Besov Regularity for the Stokes and the Navier-Stokes System in Polyhedral Domains

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22.04.2013

Abstract

In this paper we study the regularity of solutions to the Stokes and the Navier-Stokes system in polyhedral domains contained in $\mathbb{R}^3$. We consider the scale $B^s_\tau(L_\tau)$, $1/\tau = s/3 + 1/2$ of Besov spaces which determines the approximation order of adaptive numerical wavelet schemes and other nonlinear approximation methods. We show that the regularity in this scale is large enough to justify the use of adaptive methods. The proofs of the main results are performed by combining regularity results in weighted Sobolev spaces with characterizations of Besov spaces by wavelet expansions.

Subject Classification: 30H25, 35B65, 42C40, 46E35, 65T60, 76D07

Keywords: Stokes system, Besov spaces, weighted Sobolev spaces, wavelets, characterization of function spaces, nonlinear and adaptive approximation.

1 Introduction

In this paper we are concerned with the $3D$-Navier-Stokes system

$$-\nu\Delta u + \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} + \nabla p = f \quad \text{in } \mathcal{G}$$
$$\text{div } u = g \quad \text{in } \mathcal{G}$$

$$u = 0 \text{ on } \Gamma_j,\ j = 1, ..., N$$

and the $3D$-Stokes system

$$\begin{aligned} -\Delta u + \nabla p &= f \text{ in } \mathcal{G} \\ \text{div } u &= g \text{ in } \mathcal{G} \\ u &= 0 \text{ on } \Gamma_j,\ j = 1, ..., N \end{aligned}$$

on a polyhedral domain $\mathcal{G} \subset \mathbb{R}^3$ where Γ_j are the faces of the domain. The Navier-Stokes equations and its linearized version, the Stokes equations, describe the motion of a viscous fluid. Here $\Delta := \sum_{k=1}^{3} \frac{\partial^2}{\partial^2 x_i}$ is the Laplace operator and by $\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)^T$ we denote the gradient. As usual, $u(\cdot) = (u_1(\cdot), u_2(\cdot), u_3(\cdot))$ denotes the velocity field and p stands for the pressure field. Our aim is to prove regularity results for each component of the solution (u, p) in the specific scale of Besov spaces $B^s_\tau(L_\tau(\mathcal{G}))$, $1/\tau = s/3 + 1/2$. This specific scale comes into play when studying the convergence rate of adaptive numerical shemes. We will explain the relationship very briefly in the following. Let us for the sake of simplicity assume $g = 0$, then the weak formulation of the Stokes problem is given by

$$\begin{gathered} a(u, v) + b(p, v) = f(v) \text{ for all } v \in H^1_0(\mathcal{G})^3, \\ b(q, u) = 0 \text{ for all } q \in L_{2,0}(\mathcal{G}) \end{gathered}$$

with

$$a(u, v) := \int_{\mathcal{G}} \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial v_j}{\partial x_j} \mathrm{dx},$$

$$b(p, v) := -\int_{\mathcal{G}} p(x)(\mathrm{div} v)(x)\mathrm{d}x$$

and

$$f(v) := \int_{\mathcal{G}} \langle f, v \rangle \,\mathrm{d}x.$$

$H^1_0(\mathcal{G})$ is the closure of $\mathcal{C}^\infty_0(\mathcal{G})$ with respect to the $H^1(\mathcal{G})$-Sobolev norm and $L_{2,0}(\mathcal{G}) := \{p \in L_2(\mathcal{G}) : \int_{\mathcal{G}} p(x)\mathrm{d}x = 0\}$. To treat the equation numerically we use the Galerkin approach, i.e. we consider a nested sequence $\{S_j \times \tilde{S}_j\}_{j \geq 0}$ of finite dimensional linear subspaces of $H^1_0(\mathcal{G})^3 \times L_{2,0}(\mathcal{G})$ such that the union is dense in $H^1_0(\mathcal{G})^3 \times L_{2,0}(\mathcal{G})$. This leads to the problems

$$a(u_j, v) + b(p_j, v) = f(v) \text{ for all } v \in S_j,$$

$$b(q, u_j) = 0 \text{ for all } q \in \tilde{S}_j.$$

In many cases, the approximation spaces S_j and $\tilde{S}_j$ are constructed by means of a uniform grid refinement strategy. This kind of approximation is called *linear approximation*. It is well-known that the performance usually depends on the Sobolev regularity of the solution. For details we refer to [3],[9],[12] and [13]. However, in practice, due to singularities at the boundary of the domain, this Sobolev regularity might not be very high and therefore the approximation rate of uniform schemes drops down. In this setting, the use of adaptive strategies seems to be reasonable. Roughly speaking, an adaptive scheme corresponds to nonuniform grid refinement where the underlaying space is only refined in regions where the current approximation is still far away from the exact solution. In this paper we are in particular interested in adaptive wavelet algorithms. In this setting, an adaptive scheme can be interpreted as a nonlinear approximation scheme, and for that reason best n-term approximation serves as a benchmark for adaptive strategies (see [1],[3] for further information): Instead of linear spaces one uses the nonlinear manifold $\mathcal{M}_n$ of all functions

$$S = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda, \quad |\Lambda| \leq n,$$

where $\{\psi_\lambda\}_{\lambda \in \mathcal{J}}$ is a suitable wavelet basis. We define the approximation error

$$\sigma_n(u)_{L_2} := \inf_{S \in \mathcal{M}_n} ||u - S||_{L_2}.$$

In contrast to linear approximation schemes, the order of convergence for best n-term wavelet approximation does not depend on the Sobolev regularity, but on the Besov smoothness, i.e.

$$\sum_{n=1}^{\infty} [n^{s/d} \sigma_n(u)_{L_2}]^\tau \frac{1}{n} < \infty \Longleftrightarrow u \in B^s_\tau(L_\tau), \ 1/\tau = s/d + 1/2,$$

see [9], [10] for further details. As suggested above this shows that it is profitable to use adaptive shemes if the Besov regularity of the solution in this specific scale is higher than the Sobolev regularity. It is known that in smooth domains the Sobolev regularity of the solution increases if the Sobolev regularity of f and g increase (see e.g. [13] for details for the Stokes system). If the domain is only Lipschitz, this conclusion is no longer true due to singularities at the boundary (see Proposition 5.1), but there is some hope that these singularities do not influence the Besov smoothness in the scale $1/\tau = s/d + 1/2$. Indeed there are already some positive results in this direction for a large class of partial differential equations: In [2] it was shown that the Besov regularity of the 2D-Stokes system in a polygonal domain is under some technical conditions higher than the Sobolev regularity. In [4] the Besov regularity of the solution to the Dirichlet problem for harmonic functions and for the Poisson equation in Lipschitz domains was investigated. A result which is similiar to our main statement was proven in [5] for Poisson equation. In many cases these results are proven by using the caracterization of Besov spaces by means of weighted sequence norms of coefficients related to the wavelet decomposition of the solution. Similiar to the investigation in [5] we estimate the wavelet coefficient of the solution by exploiting regularity results related to weighted Sobolev spaces introduced by Maz'ya and Rossmann (see [16]). Furthermore there are also results for nonlinear partial differential equations, see [6]. In this paper we consider the Navier-Stokes system and the Stokes system on a polyhedral domain where singularities at the vertices and on the edges might occur. To prove regularity results we need certain weighted Sobolev spaces which take these singularities into account. We denote these spaces by $W^{l,2}_{\vec{\beta},\vec{\delta}}$, for details see Section 2 and Section 3. In this paper we establish a result which shows that under certain technical conditions the Besov regularity to the solution of the Navier-Stokes respectively the Stokes problem is higher than the Sobolev regularity if additionally the parameter l is not so small: For suitable values of l the Besov regularity is at least 3/2 times higher than the Sobolev regularity. For details, we refer to Theorem 2.1, Theorem

3.1 and Theorem 3.2, respectively.

This paper is organized as follows: In the second section we state and prove a result for the Navier-Stokes system on a polyhedral domain. In the third section we show analog results for the Stokes system. As mentioned above we use weighted Sobolev estimates. In Section 4 we discuss some norm estimates for the solution of the considered Navier-Stokes and Stokes equations. In Appendix 5, we discuss the Sobolev regulartiy and results for weighted Sobolev regulartiy of the solution as far as they are needed for our purposes. In the last section we recall the definition of Besov and Sobolev spaces and explain the connection between the Besov regularity of a distribution and the decay of its wavelet coefficients.

2 Besov Regularity for the Navier-Stokes System in Polyhedral Domains

In this section we state and prove the main result of this paper: We will show that under some technical assumptions the Besov regularity of the solution to

$$\begin{aligned} -\nu\Delta u + \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} + \nabla p = f \quad &\text{in } \mathcal{G} \\ \text{div } u = g \quad &\text{in } \mathcal{G} \\ u = 0 \quad &\text{on } \Gamma_j,\ j = 1, ..., N \end{aligned} \tag{2.1}$$

in the scale $1/\tau = s/3 + 1/2$ is 3/2 times higher than its Sobolev regularity. We consider the Navier-Stokes equation on polyhedral domains. The basic type of a polyhedral domain is a polyhedral cone: Let

$$\mathcal{K} = \{x \in \mathbb{R}^3 : x = \rho(x) \cdot \omega(x),\ 0 < \rho(x) < \infty, \omega(x) \in \Omega\} \tag{2.2}$$

be a polyhedral cone with vertex at the origin where Ω is a curvilinear polygon on the unit sphere bounded by the arcs $\gamma_1, ..., \gamma_d$. Suppose that the boundary $\partial\mathcal{K}$ consists of the vertex $x = 0$, the edges $M_1, ..., M_d$ and the faces $\Gamma_j := \{x : x/|x| \in \gamma_j\}$, $j = 1, ..., d$. The angle at edge M_j will be denoted by θ_j. Furthermore we define for $x \in \mathcal{K}$ the function $r_j(x) := \text{dist}(x, M_j)$. By $\mathcal{K}_0$ we denote an arbitrary truncated cone, i.e. there exists a positive real number r_0 such that

$$\mathcal{K}_0 = \{x \in \mathcal{K} : |x| < r_0\}.$$

Our technique requires regularity assertions in weighted Sobolev spaces. Following Maz'ya and Rossmann we define these spaces for cones (see [16] for details): Let l be a nonnegative integer, $\beta \in \mathbb{R}$ and $\vec{\delta} = (\delta_1, ..., \delta_d) \in \mathbb{R}^d$, $\delta_j > -1$ for $j = 1, ..., d$. We define the space $W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})$ as the closure of the set $\mathcal{C}_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$ with respect to the norm

$$||u||_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})} := \left(\int_{\mathcal{K}} \sum_{|\alpha|\le l} \rho(x)^{2(\beta-l+|\alpha|)} \prod_{k=1}^{d} \left(\frac{r_k(x)}{\rho(x)}\right)^{2\delta_k} |D^\alpha u(x)|^2 \mathrm{d}x\right)^{1/2}.$$

General polyhedral domains are usually defined by means of diffeomorphism which maps the domain local to a polyhedral cone (see [16, Chapter 8.1] for details):

(i) The boundary $\partial\mathcal{G}$ consists of smooth open two-dimensional manifolds Γ_j ($j = 1, ..., N$), smooth curves M_k ($k = 1, ..., d$) and vertices $x^{(1)}, ..., x^{(d')}$.

(ii) For every $\xi \in M_k$ there exist a neighborhood U_ξ and a diffeomorphism κ_ξ wich maps $\mathcal{G} \cap U_\xi$ onto $D_\xi \cap B_1$ where D_ξ is a dihedron and B_1 is the unit ball.

(iii) For every vertex $x^{(i)}$ there exist a neighborhood U_i and a diffeomorphism κ_i mapping $\mathcal{G} \cap U_i$ onto $\mathcal{K}_i \cap B_1$ where $\mathcal{K}_i$ is a polyhedral cone with vertex at the origin.

We will restrict ourselves to the case of

$$\kappa_j : \mathcal{G} \cap U_j \to \mathcal{K}_j \cap B_j, x \mapsto A_j x + b,$$

where $A_j \in SO(3)$ and b is a vector in $\mathbb{R}^3$ independent of x.

Now we recall the definition of weighted Sobolev spaces corresponding to polyhedral domains (see again [16] for details). We put

$$r_k(x) := \mathrm{dist}(x, M_k),\ k = 1, ..., d,$$

$$\rho_j(x) := \mathrm{dist}(x, x^{(j)}).\ j = 1, ..., d'.$$

With X_j we denote the set of indices k such that $x^{(j)}$ is an end point of the edge M_k. Let $U_1, ..., U_{d'}$ be domains in $\mathbb{R}^3$ such that

$$U_1 \cup ... \cup U_{d'} \supset \overline{G} \text{ and } \overline{U_j} \cap \overline{M_k} = \emptyset \text{ if } k \notin X_j.$$

For $l \in \mathbb{N}_0$, $\beta = (\beta_1, ..., \beta_{d'}) \in \mathbb{R}^{d'}$ and $\delta := (\delta_1, ..., \delta_d) \in \mathbb{R}^d$ with $\delta_k > -1$ for $k = 1, ..., d$ we define the weighted Sobolev space $W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$ to be the closure of the set $\mathcal{C}_0^\infty(\overline{\mathcal{G}} \backslash \{x^{(1)}, ..., x^{(d')}\})$ with respect to the norm

$$||u||_{W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})} = \left(\sum_{j=1}^{d'} \int_{G \cap U_j} \sum_{|\alpha| \le l} \rho_j(x)^{2(\beta_j - l + |\alpha|)} \prod_{k \in X_j} \left(\frac{r_k(x)}{\rho_j(x)} \right)^{2\delta_k} |D^\alpha u(x)|^2 \mathrm{d}x \right)^{1/2}.$$

In our case we consider polyhedral domains for which we can find a partition of unity $\{\sigma_j\}_{j=1}^{d'}$ related to the domain decomposition $\mathcal{G} = \bigcup_{j=1}^{d'} \mathcal{G} \cap U_j$ which fulfills

$$||\sigma_j v||_{B^p_s(L_p(G \cap U_j))} \lesssim ||v||_{B^p_s(L_p(G))},\ \ 1/p = s/3 + 1/2, \tag{2.3}$$

uniformely for all $v \in B^p_s(L_p(G))$. In many cases this condition is fulfilled. For example investigations for the L-shaped domain can be found in [19, Section 4.2]. Now we can formulate and prove the following result:

Theorem 2.1. *It exists a countable set $E \subset \mathbb{C}$ such that for all $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ with*

$$\beta^* := \max_{j=1,\dots,d'} \beta_j < 1,$$

$$Re\ \lambda \neq 1/2 - \beta_j \quad for\ all\ \lambda \in E \tag{2.4}$$

and

$$\max\left(0, 1 - \frac{\pi}{\theta_k}\right) < \delta_k < 1, \ \ k = 1, ..., d,$$

the following holds: If $(f, g) \in W^{0,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$, *g fulfills the compatibility condition*

$$g|_{M_k} = 0, \ \ k = 1, ..., d$$

and if a solution (u, p) *of (2.1) is contained in* $H^{s_0}(\mathcal{G})^3 \times H^{t_0}(\mathcal{G})$ *then*

$$u \in \left[B^{s_1}_{\tau}(L_{\tau_1}(\mathcal{G}))\right]^3, \ \ \frac{1}{\tau_1} = \frac{s_1}{3} + \frac{1}{2}, \ \ s_1 < \min\left(2, 3/2 \cdot s_0, 3 \cdot (2 - |\vec{\delta}|)\right), \tag{2.5}$$

$$p \in B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0)), \ \ \frac{1}{\tau_2} = \frac{s_2}{3} + \frac{1}{2}, \ \ s_2 < \min\left(1, 3/2 \cdot t_0, 3 \cdot (1 - |\vec{\delta}|)\right) \tag{2.6}$$

Proof. : To prove the theorem we will study each component of the solution $(u, p) = (u_1, u_2, u_3, p)$ to (2.1) separatly. Let v be one of the functions u_1, u_2, u_3 or p, respectively. Moreover we define

$$\mu := \begin{cases} 2 & v = u_i \text{ for } i = 1, 2 \text{ or } 3 \\ 1 & v = p \end{cases}$$

and

$$\alpha := \begin{cases} s_0 & v = u_i \text{ for } i = 1, 2 \text{ or } 3 \\ t_0 & v = p \end{cases}. \tag{2.7}$$

From Proposition 5.6 we obtain $v \in W^{\mu,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$. Using the transformation $\kappa_j = A_j(\cdot) + b$ introduced in the beginning of this section we define the function

$$v_j := v \circ \kappa_j^{-1} : \mathcal{K}_j \cap B_j \to \mathbb{R}.$$

For the sake of notation simplicity we denote v_j by v, $\mathcal{K}_j$ by $\mathcal{K}$ and $\mathcal{K}_j \cap B_j$ by $\mathcal{K}_0$. Using the transformation formular and the fact that the Matrix A_j defining κ_j is an element of $SO(3)$ we obtain

$$\left(\int_{\mathcal{K}_0} \sum_{|\alpha| \leq l} \rho(x)^{2(\beta - l + |\alpha|)} \prod_{k=1}^{d} \left(\frac{r_k(x)}{\rho(x)}\right)^{2\delta_k} |D^\alpha v(x)|^2 \mathrm{d}x\right)^{1/2} < \infty \text{ and } v \in H^\alpha(\mathcal{K}_0), \tag{2.8}$$

with the abbreviation $\beta := \beta_j$. The proof uses the characterizations of Besov spaces by wavelet expansions. Therefore we estimate the wavelet coefficients of v in order to show that the equivalent quasi-norm as outlined in Proposition 6.1 is bounded. We make the following agreements concerning the wavelet characterization of Besov spaces on $\mathbb{R}^3$: For the sake of simplicity we associate to each dyadic cube $I := 2^{-j}k + 2^{-j}[0, 1]^3$ the functions

$$\eta_I := \tilde{\psi}_{i,j,k}, \ \ j \in \mathbb{N}_0, \ \ k \in \mathbb{Z}^3, \ i = 1, ..., 7,$$

see Section 6.3 for details. Note that we disregard the dependence on i. By η^*_I we denote the corresponding element of the primal basis. Because the supports of the wavelets are assumed to be compact there exists a cube Q centered at the origin such that

$$Q(I) := 2^{-j}k + 2^{-j}Q$$

contains the support of η_I and η_I^* for all I. We will prove the result in three steps: In a first step we will estimate the coefficients $|\langle v, \eta_I\rangle|$ for with $Q(I)$ is contained in the truncated cone and the distance from $Q(I)$ to the origin is not to small. We will specifiy this later. In a second step we look for the coefficients for which $Q(I)$ is contained in $\mathcal{K}_0$ but $Q(I)$ can be located arbitrarily close zu the origin. In the last step we consider the coefficients for which the intersection of $Q(I)$ and the boundary of $\mathcal{K}_0$ is not empty.

step 1: We start by estimating the coefficients $|\langle v, \eta_I\rangle|$ with $Q(I) \subset \mathcal{K}_0$. We put

$$\rho_I := \mathrm{dist}(Q(I), 0)$$

and

$$r_I := \min_{j=1,\ldots,d} \min_{x \in Q(I)} r_j(x).$$

For $j \in \mathbb{N}_0$ consider the set of indices:

$$\Lambda_j := \{I : Q(I) \subset \mathcal{K}_0,\ 2^{-3j} \leq |I| \leq 2^{-3j+2}\}.$$

Then we define a subset of Λ_j for $k \in \mathbb{N}$:

$$\Lambda_{j,k} := \{I \in \Lambda_j : k2^{-j} \leq \rho_I < (k+1)2^{-j}\}.$$

Further we put for $m \in \mathbb{N}$

$$\Lambda_{j,k,m} := \{I \in \Lambda_{j,k} : m2^{-j} \leq r_I < (m+1)2^{-j}\}.$$

We observe the following facts:

- There exists a general number C such that

$$\Lambda_{j,k} = \emptyset, k > C2^j. \tag{2.9}$$

- For the cardinality $|\Lambda_{j,k}|$ of $\Lambda_{j,k}$ holds

$$|\Lambda_{j,k}| \lesssim k^2,\ \ k \in \mathbb{N}. \tag{2.10}$$

- It holds

$$|\Lambda_{j,k,m}| \lesssim m,\ \ m \in \mathbb{N}. \tag{2.11}$$

In every case the constant is independent of j, k and m. Recall that

$$|v|_{W^\mu(L_2(Q(I)))} := \left(\int_{Q(I)} |\nabla^\mu v(x)|^2 \mathrm{d}x\right)^{1/2},$$

which is well defined because of (2.8). The vector space of polynomials of order at most μ is finite dimensional so there exists a polynomial P_I such that

$$||v - P_I||_{L_2(Q(I))} = \inf\left\{||v - P||_{L_2(Q(I))} : P \text{ is a polynomial of degree } \leq \mu\right\}.$$

The vanishing moment property of wavelets, see Subsection 6.3, Hölder's inequality and a classical Whitney-estimate (see [11, Theorem 3.4]) lead to

$$
\begin{aligned}
|\langle v,\eta_I\rangle| &\le ||v-P_I||_{L_2(Q(I))}||\eta_I||_{L_2(Q(I))} \\
&\lesssim |I|^{\mu/3}\cdot |v|_{W^\mu(L_2(Q(I)))}.
\end{aligned}
$$

For $I\in\Lambda_j$ we obtain

$$
|\langle v,\eta_I\rangle| \lesssim 2^{-\mu j}|v|_{W^\mu(L_2(Q(I)))}.
$$

Let $0<\tau<2$. Summing up over $I\in\Lambda_{j,k}$ yields

$$
\begin{aligned}
\sum_{I\in\Lambda_{j,k}} |\langle v,\eta_I\rangle|^\tau &\lesssim \sum_{I\in\Lambda_{j,k}} 2^{-\mu j\tau}\left(\int_{Q(I)}|\nabla^\mu v(x)|^2\mathrm{d}x\right)^{\tau/2} \\
&\lesssim \sum_{I\in\Lambda_{j,k}} 2^{-\mu j\tau} r_I^{-\tau|\vec{\delta}|}\rho_I^{-\tau(\beta-|\vec{\delta}|)}\left(\int_{Q(I)}\rho^{2(\beta-|\vec{\delta}|)}\left(\prod_{\nu=1}^d r_\nu^{\delta_\nu}\right)^2|\nabla^\mu v(x)|^2\mathrm{d}x\right)^{\tau/2}.
\end{aligned}
$$

We define

$$
v_I := \int_{Q(I)}\rho^{2(\beta-|\vec{\delta}|)}\left(\prod_{\nu=1}^d r_\nu^{\delta_\nu}\right)^2|\nabla^\mu v(x)|^2\mathrm{d}x.
$$

Now we focus on the coefficients belonging to $\Lambda_{j,k,m}$. We now have to consider the cases $\beta>|\vec{\delta}|$ and $|\vec{\delta}|\ge\beta$ separatly. If $\beta-|\vec{\delta}|>0$ we can conclude $\rho_I^{-\tau(\beta-|\vec{\delta}|)}\lesssim(k2^{-j})^{-\tau(\beta-|\vec{\delta}|)}$. Otherwise we get $\rho_I^{-\tau(\beta-|\vec{\delta}|)}\lesssim((k+1)2^{-j})^{-\tau(\beta-|\vec{\delta}|)}$. We will only discuss the case $\beta>|\vec{\delta}|$ in detail. The second case can be treated analogously. Using Hölder's inequality with $q=2/\tau$, $q'=2/(2-\tau)$ results in

$$
\begin{aligned}
\sum_{I\in\Lambda_{j,k,m}} |\langle v,\eta_I\rangle|^\tau &\lesssim 2^{-\mu\tau j}(k2^{-j})^{-\tau(\beta-|\vec{\delta}|)}\left(\sum_{I\in\Lambda_{j,k,m}} r_I^{-\tau|\vec{\delta}|\frac{2}{2-\tau}}\right)^{\frac{2-\tau}{2}}\cdot\left(\sum_{I\in\Lambda_{j,k,m}} v_I\right)^{\frac{\tau}{2}} \\
&\lesssim 2^{-\mu\tau j}(k2^{-j})^{-\tau(\beta-|\vec{\delta}|)}\left(\sum_{I\in\Lambda_{j,k,m}} (m2^{-j})^{-\tau|\vec{\delta}|\frac{2}{2-\tau}}\right)^{\frac{2-\tau}{2}}\left(\sum_{I\in\Lambda_{j,k,m}} v_I\right)^{\frac{\tau}{2}}
\end{aligned}
$$

Together with (2.11) we obtain

$$
\sum_{I\in\Lambda_{j,k,m}} |\langle v,\eta_I\rangle|^\tau \lesssim 2^{\tau j(\beta-\mu)}k^{-\tau(\beta-|\vec{\delta}|)}m^{-\tau|\vec{\delta}|+\frac{2-\tau}{2}}\left(\sum_{I\in\Lambda_{j,k,m}} v_I\right)^{\frac{\tau}{2}}.
$$

We continue by using the fact that there are of order k sets $\Lambda_{j,k,m}$ in each layer $\Lambda_{j,k}$. Together with Hölders inequality, this gives

$$
\sum_{I\in\Lambda_{j,k}} |\langle v,\eta_I\rangle|^\tau \lesssim 2^{j\tau(\beta-\mu)}k^{-\tau(\beta-|\vec{\delta}|)}\left(\sum_{m=1}^{Ck} m^{-\tau|\vec{\delta}|\frac{2}{2-\tau}+1}\right)^{\frac{2-\tau}{2}}\left(\sum_{I\in\Lambda_{j,k}} v_I\right)^{\frac{\tau}{2}}. \tag{2.12}
$$

Note, that the constant C only depends on $\mathcal{K}_0$. Together with

$$\left(\sum_{m=1}^{Ck} m^{-\tau|\vec{\delta}|\frac{2}{2-\tau}+1}\right)^{\frac{2-\tau}{2}} \lesssim \begin{cases} k^{-\tau|\vec{\delta}|+2-\tau} & 2 > \tau(1+|\vec{\delta}|), \\ (\log(1+k))^{\frac{2-\tau}{2}} & 2 = \tau(1+|\vec{\delta}|), \\ 1 & 2 < \tau(1+|\vec{\delta}|). \end{cases}$$

we obtain from (2.12)

$$\sum_{I\in\Lambda_{j,k}} |\langle v, \eta_I\rangle|^\tau \lesssim 2^{j\tau(\beta-\mu)} \left(\sum_{I\in\Lambda_{j,k}} v_I\right)^{\frac{\tau}{2}} \times \begin{cases} k^{-\tau(\beta+1)+2} & 2 > \tau(1+|\vec{\delta}|), \\ k^{-\tau(\beta-|\vec{\delta}|)}(\log(1+k))^{\frac{2-\tau}{2}} & 2 = \tau(1+|\vec{\delta}|), \\ k^{-\tau(\beta-|\vec{\delta}|)} & 2 < \tau(1+|\vec{\delta}|). \end{cases}$$

To simplify the notation we denote these functions of k in the second line by a_k. Employing (2.9) and Hölder's inequality we get

$$\sum_{I\in\Lambda_j} |\langle v, \eta_I\rangle|^\tau \lesssim 2^{j\tau(\beta-\mu)} \left(\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}\right)^{\frac{2-\tau}{2}} \left(\sum_{I\in\Lambda_j} v_I\right)^{\frac{\tau}{2}}.$$

From (2.8) we conclude that the last factor is bounded. To complete the estimate we have to sum with respect to $j\in\mathbb{N}_0$: We first consider the sum $\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}$ and derive estimates depending on β. Then we study the convergence of

$$\sum_{j\geq 0}\left(2^{j\tau(\beta-\mu)}\left(\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}\right)^{\frac{2}{2-\tau}}\right).$$

More detailed we get the following cases:

$$\begin{aligned}
3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right) &< \mu && \text{if } \tau(1+|\vec{\delta}|) < 2 \quad \text{and} \quad \beta < 3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right), \\
\beta &< \mu && \text{if } \tau(1+|\vec{\delta}|) < 2 \quad \text{and} \quad \beta \geq 3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right), \\
\tfrac{3}{2}|\vec{\delta}| &< \mu && \text{if } \tau(1+|\vec{\delta}|) = 2 \quad \text{and} \quad \beta < \tfrac{3}{2}|\vec{\delta}|, \\
\beta &< \mu && \text{if } \tau(1+|\vec{\delta}|) = 2 \quad \text{and} \quad \beta \geq \tfrac{3}{2}|\vec{\delta}|, \\
\tfrac{1}{\tau}-\tfrac{1}{2} &< \mu - |\vec{\delta}| && \text{if } \tau(1+|\vec{\delta}|) > 2 \quad \text{and} \quad \tfrac{1}{\tau}-\tfrac{1}{2} > \beta - |\vec{\delta}|, \\
\beta &< \mu && \text{if } \tau(1+|\vec{\delta}|) > 2 \quad \text{and} \quad \tfrac{1}{\tau}-\tfrac{1}{2} \leq \beta - |\vec{\delta}|.
\end{aligned}$$

Now we want to derive from these six cases sufficient conditions for $s := 3\left(\frac{1}{\tau}-\frac{1}{2}\right)$ such that

$$v^* := \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}}\sum_{I\in\Lambda_{j,k}} \langle v, \eta_I\rangle\, \eta_I^*$$

belongs to $B^s_\tau(L_\tau(\mathbb{R}^3))$. We find that $\beta < \mu$ is necessary in all six cases. First we consider the case $|\vec{\delta}| < \frac{2}{3}\mu$. If we require

$$s < \mu$$

we can conclude (depending on the value of $\tau(1+|\vec{\delta}|)$) from the first, the third respectively the fifth case the convergence ot the above series if additionaly $s > \beta$ is fulfilled. For the regularity result this is no relevant restriction. Next we look for the case $|\vec{\delta}| \geq \frac{2}{3}\mu$. From the fifth case again we conclude

$$\beta - |\vec{\delta}| < s < 3(\mu - |\vec{\delta}|).$$

Since we have already found $s < \mu$ we actually obtain from $|\vec{\delta}| \geq \frac{2}{3}\mu$ the condition $s < \frac{3}{2}|\vec{\delta}|$. But $|\vec{\delta}| \geq \frac{2}{3}\mu$ implies $3(\mu - |\vec{\delta}|) \leq 3/2|\vec{\delta}|$. Finally we have found the second restriction in (2.5), (2.6).

step 2: In the next step we have to estimate the coefficients in

$$\Lambda_{j,0} := \{I \in \Lambda_j : 0 < \rho_I < 2^{-j}\}.$$

If $\Lambda_{j,0}$ is empty, there is nothing to do. Otherwise we argue as follows. From the Lipschitz character of $\mathcal{K}_0$ follows

$$|\Lambda_{j,0}| \lesssim 2^{2j}, \quad j \in \mathbb{N}_0.$$

Hence for $0 < q < 2$ we obtain together with Hölder's inequality:

$$\sum_{I \in \Lambda_{j,0}} |\langle v, \eta_I \rangle|^q \lesssim 2^{j2(1-q/2)} \left(\sum_{I \in \Lambda_{j,0}} |\langle v, \eta_I \rangle|^2 \right)^{\frac{q}{2}}.$$

Summing up over $j \in \mathbb{N}_0$ yields

$$\begin{aligned} \sum_{j \geq 0} 2^{j(s+3(1/2-1/q))q} \sum_{I \in \Lambda_{j,0}} |\langle v, \eta_I \rangle|^q &\lesssim \sum_{j \geq 0} 2^{j(s+3(1/2-1/q))q} 2^{j(2/q-1)q} \left(\sum_{I \in \Lambda_{j,0}} |\langle v, \eta_I \rangle|^2 \right)^{\frac{q}{2}} \\ &\lesssim ||v||^q_{B_q^{s+\frac{1}{2}-\frac{1}{q}}(L_2(\mathbb{R}^3))} \\ &\lesssim ||v||^q_{B_2^{s+\frac{1}{2}-\frac{1}{q}}(L_2(\mathbb{R}^3))}. \end{aligned}$$

The last step is a consequence of $q < 2$, see [17, Chapter 2.2.1]. We choose s and q such that

$$s := \frac{3\alpha}{2} \quad \text{and} \quad \frac{1}{q} := \frac{s}{3} + \frac{1}{2}, \quad \text{i.e. } s = 3\left(\frac{1}{q} - \frac{1}{2}\right),$$

see (2.7) for the definition of α. We obtain $\alpha = \frac{2}{q} - 1$, i.e. $\alpha > 0$ is insured. Additionally we get $\alpha = s + \frac{1}{2} - \frac{1}{q}$. That means $||v||^q_{B_2^{s+\frac{1}{2}-\frac{1}{q}}(L_2(\mathbb{R}^3))} < \infty$. We get that

$$v^{**} := \sum_{j \geq 0} \sum_{I \in \Lambda_{j,0}} \langle v, \eta_I \rangle \, \eta_I^*$$

belongs to $B^{3/2\alpha}_q(L_q(\mathbb{R}^3))$.

step 3: Finally we have to estimate the coefficients for which the supports of the appendant wavelets intersect with the boundary of the truncated cone. More precisely, we consider the set

$$\Lambda_j^{\#} := \{I |\, Q(I) \cap \partial\mathcal{K}_0 \neq \emptyset,\ 2^{-3j} \leq |I| \leq 2^{-3j+2}\}, \quad j \in \mathbb{N}_0.$$

Since $\mathcal{K}_0$ is a bounded Lipschitz domain there exists a linear and bounded extension operator

$$\mathcal{E} : H^\alpha(\mathcal{K}_0) \to H^\alpha(\mathbb{R}^3).$$

which is simultaneously a bounded operator from $B^s_q(L_p(\mathcal{K}_0))$ to $B^s_q(L_p(\mathbb{R}^3))$ not depending on s, p and q. We refer to [18] for further details. We define

$$v^{\#} := \sum_{j=0}^{\infty} \sum_{I \in \Lambda_j^{\#}} \langle \mathcal{E}v, \eta_I \rangle\, \eta_I^*.$$

We recognize that

$$|\Lambda_j^{\#}| \lesssim 2^{2j}, \quad j \in \mathbb{N}_0.$$

So we can argue as in step 2, this yields:

$$||v^{\#}||^q_{B^{3/2\alpha}_q(L_q(\mathbb{R}^3))} \lesssim ||\mathcal{E}v||^q_{B^\alpha_2(L_2(\mathbb{R}^3))} \lesssim ||v||^q_{B^\alpha_2(L_2(\mathcal{K}_0))} \lesssim ||v||^q_{H^\alpha(\mathcal{K}_0)}.$$

We end with summing up the functions v^*, v^{**} and $v^{\#}$ and obtain a function belonging to $B^s_\tau(L_\tau(\mathbb{R}^3))$ where $s < (\mu, 3/2 \cdot \alpha, 3 \cdot (\mu - |\vec{\delta}|))$ and $1/\tau = s/3 + 1/2$. This shows that $v \in B^s_\tau(L_\tau(\mathcal{K}_0))$.

In order to ensure that a solution has the postulated Besov smoothness in the domain $\mathcal{G}$ we use beside (2.3) the following lemma. We will prove the lemma afterwards.

Lemma 2.2. *Suppose $A \in SO(3)$ and $b \in \mathbb{R}^n$ a fixed vector. Let U be an domain in $\mathbb{R}^d$. Define the functional $F : V \to U, F(x) := Ax + b$ with $V = F^{-1}(U)$. Then, if $u \in B^s_\tau(L_\tau(U))$ the same holds for*

$$v : V \to \mathbb{R},\ x \mapsto (u \circ F)(x),$$

i.e., $v \in B^s_\tau(L_\tau(V))$.

Now we can transform v back to the polyhedral domain and the Besov smoothness persists. This shows the Theorem.

□

We still have to show the statement of Lemma 2.2:

Proof of Lemma 2.2: We consider the Besov norm

$$||v||_{B^s_\tau(L_\tau(U))} = ||v||_{L_\tau(U)} + \left(\int_0^\infty \left[t^{-s} \omega_r(v, t)_{L_\tau(U)} \right]^\tau dt/t \right)^{1/\tau},$$

where $\omega_r(f,t)_{L_\tau(U)}$ is the rth moduli of smoothness and $r = [s]+1$. Using the transformation formular we obtain

$$\begin{aligned}\omega_r(v,t) &= \sup_{|h|\le t} ||(\Delta_h^r v)||_{L_\tau(V_{rh})} \\ &= \det A^{-1} \sup_{|h|\le t} ||(\Delta_{Ah}^r u)||_{L_\tau(U_{r\cdot Ah})},\end{aligned}$$

where V_{rh} denotes the set of all $x \in V$ such that the line segment $[x, x + r\cdot h]$ is contained in V. Since $A \in SO(3)$ we achieve

$$\begin{aligned}\omega_r(v,t) &= \sup_{|h|\le t} ||(\Delta_h^r u)||_{L_\tau(U_{r\cdot h})} \\ &= \omega_r(u,t)\end{aligned}$$

This proves the lemma. □

Remark 2.3. (i) If we consider arbitrary functions (u,p) in $\left[W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{s_0}(\mathcal{K}_0)\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{t_0}(\mathcal{K}_0)$, $l \ge 2$, $\beta \in \mathbb{R}, \beta < l-1$, $\vec{\delta} \in \mathbb{R}^d$ we achieve by applying the arguments in the proof of Theorem 2.1 the estimate

$$\begin{aligned}&||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))} \lesssim \\ &||u||_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})^3} + ||u||_{H^{s_0}(\mathcal{K}_0)^3} + ||p||_{W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})} + ||p||_{H^{t_0}(\mathcal{K}_0)}.\end{aligned}$$

Note that this estimate is true independet of problem (2.1). That means we have a continious imbedding from $\left[W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{s_0}(\mathcal{K}_0)\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{t_0}(\mathcal{K}_0)$ into $B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3 \times B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))$. We will use this imbedding in Section 4 in order to show norm estimates for the solution of the Navier-Stokes and the Stokes system on polyhedral domains.

(ii) It can be shown that the set E in the theorem is the set of eigenvalues of the operator pencil related to (2.1). It is known that E consists of isolated points, see [14],[16] for details. Therefore by a minor modification of $\vec{\beta}$, condition (2.4) is satisfied and our arguments in the proof below also work with this minor modification. That shows that condition (2.4) is not as restrictive as it seems to be.

(iii) Assume that the weight $\vec{\delta} \in \mathbb{R}^d$ is chosen such that $|\vec{\delta}| < 2/3$ then the Besov regularity of u is bounded by the minimum of 2 and $3/2 \cdot s_0$ and the Besov regularity of p is bounded by the minimum of 1 and $3/2 \cdot t_0$. Therefore according to our motivation explained in the introduction the use of adaptive schemes is justified if $s_0 < 4/3$ and $t_0 < 2/3$. These bounds for the Sobolev regularity depend on the smoothness index related to the weighted Sobolev spaces: If Proposition 5.6 held for a higher smoothness index l (as it does for the Stokes problem, see Proposition 5.4) then the use of adaptive schemes would be justified even for higher Sobolev regularity of the solution of (2.1): In this case we get that the Besov smoothness s_1 of u is bounded by

$$\min\left(l, \frac{3}{2}\cdot s_0, 3\cdot(l-|\vec{\delta}|)\right),$$

and the Besov smoothness s_2 of p is bounded by

$$\min\left(l-1, \frac{3}{2}\cdot t_0, 3\cdot(l-(|\vec{\delta}|+1))\right).$$

This can be derived from the arguments in the proof of Theorem 2.1, see also part (i) of this remark.

Proposition 5.5 yields a result about the Sobolev regularity for the solution u of problem (2.1). For p we only know that it is contained in $L_2(\mathcal{G})$ so we can not achieve a result for the Besov regularity. Applying Proposition 5.5 together with Theorem 2.1 we get the following result.

Corollary 2.4. *It exists a countable set $E \subset \mathbb{C}$ such that for all $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ with*

$$\beta^* := \max_{j=1,\dots,d'} \beta_j < 1,$$

$$\mathit{Re}\,\lambda \neq 1/2 - \beta_j \quad \textit{for all } \lambda \in E$$

and

$$\max\left(0, 1-\frac{\pi}{\theta_k}\right) < \delta_k < 1, \;\; k = 1, ..., d,$$

the following holds: If $(f,g) \in \left[W^{0,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})\right]^3 \times W^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap L_2(\mathcal{G})$, g fulfills the compatibility condition

$$g|_{M_k} = 0, \;\; k = 1, ..., d$$

and the functional defined in (5.2) fulfills $F \in \mathcal{H}^*$ and

$$||F||_{\mathcal{H}^*} + ||g||_{L_2(\mathcal{G})}$$

is sufficiently small then a solution (u,p) of problem (2.1) satisfies

$$u \in \left[B^s_\tau(L_\tau(\mathcal{G}))\right]^3, \;\; \frac{1}{\tau} = \frac{s}{3} + \frac{1}{2}, \;\; s < \min\left(3/2, 3\cdot(2-|\vec{\delta}|)\right).$$

Remark 2.5. (i) From Proposition 5.5 we conclude that u is unique on the set of all functions whith H^1-norm less than a certain positive ε and p is unique up to a constant.

(ii) Regarding our explanation in Remark 2.3, (iii) we see since $s_0 = 1 < 4/3$ the use of adaptive wavelet schemes to determine the solution u is justified.

3 Besov Regularity for the Stokes System in Polyhedral Domains

In this section we study the Besov regularity of the stationary Stokes system on polyhedral domains. We start with the investigation for the Stokes system on polyhedral cones and then we use these results to prove analog results for the general case of a polyhedral domain. The Stokes equations on a polyhedral cone are defined by

$$\begin{aligned} -\Delta u + \nabla p &= f \text{ in } \mathcal{K}, \\ \operatorname{div} u &= g \text{ in } \mathcal{K}, \\ u &= 0 \text{ on } \Gamma_j,\ j = 1, ..., d, \end{aligned} \tag{3.1}$$

where $\Gamma_j,\ j = 1, ..., d$ are the faces of the cone. Using the estimate in Remark 2.3, Proposition 5.1 and Proposition 5.2 we immediately achieve the following regularity result:

Theorem 3.1. *Fix an integer* $l \geq 2$ *and a real number* $0 < \alpha_0 < 0.5$. *It exists a countable set* $E \subset \mathbb{C}$ *such that for all* $\beta \in \mathbb{R}$, $\vec{\delta} \in (\mathbb{R}\backslash\mathbb{Z})^d$ *with*

$$\begin{aligned} &\beta < l - 1, \\ &Re\lambda \neq l - \beta - \frac{3}{2} \quad \textit{for all } \lambda \in E \end{aligned} \tag{3.2}$$

and

$$\max\left(0, l - 1 - \frac{\pi}{\theta_k}\right) < \delta_k < l - 1, \quad k = 1, ..., d$$

the following holds: If $(f, g) \in \left[W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap L_2(\mathcal{K}_0)\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{\alpha_0}(\mathcal{K}_0)$ *and* g *fulfills the compatibility condition*

$$\int_{\mathcal{K}_0} g(x)\,dx = 0,$$

then the unique solution (u, p) *of problem (3.1) satisfies*

$$u \in \left[B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))\right]^3, \quad \frac{1}{\tau_1} = \frac{s_1}{3} + \frac{1}{2}, \ \ s_1 < \min\left(l, \frac{3}{2}\cdot(\alpha_0 + 1), 3\cdot(l - |\vec{\delta}|)\right),$$

$$p \in B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0)), \quad \frac{1}{\tau_2} = \frac{s_2}{3} + \frac{1}{2}, \ \ s_2 < \min\left(l - 1, \frac{3}{2}\cdot\alpha_0, 3\cdot(l - (|\vec{\delta}| + 1))\right)$$

Next we investigate the Stokes System on a polyhedral domain:

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \mathcal{G} \\ \operatorname{div} u &= g \quad \text{in } \mathcal{G} \\ u &= 0 \quad \text{on } \Gamma_j,\ j = 1, ..., N. \end{aligned} \tag{3.3}$$

In the proof of Theorem 2.1 we have reduced the problem on a general polyhedral domain to a polyhedral cone. Using these arguments once again we obtain toghether with Theorem 3.1 the following result:

Theorem 3.2. *Fix an integer $l \geq 2$ and a real number $0 < \alpha_0 < 0.5$. It exists a countable set $E \subset \mathbb{C}$ such that for all $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ with*

$$\beta^* := \max_{j=1,...,d'} \beta_j < l - 1, \tag{3.4}$$

$$Re\ \lambda \neq l - \beta_j - \frac{3}{2} \quad \text{for all } \lambda \in E$$

and

$$\max\left(0, l - 1 - \frac{\pi}{\theta_k}\right) < \delta_k < l - 1, \quad k = 1, ..., d$$

the following holds: If $(f, g) \in \left[W^{l-2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap L_2(\mathcal{G})\right]^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap H^{\alpha_0}(\mathcal{G})$ and g fulfills the compatibility conditions

$$\int_{\mathcal{G}} g(x)dx = 0, \quad g|_{M_k} = 0, \ k = 1, ..., d$$

then the unique solution (u, p) of problem (3.3) satisfies

$$u \in \left[B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))\right]^3, \quad \frac{1}{\tau_1} = \frac{s_1}{3} + \frac{1}{2}, \ s_1 < \min\left(l, \frac{3}{2} \cdot (\alpha_0 + 1), 3 \cdot (l - |\vec{\delta}|)\right), \tag{3.5}$$

$$p \in B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G})), \quad \frac{1}{\tau_2} = \frac{s_2}{3} + \frac{1}{2}, \ s_2 < \min\left(l - 1, \frac{3}{2} \cdot \alpha_0, 3 \cdot (l - (|\vec{\delta}| + 1))\right) \tag{3.6}$$

Remark 3.3. (i) In the case of Theorem 3.1 and Theorem 3.2 we have a unique solution in $H^1(\mathcal{G})^3 \times L_2(\mathcal{G})$. M. Dauge could show that under the conditions formulated in Proposition 5.1 the unique solution (u, p) is contained in $H^{\alpha_0+1}(\mathcal{G})^3 \times H^{\alpha_0}(\mathcal{G})$, i.e. the solution (u, p) found in Theorem 3.1 has Sobolev regularity $\alpha_0 + 1$ or α_0 respectively. Therefore for valid parameters l and $\vec{\delta}$ the Besov regularity in the specific scale we are interested in is 3/2 times higher than its Sobolev regularity. Consequently the use of adaptive schemes is justified also in this case.

(ii) Remark 2.3, (ii) applies analogously for set set E in Theorem 3.1, Theorem 3.2.

4 Norm Estimates for Navier-Stokes and Stokes Equations on Polyhedral Domains

Analyzing the convergence of adaptive wavelet schemes we obersve that the error of the approximation can be estimated by a term in which the Besov norm of the exact solution occurs. A typical estimate is of the form

$$\sigma_{m.t}(v) \leq C||v||_{B^s_q(L_q(\Omega))} m^{-\frac{s-t}{d}},$$

where $\sigma_{m,t}$ deontes the error of the best m-term approximation measured in the H^t-norm. We refer to [3] and [9] for details. Therefore it is worthwile to ensure that the Besov norm of the exact solution can be estimated by terms depending only on the right hand side of the partial differential equation. Otherwise we may have a certain convergence rate but this is only asymptotic and for practice it is not applicable. We will derive norm estimates by exploiting Remark 2.3. We begin with the Stokes system on a polyhedral cone.

Theorem 4.1. *If the assumptions of Theorem 3.1 are fulfilled we obtain*

$$||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))} \lesssim \tag{4.1}$$
$$\Big(||F||_{\mathcal{H}^*_{l-1-\beta}} + ||g||_{V^{0,2}_{\beta-l+1}(\mathcal{K})} + ||f||_{W^{l-2,2}_{\beta,\delta}(\mathcal{K})^3} + ||g||_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})} + ||f||_{L_2(\mathcal{K}_0)^3} + ||g||_{H^{\alpha}(\mathcal{K}_0)} \Big).$$

Proof:
From Remark 5.3 we obtain an estimate for the weighted Sobolev norm and the usual Sobolev norm which occurs on the right side of the estimate in Remark 2.3. So, if the assumptions in Theorem 3.1 are fulfilled the solution of the Stokes problem on a polyhedral cone fulfills the stated estimate. □

For the Stokes system on a polyhedral domain as considered in Section 2 and Section 3 we obtain a norm estimate by reducing it to the estimate (4.1). Using the notation $\varphi_j := \kappa_j^{-1} : \mathcal{K}_j \cap B_j \to G \cap U_j$ we find by exploiting (2.3):

$$\begin{aligned} &||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G}))} \\ &\lesssim \sum_{j=1}^{d'} \Big(||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}\cap U_j))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G}\cap U_j))} \Big) \\ &\lesssim \sum_{j=1}^{d'} \Big(||u\circ\varphi_j||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_j\cap B_j))^3} + ||p\circ\varphi_j||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_j\cap B_j))} \Big) \end{aligned}$$

Then (4.1) leads to

$$\begin{aligned} &||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G}))} \\ &\lesssim \sum_{j=1}^{d'} (||u\circ\varphi_j||_{W^{l,2}_{\beta_j,\vec{\delta}}(\mathcal{K}_j)} + ||u\circ\varphi_j||_{H^{\alpha_0+1}(\mathcal{K}_j\cap B_j)} \\ &\qquad + ||p\circ\varphi_j||_{W^{l-1,2}_{\beta_j,\vec{\delta}}(\mathcal{K}_j)} + ||p\circ\varphi_j||_{H^{\alpha_0}(\mathcal{K}_j\cap B_j)}) \\ &\lesssim \sum_{j=1}^{d'} (||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g\circ\varphi_j||_{V^{0,2}_{\beta_j-l+1}(\mathcal{K}_j)} + ||f\circ\varphi_j||_{W^{l-2,2}_{\beta_j,\vec{\delta}}(\mathcal{K}_j)^3} + ||g\circ\varphi_j||_{W^{l-1,2}_{\beta_j,\vec{\delta}}(\mathcal{K}_j)}) \\ &\qquad + ||f||_{L_2(\mathcal{G})^3} + ||g||_{H^{\alpha_0}(\mathcal{G})}. \end{aligned}$$

We finally get

Theorem 4.2. *If the assumptions of Theorem 3.2 are fulfilled we obtain*

$$\begin{aligned} &||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G}))} \\ &\lesssim \sum_{j=1}^{d'} \Big(||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g\circ\varphi_j||_{V^{0,2}_{\beta_j-l+1}(\mathcal{K}_j)} \Big) \\ &\quad + ||f||_{W^{l-2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3} + ||f||_{L_2(\mathcal{G})^3} + ||g||_{W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})} + ||g||_{H^{\alpha_0}(\mathcal{G})}. \end{aligned}$$

In a last step we want to deduce an estimate for the Navier-Stokes system which we have investigated in Corollary 2.4. Especially this means we only obtain an estimate for the Besov norm of u. In the proof of existence of a weak solution for the Navier Stokes equation (see [16, Theorem 11.2.1]) we see that the solution exists on a set with bounded H^1-Norm. Hence it exists a constant $\eta > 0$ such that for the solution u of problem (2.1) we have

$$||u||_{H^1(\mathcal{G})^3} < \eta.$$

From [16, Lemma 8.1.1, Theorem 11.2.8] we conclude for arbitrary small $\varepsilon > 0$

$$||u||_{W^{2,2}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} + ||p||_{W^{1,2}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} \lesssim$$

$$(1+\eta+\eta^2)\left(\sum_{j=1}^{d'}\left(||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g\circ\varphi_j||_{V^{0,2}_{\beta_j-l+1}(\mathcal{K}_j)}\right) + ||f||_{W^{0,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} + ||g||_{W^{1,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})}\right)$$
$$+\eta^3||u||_{W^{2,2}_{\vec{\delta},\vec{\beta}}(\mathcal{G})}.$$

If we assume $\eta \in (0,1)$ we obtain with $\mu := \frac{1+\eta+\eta^2}{1-\eta^3}$ the estimate

$$\begin{aligned}&||u||_{W^{2,2}_{\vec{\delta},\vec{\beta}}(\mathcal{G})^3}\\ &\lesssim \mu\left(\sum_{j=1}^{d'}\left(||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g\circ\varphi_j||_{V^{0,2}_{\beta_j-l+1}(\mathcal{K}_j)}\right) + ||f||_{W^{0,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} + ||g||_{W^{1,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})}\right).\end{aligned}$$

Exploiting Remark 2.3 we achieve

Theorem 4.3. *If the assumptions of Corollary 2.4 are fulfilled then*

$$\begin{aligned}&||u||_{B^s_\tau(L_\tau(\mathcal{G}))^3}\\ &\lesssim \mu\left(\sum_{j=1}^{d'}\left(||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g\circ\varphi_j||_{V^{0,2}_{\beta_j-l+1}(\mathcal{K}_j)}\right) + ||f||_{W^{0,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} + ||g||_{W^{1,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})}\right) + \eta.\end{aligned}$$

5 Appendix A - Sobolev and Weighted Sobolev Regularity of Solutions of the Stokes and the Navier-Stokes System

In this section we state several results which play a fundamental role in the proof of the main theorems. First of all we recall a result concerning the Sobolev regularity for the Stokes system on a polyhedral domain, see [8, Theorem 9.20].

Proposition 5.1. *Let $\mathcal{G} \subset \mathbb{R}^3$ be a polyhedral domain. Consider problem (3.3). Assume that $(f,g) \in L_2(\mathcal{G})^3 \times H^{\alpha_0}(\mathcal{G})$ for $0 < \alpha_0 < 0.5$. Furthermore let g fulfill the compatibility condition*

$$\int_{\mathcal{G}} g(x)\,dx = 0. \tag{5.1}$$

Then there exists a unique solution $(u,p) \in H^{\alpha_0+1}(\mathcal{G})^3 \times H^{\alpha_0}(\mathcal{G})$.

Of course this theorem is true for the special case that $\mathcal{G}$ is a polyhedral cone. Next we cite a regularity result for solutions of the Stokes System in weighted Sobolev spaces defined for polyhedral cones, see [16, Theorem 10.3.2].

Proposition 5.2. *Let $\mathcal{K}$ be a polyhedral cone as defined in (2.2). Suppose $(f,g) \in W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K})^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})$ where $l \geq 2$ is an integer. Then there exists a countable set $E \subset \mathbb{C}$ such that the following holds. If $\beta \in \mathbb{R}$ and the vector $\vec{\delta} \in (\mathbb{R}\backslash\mathbb{Z})^d$ are chosen such that*

$$Re\ \lambda \neq l - \beta - \frac{3}{2} \quad \text{for all } \lambda \in E$$

and

$$\max\left(0, l-1-\frac{\pi}{\theta_k}\right) < \delta_k < l-1, \quad k = 1, ..., d,$$

then there exists a uniquely determined solution of (3.1)

$$(u,p) \in W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})$$

Remark 5.3. Following Maz'ya and Rossmann (see [16, Chapter 10.2, 10.3]) we use the notation $V^{l,2}_{\beta}(\mathcal{K}) := W^{l,2}_{\beta,0}(\mathcal{K})$ and define the space

$$\mathcal{H}_\beta := \left\{u \in V^{1,2}_{\beta}(\mathcal{K})^3 : u = 0 \text{ on } \Gamma_j,\ j = 1, ..., d\right\}.$$

If the assumptions from Proposition 5.2 are fulfilled the functional

$$F(v) := \int_{\mathcal{K}} (f + \nabla g) \cdot v \mathrm{d}x$$

defines a linear and continuous mapping on $\mathcal{H}_{l-1-\beta}$. The solution (u,p) found in Proposition 5.2 fulfills

$$||u||^2_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})^3} + ||p||^2_{W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})} \lesssim \left(||F||^2_{\mathcal{H}^*_{l-1-\beta}} + ||g||^2_{V^{0,2}_{\beta-l+1}} + ||f||^2_{W^{l-2,2}_{\beta,\delta}} + ||g||^2_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})}\right).$$

Moreover we obtain from [8] the estimate

$$||u||_{H^{\alpha_0+1}(\mathcal{K}_0)^3} + ||p||_{H^{\alpha_0}(\mathcal{K}_0)} \lesssim ||f||_{L_2(\mathcal{K}_0)^3} + ||g||_{H^{\alpha_0}(\mathcal{K}_0)}.$$

Analogously to Proposition 5.2 we cite a result for general polyhedral domains (see [16, Theorem 11.1.5]):

Proposition 5.4. *Let $\mathcal{G}$ be a polyhedral domain. Suppose $(f,g) \in W^{l-2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$ where $l \geq 2$ is an integer and $g|_{M_k} = 0$ for $k = 1, ..., d$. Then there exists a countable set $E \subset \mathbb{C}$ such that the following holds: If $\vec{\beta} \in \mathbb{R}^{d'}$ and $\vec{\delta} \in \mathbb{R}^d$ are chosen such that*

$$Re\lambda \neq l - \beta_j - 3/2 \text{ for all } \lambda \in E,\ j = 1, ..., d'$$

and

$$\max\left(0, l-1-\frac{\pi}{\theta_k}\right) < \delta_k < l-1,\ k = 1, ..., d,$$

then $(u,p) \in W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$.

For the Navier-Stokes System (2.1) we state the following result concerning the Sobolev regularity (see [16, Theorem 11.2.1]). Therefore we consider the functional

$$F : \mathcal{H} := \{u \in H^1(\mathcal{G})^3 : u|_{\Gamma_j} = 0 \text{ für } j = 1, ..., N\} \to \mathbb{R}$$
$$F(v) = \int_{\mathcal{G}} (f(x) + \nabla g(x)) \cdot v(x) \mathrm{d}x \tag{5.2}$$

Proposition 5.5. *Let $g \in L_2(\mathcal{G})$. Assume that*

$$||F||_{\mathcal{H}^*} + ||g||_{L_2(\mathcal{G})}$$

is sufficiently small. Then there exists a solution $(u, p) \in H^1(\mathcal{G})^3 \times L_2(\mathcal{G})$ of (2.1). Here u is unique on the set of all functions with norm less than a certain positive ε, p is unique up to a constant.

We cite the analog to Proposition 5.4 for the nonlinear problem (2.1) in the case $l = 2$, see [16, Theorem 11.2.8].

Proposition 5.6. *Let $(u, p) \in H^1(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a solution of the problem (2.1), where $g \in W^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$, $g|_{M_k} = 0$ for $k = 1, ..., d$ and $F \in \mathcal{H}^*$ with given $f \in W^{0,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3$. Then there exists a countable set $E \subset \mathbb{C}$ such that the following holds: If $\vec{\beta} \in \mathbb{R}^{d'}$ and $\vec{\delta} \in \mathbb{R}^d$ are chosen such that*

$$Re\ \lambda \neq 1/2 - \beta_j \quad \text{for all } \lambda \in E$$

and that

$$\max\left(0, 1 - \frac{\pi}{\theta_k}\right) < \delta_k < 1, \ \ k = 1, ..., d,$$

then $u \in W^{2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3$ and $p \in W^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$

6 Appendix B - Function Spaces

We assume that the definition of Besov and Sobolev spaces on $\mathbb{R}^n$ are known. For details see [20]. We now recall the definition of the Besov and Sobolev spaces on a bounded open nonempty set $\Omega \subset \mathbb{R}^n$.

6.1 Besov Spaces on Domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded open nonempty set. Then we define $B^s_q(L_p(\Omega))$ to be the collection of all distributions $f \in D'(\Omega)$ such that there exists a tempered distribution $g \in B^s_q(L_p(\mathbb{R}^n))$ satisfying

$$f(\varphi) = g(\varphi) \text{ for all } \varphi \in D(\Omega),$$

i.e. $g|\Omega = f$ in $\mathbb{D}'(\Omega)$. We put

$$||f||_{B^s_q(L_p(\Omega))} := \inf ||g||_{B^s_q(L_p(\mathbb{R}^n))},$$

where the infimum is taken with respect to all distributions g as above.

6.2 Sobolev Spaces on Domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. As usual with $H^m(\Omega)$ we denote the collection of all functions f such that the distributional derivatives D^α of order $|\alpha| \leq m$ belong to $L_2(\Omega)$. We put

$$||f||_{H^m(\Omega)} := \sum_{|\alpha|\leq m} ||D^\alpha f||_{L_2(\Omega)}.$$

It holds $H^m(\mathbb{R}^n) = B_2^m(L_2(\mathbb{R}^n))$ in the sense of equivalent norms (see e.g. [20]) and because of the existense of a bounded and linear extension operator for Sobolev spaces on bounded Lipschitz domains (see [18]) we also know

$$H^m(\Omega) = B_2^m(L_2(\Omega))$$

in the sense of equivalent norms. For fractional $s > 0$ we introduce the Sobolev Spaces by complex interpolation. For $0 < s < m$, $m \in \mathbb{N}$, $s \notin \mathbb{N}$ we put

$$H^s(\Omega) := \left[H^m(\Omega), L_2(\Omega)\right]_\theta, \quad \theta = 1 - \frac{1}{m},$$

see [15] for details. This definition does not depend on m in the sense of equivalent norms, cf. [21]. Moreover it can be shown that

$$H^s(\Omega) = B_2^s(L_2(\Omega)),$$

cf. [21], [22] for details.

6.3 Besov Spaces and Wavelets

In this section we impose the notations concerning the wavelets. Moreover we state the result which provides a characerization of the Besov spaces by the coefficients of the wavelet expansion. For the construction of wavelets see, e.g., [7]. Let φ be a compactly supported scaling function of sufficiently high regularity and let ψ, $i = 1, ..., 2^n - 1$ be corresponding wavelets. More detailed we require for some $N > 0$ and $r \in \mathbb{N}$:

- supp φ, supp $\psi_i \subset [-N, N]$, $i = 1, ..., 2^n - 1$.
- $\varphi, \psi_i \in \mathcal{C}^r(\mathbb{R}^n)$, $i = 1, ..., 2^n - 1$.
- The wavelets have the vanishing moments property:

  $$\int x^\alpha \psi_i(x) dx = 0$$

  for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$, $i = 1, ..., 2^n - 1$.
- We use the standard abbreviations $\varphi_k(x) := \varphi(x - k)$ and $\psi_{i,j,k}(x) := 2^{jn/2}\psi_i(2^j x - k)$. We assume that

  $$\{\varphi_k,\ \psi_{i,j,k} : (i, j, k) \in \{1, ..., 2^n - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^n\}$$

  is a Riesz basis in $L_2(\mathbb{R}^n)$.

Further, the dual Riesz basis should fulfill the same requirements, i.e. there exist functions $\tilde{\varphi}$ and $\tilde{\psi}_i$, $i = 1, ..., 2^n - 1$, such that

- $\langle \tilde{\varphi}_k, \psi_{i,j,k} \rangle = \left\langle \tilde{\psi}_{i,j,k}, \varphi_k \right\rangle = 0$,
- $\langle \tilde{\varphi}_k, \varphi_l \rangle = \delta_{k,l}$,
- $\left\langle \tilde{\psi}_{i,j,k}, \psi_{u,v,l} \right\rangle = \delta_{i,u}\delta_{j,v}\delta_{k,l}$,
- $\text{supp } \tilde{\varphi}, \text{supp } \tilde{\psi}_i \subset [-N, N]$, $i = 1, ..., 2^n - 1$.
- $\tilde{\varphi}, \tilde{\psi}_i \in \mathcal{C}^r(\mathbb{R}^n)$, $i = 1, ..., 2^n - 1$.
- $\int x^\alpha \tilde{\psi}_i(x)dx = 0$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$, $i = 1, ..., 2^n - 1$.

Next we state a result which allows to prove Besov regularity by estimating the coefficients of the wavelet expansion. More detailed, it holds:

Proposition 6.1. *Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Suppose*

$$r > \max\left(s, n \max\left(0, \frac{1}{p} - 1\right) - s\right).$$

Then $B_q^s(L_p(\mathbb{R}^n))$ is the collection of all tempered distributions f such that f is representable as

$$f = \sum_{k \in \mathbb{Z}^n} a_k \varphi_k + \sum_{i=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} a_{i,j,k} \psi_{i,j,k}$$

with

$$||f||^*_{B_q^s(L_p(\mathbb{R}^n))} := \left(\sum_{k \in \mathbb{Z}^n} |a_k|^p\right)^{1/p} + \left(\sum_{i=1}^{2^n-1} \sum_{j=0}^{\infty} 2^{j(s+n(1/2-1/p))q} \left(\sum_{k \in \mathbb{Z}^n} |a_{i,j,k}|^p\right)^{q/p}\right)^{1/q} < \infty$$

if $q < \infty$ and

$$||f||^*_{B_\infty^s(L_p(\mathbb{R}^n))} := \left(\sum_{k \in \mathbb{Z}^n} |a_k|^p\right)^{1/p} + \sup_{i=1,...,2^n-1} \sup_{j \geq 0} 2^{j(s+n(1/2-1/p))} \left(\sum_{k \in \mathbb{Z}^n} |a_{i,j,k}|^p\right)^{1/p} < \infty.$$

The representation is unique and

$$a_k = \langle f, \tilde{\varphi}_k \rangle \text{ and } a_{i,j,k} = \left\langle f, \tilde{\psi}_{i,j,k} \right\rangle$$

hold. Further $J : f \mapsto \{\langle f, \tilde{\varphi}_k \rangle, \left\langle f, \tilde{\psi}_{i,j,k} \right\rangle\}$ is an isomorphic map of $B_q^s(L_p(\mathbb{R}^n))$ onto the sequence space (equipped with the quasi-norm $|| \cdot ||^*_{B_q^s(L_p(\mathbb{R}^n))}$), i.e. $|| \cdot ||^*_{B_q^s(L_p(\mathbb{R}^n))}$ may serve as an equivalent quasi-norm on $B_q^s(L_p(\mathbb{R}^n))$.

A proof of this proposition can be found in [23].

Acknowledgement: I would like to thank Ms. Monique Dauge for several helpful remarks.

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