Approximations of Ω by a polyhedral Ω^h allow only $\mathcal{O}(h^2)$ results for our FEM. But Davydov's extension to curved approximations will yield $\mathcal{O}(h^p)$ results with $p = 5$. These $\mathcal{S}_d^1(\mathcal{T}_c^h)-$results will be available in the near future for $\mathbf{R}^2$. For polygonal triangulations $\mathcal{T}^h$ and $\mathcal{S}_d^1(\mathcal{T}^h)$, nested sequences of quasi-uniform triangulations $\mathcal{T}_1^h \sqsubset \mathcal{T}_2^h \sqsubset \cdots$ with hierarchical $\mathcal{S}_d^1(\mathcal{T}_1^h) \sqsubset \mathcal{S}_d^1(\mathcal{T}_2^h) \sqsubset \cdots$ are known, as e.g., in Davydov, Stevenson, [27]. They allow formulating multi resolution techniques.

Remark 6.1 *Thus, we have documented an approximation theoretic gap for the case* $m > 1$. *In this situation, we have chosen a compromise: We formulate in Subsection 6.2, the results, necessary for our later proofs, for the case* $\mathcal{S}_d^{2m-1}(\mathcal{T}_c^h)$ *in* $\mathbf{R}^n$. *For the present computations, there is one standard way around these difficulties for* $m > 1$. *A system of order* $2m$ *with* q *equations can be transformed into an equivalent system of order* 2 *with* mq *equations. Then the* $\mathcal{S}_d^1(\mathcal{T}_c^h)-$*results admit realizing the numerical solution of the equivalent problem.*

We combine the Davydov results with a new idea of Tiihonen, [54]. He decomposes, for linear sample problems, the error analysis for FEMs by introducing an auxiliary problem defined in a polyhedral domain approximating the original *smooth* domain. Then a standard FEM on a polyhedral domain with its known error estimates is combined with an estimate of the "geometric part of the error", caused by the approximation of the domain. We modify and generalize his approach to our fully nonlinear equations.

Smooth FEs $\mathcal{S}_d^{2m-1}(\mathcal{T}^h)$ are not too familiar in the FE community, so we provide a short summary of Davydov's [25] for polygonal and its forthcoming generalization to $\mathbf{R}^n$ and to curved approximations of Ω. In the corresponding triangulation $\mathcal{T}_c^h$, the simplices at the boundary $\partial\Omega$ usually have plane or curved edges or facets, approximating the corresponding part of $\partial\Omega$. $\mathcal{T}_c^h$ has to satisfy the standard conditions, see [25]: It is a finite set of nondegenerate open pairwise disjoint n-simplices such that $\Omega \approx \cup_{T\in\mathcal{T}_c^h}\overline{T}$. Each facet of a simplex $\overline{T}$ for $T \in \mathcal{T}_c^h$ either is a common face of, in $\mathbf{R}^2$ exactly two, for $\mathbf{R}^n, n > 2$, only few simplices in $\mathcal{T}_c^h$ or it approximates, as plane or curved surface, a part of $\partial\Omega$.

In Figure 1, for a polygonal Ω^h and a curved Ω_c^h, a triangulation, $\mathcal{T}_c^h$, is obtained as interior of the boundary defined by connecting the dotted points along $\partial\Omega$ with straight lines or nonstraight curves, resp. In the final $\mathcal{T}_c^h$ we maintain all "interior" edges with at

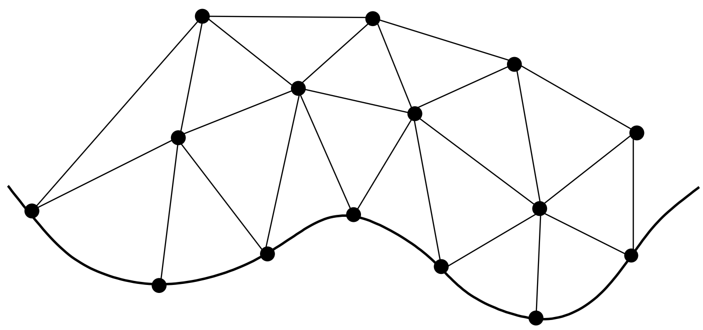

Figure 1: Admissible triangulation for a curved boundary

most one boundary point. As boundary edges, see Figure 2, we choose either ∂T or ∂T_c approximating $\partial\Omega$ between the points P_i and P_e.

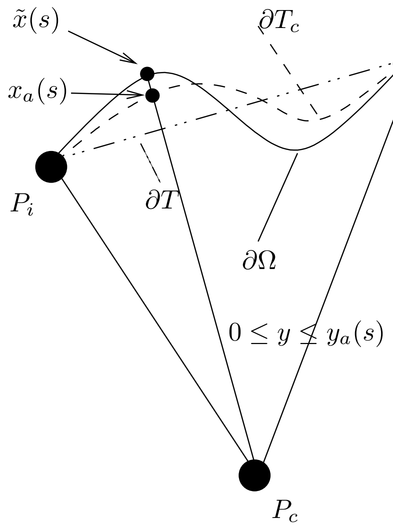

Figure 2: Approximation of the $\partial\Omega$.

We introduce the necessary notations: $\mathcal{T}_c^h$ is called *quasi-uniform* and *non-degenerate,* if $\chi > 0$ exists s.t. for small enough $h \in (0,1)$, and B_T, the largest ball contained in $\overline{T}$,

$$\min_{T\in\mathcal{T}^h} \frac{\text{diam } B_T}{h} \ge \chi \text{ and } \frac{1}{\omega_{\mathcal{T}_c^h}} := \min_{T\in\mathcal{T}^h} \frac{\text{diam } B_T}{\text{diam} T} \ge \chi \quad \forall T \in \mathcal{T}^h, \text{ resp.} \tag{6.3}$$

$\omega_{\mathcal{T}_c^h}$ is called the *shape regularity constant* of the triangulation $\mathcal{T}_c^h$. The *star* of a vertex, v, of $\mathcal{T}_c^h$, denoted by star(v), is the union of all n-simplices $\overline{T} \in \mathcal{T}_c^h$ attached to v. We set $\text{star}^1(v) := \text{star}(v)$, and define $\text{star}^\gamma(v)$, $\gamma \geq 2$, recursively as the union of the stars of all vertices of $\mathcal{T}_c^h$ contained in $\text{star}^{\gamma-1}(v)$.

The $\mathcal{S}_d^{2m-1}(\mathcal{T}_c^h) \not\subset H^k(\Omega, \mathbf{R}^q), 2m-1 < k-1$, are subsets of the following spaces:

$$\mathcal{W}_c^k := H^k(\Omega_c^h, \mathbf{R}^q) := \{\vec{u}^h : \Omega_c^h \to \mathbf{R}^q : \ \forall\, T \in \mathcal{T}_c^h \ \ \vec{u}^h|_T \in H^k(T, \mathbf{R}^q)\},\ W^{k,p}(\mathcal{T}_c^h), C^k(\mathcal{T}_c^h)$$

$$\text{with } (\vec{u}^h, \vec{u}_1^h)^h_{\mathcal{W}_c^k} := (\vec{u}^h, \vec{u}_1^h)^h_{H^k(\Omega_c^h, \mathbf{R}^q)} := \Big(\sum_{T \in \mathcal{T}_c^h} \sum_{j=0}^{k} (D^j \vec{u}^h|_T, D^j \vec{u}_1^h|_T)_{L^2(T, \mathbf{R}^q)} \Big),$$

$$\text{and, e.g., } \|\vec{u}^h\|^h_{C^k(\Omega_c^h, \mathbf{R}^q)}, \|\vec{u}^h\|_{\mathcal{U}^h} := \|\vec{u}^h\|^h_{\mathcal{W}_c^2}, \|\vec{w}^h\|_{\mathcal{W}^h} := \|\vec{w}^h\|^h_{\mathcal{W}_c^1}, \|\vec{v}^h\|_{\mathcal{V}^h} := \|\vec{v}^h\|^h_{\mathcal{W}_c^0}.$$

6.2 Estimates in and Splitting of $\mathcal{S}_d^{2m-1}$

Now we are able to formulate the inverse and interpolation error estimates and the splitting of the FE space into "boundary and interior functions".

Theorem 6.1 *For $0 \leq k < \mu \leq d$, we have*

$$|\vec{s}|_{H^\mu(T)} \leq \frac{K}{(\text{diam } T)^{\mu-k}} |\vec{s}|_{H^k(T)}, \qquad \vec{s} \in S_d(\mathcal{T}_c^h), \quad T \in \mathcal{T}_c^h, \tag{6.4}$$

where the constant K depends only on $n, d, \omega_{\mathcal{T}_c^h}$. For a quasi-uniform triangulation $\mathcal{T}_c^h$ the $|\vec{s}|_{H^j(T)}, j = \mu, k$, can be replaced by $|\vec{s}|_{H^j(\Omega_c^h, \mathbf{R}^q)}$.

Below, we generate $\vec{s} \in \mathcal{S}_d(\mathcal{T}_c^h) = \mathcal{S}_d(\mathcal{T}_c^h, \mathbf{R}^q)$ by applying the interpolation operator, I^h, to the q components of a function $\vec{u}$. We restrict the discussion to the most important case, using the same systems $S_1^h, S_1^{h'}$ for each component. So, we begin introducing a *stable and local basis* for the real-, not vector-valued, S_1^h and its dual $S_1^{h'}$, requiring

$$\begin{aligned} \mathcal{S}_d(\mathcal{T}_c^h, \mathbf{R}) \supset \mathcal{S}_d^{2m-1}(\mathcal{T}_c^h, \mathbf{R}) = S_1^h := \text{span}[s^1, \dots, s^N] \text{ and } S_1^{h'} := \text{span}[\lambda^1, \dots, \lambda^N] \quad (6.5)\\ \text{with } S^h := \mathcal{S}_d^{2m-1}(\mathcal{T}_c^h, \mathbf{R}^q) =: S_q^h := \text{span}[s_q^1, \dots, s_q^N] \text{ and } S_q^{h'} := \text{span}[\lambda_q^1, \dots, \lambda_q^N]\\ \text{where } s_q^1 := (s^1, \dots, s^1)^{\mathbf{T}} \in S_q^h = S_1^h \times \dots \times S_1^h, \ \ \lambda_q^1 := (\lambda^1, \dots, \lambda^1)^{\mathbf{T}} \in S_q^{h'} \end{aligned}$$

The next Theorem requires that for each $k = 1, \dots, N$, there is a set $E_k \subset \Omega_c^h$ s.t.

$$\text{supp}\, s^k \subset E_k, \text{ and } E_k \subset \text{star}^\gamma(v_k) \text{ for an appropriate vertex } v_k, \tag{6.6}$$

$$\|s^k\|_{L_\infty(\Omega_c^h)} \leq C_1, \text{ and } |\lambda^k s| \leq C_2 \|s\|_{L_\infty(E_k)}, \quad \forall s \in S^h, \ \lambda^k \text{in (6.5)}, \tag{6.7}$$

for some C_1, C_2 and γ, see [25] (A3.1) - (A3.4) and (A3.13). Hence the norms

$$\|(c_\mu)_{\mu=1}^N\|_p \text{ and } \|\sum_{\mu=1}^N c_\mu s^\mu\|_{L^p(\Omega_c^h)} \ \forall 1 \leq p \leq \infty \text{ are equivalent after renorming the } s^\mu.$$

This property is important for stable interpolations. It is proved for $S_d^r(\mathcal{T}_c^h, \mathbf{R})$ and subspaces of it if $d \geq r2^n + 1$, in Davydov, [26, 25] and references therein, for FE subspaces, see e.g. [17]. Generalizing (6.1), let the $S_d^r(\mathcal{T}_c^h, \mathbf{R})$ satisfy

$$\Pi_{\ell-1}^n \subset S_d(\mathcal{T}_c^h, \mathbf{R}) = S_1^h \quad \text{for some } 0 \leq \ell - 1 \leq d. \tag{6.8}$$

Theorem 6.2 *Under the conditions (6.5) - (6.8), (6.12), there exists a linear operator* $I^h : L_1(\Omega_c^h, \mathbf{R}^q) \to S^h = S_d^{2m-1}(\mathcal{T}_c^h, \mathbf{R}^q)$*, such that for any* $T \in \mathcal{T}_c^h$, $0 \le |\alpha| \le \ell - 1$*, and* $\vec{u} \in L_1(\Omega_c^h, \mathbf{R}^q)$ *with* $|\vec{u}|_{H^\ell(\Omega_T^\gamma, \mathbf{R}^q)} < \infty$*,*

$$\|D^\alpha(\vec{u} - I^h\vec{u})\|_{L^2(T)} \le K h_T^{\ell-|\alpha|} |\vec{u}|_{H^\ell(\Omega_T^\gamma, \mathbf{R}^q)}, \text{ with } \Omega_T^\gamma := \cup_{v \in T} \mathrm{star}^{2\gamma-1}(v), \tag{6.9}$$

with $K = K(n, d, \omega_{\mathcal{T}_c^h}, \gamma, C_1C_2, L_{\partial\Omega_c^h})$*,* $L_{\partial\Omega_c^h}$ *the Lipschitz constant of the boundary* $\partial\Omega_c^h$ *of* Ω_c^h*. As a consequence, if* $\vec{u} \in H^\ell(\Omega_c^h, \mathbf{R}^q)$*, then for all* $0 \le |\alpha|, k \le \ell - 1$*,*

$$\|D^\alpha(\vec{u} - I^h\vec{u})\|_{L^2(\Omega_c^h, \mathbf{R}^q)}^h \le K h^{\ell-|\alpha|} |\vec{u}|_{H^\ell(\Omega_c^h, \mathbf{R}^q)} \Rightarrow \lim_{h \to 0} \|I^h\vec{u}\|_{H^k(\Omega_c^h)}^h = \|\vec{u}\|_{H^k(\Omega_c^h)}. \tag{6.10}$$

Rewriting (6.10) for Sobolev norms, we get for all $0 \le k \le \ell - 1$*, and along* $\partial\Omega_c^h$

$$\|\vec{u} - I^h\vec{u}\|_{H^k(\Omega_c^h, \mathbf{R}^q)}^h \le K h^{\ell-k} |\vec{u}|_{H^\ell(\Omega_c^h, \mathbf{R}^q)} \text{ and by the Trace Theorem} \tag{6.11}$$

$$\|\vec{u} - I^h\vec{u}\|_{H^k(\partial\Omega_c^h, \mathbf{R}^q)}^h \le K h^{\ell-k-1/2} |\vec{u}|_{H^\ell(\Omega_c^h, \mathbf{R}^q)} \Longrightarrow \lim_{h \to 0} \|I^h\vec{u}\|_{H^k(\partial\Omega_c^h, \mathbf{R}^q)}^h = \|\vec{u}\|_{H^k(\partial\Omega_c^h, \mathbf{R}^q)}.$$

For the later discretization methods we choose $\mathbf{R}^q-$valued spline functions, see (6.5),

$$S^h = S_q^h = S_d^{2m-1}(\mathcal{T}_c^h, \mathbf{R}^q) \text{ and have to assume } d \ge (2m-1)2^n + 1. \tag{6.12}$$

$S^h = S_q^h$ and $S_q^{h'}$ are split into the FEs, with function values and its first $m-1$ normal derivatives vanishing on the boundary of Ω_c^h,

$$S_{0,q}^h = S_0^h = S^h \cap H_0^m(\Omega_c^h, \mathbf{R}^q), \text{ and the complementary set } S_{b,q}^h = S_b^h. \tag{6.13}$$

As in (6.5) we choose, for this splitting of S^h the corresponding bases for the real-valued S_1^h and its dual $S_1^{h'}$, s.t.,

$$\begin{array}{l}
\mathrm{span}\{s^1, \ldots, s^N\} = S_1^h = S_d^1(\mathcal{T}_c^h, \mathbf{R}) = S_0^h \cup S_b^h, S_0^h := S^h \cap H_0^m(\Omega_c^h), S_0^h \cap S_b^h = \{0\}, \\
S_0^h = \mathrm{span}\{s^1, \ldots, s^{N_0}\}, \; S_b^h = \mathrm{span}\{s^{1+N_0}, \ldots, s^N\}, \; N_0 + N_b = N \text{ and } \lambda^i s^j = \delta_{i,j}, \\
\mathrm{span}\{\lambda^1, \ldots, \lambda^{N_0}, \lambda^{1+N_0}, \ldots, \lambda^N\} = S_1^{h'} = S_0^{h'} \cup S_b^{h'}, \quad S_0^{h'} \cap S_b^{h'} = \{0\}, \text{ s.t. for} \\
i = 1, \ldots, N: \quad s^j \in S_0^h, j = 1, \ldots, N_0: \; \lambda^i s^j = \left\{ \begin{array}{ll} 1, & \text{, if } i = j, \\ 0, & \text{, otherwise} \end{array} \right. \\
\text{and } s_b^j = s^{j+N_0} \in S_b^h, \; \lambda_b^j = \lambda^{j+N_0}, \; j = 1, \ldots, N_b: \quad \lambda^i s_b^j = \left\{ \begin{array}{ll} 1, & \text{, if } j + N_0 = i, \\ 0, & \text{, otherwise} \end{array} \right.
\end{array} \tag{6.14}$$

For the remaining part of this Subsection we apply, for simplicity, I^h only to functions s.t. the functionals λ^i are applicable for every component $(f_1, \ldots, f_q) = \vec{f}$, so

$$\lambda^i, \quad i = 1, \ldots, N, \text{ are directly applicable, e.g., to } \vec{f} \in C^{k_0}(\Omega_c^h, \mathbf{R}^q). \tag{6.15}$$

Instead of interpolating values of functions or derivatives as I^h in (6.9) or $P^h = E_c I^h$, the P^h may be based upon some kind of quasi–interpolation or best approximation operator, see e.g., de Boor, [28], Scott, Zhang, [44], Davydov, [25]. Otherwise averaged Taylor polynomials or extension theorems could be used, see the proof of Theorem A3.1. of [25].

We discuss our main example, the original Argyris FE, see [25], Theorem A 4.1, for $s \in C^2(\Omega^h), k_0 = 2$, with the functionals

$$S_1^h = \{s \in S_5^1(\mathcal{T}_c^h, \mathbf{R}) : s \text{ is } C^2 \text{ smooth at any vertex } v \text{ of } \mathcal{T}^h\}, \text{with } \{\lambda^1, \cdots, \lambda^N\} :=$$

$$= \Big\{s(v), h_T \frac{\partial s}{\partial x_1}(v), h_T \frac{\partial s}{\partial x_2}(v), h_T^2 \frac{\partial^2 s}{\partial x_1^2}(v), h_T^2 \frac{\partial^2 s}{\partial x_2^2}(v), h_T^2 \frac{\partial^2 s}{\partial x_1 \partial x_2}(v), \ \forall v \in T \in \mathcal{T}^h \quad (6.16)$$

$$\text{and } h_T \frac{\partial s}{\partial n}(\frac{v_1 + v_2}{2}) \text{ the midpoints } \frac{v_1 + v_2}{2} \text{ of neighbors } v_1, v_2 \in T \in \mathcal{T}^h\Big\}. \quad (6.17)$$

By [25], this system satisfies (6.6) - (6.8) for $\gamma = 1$ and any $d = \ell - 1 \leq 5$, but violates (**??**), (6.14). The sets $E_k = \operatorname{supp} s^k$ are either star(v) for the functionals of type (6.16), or the unions of two triangles sharing the edge $[v_1, v_2]$ in case (6.17).

It is decisive, that a modified Argyris FE, the

$$S = \{s \in S_5^1(\mathcal{T}^h) : s \text{ is } C^2 \text{ smooth at any } \textit{interior} \text{ vertex } v \text{ of } \mathcal{T}^h\} \quad (6.18)$$

and a modified basis can be chosen yielding a stable splitting. This is worked out for a polygonal $\Omega^h, \mathcal{T}^h, \mathbf{R}^2$, and the Argyris FE with $d = 5$ in [25]. For $\mathbf{R}^n, n > 2, d \geq 5$ it is formulated as conjecture in the final Remark. We have to modify the above functionals. The new set $\{\lambda_1, \ldots, \lambda_N\}$ is defined by including the functionals (6.16) for all *interior* vertices v of $\mathcal{T}^h$, and the functionals (6.17) for all edges of $\mathcal{T}^h$. The functionals in (6.16) for the *boundary* vertices v of $\mathcal{T}^h$ are split into two groups, both making sense by (6.18):

(c1) $h_T^2 \frac{\partial^2 s}{\partial e_1^2}(v), \ldots, h_T^2 \frac{\partial^2 s}{\partial e_{\ell-1}^2}(v), h_T^2 \frac{\partial^2 s}{\partial e_0 \partial e_1}(v)$, and, in addition, $h_T \frac{\partial s}{\partial e_0^\perp}(v)$ for collinear boundary vertices, e_0 and e_ℓ

(c2) $s(v), h_T \frac{\partial s}{\partial e_0}(v), h_T^2 \frac{\partial^2 s}{\partial e_0^2}(v), h_T^2 \frac{\partial^2 s}{\partial e_\ell^2}(v)$ the latter independent by (6.18) and, in addition, $h_T \frac{\partial s}{\partial e_0^\perp}(v)$ for noncollinear boundary vertices, e_0 and e_ℓ.

The $e_0, \ldots, e_\ell$ are all edges of $\mathcal{T}^h$ emanating from v in counterclockwise order, with the boundary edges e_0 and e_ℓ. The $\partial/\partial e$ denotes the usual directional derivative in the direction of edge e. For a detailed discussion see [25]. Finally, we collect nearly all above functionals into $\{\lambda^1, \ldots \lambda^{N_0}\}$. The exceptions, defining its complement, are the boundary functionals

$$\{\lambda^{1+N_0}, \ldots \lambda^N\} = \{\text{functionals in (?????????????????)} \quad \forall \text{ boundary vertices } v \text{ in } \mathcal{T}^h\},$$

To generalize the results in [25], we introduce the two interpolation operators, I^h, I_b^h, for the interior and the boundary, see (6.10), componentwise, mind that $\lambda^i(\vec{f}) \in \mathbf{R}^q$,

$$I^h, I_b^h : C^{k_0}(\Omega_c^h, \mathbf{R}^q) \to S_q^h, I^h(\vec{f}) = \sum_{i=1}^{N} \lambda^i(\vec{f}) s^i \in S_q^h, I_b^h(\vec{f}) = \sum_{i=1+N_0}^{N} \lambda^i(\vec{f}) s^i \in S_{b,q}^h. \quad (6.19)$$

Theorem 6.3 *Compare Remark 6.1: Under the conditions (6.5) - (6.8), (6.12), there exists a subspace* S^h *of* $\mathcal{S}_d^{2m-1}, (\mathcal{T}_c^h, \mathbf{R}^q)$ *in* $\mathbf{R}^n, n \geq 2, d \geq (2m-1)2^n+1$*, and a stable splitting of a local basis,* $\{s^1, \ldots, s^N\}$ *for* $S^h = S_1^h$ *and its dual* $\{\lambda^1, \ldots, \lambda^N\}$*, into stable subsets, the* $\{s^1, \ldots, s^{N_0}\}$ *and* $\{\lambda^1, \ldots, \lambda^{N_0}\}$ *for* S_0^h*, and the complements,* $\mathrm{span}\{s^{1+N_0}, \ldots, s^N\}$ *with* $\mathrm{span}\{\lambda^{1+N_0}, \ldots, \lambda^N\}$ *for* S_b^h.

The above interpolation operators $I : C^{k_0}(\Omega_c^h, \mathbf{R}^q) \to S_q^h$ *and* $I_b : C^{k_0}(\Omega_c^h, \mathbf{R}^q) \to S_{b,q}^h$, *see (6.13), (6.14), (6.15) and (6.19), have the following properties.*

1. *If* $B_D\vec{f} = 0$*, then* $I(\vec{f}) \in S_{0,q}^h$.
2. *For* $\vec{f} \in C^{k_0}(\Omega_c^h, \mathbf{R}^q)$ *and* $\ell \leq d+1$*, let* $H^\ell(\Omega_c^h, \mathbf{R}^q) \subset C^{k_0}(\Omega_c^h, \mathbf{R}^q)$*. Then*

$$\begin{aligned} &\|D^\alpha(\vec{f} - I(\vec{f}))\|_{L_2(\Omega_c^h, \mathbf{R}^q)}^h \leq K h^{\ell-|\alpha|} |\vec{f}|_{H^\ell(\Omega_c^h, \mathbf{R}^q)} 0 \ \forall 0 \leq |\alpha| \leq \ell - 1, \text{ moreover,} \qquad (6.20)\\ &I_b(\vec{f})|_{\partial\Omega_c^h} = I(\vec{f})|_{\partial\Omega_c^h} \text{ and } \|D^\alpha(\vec{f} - I_b(\vec{f}))\|_{L_p(\partial\Omega_c^h)} \leq K h^{\ell-|\alpha|-1/2} |\vec{f}|_{H^\ell(\Omega_c^h, \mathbf{R}^q)}. \end{aligned}$$

The constant K *depends only on* p, $\omega_{\mathcal{T}_c^h}$*, and the Lipschitz constant* $L_{\partial\Omega_c^h}$.

Restricting $\vec{u}^h \in S_q^h$ *to the Dirichlet boundary conditions on* $\partial\Omega_c^h$*, yields for any* $\vec{u}^h \in S_q^h$

$$B_D^h \vec{u}^h = \vec{0} \Leftrightarrow (B_D^h \vec{u}^h, B_D^h \vec{v}^h)_{L^2(\partial\Omega_c^h)} = 0 \Leftrightarrow (B_D^h \vec{u}^h, \vec{v}^h|_{\partial\Omega_c^h})_{L^2(\partial\Omega_c^h)} = 0 \ \forall \vec{v}^h \in S_{b,q}^h. \quad (6.21)$$

For the discretization methods, we introduce two *identical sequences of FE spaces* and a splitting into subspaces with the norms in (6.4)

$$\begin{aligned} &\mathcal{U}^h = \mathcal{W}^h = \mathcal{S}_d^{2m-1}(\mathcal{T}_c^h, \mathbf{R}^q) = S_0^h \oplus S_b^h \ = \ \mathcal{V}^h \oplus \mathcal{V}_b^h, \text{ with } d \geq (2m-1)2^n + 1, \quad (6.22)\\ &\|\vec{u}^h\|_{\mathcal{U}^h}, \|\vec{w}^h\|_{\mathcal{W}^h}, \|\vec{v}^h\|_{\mathcal{V}^h}, \|\vec{v}_b^h\|_{\mathcal{V}_b^h} := \|\vec{v}_b^h\|_{L^2(\partial\Omega_c^h, \mathbf{R}^q)}^h \forall \vec{u}^h \in \mathcal{U}^h, \vec{w}^h \in \mathcal{W}^h, \vec{v}^h \in \mathcal{V}^h, \vec{v}_b^h \in \mathcal{V}_b^h. \end{aligned}$$

The $\|\vec{u}^h\|_{\mathcal{U}^h}$, $\|\vec{w}^h\|_{\mathcal{W}^h}$ norms are equivalent for a fixed h, but are nonequivalent for the limit process $h \to 0$, essential for our goal, the convergence.

6.3 An extension theorem and error estimates for $\Omega \in C^2$

The preceding FE spaces are defined on Ω^h or $\Omega_c^h \neq \Omega$. In contrast to the Ciarlet, Raviart, Lenoir, Bernadou approaches, [21, 20, 40, 8], we apply general extension operators, E_c, see Theorem 6.4, to the original functions. This allows comparing functions $u \in \mathcal{U}$ and $\vec{u}^h \in \mathcal{U}^h$ a.s.o., e.g., for the interpolation errors. Correspondingly, the boundary conditions $B_D\vec{u} = \vec{u}|_{\partial\Omega} = 0$ are violated into $B_D^h\vec{u} = (E_c\vec{u})|_{\partial\Omega_c^h} = \mathcal{O}(h^p)$ with $p = 2$ for polygonal Ω^h and $p > 2$ for curved Ω_c^h.

We formulate only one type of extension theorem. Its extension operators E_c fit well to the above approximation theoretic results. Theorems 6.5 and 6.6 correctly formulate estimates for the distance between the boundaries $\mathrm{dist}\{\partial\Omega, \partial\Omega_c^h\}$ and for the interpolation error $\|E_c\vec{u} - P^h\vec{u}\|_{H^\ell(\Omega_c^h, \mathbf{R}^q)}^h$. Other possible results, fitting better to the available regularity results for fully nonlinear problems and extending boundary functions for $C^{k,\alpha}$ domains are discussed in Gilbarg, Trudinger, [32], p 136, [61],p305-306.

Theorem 6.4 Extension from $H^k(\Omega, \mathbf{R}^q)$ to $H^k(\mathbf{R}^n, \mathbf{R}^q)$, *Stein, [47],[17], p 31: Let Ω (and $\Omega_c^h \subset \mathbf{R}^n$) have a Lipschitz boundary and let $k \in \mathbf{N}$ be given. Then there exists a linear bounded extension $E_c : H^k(\Omega, \mathbf{R}^q) \to H^k(\mathbf{R}^n, \mathbf{R}^q)$ (and $E_c : H^k(\Omega_c^h, \mathbf{R}^q) \to H^k(\mathbf{R}^n, \mathbf{R}^q)$) and a constant, $C = C(k, \Omega, \Omega')$ with $\Omega' \supset \Omega \cup \Omega_c^h$, s.t.*

$$E_c v|_\Omega = v \text{ on } \Omega \text{ and } \|E_c v\|_{H^k(\mathbf{R}^n,\mathbf{R}^q)} \leq C\|v\|_{H^k(\Omega,\mathbf{R}^q)} \quad \forall v \in H^k(\Omega, \mathbf{R}^q). \tag{6.23}$$

Remark 6.2 *Theorem 6.4 is applicable to functions defined Ω and Ω_c^h, since both have Lipschitz boundaries. For the discussion in Section 7 it is appropriate to define an $E_c : H^k(\Omega_c^h, \mathbf{R}^q) \to H^k(\Omega \cup \Omega_c^h, \mathbf{R}^q)$ for $k \in \mathbf{Z}$ modifying (6.23) as follows. There exists a constant, $C = C(k, \Omega, \Omega' \supset \Omega \cup \Omega_c^h)$ s.t.*

$$E_c v_c|_{\Omega_c^h} = v_c \text{ and } \|E_c v_c\|_{H^k(\Omega\cup\Omega_c^h)} \leq C\|v_c\|_{H^k(\Omega_c^h,\mathbf{R}^q)} \quad \forall v_c \in H^k(\Omega_c^h, \mathbf{R}^q). \tag{6.24}$$

Analogously for $E_c : H^k(\Omega, \mathbf{R}^q) \to H^k(\Omega, \mathbf{R}^q \cup \Omega_c^h)$. For $k \in \mathbf{N}$ this is a consequence of (6.23), for $k \in \mathbf{Z}, k \leq 0$ we define $E_c v_c|_{(\Omega' \setminus \Omega_c^h)} \equiv 0$, again yielding (6.24) for $k = 0$. The cases $k < 0$ follow from the dense embedding of $L^p(\Omega, \mathbf{R}^q)$ into $H^k(\Omega, \mathbf{R}^q)$ for $k < 0$.

Theorem 6.5 *Let $\partial\Omega \in C^p, p \geq 2$. Then the polyhedral Ω^h and piecewise curved approximation Ω_c^h for Ω yield*

$$\operatorname{dist}\{\partial\Omega, \partial\Omega^h\} \leq Ch^2\|\partial\Omega\|_{C^p} \text{ and } \operatorname{dist}\{\partial\Omega, \partial\Omega_c^h\} \leq Ch^p\|\partial\Omega\|_{C^p}, \text{ resp.} \tag{6.25}$$

Theorem 6.6 Interpolation errors: *Compare Remark 6.1: Under the conditions (6.5) - (6.8), (6.12) and of Theorem 6.3, and for Ω with Lipschitz boundary, a positive constant $K = K(n, d, \omega_{\mathcal{T}_c^h}, \gamma, C := C_1C_2, L_{\partial\Omega^h})$, exists, s.t. for for the operator $P^h = I^h E_c$, $\vec{u} \in H^\ell(\Omega, \mathbf{R}^q)$, and $0 \leq s \leq \ell - 1$*

$$\|E_c\vec{u} - P^h\vec{u}\|^h_{H^s(\Omega_c^h,\mathbf{R}^q)} \leq C\ h^{\ell-s}|\vec{u}|_{H^\ell(\Omega,\mathbf{R}^q)}, \text{ and } \|P^h\vec{u}\|^h_{H^s(\Omega_c^h,\mathbf{R}^q)} \leq C\|\vec{u}\|_{H^\ell(\Omega,\mathbf{R}^q)} \tag{6.26}$$
$$\|E_c\vec{u} - P^h\vec{u}\|^h_{H^s(\partial\Omega_c^h)} \leq C\ h^{\ell-s-1/2}|\vec{u}|_{H^\ell(\Omega,\mathbf{R}^q)}, \text{ and } \|P^h\vec{u}\|^h_{H^s(\partial\Omega_c^h)} \leq C\|\vec{u}\|_{H^\ell(\Omega,\mathbf{R}^q)} \text{ or}$$

equivalently $\|\vec{u} - E_c P^h\vec{u}\|^h_{H^s(\Omega,\mathbf{R}^q)}$. The $\mathcal{U}^h, \mathcal{V}_b^h$ approximate the $H^s(\Omega, \mathbf{R}^q), H^s(\partial\Omega, \mathbf{R}^q)$,e.g.

$$dist(E_c\vec{u}, \mathcal{U}^h) := \inf_{\vec{u}^h \in \mathcal{U}^h} ||E_c\vec{u} - \vec{u}^h||^h_{H^s(\Omega_c^h,\mathbf{R}^q)} \to 0\ \forall\ \vec{u} \in H^s(\Omega, \mathbf{R}^q) \text{ for } h \to 0. \tag{6.27}$$

7 An Abstract Discretization Theory

In Section 3 we have seen that the standard FEM approaches do not seem to be possible for the general case of fully nonlinear elliptic equations. So we formulated the new FEM for fully nonlinear equations of second order with trivial Dirichlet boundary conditions in Section 2, see (2.5). For the general case, we use the notations in(4.1) - (4.5) (5.1),(5.3). Let (5.2) have a locally unique solution, see Section 5. Hence, according to the above discussion, we assume a linearized boundedly invertible operator $F'(\vec{u}_0)$.

We recall, that our programme for proving convergence requires several steps. We have summarized them already in Section 2, **1.- 4.**: Until now we have worked through the first two bullets of **1.**, and turn, in this and the next Section, to the last three bullets of **1.**. We update the standard form of general discretization methods, e.g., in Stetter, [48], Böhmer, [13, 11]. The *sequences of approximating nonconforming subspaces* $\mathcal{U}^h, \mathcal{W}^h, \mathcal{V}^h$, see (7.6), defined on the approximate Ω_c^h, imply violated boundary conditions on the exact boundary. They are combined with the $\mathcal{U}, \mathcal{W}, \mathcal{V} = \mathcal{V}'$, defined on the original Ω. This FEM is a so called *nonconforming Petrov–Galerkin method.*

We have to generalize our FEM formulated for a second order equation in (2.5) to equations and systems of orders $2m$. The basic facts for the success are the existence of C^{2m-1} FEs with local support , combined with an unusual splitting and variational characterization of the discrete solution. This implies, for (trivial) Dirichlet boundary conditions, coinciding strong and weak bilinear forms, $a_s^h(\vec{u}^h, \vec{v}^h) = a^h(\vec{u}^h, \vec{v}^h)\ \forall \vec{u}^h \in \mathcal{U}^h, \vec{v}^h \in \mathcal{V}^h$, see (**??**). The regularity results in Lemma 9.1 below, allow transforming the stability estimates w.r.t. the $H^m(\Omega)$-norm into analogues w.r.t. the $H^{2m}(\Omega)$-norm. The splitting of F into differential and boundary operators allows the proof of consistency, see Section 8, and thus convergence.

For elliptic systems and equations of higher order, the simple relation between ellipticity (4.9) and $\mathcal{W}_0$-coercivity of the principle part, see (4.8), is no longer correct. This has been discussed in Sections 4 and 5, see Definition 4.1, Theorem 4.1,4.2, Remark 4.1 and the definition of ellipticity following (5.4), and has to be considered in the following discussion.

7.1 Elliptic systems and their discretization

We have indicated above, that the violation of the differential operator and the boundary conditions by the FEM is a deciding step of this method. In particular, the standard approaches for violated boundary conditions via the interplay of weak and strong forms do not work for the fully nonlinear problem, considered here. Hence, the discretization method has to care about both aspects. This is achieved by recalling our new operator, F, with the differential operator, G, and the boundary operator, B_D, as its two components, see (2.1),(2.2), and (5.14), (5.15), and the corresponding spaces in (4.5). In addition to the spaces defined on Ω we need the corresponding spaces defined on Ω_c^h :

$$\begin{aligned}
&\mathcal{U} = H^{2m}(\Omega, \mathbf{R}^q) \subset \mathcal{W} \subset \mathcal{V},\ \mathcal{W}_0,\ \mathcal{U}_0 := \mathcal{W}_0 \cap \mathcal{U},\ \mathcal{W}',\quad \text{or } \mathcal{V}_D \subset \mathcal{W}_D \subset \mathcal{V}_b \text{ on } \Omega \text{ or } \partial\Omega,\\
&\quad \mathcal{U}_c \subset \mathcal{W}_c \subset \mathcal{V}_c,\ \mathcal{W}_{c,0},\ \mathcal{U}_{c,0} := \mathcal{W}_{c,0} \cap \mathcal{U},\ \mathcal{W}_c',\quad \text{or } \mathcal{V}_{c,D} \subset \mathcal{W}_{c,D} \text{ defined on } \Omega_c^h \text{ or } \partial\Omega_c^h,\\
&\quad\quad G(\vec{u}(\cdot)) := G^w(\cdot, \vec{u}, D\vec{u}, \ldots, D^{2m}\vec{u})(\cdot), G : \mathcal{D}(G) \subseteq \mathcal{U} \to \mathcal{V}, G : \mathcal{D}(G) \subseteq \mathcal{U}_c \to \mathcal{V}_c, \quad (7.1)\\
&\quad\quad\quad B_D : \mathcal{U} \to \mathcal{V}_D \text{ or } B_D : \mathcal{W} \to \mathcal{W}_D, B_D\vec{u}(x) := \Big(\frac{\partial^j \vec{u}(x)}{\partial \nu^j}|_{\partial\Omega}\ \forall 0 \le j \le m-1\Big). \quad (7.2)
\end{aligned}$$

We combine the differential and the boundary operators, G and B_D, to define F, called elliptic simultaneously with G, mapping $\mathcal{U}$ into the Cartesian product $\mathcal{V}_\Pi$, see (5.3), (5.16),

$$F := (G, B_D) : \mathcal{U} \to \mathcal{V}_\Pi := \mathcal{V} \times \mathcal{V}_D \subset \mathcal{V} \times \mathcal{V}_b; F(\vec{u}_0) := \big(G(\vec{u}_0), B_D\vec{u}_0\big) = \vec{0}, \qquad (7.3)$$

with a locally unique solution, $\vec{u}_0$, and a boundedly invertible $F'(\vec{u}_0) \in \mathcal{L}(\mathcal{U}, \mathcal{V} \times \mathcal{V}_D)$.

We mainly consider trivial right hand sides, $\vec{0} = (\vec{0}, \vec{0})$. Thus we characterize the vanishing of $G(\vec{u}_0)$ and $B_D\vec{u}_0$ by testing with elements $\vec{v} \in \mathcal{V}$ and $\vec{v}_D' \in \mathcal{V}_D'$. Since $\mathcal{V}_D, \mathcal{W}_D$ are densely embedded into $\mathcal{V}_b = \mathcal{V}_b'$ we can test with $\vec{v} \in \mathcal{V}_b$ as well.

For nontrivial Dirichlet conditions, we redefine as usual B_D, G in (**??**), (7.3), reducing the problem to trivial Dirichlet conditions. For the discretization we define analogously new G^h, B_D^h in (7.8), (7.9). This will be combined in Theorems 8.1, 9.3, 9.5 to obtain stability and convergence for the inhomogeneous case as well.

The transformation to homogeneous Dirichlet conditions considerably reduces the technicalities. Much more important is the equality of weak and strong bilinear FE-forms for homogeneous conditions, see Proposition 9.1. This allows using all the known highly efficient solvers for weak bilinear forms, however, with C^0-FEs replaced by $C^{2m-1}-$FEs.

For applying general discretization methods, e.g. Stetter, [48], Definition 1.1.2, and Böhmer, [11], we introduce the discrete F^h for F, and the corresponding projectors $P^h \in \mathcal{L}(\mathcal{U},\mathcal{U}^h)$, and $Q_\Pi^h \in \mathcal{L}(\mathcal{V}_\Pi, \mathcal{V}_\Pi^h)$, $Q_{c,\Pi}^h \in \mathcal{L}(\mathcal{V}_{c,\Pi}, \mathcal{V}_\Pi^h)$ fitting to F^h, see (7.16). This yields the standard "consistency and stability implies convergence" result. More precisely we need *sequences*

$$\{\mathcal{U}^h, \mathcal{V}_\Pi^h = \mathcal{V}^h \times \mathcal{V}_b^h, P^h, Q_\Pi^h, Q_{c,\Pi}^h, F^h\}_{h\in H}, \text{ with } \inf\{0 < h \in H\} = 0 \text{ for } \lim_{h\to 0}. \quad (7.4)$$

The product operator $F : \mathcal{U} \to \mathcal{V} \times \mathcal{V}_b$ and the corresponding $F^h : \mathcal{U}^h \to \mathcal{V}^{h'} \times \mathcal{V}_b^h$ cause some unpleasant technicalities in the definition of Q_Π^h, $Q_{c,\Pi}^h$. Subsequently, the consistency theory becomes relatively simple again, see figure (7.20) and Section 8. These technicalities are not surprising compared to some other approaches for violated boundary conditions, e.g. in Lenoir, Bernadou, [8, 9, 40]. The unavoidable complications are piled up in Section 9.

By (5.3), (**??**), (6.1), (7.7), the terms $\vec{u}^h$, $G(\vec{u}^h)$ and $B_D\vec{u}^h$ are defined on Ω_c^h and $\partial\Omega_c^h$ instead of the original Ω and $\partial\Omega$. So we have to choose our projectors s.t. this discrepancy can be handled. We combine the extension operators E_c introduced in Theorem 6.4 and Remark 6.2 with inverse estimates and the estimates for interpolation errors on Ω in Theorems 6.1, 6.2, 6.3, 6.5, 6.6. The rather arbitrary *sequence* of P^h has to satisfy

$$P^h \in \mathcal{L}(\mathcal{U},\mathcal{U}^h): \quad \forall\, \vec{u} \in \mathcal{U} \ \lim_{h\to 0} \|P^h\vec{u}\|_{\mathcal{U}^h} = \|\vec{u}\|_{\mathcal{U}}, \text{ we choose } P^h\vec{u} := I^h E_c\vec{u}. \qquad (7.5)$$

We include E_c, unnecessary for our above Argyris FEs, however possibly needed for other schemes. Theorems 6.2 and 6.6 show that the sequences, defined on Ω_c^h, see (6.27),

$$\{\mathcal{S}_d^{2m-1}(\mathcal{T}_c^h, \mathbf{R}^q) = \mathcal{S}_d^{2m-1}(\mathcal{T}_c^h) \supset \mathcal{U}^h = \mathcal{W}^h = S^h = S_0^h \oplus S_b^h; \mathcal{V}^h = S_0^h = S^h \cap \mathcal{W}_{c,0},$$
$$\mathcal{V}_b^h := (S_b^h|_{\partial\Omega_c^h})^m, \text{or } \mathcal{V}_b^h := B_D^h S_b^h, \ d \geq (2m-1)2^n + 1\}_{h\in H}, \text{approximate } \mathcal{U}, \mathcal{W}, \mathcal{V}, \mathcal{V}_b, \ (7.6)$$

defined on Ω and $\partial\Omega$. The $\vec{u}^h \in \mathcal{S}_d^{2m-1}(\mathcal{T}_c^h)$, and the $\vec{v}^h \in \mathcal{V}_b^h$, violate the original boundary conditions. We use the equivalence (6.21) for monitoring the boundary condition $B_D^h\vec{u}_b^h = \vec{0} \in \mathcal{V}_b^h$. By the preceding (7.2) the above $\mathcal{V}_b^h$ indeed can be reduced to $\mathcal{V}_b^h = B_D^h S_b^h$.

More delicate than P^h are the projectors to the image spaces, e.g. the $Q_\Pi^h \in \mathcal{L}(\mathcal{V}_\Pi, \mathcal{V}_\Pi^{h'})$, motivated by F^h. Thus we start updating the preliminary FE version (2.4) of the differential equation. With G^w and $\mathcal{D}(G^w)$ in (5.1), and the above $\mathcal{U}^h, \mathcal{V}_c$, let

$$\mathcal{D}(G^h) := \{\vec{u}^h \in \mathcal{U}^h : \vec{w}_{\vec{u}^h}(x) := (x, \vec{u}^h(x), \ldots, D^{2m}\vec{u}^h(x)) \in \mathcal{D}(G^w) \forall x \in \Omega', G(\vec{u}^h) \in \mathcal{V}_c\}$$
$$\mathcal{D}(G) := \{\vec{u} \in \mathcal{U} : \vec{w}_{\vec{u}}(x) := (x, \vec{u}(x), \ldots, D^{2m}\vec{u}(x)) \in \mathcal{D}(G^w) \forall x \in \Omega', G(\vec{u}) \in \mathcal{V}_c\} \ (7.7)$$

This extended definition of G^w, compared to (5.3), was the reason for introducing the Ω' there. We replace the differential equation $G(\vec{u}_0) = \vec{0} \in \mathcal{V}, \vec{u}_0 \in \mathcal{D}(G)$, in (7.3), by the FE equations in equivalent formulations, see (2.4), (2.5), (7.3): Define for $\vec{u}_0^h \in \mathcal{U}^h$ the

$$\begin{aligned} &\text{sequence } G^h(\vec{u}_0^h) = \vec{0} \Longleftrightarrow (G(\vec{u}_0^h), \vec{v}^h)_{\mathcal{V}_c} = 0 \ \forall \vec{v}^h \in \mathcal{V}^h \subset \mathcal{V}_c \\ &\Longleftrightarrow \mathcal{V}^h \perp_{\mathcal{V}_c} G^h(\vec{u}_0^h) \neq G(\vec{u}_0^h) \not\ni \mathcal{V}^h \text{ with } \vec{u}_0^h \in \mathcal{D}(G^h) \subset \mathcal{U}_c, \end{aligned} \tag{7.8}$$

Similarly, the boundary conditions (7.3) are tested, see the equivalence (6.21), as

$$B_D^h \vec{u}_0^h := \big(\frac{\partial^j \vec{u}_0^h}{\partial \nu^j}|_{\partial\Omega_c^h} \ \forall 0 \le j \le m-1\big) = \vec{0} \in \mathcal{V}_b^h \Longleftrightarrow (\vec{v}_b^h, B_D^h \vec{u}_0^h)_{\mathcal{V}_{c,b}} = 0 \ \forall \vec{v}_b^h \in \mathcal{V}_b^h. \tag{7.9}$$

7.2 The necessary projectors

Hence, projectors represent an appropriate tool for both G^h and B_D^h. The projectors on the product space, $Q_\Pi^h \in \mathcal{L}(\mathcal{V}_\Pi, \mathcal{V}_\Pi^{h'})$, live on the original and discrete image spaces, listed in (7.10), with their duals, scalar products and pairings, compare (7.1) - (7.3). At the other side, we have to project, as in (7.8), functions $G(\vec{u}^h) \notin \mathcal{V}^h$ defined on Ω_c^h. So we start summarizing and defining the product spaces on Ω and Ω_c^h. The following indices $_b$ and $_c$ and $_{c,b}$ and $_\Pi$ or $_{c,\Pi}$ below, indicate functions and functionals defined on $\partial\Omega$ and Ω_c^h and $\partial\Omega_c^h$, and on the product of domain and boundary, resp., and we test $F(\vec{u}_0) = \vec{0}$ as in (7.8). The test spaces are identified with their duals, e.g. $\mathcal{V} = \mathcal{V}' = L^2(\Omega)$, as far as possible, allowing simplified notations. We keep the concepts for the original and the discrete problem as close as possible, in particular for the boundary conditions.

$$\begin{aligned} &\mathcal{V}_\Pi := \mathcal{V} \times \mathcal{V}_D \subset \mathcal{V} \times \mathcal{V}_b, \mathcal{V}_D = B_B\mathcal{U}, \mathcal{V}_b := (L^2(\partial\Omega))^m \text{ with } \mathcal{V} = \mathcal{V}', \mathcal{V}_b = \mathcal{V}_b' \supset \mathcal{V}_D \\ &\mathcal{V}_{c,\Pi} := \mathcal{V}_c \times \mathcal{V}_{c,D} \subset \mathcal{V}_c \times \mathcal{V}_{c,b}, \mathcal{V}_{c,D} = B_D^h\mathcal{U}_c, \mathcal{V}_{c,b} := (L^2(\partial\Omega_c^h))^m, \mathcal{V}_c = \mathcal{V}_c', \dots, \\ &\text{with } \big((\vec{u}, \vec{u}_b), (\vec{v}^h, \vec{v}_b)\big)_{\mathcal{V}_\Pi} := (\vec{u}, \vec{v})_{\mathcal{V}} + (\vec{u}_b, \vec{v}_b)_{\mathcal{V}_b} \text{ and } \big((\vec{u}, \vec{u}_b), (\vec{v}^h, \vec{v}_b)\big)_{\mathcal{V}_{c,\Pi}} \text{ versus} \\ &\mathcal{V}_\Pi^h = \mathcal{V}^h \times \mathcal{V}_b^h := S_0^h \times (S_b^h|_{\partial\Omega_c^h})^m, \mathcal{V}_\Pi^{h'} = \mathcal{V}^{h'} \times \mathcal{V}_b^h := S_0^{h'} \times (S_b^h|_{\partial\Omega_c^h})^m \neq \mathcal{V}_\Pi^h \text{ with} \\ &\mathcal{V}^{h'} \neq \mathcal{V}^h \text{ and } \langle(\vec{u}^{h'}, \vec{u}_b^h), (\vec{v}^h, \vec{v}_b^h)\rangle_{\mathcal{V}_\Pi^{h'} \times \mathcal{V}_\Pi^h} := \langle \vec{u}^{h'}, \vec{v}^h\rangle_{\mathcal{V}^{h'} \times \mathcal{V}^h} + (\vec{u}_b^h, \vec{v}_b^h)_{\mathcal{V}_b^h}, \end{aligned} \tag{7.10}$$

see (6.13), (6.14), Theorem 6.3, (7.6), (7.9). In (7.3), and (7.8), (7.9) we have satisfied the differential equation and boundary condition or their discrete counterparts by testing with $\mathcal{V}$ and $\mathcal{V}_b$ or, see the equivalence (6.21), (7.9), with $\mathcal{V}^h$ and $\mathcal{V}_b^h$. The two components in (7.3) and (7.8), (7.9) of F and F^h are now combined with the extension operators E_c in (6.23), (6.24), for motivating the appropriate definitions of the projection operators. For the discrete spaces $\mathcal{V}^{h'} \neq \mathcal{V}^h$, and $\mathcal{V}_b^{h'} = \mathcal{V}_b^h$ the FE equations always have to be tested with $\vec{v}^h \in \mathcal{V}^h$ and $\vec{v}_b^h \in \mathcal{V}_b^h$. Thus corresponding to the product spaces $\mathcal{V}_\Pi$ and $\mathcal{V}_{c,\Pi}$ we have to introduce two projectors $Q_\Pi^h := (Q^h, Q_{c,b}^h) \in \mathcal{L}(\mathcal{V}_\Pi, \mathcal{V}_\Pi^{h'})$ and $Q_{c,\Pi}^h := (Q_c^h, Q_{c,b}^h) \in \mathcal{L}(\mathcal{V}_{c,\Pi}, \mathcal{V}_\Pi^{h'})$. We start with (7.8), mind $\mathcal{V} = \mathcal{V}', \mathcal{V}_b = \mathcal{V}_b'$, and define, see (6.23), (6.24), and Theorem 6.6,

$$\begin{aligned} &Q^h \in \mathcal{L}(\mathcal{V}, \mathcal{V}^{h'}), \text{ by } \vec{f} \in \mathcal{V}: \ \langle Q^h \vec{f}, \vec{v}^h\rangle_{\mathcal{V}^{h'} \times \mathcal{V}^h} - ((E_c\vec{f}), \vec{v}^h)_{\mathcal{V}_c} = \vec{0} \ \ \forall \vec{v}^h \in \mathcal{V}^h \ \text{ and} \\ &Q_c^h \in \mathcal{L}(\mathcal{V}_c, \mathcal{V}^{h'}), \text{ by } \vec{f}_c \in \mathcal{V}_c: \ \langle Q_c^h \vec{f}_c, \vec{v}^h\rangle_{\mathcal{V}^{h'} \times \mathcal{V}^h} - (\vec{f}_c, \vec{v}^h)_{\mathcal{V}_c} = \vec{0} \ \ \forall \vec{v}^h \in \mathcal{V}^h \ \text{ with} \\ &\lim_{h\to 0} \|Q^h \vec{f}\|_{\mathcal{V}^{h'}} = \|\vec{f}\|_{\mathcal{V}}, \text{ and } \lim_{h\to 0} \|Q_c^h \vec{f}_c\|_{\mathcal{V}^{h'}} = \lim_{h\to 0} \|\vec{f}_c\|_{\mathcal{V}_c} = \lim_{h\to 0} \|E_c\vec{f}_c|_\Omega\|_{\mathcal{V}}. \end{aligned} \tag{7.11}$$

The last limits are obtained as a consequence of (6.10). Prima vista, it is surprising that for a function, e.g., $\vec{f} \in \mathcal{V}$ the $Q^h\vec{f} \in \mathcal{V}^{h'}$ is no function. By the above observation $G(\vec{u}^h) \notin \mathcal{V}^h$ this is in fact unavoidable for the testing process. However, since the S_0^h and $S_0^{h'}$ form a pair of dual bases, see Theorem 6.3, we easily can formulate corresponding projectors, see Theorem 6.6, e.g.,

$$Q_d^h \in \mathcal{L}(\mathcal{V}, \mathcal{V}^h), \text{ by } \vec{f} \in \mathcal{V}: \ Q_d^h\vec{f} := \sum_{j=1}^{m_0} \langle Q^h\vec{f}, \vec{s}^j\rangle_{\mathcal{V}^{h'}\times\mathcal{V}^h}\vec{s}^j \in \mathcal{V}^h \ \text{ again with}$$
$$\lim_{h\to 0}\|Q_d^h\vec{f}\|_{\mathcal{V}^h} = \|\vec{f}\|_{\mathcal{V}}, \text{ and } \lim_{h\to 0}\|Q_{c,d}^h\vec{f}_c\|_{\mathcal{V}^h} = \lim_{h\to 0}\|\vec{f}_c\|_{\mathcal{V}_c} = \lim_{h\to 0}\|E_c\vec{f}_c|_\Omega\|_{\mathcal{V}}. \quad (7.12)$$

We turn to the projectors for the traces on the boundary. The approximation property for $\mathcal{V}_b$ in (7.6), is combined in the last lines of the proof for consistency, see Theorem 8.1, with the equivalence (6.21). There we need the limits of the norms in (7.14), $B_D^h\vec{u}^h \equiv \vec{0}$ and an estimate for $\|B_D^hE_c\vec{u}_0\|_{\mathcal{V}_{c,b}}$ are necessary. We always have to recall that the $u_b \in \mathcal{V}_D$ and $u_b^h \in \mathcal{V}_b^h$ have the forms $u_b = B_Du, u \in \mathcal{U}$ and $u_b^h = B_D^hu^h, u^h \in \mathcal{S}_b^h$, resp. Again it satisfies testing vanishing boundary conditions w.r.t. the L^2 scalar product $(\cdot,\cdot)_{\mathcal{V}_{c,b}}$. This and the dense embeddings $\mathcal{V}_D \subset \mathcal{V}_b$ and $\mathcal{V}_{c,D} \subset \mathcal{V}_{c,b}$ motivate the following projectors

$$Q_b^h \in \mathcal{L}(\mathcal{V}_D, \mathcal{V}_b^h), \text{ for } \vec{u}_b := B_D\vec{u} \in \mathcal{V}_D \subset \mathcal{V}_b \text{ as } (Q_b^h\vec{u}_b, \vec{v}_b^h)_{\mathcal{V}_{c,b}} - ((B_D^hE_c\vec{u}, \vec{v}_b^h)_{\mathcal{V}_{c,b}} = 0 \text{ and}$$
$$Q_{c,b}^h \in \mathcal{L}(\mathcal{V}_{c,b}, \mathcal{V}_b^h), \vec{u}_{c,b} := B_D^hE_c\vec{u} \text{ or } := B_D^h\vec{u}^h \in \mathcal{V}_{c,D} \text{ as } (Q_{c,b}^h\vec{u}_{c,b}, \vec{v}_b^h)_{\mathcal{V}_{c,b}} - (\vec{u}_{c,b}, \vec{v}_b^h)_{\mathcal{V}_{c,b}}$$
$$= 0 \ \forall \vec{v}_b^h \in \mathcal{V}_b^h, \text{ now with } Q_{c,b}^hB_D^h\vec{u}^h = B_D^h\vec{u}^h \ \forall \vec{u}^h \in \mathcal{U}^h. \quad (7.13)$$

We have restricted the definition of $Q_b^h, Q_{c,b}^h$ to the only necessary $Q_b^hB_D\vec{u}, Q_{c,b}^hB_D^hE_c\vec{u}$, $Q_{c,b}^hB_D^h\vec{u}^h$. For a general $\vec{u}_b \in \mathcal{V}_b$ a boundary extension operator E_b would yield $(Q_b^h\vec{u}_b, \vec{v}_b^h)_{\mathcal{V}_{c,b}} - (B_D^hE_b\vec{u}_b, \vec{v}_b^h)_{\mathcal{V}_{c,b}} = 0$. The convergence results in (6.11), Theorems 6.3, 6.6, for the interpolation along $\partial\Omega_c^h$ and the above $\vec{u}_b, \vec{u}_{c,b}$ and (6.21) imply the limits of the boundary norms

$$\lim_{h\to 0}\|Q_{c,b}^h\vec{u}_{c,b}\|_{\mathcal{V}_b^h} = \lim_{h\to 0}\|\vec{u}_{c,b}\|_{\mathcal{V}_{c,b}}, \quad \lim_{h\to 0}\|Q_b^h\vec{u}_b\|_{\mathcal{V}_b^h} = \lim_{h\to 0}\|\vec{u}_b\|_{\mathcal{V}_b}. \quad (7.14)$$

This allows defining the product projectors, see (7.10), and we obtain for their norms

$$\begin{aligned}
Q_\Pi^h &:= (Q^h, Q_b^h) \in \mathcal{L}\big(\mathcal{V}\times\mathcal{V}_D, \mathcal{V}^{h'}\times\mathcal{V}_b^h\big) = \mathcal{L}\big(\mathcal{V}\times\mathcal{V}_D, \mathcal{V}_\Pi^{h'}\big), \text{ and}\\
Q_{c,\Pi}^h &:= (Q_c^h, Q_{c,b}^h) \in \mathcal{L}\big(\mathcal{V}_c\times\mathcal{V}_{c,D}, \mathcal{V}_\Pi^{h'}\big), \text{e.g., } Q_{d,\Pi}^h := (Q_d^h, Q_b^h) \in \mathcal{L}\big(\mathcal{V}\times\mathcal{V}_D, \mathcal{V}_\Pi^h\big)\\
\text{with} \quad & \lim_{h\to 0}\|Q_\Pi^h(\vec{f}, B_D\vec{u})\|_{\mathcal{V}_\Pi^{h'}} = \lim_{h\to 0}\|Q_{d,\Pi}^h(\vec{f}, B_D^hE_c\vec{u})\|_{\mathcal{V}_\Pi^h} = \|(\vec{f}, B_D\vec{u})\|_{\mathcal{V}_\Pi}, \quad (7.15)\\
\text{and} \quad & \lim_{h\to 0}\|Q_{c,\Pi}^h(\vec{f}_c, B_D^hE_c\vec{u})\|_{\mathcal{V}_\Pi^{h'}} = \lim_{h\to 0}\|(\vec{f}_c, B_D^hE_c\vec{u})\|_{\mathcal{V}_{c,\Pi}} = \lim_{h\to 0}\|(E_c\vec{f}_c, B_D\vec{u})\|_{\mathcal{V}_\Pi}.
\end{aligned}$$

The choice of $P^h, Q^h, Q_c^h, \ldots,$ for the given $\mathcal{U}^h, \mathcal{V}^h, \ldots,$ and problem (2.4) is certainly not unique. However the combination $\mathcal{U}, \mathcal{U}^h, P^h, Q^h, \ldots,$ should be chosen appropriately to yield classical consistency with the highest possible order, see below, e.g., (7.22).

7.3 Projectors and discretization

For the nonlinear operator, $F : \mathcal{U} \to \mathcal{V} \times \mathcal{V}_D$, in (7.3), the previously introduced projectors $P^h, Q^h_c, Q^h_{c,b}, \ldots,$ allow a short hand definition of the corresponding discrete operator, $F^h : \mathcal{U}^h \to \mathcal{V}^{h'} \times \mathcal{V}^h_b$, see (7.8), (7.9), (7.10), (7.15). We determine and summarize

$$\begin{aligned}\vec{u}^h_0 \in \mathcal{U}^h \text{ s.t. } F^h(\vec{u}^h_0) = \vec{0} &\iff \langle F^h(\vec{u}^h_0), \vec{v}^h_\Pi\rangle_{\mathcal{V}^{h'}_\Pi \times \mathcal{V}^h_\Pi} = (F(\vec{u}^h_0), \vec{v}^h_\Pi)_{\mathcal{V}_{c,\Pi}} = \vec{0} \forall \vec{v}^h_\Pi \in \mathcal{V}^h_\Pi \quad (7.16)\\ &\iff (G(\vec{u}^h_0), \vec{v}^h)_{\mathcal{V}_c} + (B^h_D \vec{u}^h_0, \vec{v}^h_b)_{\mathcal{V}^h_b} = 0 \ \forall \vec{v}^h \in \mathcal{V}^h \subset \mathcal{V}_c \ \forall \vec{v}^h_b \in \mathcal{V}^h_b, \text{ where}\\ &F^h : \mathcal{D}(F^h) = \mathcal{D}(G^h) \subset \mathcal{U}^h \to \mathcal{V}^{h'} \times \mathcal{V}^h_b \text{ with } F^h(\vec{u}^h) = (G^h(\vec{u}^h), B^h_D \vec{u}^h)\\ &\qquad = Q^h_{c,\Pi} F(\vec{u}^h) = Q^h_{c,\Pi}(G(\vec{u}^h), B^h_D \vec{u}^h).\end{aligned}$$

Sometimes it is appropriate transforming $G(\vec{u}^h)$ by Q^h_d and using,

$$\begin{aligned}\big((Q^h_d G(\vec{u}^h), Q^h_{c,b} B^h_D \vec{u}^h), \vec{v}^h_\Pi\big)_{\mathcal{V}_{c,\Pi}} &= \langle F^h(\vec{u}^h), \vec{v}^h_\Pi\rangle_{\mathcal{V}^{h'}_\Pi \times \mathcal{V}^h_\Pi}\\ &= \langle F(\vec{u}^h), \vec{v}^h_\Pi\rangle_{\mathcal{V}^{h'}_\Pi \times \mathcal{V}^h_\Pi} = (F^h(\vec{u}^h), \vec{v}^h_\Pi)_{\mathcal{V}_{c,\Pi}}. \quad (7.17)\end{aligned}$$

This brings about a mapping

$$\Phi^h \text{ defined by } \quad \Phi^h(F) := F^h := Q^h_{c,\Pi} F|_{\mathcal{U}^h}. \quad (7.18)$$

By our combination of the $\mathcal{U}, \mathcal{V}, \mathcal{V}_b$, the $\mathcal{U}^h, \mathcal{V}^h, \mathcal{V}^h_b$, the operators $P^h, Q^h_c, Q^h_{c,\Pi}, \ldots,$ and Φ^h, we have thus defined a *nonconforming Petrov–Galerkin discretization method applicable* to every F in (7.3). This Φ^h is even linear s.t. for F, F^1, F^2 in (7.3),

$$\Phi^h(c_1 F^1 + c_2 F^2) = c_1 \Phi^h(F^1) + c_2 \Phi^h(F^2) \ \forall c_1, c_2 \in \mathbf{R}, \ \forall F = F^1, F^2. \quad (7.19)$$

This property is essential for the mesh independence principle in Böhmer, [15], allowing the proof of a quadratically convergent discrete Newton's method for the highly nonlinear FE equations. Our modification of Stetter, [48], Vainikko, [55], emphasizes the testing with $\mathcal{V} \times \mathcal{V}_b$, $\mathcal{V}^h \times \mathcal{V}^h_b$: Summarizing yields the *diagram* with uniformly bounded P^h, Q^h_Π.

$$\begin{array}{ccccccc} \mathcal{U} & \xrightarrow{F} & \mathcal{V}_\Pi = \mathcal{V} \times \mathcal{V}_D & \overset{\text{tested by}}{\longleftrightarrow} & \mathcal{V}_\Pi = \mathcal{V} \times \mathcal{V}_b \\ P^h \big\downarrow & \Phi^h \big\downarrow & Q^h_\Pi \big\downarrow & & \\ \mathcal{U}^h & \xrightarrow{F^h} & \mathcal{V}^{h'}_\Pi = \mathcal{V}^{h'} \times \mathcal{V}^h_b & \overset{\text{tested by}}{\longleftrightarrow} & \mathcal{V}^h_\Pi = \mathcal{V}^h \times \mathcal{V}^h_b \end{array} \quad (7.20)$$

A method is called *convergent* and *convergent of order k* if the solutions $\vec{u}_0$ and $\vec{u}^h_0$ of $F\vec{u}_0 = \vec{0}$ and $F^h \vec{u}^h_0 = \vec{0}$ in (7.3) and (7.16), satisfy

$$\|\vec{u}^h_0 - P^h \vec{u}_0\|_{\mathcal{U}^h} \to 0 \text{ and } \|\vec{u}^h_0 - P^h \vec{u}_0\|_{\mathcal{U}^h} = \mathcal{O}(h^k) \text{ for } h \to 0, \text{ resp.} \quad (7.21)$$

These $\mathcal{O}(h^k)$ are usually estimated by $Ch^k \|\vec{u}_0\|_{H^{k'}(\Omega)}$ with appropriate k, k'. For convergence, we need the *classical consistency errors* for a general $\vec{u}$, see (7.3), Theorem 6.3,

$$\begin{aligned}&\|F^h P^h \vec{u} - Q^h_\Pi F \vec{u}\|_{\mathcal{V}^{h'}_\Pi} = \|\big((Q^h_c G(P^h \vec{u}) - Q^h G(\vec{u})), (B^h_D P^h \vec{u} - Q^h_b B_D \vec{u})\big)\|_{\mathcal{V}^{h'}_\Pi}\\ &\text{and for } \vec{u} = \vec{u}_0 : \|F^h P^h \vec{u}_0 - Q^h_\Pi F \vec{u}_0\|_{\mathcal{V}^{h'}_\Pi} = \|\big(Q^h_c G(P^h \vec{u}_0), -Q^h_b B_D \vec{u}_0\big)\|_{\mathcal{V}^{h'}_\Pi}, \quad (7.22)\end{aligned}$$

where these residuals can be used for adaptive methods. A discretization method for $F(\vec{u}) = \vec{0}$ is called *consistent* and *consistent of order k* in $\vec{u}$, if, resp.,

$$\|F^h P^h \vec{u} - Q^h_\Pi F\vec{u}\|_{\mathcal{V}^{h'}} \to 0 \text{ and } \|F^h P^h \vec{u} - Q^h_\Pi F\vec{u}\|_{\mathcal{V}^{h'}} = \mathcal{O}(h^k) \text{ for } h \to 0. \quad (7.23)$$

To formulate *stability,* we start with a fixed sequence $\{\vec{u}^h \in \mathcal{D}(F^h)\}_{h\in H}$ and our sequence of $F^h = Q^h_{c,\Pi} F|_{\mathcal{U}^h}$. Assume there exist constants, $h_0, r, S \in \mathbf{R}_+$, *independent of h,* s.t., *uniformly* $\forall\, h \in H, h < h_0$:

$$\forall\, \vec{u}^h_i \in B_r(\vec{u}^h) \subset \mathcal{D}(F^h), i = 1, 2, \Longrightarrow \|\vec{u}^h_1 - \vec{u}^h_2\|_{\mathcal{U}^h} \le S\|F^h(\vec{u}^h_1) - F^h(\vec{u}^h_2)\|_{\mathcal{V}^{h'}_\Pi},$$
$$\text{with } B_r(\vec{u}^h) = \{\vec{v}^h \in \mathcal{U}^h : \|\vec{v}^h - \vec{u}^h\|_{\mathcal{U}^h} < r\} \text{ sometimes } B_{r,\mathcal{U}^h}(\vec{u}^h). \quad (7.24)$$

Similarly $B_r(\vec{u}), B_{r,\mathcal{U}}(\vec{u})$ are defined. We always will assume that, e.g., these $B_r(\vec{u}) \subset \mathcal{D}(F)$. Then F^h is called *stable in the sequence* $\{\vec{u}^h\}_{h\in H}$ or for short *stable in* $\vec{u}^h$ and the S and r are called *stability bound and stability threshold,* resp. Often we choose $\vec{u}^h := P^h \vec{u}_0$. Combining Stetter's, [48], Theorems 1.2.3. and 1.2.4. we get:

Theorem 7.1 *Let the original problem (7.3) have the exact solution* $\vec{u}_0 \in \mathcal{U}$. *Let its discretization* $F^h = Q^h_{c,\Pi} F|_{\mathcal{U}^h} : \mathcal{U}^h \to \mathcal{V}^{h'}_\Pi$ *in (7.16) satisfy*

1. $F^h : \mathcal{U}^h \to \mathcal{V}^{h'}_\Pi$ *is defined and continuous in* $B_r(P^h \vec{u}_0)$, $r > 0, h-$*independent,*
2. F^h *is consistent with* F *in* $P^h \vec{u}_0$,
3. F^h *is stable for* $P^h \vec{u}_0$.

Then the discrete problem $F^h(\vec{u}^h) = \vec{0}$ *possesses a unique solution* $\vec{u}^h_0 \in \mathcal{U}^h$ *near* $\vec{u}_0$ *for all sufficiently small* $h \in H$ *and* $\vec{u}^h_0$ *converges to* $\vec{u}_0$. *If* F^h *is consistent and consistent of order p, then* $\vec{u}^h_0$ *converges and converges of order p, resp.:*

$$\|\vec{u}^h_0 - P^h \vec{u}_0\|_{\mathcal{U}^h} \le S\|Q^h_{c,\Pi} F(P^h \vec{u}_0)\|_{\mathcal{V}^{h'}_\Pi} \to 0 \text{ and } \|\vec{u}^h_0 - P^h \vec{u}_0\|_{\mathcal{U}^h} \le \mathcal{O}(h^p). \quad (7.25)$$

Remark 7.1 *Stetter, [48], requires for his theory linear bounded operators* P^h *and* $Q^h_{c,\Pi} := (Q^h_c, Q^h_{c,b})$ *in (7.5) and (7.15). But in fact he only does need equicontinuous operators and the limits of the norms listed above, see also (7.11), (7.14), For our FEM, this is satisfied by the listed results in (7.5) and (7.11), (7.14), (7.15). The convergence properties in (7.6) are not required here, since usually 2. is impossible without them. Condition 1. is correct, if* F *is continuous in* $B_r(P^h \vec{u}_0)$ *or* $B_r(\vec{u}_0)$. *The consistency, 2., will be verified in the following Section 8 in Theorem 8.1. Due to the careful preparations this is a relatively simple proof. Then 3. is the only missing and hard condition in the previous Theorem. In our approach the stability of the linearized problem and its regularity are the essential tools for the proof of stability, see Theorem 9.1. This only makes sense, if the original* F *is continuously differentiable. Then* F *is Lipschitz continuous as well.*

8 Consistency for the Nonconforming FEM

For verifying the conditions of Theorem 7.1 we start with the easier consistency. Similarly to the Ciarlet, Raviart, Lenoir, Bernadou approaches, [21, 20, 40, 8], we have already and will furthermore use extension operators, the above E_c. They allow estimates of the interpolation errors on Ω and $\partial\Omega$, as in (7.22), and thus are essential for

Theorem 8.1 *We require the conditions in Theorems 6.2, 6.3, an exact solution* $\vec{u}_0 \in H^\ell(\Omega, \mathbf{R}^q), \ell > 2m$, *a Lipschitz continuous* G^w *in (5.1) with a global constant* L,

$$G^w(\cdot) \in C_L(\mathcal{D}(G^w)), \text{ for } \mathcal{D}(G^w) \text{ see (5.14) with} \\ \big(x, (E_c\vec{u})(x), \ldots, D^{2m}(E_c\vec{u})(x)\big) \in \mathcal{D}(G^w), \forall x \in \overline{\Omega'}, \vec{u} \in \overline{B_{r,\mathcal{U}}(\vec{u}_0)} \tag{8.1}$$

and $d \geq (2m-1)n^2 + 1$. *Then* F^h *is consistent with* F *in* $P^h\vec{u}_0$ *and, with* $p = 2$ *for* Ω^h *and* $p > 2$ *for* Ω_c^h,

$$\|F^hP^h\vec{u}_0 - Q_\Pi^hF\vec{u}_0\|_{\mathcal{V}_\Pi^{h\prime}} = \|\big(Q_c^hG(P^h\vec{u}_0), -Q_b^hB_D\vec{u}_0\big)\|_{\mathcal{V}_\Pi^{h\prime}} \\ \leq CLh^{\min\{\ell-2m,p\}}\|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^q)}. \tag{8.2}$$

Remark 8.1 $d \geq (2m-1)n^2+1$, *see (6.12), and* ℓ *are related by (6.8). For inhomogeneous boundary conditions in (7.3), the estimate (8.2) remains valid.*

Proof: We estimate the first term with (7.11), L in (8.1), $G(\vec{u}_0) \equiv \vec{0}$ and (6.26):

$$\begin{aligned} &\|Q_c^hG(P^h\vec{u}_0) - Q^hG(\vec{u}_0)\|_{\mathcal{V}^{h\prime}} = \|Q_c^hG(P^h\vec{u}_0) - \vec{0}\|_{\mathcal{V}^{h\prime}} \\ &\leq \|Q_c^hG(P^h\vec{u}_0) - Q_c^hG(E_c\vec{u}_0)\|_{\mathcal{V}^{h\prime}} + \|Q_c^hG(E_c\vec{u}_0)\|_{\mathcal{V}^{h\prime}} \\ &\leq CL\|P^h\vec{u}_0 - E_c\vec{u}_0\|_{\mathcal{U}^h} + \|Q_c^hG(E_c\vec{u}_0)\|_{\mathcal{V}^{h\prime}} \text{ by (6.20)} \\ &\qquad \leq CLh^{\ell-2m}\|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^q)} + \|Q_c^hG(E_c\vec{u}_0)\|_{\mathcal{V}^{h\prime}}. \end{aligned} \tag{8.3}$$

We test $Q_c^hG(E_c\vec{u}_0) - Q^hG(\vec{u}_0)$ with $\vec{v}^h \in \mathcal{V}^h$, combine it with (5.1), (7.11), and get

$$\begin{aligned} &\big|\langle Q_c^hG(E_c\vec{u}_0), \vec{v}^h\rangle_{\mathcal{V}^{h\prime}\times\mathcal{V}^h}\big| = \big|\int_{\Omega_c^h}\big[G(E_c\vec{u}_0)(x)\big]\vec{v}^h(x)dx\big| \ \forall \vec{v}^h \in \mathcal{V}^h \\ \leq\ & \big|\int_{\Omega_c^h}\big[G(E_c\vec{u}_0)(x)\big]\vec{v}^h(x)dx - \int_\Omega\big[\vec{0} \equiv G(\vec{u}_0)(x)\big]E_c\vec{v}^h(x)dx\big| \\ \leq\ & \int_{\Omega_c^h\setminus\Omega}\big|G(E_c\vec{u}_0)(x)\vec{v}^h(x)\big|dx, \text{ with } E_c\vec{v}^h \text{ well defined by } \vec{v}^h \in L^2(\Omega_c^h), \end{aligned} \tag{8.4}$$

since $(\int_{\Omega_c^h} - \int_\Omega)|_{\Omega_c^h\cap\Omega} = \vec{0}$. Now (8.1) and $\vec{u}_0 \in H^\ell(\Omega, \mathbf{R}^q)$ allow integration $\perp \partial(\Omega \cup \Omega_c^h$:

$$\vec{f}(x) := G(E_c\vec{u}_0)(x) \text{ satisfies } \vec{f}(x) \equiv \vec{0} \text{ in } \overline{\Omega}, \text{hence } \operatorname{dist}(\Omega_c^h, \Omega) \leq Ch^p, \text{ see (6.25)} \\ \Rightarrow \|\vec{f}\|_{L^2(\Omega_c^h\setminus\Omega)} = \|G(E_c\vec{u}_0)\|_{L^2(\Omega_c^h\setminus\Omega)} \leq C'h^p\|E_c\vec{u}_0\|_{H^3(\Omega_c^h\setminus\Omega)} \leq C'h^p\|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^q)}. \tag{8.5}$$

We combine (6.20), (8.3), (8.4), (8.5), (6.25), with the Cauchy-Schwarz inequality

$$\|Q_c^hG(P^h\vec{u}_0)\|_{\mathcal{V}^h} \leq CLh^{\ell-2m}\|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^q)} + \|Q_c^hG(E_c\vec{u}_0)\|_{\mathcal{V}^h} \leq CLh^{\ell-2m}\|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^q)} \\ +C'h^p\|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^q)} \leq CLh^{\min\{\ell-2m,p\}}\|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^q)}.$$

hence, the first term. Similarly to (8.5), now replacing $\vec{f}(x) \equiv \vec{0}$ in $\overline{\Omega}$ by $B_D\vec{u}_0 \equiv \vec{0}$ with $u_0 \in \mathcal{U}$ and integration along rays $\perp \partial(\Omega \cup \Omega_c^h)$, and testing as in (7.13), we obtain for the second term

$$\|B_D^h E_c \vec{u}_0\|_{\mathcal{V}_{c,b}} \leq C'' h^p \|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^q)}.$$

If inhomogeneous boundary conditions are considered in (7.3), we employ the extension of $B_D\vec{u} = \vec{\phi} \in \mathcal{V}_D$ to $\vec{\phi}^e \in H^{2m}(\Omega,\mathbf{R}^q) \times \cdots H^{m+1}(\Omega,\mathbf{R}^q)$, or assume right away $\vec{\phi} = \vec{\phi}^e|_{\partial\Omega}$, we still get a slightly changed $G(\vec{u}_0) = \vec{0}$, so (8.3), (8.4) remain correct. Boundary conditions for the exact and the discrete problem are $B_D\vec{u}_0 - \vec{\phi} \equiv \vec{0}$ and $B_D^h\vec{u}_0^h - P^h\vec{\phi} \equiv \vec{0}$. We replace the exact boundary conditions by $B_D\vec{u}_0 - E_c(P^h\vec{\phi}) \equiv \vec{0}$ and use

$$\|\vec{\phi} - E_c(P^h\vec{\phi})\|_{\partial\Omega} \leq C'' h^p \|\vec{\phi}\|_{H^1(\Omega,\mathbf{R}^q)} \leq C''' h^p \|\vec{u}_0\|_{H^{2m+1}(\Omega,\mathbf{R}^q)}.$$

Then a combination with the stability results for linearized elliptic operators w.r.t. the right hand side $(\vec{f},\vec{\phi}) \in \mathcal{V} \times \mathcal{V}_D$, similarly to e.g. Hackbusch, [33] for $(\vec{f},\vec{\phi}) \in \mathcal{W}' \times \mathcal{W}_D$, yields again (8.2). □

9 Convergence via Stability for Linearized Operators

A standard result strongly simplifies the proof of stability, see, e.g., Stetter, [48], Theorem 1.2.5., similarly Stummel, [49, 50], and Keller, [39]: We need a Corollary of Stetter's Theorem 1.2.5. It replaces the condition of a continuous and differentiable discretization, F^h, in $B_r(\vec{u}^h)$ by the continuous differentiability in $B_r(\vec{u}^h)$. This slightly enforced condition is usually satisfied by the equations considered here. It is even simplified in our case, since $F \in C^1(B_r(\vec{u}))$ implies $F^h \in C^1(B_{r-\delta}(P^h\vec{u}))$ with a small $\delta > 0$ for $h_0 > h \in H$.

Theorem 9.1 *Let the nonlinear F be continuously differentiable in $B_r(\vec{u})$, and let the sequence of linearized $(F^h)'$ be stable at $P^h\vec{u}$, satisfying*

$$\|\big((F^h)'(P^h\vec{u})\big)^{-1}\|_{\mathcal{U}^h \leftarrow \mathcal{V}_{\Pi}^{h\prime}} \leq S \ \textit{uniformly} \ \ \forall\ h \in H, h_0 > h. \tag{9.1}$$

Then the nonlinear F^h is stable at $P^h\vec{u}$ as well with stability bound, $2S$, and threshold $r_0 = r/(2S)$.

According to Section 2, **3.**, Theorems 7.1, 8.1 and 9.1 and Remark 7.1 show that we only need proving the stability for the linearized operator for a sequence $\vec{u}^h = P^h\vec{u}$, here $\vec{u} = \vec{u}_0$, yielding convergence. So we have to elaborate the programme in Section 2 in **3.** For our $F(\vec{u}) = (G(\vec{u}), B_D\vec{u})$ with $F'(\vec{u}_1)\vec{u} = (G'(\vec{u}_1)\vec{u}, B_D\vec{u})$ the linearization of G is essential. Thus a major part of the following discussion is concerned with $G'(\vec{u}_1)$. We get the claimed stability $\|\big((F^h)'(\vec{u}_1^h)|_{\mathcal{U}^h \to \mathcal{V}_{\Pi}^{h\prime}}\big)^{-1}\|_{\mathcal{V}_{\Pi}^{h\prime} \leftarrow \mathcal{U}^h} \leq C$ by the end of this Section.

9.1 The linearized operator

We assume a continuously differentiable G or G^w. Here we sometimes integrate the boundary conditions into the spaces $\mathcal{U}_0, \mathcal{W}_0$ and $\mathcal{U}_0^h, \mathcal{W}_0^h$. In the literature the same notation $G'(\vec{u}_0) : \mathcal{U} \to \mathcal{V}$ and $G'(\vec{u}_0) : \mathcal{W} \to \mathcal{W}'$ is used for the strong and weak forms of the linearized operators, $G'_s(\vec{u}_0) = G'(\vec{u}_0)$ and $G'(\vec{u}_0)$, resp. For $\mathcal{U} \subset \mathcal{W} \subset \mathcal{V}$ see (4.4) and (4.5). For a smooth enough situation, the strong form is just the restriction of the weak form, so $G'(\vec{u}_0)|_{\mathcal{U}} = G'_s(\vec{u}_0) : \mathcal{U} \to \mathcal{V}$ under the condition (9.4). The proofs in this Section fundamentally depend upon a systematic interplay between these strong and weak forms of the linearized operators. To avoid misunderstandings, it is necessary to distinguish at least the strong and weak bilinear forms as $a_s(\cdot,\cdot)$ and $a(\cdot,\cdot)$. In diagram (9.38) we use the notations $A_{\mathcal{W}}, A_{\mathcal{U}}$ and $A^h_{\mathcal{W}}, A^h_{\mathcal{U}}$ for weak and strong linear operators. Correspondingly, we formulate two different FEMs for the linearized problem. To its strong form, $a_s(\cdot,\cdot)$, we apply the modified FEM (9.19), to its weak form, $a(\cdot,\cdot)$, the standard FEM, (9.20). We will see that for the linear case both bilinear forms coincide, so $a_s(\vec{u}^h,\vec{v}^h) = a(\vec{u}^h,\vec{v}^h)\ \forall \vec{u}^h \in \mathcal{U}^h = \mathcal{W}^h, \vec{v}^h \in \mathcal{V}^h$, see (9.6) and Proposition 9.1.

For the convenience of the reader, we reformulate equations of order 2 with $m = q = 1$ and systems of order $2m$ with $m, q > 1$, their linearization and their FE versions. Equations of order $2m$ with $m > q = 1$ and systems of order 2 with $1 = m < q$ are then simply obtained by replacing the $\vec{u}$ by u, a.s.o., and vice versa. Basic are the function spaces $\mathcal{U}, \mathcal{W}_0, \ldots$ in (4.5)(7.1), and the FE spaces $\mathcal{U}^h, \mathcal{W}_0^h, \ldots$ in (7.6), e.g., for $1 = m < q$.

Second order equations with $1 = m = q$: We obtain the solutions u_0 and u_0^h from

$$F(u_0) = (G(u_0)(\cdot) = G^w(\cdot, u_0(\cdot), Du_0(\cdot), D^2u_0(\cdot)), u_0|_{\partial\Omega}) = 0 \in \mathcal{V} \times \mathcal{V}_D, \quad \text{and} \tag{9.2}$$
$$F^h(u_0^h) := (G^h(u_0^h), u_0^h|_{\partial\Omega_c^h}) = 0 \Leftrightarrow (G(u_0^h), v^h)_{L^2(\Omega_c^h)} + (u_0^h|_{\partial\Omega_c^h}, v_b^h)_{L^2(\partial\Omega_c^h)} = 0\ \forall v_\Pi^h \in \mathcal{V}_\Pi^h,$$

For the proof of stability we need the linearization $(F^h)'(u_0^h)$. Since the boundary term is linear, it suffices to determine $G'(E_c u_0)$ applied to u^h. With $w_0^h := w^h_{E_c u_0}$ and $a_{ij}^h = a_{ij}^h(w_0^h)$, we reinterpret $G'(E_c u_0)$, see (5.4), and get for the FE bilinear forms

$$G'(E_c u_0)u^h = \frac{\partial G^w}{\partial z}(w_0^h)u^h + \ldots + \sum_{i,j=1}^n \frac{\partial G^w}{\partial r_{ij}}(w_0^h)\partial^i\partial^j u^h = \sum_{i,j=0}^n (-1)_{j>0}\partial^j(a_{ij}^h \partial^i\ u^h), \tag{9.3}$$
$$a_s(u^h, v^h) := (G'(E_c u_0)u^h, v^h)_{\mathcal{V}_c}, a(u^h, v^h) := \int_{\Omega_c^h} \sum_{i,j=0}^n a_{ij}^h \partial^i u^h \partial^j v^h dx \ =: \langle G'(E_c u_0)u^h, v^h\rangle,$$

where we admit $\forall u^h \in \mathcal{U}^h, v^h \in \mathcal{V}^h$ and define the weak $G'(E_c u_0)$ by the last equation with $\langle G'(E_c u_0)u^h, v^h\rangle = \langle G'(E_c u_0)u^h, v^h\rangle_{\mathcal{W}_c' \times \mathcal{W}_c}$.

Systems of order $2m$ *with* $1 \le m, 1 < q$: Now we determine the solutions $\vec{u}_0$ and $\vec{u}_0^h$ by

$$F(\vec{u}_0) = (G(\vec{u}_0)(\cdot) = G^w(\cdot, \vec{u}_0(\cdot), \ldots, D^{2m}\vec{u}_0(\cdot)), B_D\vec{u}_0) = 0 \in \mathcal{V} \times \mathcal{V}_D, \quad \text{and}$$
$$F^h(\vec{u}_0^h) := (G^h(\vec{u}_0^h), B_D^h\vec{u}_0^h) = 0 \Leftrightarrow (G(\vec{u}_0^h), \vec{v}^h)_{L^2(\Omega_c^h, \mathbf{R}^q)} + (B_D^h\vec{u}_0^h, \vec{v}_b^h)_{\mathcal{V}_b^h} = 0\ \forall \vec{v}_\Pi^h \in \mathcal{V}_\Pi^h,$$

For $(F^h)'(\vec{u}_0^h)\vec{u}^h = (G'(E_c\vec{u}_0)\vec{u}^h, B_D^h\vec{u}^h)$ we consider $G'(E_c\vec{u}_0)$. With $\vec{w}_0^h := \vec{w}^h_{\vec{u}_0^h}$ and $A^h_{\alpha\beta} =$

$A^h_{\alpha\beta}(\vec{w}^h_0)$, we reinterpret $G'(E_c\vec{u}_0)$, see (5.4), and get the FE bilinear forms

$$G'(E_c\vec{u}_0)\vec{u}^h = \sum_{|\gamma|\le 2m} \partial_{\vartheta^\gamma}\big(G^w(\cdot,\vec{\vartheta}^\beta,|\beta|\le 2m)\big)(\vec{w}^h_0)\partial^\gamma\vec{u}^h = \sum_{|\alpha|,|\beta|\le m}(-1)^{|\alpha|}\partial^\alpha(A^h_{\alpha\beta}\partial^\beta\vec{u}^h)$$

$$a_s(\vec{u}^h,\vec{v}^h) = (G'(E_c\vec{u}_0)\vec{u}^h,\vec{v}^h)_{\mathcal{V}_c} \text{ and } a(\vec{u}^h,\vec{v}^h) := \int_{\Omega^h_c}\sum_{|\alpha|,|\beta|\le m}(A^h_{\alpha\beta}\partial^\beta\vec{u}^h,\partial^\alpha\vec{v})_q dx = \quad (9.4)$$

$$\langle G'(E_c\vec{u}_0)\vec{u}^h,\vec{v}^h\rangle_{\mathcal{W}'_c\times\mathcal{W}_c}\ \forall \vec{u}^h\in\mathcal{U}^h,\vec{v}^h\in\mathcal{V}^h \text{ with } A_{\alpha\beta}\in W^{|\alpha|,\infty} \text{ and } A_{\alpha\beta}\in L^\infty(\Omega^h_c,\mathbf{R}^{q\times q}),$$

for the strong and the weak bilinear forms, see (4.1), (4.6), (4.7). The imposed conditions $A_{\alpha\beta}\in W^{|\alpha|,\infty}$ and $A_{\alpha\beta}\in L^\infty(\Omega^h_c,\mathbf{R}^{q\times q})$ imply well defined continuous bilinear forms $a_s(\cdot,\cdot)$ and $a(\cdot,\cdot)$, resp. As for G, we use here the same notation for the discrete $a_s(\vec{u}^h,\vec{v}^h), a(\vec{u}^h,\vec{v}^h)$ and the original $a_s(\vec{u},\vec{v}), a(\vec{u},\vec{v})$, and only sometimes indicate the extensions of, e.g., the coefficients $E_cA_{\alpha\beta}$ instead of the $A_{\alpha\beta}$ used in (9.4).

The original $a(u,v)$, inducing $A = G'(u_0)$, is the sum of its principal part, $a_p(u,v)$, inducing B, and its complement $c(u,v)$, inducing C, a compact perturbation of $A = G'(u_0) = B + C$, with $A,B,C\in\mathcal{L}(\mathcal{W},\mathcal{W}')$,

$$a(u,v) = a_p(u,v) + c(u,v) = \langle G'(u_0)u,v\rangle_{\mathcal{W}'\times\mathcal{W}} = \langle (B+C)u,v\rangle_{\mathcal{W}'\times\mathcal{W}}. \quad (9.5)$$

B is boundedly invertible on $\mathcal{W}_0$ by Theorem 4.2. Thus $G'(u_0)$ satisfies the *Fredholm alternative* on $\mathcal{W}_0$. This compact perturbation $C\in\mathcal{L}(\mathcal{W},\mathcal{W}')$ will play a dominant role for the following stability results.

In addition to the above approximating spaces, see (3.1), (6.22), we need for $\mathcal{W}_0$

$$\mathcal{U}^h = \mathcal{W}^h \supset \mathcal{V}^h \text{ and } \mathcal{U}^h_0 = \mathcal{W}^h_0 := \mathcal{V}^h = \mathcal{U}^h\cap H^m_0(\Omega^h_c,\mathbf{R}^q), \quad (9.6)$$

again with the corresponding nonequivalent norms $\|\cdot\|_{\mathcal{U}^h}, \|\cdot\|_{\mathcal{W}^h}, \|\cdot\|_{\mathcal{V}^h}$.

From now on, we will mainly use the strong and weak bilinear forms in (9.4) for the linearized problem. We use the same symbol $\vec{u}^h$ for the earlier u^h and $\vec{u}^h$. The different ellipticity conditions guaranteeing the $\mathcal{W}_0-$coercivity of the principle part of $G'((\vec{u}^0))$ are discussed in Theorem 4.2. We want to contrast the two different FEMs for the linear problem. We start with the *strong* $a_s(\cdot,\cdot)$ and then turn to the *weak bilinear form* $a(\cdot,\cdot)$ and its standard FEM. In both cases we have to determine $\vec{u}^h_1\in\mathcal{U}^h_0$ and $\vec{u}^h_1\in\mathcal{W}^h_0$ s.t.

$$\vec{u}^h_1\in\mathcal{U}^h_0 : a_s(\vec{u}^h_1,\vec{v}^h) = (G'(E_c\vec{u}_0)\vec{u}^h_1,\vec{v}^h)_{\mathcal{V}_c} = (E_c\vec{f},\vec{v}^h)_{\mathcal{V}_c} \quad \forall\ \vec{v}^h\in\mathcal{V}^h, \text{ with } \vec{f}\in\mathcal{V}, (9.7)$$

$$\vec{u}^h_1\in\mathcal{W}^h_0 : a(\vec{u}^h_1,\vec{v}^h) = \langle G'(E_c\vec{u}_0)\vec{u}^h_1,\vec{v}^h\rangle_{\mathcal{W}'_c\times\mathcal{W}_c} = \langle E_c\vec{f},\vec{v}^h\rangle_{\mathcal{W}'_c\times\mathcal{W}_c}\ \forall\vec{v}^h\in\mathcal{W}^h_0,\ \vec{f}\in\mathcal{W}'. (9.8)$$

The equations (9.7), (9.8) can and will be similarly re-interpreted by using the above projectors, $Q^h_\Pi,\ldots$, and defining new ones $Q^{h'}_\Pi,\ldots$.

9.2 Weak and strong linear operators coincide on $\mathcal{U}^h$

The techniques of proofs in the remainder of this Section is strongly related to the standard approach in spectral methods for collocation, see Canuto, Hussaini, Quarteroni, Zang, [18], and to regularity for difference equations, e.g., in Hackbusch, [33], Theorem 9.2.26.

Proposition 9.1 *Choose* $\mathcal{U}^h, \mathcal{V}^h$ *as in (7.6) and* $A_{\alpha,\beta} \in W^{|\alpha|,\infty}(\Omega, \mathbf{R}^{q\times q})$. *Then*

$$a_s(\vec{u}^h, \vec{v}^h) = (G'(E_c\vec{u}_0)\vec{u}^h, \vec{v}^h)_{\mathcal{V}_c} = a(\vec{u}^h, \vec{v}^h) = \langle G'(E_c\vec{u}_0)\vec{u}^h, \vec{v}^h\rangle_{\mathcal{W}_c'\times\mathcal{W}_c} \forall \vec{u}^h \in \mathcal{U}^h, \vec{v}^h \in \mathcal{V}^h \quad (9.9)$$

Proof: We apply partial integration for every $T \in \mathcal{T}_c^h$ with $\overline{\Omega} = \cup_{T\in\mathcal{T}_c^h}\overline{T}$. With $\nu = \nu_T$, the outer unit normal vector of T, her along $\partial\Omega_c^h$, and $\vec{v}^h, \vec{w}^h \in \mathcal{W}^h$ we obtain

$$\int_{\Omega_c^h} (\vec{w}^h, \partial\vec{v}^h/\partial x_i)_q dx + \int_{\Omega_c^h} (\vec{v}^h, \partial\vec{w}^h/\partial x_i)_q dx = \int_{\partial\Omega_c^h} (\vec{v}^h, \vec{w}^h)_q \nu_i ds. \quad (9.10)$$

This can be extended to $\vec{v}^h, \vec{w}^h$ as well and applied to

$$a(\vec{u}^h, \vec{v}^h) = \int_\Omega \sum_{|\alpha|,|\beta|\le m} (A_{\alpha\beta}\partial^\beta\vec{u}^h, \partial^\alpha\vec{v}^h)_q dx$$

in (9.4) to decrease the order of derivative in the $\partial^\alpha\vec{v}^h$ and to increase the order in $\partial^\beta\vec{u}^h$ by one in each step. To avoid too technical details, we demonstrate that for a second order system and for $\mathcal{W}$:

Let $B_a\vec{u}$ be the boundary operator, induced by $A = G'(\vec{u}_0)$, see e.g. [33],

$$B_a\ \vec{u} = \sum_{i,j=1}^n \nu_j A_{ij}\partial^i\ \vec{u} + \sum_{j=1}^n \nu_j A_{0j}\ \vec{u},\ \text{e.g., } B_a\ \vec{u} = \partial\vec{u}/\partial\nu \text{ for } A_s\vec{u} = -\Delta\vec{u}. \quad (9.11)$$

All ∂T have a *positive orientation* w.r.t. T. Hence in (9.13) below every interior edge $e \in \overline{T}$ will be obtained twice in opposite directions. Edges $e \subset \partial\Omega$ will appear once. Let $\nu = \nu_T$ be one of the normal vectors for an interior edge $e \subset \overline{T_r}$. It is oppositely oriented for neighboring T_r , $T_l \in \mathcal{T}_c^h$ with $e \subset \overline{T_r} \cap \overline{T_l}$ and ν the outer normal [4] for $e \subset \partial\Omega$. To consider the transition from a triangle T_l to its neighbor T_r, we introduce the restriction of $\vec{v}$ to $\overline{T_l}, \overline{T_r}, \partial\Omega$ and the standard notation, see [57],

$$\vec{v}_l = \vec{v}|_{\overline{T_l}},\ \vec{v}_r = \vec{v}|_{\overline{T_r}},\ [\vec{v}] := \vec{v}_l|_e - \vec{v}_r|_e \text{ and } \{\vec{v}\} := (\vec{v}_l|_e + \vec{v}_r|_e)/2, \quad (9.12)$$

for the corresponding jumps and arithmetic means of $\vec{v}$ across an interior e, and

$$[\vec{v}] := \{\vec{v}\} := \vec{v}|_{\partial\Omega} \text{ along } e \in \partial\Omega, \quad \text{and} \quad \vec{v} \ \text{ arbitrary in } \ \mathbf{R}^2\backslash\Omega.$$

Then (9.10) or the n- dimensional Green's formula or the Gauss integral theorem yields

$$\begin{aligned}
a(\vec{u}^h, \vec{v}^h) &= \ < G'(E_c\vec{u}_0)\vec{u}^h, \vec{v}^h >_{\mathcal{W}'\times\mathcal{W}} \ = \sum_{T\in\mathcal{T}^h} \Big(\int_T \sum_{i,j=0}^n a_{ij}(\partial^i\ \vec{u}^h, \partial^j\ \vec{v}^h)_q dx \Big) \\
&= \sum_{T\in\mathcal{T}^h} \Big(\int_T \sum_{i,j=0}^n (-1)_{j>0} (\partial^j(a_{ij}\partial^i\ \vec{u}^h), \vec{v}^h)_q dx \Big) + \int_{\partial\Omega_c^h} (\vec{v}_l^h, B_a\vec{u}^h)_q\, ds \qquad (9.13) \\
&+ \sum_{e\in\mathcal{T}^h\backslash\partial\Omega_c^h} \int_e \big(\{\vec{v}^h\}, \big[B_a\vec{u}^h\big]\,\big)_q + \big([\vec{v}^h], \{B_a\vec{u}^h\}\,\big)_q ds = \int_{\Omega_c^h} ((G'(E_c\vec{u}_0)_s\ \vec{u}^h), \vec{v}^h)_q dx \\
&= a_s(\vec{u}^h, \vec{v}^h) \text{ since } \vec{u}^h \in \mathcal{U}^h, \vec{v}^h \in \mathcal{V}^h, \mathcal{U}^h \subset C_0^1(\Omega), \text{ hence,} \\
a(\vec{u}^h, \vec{v}^h) &= a_s(\vec{u}^h, \vec{v}^h)\ \forall\ \vec{u}^h \in \mathcal{U}^h, \vec{v}^h \in \mathcal{V}^h. \qquad (9.14)
\end{aligned}$$

[4] we will use the notation $e \in \partial\Omega$, although mostly $e \subset \partial\Omega$, similarly $e \in \mathcal{T}^h \setminus \partial\Omega$

Obviously this would fail if only $\mathcal{U}^h, \mathcal{V}^h \subset C_0^0(\Omega)$ instead of our $\mathcal{U}^h, \mathcal{V}^h \subset C_0^{2m-1}(\Omega)$, here $m = 1$. □

9.3 Stability of the weak linear operator

The last Subsections play the role of preparing this stability proof. We essentially have confined the discussion there to the differential equation and indicated the boundary conditions either by the index $_0$ or using $\vec{v}^h \in \mathcal{V}^h$. Instead of the previous strong and weak linear operators, $A_s : \mathcal{U} \to \mathcal{V}$ and $A : \mathcal{W} \to \mathcal{W}'$, see (7.3), (7.8), (7.9), we now need their two component versions

$$\begin{aligned} A_{\mathcal{U}} &:= (A_s, B_D) : \mathcal{U} \to \mathcal{V} \times \mathcal{V}_D \subset \mathcal{V}_\Pi = \mathcal{V} \times \mathcal{V}_b,\ A_{\mathcal{U}} \in \mathcal{L}(\mathcal{U}, \mathcal{V} \times \mathcal{V}_D) \text{ and} \qquad (9.15)\\ A_{\mathcal{W}} &:= (A, B_D) : \mathcal{W} \to \mathcal{W}' \times \mathcal{W}_D \subset \mathcal{W}'_\Pi = \mathcal{W}' \times \mathcal{V}_b,\ A_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}, \mathcal{W}' \times \mathcal{W}_D), \end{aligned}$$

compare (7.3). With the dense (compact) embedding $\mathcal{W}_D \hookrightarrow \mathcal{V}_b = (L^2(\partial\Omega))^m$ we test the image space $\mathcal{W}' \times \mathcal{W}_D \subset \mathcal{W}'_\Pi$ by the following $\mathcal{W}_\Pi$. Except $\mathcal{V}_b = (L^2(\partial\Omega))^m$ the next spaces are different from their duals. We emphasize that the splitting $S^h = S_0^h + S_b^h$ in Theorem 6.3 plays a dominant role in the proofs. We choose, see (6.20), (6.21), (7.10), (7.1), and introduce the weak spaces, again $\mathcal{W}_\Pi^h = \mathcal{V}_\Pi^h$ with nonequivalent norms,

$$\begin{aligned} &\mathcal{W} \times \mathcal{W}_D \subset \mathcal{W}_\Pi := \mathcal{W} \times \mathcal{V}_b \neq \mathcal{W}'_\Pi = \mathcal{W}' \times \mathcal{V}_b, \text{ with } \mathcal{W} \neq \mathcal{W}', \mathcal{V}_b \supset \mathcal{W}_D\\ &\text{and on } \Omega_c^h \text{ the } \mathcal{W}_{c,\Pi} = \mathcal{W}_c \times \mathcal{V}_{c,b} \neq \mathcal{W}'_{c,\Pi} \text{ with } \langle(\vec{u}', \vec{u}_b), (\vec{v}, \vec{v}_b)\rangle_{\mathcal{W}'_\Pi \times \mathcal{W}_\Pi} :=\\ &\quad = \langle \vec{u}', \vec{v}\rangle_{\mathcal{W}' \times \mathcal{W}} + (\vec{u}_b, \vec{v}_b)_{\mathcal{V}_b}, \text{ and } \langle(\vec{u}', \vec{u}_b), (\vec{v}, \vec{v}_b)\rangle_{\mathcal{W}'_{c,\Pi} \times \mathcal{W}_{c,\Pi}} \text{ versus} \qquad (9.16)\\ &\mathcal{W}_\Pi^h = \mathcal{W}^h \times \mathcal{V}_b^h := S^h \times (S_b^h|_{\partial\Omega_c^h})^m = \mathcal{V}_\Pi^h,\ \mathcal{W}_\Pi^{h'} := \mathcal{W}^{h'} \times \mathcal{V}_b^h := S^{h'} \times (S_b^h|_{\partial\Omega_c^h})^m =\\ &\quad = \mathcal{V}_\Pi^{h'}, \text{ and } \langle(\vec{u}^{h'}, \vec{u}_b^h), (\vec{v}^h, \vec{v}_b^h)\rangle_{\mathcal{W}_\Pi^{h'} \times \mathcal{W}_\Pi^h} := \langle \vec{u}^{h'}, \vec{v}^h\rangle_{\mathcal{W}^{h'} \times \mathcal{W}^h} + (\vec{u}_b^h, \vec{v}_b^h)_{\mathcal{V}_b^h}. \end{aligned}$$

The exact and discrete linear boundary conditions remain nearly unchanged, except replacing $\mathcal{V}_D$ by $\mathcal{W}_D$, so they can be treated as in (7.9), (7.13).

To study the weak differential operator, we introduce a modified Ritz operator. Again, we use the above extension operator E_c, directly defined for special $\vec{f} \in \mathcal{W}'$, e.g., for a right hand side $\vec{f} \in \mathcal{V}$ or for $\vec{f} = A\vec{u} \in \mathcal{W}', A = G'(\vec{u}_0)$ in (9.4), where each term $A_{\alpha,\beta}\partial^\beta \vec{u} \in \mathcal{V}$. Otherwise we use extensions as in Remark 6.2. We define the projectors, $Q^{h'}, Q_c^{h'}$, a.s.o., indicating the weak form by the exponent $^{h'}$, in contrast to the strong projectors Q^h a.s.o. with the exponent ^{h} in (7.15), compare (6.10), (6.11), (6.20), (6.21):

$$\begin{aligned} &Q^{h'} \in \mathcal{L}(\mathcal{W}', \mathcal{W}^{h'}) \text{ for } \vec{f} \in \mathcal{W}' \text{ by } \langle Q^{h'}\vec{f}, \vec{v}^h\rangle_{\mathcal{W}^{h'} \times \mathcal{W}^h} - \langle (E_c\vec{f}), \vec{v}^h\rangle_{\mathcal{W}'_c \times \mathcal{W}_c} = 0 \text{ and}\\ &Q_c^{h'} \in \mathcal{L}(\mathcal{W}'_c, \mathcal{W}^{h'}) : \vec{f}_c \in \mathcal{W}'_c :\ \langle Q_c^{h'}\vec{f}_c, \vec{v}^h\rangle_{\mathcal{W}^{h'} \times \mathcal{W}^h} - \langle \vec{f}_c, \vec{v}^h\rangle_{\mathcal{W}'_c \times \mathcal{W}_c} = 0\ \ \forall \vec{v}^h \in \mathcal{W}^h,\\ &\lim_{h\to 0} \|Q^{h'}\vec{f}\|_{\mathcal{W}^{h'}} = \|\vec{f}\|_{\mathcal{W}'}, \text{and} \lim_{h\to 0} \|Q_c^{h'}\vec{f}_c\|_{\mathcal{W}^{h'}} = \lim_{h\to 0} \|\vec{f}_c\|_{\mathcal{W}'_c} = \lim_{h\to 0} \|E_c\vec{f}_c\|_{\mathcal{W}'}. \qquad (9.17) \end{aligned}$$

This allows to formulate the product projectors and we obtain

$$\begin{aligned} &Q_\Pi^{h'} := (Q^{h'}, Q_b^h) \in \mathcal{L}\big(\mathcal{W}'_\Pi, \mathcal{W}_\Pi^{h'}\big),\ Q_{c,\Pi}^{h'} := (Q_c^{h'}, Q_{c,b}^h) \in \mathcal{L}\big(\mathcal{W}'_{c,\Pi}, \mathcal{W}_\Pi^{h'}\big), \text{ with}\\ &\lim_{h\to 0} \|Q_\Pi^{h'}(\vec{f}, B_D\vec{u}_1)\|_{\mathcal{W}_\Pi^{h'}} = \|(\vec{f}, B_D\vec{u}_1)\|_{\mathcal{W}'_\Pi}, \text{ and } \lim_{h\to 0} \|Q_{c,\Pi}^{h'}(\vec{f}_c, B_D(E_c\vec{u}))\|_{\mathcal{W}_\Pi^{h'}}\\ &\qquad = \lim_{h\to 0} \|(\vec{f}_c, B_D(E_c\vec{u}))\|_{\mathcal{W}'_{c,\Pi}} = \lim_{h\to 0} \|(E_c\vec{f}_c, B_D(E_c\vec{u}))\|_{\mathcal{W}'_\Pi} \qquad (9.18) \end{aligned}$$

Similarly to the above $F^h = Q^h_{c,\Pi} F|_{\mathcal{U}^h}$ we reformulate (9.7) and (9.8) for the strong and weak linear operators $A_{\mathcal{U}}$ and $A_{\mathcal{W}}$. We determine the discrete solution $\vec{u}^h_1$ as

$$A^h_{\mathcal{U}} = Q^h_{c,\Pi} A_{\mathcal{U}}|_{\mathcal{U}^h} : \vec{u}^h_1 \in \mathcal{U}^h : A^h_{\mathcal{U}} \vec{u}^h_1 = Q^h_{c,\Pi}(A_s \vec{u}^h_1, B^h_D \vec{u}^h_1) = Q^h_\Pi(\vec{f}, \vec{\phi}), \quad (9.19)$$
$$A^h_{\mathcal{W}} = Q^{h'}_{c,\Pi} A_{\mathcal{W}}|_{\mathcal{W}^h} : \vec{u}^h_1 \in \mathcal{W}^h : A^h_{\mathcal{W}} \vec{u}^h_1 = Q^{h'}_{c,\Pi}(A \vec{u}^h_1, B^h_D \vec{u}^h_1) = Q^{h'}_\Pi(\vec{f}, \vec{\phi}) \Longleftrightarrow \quad (9.20)$$
$$\langle Q^{h'}_{c,\Pi} A_{\mathcal{W}} \vec{u}^h_1 - Q^{h'}_\Pi(\vec{f}, \vec{\phi}), \vec{v}^h_\Pi \rangle_{\mathcal{W}^{h'}_\Pi \times \mathcal{W}^h_\Pi} = 0 \forall \vec{v}^h_\Pi \in \mathcal{W}^h_\Pi, (\vec{f}, \vec{\phi}) \in \mathcal{W}'_\Pi, A_{\alpha,\beta} \in C_L(\Omega \cup \Omega^h_c),$$

see (9.15). We summarize the approximating spaces $\mathcal{W}^h \times \mathcal{V}^h_b$ and uniformly bounded $P^h, Q^{h'}_{c,\Pi}$ in the new *diagram*, different from (7.20)

$$\begin{array}{ccccc} \mathcal{W} & \xrightarrow{A_{\mathcal{W}}} & \mathcal{W}'_\Pi = \mathcal{W}' \times \mathcal{W}_D & \overset{\textit{tested by}}{\longleftrightarrow} & \mathcal{W}_\Pi = \mathcal{W} \times \mathcal{V}_b \\ P^h \Big\downarrow & \Phi^h_{\mathcal{W}} \Big\downarrow & Q^{h'}_\Pi \Big\downarrow & & \\ \mathcal{W}^h & \xrightarrow{A^h_{\mathcal{W}}} & \mathcal{W}^{h'}_\Pi = \mathcal{W}^{h'} \times \mathcal{V}^h_b & \overset{\textit{testedby}}{\longleftrightarrow} & \mathcal{W}^h_\Pi = \mathcal{W}^h \times \mathcal{V}^h_b \end{array} \quad (9.21)$$

Before we prove the stability result for $(A^h_{\mathcal{W}})'$ in Theorem 9.3 we consider the principle part. We replace A by B, see (9.5).

The principle part $a_p(\vec{u}, \vec{v})$ of $a(\vec{u}, \vec{v})$ induced by $G'(\vec{u}^0)$, is $\mathcal{W}_0$- coercive under the conditions of Theorem 4.2. So $a(\vec{u}, v)$ is the sum of $a_p(\vec{u}, \vec{v})$ and its complement $c(\vec{u}, \vec{v})$, inducing $B \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$, and the compact perturbation $C \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$, of $A = G'(\vec{u}_0) = B + C$,

$$a(\vec{u}, \vec{v}) = a_p(\vec{u}, \vec{v}) + c(\vec{u}, \vec{v}) = < G'(\vec{u}_0)\vec{u}, \vec{v} >_{\mathcal{W}' \times \mathcal{W}} = < (B + C)\vec{u}, \vec{v} >_{\mathcal{W}' \times \mathcal{W}} . \quad (9.22)$$

B is boundedly invertible on $\mathcal{W}_0$, since $a_p(\vec{u}, \vec{v})$ is $\mathcal{W}_0$-*coercive*, hence $(a_p(\vec{u}, \vec{u}))^{1/2}$ and $\|\vec{u}\|_{\mathcal{W}_0}$ are equivalent norms on $\mathcal{W}_0$. Thus $G'(\vec{u}_0)$ satisfies the *Fredholm alternative.* This compact perturbation $C \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$ will play a dominant role in the following stability result. Now, we combine the A, B, C in (9.22) with the standard FEM in (9.20). Here and in this Section we use, for the two component operators, the notations $A_{\mathcal{W}} = (A, B_D), B_{\mathcal{W}} = (B, B_D), C_{\mathcal{W}} = (C, B_D)$. Under the conditions of Theorem 9.2, the $A_{\mathcal{W}}, B_{\mathcal{W}}, C_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}, \mathcal{W}' \times \mathcal{W}_D)$, see (9.22) and their discretizations, see (9.20), the $A^h_{\mathcal{W}}, B^h_{\mathcal{W}}, C^h_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}^h, \mathcal{W}^{h'} \times \mathcal{V}^h_b)$ in (9.8), are bounded and equi-bounded.

Theorem 9.2 Stability and convergence for the principle part : *Assume the conditions of Theorem 4.2, yielding the $\mathcal{W}_0$−coercivity of the main part, the approximation spaces in (9.6) with the norm limits in (9.17) and (9.18), and $B_{\mathcal{W}}$ and $B^h_{\mathcal{W}}$ as in (9.15), (9.20). Then $B_{\mathcal{W}}$ is boundedly invertible, $B^h_{\mathcal{W}}$ is stable and the unique solutions $\vec{u}^h_1$, of $B^h_{\mathcal{W}} \vec{u}^h_1 = Q^{h'}_\Pi(\vec{f}, \vec{\phi}) \in \mathcal{W}^{h'}_{c,\Pi}$ converge to $\vec{u}_1$, the unique solution of $B_{\mathcal{W}} \vec{u}_1 = (\vec{f}, \vec{\phi}) \in \mathcal{W}'_\Pi$, s.t. $\|\vec{u}^h_1 - E_c \vec{u}_1\|_{\mathcal{W}^h} \to 0$ for $h \to 0$.*

Proof: (9.4) shows that $a^h_p(\vec{u}^h, \vec{v}^h)$ is bounded on $\mathcal{W}^h$. The $\mathcal{W}_0$−coercivity conditions of Theorem 4.2, guarantee, that

$$a^h_p(\vec{u}^h, \vec{u}^h) \geq \lambda \int_{\Omega^h_c} \sum_{i=1}^n (\partial^i \vec{u}^h)^2 dx = \lambda \|\vec{u}^h\|^2_{H^1_0(\Omega^h_c)} \geq \alpha \|\vec{u}^h\|^2_{\mathcal{W}^h} \ \ \forall \vec{u}^h \in \mathcal{W}^h_0. \quad (9.23)$$

To reduce the problem

$$B^h_{\mathcal{W}}\vec{u}^h_1 = (E_c\vec{f} \in \mathcal{W}^{h'}, (\vec{u}^h_1 - E_c\vec{\phi})|_{\partial\Omega^h_c} = \vec{0} \in \mathcal{V}^h_b) \in \mathcal{W}^{h'}_{c,\Pi}$$

to $\mathcal{W}^h_0$, we determine, in the standard way, a $\vec{u}^h_2 \in \mathcal{W}^h$ with $(\vec{u}^h_2 - E_c\vec{\phi})|_{\partial\Omega^h_c} = \vec{0} \in \mathcal{V}^h_b$ and $\vec{f}_2 := B_s\vec{u}^h_2$. Then $\vec{u}^h_1 - \vec{u}^h_2 \in \mathcal{W}^h_0$ is, by (9.23), the uniquely existing solution for the problem

$$B^h_{\mathcal{W}}(\vec{u}^h_1 - \vec{u}^h_2) = (E_c\vec{f} - E_c\vec{f}_2 \in \mathcal{W}^{h'}, \vec{0}|_{\partial\Omega^h_c} \in \mathcal{V}^h_b) \in \mathcal{W}^{h'}_{c,\Pi}.$$

This guarantees, by with (9.23) and Theorems 7.1 and 8.1 the stability, consistency, hence, existence of a unique discrete solution $\vec{u}^h_1 - \vec{u}^h_2$ and its convergence to the exact solution $\vec{u}_1 - \vec{u}_2$, and hence of $\vec{u}^h_1$ as well. □

Theorem 9.3 Stability of $A^h_{\mathcal{W}} = (A = G'(\vec{u}_0), B_D)^h \in \mathcal{L}(\mathcal{W}^h, \mathcal{W}^{h'} \times \mathcal{V}^h_b)$ for invertible $A_{\mathcal{W}} = (A = G'(\vec{u}_0), B_D)$ *Under the conditions of Theorems 9.2 and 4.2, assume a boundedly invertible $A_{\mathcal{W}}$ in (9.15). Then $A^h_{\mathcal{W}}$ in (9.20) is stable.*

Proof: This is a generalization to nonconforming FEMs of the Böhmer and Sassmannshausen results for conforming FEMs, [43, 13, 12]. For an arbitrary $\vec{u} \in \mathcal{W}$ choose $\vec{v}_w := C_{\mathcal{W}}\vec{u} \in \mathcal{W}'_\Pi = \mathcal{W}' \times \mathcal{W}_D$. By assumption, unique exact and discrete solutions, $\hat{\vec{u}}$ and $\hat{\vec{u}}^h$, exist for the equations

$$B_{\mathcal{W}}\hat{\vec{u}} = \vec{v}_w \text{ and } B^h_{\mathcal{W}}\hat{\vec{u}}^h = Q^{h'}_{c,\Pi}B_{\mathcal{W}}|_{\mathcal{W}^h}\hat{\vec{u}}^h = Q^{h'}_\Pi B_{\mathcal{W}}\hat{\vec{u}} = Q^{h'}_\Pi\vec{v}_w \in \mathcal{W}^{h'}_\Pi. \tag{9.24}$$

With the notations

$$T_w := B^{-1}_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}'_\Pi, \mathcal{W}) \text{ and } T^h_w := (B^h_{\mathcal{W}})^{-1}Q^{h'}_\Pi \in \mathcal{L}(\mathcal{W}'_\Pi, \mathcal{W}^h),$$

Theorem 9.2 implies

$$\|E_c\hat{\vec{u}} - \hat{\vec{u}}^h\|_{\mathcal{W}^h} = \|(E_cT_w - T^h_w)C_{\mathcal{W}}\vec{u}\|_{\mathcal{W}^h} \to 0 \text{ for } h \to 0 \text{ and } \forall\vec{u} \in \mathcal{W}.$$

Since, by assumption, $C_{\mathcal{W}}$ is compact and $(E_cT_w - T^h_w)$ are equibounded, this implies

$$\|(E_cT_w - T^h_w)C_{\mathcal{W}}\|_{\mathcal{W}\hookleftarrow\mathcal{W}} \to 0 \quad \text{for } h \to 0. \tag{9.25}$$

Now let $\vec{u}^h \in \mathcal{W}^h \subset \mathcal{W}_c$. Because $A_{\mathcal{W}}$ has been extended to yield

$$A_{\mathcal{W}} : \mathcal{W}_c \to \mathcal{W}'_{c,\Pi} = \mathcal{W}'_c \times \mathcal{W}_{c,D} \text{ still boundedly invertible with } \mathcal{W}^h \subset \mathcal{W}_c, \tag{9.26}$$

we can estimate

$$\begin{aligned}\|\vec{u}^h\|_{\mathcal{W}^h} &\le \|A^{-1}_{\mathcal{W}}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}'_{c,\Pi}}\|A_{\mathcal{W}}\vec{u}^h\|_{\mathcal{W}'_{c,\Pi}} = \|A^{-1}_{\mathcal{W}}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}'_{c,\Pi}}\|B_{\mathcal{W}}(I + T_wC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}'_{c,\Pi}}\\ &\le \|A^{-1}_{\mathcal{W}}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}'_{c,\Pi}}\|B_{\mathcal{W}}\|_{\mathcal{W}'_{c,\Pi}\hookleftarrow\mathcal{W}^h}\|(I + T_wC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}^h}, \text{ hence,}\\ \|(I + T_wC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}^h} &\ge (\|A^{-1}_{\mathcal{W}}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}'_{c,\Pi}}\|B_{\mathcal{W}}\|_{\mathcal{W}'_{c,\Pi}\hookleftarrow\mathcal{W}^h})^{-1}\|\vec{u}^h\|_{\mathcal{W}^h}.\end{aligned} \tag{9.27}$$

The $Q_{\Pi}^{h'}B_{\mathcal{W}}\vec{u}$ is defined for every $\vec{u}\in\mathcal{W}$. We apply the stability of $B_{\mathcal{W}}^h$ to $w^h := B_{\mathcal{W}}^h\vec{u}^h$

$$\|\vec{u}^h\|_{\mathcal{W}^h} \le \|(B_{\mathcal{W}}^h)^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}_{\Pi}^{h'}}\cdot\|w^h\|_{\mathcal{W}_{\Pi}^{h'}}.$$

Furthermore

$$\begin{aligned}\|(I+T_w^hC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}^h} &= \|(B_{\mathcal{W}}^h)^{-1}B_{\mathcal{W}}^h(I+T_w^hC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}^h}\\ &\le \|(B_{\mathcal{W}}^h)^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}_{\Pi}^{h'}}\|B_{\mathcal{W}}^h(I+T_w^hC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}}\end{aligned}$$

implies

$$\|B_{\mathcal{W}}^h(I+T_w^hC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}} \ge \frac{1}{\|(B_{\mathcal{W}}^h)^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}_{\Pi}^{h'}}}\cdot\|(I+T_w^hC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}^h}\ . \tag{9.28}$$

We combine (9.24), (9.25), (9.27), (9.28) and use the fact that $A_{\mathcal{W}}, B_{\mathcal{W}}, C_{\mathcal{W}}$ are extended to $\mathcal{W}^h$, s.t. $Q_{c,\Pi}^{h'}A_{\mathcal{W}}\vec{u}^h$ is well defined to estimate

$$\begin{aligned}\|A_{\mathcal{W}}^h\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}} &= \|Q_{c,\Pi}^{h'}A_{\mathcal{W}}|_{\mathcal{W}^h}\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}} = \|Q_{c,\Pi}^{h'}A_{\mathcal{W}}\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}} \qquad \forall\vec{u}^h\in\mathcal{W}^h\\
&= \|Q_{c,\Pi}^{h'}B_{\mathcal{W}}(I+T_wC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}} \overset{(9.24)}{=} \|B_{\mathcal{W}}^h(I+T_wC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}}\\
&\ge \|B_{\mathcal{W}}^h(I+T_w^hC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}} - \|B_{\mathcal{W}}^h(E_cT_w-T_w^h)C_{\mathcal{W}}\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}}\\
&\overset{(9.28)}{\ge} \|(I+T_w^hC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}^h}/\|(B_{\mathcal{W}}^h)^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}_{\Pi}^{h'}} - \|B_{\mathcal{W}}^h(E_cT_w-T_w^h)C_{\mathcal{W}}\vec{u}^h\|_{\mathcal{W}_{\Pi}^{h'}}\\
&\ge (\|(I+T_wC_{\mathcal{W}})\vec{u}^h\|_{\mathcal{W}^h} - \|(E_cT_w-T_w^h)C_{\mathcal{W}}\vec{u}^h\|_{\mathcal{W}^h})/\|(B_{\mathcal{W}}^h)^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}_{\Pi}^{h'}}\\
&\qquad - \|B_{\mathcal{W}}^h(E_cT_w-T_w^h)C_{\mathcal{W}}\|_{\mathcal{W}_{\Pi}^{h'}\hookleftarrow\mathcal{W}^h}\|\vec{u}^h\|_{\mathcal{W}^h}\\
&\overset{(9.27)}{\ge} \Big(\frac{1}{(\|A_{\mathcal{W}}^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}'_{c,\Pi}}\|B_{\mathcal{W}}^h\|_{\mathcal{W}_{\Pi}^{h'}\hookleftarrow\mathcal{W}^h}\|(B_{\mathcal{W}}^h)^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}_{\Pi}^{h'}})}\\
&\qquad -\mathcal{O}(\|(E_cT_w-T_w^h)C_{\mathcal{W}}\|_{\mathcal{W}_{\Pi}^{h'}\hookleftarrow\mathcal{W}^h})\Big)\|\vec{u}^h\|_{\mathcal{W}^h}.\end{aligned}$$

The last estimate shows: The $\|A_{\mathcal{W}}^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}'_{c,\Pi}}$ is the deciding factor, if $B_{\mathcal{W}}^h$ has a moderate condition number. Because of (9.25) and the stability of $(B_{\mathcal{W}}^h)_{h\in H}$ there exists a positive constant K, independent of h, such that for all $h\le h_0$ the following holds:

$$\|A_{\mathcal{W}}^h\vec{u}^h\|_{\mathcal{W}^h} = \|Q_{c,\Pi}^{h'}A_{\mathcal{W}}\vec{u}^h\|_{\mathcal{W}^h} \ge K\|\vec{u}^h\|_{\mathcal{W}^h} \quad \text{for all } \vec{u}^h\in\mathcal{W}^h.$$

This and $\dim\mathcal{W}^h<\infty$ shows that $Q_{c,\Pi}^{h'}A_{\mathcal{W}}|_{\mathcal{W}^h}$ is invertible for $h\le h_0$. Moreover, we obtain $\|(Q_{c,\Pi}^{h'}A_{\mathcal{W}}|_{\mathcal{W}^h})^{-1}\|_{\mathcal{W}^h\hookleftarrow\mathcal{W}^{h'}}\le 1/K$, i.e. $A_{\mathcal{W}}^h|_{\mathcal{W}^h}$ $\forall h\in H$ is stable. □

9.4 Regularity for FE solutions

Our next task is lifting the preceding stability result w.r.t. the weak forms back to the strong form. This is achieved by the regularity for FE solutions in this Subsection. Until now, *regularity results for the solutions of FEMs* seem to be known only as oscillation results in the sense of De Giorgi, Nash, Moser, see Aguilera and Caffarelli, [1]. They are not applicable to our problem of stability. Here we need, for the solution of a FEM applied to a differential equation or system of order 2 or $2m$, results analogous to those for the exact solution. These are estimates of the $H^{2m}(\Omega)$ instead of the usual $H^m(\Omega)$ norm for a right hand side in $L^2 \times H^{2m-1/2} \times \dots H^{m+1/2}(\Omega, \mathbf{R}^q)$ instead of the usual $H^{-m} \times H^{m-1/2} \times \dots H^{1/2}(\Omega, \mathbf{R}^q)$. To the authors knowledge, the following type of Lemma was not known for FEMs. It is a transformation to our FEM of Hackbusch's regularity result, [33], Theorem 9.2.26, for difference methods. We do need the results only for the above case. This simplifies the proof. We employ inverse and interpolation error estimates (6.4) and (6.10), (6.26), valid under the conditions of Theorems 6.1 and 6.2, 6.6.

Lemma 9.1 is based upon regularity results for the exact solution $\vec{u}_1$ of the corresponding linear problem. The basic assumptions for all cases are the bounded invertability of $A_{\mathcal{W}}$ and the coercivity of its principle part, e.g., by the conditions of Theorems 9.2 and 4.2.

$$A \text{ strongly elliptic, } \mathcal{W}_0-\text{coercive principle part, } (A_{\mathcal{W}})^{-1} \in \mathcal{L}(\mathcal{W}' \times \mathcal{W}_D, \mathcal{W}). \quad (9.29)$$

For the different cases $m \geq 1, q \geq 1$ we impose, see e.g., Hackbusch, [33], Theorem 9.1.16.

$$\text{For } m = q = 1 \text{ assume } \Omega \in C^2, a_{i,j} \in W^{1-\delta_{j0},\infty}(\Omega), 0 \leq i,j \leq n, (f,\phi) \in \mathcal{V} \times \mathcal{V}_D \quad (9.30)$$

$$\text{For } m > q = 1 \text{ assume } \Omega \in C^{2m}, \forall \alpha, \beta \text{ with } |\alpha|, |\beta| \leq m,\ \partial^\gamma a_{\alpha,\beta} \in L^\infty(\Omega) \forall\ \gamma \text{ with } |\gamma| \leq |\beta|,\ \text{else } a_{\alpha,\beta} \in L^\infty(\Omega),\ (f,\phi) \in \mathcal{V} \times \mathcal{V}_D \quad (9.31)$$

For systems of order 2, we extend results in Chen/Wu, [19]: Chapter 8, Theorem 2.6, mind our $\vec{f} \in \mathcal{V}$ corresponds to his condition.

$$\text{For } m = 1, q > 1 \text{ assume } \partial\Omega \in C^2,\ A_{kl} \in W^{1-\delta_{0k},\infty}(\Omega), \vec{f} \in \mathcal{V}\ \forall k,l. \quad (9.32)$$

For systems of order $2m$, regularity results follow from Taylor, [51], Chapter 5, Section 11. A system (4.7) with Dirichlet boundary conditions is regular, if it satisfies

$$\exists \lambda > 0 : \forall x \in \Omega, \forall \vartheta \in \mathbf{R}^n : \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta} \vartheta^\beta \vartheta^\alpha = \Big(\sum_{|\alpha|=|\beta|=m} a^{ij}_{\alpha\beta} \vartheta^\beta \vartheta^\alpha \Big)_{i,j=1}^q \geq \lambda |\vartheta|^{2m} I \quad (9.33)$$

with the identity matrix I, see [51], Chapter 5, Propositions 11.10, 11.16, compare Zeidler, [60], Section 4.17: Chapter 10, Theorems 2.1, 2.2.

$$\text{For } m,q > 1, \text{ assume } \partial\Omega \in C^\infty,\ A_{\alpha,\beta} \in C^\infty(\Omega)\ \forall \alpha, \beta, \vec{f} \in L^2(\Omega). \quad (9.34)$$

We get analytic regularity if (9.29) is satisfied for all cases $m, q \geq 1$, and (9.30) for $m = q = 1$, (9.31) for $m > 1, q = 1$, (9.32) for $m = 1, q > 1$, (9.33), (9.34) for $m, q > 1$:

$$A_{\mathcal{W}} \vec{u}_1 = (\vec{f}, \vec{\phi}) \in \mathcal{V} \times \mathcal{V}_D \Rightarrow \exists_1 \vec{u}_1 \in \mathcal{V} : \|\vec{u}_1\|_{\mathcal{U}} \leq C \big(\|\vec{f}\|_{\mathcal{V}} + \|\vec{\phi}\|_{\mathcal{V}_D} \big) = C \|(\vec{f}, \vec{\phi})\|_{\mathcal{V}_\Pi}. \quad (9.35)$$

If $A_{\mathcal{W}}$ is not boundedly invertible, one has to add $\|\vec{u}_1\|_{\mathcal{W}}$ on the right hand side.

Lemma 9.1 Regularity of solutions of FEMs *We assume the bounded strongly elliptic $A_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}, \mathcal{W}' \times \mathcal{W}_D)$ to have a $\mathcal{W}_0-$coercive main part, $A_{\mathcal{U}} \in \mathcal{L}(\mathcal{U}, \mathcal{V} \times \mathcal{V}_D)$ to be boundedly invertible, e.g., by the regularity conditions in (9.29) - (9.35). Choose the above FEs, (9.19) and (9.20) of local degree $d \geq 2m$, on a quasi-uniform triangulation, see (6.22), (9.6). Then $A^h_{\mathcal{W}}$ is regular w.r.t. stability and convergence, hence, there exist unique solutions $\vec{u}_1, \vec{u}^h_1$ and a $C' > 0$, independent of h, s.t., see (6.10),(6.26),*

$$A_{\mathcal{W}}\vec{u}_1 = (\vec{f}, \vec{\phi}) \text{ and } A^h_{\mathcal{W}}\vec{u}^h_1 = Q^{h'}_{\Pi}(\vec{f}, \vec{\phi}), (\vec{f}, \vec{\phi}) \in \mathcal{V} \times \mathcal{V}_D \Longrightarrow \|\vec{u}^h_1\|_{\mathcal{U}^h} \leq C'\|(\vec{f}, \vec{\phi}\|_{\mathcal{V}_\Pi} \quad (9.36)$$

$$\text{with } \|E_c\vec{u}_1 - \vec{u}^h_1\|_{\mathcal{U}^h} \leq C'(\|E_c\vec{u}_1 - P^h\vec{u}_1\|_{\mathcal{U}^h} + \|E_c\vec{\phi} - P^h\vec{u}_1\|_{\mathcal{V}^h_D}). \quad (9.37)$$

$A_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}, \mathcal{W}'_\Pi)$ and $A_{\mathcal{U}} \in \mathcal{L}(\mathcal{U}, \mathcal{V}_\Pi)$ simultaneously satisfy the Fredholm alternative.

Remark 9.1 *Hence, the FE error in the strong norm $\|E_c\vec{u}_1 - \vec{u}^h_1\|_{\mathcal{U}^h}$ is estimated by the interpolation errors of $\vec{u}_1$ in $\mathcal{U}^h$ and $\mathcal{V}^h_D$. The FE spaces considered here need a quasi uniform triangulation with inverse estimates in the global form, $\|\vec{v}^h\|^h_{H^j(\Omega^h_c)} \leq C\ h^{l-j}\|\vec{v}^h\|^h_{H^l(\Omega^h_c)}$, see (6.4). If inverse estimates will be available for $\mathcal{S}^1_d(\mathcal{T}^h_c)$ on non degenerate triangulations, e.g., by applying the techniques in Graham, Hackbusch and Sauter, [37], the Lemma remains valid for this case. Under the conditions of the Lemma, (9.36) even holds for noninvertible $A_{\mathcal{W}}$ with $\|(\vec{f}, \vec{\phi})\|_{\mathcal{V}_\Pi}$ replaced by $\|(\vec{f}, \vec{\phi})\|_{\mathcal{V}_\Pi} + \|\vec{u}_1\|_{\mathcal{W}}$, see [33].*

For the following diagram (9.38) and the Proof we contrast the strong and weak operators $A_{\mathcal{U}} \in \mathcal{L}(\mathcal{U} \cup \mathcal{U}_c, \mathcal{V}_\Pi)$ and $A_{\mathcal{W}} \in \mathcal{L}(\mathcal{W} \cup \mathcal{W}_c, \mathcal{W}'_\Pi)$, extended as in (7.7), (9.26), see (7.10), (9.16) with the FE versions in (7.15), (9.18). $A_{\mathcal{U}}, A^h_{\mathcal{U}}$ and the corresponding projectors $P^h, Q^h_\Pi, Q^{h'}_\Pi$ are uniformly bounded operators. The horizontal arrows in the first line of (9.38) indicate compact embeddings, in the second line the decreasingly weaker norms in the approximating spaces $\mathcal{U}^h = \mathcal{W}^h$ and $\mathcal{V}^h_\Pi \subset \mathcal{W}^{h'}_\Pi$. The $\longrightarrow$ between $\mathcal{W}, \mathcal{V}_\Pi$ are missing, since only $\mathcal{W} \longrightarrow \mathcal{V}$ would be appropriate.

$$\begin{array}{ccccccc} & \overset{A_{\mathcal{U}}}{\frown} & & \overset{A_{\mathcal{W}}}{\frown} & & & \\ \mathcal{U} & \longrightarrow & \mathcal{W} & & \mathcal{V}_\Pi = \mathcal{V}'_\Pi & \longrightarrow & \mathcal{W}'_\Pi \\ \downarrow P^h & & \downarrow P^h & & \downarrow Q^h_\Pi & & \downarrow Q^{h'}_\Pi \\ \mathcal{U}^h & \longrightarrow & \mathcal{W}^h & & \mathcal{V}^h_\Pi \neq \mathcal{V}^{h'}_\Pi & \longrightarrow & \mathcal{W}^{h'}_\Pi \\ & \underset{A^h_{\mathcal{U}}}{\smile} & & \underset{A^h_{\mathcal{W}}}{\smile} & & & \end{array} \quad (9.38)$$

Proof: For a boundedly invertible $A_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}, \mathcal{W}'_\Pi)$, Theorem 9.3 implies the stability, so bounded invertability of $A^h_{\mathcal{W}} = Q^{h'}_{c,\Pi}A_{\mathcal{W}}|_{\mathcal{W}^h} \in \mathcal{L}(\mathcal{W}^h, \mathcal{W}^{h'}_\Pi)$. This yields with the consistency in Theorem 8.1 the existence of a discrete solution $\vec{u}^h_1$ for $A^h_{\mathcal{W}}\vec{u}^h_1 = Q^{h'}_\Pi(\vec{f}, \vec{\phi}) \in \mathcal{W}^{h'}_\Pi$, by Theorem 7.1. The solution $\vec{u}^h_1 \in \mathcal{W}^h$ for $(\vec{f}, \vec{\phi}) \in \mathcal{W}'_\Pi$ satisfies the estimate $\|\vec{u}^h_1\|_{\mathcal{W}^h} \leq C\|(\vec{f}, \vec{\phi})\|_{\mathcal{W}'_\Pi}$. Now we use the same technique as in the proof of Theorem 9.2 to reduce the inhomogeneous to trivial Dirichlet boundary conditions. We have assumed $\vec{\phi} \in \mathcal{V}_D = H^{2m-1/2} \times \cdots \times H^{m+1/2}(\partial\Omega, \mathbf{R}^q)$ to be obtained as $\vec{\phi} = B_D\vec{\phi}_e$ with $\vec{\phi}_e \in \mathcal{V} = H^{2m}(\Omega)$. This is possible by combining Theorem 6.4 with Zeidler, [61]. p1030. The last lines of the proof of Theorem 8.1 can then be updated to obtain the (9.36), (9.37) results for $\vec{\phi} \not\equiv \vec{0}$, if they are correct for $\vec{\phi} \equiv \vec{0}$. So we restrict the discussion for the remaining proof to trivial Dirichlet boundary conditions.

We find for $P^h = I^h E_c$, $A_{\mathcal{U}}$ in (9.26), and the special $\vec{v}^h_\Pi = (\vec{v}^h, \vec{0}) \in \mathcal{V}^h_\Pi$,

$$P^h A_{\mathcal{U}}^{-1} : \mathcal{V}^h_\Pi \subset \mathcal{V}_{c,\Pi} \to \mathcal{U}^h \text{ or } P^h A_{\mathcal{U}}^{-1} \vec{v}^h_\Pi, \text{ is well defined } \forall \vec{v}^h_\Pi = (\vec{v}^h, \vec{0}) \in \mathcal{V}^h_\Pi.$$

$\mathcal{V}^h \subset C^1(\overline{\Omega^h_c}) \subset C^\gamma(\overline{\Omega^h_c})$ for a $0 < \gamma < 1$ implies $A_{\mathcal{U}}^{-1} \vec{v}^h_\Pi \in C^{2,\gamma}(\overline{\Omega^h_c})$ by standard regularity results, e.g. [61], Chapter 6.3, Problems 6.8, p 259, and see (9.35) and the two preceding lines. Thus $P^h A_{\mathcal{U}}^{-1} \vec{v}^h_\Pi$ is well defined, if in (6.14) derivatives up to second order are needed to define P^h, e.g. for the Argyris FEs. Higher derivatives defining P^h, require quasi interpolants as, e.g. in Theorem 6.2, Scott, Zhang, [44], and Davydov, [25], Theorem A3.1.

For the estimate in (9.41) below we need the embedding $I_{WU} : \mathcal{W}^h \to \mathcal{U}^h$, and the bounded transformation, compare (7.12), and the Appendix.

$$I_{\mathcal{V}^{h',0}_\Pi, \mathcal{V}^0_{c,\Pi}} : \mathcal{V}^{h',0}_\Pi := \mathcal{V}^{h'} \times \{\vec{0}\} \subset \mathcal{V}^{h'}_\Pi \to \mathcal{V}^0_{c,\Pi} := \mathcal{V}_c \times \{\vec{0}\} \subset \mathcal{V}^0_{c,\Pi}, (\vec{v}^{h'}, \vec{0}) \to (Q^h_d \vec{v}^{h'}, \vec{0}).$$

We assumed a smooth $(\vec{f} \in \mathcal{V}, \vec{\phi} \equiv \vec{0}) \in \mathcal{V}_\Pi$. By (7.13) and $\vec{\phi} \equiv \vec{0}$ we restrict the discussion to $\vec{u}^h \in \mathcal{W}^h \cap S^h_0 = \mathcal{U}^h_0 = \mathcal{W}^h_0 = \mathcal{V}^h$ hence, we obtain by (9.9), (7.10), (9.17), (9.16), (7.11), the following equalities between the weak and strong problems

$$\forall \vec{u}^h, \vec{v}^h \in \mathcal{U}^h_0 = \mathcal{W}^h_0 = \mathcal{V}^h, \quad \forall \vec{v}^h_\Pi := (\vec{v}^h, \vec{0}) \in \mathcal{V}^{h,0}_\Pi := \mathcal{V}^h \times \{\vec{0}\}, \text{ we find with}$$
$$\mathcal{V}^{h',0}_\Pi, \ \mathcal{W}^{h,0}_\Pi := \mathcal{W}^h \times \{\vec{0}\}, \ \mathcal{W}^{h',0}_\Pi := \mathcal{W}^{h'} \times \{\vec{0}\} \text{ that}$$
$$\langle A^h_{\mathcal{W}} \vec{u}^h, \vec{v}^h_\Pi \rangle_{\mathcal{W}^{h',0}_\Pi \times \mathcal{W}^{h,0}_\Pi} = \langle A \vec{u}^h, \vec{v}^h \rangle_{\mathcal{W}'_c \times \mathcal{W}_c} = (A_s \vec{u}^h, \vec{v}^h)_{\mathcal{V}_c} = \langle A^h_{\mathcal{U}} \vec{u}^h, \vec{v}^h_\Pi \rangle_{\mathcal{V}^{h',0}_\Pi \times \mathcal{V}^{h,0}_\Pi} \quad (9.39)$$
$$\text{and } (E_c \vec{f}, \vec{v}^h)_{\mathcal{V}_c} = \langle Q^{h',0}_\Pi (E_c \vec{f}, \vec{0}), \vec{v}^h_\Pi \rangle_{\mathcal{W}^{h',0}_\Pi \times \mathcal{W}^{h,0}_\Pi} = \langle Q^{h,0}_\Pi (E_c \vec{f}, \vec{0}), \vec{v}^h_\Pi \rangle_{\mathcal{V}^{h',0}_\Pi \times \mathcal{V}^{h,0}_\Pi}, \Longrightarrow$$
$$\langle A^h_{\mathcal{W}} \vec{u}^h_1, \vec{v}^h_\Pi \rangle_{\mathcal{W}^{h',0}_\Pi \times \mathcal{W}^{h,0}_\Pi} = (A_s \vec{u}^h_1, \vec{v}^h)_{\mathcal{V}_c} = \langle A^h_{\mathcal{U}} \vec{u}^h_1, \vec{v}^h_\Pi \rangle_{\mathcal{V}^{h',0}_\Pi \times \mathcal{V}^{h,0}_\Pi} = (E_c \vec{f}, \vec{v}^h)_{\mathcal{V}_c} \ \forall \vec{v}^h \in \mathcal{V}^h.$$

This implies that $\vec{u}^h_1$ simultaneously solves both equations

$$A^h_{\mathcal{W}} \vec{u}^h_1 = Q^{h',0}_\Pi (\vec{f}, \vec{0}), \quad (\vec{f}, \vec{0}) \in \mathcal{V}^0_\Pi := \mathcal{V}^h \times \{\vec{0}\} \Longrightarrow A^h_{\mathcal{U}} \vec{u}^h_1 = Q^{h,0}_\Pi (\vec{f}, \vec{0}). \quad (9.40)$$

We start with the following identity on $\mathcal{V}^{h',0}_\Pi$

$$(A^h_{\mathcal{U}})^{-1}_{\mathcal{U}^h_0 \hookleftarrow \mathcal{V}^{h',0}_\Pi} = \big\{ P^h_{\mathcal{U}^h_0 \hookleftarrow \mathcal{U}_{c,0}} (A_{\mathcal{U}}^{-1})_{\mathcal{U}_{c,0} \hookleftarrow \mathcal{V}^0_{c,\Pi}} - (I_{WU})_{\mathcal{U}^h_0 \hookleftarrow \mathcal{W}^h_0} \quad (9.41)$$
$$(A^h_{\mathcal{U}})^{-1}_{\mathcal{W}^h_0 \hookleftarrow \mathcal{W}^{h',0}_\Pi} [(A^h_{\mathcal{U}} P^h - Q^h_{c,\Pi} A_{\mathcal{U}})]_{\mathcal{W}^{h',0}_\Pi \hookleftarrow \mathcal{U}_{c,0}} (A_{\mathcal{U}}^{-1})_{\mathcal{U}_{c,0} \hookleftarrow \mathcal{V}^0_{c,\Pi}} \big\} I_{\mathcal{V}^0_{c,\Pi}, \mathcal{V}^{h',0}_\Pi}$$

A first estimate uses the inverse estimates (6.4), the approximation property (6.26) and the error estimates (6.10) showing, for the local degree $d \geq 2m$, the three inequalities

$$\|I_{WU}\|_{\mathcal{U}^h_0 \hookleftarrow \mathcal{W}^h_0} \leq C_1 h^{-m}, \ \|P^h\|_{\mathcal{U}^h_0 \hookleftarrow \mathcal{U}_{c,0}} \leq C_2, \ \|P^h \vec{u}_1 - E_c \vec{u}_1\|_{\mathcal{W}^h} \leq C_3 h^m \|\vec{u}_1\|_{\mathcal{U}}. \quad (9.42)$$

With the solution $\vec{u}_1$ of $A_{\mathcal{U}} \vec{u}_1 = (\vec{f}, \vec{0}) \in \mathcal{V}^0_\Pi$, the equal weak and strong bilinear forms, see(9.40), and the special $\vec{v}^h_\Pi = (\vec{v}^h \in \mathcal{V}^h, \vec{0})$, this implies the consistency estimate

$$|\langle A^h_{\mathcal{U}} P^h \vec{u}_1 - Q_\Pi A_{\mathcal{U}} \vec{u}_1, \vec{v}^h_\Pi \rangle_{\mathcal{W}^{h',0}_\Pi \times \mathcal{W}^{h,0}_\Pi}| = |(A_s P^h \vec{u}_1 - A_s E_c \vec{u}_1, \vec{v}^h)_{\mathcal{V}_c}| =$$
$$|(\sum_{|\alpha|,|\beta|=0}^{m} A_{\alpha,\beta} \partial^\beta \ (P^h \vec{u}_1 - E_c \vec{u}_1), \partial^\alpha \vec{v}^h)_{\mathcal{V}_c}| \leq C_3 h^m \|\vec{u}_1\|_{\mathcal{U}} \|\vec{v}^h\|_{\mathcal{W}^h}, \ \forall \vec{v}^h_\Pi = (\vec{v}^h, \vec{0}) \in \mathcal{V}^{h,0}_\Pi.$$

With (9.40) and the stability of $A^h_{\mathcal{W}}$, the $\|(A^h_{\mathcal{W}})^{-1}_{\mathcal{W}^h_0 \hookleftarrow \mathcal{W}^{h',0}_\Pi}\| \le C_4$, we estimate the middle term in (9.41):

$$\begin{aligned}
&\|(A^h_{\mathcal{U}})^{-1}_{\mathcal{W}^h_0 \hookleftarrow \mathcal{W}^{h',0}_\Pi}[(A^h_{\mathcal{U}}P^h - Q^h_{c,\Pi}A_{\mathcal{U}})]_{\mathcal{W}^{h',0}_\Pi \hookleftarrow \mathcal{U}_{c,0}}\|_{\mathcal{W}^h_0 \hookleftarrow \mathcal{U}_{c,0}} \\
&= \|(A^h_{\mathcal{W}})^{-1}_{\mathcal{W}^h_0 \hookleftarrow \mathcal{W}^{h',0}_\Pi}[(A^h_{\mathcal{U}}P^h - Q^h_{c,\Pi}A_{\mathcal{U}})]_{\mathcal{W}^{h',0}_\Pi \hookleftarrow \mathcal{U}_{c,0}}\|_{\mathcal{W}^h_0 \hookleftarrow \mathcal{U}_{c,0}} \\
&\le C_4 \|(A^h_{\mathcal{U}}P^h - Q^h_{\Pi}A_{\mathcal{U}})\|_{\mathcal{W}^{h',0}_\Pi \hookleftarrow \mathcal{U}_{c,0}} \\
&\le C_4 \sup_{\vec 0 \ne \vec u \in \mathcal{U}_{c,0}} \sup_{\vec 0 \ne \vec v^h \in \mathcal{W}^h_0} \frac{|\langle A^h_{\mathcal{U}}P^h u - A_{\mathcal{U}}\vec u, \vec v^h_\Pi\rangle_{\mathcal{W}^{h',0}_\Pi \times \mathcal{W}^{h,0}_\Pi}|}{\|\vec v^h\|_{\mathcal{W}^h}\|\vec u\|_{\mathcal{U}}} \le C_3 C_4 h^m.
\end{aligned}$$

With (9.42), (9.41) and a boundedly invertible $\|(A^{-1}_{\mathcal{U}}\|_{\mathcal{U}_{c,0} \hookleftarrow \mathcal{V}^0_{c,\Pi}} \le C_5$ and a bound for $\|I_{\mathcal{V}^0_{c,\Pi}, \mathcal{V}^{h',0}_\Pi}\|_{\mathcal{V}^0_{c,\Pi} \hookleftarrow \mathcal{V}^{h',0}_\Pi} \le C_6$, this yields the stability of $A^h_{\mathcal{U}} : \mathcal{U}^h_0 \to \mathcal{V}^{h',0}_\Pi$, since

$$\begin{aligned}
\|(A^h_{\mathcal{U}})^{-1}\|_{\mathcal{U}^h_0 \hookleftarrow \mathcal{V}^{h',0}_\Pi} &\le \big[\|P^h\|_{\mathcal{U}^h_0 \hookleftarrow \mathcal{U}_{c,0}}\|(A^{-1}_{\mathcal{U}})\|_{\mathcal{U}_{c,0} \hookleftarrow \mathcal{V}^0_{c,\Pi}} + \|(I_{WU})_{\mathcal{U}^h_0 \hookleftarrow \mathcal{W}^h_0}\| \\
\|(A^h_{\mathcal{U}})^{-1}_{\mathcal{W}^h_0 \hookleftarrow \mathcal{W}^{h',0}_\Pi}&[(A^h_{\mathcal{U}}P^h - Q^h_{c,\Pi}A_{\mathcal{U}})]_{\mathcal{W}^{h',0}_\Pi \hookleftarrow \mathcal{U}_{c,0}}\|_{\mathcal{W}^h_0 \hookleftarrow \mathcal{U}_{c,0}}\|A^{-1}_{\mathcal{U}}\|_{\mathcal{U}_{c,0} \hookleftarrow \mathcal{V}^0_{c,\Pi}}\big] \\
&\|I_{\mathcal{V}^0_{c,\Pi}, \mathcal{V}^{h',0}_\Pi}\|_{\mathcal{V}^0_{c,\Pi} \hookleftarrow \mathcal{V}^{h',0}_\Pi} \le \big[C_2C_5 + C_1h^{-1}C_3C_4hC_5\big]C_6 \le C
\end{aligned}$$

The combination of these inequalities yields the claim, since

$$\|\vec u^h_1\|_{\mathcal{U}^h} = \|(A^h_{\mathcal{U}})^{-1}Q^h_\Pi(\vec f, \vec 0)\|_{\mathcal{U}^h} \le \|(A^h_{\mathcal{U}})^{-1}\|_{\mathcal{U}^h_0 \hookleftarrow \mathcal{V}^{h',0}_\Pi}\|Q^h_\Pi\|_{\mathcal{V}^{h',0}_\Pi \hookleftarrow \mathcal{V}^0_\Pi}\|\vec f\|_{\mathcal{V}} \le C\|\vec f\|_{\mathcal{V}}.$$

The convergence in (9.37) is, by Theorem 7.1, an immediate consequence of the stability in (9.36) and the consistency in Theorem 8.1. □

9.5 Stability for the strong linear operator and convergence

After all these preparations we are ready to prove the stability of the linearized strong operator and thus the convergence of the method. We had to impose many different conditions for the different areas of ellipticity and $\mathcal{W}_0-$coercivity, approximation, coinciding weak and strong bilinear forms and regularity. We recall the notations $A_{\alpha\beta} = a_{\alpha\beta}$ for $m > 1, q = 1$, $A_{\alpha\beta} = A_{ij}$ for $m = 1 < q$, and $A_{\alpha\beta} = a_{ij}, i, j = 0, \ldots, n$ for $m = 1 = q$. For the operator and the domain, we imposed (4.9), the strong Legendre condition for $m \ge 1, q = 1$ and $m = 1, q \ge 1$, and (4.11) and (9.33), the strong Legendre-Hadamard and the modified Legendre condition for regularity, for $m > 1, q > 1$, summarized as

$$\begin{aligned}
&\text{for all } m, q \ge 1 : \mathcal{U}, \mathcal{U}^h \ldots, \text{ in (4.5), (7.1),(7.6), (9.16)}, d \ge (2m-1)2^n + 1 \\
&\text{and (4.9)}, \Omega \in C^{2m}, A_{\alpha\beta} \in W^{|\alpha|,\infty}(\Omega, \mathbf{R}^{q\times q})\ \forall |\alpha|, |\beta| \le m, \text{ except } m, q > 1, \\
&\text{equations of order 2}: \ m = q = 1 : \ a_{ij} \in W^{1-\delta_{i,0},\infty}(\Omega), \ \forall 0 \le i, j \le n, \qquad (9.43) \\
&\text{systems of order 2}: \ m = 1 < q : \ A_{ij} \in W^{1-\delta_{i,0},\infty}(\Omega, \mathbf{R}^{q\times q}), \forall 0 \le i, j \le n, \\
&\text{equations of order } 2m : \ m > q = 1 :, \Omega \in C^{2m}, a_{\alpha\beta} \in C(\overline{\Omega})\forall |\alpha|, |\beta| = m, \\
&\text{syst. order } 2m : m, q > 1 : \Omega \in C^\infty, (4.11), (9.33), A_{\alpha\beta} \in C^\infty(\overline{\Omega}, \mathbf{R}^{q\times q})\forall |\alpha| = |\beta| \le m.
\end{aligned}$$

These conditions imply, with the Sobolev Embedding Theorem the conditions for the coercivity of the principle part in Theorem 4.2, e.g, (4.13), for coinciding bilinear forms in Proposition 9.1, and for regular solutions of FE equations, e.g. (9.29) - (9.35) in Lemma 9.1. As a consequence of the above regularity conditions in (9.43), we obtain for the exact solution $\vec{u}_0$ of the linear problem the estimate

$$\|\vec{u}_0\|_{\mathcal{U}} \le C(\|\vec{f}\|_{\mathcal{V}} + \|\vec{\phi}\|_{\mathcal{V}_D}) \text{ for } m, q \ge 1.$$

For the consistency we needed in Theorem 8.1: an exact solution $\vec{u}_0 \in H^\ell(\Omega, \mathbf{R}^q), \ell > 2m$, and a Lipschitz continuous G^w in (5.1) with a global constant L, see (8.1).

Next, we choose the $\mathcal{U}^h = \mathcal{S}_d^{2m-1}(\Omega_c^h, \mathbf{R}^q) \supset \mathcal{V}^h$ as in (7.6), (3.1), (6.22), (9.6) with the norm limits in (9.17) and (9.18), on a quasi uniform triangulation under the conditions (6.5) - (6.8), (6.12), see Theorems 6.6 Theorem 6.3, but mind Remark 6.1 and the product spaces and projectors, e.g., $\mathcal{V}_\Pi = \mathcal{V} \times \mathcal{V}_D$ and $Q_\Pi^h \in \mathcal{L}(\mathcal{V}_\Pi, \mathcal{V}_\Pi^{h'})$.

The last condition missing for Theorems 9.3, 9.2, 4.2, is the boundedly invertible $A_{\mathcal{W}}$.

Remark 9.2 *The bounded invertability of $A_{\mathcal{U}} := (A_s := G'(\vec{u}_0), B_D) \in \mathcal{L}(\mathcal{U}, \mathcal{V} \times \mathcal{V}_D)$ is strongly related to the existence of a locally unique solution of $A_{\mathcal{U}}\vec{u}_1(\vec{f}, \vec{\phi})$, see the discussion at the end of Section 5.*

Combining Theorems 4.2 9.3, Proposition 9.1 and Lemma 9.1 we obtain

Theorem 9.4 *Assume the previous conditions starting with (9.43). Then we get, for the strong linearized operator in (9.2), (9.4), (9.15), a stable FEM, see (9.19),*

$$A_{\mathcal{U}}^h \vec{u}_1^h = Q_{c,\Pi}^h A_{\mathcal{U}} \vec{u}_1^h = Q_\Pi^h(\vec{f}, \vec{\phi}), \quad (\vec{f}, \vec{\phi}) \in \mathcal{V} \times \mathcal{V}_D.$$

Proof: The boundedness of the strong $A_{\mathcal{U}} \in \mathcal{L}(\mathcal{U}, \mathcal{V} \times \mathcal{V}_D)$ and its weak counterpart $A_{\mathcal{W}} := (A := G'(\vec{u}_0), B_D) \in \mathcal{L}(\mathcal{W}, \mathcal{W}' \times \mathcal{W}_D)$ under the conditions in (9.29) - (9.35), implying (9.4), is straight forward. We want to show that the bounded invertability of $A_{\mathcal{U}}$ implies that of $A_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}, \mathcal{W}' \times \mathcal{W}_D)$ as well. Indeed, by (9.4), the corresponding $G'(\vec{u}_0) \in \mathcal{L}(\mathcal{U}, \mathcal{V}')$ induces a strong bounded bilinear form, by Proposition 9.1 simultaneously a $\mathcal{W}$ - elliptic bilinear form, and thus finally $A_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}, \mathcal{W}' \times \mathcal{W}_D)$. Thus the Fredholm alternative applies: If $A_{\mathcal{W}} \in \mathcal{L}(\mathcal{W}, \mathcal{W}' \times \mathcal{W}_D)$ would not be boundedly invertible, each of the infinitely many solutions of $A_{\mathcal{W}}\vec{u}_1 = (\vec{f}, \vec{\phi}) \in \mathcal{V} \times \mathcal{V}_D$ for appropriate $(\vec{f}, \vec{\phi})$ would be $\vec{u}_1 \in \mathcal{U}$ by (9.29) and the above list of regularity results, see (9.29) - (9.35), could not be boundedly invertible either.

Now we apply Theorem 9.3 to guarantee the stability for the strong form: The bounded invertability of $Q_\Pi^{h'} A_{\mathcal{W}}|_{\mathcal{W}^h} \in \mathcal{L}(\mathcal{W}^h, \mathcal{W}_\Pi^{h'})$ and Lemma 9.1 guarantee the stability of $Q_\pi^h A_{\mathcal{U}}|_{\mathcal{U}^h} \in \mathcal{L}(\mathcal{U}^h, \mathcal{V}_\Pi^h)$, and thus the claim. □

Theorems 7.1 , 8.1, 9.1 combined with the last proof show that we have proved stability and convergence for the FEM in (7.16), if G, hence F as well, is continuously differentiable, see (9.44), near a locally unique solution $\vec{u}_0$ with boundedly invertible $F'(\vec{u}_0)$. So, we require, with the notations in (9.4), (9.43), continuity w.r.t. the variable $\vec{u}$, via $\vec{w}_{\vec{u}}$, see (5.3), (5.14), for Theorem 9.5 below, s.t. for the ball $B_{r,\mathcal{W}}(u_0)$ w.r.t. $\|\mathbf{u} - u_0\|_{\mathcal{W}} < r$,

$$A_{\alpha\beta}(\vec{w}_{\vec{u}}) \in C^{|\alpha|,\infty}(B_{r,\mathcal{W}}(\vec{u}_0)) \text{ and } A_{\alpha\beta}(\vec{w}_{\vec{u}}) \in C^{|\alpha|,\infty}(B_{r,\mathcal{U}}(\vec{u}_0)), \forall |\alpha|, |\beta| \le m \quad (9.44)$$
$$\text{yield } G'(u) : \mathcal{W} \to \mathcal{W}' \in C(B_{r,\mathcal{W}}(\vec{u}_0)), \text{ and } G'(u) : \mathcal{U} \to \mathcal{V}' \in C(B_{r,\mathcal{U}}(\vec{u}_0)).$$

Theorem 9.5 Main Result: Unique existence and convergence of discrete solutions: *Let the nonlinear elliptic problem in (5.16), $F(\vec{u}_0) = 0$ with $F = (G, B_D) : \mathcal{U} \to \mathcal{V}_\Pi = \mathcal{V} \times \mathcal{V}_D$, have an isolated solution $\vec{u}_0 \in \mathcal{U}$, and let F' satisfy (9.43) - (9.44), hence $F'(\vec{u}_0) : \mathcal{U} \to \mathcal{V}_\Pi$ is boundedly invertible and F is continuously differentiable in $B_r(\vec{u}_0)$. Then the FEM*

$$\vec{u}_0^h \in \mathcal{D}(F^h) \subset \mathcal{U}^h \ s.t. \quad F^h(\vec{u}_0^h) = 0, \quad F^h = Q^h_{c,\Pi} F|_{\mathcal{U}^h} : \mathcal{U}^h \to \mathcal{V}^{h'} \times \mathcal{V}^h_b,$$

see (7.16), is stable in $P^h\vec{u}_0$, consistent and convergent. It has, for small enough h, a unique solution $\vec{u}_0^h$ near $\vec{u}_0$. We get, see (8.2), for $\vec{u}_0 \in H^\ell(\Omega, \mathbf{R}^{q\times q}), \ell > 2m$, related with d by (6.8),

$$\|P^h\vec{u}_0 - \vec{u}_0^h\|_{\mathcal{U}^h} \le Ch^{\min\{\ell-2m,p\}} \|\vec{u}_0\|_{H^\ell(\Omega,\mathbf{R}^{q\times q})}, \tag{9.45}$$

with $p = 2$ for polygonal Ω^h and $p > 2$ for curved Ω^h_c.

Simultaneously $F'(\vec{u}_0) \in \mathcal{L}(\mathcal{W}, \mathcal{W}' \times \mathcal{W}_D)$ and $F'(\vec{u}_0) \in \mathcal{L}(\mathcal{U}, \mathcal{V} \times \mathcal{V}_D)$ satisfy the Fredholm alternative.

10 Approximation by Quadrature Formulas

We unfold the short outline in Section **??**, **4.**. The FEM in (7.16) requires the exact evaluation of the integrals. This is still possible for the boundary terms and for not too complicated $G(u)$, e.g., for (5.5) and simple f. For general G these integrals have to be approximated by quadrature and cubature. This is only necessary for G, not for B_D. So we consider only the first components of F and the projectors, the G and the corresponding $Q^h, Q^{h'}$. We apply the same quadrature formula to every component and return to the notation u instead of $\vec{u}$.

The quadrature formulas are employed on every $T \in \mathcal{T}^h_c$ and reproduce the integral of polynomials of a certain degree. For notational simplicity, we restrict the discussion to formulas defined only by values of functions, excluding derivatives. Although the u^h are defined by values of functions and derivatives, it is much easier using functions values than derivatives of $G(u^h)$. The notation $C(\overline{T})$ indicates continuous extensions of $C(T)$. A simple application of the Bramble-Hilbert Lemma yields :

Proposition 10.1 *Choose a quadrature formula satisfying for the interiour $T \in \mathcal{T}^h_c$*

$$\int_T f(x)dx \ \approx \ q^h_T(f) := \sum_{P_{j,T}\in\overline{T}} w_{j,T} f(P_{j,T}), \quad \int_T p(x)dx = q^h_T(p) \ \forall p \in \Pi_{k-1}, \ k \in \mathbf{N}$$

with f replaced by $E_c f$ for the boundary $T \in \mathcal{T}^h_c$, and let diam $T \le h$*. Then there exists $C = C_k$ s.t. $\forall\ 0 < h < 1, 1 \le q \le \infty, f \in W^{k,q}(T) \cap C(\overline{T})$,*

$$|\int_T f(x)dx - \sum_{P_{j,T}\in\overline{T}} w_{j,T} f(P_{j,T})| \le C_k h^{k+(1-1/q)} |f|_{W^{k,q}(T)}. \tag{10.1}$$

These $P_{j,T} \in \overline{T} \in \mathcal{T}_c^h$ are defined by affinely transforming the original points P_j in the reference domain. Applying the above q_T^h to all $T \in \mathcal{T}_c^h$, yields a quadrature formula for Ω or Ω_c^h and $f \in W^{k,q}(\Omega)$ or $f_c \in W^{k,q}(T) \cap C(\overline{T}) \forall T \in \mathcal{T}_c^h$, for short, $f_c \in W_C^{k,q}(\mathcal{T}_c^h)$,

$$\int_\Omega f(x)dx \approx q^h(E_c f) := \sum_{T\in\mathcal{T}_c^h} q_T^h(E_c f), \text{ or } \int_{\Omega_c^h} f_c(x)dx \approx q^h(f_c) \ f_c \in W_C^{k,q}(\mathcal{T}_c^h). \quad (10.2)$$

We estimate the quadrature error for (10.2) with Proposition 10.1 and for $q = 1$:

$$|q^h(f_c) - \int_{\Omega_c^h} f_c(x)dx| \le C_k h^k \sum_{T\in\mathcal{T}_c^h} |f_c|_{W^{k,1}(T)} = C_k h^k |f_c|_{W^{k,1}(\mathcal{T}_c^h)}. \quad (10.3)$$

We apply (10.2) to the terms in our FEM (7.16), defining, in particular $\tilde{G}^h, \tilde{Q}^h, \tilde{Q}_c^h, \ldots$,

$$(f_c, v^h)^{\tilde{}h} := q^h(f_c v^h) = \sum_{T\in\mathcal{T}_c^h} q_T^h(f_c\ v^h) \forall f_c \in C(\mathcal{T}_c^h), v^h \in \mathcal{V}^h,$$

$$\tilde{Q}_c^h : \mathcal{V}_c \to \mathcal{V}^{h'},\ \langle \tilde{Q}_c^h f_c - f_c, v^h\rangle^{\tilde{}h} := ((\tilde{Q}_c^h f_c - f_c), v^h)^{\tilde{}h} = 0\ \forall v^h \in \mathcal{V}^h, \quad (10.4)$$

$$\tilde{a}_s^h(u^h, v^h) := \sum_{T\in\mathcal{T}_c^h} \sum_{|\alpha|,|\beta|\le m} (-1)^{|\alpha|} q_T^h\big(\partial^\alpha(A_{\alpha\beta}\partial^\beta u^h)v^h\big) \quad (10.5)$$

$$\tilde{G}^h = \tilde{Q}_c^h G|_{\mathcal{U}^h} : \mathcal{U}^h \to \mathcal{V}^{h'},\ (G(\tilde{u}_0^h), v^h)^{\tilde{}h} := \sum_{T\in\mathcal{T}_c^h} q_T^h(G(\tilde{u}_0^h)v^h) = 0 \forall v^h \in \mathcal{V}^h, \quad (10.6)$$

where the $\tilde{a}_s^h(u^h, v^h)$ is the basis for our stability properties and $\tilde{G}^h(\tilde{u}_0^h) = 0$ represents the first component of the corresponding $\tilde{F}^h(\tilde{u}_0^h) = 0$.

In Theorems 10.1 and 10.2 we will show the consistency and stability of our approximate FEM (10.6). The stability is equivalent to showing under which conditions $\tilde{a}_s^h(\cdot,\cdot)$ inherits the boundedness and discrete inf–sup–condition from $a_s(\cdot,\cdot) = a(\cdot,\cdot)$ on $\mathcal{W}_0^h$. As above, we do not distinguish between $A_{\alpha\beta} \in W^{k+|\alpha|,\infty}(\Omega)$ and $A_{\alpha\beta} \in W^{k+|\alpha|,\infty}(\Omega_c^h)$.

Theorem 10.1 *For $m \ge 1, k > 2(d-m)$, hence, $k - d > 0$, let $A_{\alpha\beta} \in W^{k+|\alpha|,\infty}(\Omega)$, see (9.4), and $G : \mathcal{D}(G) \cap H^{k+2m}(\Omega) \to H^k(\Omega)$ and, hence, $G : \mathcal{D}(G^h) \cap H_C^{k+2m}(\mathcal{T}_c^h) \to H_C^k(\mathcal{T}_c^h)$. Choose the approximating $\mathcal{U}^h = \mathcal{S}_d^{2m-1}(\mathcal{T}_c^h, \mathbf{R}) \supset \mathcal{V}^h$, $d \ge (2m-1)2^n + 1$ as in (7.6). Let the quadrature formula (10.2) satisfy (10.3) and let $h > 0$ be small enough. Then*

$$|(f_c, v^h)_{\mathcal{V}_c} - (f_c, v^h)^{\tilde{}h}| \le C_k h^{k-d} \|f_c\|_{H^k(\mathcal{T}_c^h)} \|v^h\|_{\mathcal{V}^h},\ \forall f_c \in H_C^k(\mathcal{T}_c^h), \quad (10.7)$$

$$\|Q_c^h f_c - \tilde{Q}_c^h f_c\|_{\mathcal{V}^{h'}} \le C_k h^{k-d} \|f_c\|_{H^k(\mathcal{T}_c^h)}\ \forall f_c \in H_C^k(\mathcal{T}_c^h), \text{ similarly}$$

$$\|Q^h f - \tilde{Q}^h f\|_{\mathcal{V}^{h'}} \le C_k h^{k-d} \|f\|_{H^k(\Omega)}\ \forall f \in H^k(\Omega), \quad (10.8)$$

$$|(G(u^h), v^h)_{\mathcal{V}_c} - (G(u^h), v^h)^{\tilde{}h}| \le C h^{k-d} \|G(u^h)\|_{H_C^k(\mathcal{T}_c^h)} \|v^h\|_{\mathcal{V}^h}, \text{ or}$$

$$\|Q_c^h G(u^h) - \tilde{Q}_c^h G(u^h)\|_{\mathcal{V}} \le C h^{k-d} \|G(u^h)\|_{H_C^k(\mathcal{T}_c^h)}, \quad (10.9)$$

$$|a_s(u^h, v^h) - \tilde{a}_s^h(u^h, v^h)| \le C h^{k-2(d-m)} \max_{|\alpha|,|\beta|\le m} \|A_{\alpha\beta}\|_{W_C^{k+|\alpha|,\infty}(\Omega)} \|u^h\|_{\mathcal{U}^h}$$

$$\times \|v^h\|_{\mathcal{V}^h} \text{ or } \le C h^{k-2(d-m)} \cdot \max_{|\alpha|,|\beta|\le m} \|A_{\alpha\beta}\|_{W_C^{k+|\alpha|,\infty}(\Omega)} \|u^h\|_{\mathcal{W}^h} \|v^h\|_{\mathcal{W}^h}. \quad (10.10)$$

Proof: The above conditions $d \geq (2m-1)2^n + 1 > 2, n \geq 2$, for $\mathcal{S}_d^{2m-1}(\mathcal{T}_c^h) \subset C^{2m-1}(\Omega)$ and $k > 2(d-m)$, imply $k > 2(d-m) \geq 2^{n+1} > n/2,\ n \geq 2,\ k > d$. Hence, by the Sobolev Embedding Theorem the $W^{k,\infty}(\Omega), H^k(\Omega)$ are compactly embedded into $C(\overline{\Omega})$. So all the quadrature formulas (10.4) - (10.6) are well defined. Since the $u^h, v^h \in \mathcal{S}_d^{2m-1}(\mathcal{T}_c^h) \subset H_C^k(\mathcal{T}_c^h)$, the estimates in (10.3) can be applied to (10.4) - (10.6) under the above conditions to different choices of f_c :

$$
\begin{aligned}
|(f_c, v^h) - \tilde{(f_c, v^h)}^h| = |\int_{\Omega_c^h} f_c v^h dx - q^h(f_c v^h)| \ &\leq \ C_k h^k |f_c v^h|_{W_C^{k,1}(\mathcal{T}_c^h)} \\
\leq C_k h^k \|f_c\|_{H^k(\mathcal{T}_c^h)} \|v^h\|_{H_C^d(\mathcal{T}_c^h)} \ &\leq \ C_k' h^{k-d} \|f_c\|_{H_C^k(\mathcal{T}_c^h)} \|v^h\|_{\mathcal{V}^h}.
\end{aligned}
$$

The last inequalities follows by the discrete Hölder inequality and $\|f_c v^h\|_{L^1(\Omega_c^h)} \leq \|v^h\|_{\mathcal{V}^h}$ $\|f_c\|_{\mathcal{V}_c}$ and the inverse estimates (6.4) for the piecewise polynomials v^h of degree $\leq d$ with $d \geq 5$. (10.8) an obvious consequence of (10.7). Analogously, we get for the following (10.11) - (10.12), again for $k > d$ and with the inverse estimates (6.4)

$$
\begin{aligned}
|(G(u^h), v^h)_{\mathcal{V}_c} - \tilde{(G(u^h), v^h)}^h| &\leq C h^k \|G(u^h)\|_{H_C^k(\mathcal{T}_c^h)} \|v^h\|_{H_C^d(\mathcal{T}_c^h)} \\
&\leq C_k h^{k-d} \|G(u^h)\|_{H_C^k(\mathcal{T}_c^h)} \|v^h\|_{\mathcal{V}^h}.
\end{aligned}
\tag{10.11}
$$

To obtain (10.10) we estimate with (10.5) and the discrete Hölder inequality

$$
\begin{aligned}
&|\partial^\alpha(A_{\alpha\beta}\partial^\beta u^h) v^h|_{W^{k,1}(T)} \ \leq C \|A_{\alpha\beta}\|_{W^{k+|\alpha|,\infty}(T)} \|v^h\|_{H^d(T)} \\
&\qquad \big(\sum_{\gamma \leq \alpha} \|\partial^\gamma \partial^\beta u^h\|_{H^{d-|\gamma|-|\beta|}(T)}\big), \text{ hence,} \\
&\big|\big[\int_{\Omega_c^h} \big(\partial^\alpha(A_{\alpha\beta}\partial^\beta u^h) v^h\big) dx\big] - q^h\big(\partial^\alpha(A_{\alpha\beta}\partial^\beta u^h) v^h\big)\big| \\
&\quad \leq C_k h^k \|A_{\alpha\beta}\|_{W_C^{k+|\alpha|,\infty}(\mathcal{T}_c^h)} \sum_{T \in \mathcal{T}_c^h} \big(\|v^h\|_{H^d(T)} \sum_{\gamma \leq \alpha} \|\partial^\gamma \partial^\beta u^h\|_{H^{d-|\gamma|-|\beta|}(T)}\big) \\
&\quad \leq C_k h^k \|A_{\alpha\beta}\|_{W_C^{k+|\alpha|,\infty}(\mathcal{T}_c^h)} \|v^h\|_{H_C^d(\mathcal{T}_c^h)} \ \big(\sum_{\gamma \leq \alpha} \|\partial^\gamma \partial^\beta u^h\|_{H_C^{d-|\gamma|-|\beta|}(\mathcal{T}_c^h)}\big) \\
&\quad \leq C_k h^{k-d} \|A_{\alpha\beta}\|_{W_C^{k+|\alpha|,\infty}(\mathcal{T}_c^h)} \|v^h\|_{\mathcal{V}^h} \qquad\qquad (10.12) \\
&\qquad\qquad h^{-(d-|\alpha|-|\beta|)} \big[\sum_{j=0}^{|\alpha|+|\beta|} \big(\sum_{|\gamma| = |\alpha|-j} \|\partial^\gamma \partial^\beta u^h\|_{H_C^{d-|\alpha|-|\beta|+j}(\mathcal{T}_c^h)}\big)\big] \\
&\quad \leq C_k h^{k-(2d-|\alpha|-|\beta|)} \|A_{\alpha\beta}\|_{W^{k+|\alpha|,\infty}(\Omega)} \|u^h\|_{\mathcal{U}^h} \|v^h\|_{\mathcal{V}^h} \text{ where}
\end{aligned}
$$

the last inequalities are obtained by applying $0 < h < 1, |\alpha| + |\beta| \leq 2m$, and the inverse estimates (6.4) to the piecewise polynomials u^h, v^h of degree $\leq d$ with $d > 5$, to $\partial^\gamma \partial^\beta u^h$ for $|\gamma| = |\alpha| - j$ of degree $d - |\alpha| - |\beta| + j$. A final $\sum_{|\beta|,|\alpha| \leq m}$ yields the claim in the first line of (10.10). The second line is obtained by aiming in (10.12) for $\|v^h\|_{\mathcal{W}^h}, \|u^h\|_{\mathcal{W}^h}$ instead of the $\|v^h\|_{\mathcal{V}^h}, \|u^h\|_{\mathcal{U}^h}$ there. Then we loose and win h^m in each factor. □

Before proving the final Theorem, we introduce the quadrature projectors and the FEM, based upon $\tilde{Q}_c^h$ in (10.4) and $\tilde{G}^h$ in (10.6):

$$\tilde{Q}_{c,\Pi}^h := (\tilde{Q}_c^h, Q_b^h) : \mathcal{V}_c \times \mathcal{V}_{c,D} \to \mathcal{V}_\Pi^{h'}, \tilde{F}^h : \mathcal{U}^h \to \mathcal{V}_\Pi^{h'},\ \tilde{F}^h(\tilde{u}_0^h) := \tilde{Q}_{c,\Pi}^h F|_{\mathcal{U}^h}(\tilde{u}_0^h) = 0. \quad (10.13)$$

Theorem 10.2 *Under the conditions in Theorems 9.5 and 10.1, in particular $k > 2(d-m)$, and boundedly invertible $F'(u_0)$, the quadrature approximate FEM $\tilde{F}^h : \mathcal{U}^h \to \mathcal{V}_\Pi^{h'}$, $\tilde{F}^h(\tilde{u}_0^h) = 0$ in (10.13) has a unique solution $\tilde{u}_0^h$, is stable and consistent in $P^h u_0$, hence convergent, s.t., see (8.2),(9.45), (10.11),*

$$\|E_c u_0 - \tilde{u}_0^h\|_{\mathcal{U}^h} \le Ch^{\min\{\ell-2m,p,k-d\}} \|u_0\|_{\in H^{\min\{\ell,k+2m\}}(\Omega)}. \quad (10.14)$$

Remark 10.1 *The combination of the two inequalities for k and d require large values for k, e.g., $k > 8$ for $m = 1, n = 2$ and $k > 18$ for $m = 1, n = 3$. If the stability of $\tilde{G}^h$ is proved via inverse estimates, as in (10.12), this is unavoidable. Otherwise, the quadrature formulas could be applied to subtriangles of the T and k could be reduced.*

Proof: We show that the $a_s(\cdot,\cdot)$ and $\tilde{a}_s^h(\cdot,\cdot)$ are simultaneously bounded and, for small enough h, satisfy the discrete inf–sup–condition on $\mathcal{U}^h \times \mathcal{V}^h$. (It is straight forward generalizing Hackbusch's proof for his Lemma 6.5.3, [33], from his $A \in \mathcal{L}(\mathcal{U},\mathcal{U}')$ form to our setting $A_\mathcal{U}^h \in \mathcal{L}(\mathcal{U}^h, \mathcal{V}_\Pi^{h'})$.) By Theorem 9.4 the boundedly invertible $A_\mathcal{U} = F'(u_0)$ implies the equibounded invertability (stability) of $A_\mathcal{U}^h : \mathcal{U}^h \to \mathcal{V}_\Pi^{h'}$. This implies the stability of the corresponding $(\tilde{F}^h)'(u_0) : \mathcal{U}^h \to \mathcal{V}_\Pi^{h'}$, and hence of $\tilde{F}^h : \mathcal{D}^h \subset \mathcal{U}^h \to \mathcal{V}_\Pi^{h'}$ in $P^h u_0$. In fact, the discrete inf–sup–condition for $a_s(\cdot,\cdot)$ is correct by Lemma 9.1. We can assume and obtain with (10.10) and $k > 2(d-m)$

$$\begin{aligned}
\epsilon'\|u^h\|_{\mathcal{U}^h} &\le \inf_{0\ne v^h\in\mathcal{V}^h} |a_s(u^h,v^h)|/\|v^h\|_\mathcal{V}^h \quad \forall u^h \in \mathcal{U}^h \\
&= \inf_{0\ne v^h\in\mathcal{V}^h} |\tilde{a}_s^h(u^h,v^h) + (a_s(u^h,v^h) - \tilde{a}_s^h(u^h,v^h))|/\|v^h\|_\mathcal{V}^h \\
&\le \inf_{0\ne v^h\in\mathcal{V}^h} |\tilde{a}_s^h(u^h,v^h)|/\|v^h\|_\mathcal{V}^h \\
&\quad + \sup_{0\ne v^h\in\mathcal{V}^h} |(a_s(u^h,v^h) - \tilde{a}_s^h(u^h,v^h))|/\|v^h\|_\mathcal{V}^h \\
&\le \inf_{0\ne v^h\in\mathcal{V}^h} |\tilde{a}_s^h(u^h,v^h)|/\|v^h\|_\mathcal{V}^h + Ch\|u^h\|_{\mathcal{U}^h},
\end{aligned}$$

so finally

$$(\epsilon' - Ch)\|u^h\|_{\mathcal{U}^h} \le \inf_{0\ne v^h\in\mathcal{V}^h} |\tilde{a}_s^h(u^h,v^h)|/\|v^h\|_\mathcal{V}^h.$$

Analogously the inequalities for the $\inf_{0\ne u^h\in\mathcal{U}^h}$ are handled. This shows the discrete inf–sup–condition for $\tilde{a}^h(\cdot,\cdot)$ for sufficiently small $h > 0$.

The classical consistency error, see (7.22), is combined with the preceding (10.9) yielding with $k > d$

$$\begin{aligned}
\|\tilde{F}^h P^h u_0 - \tilde{Q}_\Pi^h F u_0\|_{\mathcal{V}_\Pi^{h'}} &= \|\big(\tilde{Q}_c^h G(P^h u_0), -Q_b^h B_D u_0\big)\|_{\mathcal{V}_\Pi^{h'}} \\
\le \|\big((\tilde{Q}_c^h G(P^h u_0) - Q_c^h G P^h u_0) &+ Q_c^h G P^h u_0, -Q_b^h B_D u_0\big)\|_{\mathcal{V}_\Pi^{h'}} \\
&\le Ch^{k-d}\|u_0\|_{H^{k+2m}(\Omega} + CLh^{\min\{\ell-2m,p\}}\|u_0\|_{H^\ell(\Omega}. \\
&\le C'h^{\min\{\ell-2m,p,k-d\}}\|u_0\|_{H^{\min\{\ell,k+2m\}}(\Omega)}.
\end{aligned}$$

and hence, the claimed convergence.

We finally discuss the implications of the two inequalities for k and d. We consider $m \geq 1$ and obtain

$$\begin{aligned} d &\geq (2m-1)2^n + 1 \text{ to guarantee } \mathcal{S}_d^{2m-1}(\mathcal{T}_c^h) \subset C^{(2m-1)}(\Omega) \text{ and} && (10.15)\\ k &\geq 2(d-m)+1, m \geq 1 \Longrightarrow \text{ inf–sup–condition for } \tilde{a}_s^h(u^h, v^h), && (10.16)\\ &\text{consequently } k-d \overset{(10.16)}{\geq} d-2m+1 \overset{(10.15)}{>} 0. && (10.17)\end{aligned}$$

This indeed requires the large values for k mentioned above. □

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