Measurement of Long-Range Wave-Function Correlations in an Open Microwave Billiard


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We investigate the statistical properties of wave functions in an open chaotic cavity. When the number of channels in the openings of the billiard is increased by varying the frequency, wave functions cross over from real to complex. The distribution of the phase rigidity, which characterizes the degree to which a wave function is complex, and long-range correlations of intensity and current density are studied as a function of the number of channels in the openings. All measured quantities are in perfect agreement with theoretical predictions.

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From a statistical point of view, eigenvalues and eigenfunctions of the wave equation in a chaotic billiard are well described by random matrix theory [1]: Depending on the presence or absence of time-reversal symmetry, they show the same distribution as eigenvalues and eigenvectors of a large Hermitian matrix with random Gaussian distributed real or complex elements. Whereas random matrix theory predicts a very characteristic eigenvalue distribution, with eigenvalue repulsion and spectral rigidity [2], its predictions for eigenvectors are rather “uninteresting”: eigenvector elements are independent Gaussian distributed real or complex random numbers.

The situation is different for the eigenvectors of a random matrix that interpolates between the standard ensembles with real and complex matrix elements. An example is the “Pandey-Mehta” Hamiltonian [3]

$H(\alpha) = H_0 + \alpha H_1,$

(1)

where $H_0$ and $H_1$ are real and complex random Hermitian matrices, respectively, and $\alpha$ is a crossover parameter. For such an ensemble the eigenvector elements have a non-Gaussian distribution [4,5] and acquire correlations, both between elements of the same eigenvector [5,6] and between different eigenvectors [7].

Experimental verification of this eigenvector distribution has proven problematic because the long-range correlations and the deviations from Gaussian distributions are only of the order of a few percent [4–7]. An additional complication arises from the fact that the crossover parameter $\alpha$ needs to be fitted to the experiment. These complications may explain why measurements of wave-function distributions in two-dimensional microwave cavities in which time-reversal symmetry was broken using magneto-optical effects were inconclusive with respect to the functional form of the probability distribution and did not reveal long-range wave-function correlations [8].

An alternative method to observe the real-to-complex crossover is either to introduce dissipation [9] or to consider traveling waves in a billiard that is opened to the outside world [10]. For microwaves, such an open billiard is obtained by connecting a two-dimensional microwave cavity to waveguides. The parameter governing the crossover from real to complex wave functions is the total number of channels $N$ in the two waveguides. This is true for ensemble averages only. For individual systems such a crossover may be observed already in a billiard with only two attached waveguides [11]. As was shown by one of the authors [12], wave functions in the crossover regime have a non-Gaussian distribution and long-range correlations, just like the eigenvectors of the Pandey-Mehta Hamiltonian (1). The main difference, however, is that the crossover parameter $N$ is discrete and can be measured independently. This allows a fit-parameter free comparison of theory and experiment.

We here report on the first measurement of long-range wave-function correlations in the real-to-complex crossover. The basic principles of the experiment can be found in Ref. [13]. We used a rounded rectangular cavity (21 cm × 16 cm) coupled to two waveguides of width 3 cm with a cutoff frequency at $\nu_c = 5$ GHz. To break the symmetry and to block direct transport, two half disks with a radius of 3 cm were placed in the resonator. Absorbers were placed at the ends of the leads to avoid reflection. We scanned the billiard on a square grid of 5 mm with a movable antenna $A_1$ and measured transmission $S_{12}$ in the range of $4–18$ GHz from a fixed antenna $A_2$ in the end of the right lead. The fixed antenna had a metallic core of diameter 1 mm and a Teflon coating while the probe antenna $A_1$ was a thin wire of diameter 0.2 mm to minimize the leakage current. The lengths of the antenna $A_1$ and antenna $A_2$ were 4 and 5 mm, respectively.

For microwave frequencies $\nu < c/2d = 18.75$ GHz, where $c$ is the velocity of light and $d$ is the resonator height, the billiard is quasi-two-dimensional. In this regime there is an exact correspondence between electrodynamics and quantum mechanics, where the component of the electric field perpendicular to the plane of the microwave billiard $E_z$ corresponds to the quantum-mechanical wave function $\psi$. We normalize the wave function $\psi$ such that $\int d|\psi|^2 = 1$. Then the square of the electric field and...
the Poynting vector map to the normalized “intensity” and “current density,” respectively,

\[ I(r) = A|\psi(r)|^2, \quad j(r) = \frac{A}{\hbar} \text{Im}[\psi^*(r)\nabla \psi(r)], \quad (2) \]

where \( A \) is the area of the billiard and \( \hbar \) the wave number. Figure 1 shows typical intensity and current patterns thus obtained.

The correspondence between the perpendicular component of the electric field \( E \), and the wave function \( \psi \) has been used previously to study the spatial distributions and correlation functions of currents and vortices in open billiards [14,15]. In the present work we study spatial correlation functions of currents and vortices in open billiards [14,15]. In the present work we study spatial correlation functions of the squares of intensities and currents. The statistical average is taken over both position and frequency. It is for these quantities and for this full ensemble average that long-range correlations were predicted [12]. Before we describe the experimental results, we briefly summarize the conclusions of Ref. [12].

In Ref. [12] the wave-function distribution was described as the convolution of a Gaussian distribution with correlated real and imaginary parts and that of a single complex number \( \rho \), the dot product of the wave function \( \psi \) and its time reversed,

\[ \rho = \int d\mathbf{r} \psi(\mathbf{r})^2. \quad (3) \]

The absolute value \( |\rho|^2 \) is known as the “phase rigidity” of \( \psi \) [6]. The Gaussian wave-function distribution at a fixed value of \( \rho \) implies a generalized Porter-Thomas distribution for the intensity,

\[ P_{\rho}(I) = \frac{1}{\sqrt{1-|\rho|^2}} \exp\left[-\frac{I}{1-|\rho|^2}\right] I_0\left[\frac{|\rho|^2}{1-|\rho|^2}\right], \quad (4) \]

so that the full intensity distribution is obtained by convolution of Eq. (4) and the distribution \( p(\rho) \) of \( \rho \),

\[ P(I) = \int d\rho p(\rho) P_{\rho}(I). \quad (5) \]

The distribution \( p(\rho) \) was calculated in Ref. [12] using random matrix theory.

In order to explain the origin of long-range correlations of intensity and current density in an open chaotic billiard, we consider the joint distribution of intensities at points \( \mathbf{r} \) and \( \mathbf{r}' \) with separation \( k|\mathbf{r} - \mathbf{r}'| \gg 1 \),

\[ P[I(r), I(r')], \quad (6) \]

For an open billiard, \( \rho \) has a nontrivial distribution, hence the long-range correlations of \( P[I(r), I(r')] \). Whereas random matrix theory predicts the long-range correlations of intensities through Eq. (6), it cannot alone predict the long-range correlations of current densities and the precise dependence of these correlators on the separation \( |\mathbf{r} - \mathbf{r}'| \) for \( k|\mathbf{r} - \mathbf{r}'| \) of order unity. However, as shown in Ref. [12], the latter can be obtained from the random matrix result by making use of Berry’s ansatz [16], which expresses \( \psi \) as a random sum over plane waves,

\[ \psi(\mathbf{r}) = \sum_{\mathbf{k}} a(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}. \quad (7) \]

Here the plane wave amplitudes \( a(\mathbf{k}) \) have a Gaussian distribution with zero mean and with variance

\[ \langle a(\mathbf{k})a(-\mathbf{k}) \rangle = \rho \langle a(\mathbf{k})a^*(\mathbf{k}) \rangle, \quad (8) \]

where \( \rho \) is the (random) phase rigidity of \( \psi \). Performing the ensemble average using Eqs. (7) and (8), correlators of intensity and current density are then expressed in terms of moments of the phase rigidity \( \langle |\rho|^2 \rangle \). Long-range correlations are found for correlators involving the square of the intensity and the current density,

\[ \langle I(r)^2 I(r')^2 \rangle_c = \text{var}|\rho|^2 + 4f^2(1 + 13|\rho|^2 + |\rho|^4) \]

\[ + 4f^4(1 + 4|\rho|^2 + |\rho|^4), \quad \langle I(r)^2 J(r')^2 \rangle_c = -\frac{1}{2}\text{var}|\rho|^2 + f^2(2 - |\rho|^2 - |\rho|^4), \quad \langle J(r)^2 J(r')^2 \rangle_c = \frac{1}{2}\text{var}|\rho|^2 + \frac{1}{2}f^2(1 - 2|\rho|^2 + |\rho|^4) \quad + \frac{1}{2}f^4(3 - 5|\rho|^2 + 2|\rho|^4). \quad (9) \]

Here \( f = J_0(k|\mathbf{r} - \mathbf{r}'|), J_0 \) being the Bessel function, and the subscript “\( c \)” refers to the connected correlator, \( \langle AB \rangle_c = \langle AB \rangle - \langle A \rangle \langle B \rangle \). The relevant moments of the phase rigidity \( \langle |\rho|^2 \rangle \) were calculated in Ref. [12]: For \( N = 2, 4, \) and \( 6, \) one has \( \langle |\rho|^2 \rangle = 0.7268, 0.5014, \) and \( 0.3918, \) and \( \langle |\rho|^4 \rangle = 0.6064, 0.3285, \) and \( 0.2155, \) respectively.

We now describe the measured wave-function distributions. We first discuss wave-function distributions measured at a fixed frequency, and compare to the theory for the corresponding fixed value of \( \rho \). A statistical distribution at a fixed frequency is obtained by varying the position of the antenna only. The phase rigidity \( |\rho|^2 \) can be determined directly from the variances of real and imaginary parts of the wave functions using Eq. (3). Figure 2 shows the intensity distribution for the intensity pattern shown in the left panel of Fig. 1 together with the theory of Eq. (4), using the measured value of \( |\rho|^2 = 0.5202. \)
As was discussed in Ref. [14], there are frequency regimes where the leakage to the probe antenna becomes intolerably high, either since the transport through the cavity is small or because of some strongly scarred wave functions [17]. In all such cases there were strong deviations from the generalized Porter-Thomas behavior described by Eq. (4). We therefore used only the frequency regimes where \( P(|\psi|^2) \) was in agreement with theory on a confidence level of 90%. Altogether about 60% of the data had to be discarded due to this criterium. To have a well defined number of transversal modes, we investigated the frequency regimes 5–9.5, 10–14.5, and 15–18 GHz, corresponding to a total number of channels \( N = 2, 4, \) and 6, respectively, (i.e., 1, 2, and 3 propagating modes in each waveguide). Only regions more than 1 cm apart from the boundary were considered. To check for boundary effects, we varied the width of the boundary region over 2 cm, but found no difference.

We now describe our results for a full ensemble average, in which both the position of the detector antenna and the frequency are varied. Figure 3 shows the measured phase rigidity distribution \( P(|\rho|^2) \) together with the theory of Ref. [12], for \( N = 2, 4, \) and 6. Good agreement is found between experiment and theory, especially as there is no free parameter. The second and fourth moments of \( \rho \) obtained from the experimental \( P(|\rho|^2) \) are in accordance with theory up to deviations of some percent. Figures 4–6 show measurement and theoretical prediction for the correlation functions of the squared intensity and the squared current density at positions \( \mathbf{r} \) and \( \mathbf{r}' \). Since these correlation functions depend on the positions \( \mathbf{r} \) and \( \mathbf{r}' \) through the combination \( k|\mathbf{r} - \mathbf{r}'| \) only, results from different frequency regimes can be superimposed by a proper scaling. Since the billiard is open, there are net currents, but they do not contribute to the correlators discussed here. In our
The correlations scale with the inverse of the sample conductance, whereas in the present work there is no small parameter that sets the size of the long-range correlations, irrespective of sample size. In that sense, only the latter correlations are truly long range.

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