

# ON THE ATOMIC DECOMPOSITION OF COORBIT SPACES WITH NON-INTEGRABLE KERNEL

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ABSTRACT. This paper is concerned with recent progress in the context of coorbit space theory. Based on a square integrable group representation, the coorbit theory provides new families of associated smoothness spaces, where the smoothness of a function is measured by the decay of the associated voice transform. Moreover, by discretizing the representation, atomic decompositions and Banach frames can be constructed. Usually, the whole machinery works well if the associated reproducing kernel is *integrable* with respect to a weighted Haar measure on the group. In recent studies, it has turned out that to some extent coorbit spaces can still be established if this condition is violated. In this paper, we clarify in which sense atomic decompositions and Banach frames for these generalized coorbit spaces can be obtained.

## 1. INTRODUCTION

This paper is concerned with specific problems arising in the context of signal analysis. The overall goal in signal analysis is the efficient extraction of the relevant information one is interested in. For this, the signal—usually modeled as an element in a suitable function space—has to be processed, denoised, compressed, etc. The first step is always to decompose the signal into appropriate building blocks. This is performed by an associated transform, such as the wavelet transform, the Gabor transform or the shearlet transform, just to name a few. Which transform to choose clearly depends on the type of information one wants to extract from the signal. In recent years, it has turned out that group theory—in particular representation theory—acts as a common thread behind many transforms. Indeed, many transforms are related with square-integrable representations of certain locally compact groups. For instance, the wavelet transform is associated with the affine group whereas the Gabor transform stems from the Weyl-Heisenberg group. We refer e.g., to [12, 13] for details.

This connection with group theory paves the way to the application of another very important concept, namely coorbit theory. This theory has been developed by Feichtinger and Gröchenig already in the late 1980's, see [12, 13, 14, 21]. In recent years, coorbit theory has experienced a real renaissance. Among other things, the connections to the various shearlet transforms [6] and to the concept of decomposition spaces [28, 18] have been investigated.

Based on a square integrable group representation, by means of coorbit space theory it is possible to construct canonical smoothness spaces, the coorbit spaces,

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by collecting all functions for which the associated voice transform has a certain decay. Moreover, by discretizing the underlying representation, it is possible to obtain atomic decompositions for the coorbit spaces. Moreover, also Banach frames can be constructed.

The coorbit space theory is based on certain assumptions. In particular, it is not enough that the representation is square-integrable, it must also be *integrable*, i.e., the reproducing kernel must be contained in a weighted  $L_1$ -space on the group. Unfortunately, this condition is restrictive, and even in very simple settings such as for the case of band-limited functions, it is not satisfied. Nevertheless, in [5], it has been shown that there is a way out. Instead of using a classical  $L_1$ -space as the space of generalized test functions, one can work with the weaker concept of Fréchet spaces. Then, more or less all the basic steps to establish the associated coorbit spaces can be performed. We refer to Sect. 2 for brief discussion of this approach.

However, in [5] one issue remained open, namely the construction of atomic decompositions for the resulting coorbit spaces. This is exactly the problem we are concerned with here. As a surprise, it turns out that this part of the coorbit space theory does not directly carry over to the Fréchet setting. There are two essential differences: First of all, a synthesis map can be constructed, but only at the price that the integrability parameters of the discrete norms on the coefficient spaces and of the coorbit norms are different. At first sight, this might look strange, but in the setting of non-integrable kernels this is in a certain sense not too surprising. Indeed, in the context of coorbit space theory, sooner or later convolution estimates of Young type have to be employed, which yield bounded mappings between  $L_p$ -spaces with *different* integrability exponents for domain and codomain if the convolution kernel is not in  $L_1$ . Concerning the atomic decomposition part, the situation is even more involved. It turns out that for any element in the coorbit space a suitable approximation by linear combinations of the atoms can be derived, but at the price that the weighted sequence norms of the expansion coefficients cannot be uniformly bounded by the coorbit norm. These results will be stated and proved in Sect. 3, see in particular Theorem 3.16.

Looking at these results, the inclined reader might have the impression that the authors were simply unable to prove sharper results, whereas such results might still be true, and provable with a more refined analysis. This might be true, but only partially. Indeed, in Sect. 4 we prove an additional result which shows that, under some very natural conditions, *uniform* bounds can only be obtained if the kernel operator acts as a bounded operator on the weighted  $L_p$ -spaces, that is, this additional assumption is necessary for obtaining uniform bounds. These facts strongly indicate that with the decomposition results stated in Sect. 3 we have almost reached the ceiling. However, there is still a little bit of flexibility which we can use to improve our results. Indeed, in Sect. 5 we prove that if there exists a second kernel  $W$  that satisfies additional smoothness assumptions and acts as the identity by left and right convolution on the reproducing kernel of the representation, then *uniform* bounds for both, the synthesis and the analysis part, can be obtained. Fortunately, in one important practical application given by the Paley-Wiener spaces such a kernel can be found.

This paper is organized as follows. First of all, in Sect. 2, we recall the construction of coorbit spaces based on non-integrable kernels. We keep the explanation

as short as possible and refer to [5] for further details. Then, in Sect. 3, we provide first discretization results for the associated coorbit spaces; the main result is Theorem 3.16. Then, in Sect. 4, we are concerned with ‘negative’ results. Indeed, in Theorem 4.1 we show that stable decompositions can only be obtained if the right convolution by the reproducing kernel is bounded on the underlying  $L_p$ -spaces. Finally, in Sect. 5 we present satisfactory discretization results with the aid of an additional kernel  $W$ . Indeed, in the Theorems 5.12 and 5.18, respectively, we show that atomic decompositions and Banach frames with uniform bounds can be constructed, just as in the context of the classical coorbit theory.

## 2. AN OVERVIEW

Throughout this paper,  $G$  denotes a fixed locally compact second countable group with left *Haar measure*  $\beta$  and *modular function*  $\Delta$ . For a definition of these terms, we refer to [15]. We simply write  $\int_G f(x) dx$  instead of  $\int_G f(x) d\beta(x)$  and we denote by  $L_0(G)$  the space of Borel-measurable functions. Given  $f \in L_0(G)$  the functions  $\check{f}$  and  $\bar{f}$  are

$$\check{f}(x) = f(x^{-1}), \quad \bar{f}(x) = \overline{f(x)},$$

and for all  $x \in G$  the left and right regular representations  $\lambda$  and  $\rho$  act on  $f$  as

$$\begin{aligned} \lambda(x)f(y) &= f(x^{-1}y) && \text{a.e } y \in G, \\ \rho(x)f(y) &= f(yx) && \text{a.e } y \in G. \end{aligned}$$

Finally, the convolution  $f * g$  between  $f, g \in L_0(G)$  is the function

$$f * g(x) = \int_G f(y)g(y^{-1}x) dy = \int_G f(y) \cdot (\lambda(x)\check{g})(y) dy \quad \text{a.e. } x \in G,$$

provided that, for almost all  $x \in G$ , the function  $y \mapsto f(y) \cdot (\lambda(x)\check{g})(y)$  is integrable.

Furthermore, given two functions  $f, g \in L_0(G)$ , with slight abuse of notations, we write

$$\langle f, g \rangle_{L_2} = \int_G f(x)\overline{g(x)} dx,$$

provided that the function  $fg$  is integrable.

We fix a continuous weight  $w : G \rightarrow (0, \infty)$  satisfying

$$(1a) \quad w(xy) \leq w(x)w(y),$$

$$(1b) \quad w(x) = w(x^{-1})$$

for all  $x, y \in G$ . As a consequence, it also holds that

$$(1c) \quad \inf_{x \in G} w(x) \geq 1.$$

The symmetry (1b) can always be satisfied by replacing  $w$  with  $w + \check{w}$ , where the latter weight is easily seen to still satisfy the submultiplicativity condition (1a).

For all  $p \in [1, \infty)$  define the separable Banach space

$$L_{p,w}(G) = \left\{ f \in L_0(G) \mid \int_G |w(x)f(x)|^p dx < \infty \right\}$$

with norm

$$\|f\|_{L_{p,w}}^p = \int_G |w(x)f(x)|^p dx,$$

and the obvious modifications for  $L_\infty(G)$ , which however is not separable. When  $w \equiv 1$  we simply write  $L_p(G)$ .

With terminology as in [5] we choose, as a *target space* for the coorbit space theory, the following space

$$\mathcal{T}_w = \bigcap_{1 < p < \infty} L_{p,w}(G).$$

We recall some basic properties of  $\mathcal{T}_w$ ; for proofs we refer to Theorem 4.3 of [5], which is based on results in [7]. We endow  $\mathcal{T}_w$  with the (unique) topology such that a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $\mathcal{T}_w$  converges to 0 if and only if  $\lim_{n \rightarrow +\infty} \|f_n\|_{L_{p,w}} = 0$  for all  $1 < p < \infty$ . With this topology,  $\mathcal{T}_w$  becomes a reflexive Fréchet space. The (anti)-linear dual space of  $\mathcal{T}_w$  can be identified with

$$\mathcal{U}_w = \text{span} \bigcup_{1 < q < \infty} L_{q,w^{-1}}(G)$$

under the pairing

$$(2) \quad \int_G \Phi(x) \overline{f(x)} dx = \langle \Phi, f \rangle_w, \quad \Phi \in \mathcal{U}_w, f \in \mathcal{T}_w.$$

**Remark 2.1.** *The space  $\mathcal{U}_w$  is endowed with one of the following equivalent topologies, both compatible with the pairing (2).*

- i) *The finest topology making the inclusions  $L_{q,w^{-1}}(G) \hookrightarrow \mathcal{U}_w$  continuous for all  $1 < q < \infty$ .*
- ii) *The topology induced by the family of semi-norms  $(\|\cdot\|_{p,r})_{1 < p < r < \infty}$ , where*

$$\|\Phi\|_{p,r} = \sup \left\{ |\langle \Phi, f \rangle_w| \mid f \in \mathcal{T}_w \text{ and } \max \left\{ \|f\|_{L_{p,w}}, \|f\|_{L_{r,w}} \right\} \leq 1 \right\},$$

for  $\Phi \in \mathcal{U}_w$ .

The representation  $\lambda$  leaves invariant both  $\mathcal{T}_w$  and  $\mathcal{U}_w$ , it acts continuously on  $\mathcal{T}_w$ , and the contragradient representation  ${}^t\lambda$  of  $\lambda|_{\mathcal{T}_w}$ , given by

$$\langle {}^t\lambda_g \Phi, f \rangle_w = \langle \Phi, \lambda_{g^{-1}} f \rangle_w \quad \text{for } \Phi \in \mathcal{U}_w \text{ and } f \in \mathcal{T}_w,$$

is simply  ${}^t\lambda = \lambda|_{\mathcal{U}_w}$ .

Take  $g \in \mathcal{T}_w$  with  $\check{g} \in \mathcal{T}_w$ . For all  $f \in \mathcal{T}_w$  the convolution  $f * g$  is in  $\mathcal{T}_w$  and the map

$$f \mapsto f * g$$

is continuous from  $\mathcal{T}_w$  into  $\mathcal{T}_w$ . Furthermore, for all  $\Phi \in \mathcal{U}_w$  the convolution  $\Phi * g$  is in  $\mathcal{U}_w$  and the map

$$\Phi \mapsto \Phi * g$$

is continuous from  $\mathcal{U}_w$  into  $\mathcal{U}_w$ .

Take now a (strongly continuous) unitary representation  $\pi$  of  $G$  acting on a separable complex Hilbert space  $\mathcal{H}$  with scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  linear in the first entry. We assume that  $\pi$  is reproducing, namely there exists a vector  $u \in \mathcal{H}$  such that the corresponding voice transform

$$Vv(x) = \langle v, \pi(x)u \rangle_{\mathcal{H}}, \quad v \in \mathcal{H}, x \in G,$$

is an isometry from  $\mathcal{H}$  into  $L_2(G)$ . We observe that this implies that  $V$  is injective, whence  $\text{span} \{\pi(x)u\}_{x \in G}$  is dense in  $\mathcal{H}$ .

We denote by  $K$  the reproducing kernel

$$(3) \quad K(x) = Vu(x) = \langle u, \pi(x)u \rangle_{\mathcal{H}}, \quad x \in G, v \in \mathcal{H},$$

which is a bounded continuous function and enjoys the following basic properties

$$(4a) \quad \overline{K} = \check{K},$$

$$(4b) \quad \sum_{i,j=1}^n c_i \overline{c_j} K(x_i^{-1} x_j) \geq 0, \quad c_1, \dots, c_n \in \mathbb{C}, x_1, \dots, x_n \in G,$$

$$(4c) \quad K * K = K \in L_2(G).$$

In general,  $\pi$  is not assumed to be irreducible, but the reproducing assumption implies that  $u$  is a cyclic vector. Properties (4a) and (4b) uniquely define the representation  $\pi$  up to a unitary equivalence, see Theorem 3.20 and Proposition 3.35 of [15]. Equation (4c) states that  $\pi$  is equivalent to the sub-representation of the left-regular representation (on  $L_2(G)$ ) having  $K$  as a cyclic vector. Conversely, if a bounded continuous function  $K$  satisfies (4a), (4b) and (4c), then there exists a unique (up to a unitary equivalence) reproducing representation  $\pi$  whose reproducing kernel is  $K$ .

For the remainder of the paper, we will always impose the following basic assumption:

**Assumption 2.2.** *We assume  $K \in \mathcal{T}_w$ , i.e.,*

$$(5) \quad K \in L_{p,w}(G) \text{ for all } 1 < p < \infty.$$

We add some remarks.

**Remark 2.3.** *i) Since  $w(x) \geq 1$ , Assumption (5) implies that  $K \in L_p(G)$  for all  $p > 1$ . If  $\pi$  is irreducible, this last fact gives that  $V$  is an isometry up to a constant, so that  $\pi$  is always a reproducing representation. If  $\pi$  is reducible, condition (5) is not sufficient to ensure that  $\pi$  is reproducing; however if  $K * K = K$ , then  $\pi$  is always reproducing.*

*ii) If  $w^{-1}$  belongs to  $L_q(G)$  for some  $1 < q < \infty$ , then Hölder's inequality shows  $K \in L_1(G)$ , but in general  $K \notin L_{1,w}(G)$ . However in many interesting examples  $w$  is independent of one or more variables, so that  $w^{-1} \notin L_q(G)$  for all  $1 < q < \infty$ .*

We now define the *test space*  $\mathcal{S}_w$  as

$$(6) \quad \mathcal{S}_w = \left\{ v \in \mathcal{H} \mid Vv \in L_{p,w}(G) \text{ for all } 1 < p < \infty \right\},$$

which becomes a locally convex topological vector space under the family of seminorms

$$(7) \quad \|v\|_{p,\mathcal{S}_w} = \|Vv\|_{L_{p,w}}.$$

We recall the main properties of  $\mathcal{S}_w$ .

**Theorem 2.4** (Theorem 4.4 of [5]). *Under Assumption (5), the following hold:*

- i) the space  $\mathcal{S}_w$  is a reflexive Fréchet space, continuously and densely embedded in  $\mathcal{H}$ ;*
- ii) the representation  $\pi$  leaves  $\mathcal{S}_w$  invariant and its restriction to  $\mathcal{S}_w$  is a continuous representation;*
- iii) the space  $\mathcal{H}$  is continuously and densely embedded into the (anti)-linear dual  $\mathcal{S}'_w$ , where both spaces are endowed with the weak topology;*

iv) the restriction of the voice transform  $V : \mathcal{S}_w \rightarrow \mathcal{T}_w$  is a topological isomorphism from  $\mathcal{S}_w$  onto the closed subspace  $\mathcal{M}^{\mathcal{T}_w}$  of  $\mathcal{T}_w$ , given by

$$\mathcal{M}^{\mathcal{T}_w} = \{f \in \mathcal{T}_w \mid f * K = f\},$$

and it intertwines  $\pi$  and  $\lambda$ ;

v) for every  $f \in \mathcal{T}_w$ , there exists a unique element  $\pi(f)u \in \mathcal{S}_w$  such that

$$\langle \pi(f)u, v \rangle_{\mathcal{H}} = \int_G f(x) \langle \pi(x)u, v \rangle_{\mathcal{H}} dx = \int_G f(x) \overline{Vv(x)} dx, \quad v \in \mathcal{H}.$$

Furthermore, it holds that

$$V\pi(f)u = f * K,$$

and the map

$$\mathcal{T}_w \ni f \mapsto \pi(f)u \in \mathcal{S}_w$$

is continuous and its restriction to  $\mathcal{M}^{\mathcal{T}_w}$  is the inverse of  $V$ .

Here and in the following the notation  $\pi(f)u$  is motivated by the following fact.

**Remark 2.5.** In the framework of abstract harmonic analysis, any function  $f \in L_1(G)$  defines a bounded operator  $\pi(f)$  on  $\mathcal{H}$ , which is weakly given by

$$\langle \pi(f)v, v' \rangle_{\mathcal{H}} = \int_G f(x) \langle \pi(x)v, v' \rangle_{\mathcal{H}} dx, \quad v, v' \in \mathcal{H},$$

see for example Sect. 3.2 of [15]. However, if  $f \notin L_1(G)$ , then in general  $\pi(f)v$  is well defined only if  $v = u$ , where  $u$  is an admissible vector for the representation  $\pi$ .

Recalling that the (anti-)dual of  $\mathcal{T}_w$  is  $\mathcal{U}_w$  under the pairing (2), we denote by  ${}^tV$  the contragradient map  ${}^tV : \mathcal{U}_w \rightarrow \mathcal{S}'_w$  given by

$$\langle {}^tV\Phi, v \rangle_{\mathcal{S}'_w} = \langle \Phi, Vv \rangle_w, \quad \Phi \in \mathcal{U}_w, v \in \mathcal{S}_w.$$

As usual, we extend the voice transform from  $\mathcal{H}$  to the (anti-)dual  $\mathcal{S}'_w$  of  $\mathcal{S}_w$ , where  $\mathcal{S}'_w$  plays the role of the space of distributions. For all  $T \in \mathcal{S}'_w$  we set

$$(8) \quad V_e T(x) = \langle T, \pi(x)u \rangle_{\mathcal{S}'_w}, \quad x \in G,$$

which is a continuous function on  $G$  by item ii) of the previous theorem and  $\langle \cdot, \cdot \rangle_{\mathcal{S}'_w}$  denotes the pairing between  $\mathcal{S}_w$  and  $\mathcal{S}'_w$ , whereas  $\langle \cdot, \cdot \rangle_w$  is the pairing between  $\mathcal{T}_w$  and  $\mathcal{U}_w$ .

We summarize the main properties of the extended voice transform in the following theorem.

**Theorem 2.6** (Theorem 4.4 of [5]). *Under assumption (5), the following hold:*

i) for every  $\Phi \in \mathcal{U}_w$  there exists a unique element  $\pi(\Phi)u \in \mathcal{S}'_w$  such that

$$\langle \pi(\Phi)u, v \rangle_{\mathcal{S}'_w} = \int_G \Phi(x) \langle \pi(x)u, v \rangle_{\mathcal{H}} dx = \int_G \Phi(x) \overline{Vv(x)} dx, \quad v \in \mathcal{S}_w.$$

Furthermore, it holds that

$$V_e \pi(\Phi)u = \Phi * K;$$

ii) for all  $T \in \mathcal{S}'_w$  the voice transform  $V_e T$  is in  $\mathcal{U}_w$  and satisfies

$$(9) \quad V_e T = V_e T * K,$$

$$(10) \quad \langle T, v \rangle_{\mathcal{S}'_w} = \langle V_e T, Vv \rangle_w, \quad v \in \mathcal{S}_w;$$

iii) the extended voice transform  $V_e$  is injective, continuous from  $\mathcal{S}'_w$  into  $\mathcal{U}_w$  (when both spaces are endowed with the strong topology), its range is the closed subspace

$$(11) \quad \mathcal{M}^{\mathcal{U}_w} = \{\Phi \in \mathcal{U}_w \mid \Phi * K = \Phi\} = \text{span} \bigcup_{p \in (1, \infty)} \mathcal{M}^{L^p, w(G)} \subset L_{\infty, w^{-1}}(G)$$

and it intertwines the contragradient representation of  $\pi|_{\mathcal{S}_w}$  and  $\lambda|_{\mathcal{U}_w}$ ;

iv) the map

$$\mathcal{M}^{\mathcal{U}_w} \ni \Phi \mapsto \pi(\Phi)u \in \mathcal{S}'_w$$

is the left inverse of  $V_e$  and coincides with the restriction of the map  ${}^tV$  to  $\mathcal{M}^{\mathcal{U}_w}$ , namely

$$(12) \quad V_e({}^tV\Phi) = V_e\pi(\Phi)u = \Phi, \quad \Phi \in \mathcal{M}^{\mathcal{U}_w};$$

v) regarding  $\mathcal{S}_w \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{S}'_w$ , it holds

$$\mathcal{S}_w = \{T \in \mathcal{S}'_w \mid V_e T \in \mathcal{T}_w\} = \{\pi(f)u \mid f \in \mathcal{M}^{\mathcal{T}_w}\}.$$

Item ii) of the previous theorem states that the voice transform of any distribution  $T \in \mathcal{S}'_w$  satisfies the reproducing formula (9) and uniquely defines the distribution  $T$  by means of the reconstruction formula (10), i.e.

$$T = \int_G \langle T, \pi(x)u \rangle_{\mathcal{S}_w} \pi(x)u \, dx,$$

where the integral is a Dunford-Pettis integral with respect to the duality between  $\mathcal{S}_w$  and  $\mathcal{S}'_w$ , see, for example, Appendix 3 of [15].

We now fix an exponent  $r \in [1, \infty)$ , and a  $w$ -moderate weight  $m$ , i.e. a continuous function  $m : G \rightarrow (0, \infty)$  such that

$$(13) \quad m(xy) \leq w(x) \cdot m(y) \quad \text{and} \quad m(xy) \leq m(x) \cdot w(y) \quad \text{for all } x, y \in G.$$

**Remark 2.7.** The definition (13) of a  $w$ -moderate weight  $m$  is equivalent to the condition

$$m(xyz) \leq w(x) \cdot m(y) \cdot w(z) \quad \text{for all } x, y, z \in G$$

up to the constant  $w(e)$ .

The result of the following lemma is used multiple times in this paper.

**Lemma 2.8.** *If  $m$  is a  $w$ -moderate weight on  $G$ , then so is  $m^{-1}$ .*

*Proof.* To prove the estimates in (13) for  $m^{-1}$  we fix  $x, y \in G$ , then by the  $w$ -moderateness of  $m$  it holds

$$m(y) = m(x^{-1}xy) \leq w(x^{-1}) \cdot m(xy) = w(x) \cdot m(xy),$$

which implies  $m(xy)^{-1} \leq w(x) \cdot m(y)^{-1}$ . Similarly we observe that

$$m(x) = m(xy y^{-1}) \leq m(xy) \cdot w(y^{-1}) = m(xy) \cdot w(y),$$

which in turn implies  $m(xy)^{-1} \leq m(x)^{-1} \cdot w(y)$ .  $\square$

With terminology as in [5], we choose as a *model space* for the coorbit space theory, the Banach space  $Y = L_{r, m}(G)$  with  $r \in (1, \infty)$ . The corresponding coorbit space is defined as

$$(14) \quad \text{Co}(Y) = \{T \in \mathcal{S}'_w \mid V_e T \in Y\}$$

endowed with the norm

$$(15) \quad \|T\|_{\text{Co}(Y)} = \|V_e T\|_Y.$$

We summarize the main properties of  $\text{Co}(Y)$  in the following proposition.

**Proposition 2.9.** *The space  $\text{Co}(Y)$  is a Banach space invariant under the action of the contragradient representation of  $\pi|_{\mathcal{S}_w}$ . The extended voice transform is an isometry from  $\text{Co}(Y)$  onto the  $\lambda$ -invariant closed subspace*

$$\mathcal{M}^Y = \{F \in Y \mid F * K = F\} \subset \mathcal{U}_w,$$

and we have

$$\text{Co}(Y) = \left\{ \pi(F)u \mid F \in \mathcal{M}^Y \right\}.$$

Furthermore

$$(16) \quad V_e \pi(F)u = F, \quad F \in \mathcal{M}^Y,$$

$$(17) \quad \pi(V_e T)u = T, \quad T \in \text{Co}(Y).$$

*Proof.* The proof is essentially an application of Theorem 3.5 in [5]. We first note that convergence with respect to  $\|\cdot\|_Y = \|\cdot\|_{L_{r,m}}$  implies convergence in measure. Furthermore, since  $m$  is  $w$ -moderate, it is not hard to see that  $Y = L_{r,m}(G)$  is  $\lambda$ -invariant, and that the restriction of  $\lambda$  to  $Y$  is a continuous representation of  $G$ . Therefore, we only need to prove that Assumption 5 and Assumption 6 in [5] are satisfied.

We first show that  $Y \subset \mathcal{U}_w$ . By (13) and (1b) we get for any  $x \in G$  that

$$(18) \quad m(e) = m(xx^{-1}) \leq m(x) \cdot w(x^{-1}) = m(x) \cdot w(x),$$

and hence  $[w(x)]^{-1} \cdot m(e) \leq m(x)$ , whence  $Y = L_{r,m}(G) \hookrightarrow L_{r,w^{-1}}(G) \subset \mathcal{U}_w$  since  $r > 1$ .

Since  $\mathcal{U}_w = \mathcal{T}_w'$  under the pairing (2), for all  $F \in Y$  and  $f \in \mathcal{T}_w$  it holds that  $Ff \in L_1(G)$ . In particular, by assumption (5),  $FK \in L_1(G)$  for all  $F \in Y$  and, by construction,  $FVv \in L_1(G)$  for all  $v \in \mathcal{S}_w$  and  $F \in \mathcal{M}^Y$ , so that Assumption 5 and Assumption 6 in [5] hold true.  $\square$

### 3. DISCRETIZATION

The aim of this section is to establish certain atomic decompositions for the coorbit spaces described in Sect. 2. In particular, we recall that

$$(19) \quad \mathcal{T}_w = \bigcap_{p \in (1, \infty)} L_{p,w}(G), \quad \mathcal{T}_w' = \mathcal{U}_w = \text{span} \bigcup_{q \in (1, \infty)} L_{q,w^{-1}}(G)$$

and for some  $1 < r < \infty$ ,

$$(20) \quad Y = L_{r,m}(G).$$

Proposition 2.9 shows that the correspondence principle holds, i.e., the extended voice transform  $V_e$  is an isometry from the associated coorbit space

$$(21) \quad \text{Co}(L_{r,m}) := \{T \in \mathcal{S}'_w \mid V_e(T) \in L_{r,m}(G)\}$$

onto the corresponding reproducing kernel Banach space

$$(22) \quad \mathcal{M}_{r,m} = \mathcal{M}^{L_{r,m}(G)} = \{f \in L_{r,m}(G) \mid f * K = f\}.$$



**Remark 3.1.** Assumption (5) on the kernel  $K$  and the fact that  $m$  is  $w$ -moderate imply that for all  $f \in L_{r,m}(G)$  the convolution  $f * K$  is well-defined; see Proposition A.2.

In this setting we can characterize the anti-dual  $\mathcal{M}'_{r,m}$  of the reproducing kernel space.

**Lemma 3.2.** The anti-dual  $\mathcal{M}'_{r,m}$  of  $\mathcal{M}_{r,m}$  is canonically isomorphic to  $L_{r',m^{-1}}(G)/\mathcal{M}_{r,m}^\perp$ , where

$$(23) \quad \mathcal{M}_{r,m}^\perp = \left\{ \tilde{F} \in L_{r',m^{-1}}(G) \mid \langle \tilde{F}, F \rangle_{L_2} = 0 \text{ for all } F \in \mathcal{M}_{r,m} \right\}$$

and  $1/r + 1/r' = 1$ . Hence, for every  $\Gamma \in \mathcal{M}'_{r,m}$  there is a  $\tilde{F} \in L_{r',m^{-1}}(G)$  such that  $\Gamma(F) = \langle \tilde{F}, F \rangle_{L_2}$  for all  $F \in \mathcal{M}_{r,m}$ .

*Proof.* Since  $\mathcal{M}_{r,m}$  is a closed subspace of  $L_{r,m}(G)$ , [25, Proposition 1.4] yields that  $\mathcal{M}'_{r,m}$  is canonically isomorphic to  $L'_{r,m}(G)/\mathcal{M}_{r,m}^\perp$ . The claim follows because  $L'_{r,m}(G)$  is canonically isomorphic to  $L_{r',m^{-1}}(G)$ .  $\square$

Some more preparations are necessary. Given a compact neighborhood  $Q \subset G$  of  $e$  with  $Q = \text{int } Q$ , the local maximal function (with respect to the right regular representation)  $M_Q^\rho f$  of  $f \in L_0(G)$  is defined by

$$(24) \quad M_Q^\rho f(x) := \|f \cdot \rho(x)\chi_Q\|_{L_\infty}, \quad \text{whence} \quad \check{M}_Q^\rho f(x) := M_Q^\rho f(x^{-1}) = \|f\|_{L_\infty(Qx)}.$$

Then, for a function space  $Y$  on  $G$ , we define

$$(25) \quad \mathcal{M}_Q^\rho(Y) := \left\{ f \in L_0(G) \mid \check{M}_Q^\rho f \in Y \right\}.$$

Now we define the  $Q$ -oscillation of a function  $f$  with respect to  $Q$  as

$$(26) \quad \text{osc}_Q f(x) := \sup_{u \in Q} |f(ux) - f(x)|.$$

The decay-properties of the  $Q$ -oscillation play an important role in view of the discretization of coorbit spaces. To this end, the following lemma is useful. Since the proof is a simple generalization of the proof of [21, Lemma 4.6], it is deferred to the appendix.

**Lemma 3.3.** Let  $w$  be a weight on  $G$ , let  $p \in (1, \infty)$ , and assume that  $f : G \rightarrow \mathbb{C}$  is continuous and that  $f \in \mathcal{M}_{Q_0}^\rho(L_{p,w})$  for some compact unit neighborhood  $Q_0$  with  $Q_0 = \overline{\text{int } Q_0}$ . Then the following hold:

- i)  $\|\text{osc}_{Q_0} f\|_{L_{p,w}} < \infty$ .
- ii) For arbitrary  $\varepsilon > 0$ , there is a unit neighborhood  $Q_\varepsilon \subset Q_0$  such that for each unit neighborhood  $Q \subset Q_\varepsilon$ , we have  $\|\text{osc}_Q f\|_{L_{p,w}} < \varepsilon$ . Put briefly,

$$\lim_{Q \rightarrow \{e\}} \|\text{osc}_Q f\|_{L_{p,w}} = 0.$$

**3.1. An Assumption on the Kernel.** From now on we make the following assumption on the reproducing kernel space.

**Assumption 3.4.** Assumption 2.2 is satisfied, and  $\text{span}\{\lambda(x)K\}_{x \in G}$  is dense in  $\mathcal{M}_{r,m}$ .

This assumption is similar to the density of  $\text{span}\{\pi(x)K\}_{x \in G}$  in  $\mathcal{H}$ —which is equivalent to  $K$  being a cyclic vector for the representation  $\pi$  on  $\mathcal{H}$ —and in  $\mathcal{M}^{\mathcal{T}_w}$ , which is Assumption 3 of [5] and fulfilled in our setting, as can be seen by combining Theorems 2.4 and 2.6

In the following we will denote with  $RC_K$  the right convolution operator  $RC_K f := f * K$ , where the space on which  $RC_K$  acts may vary depending of the context.

Before we provide a sufficient condition under which Assumption 3.4 is fulfilled (see Lemma 3.8), we need a couple of auxiliary results.

**Proposition 3.5.** *Assume that for all  $f \in L_{r,m}(G)$ ,  $f$  and  $K$  are convolvable (in the sense that  $f \cdot \lambda(x)\check{K} \in L_1(G)$  for almost all  $x \in G$ ) and  $f * K \in L_{r,m}(G)$ , then the right convolution operator*

$$RC_K : L_{r,m}(G) \rightarrow L_{r,m}(G), \quad RC_K f = f * K,$$

*is bounded.*

*Proof.* For  $r = 2$  the result is stated in [22, Proposition 3.10], whose proof holds true for any  $p$ . Indeed, by the closed graph theorem, it is enough to show that  $RC_K$  is a closed operator. Take a sequence  $(f_n)_{n \in \mathbb{N}}$  converging to  $f \in L_{r,m}(G)$  such that  $(RC_K f_n)_{n \in \mathbb{N}}$  converges to  $g \in L_{r,m}(G)$ . By a sharp version of the Riesz-Fischer theorem, see [1, Theorem 13.6], there exists a positive function  $g \in L_{r,m}(G)$  such that, possibly passing twice to a subsequence, there exist two null sets  $E, F$  such that for all  $y \in G \setminus E$  and  $x \in G \setminus F$

$$\begin{aligned} |f_n(y)| &\leq g(y), \\ \lim_{n \rightarrow \infty} f_n(y) &= f(y), \\ \lim_{n \rightarrow \infty} RC_K f_n(x) &= g(x). \end{aligned}$$

Furthermore, by definition of convolution and possibly re-defining the null set  $F$ , we get that for all  $x \in G \setminus F$  and all  $n \in \mathbb{N}$  the mappings

$$y \mapsto f_n(y)K(y^{-1}x), \quad y \mapsto g(y)K(y^{-1}x)$$

are integrable. Then, given  $x \in G \setminus F$ , for all  $y \in G \setminus E$

$$|f_n(y)K(y^{-1}x)| \leq |g(y)K(y^{-1}x)|, \quad \lim_{n \rightarrow \infty} f_n(y)K(y^{-1}x) = f(y)K(y^{-1}x).$$

For  $x \in G \setminus F$ , the function  $y \mapsto g(y)K(y^{-1}x)$  is integrable, so that by dominated convergence we obtain

$$g(x) = \lim_{n \rightarrow \infty} \int_G f_n(y)K(y^{-1}x) dy = \int_G f(y)K(y^{-1}x) dy = f * K(x),$$

so  $RC_K$  is indeed closed.  $\square$

**Proposition 3.6.** *Denote by  $r'$  the dual exponent  $1/r + 1/r' = 1$ . Assume that the right convolution operator*

$$RC_K : L_{r,m}(G) \rightarrow L_{r,m}(G), \quad RC_K f = f * K$$

*is bounded, then*

- i) the right convolution operator is bounded on  $L_{r',m^{-1}}(G)$  and it coincides with the adjoint of  $RC_K$ ;*
- ii) the operator  $RC_K$  is a projection from  $L_{r,m}(G)$  onto the reproducing kernel Banach space  $\mathcal{M}_{r,m}$ .*

Here and in the following, the duality pairing is the sesqui-linear form

$$\langle f, g \rangle_{L_2} = \int_G f(x) \overline{g(x)} dx, \quad f \in L_{r,m}(G), g \in L_{r',m^{-1}}(G).$$

*Proof.* Since  $RC_K$  is a bounded operator on  $L_{r,m}(G)$ , the adjoint is a bounded operator on  $L_{r,m}(G)'$ . Take  $g \in L_{r',m^{-1}}(G)$  and  $f \in C_c(G) \subset L_{r,m}(G)$ , then

$$\begin{aligned} \langle RC_K^* g, f \rangle_{L_2} &= \langle g, RC_K f \rangle_{L_2} = \int_G g(x) \left( \int_G \overline{f(y)K(y^{-1}x)} dy \right) dx \\ &= \int_G \left( \int_G g(x)K(x^{-1}y) dx \right) \overline{f(y)} dy \\ &= \langle g * K, f \rangle_{L_2}, \end{aligned}$$

where  $\overline{K(y^{-1}x)} = K(x^{-1}y)$ . Note that we can interchange the integral by Fubini's theorem since

$$\int_G |g(x)| \left( \int_G |f(y)K(y^{-1}x)| dy \right) dx \leq \|g\|_{L_{r',m^{-1}}} \cdot \| |f| * |K| \|_{L_{r,m}}$$

and  $|f| * |K| \in L_{r,m}(G)$  by Young's inequality (81) with  $q = 1$  and  $p = r$ ,  $f \in L_{1,m}(G)$  and  $g = K \in L_{r,w}(G)$ . Note that Fubini's theorem shows that

$$\int |f(y)| \cdot (|g| * |K|)(y) dy < \infty.$$

Since this holds for any  $f \in C_c(G)$ , we see  $|g| * |K| < \infty$  almost everywhere, so that  $g$  and  $K$  are convolvable. By density of  $C_c(G)$  in  $L_{r,m}(G)$  we get that  $RC_K^* g = g * K$ , so that  $g * K \in L_{r',m^{-1}}(G)$ . Hence the convolution operator acts continuously on  $L_{r',m^{-1}}(G)$  and it coincides with  $RC_K^*$ .

To show the second claim, observe first that for any  $f \in C_c(G) \subset \mathcal{T}_w \subset L_{r',m^{-1}}(G)$ , since  $K \in \mathcal{T}_w$ , both  $|f| * |K|$  and  $(|f| * |K|) * |K|$  exist, so that by (77d) of [5] the convolution is associative and

$$RC_K^2 f = (f * K) * K = f * (K * K) = f * K = RC_K f.$$

By density, and since  $RC_K$  is bounded on  $L_{r,m}(G)$  by assumption, we get that  $RC_K^2 = RC_K$  and hence  $\text{Ran } RC_K \subset \mathcal{M}_{r,m}$ . The other inclusion is trivial.  $\square$

As a consequence of the above result, we get the following corollary.

**Corollary 3.7.** *Denote by  $r'$  the dual exponent  $1/r + 1/r' = 1$  and assume that the right convolution operator  $RC_K$  is bounded on  $L_{r,m}(G)$ . The sesqui-linear pairing on  $\text{Co}(L_{r,m}) \times \text{Co}(L_{r',m^{-1}})$  given by*

$$\langle T, T' \rangle_{\text{Co}(L_{r,m})} = \langle V_e T, V_e T' \rangle_{L_2}$$

is such that the linear map

$$T' \mapsto \left( T \mapsto \overline{\langle T, T' \rangle_{\text{Co}(L_{r,m})}} \right)$$

is an isomorphism of  $\text{Co}(L_{r',m^{-1}})$  onto the anti-linear dual of  $\text{Co}(L_{r,m})$ .

*Proof.* We identify  $\text{Co}(L_{r,m})$  with  $\mathcal{M}_{r,m}$  by the extended voice transform  $V_e$ , so that the pairing becomes

$$\langle f, g \rangle_{L_2} = \int_G f(x) \overline{g(x)} dx, \quad f \in \mathcal{M}_{r,m}, g \in \mathcal{M}_{r',m^{-1}}.$$

Since  $g \in L_{r',m^{-1}}(G)$ , clearly  $f \mapsto \overline{\langle f, g \rangle_{L_2}}$  is a continuous anti-linear map, which we denote by  $\Gamma_g$ , on  $\mathcal{M}_{r,m}$  whose norm is

$$\begin{aligned} \|\Gamma_g\| &= \sup \left\{ |\langle f, g \rangle_{L_2}| \mid f \in \mathcal{M}_{r,m}, \|f\|_{L_{r,m}} \leq 1 \right\} \\ &\leq \sup \left\{ |\langle h, g \rangle_{L_2}| \mid h \in L_{r,m}(G), \|h\|_{L_{r,m}} \leq 1 \right\} = \|g\|_{L_{r',m^{-1}}}. \end{aligned}$$

Next, since  $L_{r,m}(G)$  is the dual of  $L_{r',m^{-1}}(G)$ , there is  $h \in L_{r,m}(G)$  with  $\|h\|_{L_{r,m}} \leq 1$  such that  $\|g\|_{L_{r',m^{-1}}} = \langle h, g \rangle_{L_2}$ . Now, setting  $c := \|RC_K\|_{L_{r,m} \rightarrow L_{r',m^{-1}}}$  and  $f = c^{-1} \cdot RC_K h$ , we have  $\|f\|_{L_{r,m}} \leq 1$  and

$$\langle f, g \rangle_{L_2} = c^{-1} \langle RC_K h, g \rangle_{L_2} = c^{-1} \langle h, RC_K g \rangle_{L_2} = c^{-1} \langle h, g \rangle_{L_2} = c^{-1} \|g\|_{L_{r',m^{-1}}}.$$

Hence,  $c^{-1} \cdot \|g\|_{L_{r',m^{-1}}} \leq \|\Gamma_g\| \leq \|g\|_{L_{r',m^{-1}}}$ .

We now prove that the map  $g \mapsto \Gamma_g$  is surjective. Take  $\Gamma$  in the anti-linear dual of  $\mathcal{M}_{r,m}$ . Since  $\mathcal{M}_{r,m}$  is a subspace of  $L_{r,m}(G)$  there exists  $g' \in L_{r',m^{-1}}(G)$  such that  $\Gamma(f) = \overline{\langle f, g' \rangle_{L_2}}$  for all  $f \in \mathcal{M}_{r,m}$ . By setting  $g = RC_K g' \in \mathcal{M}_{r',m^{-1}}$ , as above

$$\Gamma(f) = \overline{\langle f, g' \rangle_{L_2}} = \overline{\langle RC_K f, g \rangle_{L_2}} = \overline{\langle f, g \rangle_{L_2}} = \Gamma_g(f) \quad \text{for all } f \in \mathcal{M}_{r,m},$$

thus  $\Gamma = \Gamma_g$ . □

Now we can prove that in the following setting Assumption 3.4 is fulfilled.

**Lemma 3.8.** *Fix  $r \in (1, \infty)$  and assume that the right convolution operator  $RC_K$  is bounded on  $L_{r,m}(G)$ . Then the sets  $\text{span}\{\pi(x)u\}_{x \in G}$  and  $\text{span}\{\lambda(x)K\}_{x \in G}$  are dense in  $\text{Co}(L_{r,m})$  and  $\mathcal{M}_{r,m}$ , respectively. Thus, Assumption 3.4 is fulfilled.*

*Proof.* By the correspondence principle, it is enough to show the second claim. Let  $\Gamma \in \mathcal{M}'_{r,m}$  be such that for all  $x \in G$ ,

$$\Gamma(\lambda(x)K) = 0.$$

By the above corollary, there exists  $g \in \mathcal{M}_{r',m^{-1}}$  such that  $\Gamma(f) = \langle g, f \rangle_{L_2}$  for all  $f \in \mathcal{M}_{r,m}$ . In particular,

$$0 = \Gamma(\lambda(x)K) = \langle g, \lambda(x)K \rangle_{L_2} = g * K(x) = RC_K g(x)$$

for all  $x \in G$ , that is,  $RC_K g = 0$ . Since  $g \in \mathcal{M}_{r',m^{-1}}$ , this implies  $g = 0$  and then  $\Gamma = 0$ . Since this holds for any  $\Gamma \in \mathcal{M}'_{r,m}$  such that  $\Gamma(\lambda(x)K) = 0$  for all  $x \in G$ , we see that  $\text{span}\{\lambda(x)K\}_{x \in G}$  is dense in  $\mathcal{M}_{r,m}$ . □

By Young's inequality we know that the  $L_1(G)$ -integrability of  $K \cdot w$  implies that the (right) convolution operator  $RC_K$  is a bounded operator acting on  $L_{p,m}(G)$  for all  $1 < p < \infty$ . But for general  $K \in \mathcal{T}_w$  this question is unclear. As we will show in Sect. 3.3 there are kernels that act boundedly on all  $L_p(G)$  without being integrable. But in Sect. 4 we also show that there exist kernels for a very similar setting that are contained in  $\mathcal{T}_w$  but that do *not* give rise to bounded operators on  $L_{p,m}(G)$ .

**3.2. Atomic Decompositions.** This section is dedicated to finding possible atomic decompositions of coorbit spaces, provided that Assumption 3.4 is fulfilled. The main results of this section will be stated in Theorem 3.16.

But before that, we need to introduce some notation. First, for each  $n \in \mathbb{N}$ , we choose a countable subset  $Y_n = \{x_{j,n}\}_{j \in \mathcal{J}_n} \subset G$  such that

$$(27) \quad Y_n \subset Y_{n+1},$$

$$(28) \quad \overline{\bigcup_{n \in \mathbb{N}} Y_n} = G.$$

Moreover, for every  $n \in \mathbb{N}$ , we assume that there exists a compact neighborhood  $Q_n$  of the identity  $e \in G$ , such that  $Y_n$  is  $Q_n$ -dense in  $G$ , i.e.,

$$(29) \quad G = \bigcup_{j \in \mathcal{J}_n} x_{j,n} Q_n.$$

Additionally we assume each  $Y_n$  to be uniformly relatively  $Q_n$ -separated, i.e. there exists an integer  $I$ , independent of  $n$ , and subsets  $Z_{n,i} \subset Y_n$ ,  $1 \leq i \leq I$ , such that

$$(30) \quad Y_n = \bigcup_{i=1}^I Z_{n,i}$$

and for all  $x, y \in Z_{n,i}$ ,  $1 \leq i \leq I$ , it holds  $xQ_n \cap yQ_n \neq \emptyset$  if and only if  $x = y$ .

By  $\Psi_n = \{\psi_{n,x}\}_{x \in Y_n}$  we denote a *partition of unity* subordinate to the  $Q_n$ -dense set  $Y_n$ , i.e.,

$$(31) \quad 0 \leq \psi_{n,x} \leq 1,$$

$$(32) \quad \sum_{x \in Y_n} \psi_{n,x} \equiv 1,$$

$$(33) \quad \text{supp}(\psi_{n,x}) \subset xQ_n.$$

We also assume that the family  $\Psi_n = \{\psi_{n,x}\}_{x \in Y_n}$  is linearly independent as a.e. defined functions, i.e. for any finite subset  $X \subset Y_n$  and  $(\alpha_x)_{x \in X} \in \mathbb{C}^X$ , the condition

$$\sum_{x \in X} \alpha_x \psi_{n,x}(y) = 0$$

for almost all  $y \in G$  implies that  $\alpha_x = 0$  for all  $x \in X$ .

We now denote with  $X_n$  a *finite* subset of  $Y_n$ , such that

$$(34) \quad X_n \subset X_{n+1},$$

$$(35) \quad \overline{\bigcup_{n \in \mathbb{N}} X_n} = G.$$

Therefore, for every  $n \in \mathbb{N}$ , the finite set of functions  $\{\psi_{n,x}\}_{x \in X_n}$  is similar to a partition of unity subordinate to the family  $(xQ_n)_{x \in X_n}$ .

For each  $n \in \mathbb{N}$  and  $1 < r < \infty$ , set

$$(36) \quad T_n : L_{r,m}(G) \rightarrow \mathcal{M}_{r,m}, \quad T_n F := \sum_{x \in X_n} \langle F, \psi_{n,x} \rangle_{L_2} \lambda(x) K.$$

We observe that this operator is well-defined. Since the sum is finite, we only have to verify that each term of the sum is a well-defined element of  $\mathcal{M}_{r,m}$ . It is easy to verify that the reproducing identity holds for  $\lambda(x)K$ , since it holds for  $K$ . Moreover we have  $\lambda(x)K \in L_{r,m}(G)$  by Assumption 2.2 and by translation invariance of the spaces  $L_{r,m}(G)$ ; thus,  $\lambda(x)K \in \mathcal{M}_{r,m}$ . Finally, the pairing

$$\langle F, \psi_{n,x} \rangle_{L_2} = \int_G F(y) \psi_{n,x}(y) dy$$

is well-defined for all  $x \in X_n$ , since  $\psi_{n,x}$  is bounded with compact support, so that  $\psi_{n,x} \in L_{r',m^{-1}}(G)$ .

Now we define  $V_n = \text{Ran } T_n$ , which is a finite dimensional subspace of  $\mathcal{M}_{r,m}$ , as well as  $\tilde{V}_n = V_e^{-1}(V_n)$ , which is a finite dimensional subspace of  $\text{Co}(L_{r,m})$  by the correspondence principle. We show the following result concerning the structure of the spaces  $V_n$ :

**Lemma 3.9.** *The following holds for all  $n \in \mathbb{N}$ :*

$$(37) \quad V_n = \text{span} \{ \lambda(x)K \}_{x \in X_n},$$

$$(38) \quad V_n \subset V_{n+1},$$

$$(39) \quad \overline{\bigcup_{n \geq 1} V_n} = \mathcal{M}_{r,m}.$$

*Proof.* We start by showing (37). By the construction we made above,  $V_n \subseteq \text{span} \{ \lambda(x)K \}_{x \in X_n}$ .

We first observe that the map

$$F \mapsto (\langle F, \psi_{n,x} \rangle_{L_2(G)})_{x \in X_n}$$

is surjective from  $L_{r,m}(G)$  to  $\mathbb{C}^{X_n}$ . Indeed, if this was not true, there would be a nonzero family  $(\alpha_x)_{x \in X_n} \in \mathbb{C}^{X_n}$  satisfying  $\sum_{x \in X_n} \alpha_x \langle F, \psi_{n,x} \rangle_{L_2} = 0$  for all  $F \in L_{r,m}(G)$ , then  $\sum_{x \in X_n} \alpha_x \psi_{n,x} = 0$  in  $L_{r',m^{-1}}(G)$  and, hence, almost everywhere, then by assumption  $\alpha_x = 0$  for all  $x \in X_n$ , a contradiction. It follows that  $\text{span} \{ \lambda(x)K \}_{x \in X_n} \subseteq V_n$  and (37) holds true.

Equation (38) is an easy consequence of (34) and (37).

It remains to show (39). Since the sequence  $(V_n)_{n \in \mathbb{N}}$  is an increasing family of subspaces, and since  $V_n \subset \mathcal{M}_{r,m}$  for all  $n \in \mathbb{N}$ , the set  $\overline{\bigcup_{n \geq 1} V_n}$  is a subspace of the closed space  $\mathcal{M}_{r,m}$ . Hence, by the Hahn-Banach theorem, condition (39) is equivalent to the following condition: If  $\Gamma \in \mathcal{M}'_{r,m}$  satisfies

$$\langle \Gamma, F \rangle_{\mathcal{M}'_{r,m} \times \mathcal{M}_{r,m}} = 0, \quad \text{for all } F \in V_n, \quad n \in \mathbb{N},$$

then  $\Gamma = 0$  in  $\mathcal{M}'_{r,m}$ . By Lemma 3.2 we can write  $\langle \Gamma, F \rangle_{\mathcal{M}'_{r,m} \times \mathcal{M}_{r,m}} = \langle g, F \rangle_{L_2}$  for all  $F \in \mathcal{M}_{r,m}$ , for a suitable  $g \in L_{r',m^{-1}}(G)$ . Since  $\lambda(x)K \in \mathcal{M}_{r,m}$ ,  $x \in G$ , for every  $f \in L_{r',m^{-1}}(G)$  with  $f - g \in \mathcal{M}_{r,m}^\perp$ , it holds for all  $x \in G$ ,

$$(g * K)(x) = \langle g, \lambda(x)K \rangle_{L_2} = \langle \Gamma, \lambda(x)K \rangle_{\mathcal{M}'_{r,m} \times \mathcal{M}_{r,m}}.$$

Now, with  $F = T_n f$  for some  $f \in L_{r,m}(G)$ , we obtain

$$\begin{aligned} 0 &= \langle \Gamma, T_n f \rangle_{\mathcal{M}'_{r,m} \times \mathcal{M}_{r,m}} = \sum_{x \in X_n} \langle \psi_{n,x}, f \rangle_{L_2} \cdot \langle \Gamma, \lambda(x)K \rangle_{\mathcal{M}'_{r,m} \times \mathcal{M}_{r,m}} \\ &= \sum_{x \in X_n} \langle \psi_{n,x}, f \rangle_{L_2} \cdot (g * K)(x) = \left\langle \sum_{x \in X_n} (g * K)(x) \psi_{n,x}, f \right\rangle_{L_2}. \end{aligned}$$

Since this holds for any  $f \in L_{r,m}(G)$ , we get  $\sum_{x \in X_n} (g * K)(x) \psi_{n,x} = 0$  in  $L_{r',m^{-1}}(G)$  for all  $n \in \mathbb{N}$ . Because the finite family  $\{ \psi_{n,x} \}_{x \in X_n}$  is linearly independent as elements of  $L_{r',m^{-1}}(G)$ , we have  $(g * K)(x) = 0$  for all  $x \in X_n$  and  $n \in \mathbb{N}$ . Therefore, by (35), the function  $g * K$  vanishes on a dense subset of  $G$ . But since we have  $g * K(x) = \langle g, \lambda(x)K \rangle_{L_2}$  with  $g \in L_{r',m^{-1}}(G)$ , and since the map  $G \rightarrow L_{r,m}(G), x \mapsto \lambda(x)K$  is continuous, we see that  $g * K : G \rightarrow \mathbb{C}$  is a continuous functions, so that we get  $g * K \equiv 0$ , i.e.,  $\langle \Gamma, \lambda(x)K \rangle_{\mathcal{M}'_{r,m} \times \mathcal{M}_{r,m}} = 0$  for all  $x \in G$ .

By Assumption 3.4, this implies  $\Gamma = 0$  as an element of  $\mathcal{M}'_{r,m}$ , which proves (39).  $\square$

**Remark 3.10.** *By the correspondence principle, analogous results to (37), (38) and (39) hold true for  $\tilde{V}_n$ . This can be seen as follows: Since it holds  $V_e\pi(x)u = \lambda(x)K$  for all  $x \in X_n$ , by (37) we obtain*

$$(40) \quad \tilde{V}_n = \text{span} \{ \pi(x)u \}_{x \in X_n}.$$

Hence, the nesting property  $\tilde{V}_n \subset \tilde{V}_{n+1}$  analogous to (38) is straightforward. By the correspondence principle, it follows from (38) that

$$(41) \quad \overline{\bigcup_{n \in \mathbb{N}} \tilde{V}_n} = \text{Co}(L_{r,m}).$$

With the spaces  $V_n$  at hand, in the following we will turn to projections from  $\mathcal{M}_{r,m}$  onto  $V_n$  and their properties. To this end, let  $\pi_n : \mathcal{M}_{r,m} \rightarrow V_n$  be the metric projection defined by

$$(42) \quad \pi_n(F) = \text{argmin}_{g \in V_n} \|F - g\|_{\mathcal{M}_{r,m}}.$$

Since  $\mathcal{M}_{r,m}$  is a closed subspace of  $L_{r,m}(G)$  with  $1 < r < \infty$ , the space  $\mathcal{M}_{r,m}$  is a uniformly convex Banach space and every  $V_n$  is convex and closed; therefore  $\pi_n$  is a well-defined and unique function, see [19, Proposition 3.1]. Similarly, we define the projection  $\tilde{\pi}_n : \text{Co}(L_{r,m}) \rightarrow \tilde{V}_n$  by setting  $\tilde{\pi}_n = V_e^{-1}\pi_n V_e$ .

The following lemma gives us a first norm estimate for this metric projection.

**Lemma 3.11.** *Given  $\varepsilon > 0$  and  $F \in \mathcal{M}_{r,m}$ , there exists  $n^* = n_{F,\varepsilon}^* \in \mathbb{N}$  such that for all  $n \geq n^*$  it holds*

$$(43) \quad \|F - \pi_n(F)\|_{\mathcal{M}_{r,m}} \leq \varepsilon,$$

$$(44) \quad \|\pi_n(F)\|_{\mathcal{M}_{r,m}} \leq (1 + \varepsilon)\|F\|_{\mathcal{M}_{r,m}}.$$

*Proof.* If  $F = 0$  the claim is clear since  $0 \in V_n$  so that  $\pi_n(F) = 0$ . Hence, we can assume that  $F \neq 0$ . Let  $\delta := \min \{1, \|F\|_{\mathcal{M}_{r,m}}\} \cdot \varepsilon > 0$ . By (39) there exists  $n^* \geq 1$  and  $g \in V_{n^*}$  such that  $\|F - g\|_{\mathcal{M}_{r,m}} \leq \delta$ . For all  $n \geq n^*$ , by (38) we have  $g \in V_n$  and, by definition of the metric projection,

$$\|F - \pi_n(F)\|_{\mathcal{M}_{r,m}} \leq \|F - g\|_{\mathcal{M}_{r,m}} \leq \delta \leq \varepsilon.$$

The triangle inequality gives

$$\|\pi_n(F)\|_{\mathcal{M}_{r,m}} \leq \|F - \pi_n(F)\|_{\mathcal{M}_{r,m}} + \|F\|_{\mathcal{M}_{r,m}} \leq \delta + \|F\|_{\mathcal{M}_{r,m}} \leq (1 + \varepsilon)\|F\|_{\mathcal{M}_{r,m}},$$

which concludes the proof.  $\square$

The following auxiliary result establishes a first upper bound for certain coefficients related to functions  $F \in \mathcal{M}_{r,m}$ . This will be used for the atomic decomposition afterwards.

**Proposition 3.12.** *For any  $F \in L_{r,m}(G)$  and  $n \in \mathbb{N}$ , let the coefficients  $c_{n,x} \in \mathbb{C}$ ,  $x \in X_n$ , be defined via*

$$c_{n,x} := \int_G F(y)\psi_{n,x}(y) dy.$$

Then the inequality

$$(45) \quad \left( \sum_{x \in X_n} |c_{n,x}|^r m(x)^r \right)^{1/r} \leq |\mathcal{Q}_n|^{1/r'} \cdot \sup_{q \in \mathcal{Q}_n} w(q) \cdot \|F\|_{L_{r,m}}$$

holds, where  $|\mathcal{Q}_n|$  denotes the Haar measure of the set  $\mathcal{Q}_n$  and  $r'$  denotes the dual exponent of  $r$ .

*Proof.* We first note that, since  $\psi_{n,x}$  is compactly supported and bounded, the coefficient  $c_{n,x}$  is well-defined.

Next, we observe that if  $\psi_{n,x}(y) \neq 0$ , then  $y = xq_n$  for some  $q_n \in \mathcal{Q}_n$ , and hence  $m(x) = m(xq_nq_n^{-1}) \leq m(xq_n) \cdot w(q_n^{-1}) \leq m(y) \cdot \sup_{q \in \mathcal{Q}_n} w(q)$ . This shows

$$(46) \quad \begin{aligned} m(x) \cdot |c_{n,x}| &\leq m(x) \cdot \int_G |F(y)| \cdot \psi_{n,x}(y) dy \\ &\leq \sup_{q \in \mathcal{Q}_n} w(q) \cdot \int_G |(mF)(y)| \cdot \psi_{n,x}(y) dy. \end{aligned}$$

We will now further estimate the integral on the right-hand side, setting  $F_0 := m \cdot F$  for brevity.

To this end, we define the measure  $d\mu_x$  on  $G$  (for  $x \in X_n$ ) by setting

$$d\mu_x(y) = \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}} dy$$

and readily observe that  $\int_G 1 d\mu_x = 1$ . Thus, by Jensen's inequality, see [8, Theorem 10.2.6], we obtain

$$\begin{aligned} \left( \int_G |F_0(y)| \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}} dy \right)^r &= \left( \int_G |F_0(y)| d\mu_x(y) \right)^r \\ &\leq \int_G |F_0(y)|^r d\mu_x(y) \\ &= \int_G |F_0(y)|^r \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}} dy. \end{aligned}$$

By the properties of  $\Psi_n$ , see (31), (32) and (33), it holds

$$\|\psi_{n,x}\|_{L_1} = \int_G \psi_{n,x}(y) dy \leq \int_{x\mathcal{Q}_n} 1 dy = \int_{\mathcal{Q}_n} 1 dy = |\mathcal{Q}_n|.$$

Recalling (46), we thus see

$$\begin{aligned} &\sum_{x \in X_n} (m(x) \cdot |c_{n,x}|)^r \\ &\leq \sup_{q \in \mathcal{Q}_n} w(q)^r \cdot \sum_{x \in X_n} \|\psi_{n,x}\|_{L_1}^r \left( \int_G |F_0(y)| \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}} dy \right)^r \\ &\leq \sup_{q \in \mathcal{Q}_n} w(q)^r \cdot \sum_{x \in X_n} \|\psi_{n,x}\|_{L_1}^r \int_G |F_0(y)|^r \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}} dy \\ &\leq \sup_{q \in \mathcal{Q}_n} w(q)^r \cdot \sup_{x \in X_n} \|\psi_{n,x}\|_{L_1}^{r-1} \sum_{x \in X_n} \int_G |F_0(y)|^r \psi_{n,x}(y) dy \\ &\leq \sup_{q \in \mathcal{Q}_n} w(q)^r \cdot |\mathcal{Q}_n|^{r-1} \cdot \|F\|_{L_{r,m}}^r, \end{aligned}$$



which concludes the proof.  $\square$

With this at hand we are able to give a first atomic decomposition for functions  $F \in V_n$ ,  $n \in \mathbb{N}$ , as well as an estimate for the norm of the coefficients involved.

**Lemma 3.13.** *Given  $n \in \mathbb{N}$ , for all  $F \in V_n$  the following atomic decomposition holds true:*

$$(47) \quad F = \sum_{x \in X_n} c(F)_{n,x} \lambda(x) K,$$

where the coefficients  $c(F)_{n,x}$  are of the form

$$(48) \quad c(F)_{n,x} = \langle S_n F, \psi_{n,x} \rangle_{L_2},$$

where  $S_n$  denotes any linear right inverse of  $T_n : L_{r,m}(G) \rightarrow V_n$ . In particular, the coefficients depend linearly on  $F$  and they satisfy

$$(49) \quad \left( \sum_{x \in X_n} |c(F)_{n,x}|^r m(x)^r \right)^{1/r} \leq C_n \|F\|_{\mathcal{M}_{r,m}},$$

with  $C_n = \|S_n\| \cdot |Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q)$ .

*Proof.* We first observe that the operator  $T_n$  admits a bounded right inverse  $S_n : V_n \rightarrow L_{r,m}(G)$ . Indeed, by [4, Theorem 2.12] the existence of a bounded right inverse is equivalent to the existence of a topological supplement of the kernel of  $T_n$ . However, since the spaces  $V_n$  are finite dimensional, such a topological supplement exists, see [4, Example 2.4.2].

In the remainder of the proof, we denote by  $S_n$  an arbitrary linear right inverse of  $T_n$ . Thus, for all  $F \in V_n$  we have the decomposition

$$F = T_n S_n F = \sum_{x \in X_n} \langle S_n F, \psi_{n,x} \rangle_{L_2} \lambda(x) K,$$

so that (47) holds true if we define the coefficients  $c(F)_{n,x}$  as in (48). With this notation the coefficients depend linearly on  $F$ . By applying (45) we obtain the estimate

$$\left( \sum_{x \in X_n} |c(F)_{n,x}|^r m(x)^r \right)^{1/r} \leq |Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q) \cdot \|S_n F\|_{\mathcal{M}_{r,m}} \leq C_n \|F\|_{\mathcal{M}_{r,m}},$$

where  $C_n$  is as in the statement of the lemma, and where  $\|S_n\|$  is the operator norm of  $S_n$  as an operator from  $V_n$  into  $L_{r,m}(G)$ . This proves (49).  $\square$

**Remark 3.14.** *Note that if the sequence  $(|Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q) \cdot \|S_n\|)_{n \in \mathbb{N}}$  is bounded, then the constant  $C_n$  in (49) can be bounded independently of  $n$ . Naturally the question arises under which conditions this really is the case. To answer this question, it is necessary to determine the asymptotic behaviour of the operator norm of  $S_n$ . As we will show in Sect. 3.3 this task is already non-trivial for a very simple setting. Still, in Lemma A.3 we give a partial answer, as we present a technique to characterize the operator-norm in a different manner.*

The proof of the following technical lemma can be found in the appendix. We recall that the integer  $I$  is defined through assumption (30).

**Lemma 3.15.** *Let  $1 \leq p \leq \infty$  and  $(d_x)_{x \in Y_n} \in \ell_{p,m}(Y_n)$  for some  $n \in \mathbb{N}$ , then*

$$\left\| \sum_{x \in Y_n} |d_x| \chi_{xQ_n} \right\|_{L_{p,m}} \leq \mathcal{I}^{1-\frac{1}{p}} \cdot \sup_{q \in Q_n} w(q) \cdot |Q_n|^{\frac{1}{p}} \cdot \|(d_x)_{x \in Y_n}\|_{\ell_{p,m}}$$

with the convention  $\frac{1}{\infty} := 0$ .

With the auxiliary results above, we are in the position to state and prove our main result.

**Theorem 3.16.** *We assume that  $K$  satisfies (5) and that there exists  $p < r$  such that*

$$(50) \quad \begin{aligned} K &\in L_{p,w\Delta^{-1/p}}(G), \\ \text{osc}_{Q_n}(K) &\in L_{p,w}(G) \cap L_{p,w\Delta^{-1/p}}(G), \end{aligned}$$

for all  $n \in \mathbb{N}$ .

i) *Fix  $\varepsilon > 0$ ; then for any  $T \in \text{Co}(L_{r,m})$  there exists  $n^* = n_{T,\varepsilon}^* \in \mathbb{N}$  such that for all  $n \geq n^*$*

$$\left\| T - \sum_{x \in X_n} c(T)_{n,x} \pi(x) u \right\|_{\text{Co}(L_{r,m})} \leq \varepsilon,$$

where the family  $(c(T)_{n,x})_{x \in X_n}$  satisfies

$$\|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{r,m}} \leq C_n(1 + \varepsilon) \|T\|_{\text{Co}(L_{r,m})},$$

with  $C_n = |Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q) \cdot \|S_n\|$ , where  $S_n$  denotes any linear right inverse to the operator  $T_n : L_{r,m}(G) \rightarrow V_n$  defined in (36).

ii) *Let  $n \in \mathbb{N}$ , and let  $d = (d_x)_{x \in Y_n} \in \ell_{q,m}(Y_n)$ . Then  $T = \sum_{x \in Y_n} d_x \pi(x) u$  is in  $\text{Co}(L_{r,m})$ . Furthermore the estimate*

$$\|T\|_{\text{Co}(L_{r,m})} \leq D_n \|(d_x)_{x \in Y_n}\|_{\ell_{q,m}}$$

holds, where  $1/q + 1/p = 1 + 1/r$ , and

$$(51) \quad D_n := |Q_n|^{\frac{1}{q}-1} \cdot \mathcal{I}^{1-\frac{1}{q}} \cdot \sup_{q \in Q_n} w(q) \cdot \theta_n$$

with  $\theta_n := \max \left\{ \|\text{osc}_{Q_n}(K) + |K|\|_{L_{p,w}}, \|\text{osc}_{Q_n}(K) + |K|\|_{L_{p,w\Delta^{-1/p}}} \right\}$ .

*Proof.* To prove i), choose  $n^* = n_{F,\varepsilon}^*$  as in Lemma 3.11 with  $F = V_e T \in \mathcal{M}_{r,m}$ . By applying (47) and (48) to  $\pi_n(F) \in V_n$  we obtain the atomic decomposition

$$\begin{aligned} \tilde{\pi}_n(T) &= V_e^{-1} \pi_n(F) = V_e^{-1} \left( \sum_{x \in X_n} \langle S_n \pi_n(F), \psi_{n,x} \rangle_{L_2} \cdot \lambda(x) K \right) \\ &= \sum_{x \in X_n} \langle S_n \pi_n V_e(T), \psi_{n,x} \rangle_{L_2} \cdot V_e^{-1} \lambda(x) K = \sum_{x \in X_n} c(T)_{n,x} \pi(x) u, \end{aligned}$$

where  $c(T)_{n,x} = \langle S_n \pi_n V_e(T), \psi_{n,x} \rangle_{L_2}$ . Using (43) and the correspondence principle we derive

$$\begin{aligned} \left\| T - \sum_{x \in X_n} c(T)_{n,x} \pi(x) u \right\|_{\text{Co}(L_{r,m})} &= \|T - \tilde{\pi}_n(T)\|_{\text{Co}(L_{r,m})} = \|V_e T - V_e \tilde{\pi}_n(T)\|_{\mathcal{M}_{r,m}} \\ &= \|F - \pi_n(F)\|_{\mathcal{M}_{r,m}} \leq \varepsilon. \end{aligned}$$

Now (49) and (44) yield the estimate

$$\begin{aligned} \left( \sum_{x \in X_n} |c(T)_{n,x}|^r m(x)^r \right)^{\frac{1}{r}} &\leq C_n \|\pi_n(V_e T)\|_{\mathcal{M}_{r,m}} \leq C_n(1 + \varepsilon) \|V_e T\|_{\mathcal{M}_{r,m}} \\ &= C_n(1 + \varepsilon) \|T\|_{\text{Co}(L_{r,m})} \end{aligned}$$

for any  $n \geq n^*$ .

It remains to prove ii). In [6, Chap. 3, p. 100] the following pointwise estimate for  $y \in G$  has been established:

$$\left| \sum_{x \in Y_n} d_x \lambda(x) K(y) \right| \leq \left( \sum_{x \in Y_n} |d_x| \frac{\chi_{xQ_n}}{|Q_n|} \right) * (\text{osc}_{Q_n}(K) + |K|)(y).$$

Let now  $q > 1$  such that  $1/q + 1/p = 1 + 1/r$ . By using Young's inequality, see Proposition A.2, and Lemma 3.15, we obtain

$$\begin{aligned} \left\| \sum_{x \in Y_n} d_x \pi(x) u \right\|_{\text{Co}(L_{r,m})} &= \left\| \sum_{x \in Y_n} d_x \lambda(x) K \right\|_{L_{r,m}} \\ &\leq \left\| \left( \sum_{x \in Y_n} |d_x| \chi_{xQ_n} \right) * (\text{osc}_{Q_n}(K) + |K|) \right\|_{L_{r,m}} \cdot |Q_n|^{-1} \\ &\leq \left\| \sum_{x \in Y_n} |d_x| \chi_{xQ_n} \right\|_{L_{q,m}} \cdot |Q_n|^{-1} \\ &\quad \cdot \max \left\{ \|\text{osc}_{Q_n}(K) + |K|\|_{L_{p,w}}, \|\text{osc}_{Q_n}(K) + |K|\|_{L_{p,w\Delta^{-1/p}}} \right\} \\ &\leq |Q_n|^{\frac{1}{q}-1} \cdot \mathcal{I}^{1-\frac{1}{q}} \cdot \sup_{q \in Q_n} w(q) \cdot \|(d_x)\|_{\ell_{q,m}} \\ &\quad \cdot \max \left\{ \|\text{osc}_{Q_n}(K) + |K|\|_{L_{p,w}}, \|\text{osc}_{Q_n}(K) + |K|\|_{L_{p,w\Delta^{-1/p}}} \right\}. \end{aligned}$$

By the assumption (50) the expression on the right-hand side is finite.  $\square$

**Remark 3.17.** *The coefficients  $c(T)_{n,x}$ ,  $x \in X_n$ , in Theorem 3.16 i) depend linearly on  $T$  if and only if the projection  $\pi_n$  from (42) is linear.*

The following proposition presents a slight variation of Theorem 3.16.

**Proposition 3.18.** *Under the same assumptions as in Theorem 3.16 the following holds: Fix  $\varepsilon > 0$  and  $T \in \text{Co}(L_{r,m})$ ; then there exists  $n^* = n_{T,\varepsilon}^* \in \mathbb{N}$  such that for all  $n \geq n^*$*

$$(52) \quad \frac{1}{\tau_n(1 + \varepsilon)} \cdot \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{q,m}} \leq \|T\|_{\text{Co}(L_{r,m})}$$

and

$$(53) \quad \|T\|_{\text{Co}(L_{r,m})} \leq \varepsilon + D_n \cdot \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{q,m}},$$

where  $D_n$  as in (51) and  $\tau_n := C_n \cdot |X_n|^{\frac{1}{q}-\frac{1}{r}}$ ,  $|X_n|$  is the cardinality of  $X_n$  and  $1/q + 1/p = 1 + 1/r$ .

*Proof.* Throughout this proof we use the same notations as in Theorem 3.16. We first note that for any finite sequence  $(d_x)_{x \in X_n}$ ,  $n \in \mathbb{N}$ , by Hölder's inequality, it holds that

$$\|(d_x)_{x \in X_n}\|_{\ell_{q,m}} \leq \|(d_x)_{x \in X_n}\|_{\ell_{r,m}} \cdot \|1_{X_n}\|_{\ell_{\frac{r}{r-q}}},$$

where  $1_{X_n}$  is a sequence of ones only. Furthermore it holds

$$\|1_{X_n}\|_{\ell_{\frac{r}{r-q}}} = |X_n|^{\frac{1}{q} - \frac{1}{r}}.$$

With  $\tau_n := C_n \cdot |X_n|^{\frac{1}{q} - \frac{1}{r}}$  we then obtain from Theorem 3.16 i) the estimate

$$\frac{1}{\tau_n(1 + \varepsilon)} \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{q,m}} \leq \frac{1}{C_n(1 + \varepsilon)} \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{r,m}} \leq \|T\|_{\text{Co}(L_{r,m})},$$

which proves (52).

It remains to show the second inequality (53). For this we note that the sequence  $(c(T)_{n,x})_{x \in X_n}$  can be understood as a sequence over the index set  $Y_n$  with only finitely many non-zero entries. Therefore, by (43) and Theorem 3.16 ii), this yields

$$\begin{aligned} \|T\|_{\text{Co}(L_{r,m})} &\leq \varepsilon + \left\| \sum_{x \in X_n} c(T)_{n,x} \pi(x) u \right\|_{\text{Co}(L_{r,m})} \\ &\leq \varepsilon + |Q_n|^{\frac{1}{q} - 1} \cdot \theta_n \cdot \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{q,m}}, \end{aligned}$$

which concludes the proof.  $\square$

**3.3. An Example: Coorbit Theory for Paley-Wiener Spaces.** As an example we will discuss the case of band-limited functions on the real line. This case cannot be handled with the classical coorbit theory, since the reproducing kernel that arises is the sinc function, which is not integrable. Thus, the band-limited functions are a suitable example for our setting.

We will briefly recall the setting following the lines of Sect. 4.2 in [5]. Let  $G$  denote the additive group  $\mathbb{R}$  whose Haar measure is the Lebesgue measure  $dx$ . Since the group is abelian,  $\mathbb{R}$  is unimodular. We denote by  $\mathcal{S}(\mathbb{R})$  the Schwartz space of smooth, rapidly decaying functions and by  $\mathcal{S}'(\mathbb{R})$  the space of tempered distributions. The Fourier transform on  $\mathcal{S}(\mathbb{R})$  and  $\mathcal{S}'(\mathbb{R})$ —defined for  $f \in L^1(\mathbb{R})$  as  $\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$ —is denoted by  $\mathcal{F}$ . If  $v \in \mathcal{S}'(\mathbb{R})$  we also set  $\widehat{v} = \mathcal{F}v$ .

The Hilbert space  $\mathcal{H}$  we are interested in is the Paley-Wiener space of functions with band in the fixed set  $\Omega \subset \mathbb{R}$ , namely

$$\mathcal{H} = B_{\Omega}^2 = \{v \in L_2(\mathbb{R}) \mid \text{supp}(\widehat{v}) \subseteq \Omega\}$$

equipped with the  $L_2(\mathbb{R})$  scalar product. Then, by defining  $\pi$  for  $b \in \mathbb{R}$  as

$$\pi(b)v(x) = v(x - b), \quad v \in B_{\Omega}^2,$$

$\pi$  becomes a unitary representation of the group  $\mathbb{R}$  acting on  $B_{\Omega}^2$ . With this definition of  $\pi$ , on the frequency side  $\widehat{\pi} = \mathcal{F}\pi\mathcal{F}^{-1}$  acts on  $\mathcal{F}\mathcal{H} = L_2(\Omega)$  by modulations:

$$\widehat{\pi}(b)\widehat{v}(\xi) = e^{2\pi i b \xi} \widehat{v}(\xi), \quad v \in B_{\Omega}^2.$$

From now on we set  $\Omega$  to be a symmetrical interval,  $\Omega = [-\omega, \omega]$ . Proposition 4.6 in [5] then shows that by choosing as admissible vector the function  $u = \mathcal{F}^{-1}\chi_{\Omega} \in$

$B_\Omega^2$ , the resulting kernel  $K$  as defined in (3) is the sinc function

$$(54) \quad K(b) = \mathcal{F}^{-1} \chi_\Omega(b) = 2\omega \operatorname{sinc}(2\omega\pi b) = \frac{\sin(2\omega\pi b)}{\pi b},$$

where  $\operatorname{sinc} x = \sin x/x$ . Clearly,  $K$  is not in  $L_1(\mathbb{R})$ , but it belongs to  $L_p(\mathbb{R})$  for every  $p > 1$ . Therefore we choose the weight  $w = 1$  and take

$$\mathcal{T} = \bigcap_{1 < p < \infty} L_p(\mathbb{R})$$

as a target space to construct coorbits, see (19). As above, the (anti)-dual of  $\mathcal{T}$  can be identified with

$$\mathcal{T}' = \mathcal{U} = \operatorname{span} \bigcup_{1 < q < \infty} L_q(\mathbb{R}).$$

For  $p \in [1, \infty)$ , we define the Paley-Wiener  $p$ -spaces

$$B_\Omega^p := \left\{ f \in L_p(\mathbb{R}) \mid \operatorname{supp}(\mathcal{F}f) \subseteq \Omega \right\}.$$

Recall that the Fourier transform maps  $L_p(\mathbb{R})$  to  $L_{p'}(\mathbb{R})$  for  $p \leq 2$ , which follows from the Hausdorff-Young inequality. In contrast, for  $p > 2$  the space  $\mathcal{F}L_p(\mathbb{R})$  contains distributions that in general are not functions, see [23, Theorem 7.6.6].

The spaces  $B_\Omega^p$  are sometimes defined in the literature as the spaces of the entire functions of fixed exponential type whose restriction to the real line is in  $L_p(\mathbb{R})$ . This definition is equivalent to ours since a Paley-Wiener theorem holds for all  $p \in [1, \infty)$ . In particular, all these functions are infinitely differentiable on  $\mathbb{R}$ . Moreover, if  $f \in B_\Omega^p$  with  $p < \infty$ , then  $f(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ , and hence

$$B_\Omega^p \subset C_0^\infty(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(x) \rightarrow 0 \text{ as } x \rightarrow \pm\infty\}, \quad 1 \leq p < \infty.$$

Consequently, the Paley-Wiener spaces are nested and increase with  $p$ :

$$B_\Omega^p \subseteq B_\Omega^q, \quad 1 \leq p \leq q < \infty.$$

**Proposition 3.19** (Proposition 4.8 of [5]). *Let  $\Omega = [-\omega, \omega]$  and define  $u := K := \mathcal{F}^{-1} \chi_\Omega$ . The “test space” (as defined in (6)) is*

$$\mathcal{S} = \bigcap_{p \in (1, \infty)} B_\Omega^p$$

and its dual space is

$$\mathcal{S}' = \bigcup_{p \in (1, \infty)} B_\Omega^p.$$

The extended voice transform is the inclusion

$$V_e : \mathcal{S}' \hookrightarrow \mathcal{U}$$

and the following identification holds:

$$\operatorname{Co}(L_p(\mathbb{R})) = \mathcal{M}^p = B_\Omega^p.$$

To obtain a discretization as laid out in Sect. 3, we first need to show that Assumption 3.4 is fulfilled. By Lemma 3.8 it suffices to show that the convolution operator associated to  $K$  is a bounded operator on  $L_p(\mathbb{R})$ .

**Corollary 3.20.** *Let  $1 < p < \infty$ , then  $RC_K$  is a bounded operator on  $L_p(\mathbb{R})$ .*

*Proof.* Since  $K = \mathcal{F}^{-1} \chi_\Omega$ , the convolution with  $K$  is a bounded operator on  $L_p(\mathbb{R})$  if and only if  $\chi_\Omega$  is a Fourier multiplier on  $L_p(\mathbb{R})$ . By [20, Example 2.5.15] this is true if and only if  $\chi_{[0,1]}$  is a Fourier multiplier on  $L_p(\mathbb{R})$ . However, it is well-known that this is true because the Hilbert transform is bounded as an operator acting on  $L_p(\mathbb{R})$ , see [20, Theorem 5.1.7].  $\square$

We will now apply the analysis outlined in Sect. 3.2 to obtain a discretization for these spaces. To this end, for  $n \in \mathbb{N}$ , let

$$(55) \quad Y_n := \{2^{-n}k\}_{k \in \mathbb{Z}} \subset \mathbb{R}.$$

Furthermore we fix

$$(56) \quad Q_n = [-2^{-n-1}, 2^{-n-1}],$$

which is a compact neighborhood of zero, and we set

$$(57) \quad \psi_{n,k} := \chi_{[-2^{-n-1}, 2^{-n-1}]}(\cdot - 2^{-n}k)$$

for  $n \in \mathbb{N}$  and  $k \in \mathbb{Z}$ , where  $\chi$  denotes the characteristic function. Then, the verification of (31)–(33) is straightforward. For later use we note that  $|Q_n| = 2^{-n}$ . Furthermore the system  $\{\psi_{n,k}\}_{k \in \mathbb{Z}}$ ,  $n \in \mathbb{N}$  fixed, is orthogonal with  $\|\psi_{n,k}\|_{L_2(\mathbb{R})}^2 = |Q_n|$ . As a finite subset of  $Y_n$ ,  $n \in \mathbb{N}$ , we set

$$(58) \quad X_n := \{2^{-n}k \mid -N(n) \leq k \leq N(n)\},$$

where  $N(n) \in \mathbb{N}$  is chosen such that (34) and (35) are fulfilled. A possible choice is  $N = N(n) = n \cdot 2^n$ .

According to (36) the operator  $T_n : B_\Omega^p \rightarrow V_n \subset B_\Omega^p$  is defined via

$$T_n f(x) = \sum_{k=-N(n)}^{N(n)} \langle f, \psi_{n,k} \rangle_{L_2} K(x - 2^{-n}k),$$

for  $f \in B_\Omega^p$ , where

$$\langle f, \psi_{n,k} \rangle_{L_2} = \int_{2^{-n}(k-1/2)}^{2^{-n}(k+1/2)} f(y) dy.$$

By (37) this means

$$V_n = \text{span} \{ \text{sinc}(2\pi\omega(\cdot - 2^{-n}k)) \mid -N(n) \leq k \leq N(n) \}.$$

In order to apply Theorem 3.16 we first need to show the following:

**Lemma 3.21.** *It holds  $K \in \mathcal{M}_{Q_n}^p(L_r)$ , and therefore  $\text{osc}_{Q_n} f \in L_r(\mathbb{R})$  for all  $n \in \mathbb{N}$ .*

*Proof.* We have  $|K(y)| \leq 2\omega$  for all  $y \in \mathbb{R}$ , and  $|K(b)| \leq 1/(\pi|b|)$  for all  $b \neq 0$ . This implies

$$|K(y)| \leq \frac{1 + 4\omega}{1 + |y|}.$$

Indeed, for  $|y| \leq 1$  we have  $|K(y)| \leq 2\omega \leq \frac{4\omega}{1+|y|}$ , while for  $|y| \geq 1$ , we have  $1/|y| \leq 2/(1+|y|)$ , and thus  $|K(y)| \leq \frac{2}{\pi} \frac{1}{|y|} \leq \frac{1+4\omega}{1+|y|}$ .

Now, for  $y \in x + Q_n \subset x + [-1, 1]$  we have  $1 + |x| \leq 2 + |y| \leq 2(1 + |y|)$ , so that  $|K(y)| \leq \frac{1+4\omega}{1+|y|} \leq \frac{2+8\omega}{1+|x|}$ . Hence,

$$\sup_{y \in x + Q_n} |K(y)|^r \leq \left( \frac{2 + 8\omega}{1 + |x|} \right)^r,$$

and thus

$$\int_{\mathbb{R}} \sup_{y \in x + Q_n} |K(y)|^r dx < \infty,$$

concluding the proof.  $\square$

With this at hand we can discretize the Paley-Wiener  $p$ -spaces according to Theorem 3.16.

**Proposition 3.22.** *Let  $1 < p < \infty$ .*

*i) Fix  $\varepsilon > 0$ ; then for any  $f \in B_{\Omega}^p$  there exists an integer  $n^* = n_{f,\varepsilon}^* \in \mathbb{N}$ , such that for all  $n \geq n^*$*

$$\left\| f - \sum_{k=-N(n)}^{N(n)} c(f)_{n,k} K(\cdot - 2^{-n}k) \right\|_{L_p} \leq \varepsilon,$$

*where the family of coefficients  $(c(f)_{n,k})_{-N(n) \leq k \leq N(n)}$  satisfies*

$$\|(c(f)_{n,k})_{-N(n) \leq k \leq N(n)}\|_{\ell_p} \leq 2^{-n/p'} (1 + \varepsilon) \|S_n\| \cdot \|f\|_{L_p}.$$

*Here, as usual,  $S_n$  is a linear right inverse for the operator  $T_n$  defined in (36).*

*ii) For any sequence  $(d_x)_{x \in Y_n} \in \ell_q(Y_n)$ ,  $n \in \mathbb{N}$ , the function  $f$  defined by  $f = \sum_{k \in \mathbb{Z}} d_{2^{-n}k} K(\cdot - 2^{-n}k)$  is in  $B_{\Omega}^p$  with*

$$\|f\|_{L_p(\mathbb{R})} \leq C \cdot 2^{n(1-1/q)} \|(d_x)_{x \in Y_n}\|_{\ell_q},$$

*where  $C = C(p, q) > 0$  is a constant and  $q < p$ .*

*Proof.* i) is an application of Theorem 3.16 i), with  $|Q_n| = 2^{-n}$ .

It remains to prove ii). Again, we can apply Theorem 3.16 ii) and note that, by Lemma 3.21, the assumption (50) is fulfilled. Moreover, Lemma 3.21 shows that  $\|\text{osc}_{Q_n}(K) + |K|\|_{L_r}$  can be estimated from above by a constant  $C > 0$  independent of  $n \in \mathbb{N}$ .  $\square$

As stated in Remark 3.14 the asymptotic behaviour of the operator norm of  $S_n$  is crucial. In the following we apply Lemma A.3 to obtain a useful characterization of  $\|S_n\|$ .

For this we restrict ourselves to the case  $p = 2$  and obtain with the notation of Lemma A.3

$$\begin{aligned} \|S_n\|^{-1} &= \varepsilon = \inf \left\{ \frac{\|T_n f\|_{L_2}}{\|f\|_{L_2}} \mid f \in (\text{Ker } T_n)^\perp \right\} \\ &= \inf \left\{ \frac{\langle T_n^* T_n f, f \rangle_{L_2}}{\langle f, f \rangle_{L_2}} \mid f \in (\text{Ker } T_n)^\perp \right\}^{1/2} = \lambda_{\min}(U_n)^{1/2}, \end{aligned}$$

where  $\lambda_{\min}(U_n)$  denotes the smallest eigenvalue of the operator

$$U_n := T_n^* T_n : (\text{Ker } T_n)^\perp \rightarrow (\text{Ker } T_n)^\perp.$$

Here, we used the well-known inclusion  $\text{Ran } A^* \subset (\text{Ker } A)^\perp$  which guarantees that  $U_n$  is well-defined.

We have thus shown that the asymptotic behaviour of the smallest eigenvalue of  $U_n$  is equivalent to the asymptotic behaviour of  $\|S_n\|$ .

By using

$$T_n f = \sum_{j=-N(n)}^{N(n)} \langle f, \psi_{n,j} \rangle_{L_2} K(\cdot - 2^{-n}j), \quad T_n^* g = \sum_{k=-N(n)}^{N(n)} \langle g, K(\cdot - 2^{-n}k) \rangle_{L_2} \psi_{n,k},$$

we can rewrite  $U_n$  as

$$\begin{aligned} U_n f &= \sum_{j,k=-N(n)}^{N(n)} \langle f, \psi_{n,j} \rangle_{L_2(\mathbb{R})} \langle K(\cdot - 2^{-n}j), K(\cdot - 2^{-n}k) \rangle_{L_2(\mathbb{R})} \psi_{n,k} \\ &= \sum_{j,k=-N(n)}^{N(n)} \langle f, \psi_{n,j} \rangle_{L_2(\mathbb{R})} K(2^{-n}(k-j)) \psi_{n,k} \end{aligned}$$

for  $f \in (\text{Ker } T_n)^\perp$ .

We set  $W_n := \text{span}\{\psi_{n,k} \mid -N(n) \leq k \leq N(n)\}$  and obtain the relation  $W_n^\perp \subset \text{Ker } T_n$ ; thus  $(\text{Ker } T_n)^\perp \subset W_n$ . Next, we note that the family  $\{\lambda(x)K\}_{x \in \mathbb{R}}$  is linearly independent; indeed, we have  $\mathcal{F}(\lambda(x)K) = e^{-2\pi i x \cdot} \chi_{[-\omega, \omega]}$ , and by analyticity these functions are linearly independent if and only if the functions  $(\mathbb{R} \rightarrow \mathbb{C}, \xi \mapsto e^{-2\pi i x \xi})$  are. But each of these functions is an eigenvector of the differential operator  $d/d\xi$  with pairwise distinct eigenvalues  $2\pi i x, x \in \mathbb{R}$ , which yields the linear independence. From this and from Lemma 3.9, we see that  $\text{Ran } T_n = V_n = \text{span}\{\lambda(x)K\}_{x \in X_n}$  satisfies  $\dim \text{Ran } T_n = |X_n| = 1 + 2N(n)$ . But since  $T_n : (\text{Ker } T_n)^\perp \rightarrow V_n$  is an isomorphism, we see  $\dim(\text{Ker } T_n)^\perp = 1 + 2N(n)$  as well, so that we finally see  $W_n = (\text{Ker } T_n)^\perp$  by comparing dimensions. Hence,  $U_n : W_n \rightarrow W_n$ .

Moreover, by the orthogonality of the family  $\{\psi_{n,k}\}$ , we see that

$$(59) \quad U_n \psi_{n,k} = \|\psi_{n,k}\|_{L_2}^2 \sum_{\ell=-N(n)}^{N(n)} K(2^{-n}(\ell-k)) \psi_{n,\ell}$$

for any  $-N(n) \leq k \leq N(n)$ . Since  $\dim W_n = 2N(n) + 1 < \infty$ , we may define an isomorphism

$$(60) \quad P_n : W_n \rightarrow \mathbb{R}^{2N(n)+1}, \quad P_n(\psi_{n,k}) = \|\psi_{n,k}\|_{L_2} e_k,$$

where  $e_k$  denotes the  $k$ -th canonical unit vector of  $\mathbb{R}^{2N(n)+1}$ . Note that  $P_n$  maps the orthonormal basis  $(\psi_{n,k}/\|\psi_{n,k}\|_{L_2(\mathbb{R})})$  to the orthonormal basis  $(e_k)_{k \in \mathbb{N}}$ , so that  $P_n$  is unitary.

The linear map  $P_n U_n P_n^{-1} : \mathbb{R}^{2N(n)+1} \rightarrow \mathbb{R}^{2N(n)+1}$  is represented by a matrix  $M_n$ , whose entries are given via

$$\begin{aligned} (M_n)_{j,k} &= \langle P_n U_n P_n^{-1} e_k, e_j \rangle_{\mathbb{R}^{2N(n)+1}} = \langle U_n \frac{\psi_{n,k}}{\|\psi_{n,k}\|_{L_2}}, \frac{\psi_{n,j}}{\|\psi_{n,j}\|_{L_2(\mathbb{R})}} \rangle_{L_2} \\ &= \frac{\|\psi_{n,k}\|_{L_2}}{\|\psi_{n,j}\|_{L_2}} \sum_{\ell=-N(n)}^{N(n)} K(2^{-n}(\ell-k)) \langle \psi_{n,\ell}, \psi_{n,j} \rangle_{L_2} \\ &= \|\psi_{n,k}\|_{L_2} \|\psi_{n,j}\|_{L_2} K(2^{-n}(j-k)) \\ &= 2^{-n} K(2^{-n}(j-k)), \end{aligned}$$

$1 \leq j, k \leq 2N(n) + 1$ . Since  $K$  is real, the matrix  $M_n$  is a *symmetric Toeplitz matrix*, which means that the entries of  $M_n$  only depend on the quantity  $|k-j|$ , thus yielding a band-structure. Since the eigenvalues of  $M_n$  coincide with those of the map  $U_n$ ,



finding the smallest eigenvalue of  $U_n$  is equivalent to finding the smallest eigenvalue of the Toeplitz matrix  $M_n$ .

Unfortunately, this task is very difficult. To the best knowledge of the authors, it is not possible to properly characterize the asymptotic behaviour of the smallest eigenvalue of such a Toeplitz matrix. We further refer to [3], where the authors were told that leading experts on the field of Toeplitz matrices are unaware of these asymptotics.

Since there are already big obstacles in understanding the asymptotic behaviour of  $\|S_n\|$  in this rather simple setting, one cannot hope that easy answers are available when turning to more complex groups and their associated coorbit spaces.

#### 4. OBSTRUCTIONS TO DISCRETIZATION FOR NON-INTEGRABLE KERNELS

In classical coorbit theory, the kernel  $K(x) = Vu(x) = \langle u, \pi(x)u \rangle_{\mathcal{H}}$  is assumed to be integrable; in other words, it has to satisfy  $K \in L_{1,w}(G)$  for a suitable weight  $w \geq 1$  on  $G$ . This assumption is introduced in order to guarantee two independent properties: First, it ensures that one can construct a suitable reservoir of “distributions,” and thus obtains well-defined coorbit spaces. Second, it ensures that the right convolution operator  $f \mapsto f * K$  acts boundedly on the function space  $Y$  which is used to define the coorbit space  $\text{Co}(Y)$ . For instance, this is the case if  $Y = L_{r,m}(G)$  with a  $w$ -moderate weight  $m$ .

Replacing the integrability condition  $K \in L_{1,w}(G)$  by the weaker assumption  $K \in \bigcap_{1 < p < \infty} L_{p,w}(G)$ , one can still define a suitable reservoir and obtains well-defined decomposition spaces, as we saw in Sect. 2. However, we will see in the present section—precisely, in Proposition 4.5—that the modified assumption  $K \in \bigcap_{1 < p < \infty} L_{p,w}(G)$  is in general too weak to ensure that right convolution with  $K$  defines a bounded operator on  $L_{r,m}(G)$ . In other words, a given kernel  $K$  satisfying the weak integrability assumption might or might not act boundedly on  $L_{r,m}(G)$  by right convolution.

For such “bad” kernels that do not act boundedly, no discretization results similar to those from classical coorbit theory can hold, as we will prove in the present section. Therefore, if such discretization results for the coorbit space  $\text{Co}(L_{r,m})$  are desired, one needs to assume that  $K \in \bigcap_{1 < p < \infty} L_{p,w}(G)$  and additionally that  $f \mapsto f * K$  defines a bounded operator on  $L_{r,m}(G)$ . This second condition is highly nontrivial to verify in many cases where the kernel  $K$  is not integrable. However, it is possible in the setting of the group  $(\mathbb{R}, +)$  as discussed in Sect. 3.3 above.

Since we aim to show that no discretization as for classical coorbit theory is possible, we briefly recall these results: Assuming the kernel  $K$  to be well behaved, a combination of Lemma 3.5 v) and Theorem 6.1 in [13] shows that the *synthesis operator*

$$\text{Synth}_X : \ell_{r,m_X}(I) \rightarrow \text{Co}(L_{r,m}), (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \cdot \pi(x_i)u \quad \text{with} \quad (m_X)_i = m(x_i)$$

is well-defined and bounded, for each  $r \in (1, \infty)$ , each  $w$ -moderate weight  $m$ , and each family  $X = (x_i)_{i \in I}$  in  $G$  that is sufficiently *separated*—similar to  $\delta\mathbb{Z}^d$  in  $G = \mathbb{R}^d$ . The operator  $\text{Synth}_X$  even has a bounded linear right inverse, provided that the family  $X$  is sufficiently dense in  $G$ , where the required density only depends on  $w, u$ . If  $\text{Synth}_X$  indeed has a bounded linear right inverse, the family  $(\pi(x_i)u)_{i \in I}$  is called a family of atoms for  $\text{Co}(L_{r,m})$  with coefficient space  $\ell_{r,m_X}(I)$ .

Dual to the concept of atomic decompositions is the notion of *Banach frames*, which was introduced in [21]. By definition, the family  $(\pi(x_i)u)_{i \in I}$  is a Banach frame for  $\text{Co}(L_{r,m})$  with coefficient space  $\ell_{r,m_X}(I)$  if the *analysis operator*

$$A_X : \text{Co}(L_{r,m}) \rightarrow \ell_{r,m_X}(I), f \mapsto (\langle f, \pi(x_i)u \rangle_{S_w})_{i \in I}$$

is well-defined and bounded and has a bounded linear left inverse. As shown in [21, Theorem 5.3], this is satisfied if the sampling points  $X = (x_i)_{i \in I}$  satisfy the same properties as above: they should be sufficiently separated and dense enough in  $G$ , where these conditions only depend on  $w$  and  $u$ , but not on the integrability exponent  $r$  or the  $w$ -moderate weight  $m$ . Provided that  $(\pi(x_i)u)_{i \in I}$  is a Banach frame for  $\text{Co}(L_{r,m})$ , we have in particular  $\|A_X f\|_{\ell_{r,m_X}} \asymp \|f\|_{\text{Co}(L_{r,m})}$  for all  $f \in \text{Co}(L_{r,m})$ ; but in general this latter property is weaker than the Banach frame property.

The preceding statements hold for all  $w$ -moderate weights  $m$  and for all exponents  $r \in (1, \infty)$ . Since the reciprocal  $m^{-1}$  of a  $w$ -moderate weight  $m$  is again  $w$ -moderate, see Lemma 2.8, it follows that if the above properties hold for  $L_{r,m}(G)$ , then they also hold for  $L_{r',m^{-1}}(G)$ . Therefore, classical coorbit theory provides discretization results that are stronger than the assumptions of the following theorem. The following theorem thus shows that discretization results as in classical coorbit theory can only hold if the kernel  $K$  acts boundedly on  $L_{r,m}(G)$  via right convolutions.

**Theorem 4.1.** *Let  $r \in (1, \infty)$  be arbitrary. Assume that Assumption 2.2 is satisfied, and let  $m : G \rightarrow (0, \infty)$  be a  $w$ -moderate weight. Furthermore, assume that for some family  $(x_i)_{i \in I}$  in  $G$  and for some weight  $\theta = (\theta_i)_{i \in I}$  on the index set  $I$ , the following hold:*

i) “*Weak Banach frame condition for  $\text{Co}(L_{r,m})$ ”:* The analysis map

$$A : \text{Co}(L_{r,m}) \rightarrow \ell_{r,\theta}(I), \varphi \mapsto (\langle \varphi, \pi(x_i)u \rangle_{S_w})_{i \in I}$$

is well-defined and bounded, with

$$(61) \quad \|A\varphi\|_{\ell_{r,\theta}} \asymp \|\varphi\|_{\text{Co}(L_{r,m})} \quad \text{for all } \varphi \in \text{Co}(L_{r,m}).$$

ii) “*Weak atomic decomposition condition for  $\text{Co}(L_{r',m^{-1}})$ ”:* The synthesis map

$$S : \ell_{r',\theta^{-1}}(I) \rightarrow \text{Co}(L_{r',m^{-1}}), (c_i)_{i \in I} \mapsto \sum_{i \in I} [c_i \cdot \pi(x_i)u]$$

is well-defined and bounded.

Then the right convolution operator  $RC_K : f \mapsto f * K$  defines a bounded linear operator on  $L_{r,m}(G)$ .

For the proof of this theorem, we will need several technical lemmata. Having shown in Sect. 2 that the voice transform can be extended from  $\mathcal{H}$  to the reservoir  $S'_w$  (and thus to the coorbit spaces  $\text{Co}(L_{r,m})$ ), our first lemma shows that one can also define a version of the voice transform on the (anti)-dual space  $[\text{Co}(L_{r,m})]'$ .

**Lemma 4.2.** *If Assumption 2.2 is satisfied for  $r \in (1, \infty)$ , and if  $m : G \rightarrow (0, \infty)$  is  $w$ -moderate, then there is a constant  $C = C(m, r, w, K) > 0$  such that*

$$\text{for all } x \in G : \quad \pi(x)u \in \text{Co}(L_{r,m}) \quad \text{and} \quad \|\pi(x)u\|_{\text{Co}(L_{r,m})} \leq C \cdot w(x).$$

Therefore, for any (antilinear) continuous functional  $\varphi \in [\text{Co}(L_{r,m})]'$ , the special voice transform

$$V_{\text{sp}} \varphi : G \rightarrow \mathbb{C}, x \mapsto \varphi(\pi(x)u) = \langle \varphi, \pi(x)u \rangle_{[\text{Co}(L_{r,m})]' \times \text{Co}(L_{r,m})}$$

is a well-defined function.

*Proof.* First, let us set  $C_1 := m(e)$ , where  $e$  is the unit element of  $G$ . Since  $m$  is  $w$ -moderate (see (13)), we have

$$m(x) = m(x \cdot e) \leq w(x) \cdot m(e) \leq C_1 \cdot w(x) \quad \text{for all } x \in G.$$

Furthermore,

$$w(y) = w(xx^{-1}y) \leq w(x) \cdot w(x^{-1}y) = w(x) \cdot (\lambda(x)w)(y).$$

Now, recall from Sect. 2 the embedding  $\mathcal{H} \hookrightarrow \mathcal{S}'_w$ , and that the extended voice transform  $V_e$  coincides with the usual voice transform on  $\mathcal{H}$ . Therefore, since  $\pi(x)u \in \mathcal{H}$ , and since  $K = Vu$ , we get

$$\begin{aligned} \|V_e [\pi(x)u]\|_{L_{r,m}} &= \|V [\pi(x)u]\|_{L_{r,m}} \leq C_1 \cdot \|V [\pi(x)u]\|_{L_{r,w}} \\ &= C_1 \cdot \|w \cdot \lambda(x) [Vu]\|_{L_r} \leq C_1 \cdot w(x) \cdot \|\lambda(x) [w \cdot Vu]\|_{L_r} \\ &= C_1 \cdot w(x) \cdot \|w \cdot Vu\|_{L_r} = C_1 \cdot w(x) \cdot \|K\|_{L_{r,w}} = C \cdot w(x), \end{aligned}$$

where  $C := C_1 \cdot \|K\|_{L_{r,w}}$  is finite thanks to Assumption 2.2. This proves the first part of the lemma, which then trivially implies that  $V_{\text{sp}} \varphi$  is a well-defined function, for any  $\varphi \in [\text{Co}(L_{r,m})]'$ .  $\square$

Our next lemma shows that if the right convolution with  $K$  does *not* act boundedly on  $L_{r',m^{-1}}(G)$ , then there exist certain pathological functionals on  $\text{Co}(L_{r,m})$ .

**Lemma 4.3.** *Assume that Assumption 2.2 is satisfied, and let  $r \in (1, \infty)$ . If the right convolution operator  $RC_K : f \mapsto f * K$  does not yield a well-defined bounded linear operator on  $L_{r',m^{-1}}(G)$ , then there is an (antilinear) continuous functional  $\varphi \in [\text{Co}(L_{r,m})]'$  satisfying  $V_{\text{sp}} \varphi \notin L_{r',m^{-1}}(G)$ .*

*Proof.* We first claim that there is some  $\Phi \in L_{r',m^{-1}}(G)$  with  $\Phi * K \notin L_{r',m^{-1}}(G)$ ; that is, we claim that  $RC_K : L_{r',m^{-1}}(G) \rightarrow L_{r',m^{-1}}(G)$  is not well-defined.

To see this, recall from Assumption 2.2 that  $K \in \bigcap_{1 < p < \infty} L_{p,w}(G)$ . Thus, since  $m^{-1}$  is  $w$ -moderate (see Lemma 2.8), Young's inequality (see Proposition A.2) shows that the right convolution operator  $RC_K$  is bounded as a map  $RC_K : L_{r',m^{-1}}(G) \rightarrow L_{q,m^{-1}}(G)$  for any  $q \in (r', \infty)$ . Therefore, if  $RC_K : L_{r',m^{-1}}(G) \rightarrow L_{r',m^{-1}}(G)$  was well-defined, then the closed graph theorem would imply that  $RC_K : L_{r',m^{-1}}(G) \rightarrow L_{r',m^{-1}}(G)$  is bounded, contradicting our assumptions. Hence, there is a function  $\Phi$  as desired.

Now, define the antilinear functional

$$\varphi : \text{Co}(L_{r,m}) \rightarrow \mathbb{C}, \quad f \mapsto \int_G \Phi(y) \cdot \overline{V_e f(y)} dy.$$

It is easy to see that  $\varphi$  is well-defined and bounded; in fact,

$$|\varphi(f)| \leq \|\Phi\|_{L_{r',m^{-1}}} \cdot \|V_e f\|_{L_{r,m}} = \|\Phi\|_{L_{r',m^{-1}}} \cdot \|f\|_{\text{Co}(L_{r,m})}.$$

Finally, note for all  $x \in G$  that

$$\begin{aligned} V_{\text{sp}} \varphi(x) &= \langle \varphi, \pi(x)u \rangle_{[\text{Co}(L_{r,m})]' \times \text{Co}(L_{r,m})} = \int_G \Phi(y) \cdot \overline{V_e [\pi(x)u](y)} dy \\ &= \int_G \Phi(y) \cdot \overline{\langle \pi(x)u, \pi(y)u \rangle_{\mathcal{H}}} dy = \int_G \Phi(y) \cdot \langle u, \pi(y^{-1}x)u \rangle_{\mathcal{H}} dy \\ &= \int_G \Phi(y) \cdot K(y^{-1}x) dy = (\Phi * K)(x) \end{aligned}$$

with  $\Phi * K \in L_{q,m^{-1}}(G)$  for all  $q \in (r', \infty)$ . But by our choice of  $\Phi$ , we have  $V_{\text{sp}} \varphi = \Phi * K \notin L_{r',m^{-1}}(G)$ , as desired.  $\square$

Our next lemma shows that the assumptions of Theorem 4.1 exclude the existence of pathological functionals as in the preceding lemma.

**Lemma 4.4.** *Under the assumptions of Theorem 4.1 and with notation as in Lemma 4.2, every antilinear continuous functional  $\varphi \in [\text{Co}(L_{r,m})]'$  satisfies  $V_{\text{sp}} \varphi \in L_{r',m^{-1}}(G)$ .*

*Proof.* Let  $\varphi \in [\text{Co}(L_{r,m})]'$  be arbitrary, and let the analysis operator  $A$  be as in the assumptions of Theorem 4.1. Using this operator, we define the (antilinear) functional

$$\Lambda_0 : A(\text{Co}(L_{r,m})) \rightarrow \mathbb{C}, Af \mapsto \varphi(f).$$

Note that this is well-defined, since (61) ensures that  $A$  is injective. Furthermore, with  $A(\text{Co}(L_{r,m}))$  considered as a subspace of  $\ell_{r,\theta}(I)$ , the functional  $\Lambda_0$  is bounded, since (61) yields a constant  $C > 0$  such that each  $c = Af \in A(\text{Co}(L_{r,m}))$  satisfies

$$\begin{aligned} |\Lambda_0(c)| &= |\varphi(f)| \leq \|\varphi\|_{[\text{Co}(L_{r,m})]'} \cdot \|f\|_{\text{Co}(L_{r,m})} \\ &\leq C \|\varphi\|_{[\text{Co}(L_{r,m})]'} \cdot \|Af\|_{\ell_{r,\theta}} = C \|\varphi\|_{[\text{Co}(L_{r,m})]'} \cdot \|c\|_{\ell_{r,\theta}}. \end{aligned}$$

With  $\Lambda_0$  being bounded, an antilinear version of the Hahn-Banach theorem yields a bounded (antilinear) extension  $\Lambda : \ell_{r,\theta}(I) \rightarrow \mathbb{C}$  of  $\Lambda_0$ . Therefore, an antilinear version of the Riesz representation theorem for the dual of  $\ell_{r,\theta}(I)$  ensures the existence of  $\varrho = (\varrho_i)_{i \in I} \in \ell_{r',\theta^{-1}}(I)$  satisfying  $\Lambda(c) = \langle \varrho, c \rangle_{\ell_{r',\theta^{-1}} \times \ell_{r,\theta}}$  for all  $c \in \ell_{r,\theta}(I)$ . Here, the pairing between  $\ell_{r',\theta^{-1}}(I)$  and  $\ell_{r,\theta}(I)$  is given by  $\langle (c_i)_{i \in I}, (e_i)_{i \in I} \rangle_{\ell_{r',\theta^{-1}} \times \ell_{r,\theta}} = \sum_{i \in I} c_i \cdot \bar{e}_i$ .

Having constructed the sequence  $\varrho \in \ell_{r',\theta^{-1}}(I)$ , we can now apply the second assumption of Theorem 4.1—the boundedness of the synthesis operator  $S$ —to define  $g := S\varrho \in \text{Co}(L_{r',m^{-1}})$ . Furthermore, for arbitrary  $x \in G$ , we recall from Lemma 4.2 that  $\pi(x)u \in \text{Co}(L_{r,m})$ , so that

$$c^{(x)} = (c_i^{(x)})_{i \in I} := A(\pi(x)u) = (\langle \pi(x)u, \pi(x_i)u \rangle_{\mathcal{H}})_{i \in I} \in \ell_{r,\theta}(I)$$

is well-defined. Combining our preceding observations, we see

$$\begin{aligned} (62) \quad V_{\text{sp}} \varphi(x) &= \varphi(\pi(x)u) = \Lambda_0(A(\pi(x)u)) = \Lambda(c^{(x)}) = \langle \varrho, c^{(x)} \rangle_{\ell_{r',\theta^{-1}} \times \ell_{r,\theta}} \\ &= \sum_{i \in I} [\varrho_i \cdot \overline{\langle \pi(x)u, \pi(x_i)u \rangle_{\mathcal{H}}}] = \sum_{i \in I} [\varrho_i \cdot \langle \pi(x_i)u, \pi(x)u \rangle_{\mathcal{S}_w}] \\ &\stackrel{(*)}{=} \left\langle \sum_{i \in I} (\varrho_i \cdot \pi(x_i)u), \pi(x)u \right\rangle_{\mathcal{S}_w} = \langle S\varrho, \pi(x)u \rangle_{\mathcal{S}_w} = [V_e g](x). \end{aligned}$$

This identity—which will be fully justified below—completes the proof, since we have  $g = S\varrho \in \text{Co}(L_{r',m^{-1}})$ , that is  $V_e g \in L_{r',m^{-1}}(G)$ . Therefore, (62) implies  $V_{\text{sp}} \varphi = V_e g \in L_{r',m^{-1}}(G)$ , as claimed.

It remains to justify the step marked with  $(*)$  in (62). At that step, we used on the one hand that  $S\varrho = \sum_{i \in I} [\varrho_i \cdot \pi(x_i)u]$  with unconditional convergence in  $\text{Co}(L_{r',m^{-1}})$ . To see that this indeed holds, recall that  $r' < \infty$ , so that  $\varrho = \sum_{i \in I} \varrho_i \delta_i$ , with unconditional convergence in  $\ell_{r',\theta^{-1}}(I)$ ; by the boundedness of  $S$ , this implies the claimed identity. On the other hand, we also used at  $(*)$  that  $\text{Co}(L_{r',m^{-1}}) \rightarrow$

$\mathbb{C}, f \mapsto \langle f, \pi(x)u \rangle_{\mathcal{S}_w}$  is a bounded linear functional. Indeed, (10) and Lemma 4.2 imply

$$\begin{aligned} |\langle f, \pi(x)u \rangle_{\mathcal{S}_w}| &= |\langle V_e f, V[\pi(x)u] \rangle_{L_2}| \leq \|V_e f\|_{L_{r',m^{-1}}} \cdot \|V[\pi(x)u]\|_{L_{r,m}} \\ &= \|f\|_{\text{Co}(L_{r',m^{-1}})} \cdot \|V[\pi(x)u]\|_{L_{r,m}} \leq C \cdot \|f\|_{\text{Co}(L_{r',m^{-1}})} \cdot w(x), \end{aligned}$$

with  $C = C(m, w, u, r)$ .  $\square$

We can now finally prove Theorem 4.1.

*Proof of Theorem 4.1.* Assume towards a contradiction that the right convolution operator  $RC_K : L_{r,m}(G) \rightarrow L_{r,m}(G)$  is not bounded. By Proposition 3.6, and since the prerequisites of Theorem 4.1 include Assumption 2.2, this implies that  $RC_K : L_{r',m^{-1}}(G) \rightarrow L_{r',m^{-1}}(G)$  is also not bounded. Therefore, Lemma 4.3 yields an antilinear continuous functional  $\varphi \in [\text{Co}(L_{r,m})]'$  with  $V_{\text{sp}}\varphi \notin L_{r',m^{-1}}(G)$ . In view of Lemma 4.4, this yields the desired contradiction.  $\square$

Before closing this section, we show that the “weak integrability assumption”  $K \in \bigcap_{1 < p < \infty} L_{p,w}(G)$  does *not* imply in general that the right convolution operator  $RC_K : f \mapsto f * K$  acts boundedly on any  $L_p$ -space with  $p \neq 2$ .

To this end, we consider as in Sect. 3.3 the Paley-Wiener space

$$(63) \quad \mathcal{H} = B_{\Omega}^2 = \{f \in L_2(\mathbb{R}) : \widehat{f} \equiv 0 \text{ almost everywhere on } \mathbb{R} \setminus \Omega\}$$

for a fixed measurable subset  $\Omega \subset \mathbb{R}$  of finite measure. As seen in Sect. 3.3, the group  $G = \mathbb{R}$  acts on this space by translations; that is, if we set  $\pi(x)f = \lambda(x)f$  for  $f \in B_{\Omega}^2$ , then  $\pi$  is a unitary representation of  $\mathbb{R}$ . Setting  $u := \mathcal{F}^{-1}\chi_{\Omega} \in B_{\Omega}^2$ , using Plancherel’s theorem, and noting  $\widehat{f} = \widehat{f} \cdot \chi_{\Omega} = \widehat{f} \cdot \widetilde{u}$  for  $f \in B_{\Omega}^2$ , we see that the associated voice transform is given by

$$Vf(x) = \langle f, \pi(x)u \rangle_{L_2} = \langle \widehat{f}, e^{-2\pi ix} \widehat{u} \rangle_{L_2} = \int_{\mathbb{R}} \widehat{f}(\xi) \cdot e^{2\pi ix\xi} d\xi = (\mathcal{F}^{-1}\widehat{f})(x) = f(x).$$

Thus,  $V : B_{\Omega}^2 \rightarrow L_2(\mathbb{R})$  is an isometry and the reproducing kernel  $K$  is simply given by  $K(x) = Vu(x) = u(x)$  for  $x \in \mathbb{R}$ . In view of these remarks, the following proposition shows that there is a reproducing kernel that satisfies the weak integrability assumption, but for which the associated right convolution operator does *not* act boundedly on  $L_p(\mathbb{R})$  for any  $p \neq 2$ .

**Proposition 4.5.** *There is a compact set  $C \subset [0, 1]$  with the following properties:*

- i)  $\mathcal{F}^{-1}\chi_C \in \bigcap_{1 < p \leq \infty} L_p(\mathbb{R})$ .
- ii) *For any  $p \in (1, \infty) \setminus \{2\}$ , the convolution operator  $f \mapsto f * \mathcal{F}^{-1}\chi_C$  is not bounded, and by Proposition 3.5 not well-defined, as an operator on  $L_p(\mathbb{R})$ .*

Since the construction of the set  $C$  is quite technical, we defer the proof to the appendix.

## 5. IMPROVED DISCRETIZATION RESULTS UNDER ADDITIONAL ASSUMPTIONS

In the preceding section we have seen that there are limitations to the possible discretization theory for coorbit spaces with “bad” kernels, that is, for kernels  $K$  for which the right convolution with  $K$  does not act boundedly on  $L_{r,m}(G)$ .

But even if this right convolution operator *does* act boundedly, the results in the preceding sections only yield discretization results that are weaker than those

that one would expect to hold when coming from classical coorbit theory. In the present section we will see that a “proper” discretization theory is possible even for relatively bad (i.e., non-integrable) kernels, as long as the kernel in question acts boundedly on  $L_{r,m}(G)$  and is compatible with another “well-behaved” kernel  $W : G \rightarrow \mathbb{C}$ , in the sense that it satisfies  $K * W = K$  for the construction of Banach frames, or  $W * K = K$  for the construction of atomic decompositions.

We emphasize that we do *not* assume that the kernel  $W$  satisfies  $W * W = W$ , thereby allowing a larger freedom in the choice of  $W$ . To see that the property  $K * K = K$  is indeed quite restrictive, let us consider the case when  $G = \mathbb{R}$  is the real line. Then  $K * K = K$  implies that  $\widehat{K} = \widehat{K} \cdot \widehat{K}$ , so that  $\widehat{K} = \chi_\Omega$  must be the indicator function of a (measurable) set, see also [5]. In particular,  $K \notin L_1(\mathbb{R})$  (unless  $K \equiv 0$ ), since otherwise  $\widehat{K}$  would be continuous. In stark contrast, at least if the set  $\Omega$  is bounded, one can choose a Schwartz function  $\psi$  with  $\psi \equiv 1$  on  $\Omega$ , so that  $W := \mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R})$  satisfies  $\widehat{W * K} = \widehat{W} \cdot \chi_\Omega = \chi_\Omega = \widehat{K}$ , and thus  $K * W = W * K = K$ . It is worth noting that a related approach has been established in [17].

The section is structured as follows: In the first subsection, we recall some basic notions from classical coorbit theory: Relatively separated sets, BUPUs, etc. Then, in Subsect. 5.2 we discuss conditions on the well-behaved kernel  $W$  which guarantee the existence of Banach frames for the coorbit spaces. The existence of atomic decompositions, under similar but different conditions on  $W$ , is discussed in Subsect. 5.3. In the last subsection we apply the abstract results to the setting of Paley-Wiener spaces.

Finally, we should mention that most of the proofs in this section are heavily inspired by the original coorbit papers [12, 13, 14, 21]. The main novel ingredient here is the observation that instead of the idempotent reproducing formula  $K * K = K$ , it suffices to have  $K * W = K$  or  $W * K = K$  for potentially different kernels  $K, W$ .

**Remark 5.1.** *Most of the results in this section can also be obtained for coorbit spaces  $\text{Co}(Y)$  where  $Y$  is a solid Banach space continuously embedded into  $L_0(G)$ . For simplicity, we restrict our attention to the case  $Y = L_{r,m}(G)$  as in the rest of the paper.*

**5.1. Required Notions from Classical Coorbit Theory.** We would like to sample the continuous frame  $(\pi(x)u)_{x \in G}$  to obtain a discrete (Banach) frame  $(\pi(x_i)u)_{i \in I}$ . In order for this to succeed, the family of sampling points  $(x_i)_{i \in I}$  needs to be sufficiently well distributed in  $G$ . This intuition is made precise in the following definition. The reader might compare this to the definitions in the beginning of Sect. 3.2.

**Definition 5.2.** (cf. [13, Definition 3.2])

Let  $X = (x_i)_{i \in I}$  be a family in  $G$ .

- i)  $X$  is  $V$ -dense in  $G$ , for a unit neighborhood  $V \subset G$ , if  $G = \bigcup_{i \in I} x_i V$ .
- ii)  $X$  is  $V$ -separated, for a unit neighborhood  $V \subset G$ , if the family  $(x_i V)_{i \in I}$  is pairwise disjoint.
- iii)  $X$  is relatively separated if for every compact unit neighborhood  $Q \subset G$  there is a constant  $N = N(X, Q) \in \mathbb{N}$  with

$$\sum_{i \in I} \chi_{x_i Q}(x) \leq N \quad \text{for all } x \in G.$$

- iv)  $X$  is  $V$ -well-spread for a unit neighborhood  $V \subset G$  if  $X$  is relatively separated and  $V$ -dense.

- Remark 5.3.** *i) Since we always assume the underlying group  $G$  to be second countable,  $G$  is in particular  $\sigma$ -compact. Therefore, [28, Lemma 2.3.10] shows that (the index set of) every relatively separated family in  $G$  is countable.*
- ii) Usually,  $X$  is called relatively separated if  $X$  is a finite union of  $V$ -separated sets, for some compact unit neighborhood  $V$ . The two definitions are shown to be equivalent in [11, Lemma 2.9] and [28, Lemma 2.3.11].*

Given a  $V$ -well-spread family  $X = (x_i)_{i \in I}$ , one often wants to decompose a given function  $f$  into building blocks  $f_i$  which are supported in the sets  $(x_i V)_{i \in I}$ . This can be done using suitable partitions of unity; again the reader might compare this to Sect. 3.2.

**Definition 5.4.** (cf. [13, Definition 3.6])

Let  $V \subset G$  be a compact unit neighborhood. A family  $\Psi = (\psi_i)_{i \in I}$  is called a  $V$ -BUPU (bounded uniform partition of unity) with localizing family  $X = (x_i)_{i \in I}$  if the following holds:

- i) Each  $\psi_i : G \rightarrow [0, 1]$  is a measurable function.*
- ii)  $X$  is relatively separated and  $\psi_i \equiv 0$  on  $G \setminus x_i V$  for all  $i \in I$ .*
- iii) We have  $\sum_{i \in I} \psi_i \equiv 1$  on  $G$ .*

One can find a  $V$ -BUPU for any compact unit neighborhood  $V$ :

**Lemma 5.5.** (cf. [10, Theorem 2] and [28, Lemma 2.3.12])

Let  $V \subset G$  be an arbitrary compact unit neighborhood. Then there exists a  $V$ -BUPU  $\Psi = (\psi_i)_{i \in I}$  with  $\psi_i \in C_c(G)$  for all  $i \in I$ .

The following lemma points out an important property of relatively separated families that we will use time and again:

**Lemma 5.6.** Let  $X = (x_i)_{i \in I}$  be a relatively separated family and let  $r \in [1, \infty)$ . Let further  $m : G \rightarrow (0, \infty)$  be a  $w$ -moderate weight. Define the weight  $m_X$  on the index set  $I$  by  $(m_X)_i := m(x_i)$  for  $i \in I$ .

Then for every compact unit neighborhood  $U \subset G$ , the synthesis operator

$$\text{Synth}_{X,U} : \ell_{r,m_X}(I) \rightarrow L_{r,m}(G), (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \chi_{x_i U}$$

is well-defined and bounded, with pointwise absolute convergence of the defining series.

Furthermore, if  $\Psi = (\psi_i)_{i \in I}$  is a  $U$ -BUPU with localizing family  $X$ , then the synthesis operator

$$\text{Synth}_{X,\Psi} : \ell_{r,m_X}(I) \rightarrow L_{r,m}(G), (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \psi_i$$

is well-defined and bounded, with pointwise absolute convergence of the defining series.

*Proof.* The second part of the lemma is a consequence of the first one: Since  $0 \leq \psi_i \leq 1$ , and since  $\psi_i$  vanishes outside of  $x_i U$ , we have

$$|(\text{Synth}_{X,\Psi} c)(x)| \leq \sum_{i \in I} |c_i| \psi_i(x) \leq \sum_{i \in I} |c_i| \chi_{x_i U}(x) = (\text{Synth}_{X,U} |c|)(x) < \infty$$

for all  $x \in G$  and all  $c = (c_i)_{i \in I} \in \ell_{r,m_X}(I)$ , where  $|c| = (|c_i|)_{i \in I} \in \ell_{r,m_X}(I)$  with  $\| |c| \|_{\ell_{r,m_X}} = \| c \|_{\ell_{r,m_X}}$ , so that

$$\| \text{Synth}_{X,\Psi} c \|_{L_{r,m}} \leq \| \text{Synth}_{X,U} |c| \|_{L_{r,m}} \lesssim \| |c| \|_{\ell_{r,m_X}} = \| c \|_{\ell_{r,m_X}}.$$

Thus, it remains to prove the first part of the lemma.

By definition of a relatively separated family, there is  $N = N(X, U) > 0$  with  $\sum_{i \in I} \chi_{x_i U} \leq N$ . On the one hand, this shows that for each  $x \in G$  only finitely many terms of the series defining  $(\text{Synth}_{X,U} c)(x)$  do not vanish; in particular, the defining series is pointwise absolutely convergent. On the other hand, we see

$$\begin{aligned} \left| (\text{Synth}_{X,U} c)(x) \right|^r &\leq \left( \sum_{i \in I} |c_i| \chi_{x_i U}(x) \chi_{x_i U}(x) \right)^r \leq \left( \sup_{j \in I} |c_j| \chi_{x_j U}(x) \cdot \sum_{i \in I} \chi_{x_i U}(x) \right)^r \\ &\leq N^r \cdot \sup_{j \in I} |c_j|^r \chi_{x_j U}(x) \leq N^r \cdot \sum_{i \in I} |c_i|^r \chi_{x_i U}(x). \end{aligned}$$

Thus,

$$\begin{aligned} \|\text{Synth}_{X,U} c\|_{L_{r,m}}^r &\leq N^r \cdot \int_G (m(x))^r \cdot \sum_{i \in I} |c_i|^r \chi_{x_i U}(x) dx \\ &\leq N^r \cdot \sum_{i \in I} \left( |c_i|^r \cdot \int_{x_i U} (m(x))^r dx \right). \end{aligned}$$

But for  $x = x_i u \in x_i U$ , we have  $m(x) = m(x_i u) \leq m(x_i) \cdot w(u) \leq C \cdot m(x_i)$  for  $C := \sup_{u \in U} w(u)$ , which is finite since  $U$  is compact and  $w$  is continuous. Overall, since  $|x_i U| = |U|$  for all  $i \in I$ , where  $|U|$  is the Haar-measure of  $U$ , we see

$$\|\text{Synth}_{X,U} c\|_{L_{r,m}}^r \leq N^r \cdot C^r \cdot |U| \cdot \sum_{i \in I} (m(x_i) \cdot |c_i|)^r,$$

which easily yields the boundedness of  $\text{Synth}_{X,U}$ .  $\square$

**5.2. Banach Frames.** In this subsection, we will assume the following:

**Assumption 5.7.** *We fix some  $r \in (1, \infty)$  and a  $w$ -moderate weight  $m : G \rightarrow (0, \infty)$  and assume that the kernel  $K$  from (3) satisfies the following:*

- i) Assumption 2.2 is satisfied, that is,  $K \in L_{p,w}(G)$  for all  $p \in (1, \infty)$ .*
- ii) The right convolution operator  $RC_K : f \mapsto f * K$  is well-defined, and by Proposition 3.5 bounded, as an operator on  $L_{r,m}(G)$ .*
- iii) There is some unit neighborhood  $U_0 \subset G$  such that for each unit neighborhood  $U \subset U_0$  there is a constant  $C_U > 0$  with*

$$(64) \quad \text{for all } f \in \mathcal{M}_{r,m} : \quad \|\text{osc}_U^\rho f\|_{L_{r,m}} \leq C_U \cdot \|f\|_{L_{r,m}}.$$

Here,  $\mathcal{M}_{r,m}$  is the reproducing kernel space from (22), and

$$(65) \quad \text{osc}_U^\rho f(x) := \sup_{u \in U} |f(xu) - f(x)|$$

similar to (26).

- iv) The constants  $C_U$  from the preceding point satisfy  $C_U \rightarrow 0$  as  $U \rightarrow \{e\}$ . More precisely, for every  $\varepsilon > 0$  there is a unit neighborhood  $U_\varepsilon \subset U_0$  with  $C_U \leq \varepsilon$  for all unit neighborhoods  $U \subset U_\varepsilon$ .*

At a first glance it seems that the preceding assumptions have nothing to do with the existence of a “well-behaved” kernel  $W$  which is compatible with the kernel  $K$ . But it turns out that the existence of such a kernel provides an easy way of verifying the preceding assumptions:



**Lemma 5.8.** *Assume that  $K \in L_{p,w}(G)$  for all  $p \in (1, \infty)$  and that the operator  $RC_K : L_{r,m}(G) \rightarrow L_{r,m}(G)$ ,  $f \mapsto f * K$  is well-defined and bounded.*

*Furthermore assume that there is a kernel  $W : G \rightarrow \mathbb{C}$  with the following properties:*

- i)  $W$  is continuous.*
- ii)  $K * W = K$ .*
- iii)  $M_{U_0}^\lambda W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$  for some compact unit neighborhood  $U_0 \subset G$ .*  
Here

$$(66) \quad M_{U_0}^\lambda W(x) := \|W\|_{L_\infty(xU_0)}, \quad x \in G$$

*is the local maximal function (with respect to left regular representation), similar to (24).*

*Then Assumption 5.7 is satisfied.*

*Proof.* We first note that our assumptions imply  $M_U^\lambda W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$  for every compact unit neighborhood  $U \subset G$ . Indeed, by compactness, and since  $U \subset \bigcup_{x \in G} x \operatorname{int}(U_0)$ , there is a finite family  $(x_i)_{i=1, \dots, n}$  with  $U \subset \bigcup_{i=1}^n x_i U_0$ . Therefore,  $xU \subset \bigcup_{i=1}^n xx_i U_0$ , whence

$$\begin{aligned} M_U^\lambda W(x) &= \|W\|_{L_\infty(xU)} \leq \sum_{i=1}^n \|W\|_{L_\infty(xx_i U_0)} \\ &= \sum_{i=1}^n M_{U_0}^\lambda W(xx_i) = \sum_{i=1}^n [\rho(x_i)(M_{U_0}^\lambda W)](x). \end{aligned}$$

But since  $w$  and  $w\Delta^{-1}$  are submultiplicative, both  $L_{1,w}(G)$  and  $L_{1,w\Delta^{-1}}(G)$  are invariant under right translations, and hence  $M_U^\lambda W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ .

Next, if  $V$  is an open precompact unit neighborhood and  $U := \bar{V}$ , then by continuity of  $W$ , we have  $|W(x)| \leq \sup_{v \in V} |W(xv)| = \|W\|_{L_\infty(xV)} \leq M_U^\lambda W(x)$  for all  $x \in G$ . Therefore,  $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ .

Since by assumption the right convolution operator  $RC_K$  acts boundedly on  $L_{r,m}(G)$ , Lemma 3.8 shows that the set  $X_0 := \operatorname{span}\{\lambda(x)K\}_{x \in G}$  is dense in the reproducing kernel space  $\mathcal{M}_{r,m}$ . Furthermore, the assumption  $K * W = K$  yields  $(\lambda(x)K) * W = \lambda(x)(K * W) = \lambda(x)K$  for all  $x \in G$ , and thus  $f * W = f$  for all  $f \in X_0$ . By density of  $X_0$  in  $\mathcal{M}_{r,m}$ , and since the right convolution operator  $f \mapsto f * W$  is continuous on  $L_{r,m}(G)$  thanks to  $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$  and Young's inequality (Proposition A.2), we see

$$(67) \quad f * W = f \quad \text{for all } f \in \mathcal{M}_{r,m}.$$

We now use (67) to prove (64). To this end, let  $U \subset G$  be an arbitrary compact unit neighborhood. Let  $f \in \mathcal{M}_{r,m}$ ,  $x \in G$  and  $u \in U$  be arbitrary. Then

$$\begin{aligned} |f(xu) - f(x)| &= |(f * W)(xu) - (f * W)(x)| \leq \int_G |f(y)| \cdot |W(y^{-1}xu) - W(y^{-1}x)| dy \\ &\leq \int_G |f(y)| \cdot (\operatorname{osc}_U^\rho W)(y^{-1}x) dy = (|f| * (\operatorname{osc}_U^\rho W))(x). \end{aligned}$$

Since this holds for every  $u \in U$ , we get  $\operatorname{osc}_U^\rho f(x) \leq |f| * (\operatorname{osc}_U^\rho W)(x)$  for all  $x \in G$ . By solidity of  $L_{r,m}(G)$  and in view of Young's inequality (Proposition A.2), this

implies

$$\|\operatorname{osc}_U^\rho f\|_{L_{r,m}} \leq \|f\|_{L_{r,m}} \cdot \max\{\|\operatorname{osc}_U^\rho W\|_{L_{1,w}}, \|\operatorname{osc}_U^\rho W\|_{L_{1,w\Delta^{-1}}}\}.$$

But an easy generalization of Lemma 3.3 shows that  $\|\operatorname{osc}_U^\rho W\|_{L_{1,v}} \rightarrow 0$  as  $U \rightarrow \{e\}$ , for  $v = w$  as well as for  $v = w\Delta^{-1}$ . From this it is not hard to see that the two remaining properties from Assumption 5.7 are satisfied.  $\square$

We now prove that Assumption 5.7 ensures that a sufficiently fine sampling of the continuous frame  $(\pi(x)u)_{x \in G}$  provides a Banach frame for the coorbit space  $\operatorname{Co}(L_{r,m})$ . For this, we will first show that we can sample the point evaluation functionals to obtain a Banach frame for the reproducing kernel space  $\mathcal{M}_{r,m}$ . In the end, we will then use the correspondence principle to transfer the result from the reproducing kernel space to the coorbit space.

We begin by showing that a certain sampling operator is bounded:

**Lemma 5.9.** *Let Assumption 5.7 be satisfied, and let  $X = (x_i)_{i \in I}$  be a relatively separated family in  $G$ .*

*Then, with the weight  $m_X$  as in Lemma 5.6, the sampling operator*

$$\operatorname{Samp}_X : \mathcal{M}_{r,m} \rightarrow \ell_{r,m_X}(I), f \mapsto (f(x_i))_{i \in I} = (\langle f, \lambda_{x_i} K \rangle_{L_2})_{i \in I}$$

*is well-defined and bounded.*

*Proof.* We first recall that each  $f \in \mathcal{M}_{r,m}$  satisfies  $f = f * K$ , and hence

$$f(x) = f * K(x) = \int_G f(y) \cdot K(y^{-1}x) dy = \int_G f(y) \cdot \overline{K(x^{-1}y)} dy = \langle f, \lambda(x)K \rangle_{L_2}.$$

But  $K \in L_{r',w}(G)$ , and thus also  $\lambda(x)K \in L_{r',w}(G)$ , since  $w$  is submultiplicative. Furthermore, since  $m$  is  $w$ -moderate, we have  $m(e) = m(xx^{-1}) \leq m(x)w(x^{-1})$ , and thus  $[m(x)]^{-1} \leq w(x^{-1})/m(e) = w(x)/m(e)$ , whence  $L_{r',w}(G) \hookrightarrow L_{r',m^{-1}}(G)$ . Thus, the dual pairing  $\langle f, \lambda(x)K \rangle_{L_2} \in \mathbb{C}$  is well-defined for every  $x \in \mathbb{R}$ . Therefore, each entry  $f(x_i)$  of the sequence  $\operatorname{Samp}_X f = (f(x_i))_{i \in I}$  makes sense.

Now, let  $U$  be a compact unit neighborhood with  $\|\operatorname{osc}_U^\rho f\|_{L_{r,m}} \leq C \cdot \|f\|_{L_{r,m}}$  for all  $f \in \mathcal{M}_{r,m}$ . Such a neighborhood exists by virtue of Assumption 5.7. Note that  $U^{-1}$  is also a compact unit neighborhood, so that by definition of a relatively separated family there is a constant  $N > 0$  with  $\sum_{i \in I} \chi_{x_i U^{-1}}(x) \leq N$  for all  $x \in G$ .

Next, fix any  $i \in I$  and note that  $\chi_{x_i U^{-1}}(x) \neq 0$  can only hold if  $x = x_i u^{-1}$  and thus  $x_i = xu$  for some  $u \in U$ . But in this case, we see by definition of the oscillation  $\operatorname{osc}_U^\rho f$  that

$$|f(x_i)| \leq |f(x)| + |f(x_i) - f(x)| \leq |f(x)| + (\operatorname{osc}_U^\rho f)(x) =: F(x).$$

We have thus shown  $|f(x_i)| \cdot \chi_{x_i U^{-1}}(x) \leq F(x) \cdot \chi_{x_i U^{-1}}(x)$  for all  $x \in G$ . Summing over  $i \in I$ , we see

$$\Theta_f(x) := \sum_{i \in I} |f(x_i)| \cdot \chi_{x_i U^{-1}}(x) \leq \left( \sum_{i \in I} \chi_{x_i U^{-1}}(x) \right) \cdot F(x) \leq N \cdot F(x)$$

for all  $x \in G$ .

Because of  $r \geq 1$ , we have  $\ell_1(I) \hookrightarrow \ell_r(I)$ , which implies  $\sum_{i \in I} c_i^r \leq (\sum_{i \in I} c_i)^r$  for arbitrary  $c_i \geq 0$ . Therefore,

$$\begin{aligned} \int_G (m(x))^r \cdot \sum_{i \in I} |f(x_i)|^r \cdot \chi_{x_i U^{-1}}(x) dx &\leq \|\Theta_f\|_{L_{r,m}}^r \leq N^r \cdot \|F\|_{L_{r,m}}^r \\ &\leq N^r \cdot (\|f\|_{L_{r,m}} + \|\text{osc}_U^{\rho} f\|_{L_{r,m}})^r \\ &\leq N^r \cdot (1 + C)^r \cdot \|f\|_{L_{r,m}}^r. \end{aligned}$$

Finally, if  $\chi_{x_i U^{-1}}(x) \neq 0$ , then  $x = x_i u^{-1}$  for some  $u \in U$ , and therefore  $m(x_i) = m(xu) \leq m(x) \cdot w(u) \leq C' \cdot m(x)$  for  $C' := \sup_{u \in U} w(u)$ , which is finite since  $w$  is continuous and  $U$  is compact.

Overall, we have thus shown

$$\begin{aligned} (C')^{-r} \cdot \sum_{i \in I} (m(x_i))^r \cdot |f(x_i)|^r \cdot |x_i U^{-1}| &\leq \int_G (m(x))^r \cdot \sum_{i \in I} |f(x_i)|^r \cdot \chi_{x_i U^{-1}}(x) dx \\ &\leq N^r \cdot (1 + C)^r \cdot \|f\|_{L_{r,m}}^r, \end{aligned}$$

which—because of  $|x_i U^{-1}| = |U^{-1}|$ —shows

$$\|\text{Samp}_X f\|_{\ell_{r,m_X}} \leq C' N (1 + C) \cdot |U^{-1}|^{-1/r} \cdot \|f\|_{L_{r,m}} \quad \text{for all } f \in \mathcal{M}_{r,m},$$

which finally proves that  $\text{Samp}_X$  is well-defined and bounded.  $\square$

Now we can prove that a sufficiently fine sampling of the point evaluations yields a Banach frame for the reproducing kernel space  $\mathcal{M}_{r,m}$ .

**Proposition 5.10.** *Let Assumption 5.7 be satisfied, and let  $U \subset U_0^{-1}$  be a compact unit neighborhood such that the constant  $C_{U^{-1}}$  from (64) satisfies*

$$(68) \quad \|RC_K\|_{L_{r,m} \rightarrow L_{r,m}} \cdot C_{U^{-1}} < 1.$$

Let  $X = (x_i)_{i \in I}$  be any relatively separated family in  $G$  for which there exists a  $U$ -BUPU  $\Psi = (\psi_i)_{i \in I}$  with localizing family  $X$ , and let the weight  $m_X$  be defined as in Lemma 5.6.

Then there is a bounded linear reconstruction map  $R : \ell_{r,m_X}(I) \rightarrow \mathcal{M}_{r,m}$  which satisfies  $R \circ \text{Samp}_X = \text{id}_{\mathcal{M}_{r,m}}$  for the sampling map  $\text{Samp}_X$  from Lemma 5.9.

In other words, the family  $(\delta_{x_i})_{i \in I}$  of point evaluations forms a Banach frame for  $\mathcal{M}_{r,m}$  with coefficient space  $\ell_{r,m_X}(I)$ .

**Remark 5.11.** *The proof shows that the action of the reconstruction operator is independent of the choice of  $r, m$ .*

In other words, if (68) is satisfied for  $L_{r_1, m_1}(G)$  and  $L_{r_2, m_2}(G)$  and if  $R_1 : \ell_{r_1, m_1, X}(I) \rightarrow \mathcal{M}_{r_1, m_1}$  and  $R_2 : \ell_{r_2, m_2, X}(I) \rightarrow \mathcal{M}_{r_2, m_2}$  denote the respective reconstruction operators, then  $R_1 c = R_2 c$  for all  $c \in \ell_{r_1, m_1, X}(I) \cap \ell_{r_2, m_2, X}(I)$ .

*Proof.* With the synthesis operator  $\text{Synth}_{X, \Psi}$  from Lemma 5.6, we define

$$B := \text{Synth}_{X, \Psi} \circ \text{Samp}_X : \mathcal{M}_{r,m} \rightarrow L_{r,m}(G).$$

Because of  $f(x) = \sum_{i \in I} \psi_i(x) f(x)$ , we have

$$|f(x) - Bf(x)| \leq \sum_{i \in I} \psi_i(x) \cdot |f(x) - f(x_i)|.$$

But if  $\psi_i(x) \neq 0$ , then  $x = x_i u \in x_i U$ , so that  $x_i = x u^{-1} \in x U^{-1}$ , and hence  $|f(x) - f(x_i)| = |f(x) - f(x u^{-1})| \leq \text{osc}_{U^{-1}}^\rho f(x)$ . Therefore,

$$|f(x) - Bf(x)| \leq \sum_{i \in I} \psi_i(x) \text{osc}_{U^{-1}}^\rho f(x) = \text{osc}_{U^{-1}}^\rho f(x).$$

By Proposition 3.6 the operator  $RC_K$  is a projection onto  $\mathcal{M}_{r,m}$ , therefore  $RC_K f = f$  for  $f \in \mathcal{M}_{r,m}$ . Thus, the operator  $A := RC_K \circ B : \mathcal{M}_{r,m} \rightarrow \mathcal{M}_{r,m}$  is well-defined and bounded, and we have

$$\begin{aligned} \|f - Af\|_{L_{r,m}} &= \|RC_K(f - Bf)\|_{L_{r,m}} \leq \|RC_K\|_{L_{r,m} \rightarrow L_{r,m}} \cdot \|f - Bf\|_{L_{r,m}} \\ &\leq \|RC_K\|_{L_{r,m} \rightarrow L_{r,m}} \cdot \|\text{osc}_{U^{-1}}^\rho f\|_{L_{r,m}} \leq \|RC_K\|_{L_{r,m} \rightarrow L_{r,m}} \cdot C_{U^{-1}} \cdot \|f\|_{L_{r,m}} \end{aligned}$$

for all  $f \in \mathcal{M}_{r,m}$ .

In view of (68), a Neumann series argument (see [2, Sect. 5.7]) shows that the bounded linear operator  $R_0 := \sum_{n=0}^{\infty} (\text{id}_{\mathcal{M}_{r,m}} - A)^n : \mathcal{M}_{r,m} \rightarrow \mathcal{M}_{r,m}$  satisfies

$$(R_0 \circ RC_K \circ \text{Synth}_{X,\Psi}) \circ \text{Samp}_X = R_0 \circ A = \text{id}_{\mathcal{M}_{r,m}}.$$

Thus,  $R := R_0 \circ RC_K \circ \text{Synth}_{X,\Psi} : \ell_{r,m_X}(I) \rightarrow \mathcal{M}_{r,m}$  is the desired reconstruction operator. Note that the action of this operator on a given sequence is independent of the choice of  $r, m$ , since the action of the operators  $RC_K$ ,  $\text{Synth}_{X,\Psi}$  and  $A = RC_K \circ \text{Synth}_{X,\Psi} \circ \text{Samp}_X$  is independent of  $r, m$ , so that the same holds for  $R_0 = \sum_{n=0}^{\infty} (\text{id} - A)^n$ .  $\square$

Using the correspondence principle, we can finally lift the result from the reproducing kernel space  $\mathcal{M}_{r,m}$  to the coorbit space  $\text{Co}(L_{r,m})$ .

**Theorem 5.12.** *Under the assumptions of Proposition 5.10, the sampled family  $(\pi(x_i)u)_{i \in I} \subset (\text{Co}(L_{r,m}))'$  forms a Banach frame for  $\text{Co}(L_{r,m})$  with coefficient space  $\ell_{r,m_X}(I)$ .*

*More precisely, the sampling operator*

$$\text{Samp}_{X,\text{Co}} : \text{Co}(L_{r,m}) \rightarrow \ell_{r,m_X}(I), f \mapsto (V_e f(x_i))_{i \in I} = (\langle f, \pi(x_i)u \rangle_{S_w})_{i \in I}$$

*is well-defined and bounded, and there is a bounded linear reconstruction operator  $R_{\text{Co}} : \ell_{r,m_X}(I) \rightarrow \text{Co}(L_{r,m})$  satisfying  $R_{\text{Co}} \circ \text{Samp}_{X,\text{Co}} = \text{id}_{\text{Co}(L_{r,m})}$ .*

*Finally, the action of the reconstruction operator  $R_{\text{Co}}$  is independent of the choice of  $r, m$ , that is, if the assumptions of the current theorem are satisfied for  $L_{r_1,m_1}(G)$  and for  $L_{r_2,m_2}(G)$  and if  $R_1, R_2$  denote the corresponding reconstruction operators, then  $R_1 c = R_2 c$  for all  $c \in \ell_{r_1,m_1,X}(I) \cap \ell_{r_2,m_2,X}(I)$ .*

*Proof.* The correspondence principle (Proposition 2.9) states that the extended voice transform  $V_e : \text{Co}(L_{r,m}) \rightarrow \mathcal{M}_{r,m}$  is an isometric isomorphism. Now, with the sampling map  $\text{Samp}_X$  from Proposition 5.10, we have

$$(\text{Samp}_X \circ V_e) f = (V_e f(x_i))_{i \in I} = (\langle f, \pi(x_i)u \rangle_{S_w})_{i \in I} = \text{Samp}_{X,\text{Co}} f.$$

Thus, the sampling operator  $\text{Samp}_{X,\text{Co}} = \text{Samp}_X \circ V_e : \text{Co}(L_{r,m}) \rightarrow \ell_{r,m_X}(I)$  is indeed well-defined and bounded.

Now, with the reconstruction operator  $R : \ell_{r,m_X}(I) \rightarrow \mathcal{M}_{r,m}$  from Proposition 5.10, define  $R_{\text{Co}} := V_e^{-1} \circ R : \ell_{r,m_X}(I) \rightarrow \text{Co}(L_{r,m})$ . Then

$$R_{\text{Co}} \circ \text{Samp}_{X,\text{Co}} = V_e^{-1} \circ R \circ \text{Samp}_X \circ V_e = V_e^{-1} \circ V_e = \text{id}_{\text{Co}(L_{r,m})},$$

as desired. Since the action of  $R$  is independent of the choice of  $r, m$ , so is the action of  $R_{\text{Co}}$ .  $\square$

**5.3. Atomic Decompositions.** For the case of atomic decompositions we will impose slightly different conditions compared to the case of Banach frames. In this case, our assumptions immediately refer to a “well-behaved” kernel  $W$ .

**Assumption 5.13.** *We fix some  $r \in (1, \infty)$  and some  $w$ -moderate weight  $m : G \rightarrow (0, \infty)$ , and we assume that the kernel  $K$  from (3) satisfies the following:*

- i) *Assumption 2.2 is satisfied, that is,  $K \in L_{p,w}(G)$  for all  $p \in (1, \infty)$ .*
- ii) *The right convolution operator  $RC_K : f \mapsto f * K$  is well-defined and bounded as an operator on  $L_{r,m}(G)$ .*
- iii) *There is a continuous kernel  $W : G \rightarrow \mathbb{C}$  with the following properties:*
  - a)  $W * K = K$ .
  - b)  $\check{M}_Q^\rho W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$  for some compact unit neighborhood  $Q \subset G$ .  
Here, the maximal function  $\check{M}_Q^\rho W$  is defined as in (24).

**Remark 5.14.** *We will use below that  $\check{M}_U^\rho W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$  for every compact unit neighborhood  $U \subset G$  if we assume  $\check{M}_Q^\rho W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$  for some unit neighborhood  $Q \subset G$ .*

*Indeed, by compactness of  $U$ , and since  $U \subset \bigcup_{x \in G} (\text{int } Q)x$ , there is a finite family  $(x_i)_{i=1, \dots, n}$  in  $G$  with  $U \subset \bigcup_{i=1}^n Qx_i$ . Therefore,  $Ux \subset \bigcup_{i=1}^n Qx_ix$ , whence*

$$\check{M}_U^\rho W(x) = \|W\|_{L_\infty(Ux)} \leq \sum_{i=1}^n \|W\|_{L_\infty(Qx_ix)} = \sum_{i=1}^n (\check{M}_Q^\rho W)(x_ix).$$

*By solidity and (left) translation invariance of  $L_{1,v}(G)$  for  $v = w$  or  $v = w\Delta^{-1}$ , this implies*

$$\|\check{M}_U^\rho W\|_{L_{1,v}} \leq \sum_{i=1}^n \|\lambda(x_i^{-1})(\check{M}_Q^\rho W)\|_{L_{1,v}} < \infty.$$

*Here, the left-translation invariance of  $L_{1,v}(G)$  is a consequence of the submultiplicativity of  $v$ .*

As in the preceding subsection, our first goal is to show that certain synthesis and analysis operators are bounded.

**Lemma 5.15.** *Let Assumption 5.13 be satisfied, and let  $X = (x_i)_{i \in I}$  be any relatively separated family in  $G$ . Let the weight  $m_X$  be as in Lemma 5.6. Then the following hold:*

- i) *If  $\Psi = (\psi_i)_{i \in I}$  is a  $U$ -BUPU with localizing family  $X$ , then the analysis operator*

$$\text{Ana}_{X,\Psi} : L_{r,m}(G) \rightarrow \ell_{r,m_X}(I), f \mapsto (\langle f, \psi_i \rangle_{L_2})_{i \in I}$$

*is a well-defined bounded linear map.*

- ii) *The synthesis map*

$$\text{Synth}_{X,W} : \ell_{r,m_X}(I) \rightarrow L_{r,m}(G), (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \cdot \lambda(x_i)W$$

*is a well-defined bounded linear map, where the defining series is almost everywhere absolutely convergent.*

*Proof.* For  $x = x_i u \in x_i U$  we have  $m(x_i) = m(xu^{-1}) \leq m(x)w(u^{-1}) \leq C \cdot m(x)$ , where  $C := \sup_{u \in U} w(u^{-1})$  is finite by continuity of  $w$  and compactness of  $U$ . Since we

also have  $\psi_i \equiv 0$  on  $G \setminus x_i U$ , then we see by following the lines of the proof of Proposition 3.12 and using Jensen's inequality, see [8, Theorem 10.2.6]:

$$\begin{aligned} (m(x_i))^r \cdot |\langle f, \psi_i \rangle_{L_2}|^r &\leq |x_i U|^r \cdot \left( \int_{x_i U} |f(x)| m(x_i) \psi_i(x) \frac{dx}{|x_i U|} \right)^r \\ &\leq |x_i U|^r \cdot \int_{x_i U} (|f(x)| \cdot C \cdot m(x) \psi_i(x))^r \frac{dx}{|x_i U|} \\ &\leq |U|^{r-1} \cdot C^r \cdot \int_G |(m \cdot f)(x)|^r \cdot \psi_i(x) dx, \end{aligned}$$

where the last step used the left invariance of the Haar measure and the estimate  $(\psi_i(x))^r \leq \psi_i(x)$  which holds since  $\psi_i(x) \in [0, 1]$  and  $r \geq 1$ .

Summing over  $i \in I$  and applying the monotone convergence theorem, we thus get

$$\begin{aligned} \|\text{Ana}_{X,\Psi} f\|_{\ell_{r,m_X}}^r &= \sum_{i \in I} (m(x_i) \cdot |\langle f, \psi_i \rangle_{L_2}|)^r \\ &\leq |U|^{r-1} \cdot C^r \cdot \int_G |(m \cdot f)(x)|^r \cdot \sum_{i \in I} \psi_i(x) dx \\ &= |U|^{r-1} \cdot C^r \cdot \|f\|_{L_{r,m}}^r < \infty, \end{aligned}$$

thereby proving the boundedness and well-definedness of  $\text{Ana}_{X,\Psi}$ .

We now consider the synthesis map  $\text{Synth}_{X,W}$ . Let  $V \subset G$  be any compact unit neighborhood, and set  $\underline{Q} := \text{int } V$ , so that  $U := \overline{\underline{Q}} \subset V$  is a compact unit neighborhood that satisfies  $\text{int } \overline{U} \supset \underline{Q} = U$  and hence  $U = \text{int } \overline{U}$ . As a consequence, as seen in the proof of Lemma 3.3 (see p. 50), we have  $\|W\|_{L_\infty(Ux)} = \sup_{y \in Ux} |W(y)|$  for all  $x \in G$ . Here we used that  $W$  is continuous.

Now, let  $x \in G$  and  $i \in I$  be arbitrary. For any  $y = x_i u \in x_i U$  we then have  $x_i^{-1} x = (y u^{-1})^{-1} x = u y^{-1} x \in U y^{-1} x$ , and thus

$$|W(x_i^{-1} x)| \leq \|W\|_{L_\infty(U y^{-1} x)} = (\check{M}_U^\rho W)(y^{-1} x) \quad \text{for all } x \in G \text{ and } y \in x_i U.$$

Writing  $\Theta := \check{M}_U^\rho W$ , and averaging this estimate over  $y \in x_i U$ , we get

$$(69) \quad |\lambda_{x_i} W(x)| \leq \frac{1}{|U|} \int_G \chi_{x_i U}(y) \cdot \Theta(y^{-1} x) dy \quad \text{for all } x \in G, i \in I.$$

Now, let  $c = (c_i)_{i \in I} \in \ell_{r,m_X}(I)$  be arbitrary, and set  $\Upsilon := \sum_{i \in I} |c_i| \cdot \chi_{x_i U}$ . With the notation introduced in Lemma 5.6 we get  $\Upsilon = \text{Synth}_{X,U} |c|$  with  $|c| = (|c_i|)_{i \in I}$ . This easily implies  $\Upsilon \in L_{r,m}(G)$  with

$$(70) \quad \|\Upsilon\|_{L_{r,m}} \leq C \cdot \|c\|_{\ell_{r,m_X}}$$

for a constant  $C = C(m, X, U, r)$  independent of  $c$ .

By weighting estimate (69) with  $|c_i|$  and summing over  $i \in I$ , and by invoking the monotone convergence theorem, we see for all  $x \in G$  that

$$\sum_{i \in I} |c_i| \cdot |(\lambda_{x_i} W)(x)| \leq \frac{1}{|U|} \cdot \int_G \Upsilon(y) \cdot \Theta(y^{-1} x) dy = \frac{1}{|U|} \cdot (\Upsilon * \Theta)(x).$$

But since  $\Theta = \check{M}_U^\rho W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$  (see Assumption 5.13 and Remark 5.14) and since  $\Upsilon \in L_{r,m}(G)$ , Young's inequality (Proposition A.2) shows  $\Upsilon * \Theta \in L_{r,m}(G)$ . In particular,  $\Upsilon * \Theta(x) < \infty$  almost everywhere. Therefore, we already see that the

series defining  $\text{Synth}_{X,W} c$  is almost everywhere absolutely convergent. Finally, we also see

$$\begin{aligned} \|\text{Synth}_{X,W} c\|_{L_{r,m}} &\leq \left\| \sum_{i \in I} |c_i| \cdot |\lambda(x_i)W| \right\|_{L_{r,m}} \leq \frac{1}{|U|} \cdot \|\Upsilon * \Theta\|_{L_{r,m}} \\ &\leq \frac{1}{|U|} \cdot \max\{\|\Theta\|_{L_{1,w}}, \|\Theta\|_{L_{1,w\Delta^{-1}}}\} \cdot \|\Upsilon\|_{L_{r,m}}. \end{aligned}$$

In view of (70), this proves the boundedness and well-definedness of  $\text{Synth}_{X,W}$ .  $\square$

Now we can prove the desired atomic decomposition result:

**Proposition 5.16.** *Let Assumption 5.13 be satisfied. For each compact unit neighborhood  $U \subset G$  write*

$$(71) \quad C_U := \max\{\|\text{osc}_U W\|_{L_{1,w}}, \|\text{osc}_U W\|_{L_{1,w\Delta^{-1}}}\}.$$

Assume that

$$(72) \quad C_U \cdot \|RC_K\|_{L_{r,m} \rightarrow L_{r,m}} < 1.$$

Finally, let  $X = (x_i)_{i \in I}$  be a relatively separated family for which there exists a  $U$ -BUPU  $\Psi = (\psi_i)_{i \in I}$  with localizing family  $X$ , and let the weight  $m_X$  be as defined in Lemma 5.6.

Then the family  $(\lambda(x_i)K)_{i \in I}$  forms a family of atoms for  $\mathcal{M}_{r,m}$  with associated sequence space  $\ell_{r,m_X}(I)$ . This means:

i) The synthesis operator

$$\text{Synth}_{X,K} : \ell_{r,m_X}(I) \rightarrow \mathcal{M}_{r,m}, (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \cdot \lambda(x_i)K$$

is well-defined and bounded, with unconditional convergence of the defining series. This even holds without assuming (72).

ii) There is a bounded coefficient operator

$$C : \mathcal{M}_{r,m} \rightarrow \ell_{r,m_X}(I) \quad \text{with} \quad \text{Synth}_{X,K} \circ C = \text{id}_{\mathcal{M}_{r,m}}.$$

**Remark 5.17.** i) We note that condition (72) is always satisfied for  $U$  small enough, thanks to Lemma 3.3 and Assumption 5.13.

ii) As in Proposition 5.10, the action of the coefficient operator  $C$  is independent of the choice of  $r, m$ , that is, if condition (72) is satisfied for  $L_{r_1, m_1}(G)$  and  $L_{r_2, m_2}(G)$  and if  $C_1 : \mathcal{M}_{r_1, m_1} \rightarrow \ell_{r_1, m_1, X}(I)$  and  $C_2 : \mathcal{M}_{r_2, m_2} \rightarrow \ell_{r_2, m_2, X}(I)$  are the respective coefficient operators, then  $C_1 f = C_2 f$  for all  $f \in \mathcal{M}_{r_1, m_1} \cap \mathcal{M}_{r_2, m_2}$ .

*Proof. Step 1 (Boundedness of the synthesis operator):* For this step we will not use condition (72). By Assumption 5.13,  $RC_K : L_{r,m}(G) \rightarrow L_{r,m}(G)$  is bounded, and we have  $W * K = K$ , which implies  $(\lambda(x)W) * K = \lambda(x)(W * K) = \lambda(x)K$  for all  $x \in G$ .

Furthermore, Lemma 5.15 shows that the map

$$\text{Synth}_{X,W} : \ell_{r,m_X}(I) \rightarrow L_{r,m}(G), (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \cdot \lambda(x_i)W$$

is well-defined and bounded. Because of  $r < \infty$ , each  $c = (c_i)_{i \in I} \in \ell_{r,m_X}(I)$  satisfies  $c = \sum_{i \in I} c_i \delta_i$  with unconditional convergence in  $\ell_{r,m_X}(I)$ , where  $(\delta_i)_{i \in I}$  denotes the

standard basis of  $\ell_{r,m_X}(I)$ . This implies that the series defining  $\text{Synth}_{X,W} c$  converges unconditionally in  $L_{r,m}(G)$ . Since bounded operators preserve unconditional convergence, we see

$$RC_K(\text{Synth}_{X,W} c) = RC_K\left(\sum_{i \in I} c_i \lambda(x_i) W\right) = \sum_{i \in I} c_i [(\lambda(x_i) W) * K] = \sum_{i \in I} c_i \lambda(x_i) K$$

with unconditional convergence of the series. We have thus shown that  $\text{Synth}_{X,K} = RC_K \circ \text{Synth}_{X,W} : \ell_{r,m_X}(I) \rightarrow L_{r,m}(G)$  is well-defined and bounded, with unconditional convergence of the defining series.

Since  $\lambda(x_i) K \in \mathcal{M}_{r,m}$  for all  $i \in I$ , we also see that the range of  $\text{Synth}_{X,K}$  is contained in the closed subspace  $\mathcal{M}_{r,m} \subset L_{r,m}(G)$ .

**Step 2 (An alternative reproducing formula for  $\mathcal{M}_{r,m}$ ):** In this step we will prove

$$(73) \quad f = (f * W) * K \quad \text{for all } f \in \mathcal{M}_{r,m}.$$

This is almost trivial: For  $f \in \mathcal{M}_{r,m}$ , we have  $f = f * K$  by definition of  $\mathcal{M}_{r,m}$ , and we have  $K = W * K$  by Assumption 5.13. By combining these facts, we get  $f = f * K = f * (W * K)$ . Thus, all we need to verify is that the convolution is associative in the setting that we consider here.

In light of [5, Lemma 6.3] to prove this it remains to show  $((|f| * |W|) * |K|)(x) < \infty$  for almost all  $x \in G$ . To this end, we first show  $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ . In order to see this, let  $V \subset G$  be any compact unit neighborhood, and set  $Q := \text{int } V$ , so that  $U := \overline{Q} \subset V$  is a compact unit neighborhood that satisfies  $\overline{\text{int } U} \supset \overline{Q} = U$  and hence  $U = \overline{\text{int } U}$ . As a consequence of this and of the continuity of  $W$ , as seen in the proof of Lemma 3.3 (see p. 50), we have

$$\check{M}_U^\rho W(x) = \|W\|_{L_\infty(Ux)} = \sup_{y \in Ux} |W(y)| \geq |W(x)| \quad \text{for all } x \in G.$$

Since  $\check{M}_U^\rho W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$  (see Assumption 5.13 and Remark 5.14), we see  $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ .

Now, fix some  $s \in (r, \infty)$  and let  $f \in \mathcal{M}_{r,m}$ . Because of  $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ , Proposition A.2 shows  $|f| * |W| \in L_{r,m}(G)$ . Therefore, by the second part of Proposition A.2, we see  $(|f| * |W|) * |K| \in L_{s,m}(G)$ , since  $|K(x^{-1})| = |K(x)|$  and since  $K \in L_{p,w}(G)$  for all  $p \in (1, \infty)$ . In particular,  $((|f| * |W|) * |K|)(x) < \infty$  for almost all  $x \in G$ . By the considerations from above, we thus see that (73) holds.

**Step 3 (Approximating  $f \mapsto f * W$ ):** Let  $\text{Ana}_{X,\Psi} : L_{r,m}(G) \rightarrow \ell_{r,m_X}(I)$  and  $\text{Synth}_{X,W} : \ell_{r,m_X}(I) \rightarrow L_{r,m}(G)$  be as defined in Lemma 5.15, and define  $A := \text{Synth}_{X,W} \circ \text{Ana}_{X,\Psi} : L_{r,m}(G) \rightarrow L_{r,m}(G)$ . In this step, we will show

$$(74) \quad \|f * W - Af\|_{L_{r,m}} \leq C_U \cdot \|f\|_{L_{r,m}} \quad \text{for all } f \in L_{r,m}(G),$$

with  $C_U$  as in (71).

To this end, recall from the previous step that  $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ , so that Young's inequality (Proposition A.2) shows  $|f| * |W| \in L_{r,m}(G)$  for  $f \in L_{r,m}(G)$ . In particular, this implies  $|f| * |W|(x) < \infty$  for almost all  $x \in G$ . For each such  $x \in G$ , the dominated convergence theorem and the definition of the operators  $\text{Ana}_{X,\Psi}$ ,



Synth $_{X,W}$  and  $A$  shows

$$\begin{aligned} |f * W(x) - Af(x)| &= \left| \sum_{i \in I} \int_G f(y) \psi_i(y) W(y^{-1}x) dy - \sum_{i \in I} \langle f, \psi_i \rangle_{L_2} (\lambda(x_i)W)(x) \right| \\ &\leq \sum_{i \in I} \int_G \psi_i(y) \cdot |f(y)| \cdot |W(y^{-1}x) - W(x_i^{-1}x)| dy. \end{aligned}$$

Fix  $i \in I$  for the moment. For  $y \in G$  with  $\psi_i(y) \neq 0$ , we have  $y = x_i u \in x_i U$ , and thus  $x_i^{-1}x = uy^{-1}x \in Uy^{-1}x$ . Therefore,  $|W(y^{-1}x) - W(x_i^{-1}x)| \leq (\text{osc}_U W)(y^{-1}x)$ , by definition of the oscillation  $\text{osc}_U W$  (see (26)).

If we combine this with the estimate from above and with the monotone convergence theorem, we get

$$\begin{aligned} |f * W(x) - Af(x)| &\leq \sum_{i \in I} \int_G \psi_i(y) \cdot |f(y)| \cdot (\text{osc}_U W)(y^{-1}x) dy \\ &= \int_G |f(y)| \cdot (\text{osc}_U W)(y^{-1}x) dy = (|f| * \text{osc}_U W)(x). \end{aligned}$$

In view of Young's inequality (Proposition A.2) and the definition of  $C_U$  (see (71)), we see that (74) holds true.

**Step 4 (Completing the proof):** Recall that  $RC_K : L_{r,m}(G) \rightarrow \mathcal{M}_{r,m}$  is bounded by Assumption 5.13. Thus,  $B := RC_K \circ A|_{\mathcal{M}_{r,m}} : \mathcal{M}_{r,m} \rightarrow \mathcal{M}_{r,m}$  is well-defined and bounded, with  $A$  as in the preceding step. Now, for arbitrary  $f \in \mathcal{M}_{r,m}$  our results from Steps 2 and 3 show

$$\|f - Bf\|_{L_{r,m}} = \|RC_K(f * W - Af)\|_{L_{r,m}} \leq \|RC_K\|_{L_{r,m} \rightarrow L_{r,m}} \cdot C_U \cdot \|f\|_{L_{r,m}}.$$

In view of our assumption (72), a Neumann series argument (see [2, Sect. 5.7]) shows that  $C_0 := \sum_{n=0}^{\infty} (\text{id}_{\mathcal{M}_{r,m}} - B)^n$  defines a bounded linear operator  $C_0 : \mathcal{M}_{r,m} \rightarrow \mathcal{M}_{r,m}$  with  $B \circ C_0 = \text{id}_{\mathcal{M}_{r,m}}$ .

But we saw in Step 1 that  $\text{Synth}_{X,K} = RC_K \circ \text{Synth}_{X,W}$ , so that

$$\begin{aligned} B &= RC_K \circ A|_{\mathcal{M}_{r,m}} = RC_K \circ \text{Synth}_{X,W} \circ \text{Ana}_{X,\Psi}|_{\mathcal{M}_{r,m}} \\ &= \text{Synth}_{X,K} \circ \text{Ana}_{X,\Psi}|_{\mathcal{M}_{r,m}}. \end{aligned}$$

Thus, the operator  $C := \text{Ana}_{X,\Psi}|_{\mathcal{M}_{r,m}} \circ C_0 : \mathcal{M}_{r,m} \rightarrow \ell_{r,m_X}(I)$  satisfies

$$\text{Synth}_{X,K} \circ C = B \circ C_0 = \text{id}_{\mathcal{M}_{r,m}}.$$

It is not hard to see that the action of the coefficient operator  $C$  is independent of the choice of  $r, m$ . For more details see the end of the proof of Proposition 5.10, where a similar claim is shown.  $\square$

Finally, as in the preceding section we apply the correspondence principle to obtain atomic decomposition results for the coorbit spaces from those for the reproducing kernel spaces.

**Theorem 5.18.** *Under the assumptions of Proposition 5.16, the sampled family  $(\pi(x_i)u)_{i \in I} \subset \text{Co}(L_{r,m})$  forms a family of atoms for  $\text{Co}(L_{r,m})$  with coefficient space  $\ell_{r,m_X}(I)$ .*

*More precisely, the synthesis operator*

$$\text{Synth}_{X,C_0} : \ell_{r,m_X}(I) \rightarrow \text{Co}(L_{r,m}), (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \cdot \pi(x_i)u$$

is well-defined and bounded, and there is a bounded linear coefficient operator  $C_{C_0} : \text{Co}(L_{r,m}) \rightarrow \ell_{r,m_X}(I)$  satisfying  $\text{Synth}_{X,C_0} \circ C_{C_0} = \text{id}_{\text{Co}(L_{r,m})}$ .

Finally, the action of the coefficient operator  $C_{C_0}$  is independent of the choice of  $r, m$ . In other words, if the assumptions of the current theorem are satisfied for  $L_{r_1, m_1}$  and for  $L_{r_2, m_2}$  and if  $C_1, C_2$  denote the corresponding coefficient operators, then  $C_1 f = C_2 f$  for all  $f \in \text{Co}(L_{r_1, m_1}) \cap \text{Co}(L_{r_2, m_2})$ .

*Proof.* The correspondence principle (Proposition 2.9) states that the extended voice transform  $V_e : \text{Co}(L_{r,m}) \rightarrow \mathcal{M}_{r,m}$  is an isometric isomorphism. Furthermore,

$$(V_e \pi(x) u)(y) = \langle \pi(x) u, \pi(y) u \rangle_{S_w} = \langle \pi(x) u, \pi(y) u \rangle_{\mathcal{H}} = K(x^{-1}y) = (\lambda(x)K)(y)$$

for all  $x, y \in G$ . In other words,  $V_e \pi(x) u = \lambda(x)K$  for all  $x \in G$ .

Now, since the bounded linear operator  $V_e^{-1} : \mathcal{M}_{r,m} \rightarrow \text{Co}(L_{r,m})$  preserves unconditional convergence of series, the synthesis operator  $\text{Synth}_{X,K}$  from Proposition 5.16 satisfies

$$\begin{aligned} (V_e^{-1} \circ \text{Synth}_{X,K})(c_i)_{i \in I} &= V_e^{-1} \left( \sum_{i \in I} c_i \cdot \lambda(x_i)K \right) = \sum_{i \in I} c_i \cdot V_e^{-1}(\lambda(x_i)K) \\ &= \sum_{i \in I} c_i \cdot \pi(x_i)u = \text{Synth}_{X,C_0}(c_i)_{i \in I}, \end{aligned}$$

for arbitrary  $(c_i)_{i \in I} \in \ell_{r,m_X}(I)$ , with unconditional convergence of all involved series. Thus, the operator  $\text{Synth}_{X,C_0} = V_e^{-1} \circ \text{Synth}_{X,K} : \ell_{r,m_X}(I) \rightarrow \text{Co}(L_{r,m})$  is indeed well-defined and bounded.

Now, with the coefficient operator  $C : \mathcal{M}_{r,m} \rightarrow \ell_{r,m_X}(I)$  from Proposition 5.16, define  $C_{C_0} := C \circ V_e : \text{Co}(L_{r,m}) \rightarrow \ell_{r,m_X}(I)$ . Then

$$\text{Synth}_{X,C_0} \circ C_{C_0} = V_e^{-1} \circ \text{Synth}_{X,K} \circ C \circ V_e = V_e^{-1} \circ V_e = \text{id}_{\text{Co}(L_{r,m})},$$

as desired. Since the action of  $C$  is independent of the choice of  $r, m$ , so is the action of  $C_{C_0}$ .  $\square$

**5.4. An Application: Discretization Results for General Paley-Wiener Spaces.** In this subsection we will apply the abstract results from this section to the Paley-Wiener spaces  $B_\Omega^p$ , thereby improving on the discretization results derived in Sect. 3.3. Furthermore, our proofs clearly point out those properties that the set  $\Omega \subset \mathbb{R}$  has to satisfy if one wants discretization results to hold for the associated Paley-Wiener spaces:

**Assumption 5.19.** *Let  $\Omega \subset \mathbb{R}$  be measurable, and let  $r \in (1, \infty) \setminus \{2\}$ . Assume that the following properties hold:*

- i)  $\Omega$  is bounded;
- ii) the kernel  $K := \mathcal{F}^{-1} \chi_\Omega$  satisfies  $K \in \bigcap_{1 < p < \infty} L_p(\mathbb{R})$ ;
- iii) the convolution operator  $RC_K$  is well-defined on  $L_r(\mathbb{R})$ .

**Remark 5.20.** *The last property means that the indicator function  $\chi_\Omega$  is an  $L_r(\mathbb{R})$ -Fourier multiplier, which implies that  $\chi_{\Omega^c} = 1 - \chi_\Omega$  is an  $L_r(\mathbb{R})$ -Fourier multiplier as well. Therefore, [24, Theorem 1] shows that there is an open set  $U \subset \mathbb{R}$  with  $\chi_{\Omega^c} = \chi_U$  almost everywhere, and thus  $\chi_{U^c} = \chi_\Omega$  almost everywhere. But since  $\Omega$  is bounded, we have  $\Omega \subset [-R, R]$  for some  $R > 0$ , and then  $\chi_\Omega = \chi_\Omega \chi_{[-R, R]} = \chi_{U^c} \chi_{[-R, R]} = \chi_{[-R, R] \setminus U}$  almost everywhere. Thus, by modifying  $\Omega$  on a null set, we*

can (and will) always assume that  $\Omega$  is compact. This neither changes the kernel  $K$ , nor the underlying Hilbert space

$$\mathcal{H} := B_\Omega^2 := \{f \in L_2(\mathbb{R}) : \widehat{f} \equiv 0 \text{ a.e. on } \mathbb{R} \setminus \Omega\}.$$

As seen in the discussion before Proposition 4.5, if we set  $u := K = \mathcal{F}^{-1}\chi_\Omega$ , then all standing assumptions from Sect. 2 are satisfied for  $m = w \equiv 1$ , so that the coorbit spaces  $\text{Co}(L_p)$  are well-defined for  $1 < p < \infty$ . Furthermore, we saw before Proposition 4.5 that the associated voice transform satisfies  $Vf = f$  for all  $f \in \mathcal{H} = B_\Omega^2 \subset L_2(\mathbb{R})$ . Using this identity, we can now identify the abstractly defined coorbit spaces with more concrete reproducing kernel or Paley-Wiener spaces.

**Lemma 5.21.** *Setting  $\mathcal{T} := \bigcap_{1 < p < \infty} L_p(\mathbb{R})$ , the space  $\mathcal{S}$  from (6) satisfies*

$$\mathcal{S} = \{f \in \mathcal{T} \mid f * K = f\},$$

with topology generated by the norms  $(\|\cdot\|_{L_p})_{1 < p < \infty}$ .

Furthermore, with  $\mathcal{M}_p = \{f \in L_p(\mathbb{R}) \mid f * K = f\}$  and  $\mathcal{M} := \bigcup_{1 < p < \infty} \mathcal{M}_p$ , the map

$$\iota : \mathcal{M} \rightarrow \mathcal{S}', f \mapsto \Phi_f \quad \text{with} \quad \langle \Phi_f, g \rangle_{\mathcal{S}} := \langle f, g \rangle_{L_2}$$

is a bijection. If we identify each  $\varphi \in \mathcal{S}'$  with its inverse image  $\iota^{-1}\varphi \in \mathcal{M}$  under this map, then the extended voice transform is the identity map, that is  $V_e \varphi = \iota^{-1}\varphi$ .

According to the general result, the coorbit spaces  $\text{Co}(L_p)$  are given by

$$(75) \quad \text{Co}(L_p) = \iota(\mathcal{M}_p) \quad \text{for all } p \in (1, \infty),$$

which means that if we identify  $\varphi$  with  $\iota^{-1}\varphi$ , then  $\text{Co}(L_p) = \mathcal{M}_p$ .

Finally for  $p \in (1, 2]$  we have

$$\mathcal{M}_p = B_\Omega^p := \{f \in L_p(\mathbb{R}) \mid \widehat{f} \equiv 0 \text{ a.e. on } \mathbb{R} \setminus \Omega\}.$$

Therefore, up to canonical identifications, the coorbit spaces  $\text{Co}(L_p)$  coincide with the Paley-Wiener spaces  $B_\Omega^p$ , at least for  $p \in (1, 2]$ .

**Remark 5.22.** *We do not know if in general the identity  $\mathcal{M}_p = B_\Omega^p$  with*

$$B_\Omega^p = \{f \in L_p(\mathbb{R}) \mid \text{the tempered dist. } \widehat{f} \text{ has } \text{supp}(\widehat{f}) \subset \Omega\}$$

also holds for  $p \in (2, \infty)$ . In case of  $\Omega = [-\omega, \omega]$ , it was shown in [5, Proposition 4.8] that this is true. Using this, one can show  $\mathcal{M}_p = B_\Omega^p$  even if  $\Omega$  is a finite disjoint union of compact intervals. For more general sets  $\Omega$ , however, we do not know whether  $\mathcal{M}_p = B_\Omega^p$  for  $p \in (2, \infty)$ .

of Lemma 5.21. The following proof is similar to the proof of [5, Proposition 4.8] with some significant improvements and generalizations.

The first property is an immediate consequence of the definitions, combined with  $Vf = f$  for  $f \in \mathcal{H}$ .

The map  $\iota$  is indeed well-defined, since if  $f \in \mathcal{M}_p$  for some  $p \in (1, \infty)$ , then  $|\langle f, g \rangle_{L_2}| \leq \|f\|_{L_p} \cdot \|g\|_{L_{p'}}$ , where  $\|\cdot\|_{L_{p'}}$  is a continuous norm on  $\mathcal{S}$ .

To prove the surjectivity of  $\iota$ , we first show that  $\mathcal{M}$  is a (complex) vector space. Since each  $\mathcal{M}_p$  is closed under multiplication with complex numbers, we only need to show that  $\mathcal{M}$  is closed under addition. To this end, note for  $f \in \mathcal{M}_p$  because of  $K \in L_{p'}(\mathbb{R})$  that

$$|f(x)| = |(f * K)(x)| = |\langle f, \lambda(x)K \rangle_{L_2}| \leq \|f\|_{L_p} \cdot \|\lambda(x)K\|_{L_{p'}} \leq C_p \cdot \|f\|_{L_p}$$

for all  $x \in \mathbb{R}$ , and thus  $\mathcal{M}_p \hookrightarrow L_\infty$ . This embedding implies  $\mathcal{M}_p \subset \mathcal{M}_q$  for  $p \leq q$ , and thus  $\mathcal{M}_p + \mathcal{M}_q \subset \mathcal{M}_q + \mathcal{M}_q = \mathcal{M}_q \subset \mathcal{M}$ . From this, we easily see that  $\mathcal{M}$  is a vector space.

With  $\mathcal{M}$  being a vector space, we see  $\mathcal{M} = \text{span} \bigcup_{1 < p < \infty} \mathcal{M}_p$ . With notation as in (11), this means  $\mathcal{M} = \mathcal{M}^{\mathcal{U}}$ . Hence, Theorem 2.6 shows for arbitrary  $\varphi \in \mathcal{S}'$  that  $f := V_e \varphi \in \mathcal{M}^{\mathcal{U}} = \mathcal{M}$ , and (10) shows because of  $Vg = g$  for  $g \in \mathcal{S} \subset \mathcal{H}$  that

$$\langle \Phi_f, g \rangle_{\mathcal{S}} = \langle f, g \rangle_{L_2} = \langle V_e \varphi, Vg \rangle_{L_2} = \langle \varphi, g \rangle_{\mathcal{S}}.$$

Hence,  $\varphi = \widehat{\Phi}_f = \iota f = \iota V_e \varphi$ . On the one hand, this shows that  $\iota$  is surjective, and on the other hand—once we know that  $\iota$  is bijective—it proves that the inverse of  $\iota$  is given by  $\iota^{-1} = V_e : \mathcal{S}' \rightarrow \mathcal{M}$ .

In order to prove that  $\iota$  is injective, note  $\lambda(x)K \in \mathcal{S}$  for all  $x \in \mathbb{R}$  and recall  $\overline{K(x)} = K(-x)$ . Hence,

$$\langle \Phi_f, \lambda(x)K \rangle_{\mathcal{S}} = \langle f, \lambda(x)K \rangle_{L_2} = (f * K)(x) = f(x) \quad \text{for } f \in \mathcal{M}.$$

Therefore, if  $\Phi_f = 0$ , then  $f = 0$  as well. Since the domain  $\mathcal{M}$  of  $\iota$  is a vector space, this shows that  $\iota$  is injective.

Equation (75) is seen to be true by combining the identity  $V_e = \iota^{-1}$  with the correspondence principle (see Proposition 2.9), which states that  $V_e : \text{Co}(L_p) \rightarrow \{f \in L_p(\mathbb{R}) \mid f * K = f\} = \mathcal{M}_p$  is an isomorphism.

To prove  $\mathcal{M}_p = B_\Omega^p$  for  $p \in (1, 2]$ , first note  $\mathcal{F}(f * g) = \widehat{f} \cdot \widehat{g}$  for arbitrary  $f, g \in L_2$ , see e.g. [26, p. 270]. Therefore, for  $f \in \mathcal{M}_p \hookrightarrow \mathcal{M}_2$  (here we used that  $p \leq 2$ ) we see that  $\widehat{f} = \widehat{f * K} = \widehat{f} \cdot \widehat{K} = \widehat{f} \cdot \chi_\Omega$ , where the equality holds in the sense of tempered distributions. But since both sides are  $L_2(\mathbb{R})$  functions, this implies  $\widehat{f} = \widehat{f} \cdot \chi_\Omega$  almost everywhere, and thus  $f \in B_\Omega^p$ .

Conversely, let  $f \in B_\Omega^p$  be arbitrary. Because of  $p \leq 2$ , [27, Theorem in Sect. 1.4.1] shows  $f \in L_2(\mathbb{R})$ . Furthermore, since  $\widehat{f} \equiv 0$  almost everywhere on  $\mathbb{R} \setminus \Omega$ , we have  $\mathcal{F}(f * K) = \widehat{f} \cdot \widehat{K} = \widehat{f} \cdot \chi_\Omega = \widehat{f}$ , and thus  $f = f * K$ , so that  $f \in \mathcal{M}_p$ .  $\square$

With Lemma 5.21 showing that the coorbit spaces  $\text{Co}(L_p)$  coincide with the reproducing kernel spaces  $\mathcal{M}_p$ , we will in the following concentrate on the latter spaces for proving discretization results.

In Sects. 5.2 and 5.3, we showed that the sampled frame  $(\pi(x_i)u)_{i \in I}$  forms a Banach frame or an atomic decomposition for the coorbit space  $\text{Co}(L_{r,m})$  if the family of sampling points  $(x_i)_{i \in I}$  is sufficiently dense in  $G$ . For the case of the Paley-Wiener spaces, one can state quite precisely how dense the sampling points need to be:

**Proposition 5.23.** *Suppose that Assumption 5.19 is satisfied, and choose  $R > 0$  and  $\xi_0 \in \mathbb{R}$  with  $\Omega \subset \xi_0 + [-R, R]$ .*

*Then the family  $(\lambda(k/(2R))K)_{k \in \mathbb{Z}}$  forms a Banach frame and an atomic decomposition for the reproducing kernel space  $\mathcal{M}_r$  with coefficient space  $\ell_r(\mathbb{Z})$ . More precisely, the operators*

$$\text{Samp} : \mathcal{M}_r \rightarrow \ell_r(\mathbb{Z}), f \mapsto (f(k/(2R)))_{k \in \mathbb{Z}} = (\langle f, \lambda(k/(2R))K \rangle_{L_2})_{k \in \mathbb{Z}}$$

and

$$\text{Synth} : \ell_r(\mathbb{Z}) \rightarrow \mathcal{M}_r, (c_k)_{k \in \mathbb{Z}} \mapsto \sum_{k \in \mathbb{Z}} c_k \cdot \lambda(k/(2R))K$$

are well-defined and bounded with  $\text{Synth} \circ \text{Samp} = (2R)^{-1} \cdot \text{id}_{\mathcal{M}_r}$ .

*Proof.* Since  $\Omega \subset \mathbb{R}$  is bounded, there is a Schwartz function  $\psi \in \mathcal{S}(\mathbb{R})$  with  $\psi \equiv 1$  on  $\Omega$ . We then have  $W := \mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R})$ , so that  $W$  is continuous. Furthermore,

$$\widehat{W * K} = \widehat{K * W} = \widehat{K} \cdot \widehat{W} = \chi_\Omega \cdot \psi = \chi_\Omega = \widehat{K},$$

and hence  $W * K = K * W = K$ . Since  $W$  is a Schwartz function, there is some  $C > 0$  with  $|W(x)| \leq C \cdot (1 + |x|)^{-2}$  for all  $x \in \mathbb{R}$ . Because of

$$1 + |x| \leq 2 + |x - y| \leq 2 \cdot (1 + |x - y|)$$

for any  $y \in Q := U_0 := [-1, 1]$ , this shows  $|W(x + y)| \leq 4C \cdot (1 + |x|)^{-2}$ , and hence  $\check{M}_Q^\rho W \in L_1(\mathbb{R})$  and  $M_{U_0}^\lambda W \in L_1(\mathbb{R})$ . Overall, noting that the modular function  $\Delta$  of the abelian group  $G = \mathbb{R}$  satisfies  $\Delta \equiv 1$ , we see using Lemma 5.8 that Assumptions 5.7 and 5.13 are both satisfied for  $w \equiv m \equiv 1$ .

Now, define  $I := \mathbb{Z}$  and  $x_k := k/(2R)$  for  $k \in \mathbb{Z}$ . It is not hard to see that the family  $(x_k)_{k \in \mathbb{Z}}$  is relatively separated in  $G = \mathbb{R}$ . Therefore, Lemma 5.9 and Proposition 5.16 show that the two operators from the statement of the current proposition are well-defined and bounded. It remains to show  $\text{Synth} \circ \text{Samp} = (2R)^{-1} \text{id}_{\mathcal{M}_r}$ .

For this, it suffices to show  $\text{Synth}(\text{Samp } f) = (2R)^{-1} \cdot f$  for  $f \in \mathcal{M}_r \cap L_2(\mathbb{R})$ , since Lemma 3.8 shows that  $\text{span}\{\lambda(x)K\}_{x \in \mathbb{R}} \subset \mathcal{M}_r \cap L_2(\mathbb{R})$  is dense in  $\mathcal{M}_r$ . But it is well-known that the family  $(e_k)_{k \in \mathbb{Z}} = ((2R)^{-1/2} \cdot e^{2\pi i \frac{k}{2R}})_{k \in \mathbb{Z}}$  forms an orthonormal basis of  $L_2(\Omega_0)$  where  $\Omega_0 := \xi_0 + [-R, R]$ . To make use of this orthonormal basis, first note for  $f \in \mathcal{M}_r \cap L_2(\mathbb{R})$  that  $\widehat{f} = \widehat{f * K} = \widehat{f} \cdot \widehat{K} = \chi_\Omega \cdot \widehat{f}$ . Because of  $\widehat{f} = \widehat{f} \cdot \chi_\Omega$ , we get  $\widehat{f} \equiv 0$  almost everywhere on  $\mathbb{R} \setminus \Omega \supset \mathbb{R} \setminus \Omega_0$ .

Overall, since  $\mathcal{F}(\lambda(k/(2R))K) = e^{-2\pi i \frac{k}{2R}} \chi_\Omega = (2R)^{1/2} \cdot e_{-k} \cdot \chi_\Omega$ , we see

$$\begin{aligned} \widehat{f} &= \chi_\Omega \cdot \widehat{f} = \chi_\Omega \cdot \sum_{k \in \mathbb{Z}} \langle \widehat{f}, e_k \rangle_{L_2} e_k \\ &= (2R)^{-1} \cdot \sum_{k \in \mathbb{Z}} \langle \widehat{f}, \mathcal{F}(\lambda(-k/(2R))K) \rangle_{L_2} \cdot \mathcal{F}(\lambda(-k/(2R))K) \\ &= (2R)^{-1} \cdot \mathcal{F} \left( \sum_{\ell \in \mathbb{Z}} \langle f, \lambda(\ell/(2R))K \rangle_{L_2} \cdot \lambda(\ell/(2R))K \right) \\ &= (2R)^{-1} \cdot \mathcal{F}(\text{Synth}(\text{Samp } f)), \end{aligned}$$

which implies  $f = (2R)^{-1} \cdot (\text{Synth} \circ \text{Samp})f$  for all  $f \in L_2(\mathbb{R}) \cap \mathcal{M}_r$ , as desired.  $\square$

To close this section, we show that the existence of a ‘‘well-behaved’’ kernel  $W$  with  $K * W = K$  is *independent* of the property that  $K$  acts boundedly on  $L_r(\mathbb{R})$  via right convolutions, even when we restrict to the class of reproducing kernels  $K$  which satisfy the weak integrability property  $K \in \bigcap_{1 < p < \infty} L_p(\mathbb{R})$ . In the proof of Proposition 5.23, we saw that for every *bounded* set  $\Omega \subset \mathbb{R}$ , there is a such a well-behaved kernel  $W$  associated to the reproducing kernel  $K = \mathcal{F}^{-1}\chi_\Omega$ . But the set  $C \subset [0, 1]$  that we constructed in Proposition 4.5 is bounded and the associated kernel  $K = \mathcal{F}^{-1}\chi_C$  satisfies the weak integrability property. Still,  $K$  does not act boundedly via right convolution on *any*  $L_p(\mathbb{R})$  space with  $p \neq 2$ . Conversely, the following proposition shows the existence of a kernel  $K$  that acts boundedly via right convolution on all  $L_p$  spaces for  $1 < p < \infty$ , but for which no well-behaved kernel  $W$  with  $K = W * K$  exists.

**Proposition 5.24.** *The set  $\Omega := \bigcup_{j=1}^{\infty} [3 \cdot 2^{j-2} + (0, 2^{-2j})]$  with the associated kernel  $K := \mathcal{F}^{-1} \chi_{\Omega}$  has the following properties:*

- i)  $K \in \bigcap_{1 < p < \infty} L_p(\mathbb{R})$ .
- ii) *There is no  $W \in L_1(\mathbb{R})$  with  $K = K * W$ .*
- iii) *The operator  $RC_K$  is bounded on  $L_p(\mathbb{R})$  for every  $p \in (1, \infty)$ .*

*Proof.* We first verify that the union defining  $\Omega$  is indeed disjoint. To this end, set  $I_j := 3 \cdot 2^{j-2} + (0, 2^{-2j})$  for  $j \in \mathbb{N}$ , and note  $3j - 2 > 0$ , so that  $2^{3j-2} > 1$ , and hence  $2^{-2j} < 2^{j-2}$ . This implies

$$(76) \quad 2^{j-1} = 2 \cdot 2^{j-2} < 3 \cdot 2^{j-2} < 3 \cdot 2^{j-2} + 2^{-2j} < 3 \cdot 2^{j-2} + 2^{j-2} = 2^j.$$

Therefore,  $I_j \subset (2^{j-1}, 2^j)$ , which easily yields the desired disjointness. Next, we verify the three claimed properties.

**First property:** A direct computation shows that  $F := \mathcal{F}^{-1} \chi_{(0,1)}$  satisfies  $F(x) = \frac{e^{2\pi i x} - 1}{2\pi i x}$  for  $x \neq 0$ , and hence  $F \in \bigcap_{1 < p < \infty} L_p(\mathbb{R})$ . Since  $\chi_{I_j} = \lambda(3 \cdot 2^{j-2})(\chi_{(0,1)}(2^{2j} \cdot))$ , we thus see by elementary properties of the Fourier transform that  $\mathcal{F}^{-1} \chi_{I_j}(x) = 2^{-2j} \cdot e^{6\pi i 2^{j-2} x} \cdot F(2^{-2j} x)$ . Therefore,

$$\|K\|_{L_p} = \left\| \sum_{j=1}^{\infty} \mathcal{F}^{-1} \chi_{I_j} \right\|_{L_p} \leq \sum_{j=1}^{\infty} 2^{-2j} \cdot \|F(2^{-2j} \cdot)\|_{L_p} = \|F\|_{L_p} \cdot \sum_{j=1}^{\infty} 2^{-2j(1-p^{-1})} < \infty$$

for arbitrary  $p \in (1, \infty)$ .

**Second property:** Assume towards a contradiction that  $K = K * W$  for some  $W \in L_1(\mathbb{R})$ . This implies  $\chi_{\Omega} = \widehat{K} = \widehat{K} \cdot \widehat{W} = \chi_{\Omega} \cdot \widehat{W}$  almost everywhere. In particular, there is a null-set  $N \subset \mathbb{R}$  with  $\widehat{W}(\xi) = 1$  for all  $\xi \in \Omega \setminus N$ . But the Riemann-Lebesgue lemma shows  $\lim_{\xi \rightarrow \infty} \widehat{W}(\xi) = 0$ , so that  $|\widehat{W}(\xi)| \leq 1/2$  for all  $\xi \in \mathbb{R}$  with  $|\xi| \geq 2^{j_0-2}$ , for a suitable  $j_0 \in \mathbb{N}$ . Hence, for any  $\xi$  belonging to the positive measure set  $I_{j_0} \setminus N = (3 \cdot 2^{j_0-2}, 3 \cdot 2^{j_0-2} + 2^{-2j_0}) \setminus N \subset \Omega \setminus N$ , we have  $1 = |\widehat{W}(\xi)| \leq 1/2$ , a contradiction.

**Third property:** Here, we will use the *strong Marcinkiewicz multiplier theorem* which states the following:

**Strong Marcinkiewicz multiplier theorem** (see [9, Theorem 8.3.1]) *Let  $(\Delta_j)_{j \in \mathbb{Z}}$  denote the usual dyadic decomposition of  $\mathbb{R}$ ,*

$$\Delta_j := \begin{cases} [2^{j-1}, 2^j], & \text{if } j > 0, \\ (-1, 1), & \text{if } j = 0, \\ (-2^{|j|}, -2^{|j|-1}], & \text{if } j < 0. \end{cases}$$

*Assume that  $\phi : \mathbb{R} \rightarrow \mathbb{C}$  is measurable and satisfies*

$$\sup_{\xi \in \mathbb{R}} |\phi(\xi)| < \infty \quad \text{and} \quad \sup_{j \in \mathbb{Z}} \text{Var}_{\Delta_j} \phi < \infty,$$

*where  $\text{Var}_I \phi$  denotes the total variation of the function  $\phi$  when restricted to the interval  $I$ .*

*Then  $\phi$  is an  $L_p(\mathbb{R})$ -Fourier multiplier for all  $p \in (1, \infty)$ . In other words, the map  $\mathbb{S}(\mathbb{R}) \rightarrow \mathbb{S}'(\mathbb{R}), f \mapsto \mathcal{F}^{-1}(\widehat{f} \cdot \phi)$  extends to a bounded linear operator on  $L_p(\mathbb{R})$ , for any  $p \in (1, \infty)$ .*

We want to apply this theorem for  $\phi := \chi_{\Omega}$ . To this end, first note  $\sup_{\xi \in \mathbb{R}} |\phi(\xi)| = 1 < \infty$ . Second, (76) shows for  $j \in \mathbb{Z}$  with  $j \leq 0$  that  $\phi|_{\Delta_j} \equiv 0$ , and for  $j \in \mathbb{N}$  that

$\phi|_{\Delta_j} = \chi_{I_j}$  is the indicator function of an interval. In both cases,  $\text{Var}_{\Delta_j}\phi \leq 2$ . All in all, the strong Marcinkiewicz multiplier theorem shows that the map  $\mathbb{S}(\mathbb{R}) \rightarrow \mathbb{S}'(\mathbb{R})$ ,  $f \mapsto \mathcal{F}^{-1}(\phi \cdot \widehat{f}) = f * K$  extends to a bounded linear operator on  $L_p(\mathbb{R})$  for any  $p \in (1, \infty)$ . Finally, since  $K \in \bigcap_{1 < p < \infty} L_p(\mathbb{R})$ , Young's inequality (Proposition A.2) shows that  $L_p(\mathbb{R}) \rightarrow L_q(\mathbb{R})$ ,  $f \mapsto f * K$  is well-defined and bounded for any  $q \in (p, \infty)$ . Therefore, the extended map is still given by  $L_p(\mathbb{R}) \rightarrow L_p(\mathbb{R})$ ,  $f \mapsto f * K$ .  $\square$

## APPENDIX A.

In this appendix we provide proofs for several technical auxiliary results that we used above. We first present some weighted versions of well-known facts for the reader's convenience.

The first lemma is a weighted version of Schur's test.

**Lemma A.1** (Schur's test). *Let  $K : G \times G \rightarrow \mathbb{C}$  be measurable, let  $w > 0$  denote a weight on  $G$ , and let  $p, q, r \in [1, \infty]$  with  $1 + 1/p = 1/q + 1/r$ . Assume that there is a constant  $C_K > 0$  such that*

$$(77) \quad \left\| K(x, \cdot) \cdot \frac{w(x)}{w} \right\|_{L_r} \leq C_K \quad \text{for a.e. } x \in G,$$

$$(78) \quad \left\| K(\cdot, y) \cdot \frac{w}{w(y)} \right\|_{L_r} \leq C_K \quad \text{for a.e. } y \in G.$$

If  $f \in L_{q,w}(G)$ , then the integral

$$I_f(x) = \int_G K(x, y) f(y) dy$$

converges for a.e.  $x \in G$ . The function  $I_f$  is in  $L_{p,w}(G)$  and fulfills

$$\|I_f\|_{L_{p,w}} \leq C_K \|f\|_{L_{q,w}}.$$

*Proof.* It suffices to assume  $f \geq 0$  and  $K \geq 0$ . Indeed, temporarily writing  $I_{K,f}$  instead of  $I_f$  to emphasize the role of the kernel  $K$ , we have  $|I_{K,f}| \leq I_{|K|,|f|}$ ; furthermore, if (77) and (78) hold for  $K$ , then they also hold for  $|K|$ , with the same constants, and we have  $\|f\|_{L_{q,w}} = \||f|\|_{L_{q,w}}$ . Hence, if the claim holds for  $K, f \geq 0$ , then also

$$\|I_{K,f}\|_{L_{p,w}} \leq \|I_{|K|,|f|}\|_{L_{p,w}} \leq C_{|K|} \cdot \||f|\|_{L_{q,w}} = C_K \cdot \|f\|_{L_{q,w}}.$$

Thus, we will assume in the following that  $K, f \geq 0$ . Hence, also  $I_f \geq 0$ , so that [16, Theorem 6.14] shows

$$(79) \quad \|I_f\|_{L_{p,w}} = \sup_{0 \leq h \in L_{p',w^{-1}}(G) \setminus \{0\}} \frac{\langle I_f, h \rangle_{L_2}}{\|h\|_{L_{p',w^{-1}}}}.$$

We denote with  $d(x, y)$  the product measure on  $G \times G$ . Furthermore, for brevity we set  $M_x(y) := \frac{w(x)}{w(y)} \cdot K(x, y)$  and observe  $\|M_x\|_{L_r} \leq C_K$  for almost all  $x \in G$ , thanks to (77). Likewise, (78) shows  $\|M^{(y)}\|_{L_r} \leq C_K$  for almost all  $y \in G$ , where  $M^{(y)}(x) := \frac{w(x)}{w(y)} \cdot K(x, y)$ .

We first consider a number of special cases, so that we can then concentrate on the case where  $p, q, r \in (1, \infty)$ .

**Case 1:** At least one of  $p, q, r$  is infinite. In case of  $p < \infty$ , we have  $1 < 1 + p^{-1} = q^{-1} + r^{-1}$ . But if  $q = \infty$ , then the right-hand side of this inequality is  $r^{-1} \leq 1$ , which leads to a contradiction. Similarly, we see that  $r = \infty$  leads to a contradiction. Therefore, we necessarily have  $p = \infty$  in the present case.

Because of  $1 = 1 + p^{-1} = q^{-1} + r^{-1}$ , this implies  $q = r'$ , and hence

$$w(x) \cdot I_f(x) = \int_G M_x(y) \cdot w(y) \cdot f(y) dy \leq \|M_x\|_{L_r} \cdot \|f\|_{L_{r',w}} \leq C_K \cdot \|f\|_{L_{q,w}} < \infty$$

for almost all  $x \in G$ , proving the claim in Case 1, since  $p = \infty$ .

**Case 2:** We have  $p, q, r < \infty$ , but at least one of  $p, q, r$  is equal to one. This leaves three subcases:

**Case 2-A:** We have  $p = 1$ , and hence  $2 = 1 + p^{-1} = q^{-1} + r^{-1} \leq 2$ , which implies  $q = r = 1$ . Hence, by Fubini's theorem,

$$\begin{aligned} \|I_f\|_{L_{p,w}} &= \int_G w(x) \int_G K(x, y) \cdot f(y) dy dx = \int_G w(y) \cdot f(y) \cdot \int_G M_x(y) dx dy \\ &\leq C_K \cdot \|f\|_{L_{1,w}} = C_K \cdot \|f\|_{L_{q,w}}, \end{aligned}$$

which proves the claim in Case 2-A.

**Case 2-B:** We have  $p \in (1, \infty)$ , but  $r = 1$ . Since  $1 + p^{-1} = q^{-1} + r^{-1} = 1 + q^{-1}$ , this implies  $p = q \in (1, \infty)$ . Hence, for each nonnegative  $h \in L_{p',w^{-1}}(G) \setminus \{0\}$ , Fubini's theorem and Hölder's inequality show

$$\begin{aligned} \langle I_f, h \rangle_2 &= \int_G h(x) \int_G K(x, y) f(y) dy dx \\ &= \int_{G \times G} \frac{h(x)}{w(x)} \cdot [M^{(y)}(x)]^{\frac{1}{p'}} [M^{(y)}(x)]^{\frac{1}{p}} \cdot w(y) f(y) d(x, y) \\ &\leq \left( \int_G \left( \frac{h(x)}{w(x)} \right)^{p'} \int_G M^{(y)}(x) dy dx \right)^{1/p'} \\ &\quad \cdot \left( \int_G (w(y) f(y))^p \int_G M^{(y)}(x) dx dy \right)^{1/p} \\ &\leq C_K \cdot \|h\|_{L_{p',w^{-1}}} \cdot \|f\|_{L_{p,w}}. \end{aligned}$$

In view of (79) and because of  $p = q$ , this proves the claim in Case 2-B.

**Case 2-C:** We have  $p, r \in (1, \infty)$ , but  $q = 1$ . This implies  $p = r \in (1, \infty)$ , since  $1 + p^{-1} = q^{-1} + r^{-1} = 1 + r^{-1}$ . For nonnegative  $h \in L_{p',w^{-1}}(G) = L_{r',w^{-1}}(G)$ , we thus have

$$\begin{aligned} \langle I_f, h \rangle_{L_2} &= \int_G w(y) \cdot f(y) \int_G M_x(y) \cdot \frac{h(x)}{w(x)} dx dy \\ &\leq \int_G w(y) \cdot f(y) \cdot \|M_x\|_{L_p} \cdot \|h\|_{L_{p',w^{-1}}} dy \\ &\leq C_K \cdot \|h\|_{L_{p',w^{-1}}} \cdot \|f\|_{L_{1,w}} = C_K \cdot \|h\|_{L_{p',w^{-1}}} \cdot \|f\|_{L_{q,w}}. \end{aligned}$$

In view of (79), this proves the claim in Case 2-C.

Finally, we handle the case  $p, q, r \in (1, \infty)$ . By elementary calculations one can show  $r/p + r/q' = q/p + q/r' = p'/q' + p'/r' = 1$ , where all occurring numbers  $\frac{r}{p}, \frac{r}{q'}$  and so on are elements of the interval  $(0, 1)$ . Thus, for any  $0 \leq h \in L_{p',w^{-1}}(G)$ , it



follows from Hölder's inequality and Fubini's theorem that

$$\begin{aligned}
\langle I_f, h \rangle_{L_2} &= \int_{G \times G} K(x, y) \frac{w(x)}{w(y)} \cdot f(y)w(y) \cdot h(x)w(x)^{-1} d(x, y) \\
&= \int_{G \times G} (M^{(y)}(x))^{r/p} \cdot (f(y)w(y))^{q/p} \cdot (M_x(y))^{r/q'} \\
&\quad \cdot (h(x)w(x)^{-1})^{p'/q'} \cdot (f(y)w(y))^{q/r'} \cdot (h(x)w(x)^{-1})^{p'/r'} d(x, y) \\
&\stackrel{(*)}{\leq} \left( \int_G |f(y)w(y)|^q \int_G (M^{(y)}(x))^r dx dy \right)^{1/p} \\
&\quad \cdot \left( \int_G |h(x)w(x)^{-1}|^{p'} \int_G (M_x(y))^r dy dx \right)^{1/q'} \\
&\quad \cdot \left( \int_{G \times G} |f(y)w(y)|^q |h(x)w(x)^{-1}|^{p'} d(x, y) \right)^{1/r'} \\
&\leq C_K \cdot \|f\|_{L_{q,w}} \cdot \|h\|_{L_{p',w^{-1}}} < \infty,
\end{aligned}$$

where the step marked with (\*) used  $\frac{1}{p} + \frac{1}{q'} + \frac{1}{r'} = \frac{1}{p} + 1 - \frac{1}{q} + 1 - \frac{1}{r} = 1$ . In view of (79), this proves the claim for the case  $p, q, r \in (1, \infty)$ .  $\square$

Next we present a weighted version of the classical Young's inequality.

**Proposition A.2** (Young's inequality). *Let  $m$  be a  $w$ -moderate weight on  $G$ , see (13), and let  $p, q, r \in [1, \infty]$  such that  $1 + 1/p = 1/q + 1/r$ . Then it follows for  $f \in L_{q,m}(G)$  and  $g \in L_{r,w}(G) \cap L_{r,w\Delta^{-1/r}}(G)$  that  $f * g \in L_{p,m}(G)$  and*

$$(80) \quad \|f * g\|_{L_{p,m}} \leq \max\{\|g\|_{L_{r,w}}, \|g\|_{L_{r,w\Delta^{-1/r}}}\} \cdot \|f\|_{L_{q,m}}.$$

If, instead of  $g \in L_{r,w}(G) \cap L_{r,w\Delta^{-1/r}}(G)$ , it holds  $g \in L_{r,w}(G)$  and  $|g(x)| = |g(x^{-1})|$  as well as  $w(x) = w(x^{-1})$  for all  $x \in G$ , or if  $g \in L_{r,w}(G)$  and  $G$  is unimodular, then

$$(81) \quad \|f * g\|_{L_{p,m}} \leq \|g\|_{L_{r,w}} \cdot \|f\|_{L_{q,m}}.$$

*Proof.* We apply Lemma A.1 for the case  $K(x, y) = g(y^{-1}x)$  and the weight  $m$ . It suffices to show that there exists a constant  $C_K$  that fulfills (77) and (78). We first consider the case  $r < \infty$  and use (13) and the left invariance of the Haar measure to conclude

$$\begin{aligned}
\int_G |g(y^{-1}x)|^r \cdot \frac{m(x)^r}{m(y)^r} dx &= \int_G |g(z)|^r \cdot \frac{m(yz)^r}{m(y)^r} dz \leq \int_G |g(z)|^r \cdot \frac{m(y)^r w(z)^r}{m(y)^r} dz \\
&= \int_G |g(z)|^r \cdot w(z)^r dz = \|g\|_{L_{r,w}}^r
\end{aligned}$$

for almost all  $y \in G$ . Now, using the change of variables  $z = x^{-1}y$ , and recalling the formula  $d\rho(x) = \Delta(x^{-1})dx$  (see [15, Proposition 2.31]) for the right Haar measure  $\rho$  given by  $\rho(M) = \beta(M^{-1})$ , we see

$$\begin{aligned}
\int_G |g(y^{-1}x)|^r \cdot \frac{m(x)^r}{m(y)^r} dy &= \int_G |g(z^{-1})|^r \cdot \frac{m(x)^r}{m(xz)^r} dz \leq \int_G |g(z^{-1})|^r \cdot [w(z^{-1})]^r dz \\
&= \int_G |g(y)|^r \cdot [w(y)]^r \cdot \Delta(y)^{-1} dy = \|g\|_{L_{r,w\Delta^{-1/r}}}^r
\end{aligned}$$

for almost all  $x \in G$ . By setting  $C_K = \max\{\|g\|_{L_{r,w}}, \|g\|_{L_{r,w\Delta^{-1/r}}}\} < \infty$ , Lemma A.1 yields

$$\|f * g\|_{L_{p,m}} \leq C_K \cdot \|f\|_{L_{q,m}} \quad \text{for all } f \in L_{q,m}(G),$$

which proves (80).

Finally, for the case  $r = \infty$ , observe  $m(x) = m(yy^{-1}x) \leq m(y) \cdot w(y^{-1}x)$ , so that we get

$$|g(y^{-1}x)| \cdot \frac{m(x)}{m(y)} \leq |g(y^{-1}x)| \cdot w(y^{-1}x) \leq \|g\|_{L_{\infty,w}}$$

for almost every  $x \in G$  and almost every  $y \in G$ , which establishes (77) and (78).

It remains to prove (81). If we assume  $|g(x)| = |g(x^{-1})|$  and  $w(x) = w(x^{-1})$ , the formula  $d\varrho(x) = \Delta(x^{-1})dx$  from above yields for  $r < \infty$  that

$$\begin{aligned} \|g\|_{L_{r,w\Delta^{-1/r}}}^r &= \int_G |g(y)|^r \cdot [w(y)]^r \cdot \Delta(y^{-1}) dy = \int_G |g(z^{-1})|^r \cdot [w(z^{-1})]^r dz \\ &= \int_G |g(z)|^r \cdot [w(z)]^r dz = \|g\|_{L_{r,w}}^r. \end{aligned}$$

This identity trivially holds if  $G$  is unimodular, so that  $\Delta \equiv 1$ . For  $r = \infty$ , we always have  $\|g\|_{L_{r,w\Delta^{-1/r}}} = \|g\|_{L_{r,w}}$ . In all of these cases (81) is a direct consequence of (80).  $\square$

**Lemma A.3.** *Let  $A$  be a bounded and surjective linear operator that maps a Banach space  $W$  onto a Banach space  $V$ . Suppose that the kernel of  $A$  admits a complement  $L$  in  $W$ . Set*

$$\varepsilon := \inf \left\{ \sup \{ |\langle Ax, y \rangle_{V \times V^*}| \mid y \in V^*, \|y\|_{V^*} = 1 \} \mid x \in L, \|x\|_W = 1 \right\}.$$

*Then the map  $S := (A|_L)^{-1} : V \rightarrow L \subset W$  is a linear right inverse of  $A$  with*

$$\|S\| = \varepsilon^{-1}.$$

*Proof.* It is straightforward that  $A|_L : L \rightarrow V$  is a bijection. Therefore  $S$  is indeed a linear right inverse of  $A$ , and we have

$$\begin{aligned} &\inf \{ \sup \{ |\langle Ax, y \rangle_{V \times V^*}| \mid y \in V^*, \|y\|_{V^*} = 1 \} \mid x \in L, \|x\|_W = 1 \} \\ &= \inf \{ \|Ax\|_V \mid x \in L, \|x\|_W = 1 \} \\ &= \inf \left\{ \frac{\|Ax\|_V}{\|x\|_W} \mid x \in L \setminus \{0\} \right\} = \inf \left\{ \frac{\|ASv\|_V}{\|Sv\|_W} \mid v \in V \setminus \{0\} \right\} \\ &= \left( \sup \left\{ \frac{\|Sv\|_W}{\|ASv\|_V} \mid v \in V \setminus \{0\} \right\} \right)^{-1} = \left( \sup \left\{ \frac{\|Sv\|_W}{\|v\|_V} \mid v \in V \setminus \{0\} \right\} \right)^{-1} \\ &= \|S\|^{-1}, \end{aligned}$$

which proves the claim.  $\square$

Lemma 3.3 and Lemma 3.15 as well as Proposition 4.5 were left unproven. The proofs are presented here.

*Proof of Lemma 3.3.* We start with an auxiliary observation: We claim that  $\|g\|_{L_{\infty}(Qx)} = \sup_{y \in Qx} |g(y)|$  if  $g : G \rightarrow \mathbb{C}$  is continuous and if  $Q \subset G$  is a compact unit neighborhood with  $Q = \overline{\text{int } Q}$ .

Indeed, the inequality “ $\leq$ ” is trivial. Conversely, if we set  $\alpha := \|g\|_{L_{\infty}(Qx)}$ , then the set  $M := \{y \in G \mid |g(y)| > \alpha\}$  is open, and  $M \cap Qx$  is a null-set. Hence,  $M \cap$

$(\text{int } Q)x = \emptyset$ , since this is an open null-set. In other words,  $|g(y)| \leq \alpha$  for all  $y \in (\text{int } Q)x$ . By continuity of  $g$  and since  $Q \subset \overline{\text{int } Q}$ , we see  $|g(y)| \leq \alpha$  for all  $y \in Qx$ .

In particular, this implies  $\check{M}_Q^\rho g(x) = \sup_{q \in Q} |g(qx)|$ , and thus (because of  $e \in Q$ )  $\check{M}_Q^\rho g \geq |g|$ .

To prove i) we note that  $\check{M}_{Q_0}^\rho f \in L_{p,w}(G)$  which implies  $f \in L_{p,w}(G)$ , since we just saw that  $\check{M}_{Q_0}^\rho f \geq |f|$ . We intend to show  $\|\text{osc}_{Q_0} f\|_{L_{p,w}} < \infty$ . But we have

$$\text{osc}_{Q_0} f(x) = \sup_{q \in Q_0} |f(qx) - f(x)| \leq \sup_{q \in Q_0} |f(qx)| + |f(x)| \leq |f(x)| + \check{M}_{Q_0}^\rho f(x).$$

Therefore,

$$(82) \quad \|\text{osc}_{Q_0} f\|_{L_{p,w}} \leq \|\check{M}_{Q_0}^\rho f\|_{L_{p,w}} + \|f\|_{L_{p,w}} < \infty.$$

It remains to prove ii). For this we first note that  $\text{osc}_Q f \leq \text{osc}_{Q_0} f$  if  $Q \subset Q_0$ . Furthermore, by part i) we have  $\text{osc}_{Q_0} f \in L_{p,w}(G)$ . Hence, since  $G$  is  $\sigma$ -compact, for any  $\varepsilon > 0$ , there exists a compact set  $K \subset G$  of positive measure such that

$$(83) \quad \int_{G \setminus K} |\text{osc}_Q f(x)w(x)|^p dx \leq \int_{G \setminus K} |\text{osc}_{Q_0} f(x)w(x)|^p dx < \frac{\varepsilon}{2}$$

for all unit neighborhoods  $Q \subset Q_0$ .

Next, we observe that since  $f$  is continuous, it is uniformly continuous on  $K$  in the following sense: For every  $\delta > 0$  there is a unit neighborhood  $U_\delta \subset G$  with  $|f(x) - f(ux)| < \delta$  for all  $x \in K$  and  $u \in U_\delta$ .

The uniform continuity described above simply means  $\text{osc}_{U_\delta} f(x) \leq \delta$  for all  $x \in K$ . Choosing  $\delta := \varepsilon^{1/p} / (2 \cdot |K|^{1/p} \sup_{y \in K} w(y))$ , we see for every unit neighborhood  $Q \subset Q_0 \cap U_\delta$  that

$$(84) \quad \int_K |\text{osc}_Q f(x)w(x)|^p dx \leq \int_K \frac{\varepsilon}{2|K|} \cdot \frac{w(x)^p}{\sup_{y \in K} w(y)^p} dx \leq \int_K \frac{\varepsilon}{2|K|} dx = \frac{\varepsilon}{2}.$$

Equations (83) and (84) yield  $\|\text{osc}_Q f\|_{L_{p,w}}^p < \varepsilon$ , which concludes the proof.

*Proof of Lemma 3.15.* Let  $1 \leq p < \infty$  and  $(d_x)_{x \in Y_n} \in \ell_{p,m}(Y_n)$ , then we first note that for all  $x \in G$  it holds

$$\begin{aligned} \int_{xQ_n} m(y)^p dy &= \int_{Q_n} m(xy)^p dy \leq m(x)^p \int_{Q_n} w(y)^p dy \\ &\leq \sup_{q \in Q_n} w(q)^p \cdot |Q_n| \cdot m(x)^p. \end{aligned}$$

With this at hand and since  $Y_n$  is relatively  $Q_n$ -separated, as stated in (30), we derive

$$\begin{aligned} \left\| \sum_{x \in Y_n} |d_x| \chi_{xQ_n} \right\|_{L_{p,m}} &\leq \sum_{i=1}^I \left\| \sum_{x \in Z_{n,i}} |d_x| \chi_{xQ_n} \right\|_{L_{p,m}} = \sum_{i=1}^I \left( \sum_{x \in Z_{n,i}} |d_x|^p \int_{xQ_n} m(y)^p dy \right)^{\frac{1}{p}} \\ &\leq \sum_{i=1}^I \left( \sum_{x \in Z_{n,i}} |d_x|^p m(x)^p \right)^{\frac{1}{p}} \cdot \sup_{q \in Q_n} w(q) \cdot |Q_n|^{\frac{1}{p}} \\ &\leq I^{1-\frac{1}{p}} \cdot \sup_{q \in Q_n} w(q) \cdot |Q_n|^{\frac{1}{p}} \cdot \|(d_x)_{x \in Y_n}\|_{\ell_{p,m}}. \end{aligned}$$

It remains to prove the case  $p = \infty$ . Similarly as above, we see that

$$\begin{aligned} \left\| \sum_{x \in Y_n} |d_x| \chi_{xQ_n} \right\|_{L^{\infty, m}} &\leq \sum_{i=1}^I \left\| \sum_{x \in Z_{n,i}} |d_x| \chi_{xQ_n} \right\|_{L^{\infty, m}} = \sum_{i=1}^I \sup_{x \in Z_{n,i}} \left( |d_x| \cdot \sup_{y \in xQ_n} m(y) \right) \\ &\leq I \cdot \left( \sup_{x \in Y_n} |d_x| \cdot m(x) \right) \cdot \sup_{y \in Q_n} w(y) \\ &= I \cdot \sup_{q \in Q_n} w(q) \cdot \|(d_x)_{x \in Y_n}\|_{\ell^{\infty, m}}. \end{aligned}$$

*Proof of Proposition 4.5.* We will construct  $C \subset [0, 1]$  as a certain “fat Cantor set”. In particular, we will show below that  $C$  has positive measure and fulfills the following two additional properties:

$$(85) \quad |C \cap B| < |B| \quad \text{for all open intervals } \emptyset \neq B \subset \mathbb{R},$$

and

$$(86) \quad C^c = \bigcup_{n=0}^{\infty} \bigcup_{j=0}^{2^n-1} B_j^n \quad \text{with} \quad B_j^n := \frac{a_j^{(n)} + b_j^{(n)}}{2} + \left( -\frac{\mu_{n+1}}{2}, \frac{\mu_{n+1}}{2} \right),$$

where the complement  $C^c$  is taken relative to  $[0, 1]$ , and where  $a_j^{(n)}, b_j^{(n)} \in \mathbb{R}$  are suitable, while  $\mu_n := \min\{4^{-n}, n^{-n}\}$  for  $n \in \mathbb{N}$ .

Before we provide the precise construction of such a set  $C$ , let us see how the properties (85) and (86) imply the properties of  $C$  that are stated in the proposition.

First, [24, Theorem 1] shows that if the operator  $f \mapsto f * \mathcal{F}^{-1} \chi_C$  is bounded on  $L_p(\mathbb{R})$  for some  $p \in (1, \infty) \setminus \{2\}$ , that is, if  $\chi_C$  is an  $L_p(\mathbb{R})$ -Fourier multiplier, then  $C$  would be equivalent to an open set. In other words, there would be an open set  $U \subset \mathbb{R}$  with  $\chi_C = \chi_U$  Lebesgue almost everywhere. But since  $C$  has positive measure, this is only possible if  $U$  is a *nonempty* open set. Therefore,  $U$  contains a nonempty open interval  $B \subset U$ . Since  $\chi_C = \chi_U$  almost everywhere, this implies  $|B \cap C| = |B \cap U| = |B|$ , in contradiction to (85). In summary, we have thus shown that the convolution operator  $f \mapsto f * \mathcal{F}^{-1} \chi_C$  is *not* bounded on any  $L_p(\mathbb{R})$  space for  $p \in (1, \infty) \setminus \{2\}$ . But this even implies that  $L_p(\mathbb{R}) \rightarrow L_p(\mathbb{R}), f \mapsto f * \mathcal{F}^{-1} \chi_C$  is not well-defined, by Proposition 3.5.

Second, we will see that (86) ensures  $\mathcal{F}^{-1} \chi_{C^c} \in \bigcap_{1 < p \leq \infty} L_p(\mathbb{R})$ , which then implies  $\mathcal{F}^{-1} \chi_C = \mathcal{F}^{-1} \chi_{(0,1)} - \mathcal{F}^{-1} \chi_{C^c} \in \bigcap_{1 < p \leq \infty} L_p(\mathbb{R})$ . Here, we used that  $F := \mathcal{F}^{-1} \chi_{(0,1)} \in \bigcap_{1 < p \leq \infty} L_p(\mathbb{R})$ , since a direct computation shows  $F(x) = \frac{e^{2\pi i x} - 1}{2\pi i x}$  for  $x \neq 0$ , which implies  $|F(x)| \lesssim (1 + |x|)^{-1}$ . It remains to show  $\mathcal{F}^{-1} \chi_{C^c} \in \bigcap_{1 < p \leq \infty} L_p(\mathbb{R})$ .

To this end, we set  $\xi_j^{(n)} := \frac{a_j^{(n)} + b_j^{(n)}}{2} - \frac{\mu_{n+1}}{2}$ , recall the definition of the intervals  $B_j^n = \xi_j^{(n)} + \mu_{n+1} \cdot (0, 1)$  from (86), and use standard properties of the Fourier transform to compute

$$\mathcal{F}^{-1} \chi_{B_j^n} = \mu_{n+1} \cdot M_{\xi_j^{(n)}}[(\mathcal{F}^{-1} \chi_{(0,1)})(\mu_{n+1} \cdot)] = \mu_{n+1} \cdot M_{\xi_j^{(n)}}(F(\mu_{n+1} \cdot)),$$

where  $(M_{\xi} f)(x) = e^{2\pi i x \xi} f(x)$  denotes the *modulation* with frequency  $\xi$  of a function  $f$ . Next, (86) shows

$$\mathcal{F}^{-1} \chi_{C^c} = \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \mathcal{F}^{-1} \chi_{B_j^n}.$$

Combining this with the triangle inequality for  $L_p$  and with the elementary identities  $\|M_\xi f\|_{L_p} = \|f\|_{L_p}$  and  $\|f(a \cdot)\|_{L_p(\mathbb{R})} = a^{-1/p} \|f\|_{L_p(\mathbb{R})}$  for  $a > 0$  and  $f \in L_p(\mathbb{R})$ , we see because of  $\mu_n \leq n^{-n}$  and  $1 - p^{-1} > 0$  for each fixed  $p \in (1, \infty]$  that

$$(87) \quad \begin{aligned} \|\mathcal{F}^{-1} \chi_{C^c}\|_{L_p} &\leq \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \mu_{n+1} \cdot \|M_{\xi_j^{(n)}}(F(\mu_{n+1} \cdot))\|_{L_p} \\ &\leq \|F\|_{L_p} \cdot \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \mu_{n+1}^{1-p^{-1}} \leq \|F\|_{L_p} \cdot \sum_{\ell=1}^{\infty} 2^{\ell-1} \cdot \ell^{-\ell(1-p^{-1})}. \end{aligned}$$

But for  $\ell \geq \ell_0 = \ell_0(p)$ , we have  $(1 - p^{-1}) \cdot \log_2(\ell) \geq 2$ , and thus

$$2^{\ell-1} \cdot \ell^{-\ell(1-p^{-1})} = \frac{1}{2} \cdot 2^\ell \cdot 2^{-\ell(1-p^{-1}) \cdot \log_2(\ell)} \leq 2^{\ell(1-(1-p^{-1}) \cdot \log_2(\ell))} \leq 2^{-\ell},$$

so that the series on the right-hand side of (87) converges. Hence,  $\mathcal{F}^{-1} \chi_{C^c} \in L_p(\mathbb{R})$  for every  $p \in (1, \infty]$ .

Finally, we note because of  $\mu_n \leq 4^{-n}$  that property (86) also implies

$$|C^c| = \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} |B_j^n| = \sum_{n=0}^{\infty} 2^n \mu_{n+1} \leq \sum_{n=0}^{\infty} 2^n \cdot 4^{-(n+1)} \leq \frac{1}{4} \cdot \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{2} < 1,$$

so that  $C \subset [0, 1]$  necessarily has positive measure if it satisfies properties (85) and (86). It remains to show that one can indeed construct a compact set  $C \subset [0, 1]$  that satisfies properties (85) and (86).

To this end, as for the construction of the classical Cantor set, we will set  $C := \bigcap_{n=0}^{\infty} C^n$  where the sets  $C^n := \bigcup_{j=0}^{2^n-1} C_j^n$  will be defined inductively.

For the start of the induction set  $C_0^0 := [a_1^{(0)}, b_1^{(0)}] := [0, 1]$ .

For the induction step, assume for some  $n \in \mathbb{N}_0$  that we have constructed closed intervals  $C_\ell^n = [a_\ell^{(n)}, b_\ell^{(n)}] \subset [0, 1]$ , for  $\ell = 0, \dots, 2^n - 1$ , with

$$(88) \quad 4^{-n} \leq b_\ell^{(n)} - a_\ell^{(n)} \leq 2^{-n} \quad \text{for all } 0 \leq \ell < 2^n$$

and

$$(89) \quad b_\ell^{(n)} < a_{\ell+1}^{(n)} \quad \text{for } 0 \leq \ell < 2^n - 1.$$

Now, for  $0 \leq j < 2^{n+1}$  we can write  $j = 2\ell + k$  with uniquely determined  $k \in \{0, 1\}$  and  $0 \leq \ell < 2^n$ . We then recall from after (86) that  $\mu_{n+1} = \min\{4^{-(n+1)}, (n+1)^{-(n+1)}\}$ , and define

$$(90) \quad C_j^{n+1} := [a_j^{(n+1)}, b_j^{(n+1)}] := \begin{cases} [a_\ell^{(n)}, \frac{a_\ell^{(n)} + b_\ell^{(n)}}{2} - \frac{\mu_{n+1}}{2}] \subset [a_\ell^{(n)}, b_\ell^{(n)}] = C_\ell^n & \text{if } k = 0, \\ [\frac{a_\ell^{(n)} + b_\ell^{(n)}}{2} + \frac{\mu_{n+1}}{2}, b_\ell^{(n)}] \subset [a_\ell^{(n)}, b_\ell^{(n)}] = C_\ell^n & \text{if } k = 1. \end{cases}$$

With this choice, we see from (88) and because of  $\mu_{n+1} \leq 4^{-(n+1)}$  that

$$b_j^{(n+1)} - a_j^{(n+1)} = \frac{b_\ell^{(n)} - a_\ell^{(n)}}{2} - \frac{\mu_{n+1}}{2} \geq \frac{1}{2} \cdot (4^{-n} - 4^{-(n+1)}) = \frac{3}{8} \cdot 4^{-n} \geq 4^{-(n+1)}$$

and

$$b_j^{(n+1)} - a_j^{(n+1)} = \frac{b_\ell^{(n)} - a_\ell^{(n)}}{2} - \frac{\mu_{n+1}}{2} \leq \frac{1}{2} (b_\ell^{(n)} - a_\ell^{(n)}) \leq 2^{-(n+1)},$$

thereby proving (88) for  $n + 1$  instead of  $n$ .

For the proof of (89) for  $0 \leq j < 2^{n+1} - 1$  with  $j = 2\ell + k$  and  $k \in \{0, 1\}$ , we distinguish two cases:

**Case 1:**  $k = 0$ . In this case,  $j + 1 = 2\ell + 1$ , and hence

$$b_j^{(n+1)} = \frac{a_\ell^{(n)} + b_\ell^{(n)}}{2} - \frac{\mu_{n+1}}{2} < \frac{a_\ell^{(n)} + b_\ell^{(n)}}{2} + \frac{\mu_{n+1}}{2} = a_{j+1}^{(n+1)}.$$

**Case 2:**  $k = 1$ . In this case,  $2(\ell + 1) + 0 = j + 1 < 2^{n+1}$ , so that  $1 \leq \ell + 1 < 2^n$ . Therefore, (89) shows  $b_j^{(n+1)} = b_\ell^{(n)} < a_{\ell+1}^{(n)} = a_{j+1}^{(n+1)}$ .

We have thus verified (89) for  $n + 1$  instead of  $n$ .

As indicated above, we define  $C^n := \bigcup_{j=0}^{2^n-1} C_j^n$  and observe as a consequence of (90) that each  $C^n$  is closed with  $C^{n+1} \subset C^n$  for all  $n \in \mathbb{N}_0$ . Hence,  $C := \bigcap_{n=0}^{\infty} C^n \subset C^0 = [0, 1]$  is compact.

Having defined the set  $C$ , our first goal is to prove property (85). Let  $B \subset \mathbb{R}$  be a nonempty open interval. If  $C \cap B$  is a finite set, the inequality in (85) is trivially satisfied. Hence, we can assume that  $C \cap B$  is infinite, so that there are  $x, y \in C \cap B$  with  $x < y$ . Choose  $n \in \mathbb{N}_0$  with  $2^{-n} < y - x$  and note because of  $x, y \in C \subset C^n = \bigcup_{j=0}^{2^n-1} C_j^n$  that there are  $j_x, j_y \in \{0, \dots, 2^n - 1\}$  with  $x \in C_{j_x}^n$  and  $y \in C_{j_y}^n$ . In case of  $j_y \leq j_x$ , we would get because of  $a_\ell^{(n)} \leq b_\ell^{(n)} \leq a_{\ell+1}^{(n)}$  for  $0 \leq \ell < 2^n - 1$  and because of  $b_\ell^{(n)} - a_\ell^{(n)} \leq 2^{-n}$  for  $0 \leq \ell < 2^n$  (see (88), (89)) that

$$2^{-n} < y - x \leq b_{j_y}^{(n)} - a_{j_x}^{(n)} \leq b_{j_y}^{(n)} - a_{j_y}^{(n)} \leq 2^{-n},$$

a contradiction. Hence,  $j_y > j_x$ , so that (88) and (89) show

$$B \ni x \leq b_{j_x}^{(n)} \leq b_{j_y-1}^{(n)} < a_{j_y}^{(n)} \leq y \in B,$$

and thus  $(b_{j_y-1}^{(n)}, a_{j_y}^{(n)}) \subset B \setminus C^n \subset B \setminus C$ . But since this interval has positive measure, we see  $|B| = |B \setminus C| + |B \cap C| > |B \cap C|$ , thereby proving (85).

Finally, we prove the formula (86) for the complement  $C^c$  of  $C$ , with the complement taken relative to  $[0, 1]$ . To see this, note  $C^c = \bigcup_{n=0}^{\infty} (C^n)^c$ . By disjointization, and since  $(C^0)^c = \emptyset$  and  $(C^n)^c \subset (C^{n+1})^c$ , this yields

$$C^c = \bigcup_{n=1}^{\infty} (C^n)^c \setminus (C^{n-1})^c = \bigcup_{n=1}^{\infty} C^{n-1} \setminus C^n = \bigcup_{n=0}^{\infty} C^n \setminus C^{n+1}.$$

Next, recall  $C^n = \bigcup_{j=0}^{2^n-1} C_j^n$  and also recall from (90) that  $C_{2\ell+k}^{n+1} \subset C_\ell^n$  for  $0 \leq \ell < 2^n$  and  $k \in \{0, 1\}$ . Therefore, by (90) and the definition of  $B_j^n$  in (86) it holds

$$C_j^n \setminus C^{n+1} = \bigcap_{\ell=0}^{2^n-1} \bigcap_{k=0}^1 C_j^n \setminus C_{2\ell+k}^{n+1} = \bigcap_{k=0}^1 C_j^n \setminus C_{2j+k}^{n+1} = B_j^n.$$

Putting everything together, we see that (86) holds.

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