Besov regularity of parabolic and hyperbolic PDEs

S. Dahlke* and C. Schneider†

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Abstract

This paper is concerned with the regularity of solutions to linear and nonlinear evolution equations on nonsmooth domains. In particular, we study the smoothness in the specific scale $B^r_{\tau,\tau}$, $\frac{1}{\tau} = \frac{r}{d} + \frac{1}{p}$ of Besov spaces. The regularity in these spaces determines the approximation order that can be achieved by adaptive and other nonlinear approximation schemes. We show that for all cases under consideration the Besov regularity is high enough to justify the use of adaptive algorithms.

Key Words: Parabolic evolution equations, hyperbolic equations, Besov spaces, Kondratiev spaces, adaptive algorithms.
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Contents

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1 Introduction

This paper is concerned with regularity estimates of the solutions to evolution equations in nonsmooth domains $\mathcal{O}$ contained in $\mathbb{R}^d$. In particular, we study parabolic equations of the form

$$\left\{ \begin{array}{rcl} \frac{\partial}{\partial t}u + (-1)^m L(x,t;D_x)u &=& f \quad \text{in } K \times (0,T), \\ \frac{\partial^{k-1}}{\partial \nu^{k-1}} u\Big|_{\Gamma_{j,T}} &=& 0, \quad k=1,\ldots,m, \; j=1,\ldots,n, \\ u\Big|_{t=0} &=& 0 \quad \text{in } K \end{array} \right\} \tag{1.1}$$

in polyhedral cones $K \subset \mathbb{R}^3$ as well as associated semilinear versions of (1.1) Here the partial differential operator is given by

$$L(x,t;D_x) = \sum_{|\alpha|,|\beta|=0}^{m} D_x^\alpha (a_{\alpha\beta}(x,t) D_x^\beta). \tag{1.2}$$

We will also be concerned with hyperbolic problems

$$\left\{ \begin{array}{rcl} \frac{\partial^2}{\partial t^2}u + L(x,t,D_x)u &=& f \quad \text{in } \Omega \times (0,T), \\ u(x,0) \; = \; \frac{\partial}{\partial t}u(x,0) &=& 0 \quad \text{in } \Omega, \\ u\big|_{\partial\Omega\times(0,T)} &=& 0 \end{array} \right\} \tag{1.3}$$

in specific Lipschitz domains $\Omega \subset \mathbb{R}^d$, $d > 2$, where

$$L(x,t,D_x)u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial}{\partial x_i}u\right) + \sum_{i=1}^{d} b_i(x,t)\frac{\partial}{\partial x_i}u + c(x,t)u,$$

and given function f. We study the spatial regularity of the solutions to (1.1) and (1.3) in specific non-standard smoothness, spaces, i.e., in the so-called *adaptivity scale of Besov spaces*

$$B^r_{\tau,\tau}(\mathcal{O}), \quad \frac{1}{\tau} = \frac{r}{d} + \frac{1}{p}. \tag{1.4}$$

The motivation for these kind of studies can be explained as follows. Evolution equations of the form (1.1) and (1.3) play important roles in the modelling of a lot of problems in science and engineering. As a classical example corresponding to (1.1) let us mention the heat equation that describes the variation in temperature in a given region over time. In many cases, analytic forms of the solutions are not available, so that numerical schemes for their constructive approximation are needed. When it comes to practical applications, very often systems with hundreds of thousands or even millions of degrees of freedom have to be handled. In this case, *adaptive* strategies are often unavoidable to increase efficiency. Essentially, an adaptive scheme is an updating strategy, where additional degrees of freedom are only spent in regions where the numerical approximation is still 'far away' from the exact solution. Although the underlying idea is convincing, the following general question arises: what is the order of convergence that can be achieved by adaptive algorithms, and is it higher than the convergence order of classical nonadaptive (uniform) schemes, which are much easier to design and to implement? As a role of thumb, the following statement can be made: the convergence order that can be achieved by adaptive algorithms is determined by the regularity of the exact solution in the adaptivity scale (1.4) of Besov spaces, whereas the convergence order that is available for nonadaptive schemes depends on the classical Sobolev smoothness. In particular, for adaptive wavelet schemes, these relationships can be made very rigorous, we refer, e.g., to [11, 17, 18, 49] for details. Quite recently, it has turned out that the same interrelations also hold for the very important and widespread adaptive finite element schemes, cf. [29]. Therefore, we can draw the following conclusion: adaptivity is justified, if the Besov regularity of the solution in the Besov scale (1.4) is higher than its Sobolev smoothness!

For the case of *elliptic* partial differential equations, a lot of positive results in this direction are already known [13–16, 19]. It is well–known that if the domain under consideration, the right–hand side and the coefficients are sufficiently smooth, then the problem is completely regular [1], and there is no reason why the Besov smoothness should be higher than the Sobolev regularity. However, on general Lipschitz domains and in particular in polyhedral domains, the situation changes dramatically. For then, singularities at the boundary may occur that diminish the Sobolev regularity of the solution significantly [30, 31, 34]. However, the analysis in the above mentioned papers shows that these boundary singularities do not influence the Besov regularity too much, so that the use of adaptive algorithms for elliptic PDEs is completely justified!

In this paper, we study similar questions for evolution equations of the form (1.1), (1.3) and of associated semilinear versions of them. We show that in all these cases the Besov regularity is high enough to justify the use of adaptive algorithms. To the best of our knowledge, not so many results in this direction are available so far. For parabolic equations, first results for the special case of the heat equation have been reported in [2–4], but for a slightly different scale of Besov spaces. For hyperbolic equations, let us mention the seminal paper of DeVore and Lucier [28] which is concerned with concervation laws in 1D.

The main ingredients to prove our results are regularity estimates in so–called *Kondratiev spaces*. The study of solutions to PDEs in Kondratiev spaces has already quite a long history. We refer, e.g., to [30, 34, 39, 45] (this list is clearly not complete). The basic idea is the following. As already outlined above, on nonsmooth domains, the solutions to PDEs as well as their derivatives might become highly singular as one approaches the boundary. Nevertheless, it has turned out that their strong growth can to some extent be compensated by means of specific weights that consist of the regularized distance to the singular set of the domain to some power. We refer to Section 2 for a detailed description of these spaces. Recents studies have also shown that these Kondratiev spaces are very much related with Besov spaces in the adaptivity scale (1.4) in the sense that powerful embedding results exist, see, e.g., [33]. So, Besov regularity results can be established by first studying the equation under consideration in Kondratiev spaces and then using known (or deriving new) embeddings into Besov spaces.

We carry out this program in the follwing steps. In Section 4 we consider parabolic evolution equations in polyhedral cones contained in $\mathbb{R}^3$. For these problems, regularity estimates in Kondratiev spaces have been derived in [43]. However, for our purposes, it has been necessary to generalize these results to guarantee the boundedness of the solution operator in specific Kondratiev spaces, see Section 3. Then, by a combination with embedding results from [33], we obtain our first main result. It tells us that if the right-hand side as well as its time derivatives are contained in specific Kondratiev spaces, then, for every $t \in (0, T)$ the spatial Besov smoothness of the solution to (1.1) is always larger than $2m$, provided that some technical conditions on the operator pencil are satified, see Theorem 4.5. The reader should observe that this result is independent of the shape of the cone, and that the classical Sobolev smoothness is usually limited by m, see [43]. Therefore, for every t, the spatial Besov regularity is more than twice as high than the Sobolev smoothness, which of course justifies the use of (spatial) adaptive algorithms. Moreover, for smooth domains and right–hand sides in L_2, the best one could expect would be smoothness order $2m$ in the classical Sobolev scale. So, the Besov smoothness on polyhedral cones is at least as high as the Sobolev smoothness on smooth domains.

Then, in Subsection 4.2 we generalize this result to semilinear equations of the form

$$\left\{ \begin{array}{rcll} \frac{\partial}{\partial t}u + (-1)^m L(x,t;D_x)u + \varepsilon u^M &=& f & \text{in } K_T, \\ \frac{\partial^{k-1}}{\partial \nu^{k-1}} u\Big|_{\Gamma_{j,T}} &=& 0, & k=1,\dots,m,\ j=1,\dots,n, \\ u\Big|_{t=0} &=& 0 & \text{in } K, \end{array} \right\} \tag{1.5}$$

where $\varepsilon > 0$ and $M \in \mathbb{N}$. We show that in a sufficiently small ball containing the solution of the corresponding linear equation, there exists as unique solution to (1.5) possessing the same Besov smoothness

in the scale (1.4). The proof is performed by a technically quite involved application of the Banach fixed point theorem. We show that (1.5) has a unique solution in both, the classical scale of Sobolev spaces as well as in the scale of Kondratiev spaces, and then again the result follows by an application of the embedding results from [33]. The final result is stated in Theorem 4.12.
The next natural step is to also study the regularity in time direction. We show that the mapping $t \mapsto u(t,\cdot)$ is in fact a C^l-map into the adaptivity scale of Besov spaces, precisely,

$$u \in \mathcal{C}^{l,\frac{1}{2}}((0,T), B^{\gamma}_{\tau,\infty}),$$

see Theorem 4.14.
In Section 5 we turn our attention to parabolic Besov regularity on general Lipschitz domains. Once again, our results rely on regularity estimates in Kondratiev spaces, which, for the case of general Lipschitz domains, have been derived in [36]. As in the previous section we use embedding results of Kondratiev spaces into the scale of Besov spaces for general Lipschitz domains as, e.g., obtained in [8]. Comparing the regularity results for general Lipschitz domains with the ones for polyhedral cones from Section 4, it turns out that (as expected) the results for the more specific cones are much stronger. Furthermore, it is important to note that the analysis in Section 6 is restricted to second order operators, in contrast to the differential operators of general order we considered before, cf. (1.2).
Finally, in Section 6 we study hyperbolic equations of the form (1.3). Also for these kind of equations regularity estimates in Kondratiev spaces have been derived in [44], so that it is tempting to proceed in a similar way as in the parabolic case. But then, the following problem occurs: the specific domains treated in [44] are not directly covered by the theory presented in [33]. Therefore, we proceed in a slightly different way. We use the fact that Besov spaces can be characterized by wavelet expansions, and therefore, in order to establish Besov smoothness of the solutions to (1.3), their wavelet coefficients have to be estimated. This can be done by once again exploiting the regularity estimates in Kondratiev spaces.

2 Function spaces

2.1 Preliminaries

We start by collecting some general notation used throughout the paper. As usual, we denote by $\mathbb{N}$ the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^d$, $d \in \mathbb{N}$, the d-dimensional real Euclidean space with $|x|$, for $x \in \mathbb{R}^d$, denoting the Euclidean norm of x. By $\mathbb{Z}^d$ we denote the lattice of all points in $\mathbb{R}^d$ with integer components.
We denote by c a generic positive constant which is independent of the main parameters, but its value may change from line to line. The expression $A \lesssim B$ means that $A \leq c\,B$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.
Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded. By supp f we denote the support of the function f. For a domain $\Omega \subset \mathbb{R}^d$ and $r \in \mathbb{N} \cup \{\infty\}$ we write $C^r(\Omega)$ for the space of all real-valued r-times continuously differentiable functions, whereas $C(\Omega)$ is the space of bounded uniformly continuous functions, and $\mathcal{D}(\Omega)$ for the set of test functions, i.e., the collection of all infinitely differentiable functions with support compactly contained in Ω. Moreover, $L^1_{\text{loc}}(\Omega)$ denotes the space of locally integrable functions on Ω.
For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ with with $|\alpha| := \alpha_1 + \ldots + \alpha^d = r$, $r \in \mathbb{N}_0$, and an r-times differentiable function $u : \Omega \to \mathbb{R}$, we write

$$D^{(\alpha)}u = \frac{\partial^{|\alpha|}}{\partial(x_1)^{\alpha_1}\ldots\partial(x_d)^{\alpha_d}}u$$

for the corresponding classical partial derivative as well as $u^{(k)} := D^{(k)}u$ in the one-dimensional case. Hence, the space $C^r(\Omega)$ is normed by

$$\|u|C^r(\Omega)\| := \max_{|\alpha|\leq r} \sup_{x\in\Omega} |D^{(\alpha)}u(x)| < \infty.$$

Moreover, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of rapidly decreasing functions. The set of distributions on Ω will be denoted by $\mathcal{D}'(\Omega)$, whereas $\mathcal{S}'(\mathbb{R}^d)$ denotes the set of tempered distributions on $\mathbb{R}^d$. The terms *distribution* and *generalized function* will be used synonymously. For the application of a distribution $u \in \mathcal{D}'(\Omega)$ to a test function $\varphi \in \mathcal{D}(\Omega)$ we write (u, φ). The same notation will be used if $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (and also for the inner product in $L_2(\Omega)$). Let $u \in \mathcal{D}'(\Omega)$. For $u \in \mathcal{D}'(\Omega)$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we write $D^\alpha u$ for the α-th *generalized* or *distributional derivative* of u with respect to $x = (x_1, \ldots, x_d) \in \Omega$, i.e., $D^\alpha u$ is a distribution on Ω, uniquely determined by the formula

$$(D^\alpha u, \varphi) := (-1)^{|\alpha|}(u, D^\alpha \varphi), \qquad \varphi \in \mathcal{D}(\Omega).$$

In particular, if $u \in L^1_{\text{loc}}(\Omega)$ and there exists a function $v \in L^1_{\text{loc}}(\Omega)$ such that

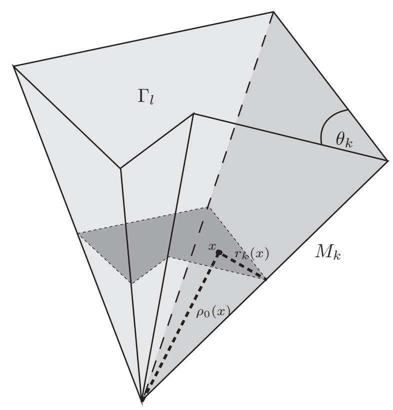
$$\int_\Omega v(x)\varphi(x)\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\varphi(x)\mathrm{d}x \qquad \text{for all} \qquad \varphi \in \mathcal{D}(\Omega),$$

we say that v is the *α-th weak derivative* of u and write $D^\alpha u = v$. We also use the notation $\frac{\partial^k}{\partial x_j^k} u := D^\beta u$ as well as $\partial_{x_j^k} := D^\beta u$, for some multi-index $\beta = (0, \ldots, k, \ldots, 0)$ with $\beta_j = k$, $k \in \mathbb{N}$. Furthermore, for $m \in \mathbb{N}_0$, we write $D^m u$ for any (generalized) m-th order derivative of u, where $D^0 u := u$ and $Du := D^1 u$. Sometimes we shall use subscripts such as D_x^m or D_t^m to emphasize that we only take derivatives with respect to $x = (x_1, \ldots, x_d) \in \Omega$ or $t \in \mathbb{R}$.
For our analysis of parabolic and hyperbolic problems we shall deal with several different types of domains, which we introduce now.
Let D denote some bounded polyhedral domain in $\mathbb{R}^d$. For $0 < T < \infty$ put $D_T = D \times (0, T)$. As a special case of a polyhedral domain in $\mathbb{R}^3$ we will consider a cone (unbounded) with edges defined as follows.

Definition 2.1 *Let*

$$K := \{x \in \mathbb{R}^3 : \ x/|x| = w \in \Omega\}$$

be a polyhedral cone in $\mathbb{R}^3$ with vertex at the origin. Suppose that the boundary ∂K consists of the vertex $x = 0$, the edges (half-lines) $M_1, \ldots, M_n$ and smooth faces $\Gamma_1, \ldots, \Gamma_n$. Moreover, $\Omega = K \cap S^2$ is a curvilinear polygon on the unit sphere bounded by the arcs $\gamma_1, \ldots, \gamma_n$. The angle at the edge M_k will be denoted by θ_k, $k = 1, \ldots, n$.
Furthermore, put $\Gamma_{j,T} = \Gamma_j \times (0, T)$ and $K_T = K \times (0, T)$.

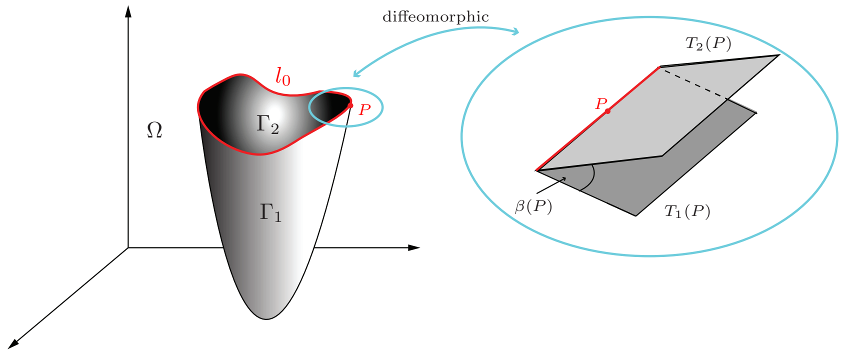


We shall also deal with the truncated cone

$$K_0 := \{x \in K : \ |x| < r_0\} \tag{2.1}$$

for some real number $r_0 > 0$ and put $K_{0,T} := K_0 \times (0, T)$.
For hyperbolic problems we will consider the following domains.

Definition 2.2 *Let $\Omega \subset \mathbb{R}^d$, $d > 2$ be a Lipschitz domain, whose boundary $\partial\Omega$ consists of two surfaces Γ_1 and Γ_2 intersecting along a manifold l_0. We assume that in a neighbourhood of each point of l_0 the set $\overline{\Omega}$ is diffeomorphic to a dihedral angle. Furthermore, $\Omega_T := \Omega \times (0, T)$.*



2.2 Sobolev and Hölder spaces

Unless stated otherwise, let $\mathcal{O}$ stand for either one of the domains D, K or Ω defined above and put $\mathcal{O}_T := \mathcal{O} \times (0,T)$. We introduce some Sobolev spaces. Let $W^m_p(\mathcal{O})$, $1 < p < \infty$, $m \in \mathbb{N}_0$, denote the Sobolev space containing all complex functions $u(x)$ defined on $\mathcal{O}$ such that

$$\|u|W^m_p(\mathcal{O})\| = \left(\sum_{|\alpha|\leq m} \int_{\mathcal{O}} |D^\alpha_x u(x)|^p \mathrm{d}x \right)^{1/p} < \infty.$$

By $\mathring{W}^m_p(\mathcal{O})$ we denote the closure of $\mathcal{D}(\mathcal{O})$ in $W^m_p(\mathcal{O})$. Moreover, $W^{-m}_p(\mathcal{O})$ stands for the dual space $\left(\mathring{W}^m_p(\mathcal{O})\right)'$ of $\mathring{W}^m_p(\mathcal{O})$. The duality pairing is denoted by (u,v) for $u \in W^{-m}_p(\mathcal{O})$ and $v \in \mathring{W}^m_p(\mathcal{O})$.

For $s \geq 0$ fractional Sobolev spaces $W^s_2(\mathbb{R}^d)$ are defined as the spaces which contain all real-valued functions $u(x)$ defined on $\mathbb{R}^d$ such that for $s = k + \lambda$ with $k \in \mathbb{N}_0$ and $\lambda \in (0,1)$ it holds

$$\|u|W^s_2(\mathbb{R}^d)\| = \|u|W^k_2(\mathbb{R}^d)\| + \sum_{|\alpha|=k} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{d+2\lambda}} \mathrm{d}x\mathrm{d}y \right)^{1/2} < \infty. \tag{2.2}$$

It is well known that an equivalent norm is given by

$$\|u|W^s_2(\mathbb{R}^d)\| \sim \| \left((1+|\xi|^2)^{s/2}\hat{u} \right)^{\vee} |L_2(\mathbb{R}^d)\|,$$

cf. [32]. Corresponding spaces on domains can be defined via restriction, i.e., we put

$$\begin{aligned} W^s_2(\mathcal{O}) &= \left\{ f \in L_2(\mathcal{O}) : \ \exists g \in W^s_2(\mathbb{R}^d), \ g\big|_{\mathcal{O}} = f \right\}, \\ \|f|W^s_2(\mathcal{O})\| &= \inf_{g|_{\mathcal{O}}=f} \|f|W^s_2(\mathbb{R}^d)\|. \end{aligned}$$

The spaces $W^m_p(I,X)$ and $\mathcal{C}^{k,\alpha}(I,X)$ Consider a Banach space X and an open interval $I = (0,T) \subset \mathbb{R}$ with $T < \infty$. We write $C(I,X)$ for the space consisting of all bounded and (uniformly) continuous functions $u : I \to X$ normed by

$$\|u|C(I,X)\| := \max_{t\in I} \|u(t)|X\|.$$

Moreover, we say that $u \in C^k(I,X)$, $k \in \mathbb{N}_0$, if u has a Taylor expansion

$$u(t+h) = u(t) + u'(t)h + \frac{1}{2}u''(t)h^2 + \ldots + \frac{1}{k!}u^{(k)}(t)h^k + r_k(t,h)$$

for all $t+h, t \in I$ such that

- $u^{(j)}(t)$ depends continuously on t for all $j = 0, \ldots, k$,

- $\lim_{|h|\to 0} \frac{\|r_k(t,h)|X\|}{|h|^k} = 0.$

The space $C^k(I,X)$ is then equipped with the following norm

$$\|u|C^k(I,X)\| := \sum_{j=0}^{k} \|u^{(j)}|C(I,X)\|.$$

Given $\alpha \in (0,1)$, we denote by $\mathcal{C}^\alpha(I,X)$ the Hölder space containing all $u \in C(I,X)$ such that

$$\begin{aligned} \|u|\mathcal{C}^\alpha(I,X)\| &:= \|u|C(I,X)\| + |u|_{C^\alpha(I,X)} \\ &= \|u|C(I,X)\| + \sup_{\substack{t,s\in I\\ t\neq s}} \frac{\|u(t)-u(s)|X\|}{|t-s|^\alpha} < \infty. \end{aligned}$$

Consequently, $\mathcal{C}^{k,\alpha}(I,X)$ contains all functions $u \in C(I,X)$ such that

$$\|u|\mathcal{C}^{k,\alpha}(I,X)\| := \|u|C^k(I,X)\| + |u^{(k)}|_{C^\alpha(I,X)} < \infty.$$

The above concepts extend to spaces $C^k(I,Y)$ and $\mathcal{C}^{k,\alpha}(I,Y)$, respectively, where Y is a quasi-Banach space. For some further comments we refer to Remark A.1 in Appendix A.
Let us briefly recall the definition of Lebesgue and Sobolev spaces for functions with values in a Banach space X. We denote by $L_p(I,X)$, $1 \le p \le \infty$, the space of (equivalence classes of) measurable functions $u : I \to X$ such that the mapping $t \mapsto \|u(t)\|_X$ belongs to $L_p(I)$, which is endowed with the norm

$$\|u|L_p(I,X)\| = \begin{cases} \left(\int_I \|u(t)|X\|^p \mathrm{d}t\right)^{1/p} & \text{if } p < \infty, \\ \operatorname{ess\,sup}_{t\in I} \|u(t)|X\| & \text{if } p = \infty. \end{cases}$$

The definition of weak derivatives of Banach-space valued distributions is a natural generalization of the one for real-valued distributions. We refer to [6, Part I, Sect. 3] in this context. Let $\mathcal{D}'(I,X) := \mathcal{L}(\mathcal{D}(I),X)$ be the space of X-valued distributions, where $\mathcal{L}(U,V)$ denotes the space of all linear continuous functions from U to V. For the application of a distribution $u \in \mathcal{D}'(I,X)$ to a test function $\varphi \in \mathcal{D}(I)$, we use the notation (u,φ). For $u \in \mathcal{D}'(I,X)$ and $k \in \mathbb{N}$, the k-th *generalized* or *distributional derivative* $\partial_{t^k} u$ is defined as a X-valued distribution satisfying

$$(\partial_{t^k} u, \varphi) := (-1)^k (u, \partial_{t^k}\varphi), \qquad \varphi \in \mathcal{D}(I).$$

In particular, if $u : I \to X$ is an integrable function and there exists an integrable function $v : I \to X$ satisfying

$$\int_I v(t)\varphi(t)\mathrm{d}t = (-1)^k \int_I u(t)\partial_{t^k}\varphi(t)\mathrm{d}t \qquad \text{for all} \quad \varphi \in \mathcal{D}(I),$$

where the integrals above are Bochner integrals, cf. [12], we say that v is the *k-th weak derivative* of u and write $\partial_{t^k} u = v$. For $m \in \mathbb{N}_0$ we denote by $W_p^m(I,X)$ the space of all functions $u \in L_p(I,X)$, whose weak derivatives of order $0 \le k \le m$ belong to $L_p(I,X)$, normed by

$$\|u\|_{W_p^m(I,X)} = \begin{cases} \left(\sum_{k=0}^{m} \|\partial_{t^k} u|L_p(I,X)\|^p\right)^{1/p} & \text{if } p < \infty, \\ \max_{0\le k\le m} \|\partial_{t^k} u|L_\infty(I,X)\| & \text{if } p = \infty. \end{cases}$$

$L_p(I,X)$ and $W_p^m(I,X)$ are Banach spaces.

In the course of this paper, we will also need a version of Sobolev's embedding theorem for Banach-space valued functions. For the reader's convenience we give a short proof in Appendix A, since a suitable reference was not found.

Theorem 2.3 (Sobolev embedding) *Let $1 < p < \infty$. Then*

$$W_p^m(I,X) \hookrightarrow \mathcal{C}^{k,\alpha}(I,X), \tag{2.3}$$

for parameters satisfying $k = m-1$, $\alpha = 1-\frac{1}{p}$, and $0 < \alpha < 1$.

Remark 2.4 In particular, we have $W_p^1(I,X) \hookrightarrow \mathcal{C}^{0,\alpha}(I,X)$ for $m = 1$. This was proven in [7, Thm. 1.4.38].

With the notation introduced above we further put for brevity

$$L_p(\mathcal{O}_T) := L_p((0,T), L_p(\mathcal{O}))$$

and

$$W_p^{m,l}(\mathcal{O}_T) := W_p^{l-1}((0,T), W_p^m(\mathcal{O})) \cap W_p^l((0,T), L_p(\mathcal{O})), \qquad l \in \mathbb{N},$$

normed by

$$\|u|W_p^{m,l}(\mathcal{O}_T)\| = \|u|W_p^{l-1}((0,T), W_p^m(\mathcal{O}))\| + \|u|W^l((0,T), L_p(\mathcal{O}))\|.$$

The space $\mathring{W}_p^{m,l}(\mathcal{O}_T)$ is the closure in $W_p^{m,l}(\mathcal{O}_T)$ of the set consisting of all functions $u \in C^\infty(\mathcal{O}_T)$, which vanish near $\partial\mathcal{O}_T := \partial\mathcal{O} \times [0,T]$.

2.3 Weighted Sobolev spaces

In the sequel we shall further consider several types of weighted Sobolev spaces. The first ones are the so-called *Babuska-Kondratiev spaces* $\mathcal{K}_{p,a}^m(\mathcal{O})$, defined as the collection of all functions $u(x)$ such that

$$\|u|\mathcal{K}_{p,a}^m(\mathcal{O})\| := \left(\sum_{|\alpha|\le m} \int_{\mathcal{O}} |\varrho(x)|^{p(|\alpha|-a)} |D_x^\alpha u(x)|^p \mathrm{d}x\right)^{1/p} < \infty, \tag{2.4}$$

where $a \in \mathbb{R}$, $1 < p < \infty$, $m \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$, and the weight function $\varrho : D \to [0,1]$ is the smooth distance to the singular set of $\mathcal{O}$, i.e., ϱ is a smooth function and in the vicinity of the singular set S it is equivalent to the distance of that set. In particular, if $\mathcal{O} = D$ or $\mathcal{O} = K$ then in 2D the singular set S consists of the vertices of the polygon whereas in 3D the set S consists of vertices and edges of the polyhedra/polyhedral cone. On the other hand if $\mathcal{O} = \Omega$, then $S = l_0$ and in the vicinity of l_0 the function ϱ is equal to $\mathrm{dist}(x, l_0)$.

Generalizing the above concept to functions depending on the time $t \in (0,T)$, we define Babuska-Kondratiev type spaces on $\mathcal{O}_T$, denoted by $L_q((0,T), \mathcal{K}_{p,a}^m(\mathcal{O}))$, which contain all functions $u(x,t)$ such that

$$\begin{aligned}
&\|u|L_q((0,T), \mathcal{K}_{p,a}^m(\mathcal{O}))\| \\
&:= \left(\int_{(0,T)} \left(\sum_{|\alpha|\le m} \int_D |\varrho(x)|^{p(|\alpha|-a)} |D_x^\alpha u(x,t)|^p \mathrm{d}x\right)^{q/p} \mathrm{d}t\right)^{1/q} < \infty,
\end{aligned} \tag{2.5}$$

with $0 < q \le \infty$ and parameters a, p, m as above.
Restricting ourselves to the cones $K \subset \mathbb{R}^3$, we introduce another refined scale of weighted Sobolev spaces $V_{p,a,\delta}^m(K)$, consisting of all functions $u(x)$ such that

$$\begin{aligned}
&\|u|V_{p,a,\delta}^m(K)\| \\
&:= \left(\sum_{|\alpha|\le m} \int_K \left(\rho_0(x)^{(|\alpha|-a)} \prod_{k=1}^n \left(\frac{r_k(x)}{\rho_0(x)}\right)^{(\delta_k+|\alpha|)}\right)^p |D_x^\alpha u(x)|^p \mathrm{d}x\right)^{1/p} < \infty,
\end{aligned} \tag{2.6}$$

where $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$, $r_k(x)$ denotes the distance of the point x from the edge M_k and $\rho_0(x) = |x|$ denotes the distance of x from the origin and m, p, a are as above.
Moreover, we generalize the concept on K_T and denote by $L_q((0,T), V^m_{p,a,\delta}(K))$ the weighted Sobolev spaces, which contain all functions $u(x,t)$ such that

$$\begin{aligned} &\|u|L_q((0,T), V^m_{p,a,\delta}(K))\|^q \\ &:= \int_{(0,T)} \left(\sum_{|\alpha| \le m} \int_K \left(\rho_0(x)^{(|\alpha|-a)} \prod_{k=1}^n \left(\frac{r_k(x)}{\rho_0(x)} \right)^{(\delta_k + |\alpha|)} \right)^p |D^\alpha_x u(x,t)|^p \mathrm{d}x \right)^{q/p} \mathrm{d}t \end{aligned} \tag{2.7}$$

is finite.
Let us now discuss the connection between the scales (2.4) and (2.6). For this we need the following lemma. The proof can be found in Appendix A.

Lemma 2.5 *Let $K \subset \mathbb{R}^3$ be a polyhedral cone. There exist positive constants C_1, C_2 independent of $x \in K$ such that*

$$C_1 \rho_0(x) \prod_{k=1}^n \frac{r_k(x)}{\rho_0(x)} \le \varrho(x) \le C_2 \rho_0(x) \prod_{k=1}^n \frac{r_k(x)}{\rho_0(x)}. \tag{2.8}$$

The above lemma now implies the following coincidences between the two weighted scales of Sobolev spaces.

Theorem 2.6 *Let $K \subset \mathbb{R}^3$ be a polyhedral cone. If $\delta = (-a, \dots, -a)$ the weighted scales of Sobolev spaces coincide, i.e.,*

$$\mathcal{K}^m_{p,a}(K) = V^m_{p,a,\delta}(K).$$

Remark 2.7

- The results established above generalize to bounded polyhedra $D \subset \mathbb{R}^3$. In this case we need to alter the neighbourhoods of the edges considered in the proof of Lemma 2.5 in an appropriate way by taking $x_3 \in (a + \epsilon', b - \epsilon')$ for some $a < b$.
- It should also be possible to extend the above considerations to general polyhedra $D \subset \mathbb{R}^d$, but then we need to be careful with the singular set S (for $d = 4$ it consists of faces, edges and vertices) and the neighbourhoods need to be chosen accordingly.

Finally, concerning the special Lipschitz domains $\Omega \subset \mathbb{R}^d$, $d > 2$, from Definition 2.2 the corresponding Sobolev spaces $\mathcal{K}^m_{p,a}(\Omega)$ we consider are defined as in (2.4) and contain all functions $u(x)$ such that

$$\|u|\mathcal{K}^m_{p,a}(\Omega)\| := \left(\sum_{|\alpha| \le m} \int_\Omega |\rho(x)|^{p(|\alpha|-a)} |D^\alpha_x u(x)|^p \mathrm{d}x \right)^{1/p} < \infty, \tag{2.9}$$

where the weight function $\rho : \Omega \to [0,1]$ now is the smooth distance to l_0, i.e., ρ is a smooth function and in the vicinity of l_0 it is equal to $\mathrm{dist}(x, l_0)$. The spaces $\mathcal{K}^{m,q}_{p,a}(\Omega_T)$ are defined in an obvious way analogously to (2.6).

Some properties of weighted Sobolev spaces

- Clearly, by definition we have the following type of embeddings

$$K^m_{p,a}(\mathcal{O}) \hookrightarrow K^{m'}_{p,a}(\mathcal{O}), \qquad K^m_{p,a}(\mathcal{O}) \hookrightarrow K^m_{p,a'}(\mathcal{O}), \tag{2.10}$$

  if $m' < m$ and $a' < a$.

- A function $\varphi \in C^m(\mathcal{O})$ is a pointwise multiplier for $\mathcal{K}^m_{p,a}(\mathcal{O})$, i.e., $\varphi u \in \mathcal{K}^m_{p,a}(\mathcal{O})$ for all $u \in \mathcal{K}^m_{p,a}(\mathcal{O})$ and

$$\|\varphi u|\mathcal{K}^m_{p,a}(\mathcal{O})\| \le c_\varphi \|u|\mathcal{K}^m_{p,a}(\mathcal{O})\|. \tag{2.11}$$

Concerning pointwise multiplication the following was proven in [20]. Note that the domains of polyhedral type considered there include our polyhedral cones $K \subset \mathbb{R}^3$ as well as bounded polyhedral domains $D \subset \mathbb{R}^d$.

Theorem 2.8 *Let $\frac{d}{2} < p < \infty$, $m \in \mathbb{N}$, and $a \geq \frac{d}{p} - 1$. Then there exists a constant c such that*

$$\|uv|\mathcal{K}^{m-1}_{a-1,p}(K)\| \leq c\|u|\mathcal{K}^{m+1}_{a+1,p}(K)\| \cdot \|v|\mathcal{K}^{m-1}_{a-1,p}(K)\|$$

holds for all $u \in \mathcal{K}^{m+1}_{a+1,p}(K)$ and $v \in \mathcal{K}^{m-1}_{a-1,p}(K)$.

Furthermore, we shall need a lifting property for Kondratiev spaces. For classical Sobolev spaces by definition it is clear that

$$u \in W^m_p \quad \Longrightarrow \quad D^\alpha u \in W^{m-|\alpha|}_p, \tag{2.12}$$

for $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq m$. For a generalization to Besov and Triebel-Lizorkin spaces we also refer to [48, p. 22, Prop. 2] in this context. In the following theorem we will study the behaviour of $u \to D^\alpha u$ in Kondratiev spaces, which turns out to be very similar as for Sobolev spaces.

Theorem 2.9 *Let $a \in \mathbb{R}$, $1 < p < \infty$, and $m \in \mathbb{N}_0$. Then for $u \in \mathcal{K}^m_{p,a}(K)$ and $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq m$, we have*

$$D^\alpha u \in \mathcal{K}^{m-|\alpha|}_{p,a-|\alpha|}(K).$$

Proof : It is sufficient to show that for the first derivatives $\partial_{x_i} u$, where $i = 1, \ldots, d$, we have

$$u \in \mathcal{K}^m_{p,a}(K) \quad \Longrightarrow \quad \partial_{x_i} u \in \mathcal{K}^{m-1}_{p,a-1}(K), \tag{2.13}$$

since the rest then follows by induction. Throughout this proof, for an arbitrary domain $\Omega \subset \mathbb{R}^d$ we shall sometimes write $\mathcal{K}^m_{a,p}(\Omega, S)$ instead of $\mathcal{K}^m_{a,p}(\Omega)$ in order to emphasize the singularity set S of the domain Ω.

Step 1: We start with a standard situation, i.e., we consider a domain $\Omega = \mathbb{R}^d \setminus \mathbb{R}^l_*$ with singularity set $S = \mathbb{R}^l_*$, where

$$\mathbb{R}^l_* := \{x_1 = \ldots = x_{d-l} = 0,\ x_i \in \mathbb{R},\ i = d-l+1, \ldots, d\}, \qquad 0 \leq l < d,$$

putting $\mathbb{R}^0_* := \{0\}$. In [21] the existence of a smooth partition of unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ of $\Omega = \bigcup_{j=0}^\infty \Omega_j$ with

$$\operatorname{supp} \varphi_j \subset \Omega_j := \{x \in \mathbb{R}^d :\ 2^{-j-1} < \varrho(x) < 2^{-j+1}\}, \qquad j \in \mathbb{N}_0,$$

is shown, satisfying $\varphi_j(x) = \varphi_1(2^{j-1}x)$. Clearly, the functions have finite overlap. With this we estimate

$$\begin{aligned}
&\|\partial_{x_i} u|\mathcal{K}^{m-1}_{p,a-1}(\mathbb{R}^d \setminus \mathbb{R}^l_*, \mathbb{R}^l_*)\|^p \\
&\quad = \sum_{|\beta| \leq m-1} \int_{\mathbb{R}^d \setminus \mathbb{R}^l_*} |\varrho(x)^{|\beta|-a+1} D^\beta \partial_{x_i} u(x)|^p \mathrm{d}x \\
&\quad \sim \sum_{j=0}^\infty \sum_{|\beta| \leq m-1} 2^{-j(|\beta|-a+1)p} \int_{2^{-j-1} < \varrho(x) \leq 2^{-j}} |D^\beta \partial_{x_i}(\varphi_j u)(x)|^p \mathrm{d}x \\
&\quad = \sum_{j=0}^\infty \sum_{|\beta| \leq m-1} 2^{-j(|\beta|-a+1)p} \int_{\Omega_j} |D^\beta \partial_{x_i}(\varphi_j u)(x)|^p \mathrm{d}x.
\end{aligned} \tag{2.14}$$

For the terms with $j \geq 1$ we obtain

$$\begin{aligned}
\sum_{|\beta|\leq m-1} & 2^{-j(|\beta|-a+1)p} \int_{\Omega_j} |D^\beta \partial_{x_i}(\varphi_j u)(x)|^p \mathrm{d}x \\
&= 2^{jap} 2^{-(j-1)d} \|\partial_{x_i}(\varphi_j u)(2^{-j+1}\cdot)|W_p^{m-1}(\Omega_1)\|^p \\
&\leq 2^{jap} 2^{-(j-1)d} \|(\varphi_j u)(2^{-j+1}\cdot)|W_p^{m}(\Omega_1)\|^p \\
&= \sum_{|\alpha|\leq m} 2^{jap} 2^{-(j-1)d} \int_{\Omega_1} |D^\alpha(\varphi_j u)(2^{-j+1}y)|^p \mathrm{d}y \\
&\sim \sum_{|\alpha|\leq m} 2^{jap} 2^{-j|\alpha|p} \int_{\Omega_j} |D^\alpha(\varphi_j u)(x)|^p \mathrm{d}x \\
&\sim \sum_{|\alpha|\leq m} \int_{\Omega_j} |\varrho(x)^{|\alpha|-a} D^\alpha(\varphi_j u)(x)|^p \mathrm{d}x, \qquad (2.15)
\end{aligned}$$

where in the first step we used the coordinate transformation $y := 2^{j-1}x$ with $\mathrm{d}x = 2^{-(j-1)d}\mathrm{d}y$. On the other hand for $j = 0$ we compute

$$\begin{aligned}
\sum_{|\beta|\leq m-1} & \int_{\Omega_0} |\varrho(x)^{|\beta|-a+1} D^\beta \partial_{x_i}(\varphi_0 u)(x)|^p \mathrm{d}x \\
&\sim \sum_{|\beta|\leq m-1} \int_{\Omega_0} |D^\beta \partial_{x_i}(\varphi_0 u)(x)|^p \mathrm{d}x \\
&= \|\partial_{x_i}(\varphi_0 u)|W_p^{m-1}(\Omega_0)\|^p \leq \|\varphi_0 u|W_p^m(\Omega_0)\|^p \\
&= \sum_{|\alpha|\leq m} \int_{\Omega_0} |D^\alpha(\varphi_0 u)(x)|^p \mathrm{d}x \\
&\sim \sum_{|\alpha|\leq m} \int_{\Omega_0} |\varrho(x)^{|\alpha|-a} D^\alpha(\varphi_0 u)(x)|^p \mathrm{d}x. \qquad (2.16)
\end{aligned}$$

Now (2.14) together with (2.15) and (2.16) yields

$$\begin{aligned}
\|\partial_{x_i} u|\mathcal{K}^{m-1}_{a-1,p}(\mathbb{R}^d \setminus \mathbb{R}^l_*, \mathbb{R}^l_*)\|^p &\lesssim \sum_{|\alpha|\leq m} \sum_{j=0}^{\infty} \int_{\Omega_j} |\varrho(x)^{|\alpha|-a} D^\alpha(\varphi_j u)(x)|^p \mathrm{d}x \\
&\sim \|u|\mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus \mathbb{R}_*, \mathbb{R}^l_*)\|^p
\end{aligned}$$

which establishes (2.13) for domains $\Omega = \mathbb{R}^d \setminus \mathbb{R}_*$.
Step 2: In this step we show that the situation for polyhedral cones K reduces to the standard situation treated in Step 1. A closer look at the proof of Lemma 2.5 reveals that the polyhedral cone K with singularity set S can be decomposed into a finite union

$$K = K^{\mathrm{reg}} \cup U_\delta^{\mathrm{vertex}} \cup \left(\bigcup_{k=1}^n U_\eta^{\mathrm{vertex-edge}}(M_k)\right) \cup \left(\bigcup_{k=1}^n U_{\varepsilon,\epsilon'}^{\mathrm{edge}}(M_k)\right) =: \bigcup_{i\in\Lambda} U_i,$$

of sets U_i (with finite overlap) denoting the regular part of the cone K^{reg} as well as the different neighbourhoods of edges $U_{\varepsilon,\epsilon'}^{\mathrm{edge}}(M_k)$, the vertex $U_\delta^{\mathrm{vertex}}$, and vertex-edges $U_\eta^{\mathrm{vertex-edge}}(M_k)$ (see the notation used in Lemma 2.5) with corresponding singularity sets S_i. In particular, for $x \in U_i$ we have

$\varrho(x) \sim \varrho_i(x) := \min\{1, \operatorname{dist}(x, S_i)\}$. With this we compute

$$\begin{aligned}
&\|u|\mathcal{K}^m_{a,p}(K,S)\| \\
&\quad = \left(\sum_{|\alpha|\le m} \int_K |\varrho(x)^{|\alpha|-a} D^\alpha u(x)|^p \mathrm{d}x \right)^{1/p} \\
&\quad = \left(\sum_{|\alpha|\le m} \int_{\bigcup_{i\in\Lambda} U_i} |\varrho(x)^{|\alpha|-a} D^\alpha u(x)|^p \mathrm{d}x \right)^{1/p} \\
&\quad \sim \left(\sum_{|\alpha|\le m} \sum_{i\in\Lambda} \int_{U_i} |\varrho_i(x)^{|\alpha|-a} D^\alpha u(x)|^p \mathrm{d}x \right)^{1/p} \sim \sum_{i\in\Lambda} \|u|\mathcal{K}^m_{a,p}(U_i,S_i)\|, \qquad (2.17)
\end{aligned}$$

thus, it remains to prove that (2.13) is true for the sets U_i. For the regular part K^{reg} since $\varrho(x) \sim 1$ we compute

$$\begin{aligned}
&\|\partial_{x_i} u|\mathcal{K}^{m-1}_{a-1,p}(K^{\mathrm{reg}})\|^p \\
&\quad = \sum_{|\beta|\le m-1} \int_{K^{\mathrm{reg}}} |\varrho(x)^{|\beta|-a+1} D^\beta \partial_{x_i} u(x)|^p \mathrm{d}x \\
&\quad \sim \sum_{|\beta|\le m-1} \int_{K^{\mathrm{reg}}} |D^\beta \partial_{x_i} u(x)|^p \mathrm{d}x \\
&\quad = \|\partial_{x_i} u|W^{m-1}_p(K^{\mathrm{reg}})\|^p \le \|u|W^m_p(K^{\mathrm{reg}})\|^p \\
&\quad = \sum_{|\alpha|\le m} \int_{K^{\mathrm{reg}}} |D^\alpha u(x)|^p \mathrm{d}x \\
&\quad \sim \sum_{|\alpha|\le m} \int_{K^{\mathrm{reg}}} |\varrho(x)^{|\alpha|-a} D^\alpha u(x)|^p \mathrm{d}x = \|u|\mathcal{K}^m_{a,p}(K^{\mathrm{reg}})\|^p, \qquad (2.18)
\end{aligned}$$

which shows that (2.13) holds in this case. We use that in [21] it was noted that Stein's extension operator extends also to Kondratiev spaces. In particular, in our situation this means

$$\begin{aligned}
\|u|\mathcal{K}^m_{a,p}(U^{\mathrm{vertex}}_\delta, \{0\})\| &\sim \|\mathrm{Ext}\ u|\mathcal{K}^m_{a,p}(\mathbb{R}^d, \{0\})\|, \\
\|u|\mathcal{K}^m_{a,p}(U^{\mathrm{vertex-edge}}_\eta, M_k)\| &\sim \|\mathrm{Ext}\ u|\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}_*)\|, \qquad (2.19) \\
\|u|\mathcal{K}^m_{a,p}(U^{\mathrm{edge}}_{\varepsilon,\epsilon'}, M_k)\| &\sim \|\mathrm{Ext}\ u|\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}_*)\|,
\end{aligned}$$

i.e., for the other sets U_i the situation reduces to the standard case already treated in Step 1. Now (2.17), (2.18), and (2.19) complete the proof. □

2.4 Besov spaces

2.4.1 Definition of Besov spaces

Besov spaces can be defined in various different ways. In this subsection, we start with a definition via higher order differences as can be found in [52]. Moreover, we recall the fact that under certain restrictions on the parameters the Besov spaces allow a characterization in terms of wavelet decompositions. In this context we refer, e.g., to [10, 47]. In particular, this wavelet characterization will turn out to be extremely useful when proving embeddings of weighted Sobolev spaces into Besov spaces from the non-linear approximation scale (1.4). For further information on Besov spaces and related function spaces as well as equivalent definitions, we refer to [54] and the references given there.
If f is an arbitrary function on $\mathbb{R}^d$, $h \in \mathbb{R}^d$ and $r \in \mathbb{N}$, then

$$(\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta^{r+1}_h f)(x) = \Delta^1_h(\Delta^r_h f)(x)$$

are the usual iterated differences. Given a function $f \in L_p(\mathbb{R}^d)$ the *r-th modulus of smoothness* is defined by

$$\omega_r(f,t)_p = \sup_{|h|\le t} \|\Delta_h^r f \mid L_p(\mathbb{R}^d)\|, \quad t>0, \quad 0<p\le\infty.$$

Definition 2.10 *Let $0<p,q\le\infty$, $s>0$, and $r\in\mathbb{N}$ such that $r>s$. Then the Besov space $B^s_{p,q}(\mathbb{R}^d)$ contains all $f\in L_p(\mathbb{R}^d)$ such that*

$$\|f|B^s_{p,q}(\mathbb{R}^d)\| = \|f|L_p(\mathbb{R}^d)\| + \left(\int_0^1 t^{-sq}\omega_r(f,t)_p^q \,\frac{\mathrm{d}t}{t}\right)^{1/q} < \infty$$

(*with the usual modification if $q=\infty$*).

Remark 2.11 Definition 2.10 is independent of r, meaning that different values of $r>s$ result in norms which are equivalent. Furthermore the spaces are quasi-Banach spaces (Banach spaces if $p,q\ge 1$). Note that we deal with subspaces of $L_p(\mathbb{R}^n)$, in particular, for $s>0$ and $0<q\le\infty$, we have the embedding

$$B^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d), \qquad 0<p\le\infty.$$

2.4.2 Wavelet characterization of Besov spaces

Wavelets are specific orthonormal bases for $L_2(\mathbb{R})$ that are obtained by dilating, translating and scaling one fixed function, the so–called *mother wavelet* ψ. The mother wavelet is usually constructed by means of a so-called *multiresolution analysis,* that is, a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of shift-invariant, closed subspaces of $L_2(\mathbb{R})$ whose union is dense in L_2 while their intersection is zero. Moreover, all the spaces are related via dyadic dilation, and the space V_0 is spanned by the translates of one fixed function ϕ, called the *generator*. In her fundamental work [24,25] I. Daubechies has shown that there exist families of compactly supported wavelets. By taking tensor products, a compactly supported orthonormal basis for $L_2(\mathbb{R}^d)$ can be constructed which will also be used in this paper.
Let ϕ be a father wavelet of tensor product type on $\mathbb{R}^d$ and let $\Psi' = \{\psi_i : \ i=1,\ldots,2^d-1\}$ be the set containing the corresponding multivariate mother wavelets such that, for a given $r\in\mathbb{N}$ and some $N>0$ the following locality, smoothness and vanishing moment conditions hold. For all $\psi\in\Psi'$,

$$\text{supp}\ \phi,\ \text{supp}\ \psi \ \subset\ [-N,N]^d, \tag{2.20}$$

$$\phi,\ \psi \ \in\ C^r(\mathbb{R}^d), \tag{2.21}$$

$$\int_{\mathbb{R}^d} x^\alpha \psi(x)\mathrm{d}x \ =\ 0 \quad \text{for all } \alpha\in\mathbb{N}_0^d \text{ with } |\alpha|\le r. \tag{2.22}$$

We refer again to the papers [24, 25] for a detailed discussion. The set of all dyadic cubes in $\mathbb{R}^d$ with measure at most 1 is denoted by

$$\mathcal{D}^+ := \left\{I\subset\mathbb{R}^d : \ I = 2^{-j}([0,1]^d+k),\ j\in\mathbb{N}_0,\ k\in\mathbb{Z}^d\right\}$$

and we set $\mathcal{D}_j := \{I\in\mathcal{D}^+ : \ |I| = 2^{-jd}\}$. For the dyadic shifts and dilations of the father wavelet and the corresponding wavelets we use the abbreviations

$$\phi_k(x) := \phi(x-k), \quad \psi_I(x) := 2^{jd/2}\psi(2^jx-k) \qquad \text{for} \quad j\in\mathbb{N}_0,\ k\in\mathbb{Z}^d,\ \psi\in\Psi'. \tag{2.23}$$

It follows that

$$\left\{\phi_k,\ \psi_I : \ k\in\mathbb{Z}^d,\ I\in\mathcal{D}^+,\ \psi\in\Psi'\right\}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$. Denote by $Q(I)$ some dyadic cube (of minimal size) such that supp $\psi_I\subset Q(I)$ for every $\psi\in\Psi'$. Then, we clearly have $Q(I) = 2^{-j}k + 2^{-j}Q$ for some dyadic cube Q. Put $\Lambda' = \mathcal{D}^+\times\Psi'$. Then, every function $f\in L_2(\mathbb{R}^d)$ can be written as

$$f = \sum_{k\in\mathbb{Z}^d}\langle f,\phi_k\rangle\phi_k + \sum_{(I,\psi)\in\Lambda'}\langle f,\psi_I\rangle\psi_I.$$

Later on, it will be convenient to include ϕ into the set Ψ'. We use the notation $\phi_I := 0$ for $|I| < 1$, $\phi_I = \phi(\cdot - k)$ for $I = k + [0,1]^d$, and can simply write

$$f = \sum_{(I,\psi)\in\Lambda} \langle f, \psi_I\rangle \psi_I, \qquad \Lambda = \mathcal{D}^+ \times \Psi, \quad \Psi = \Psi' \times \{\phi\}.$$

We describe Besov spaces on $\mathbb{R}^d$ by decay properties of the wavelet coefficients, if the parameters fulfill certain conditions.

Theorem 2.12 (Wavelet decomposition of Besov spaces) *Let $0 < p, q < \infty$ and $s > \max\{0, d(1/p - 1)\}$. Choose $r \in \mathbb{N}$ such that $r > s$ and construct a wavelet Riesz basis as described above. Then a function $f \in L_p(\mathbb{R}^d)$ belongs to the Besov space $B^s_{p,q}(\mathbb{R}^d)$ if, and only if,*

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \phi_k\rangle \phi_k + \sum_{(I,\psi)\in\Lambda'} \langle f, \psi_I\rangle \psi_I \tag{2.24}$$

(convergence in $\mathcal{S}'(\mathbb{R}^d)$) with

$$\|f|B^s_{p,q}(\mathbb{R}^d)\| \sim \left(\sum_{k\in\mathbb{Z}^d} |\langle f, \phi_k\rangle|^p\right)^{1/p} + \left(\sum_{j=0}^{\infty} 2^{j\left(s+d\left(\frac{1}{2}-\frac{1}{p}\right)\right)q} \left(\sum_{(I,\psi)\in\mathcal{D}_j\times\Psi'} |\langle f, \psi_I\rangle|^p\right)^{q/p}\right)^{1/q} < \infty \tag{2.25}$$

(with the usual modification if $q = \infty$).

Remark 2.13 (i) For parameters $q = \infty$ we use the usual modification (replacing the outer sum by a supremum), i.e.,

$$\|f|B^s_{p,\infty}(\mathbb{R}^d)\| \sim \left(\sum_{k\in\mathbb{Z}^d} |\langle f, \phi_k\rangle|^p\right)^{1/p} + \sup_{j\geq 0} 2^{j\left(s+d\left(\frac{1}{2}-\frac{1}{p}\right)\right)} \left(\sum_{(I,\psi)\in\mathcal{D}_j\times\Psi'} |\langle f, \psi_I\rangle|^p\right)^{1/p} < \infty.$$

(ii) In particular, for our adaptivity scale (1.4), i.e., $B^s_{\tau,\tau}(\mathbb{R}^d)$ with $s = d\left(\frac{1}{\tau} - \frac{1}{p}\right)$, we see that the norm (2.25) becomes

$$\|f|B^s_{\tau,\tau}(\mathbb{R}^d)\| \sim \left(\sum_{k\in\mathbb{Z}^d} |\langle f, \phi_k\rangle|^\tau\right)^{1/\tau} + \left(\sum_{j=0}^{\infty} 2^{jd\left(\frac{1}{2}-\frac{1}{p}\right)\tau} \sum_{(I,\psi)\in\mathcal{D}_j\times\Psi'} |\langle f, \psi_I\rangle|^\tau\right)^{1/\tau}. \tag{2.26}$$

Corresponding function spaces on domains can be introduced via restriction, i.e.,

$$\begin{aligned} B^s_{p,q}(\mathcal{O}) &= \left\{f \in \mathcal{D}'(\mathcal{O}) : \ \exists g \in B^s_{p,q}(\mathbb{R}^d), \ g\big|_{\mathcal{O}} = f\right\}, \\ \|f|B^s_{p,q}(\mathcal{O})\| &= \inf_{g|_{\mathcal{O}}=f} \|f|B^s_{p,q}(\mathbb{R}^d)\|. \end{aligned}$$

Alternative (different or equivalent) versions of this definition can be found, depending on possible additional properties of the distributions g (most often their support). We refer to [53] for details and references.

3 Regularity results in weighted Sobolev spaces

3.1 Parabolic regularity results

3.1.1 The fundamental problem

Let $m \in \mathbb{N}$. We consider the following first initial-boundary value problem

$$\left\{ \begin{array}{rcll} \frac{\partial}{\partial t}u + (-1)^m L(x,t;D_x)u &=& f & \text{in } K_T, \\ \frac{\partial^{k-1}u}{\partial \nu^{k-1}}\Big|_{\Gamma_{j,T}} &=& 0, & k=1,\ldots,m,\ j=1,\ldots,n, \\ u\Big|_{t=0} &=& 0 & \text{in } K. \end{array} \right\} \tag{3.1}$$

Here f is a function given on K_T, ν denotes the exterior normal to $\Gamma_{j,T}$, and the partial differential operator L is given by

$$L(x,t;D_x) = \sum_{|\alpha|,|\beta|=0}^{m} D_x^{\alpha}(a_{\alpha\beta}(x,t)D_x^{\beta}),$$

where $a_{\alpha\beta}$ are bounded real-valued functions from $C^{\infty}(K_T)$ with $a_{\alpha\beta} = (-1)^{|\alpha|+|\beta|}a_{\beta\alpha}$. Furthermore, the operator L is assumed to be strong elliptic uniformly with respect to $t \in (0,T)$, i.e.,

$$\sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}\xi^{\alpha}\xi^{\beta} \geq c|\xi|^{2m} \qquad \text{for all} \quad (x,t) \in K_T, \quad \xi \in \mathbb{R}^d.$$

Moreover, a function $u \in \mathring{W}_p^{m,1}(K_T)$ is called a *generalized solution* of problem (3.1), if and only if, $u(x,0) = 0$ for all $x \in K$, and the equality

$$(\partial_t u, v) + (-1)^m \left(L(x,t;D_x)u, v\right) = (f,v) \quad \text{a.e. } t \in [0,T],$$

holds for all $v \in \mathring{W}_2^m(K)$.
Concerning the Sobolev regularity of the generalized solution of problem (3.1) the following result may be found in [43, Thm. 2.1., L. 3.1].

Proposition 3.1 *Let $l \in \mathbb{N}_0$ and assume that*

(i) $\sup\{|(a_{\alpha\beta})_{t^k}| : \ i,j=1,\ldots,n;\ (x,t)\in K_T,\ k \leq l+1\} \leq \mu,\ \ \mu = const.,$

(ii) $f \in W_2^l((0,T),L_2(K)),\ \ \partial_{t^k} f(x,0) = 0,\ \ 0 \leq k \leq l-1$

(where the second condition in (ii) is not applicable if $l = 0$). Then problem (3.1) *has a unique generalized solution $u \in \mathring{W}_2^{m,l+1}(K_T)$ with a priori estimate*

$$\|u\|_{W_2^{m,l+1}(K_T)} \lesssim \|f\|_{W_2^l((0,T),L_2(K))}.$$

3.1.2 Operator pencils

In order to state the global regularity results in weighted Sobolev spaces of our parabolic problem we need to define operator pencils generated by the Dirichlet problem for elliptic equations in the cone K. Let us recall the basic facts, further informations on this subject may be found in [45, Sect. 2.3, Sect. 3.2.]. Let M_k be an edge of the cone $\mathcal{K}$, and let $\Gamma_{k_\pm}$ be the faces adjacent to M_k. Then by $\mathcal{D}_k$ we denote the dihedron which is bounded by the half-planes $\mathring{\Gamma}_{k_\pm}$ tangent to $\Gamma_{k_\pm}$ at M_k. Let r, φ be polar coordinates in the plane perpendicular to M_k such that

$$\mathring{\Gamma}_{k_\pm} = \left\{ x \in \mathbb{R}^3 :\ r > 0,\ \varphi = \pm\frac{\theta_k}{2} \right\}.$$

Fixing $t \in [0,T]$, we define the operator $A_k(\lambda,t)$ as follows:

$$A_k(\lambda,t)U = r^{2m-\lambda}L^0(0,t,D)(r^{\lambda}U),$$

where $u(x) = r^\lambda U(\varphi)$, $\lambda \in \mathbb{C}$, U is a function on $I_k := \left(\frac{-\theta_k}{2}, \frac{\theta_k}{2}\right)$, and

$$L^0(0,t,D) = \sum_{|\alpha|=|\beta|=m} D_x^\alpha (a_{\alpha\beta}(0,t) D_x^\beta).$$

The operator $A_k(\lambda, t)$ realizes a continuous mapping

$$W_2^{2m}(I_k) \cap \mathring{W}_2^m(I_k) \to L_2(I_k),$$

for every $\lambda \in \mathbb{C}$. A complex number λ_0 is called an eigenvalue of the pencil $A_k(\lambda, t)$ if there exists a nonzero function $U \in W_2^{2m}(I_k) \cap \mathring{W}_2^m(I_k)$ such that $A_k(\lambda_0, t)U = 0$. We denote by $\delta_\pm^{(k)}(t)$ the greatest positive real numbers such that the strip

$$m - 1 - \delta_-^{(k)}(t) < \mathrm{Re}\lambda < m - 1 + \delta_+^{(k)}(t)$$

is free of eigenvalues of the pencil $A_k(\lambda, t)$. Furthermore, we put

$$\delta_\pm^{(k)} = \inf_{t\in[0,T]} \delta_\pm^{(k)}(t), \qquad k = 1, \ldots, n.$$

Furthermore, we introduce spherical coordinates $\rho = |x|$, $\omega = \frac{x}{|x|}$ in K and define

$$\mathfrak{U}(\lambda, t)U = \rho^{2m-\lambda} L^0(0,t,D)(\rho^\lambda U), \tag{3.2}$$

where $u(x) = \rho^\lambda U(\omega)$ and U is a function on $\Omega = \{\omega : \ \omega = \frac{x}{|x|}, \ x \in K\}$. The operator $\mathfrak{U}(\lambda, t)$ realizes a continuous mapping

$$W_2^{2m}(\Omega) \cap \mathring{W}_2^m(\Omega) \to L_2(\Omega).$$

An eigenvalue of $\mathfrak{U}(\lambda, t)$ is a complex number λ_0 such that $\mathfrak{U}(\lambda_0, t)U = 0$ for some nonzero function $U \in W_2^{2m}(\Omega) \cap \mathring{W}_2^m(\Omega)$.

For the considerations concerning regularity that will be presented in the next subsection, we will need the following technical assumptions.

Assumption 3.2 *Consider the operator pencils* $\mathfrak{U}(\lambda, t)$, $t \in [0,T]$, *as defined in* (3.2). *For the parameters appearing in the weighted Sobolev spaces* $V_{p,a,\delta}^m$ *defined in* (2.6) *we assume that the closed strip between the lines* $\mathrm{Re}\lambda = m - \frac{3}{2}$ *and* $\mathrm{Re}\lambda = a - \frac{3}{2}$ *does not contain eigenvalues of the operator pencils and*

$$-\delta_+^{(k)} < \delta_k - m < \delta_-^{(k)}, \quad k = 1, \ldots, n. \tag{3.3}$$

Remark 3.3 *We recall the example given in [43, p.403]. If we consider the heat equation with* $L = \Delta$ *and* $m = 1$, *we have* $\delta_\pm^{(k)} = \frac{\pi}{\theta_k}$. *Therefore,* (3.3) *in this case reads as*

$$1 - \frac{\pi}{\theta_k} < \delta_k < 1 + \frac{\pi}{\theta_k}.$$

Using Theorem 2.6 *we see that condition* (3.3) *for the spaces* $\mathcal{K}_{p,a}^m$ *becomes*

$$-\delta_-^{(k)} < a + m < \delta_+^{(k)}, \quad k = 1, \ldots, n. \tag{3.4}$$

3.1.3 Regularity results in weighted Sobolev spaces

Concerning weighted Sobolev regularity for the parabolic problem (3.1), first fundamental results can be found in [43, Th. 3.3], which also form the basis for our investigations. However, for our purposes, it is necessary to strenghten these results in the sense that also estimates for the derivatives $\partial_{l+1}u$ and $\partial_{l+2}u$ in weighted Sobolev spaces are needed. Indeed, these additional results will be crucial when establishing the boundedness of the operator $\tilde{L}^{-1}$ in Theorem 3.8, which in turn is needed in Theorem 4.8 for proving the existence of a solution of the nonlinear problem (4.7) in weighted Sobolev spaces.

Theorem 3.4 (General weighted Sobolev regularity) *Let γ, l be nonnegative integers, $\gamma \geq 2m$, $a \in \mathbb{R}$, $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$, and $a, \delta_k \in [-m, m]$ for $k = 1, \ldots, n$. Assume that the Assumptions* 3.2 *hold and the right hand side f of* (3.1) *satisfies*

(i) $\partial_{t^k} f \in L_2(K_T) \cap L_2((0,T), V^{\gamma-2m}_{2,a,\delta}(K)), \quad k = 0, \ldots, l+1.$

(ii) $\partial_{t^k} f(x, 0) = 0, \quad k = 0, 1, \ldots, l.$

Then for the generalized solution $u \in \mathring{W}^{m,1}_2(K_T)$ of problem (3.1) *we have $\partial_{t^k} u \in L_2((0,T), V^{\gamma}_{2,a',\delta'}(K))$, where $a' = a + 2m$, $k = 0, \ldots, l+1$, and the vector δ' has components $\delta'_k = \delta_k - 2m$, $k = 1, \ldots, n$. Moreover, $\partial_{t^{l+2}} u \in L_2((0,T), V^{\gamma-2m}_{2,a,\delta}(K))$. In particular, for the derivatives $\partial_{t^k} u$ up to order $k = l$ we have the a priori estimate*

$$\sum_{k=0}^{l} \|\partial_{t^k} u | L_2((0,T), V^{\gamma}_{2,a',\delta'}(K))\| \leq C \sum_{k=0}^{l} \|\partial_{t^k} f | L_2((0,T), V^{\gamma-2m}_{2,a,\delta}(K))\| + \sum_{k=0}^{l+1} \|\partial_{t^k} f | L_2(K_T)\|, \tag{3.5}$$

where C is a constant independent of u and f.

Proof : In [43, Th. 3.3] it was already shown via induction that from the above assumptions on the initial data f it follows that $\partial_{t^k} u \in L_2((0,T), V^{\gamma}_{2,a',\delta'}(K))$ for the derivatives up to order $k = l$ that and the a priori estimate (3.5) holds.
It remains to show, why we also have $\partial_{t^{l+1}} u \in L_2((0,T), V^{\gamma}_{2,a',\delta'}(K))$. Proceeding via induction we start with $\gamma = 2m$. Differentiating the first equation in (3.1) $l+1$-times leads to

$$\partial_{t^{l+2}} u + (-1)^m \sum_{k=0}^{l+1} \binom{l+1}{k} \partial_{t^{l+1-k}} L\left(\partial_{t^k} u\right) = \partial_{t^{l+1}} f, \tag{3.6}$$

thus,

$$\partial_{t^{l+2}} u = \partial_{t^{l+1}} f - (-1)^m \sum_{k=0}^{l+1} \binom{l+1}{k} \partial_{t^{l+1-k}} L\left(\partial_{t^k} u\right). \tag{3.7}$$

From this we see that $\partial_{t^{l+2}} u \in L_2((0,T), V^{0}_{2,\delta,a}(K))$, since for the right hand side of (3.7) we know $\partial_{t^{l+1}} f \in L_2((0,T), V^{0}_{2,\delta,a}(K))$ by the assumptions and $\partial_{t^k} u \in \mathring{W}^{m,1}_2(K_T) \subset L_2((0,T), W^m_2(K)) \subset L_2((0,T), V^{0}_{2,\delta,a}(K))$ for $k = 0, \ldots, l+1$, cf. [43, Lem. 3.1]. Equation (3.6) leads to

$$L\left(\partial_{t^{l+1}} u\right) = (-1)^m \left(\partial_{t^{l+1}} f - \partial_{t^{l+2}} u\right) - \sum_{k=0}^{l} \binom{l+1}{k} \partial_{t^{l+1-k}} L\left(\partial_{t^k} u\right) =: F. \tag{3.8}$$

The right hand side is in $L_2((0,T), V^{0}_{2,\delta,a}(K))$. Fix $t \in [0,T]$, then by [43, Lem. 3.2] we obtain that the corresponding elliptic problem has a solution $\partial_{t^{l+1}} u(t) \in V^{2m}_{2,\delta',a'}(K)$ with

$$\|\partial_{t^{l+1}} u(t) | V^{2m}_{2,\delta',a'}(K)\|^2 \leq c \|F(t) | V^{0}_{2,\delta,a}(K)\|^2.$$

Now integrating with respect to t shows $\partial_{t^{l+1}} u \in L_2((0,T), V^{2m}_{2,\delta',a'}(K))$.
Let us assume now, that for some $\gamma - 1$ under the given assumptions on f it holds

$$\partial_{t^k} u \in L_2((0,T), V^{\gamma-1}_{2,a',\delta'}(K)), \qquad k = 0, \ldots, l+1.$$

Differentiating as in (3.6) now we see from (3.7) that $\partial_{t^{l+2}} u \in L_2((0,T), V^{\gamma-2m}_{2,\delta,a}(K))$, since for the right hand side we know $\partial_{t^{l+1}} f \in L_2((0,T), V^{\gamma-2m}_{2,\delta,a}(K))$ and by induction also $\partial_{t^k} u \in L_2((0,T), V^{\gamma-1}_{2,\delta',a'}(K)) \subset L_2((0,T), V^{\gamma-2m}_{2,\delta,a}(K))$ for $k = 0, \ldots, l+1$. Now in (3.8) the right hand side is in $L_2((0,T), V^{\gamma-2m}_{2,\delta,a}(K))$.

Fix $t \in [0,T]$, then by [43, Lem. 3.2] we obtain that the corresponding elliptic problem has a solution $\partial_{t^{l+1}} u(t) \in V^{\gamma}_{2,\delta',a'}(K)$ with

$$\|\partial_{t^{l+1}} u(t)|V^{\gamma}_{2,\delta',a'}(K)\|^2 \leq c\|F(t)|V^{\gamma-2m}_{2,\delta,a}(K)\|^2.$$

Integrating with respect to t gives $\partial_{t^{l+1}} u \in L_2((0,T), V^{\gamma}_{2,\delta',a'}(K))$. □

Remark 3.5

- For $l = 0$ the estimate (3.5) in Theorem 3.4 reads as

$$\|u\|_{L_2((0,T),V^{\gamma}_{2,a',\delta'}(K))} \leq C\|f\|_{L_2((0,T),V^{\gamma-2m}_{2,a,\delta}(K))} + \sum_{k=0}^{1} \|\partial_{t^k} f\|_{L_2(K_T)}.$$

- The existence of the solution $u \in \mathring{W}^{m,1}_2(K_T)$ follows from Proposition 3.1 for $l = 0$.
- In [43] the above Theorem is stated using a different scale of weighted Sobolev spaces denoted by $W^{l,2}_{\beta,\delta}(K_T)$ containing all functions $u(x,t)$ such that

$$\|u\|_{W^{l,2}_{\beta,\delta}(K_T)} := \left(\int_{K_T} \sum_{|\alpha|\leq l} \varrho^{2(\beta-l+|\alpha|)} \prod_{k=1}^{n} \left(\frac{r_k}{\varrho}\right)^{2(\delta_k-l+|\alpha|)} |D^{\alpha}_x u(x,t)|^2 \mathrm{d}x\mathrm{d}t \right)^{1/2} < \infty, \tag{3.9}$$

  with $\varrho(x)$, $r_j(x)$, δ, l as before, and $\beta \in \mathbb{R}$. Comparing this scale with our scale $L_2((0,T), V^{l}_{2,\tilde{\delta},a}(K))$, $\tilde{\delta} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_n) \in \mathbb{R}^n$, as defined in (2.7), we see that these spaces coincide if the parameters are linked via

$$\beta - l = -a, \qquad \delta_k - l = \tilde{\delta}_k. \tag{3.10}$$

  The condition $-\delta^{(k)}_{+} < \tilde{\delta}_k - l + m < \delta^{(k)}_{-}$ from [43] for the data in $W^{l-2m,2}_{\beta,\delta}(K_T)$ therefore had to be rewritten (now using (3.10) with l replaced by $l - 2m$) as

$$\tilde{\delta}_k - l + m = \delta_k + (l-2m) - l + m = \delta_k - m$$

  resulting in the assumption (3.3) above.

In view of Theorem 2.6 we can rewrite Theorem 3.4 as follows for the Kondratiev spaces.

Theorem 3.6 (Weighted Sobolev regularity) *Let γ, l be nonnegative integers, $\gamma \geq 2m$, $a \in \mathbb{R}$, and $a \in [-m,m]$. Assume that the Assumptions 3.2 hold and the right hand side f of (3.1) satisfies*

(i) $\partial_{t^k} f \in L_2(K_T) \cap L_2((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))$, $\quad k = 0, \ldots, l+1$.

(ii) $\partial_{t^k} f(x,0) = 0$, $\quad k = 0, 1, \ldots, l$.

Then for the generalized solution $u \in \mathring{W}^{m,1}_2(K_T)$ of problem (3.1) we have $\partial_{t^k} u \in L_2((0,T), \mathcal{K}^{\gamma}_{2,a'}(K))$, where $a' = a + 2m$ for $k = 0, \ldots, l+1$. Moreover, $\partial_{t^{l+2}} u \in L_2((0,T), V^{\gamma-2m}_{2,a,\delta}(K))$. In particular, for the derivatives $\partial_{t^k} u$ up to order $k = l$ we have the a priori estimate

$$\sum_{k=0}^{l} \|\partial_{t^k} u|L_2((0,T), \mathcal{K}^{\gamma}_{2,a'}(K))\| \leq C \sum_{k=0}^{l} \|\partial_{t^k} f|L_2((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))\| + \sum_{k=0}^{l+1} \|\partial_{t^k} f|L_2(K_T)\|, \tag{3.11}$$

where C is a constant independent of u and f.

Remark 3.7 For the heat equation ($m = 1$) we have $\delta_{\pm}^{(k)} = \frac{\pi}{\theta_k}$, cf. [43, Sect. 4]. In view of Theorem 2.6 we have to choose $\delta_k = -a$, $k = 1, \ldots, n$, in Theorem 3.6. Therefore, (3.3) together with the restriction $a \in [-1, 1]$ leads to

$$\max\left(-1 - \frac{\pi}{\theta_k}, -1\right) < a < \min\left(1, -1 + \frac{\pi}{\theta_k}\right)$$

which yields

$$-1 < a < \min\left(1, -1 + \frac{\pi}{\theta_k}\right) = \begin{cases} 1 & \text{if } \theta_k < \frac{\pi}{2}, \\ > 0 & \text{if } \theta_k < \pi. \end{cases}$$

From this we see that it is possible to choose $a > 0$ as long as the polyhedral cone K_0 is convex. We make some further comments in this context. Later on, in Section 4 we want to show Besov regularity for problem (4.7). This will be performed by using embedding results of Kondratiev spaces into Besov spaces. Since all functions in the adaptivity scale (1.4) of Besov spaces are locally integrable, the same must hold for the Kondratiev spaces which requires $a > 0$. For the linear problem (3.1) this is no restriction since from Theorem 3.6 it follows that our solution satisfies $u \in L_2((0,T), \mathcal{K}^{\gamma}_{2,a'}(K))$, where $a' = a+2m \geq -m+2m = m > 0$. Thus, we always have a locally integrable solution in this case. However, in order to study the nonlinear equation (4.7) later on, we need the boundedness of the operator $\tilde{L}^{-1}$, which is established in Theorem 3.8 below. For the proof of this result it is important that the spatial data space satisfies $\mathcal{K}^{\gamma-2m}_{2,a}(K) \hookrightarrow L_2(K)$, which is the case if $a \geq 0$. In conclusion, it turns out that the nonlinear problem (4.7), in particular, the heat equation disturbed with some nonlinear term, has a solution in scales of Kondratiev spaces if we assume that the angles of our polyhedral cone are sufficiently small, i.e., $\theta_k < \pi$ for all $k = 1, \ldots, n$.

We define the operator $\tilde{L}$ by putting

$$\tilde{L}u := \frac{\partial}{\partial t}u + (-1)^m Lu = \frac{\partial}{\partial t}u + (-1)^m \sum_{|\alpha|,|\beta|=0}^{m} (D^{\alpha}_x a_{\alpha\beta}(x,t) D^{\beta}_x)u. \tag{3.12}$$

Then Theorem 3.6 in fact tells us something abound the mapping properties of the operator $\tilde{L}^{-1}$. Concerning boundedness of this operator we strenghten what was obtained in Theorem 3.6 in the next theorem. The results we obtain will be crucial when dealing with Besov regularity of nonlinear problems later on.

Theorem 3.8 (Boundedness of $\tilde{L}^{-1}$) *Let γ, l be nonnegative integers, $\gamma \geq 2m$, $a \in \mathbb{R}$ with $a \in [0, m]$ and let the Assumptions 3.2 hold. Define the solution space S and data space D by*

$$S := W_2^{l+1}((0,T), \mathcal{K}^{\gamma}_{2,a'}(K)) \cap W_2^{l+2}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K)), \qquad a' = a + 2m,$$

and

$$D := W_2^{l+1}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K)).$$

Then the operator

$$\tilde{L} : S \longrightarrow D, \qquad \tilde{L}u = f,$$

is a bijective, linear and bounded operator with bounded inverse

$$\tilde{L}^{-1} : D \longrightarrow S, \qquad \tilde{L}^{-1}f = u.$$

Proof: We start by showing that

$$\tilde{L} : S \longrightarrow D$$

is bounded (linearity is clear by definition). Let $u \in S$, then $u(t) \in \mathcal{K}^{\gamma}_{2,a'}(K)$. With Theorem 2.9 we have

$$Lu(t) \in \mathcal{K}^{\gamma-2m}_{2,a'-2m}(K) = \mathcal{K}^{\gamma-2m}_{2,a}(K).$$

Since we may interchange the order of differentiation when dealing with weak derivatives, we see that $Lu \in W_2^{l+1}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))$. This together with $\partial_t u \in W_2^{l+1}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))$ gives

$$\tilde{L}u \in W_2^{l+1}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K)).$$

and

$$\begin{aligned}
\|\tilde{L}u|D\| &= \|\tilde{L}u|W_2^{l+1}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K))\| \\
&= \|\partial_t u + (-1)^m Lu|W_2^{l+1}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K))\| \\
&\leq \|\partial_t u|W_2^{l+1}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K))\| + \|Lu|W_2^{l+1}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K))\| \\
&\lesssim \|u|W_2^{l+2}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K))\| + \|u|W_2^{l+1}((0,T),\mathcal{K}_{2,a'}^{\gamma}(K))\| \\
&\lesssim \|u|W_2^{l+1}((0,T),\mathcal{K}_{2,a'}^{\gamma}(K)) \cap W_2^{l+2}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K))\| = \|u|S\|,
\end{aligned}$$

shows that $\tilde{L}$ is bounded. Concerning the inverse $\tilde{L}^{-1}$ first note that our restriction $a \geq 0$ implies $\mathcal{K}_{2,a}^{\gamma-2m}(K) \hookrightarrow L_2(K)$, therefore assumption (i) in Theorem 3.6 reduces to

$$f \in W^{l+1}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K)).$$

But then Theorem 3.6 implies

$$\tilde{L}^{-1} : W_2^{l+1}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K)) \to W_2^{l+1}((0,T),\mathcal{K}_{2,a'}^{\gamma}(K)) \cap W_2^{l+2}((0,T),\mathcal{K}_{2,a}^{\gamma-2m}(K)).$$

Now we make use of the open mapping theorem. Since $\tilde{L}$ is a bijective linear and bounded operator, its inverse $\tilde{L}^{-1}$ is bounded as well. □

3.2 Hyperbolic regularity results

In this subsection we recall regularity results of linear hyperbolic equations of second order from [44].

3.2.1 The fundamental problem

We consider the following initial-boundary value problem

$$\left\{ \begin{array}{rcll} \frac{\partial^2}{\partial t^2}u + L(x,t,D_x)u &=& f(x,t) & \text{in } \Omega_T, \\ u(x,0) \;=\; \frac{\partial}{\partial t}u(x,0) &=& 0 & \text{in } \Omega, \\ u\big|_{\partial\Omega\times(0,T)} &=& 0, & \end{array} \right\} \tag{3.13}$$

where Ω is the special Lipschitz domain from Definition 2.2 and L is a linear differential operator of second order on Ω_T of the following form

$$L(x,t,D_x)u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u,$$

where a_{ij}, $b_i(x,t)$, and $c(x,t)$ are real-valued functions on Ω_T belonging to $C^{k+1}(\Omega_T)$, $k \in \mathbb{N}_0$. Moreover, assume that the coefficients of L and their derivatives are bounded on Ω_T. Suppose that $a_{ij} = a_{ji}$ $(i,j = 1,\ldots,n)$ are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,T]$ and

$$\sum_{i,j=1}^{d} a_{ij}(x,t)\xi_i\xi_j \geq \mu_0|\xi|^2$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $(x,t) \in \Omega_T$, where μ_0 is a positive constant. It is possible to reduce the operator L with coefficients at $P \in l_0$, $t \in (0,T)$, to its canonical form

$$L_0^{(2)} := -\sum_{i,j=1}^{2} a_{ij}(P,t)\frac{\partial^2}{\partial x_i \partial x_j},$$

cf. [44, p. 460] and the references given there. Via this reduction it can be realized, that, after a linear transformation of coordinates, T_1 and T_2 go over into hyperplanes $T_1^{'}$ and $T_2^{'}$, respectively. Furthermore, the angle β at (P,t) is transformed to

$$\omega(P,t) = \arctan \frac{\left[a_{11}(P,t)a_{22}(P,t) - a_{12}^2(P,t)\right]^{1/2}}{a_{22}(P,t)\cot\beta - a_{12}(P,t)}. \tag{3.14}$$

The value $\omega(P,t)$ does not depend on the method by which $L_0^{(2)}$ is reduced to its canonical form. Moreover, the function $\omega(P,t)$ is infinitely differentiable and $\omega(P,t) > 0$. Since Ω is bounded it follows that the manifold l_0 is compact and we put

$$\omega := \max_{P\in l_0, t\in[0,T]} \omega(P,t). \tag{3.15}$$

A function $u(x,t)$ is called a *generalized solution* of problem (3.13) on $[0,T]$, if, and only if, $u \in L_2([0,T], \mathring{W}_2^1(\Omega))$, $\partial_t u \in L_2([0,T], L_2(\Omega))$, $\partial_{t^2} u \in L_2([0,T], W_2^{-1}(\Omega))$ such that $u(x,0) = \partial_t u(x,0) = 0$ and the equality

$$(\partial_{t^2} u(\cdot,t), v) + B[u(t), v; t] = (f(\cdot,t), v)$$

holds for all $v \in \mathring{W}_2^1(\Omega)$ and for all $t \in [0,T]$, where

$$B[u,v;t] = \int_\Omega \left(\sum_{i,j=1}^d a_{ij}(x,t) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^d b_i(x,t)\frac{\partial u}{\partial x_i} v + c(x,t)uv \right) \mathrm{d}x.$$

3.2.2 Regularity results in weighted Sobolev spaces

Concerning the regularity of the solution of (3.13) in weighted Sobolev spaces, a reformulation of the results from [44, Thms. 2.1, 2.2] yields the following.

Lemma 3.9 *Let $\Omega \subset \mathbb{R}^d$ be a special Lipschitz domain from Definition 2.2. Furthermore, let $m \in \mathbb{N}$, $m \geq 2$, $m-1 \leq a < \min(m, \frac{\pi}{\omega} - 1)$, and assume f satisfies*

(i) $\partial_{t^j} f \in L_\infty((0,T), \mathcal{K}_{2,a-j}^{m-j}(\Omega)),\ \ 0 \leq j \leq m-1,$

(ii) $\partial_{t^j} f(x,0) = 0,\ \ 0 \leq j \leq m-2.$

Then for the generalized solution u of problem (3.13) we have

$$\partial_{t^j} u \in L_\infty((0,T), W_2^1(\Omega)) \cap L_\infty((0,T), \mathcal{K}_{2,a-j}^{m-j}(\Omega)), \qquad 0 \leq j \leq m,$$

and the following a priori estimate holds

$$\sum_{j=0}^k \|\partial_{t^j} u | L_\infty((0,T), \mathcal{K}_{2,a-j}^{m-j}(\Omega))\| \lesssim \sum_{j=0}^{k-1} \left\|\partial_{t^j} f | L_\infty((0,T), \mathcal{K}_{2,a-j}^{m-j}(\Omega))\right\|.$$

Remark 3.10 For the restriction on a in Lemma 3.9 to make sense we require

$$m - 1 < \frac{\pi}{\omega} - 1, \quad \text{i.e.,} \quad \omega < \frac{\pi}{m}.$$

Unfortunately, this causes a restriction on the angle β of our domain Ω, i.e., β has to be small, cf. (3.14) and (3.15).

4 Parabolic Besov regularity on polyhedral cones

4.1 Besov regularity of linear parabolic PDEs

To prove our results on Besov regularity, we will heavily use embeddings of Kondratiev spaces into Besov spaces. The following theorem that can be found in [33, Sect. 5, Thm. 3] will be essential in this context.

Theorem 4.1 *Let D be a bounded polyhedral domain in $\mathbb{R}^d$. Furthermore, let $s, a \in \mathbb{R}$, $\gamma \in \mathbb{N}_0$, and suppose $\min(s,a) > \frac{\delta}{d}\gamma$, where δ denotes the dimension of the singular set (i.e. $\delta = 0$ if there are only vertex singularities, $\delta = 1$ if there are edge and vertex singularities etc.). Then there exists some $0 < \tau_0 \leq p$ such that*

$$K^{\gamma}_{p,a}(D) \cap B^{s}_{p,\infty}(D) \hookrightarrow B^{\gamma}_{\tau,\infty}(D) \hookrightarrow L_p(D), \tag{4.1}$$

for all $\tau_* < \tau < \tau_0$, where $\frac{1}{\tau_*} = \frac{\gamma}{d} + \frac{1}{p}$.

Remark 4.2 It also holds that $u \in B^{\gamma}_{\tau,\infty}(D)$ for $\tau \leq \tau_*$ but these spaces are no longer embedded into $L_p(D)$. Moreover, since we are interested in embeddings into the adaptivity scale $B^{s}_{\tau,\tau}(D)$, cf. (1.4), we later on make use of the embedding

$$B^{\gamma}_{\tau,\infty}(D) \hookrightarrow B^{\gamma-\varepsilon}_{\tau,\tau}(D), \qquad \varepsilon > 0.$$

The embedding from Theorem 4.1 immediately generalizes to the function spaces defined in (2.5) as follows:

Theorem 4.3 *Let D be some bounded polyhedral domain in $\mathbb{R}^d$ and assume $k \in \mathbb{N}_0$ and $0 < q \leq \infty$. Furthermore, let $s, a \in \mathbb{R}$, $\gamma \in \mathbb{N}_0$, and suppose $\min(s,a) > \frac{\delta}{d}\gamma$, where δ denotes the dimension of the singular set. Then there exists some $0 < \tau_0 \leq p$ such that*

$$W^k_q((0,T), K^{\gamma}_{p,a}(D)) \cap W^k_q((0,T), B^{s}_{p,\infty}(D)) \hookrightarrow W^k_q((0,T), B^{\gamma}_{\tau,\infty}(D)) \tag{4.2}$$

for all $\tau_* < \tau < \tau_0$, where $\frac{1}{\tau_*} = \frac{\gamma}{d} + \frac{1}{p}$.

Proof : Put $X_1 := K^{\gamma}_{p,a}(D)$, $X_2 := B^{s}_{p,\infty}(D)$, and $X = B^{\gamma}_{\tau,\infty}(D)$. Then Theorem 4.1 states that

$$X_1 \cap X_2 \hookrightarrow X,$$

i.e., for some $x \in X_1 \cap X_2$ we have $\|x|X\| \lesssim \|x|X_1 \cap X_2\| \sim \|x|X_1\| + \|x|X_2\|$. Using this we calculate for $I := (0,T)$ that

$$\begin{aligned}
&\|u|W^k_q((0,T), B^{\gamma}_{\tau,\infty}(D))\| \\
&= \left\|u|W^k_q(I,X)\right\| = \left(\sum_{l=0}^{k} \|\partial_{t^l} u|L_q(I,X)\|^q\right)^{1/q} \\
&\lesssim \sum_{l=0}^{k} \left(\int_I \|\partial_{t^l} u(\cdot,t)|X\|^q \mathrm{d}t\right)^{1/q} \\
&\lesssim \sum_{l=0}^{k} \left(\int_I \|\partial_{t^l} u(\cdot,t)|X_1 \cap X_2\|^q \mathrm{d}t\right)^{1/q} \\
&\sim \sum_{l=0}^{k} \left(\int_I \|\partial_{t^l} u(\cdot,t)|X_1\| \mathrm{d}t\right)^{1/q} + \sum_{l=0}^{k} \left(\int_I \|\partial_{t^l} u(\cdot,t)|X_2\| \mathrm{d}t\right)^{1/q} \\
&= \left\|u|W^k_q(I,X_1)\right\| + \left\|u|W^k_q(I,X_2)\right\| \\
&= \left\|u|W^k_q(I,X_1) \cap W^k_q(I,X_2)\right\| \\
&= \left\|u|W^k_q((0,T), K^{\gamma}_{p,a}(D)) \cap W^k_q((0,T), B^{s}_{p,\infty}(D))\right\|,
\end{aligned}$$

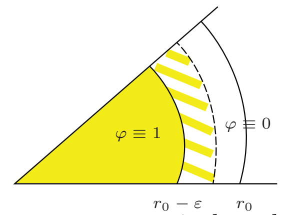
which establishes (4.2). □

Remark 4.4 For $k = 0$ the embedding (4.2) in Theorem 4.3 reads as

$$L_q((0,T), K^{\gamma}_{p,a}(D)) \cap L_q((0,T), B^{s}_{p,\infty}(D)) \hookrightarrow L_q((0,T), B^{\gamma}_{\tau,\infty}(D)). \tag{4.3}$$

Unbounded cone versus bounded polyhedral domain We wish to combine the results from [43] as stated in Theorem 3.6 with the embedding results in Theorem 4.3. The problem arises that Theorem 3.6 holds for unbounded cones $K \subset \mathbb{R}^3$ whereas the embedding results in Theorem 4.3 are true for bounded polyhedral domains $D \subset \mathbb{R}^d$. In order to avoid this problem we consider the truncated cone K_0 as defined in (2.1). Then, the additional difficulty occurs that the Kondratiev norm on the truncated cone is not just defined by restriction. Instead, the distance to the new corners produced by the truncation from considering K_0 instead of K have to be taken into account. We solve this problem by multiplying u with a radial cut-off function $\varphi \in C_0^\infty(K_0)$ satisfying

$$\varphi(x) \equiv \begin{cases} 1 & \text{on} \quad \{|x| < r_0 - \varepsilon\} \cap K_0, \\ 0 & \text{on} \quad \left\{|x| > r_0 - \frac{\varepsilon}{2}\right\} \cap K_0. \end{cases} \tag{4.4}$$



This truncation process does not induce serious restrictions for when it comes to practical applications, it is clear that only truncated cones can be considered. Then the regularity of φu corresponds to the regularity of u as stated in Theorem 3.6 and we obtain

$$\begin{aligned} \|\varphi u | L_2((0,T), \mathcal{K}^\gamma_{2,a'}(K_0))\| &\lesssim \|\varphi u | L_2((0,T), \mathcal{K}^\gamma_{2,a'}(K))\| \\ &\leq c_\varphi \|u | L_2((0,T), \mathcal{K}^\gamma_{2,a'}(K))\|, \end{aligned}$$

from (2.11). Now we are in a position to apply the embedding results from Theorem 4.3 when $k = 0$, cf. (4.3), to the function φu, which together with the regularity results for weighted Sobolev spaces from Theorem 3.6 yield maximal Besov regularity of the solution of the parabolic problem (3.1).

Theorem 4.5 (Parabolic Besov regularity) *Let $K \subset \mathbb{R}^3$ be a a polyhedral cone. Furthermore, let γ, l be nonnegative integers, $\gamma \geq 2m$, $a \in \mathbb{R}$ with $a \in [-m, m]$, and let the Assumptions 3.2 hold. Assume the right hand side f of (3.1) satisfies*

(i) $f \in W_2^{l+1}((0,T), L_2(K)) \cap W_2^{l+1}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))$,

(ii) $\partial_{t^k} f(x, 0) = 0$ *for* $k = 0, \ldots, l$.

Let φ denote the cutoff function from (4.4). Then for the generalized solution $u \in \mathring{W}_2^{m,1}(K_T)$ of problem (3.1), we have

$$\varphi u \in L_2((0,T), B^\gamma_{\tau,\infty}(K)) \qquad \textit{for all} \qquad 0 < \gamma < 3m, \quad \frac{1}{2} < \frac{1}{\tau} < \frac{\gamma}{3} + \frac{1}{2}. \tag{4.5}$$

In particular, for any γ, τ satisfying (4.5), we have the a priori estimate

$$\begin{aligned} &\|\varphi u | L_2((0,T), B^\gamma_{\tau,\infty}(K))\| \\ &\quad \lesssim \sum_{k=0}^{l} \|\partial_{t^k} f | L_2((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))\| + \sum_{k=0}^{l+1} \|\partial_{t^k} f | L_2(K_T)\|. \end{aligned}$$

Proof : According to Theorem 3.6 by our assumptions we know $\varphi u \in L_2((0,T), \mathcal{K}^\gamma_{2,a+2m}(K))$. Together with Theorem 4.3 (choosing $k = 0$) we obtain

$$\begin{aligned} \varphi u \in & L_2((0,T), \mathcal{K}^\gamma_{2,a+2m}(K)) \cap \mathring{W}_2^{m,1}(K_T) \\ & \hookrightarrow L_2((0,T), \mathcal{K}^\gamma_{2,a+2m}(K)) \cap L_2((0,T), W_2^m(K)) \\ & \hookrightarrow L_2((0,T), \mathcal{K}^\gamma_{2,a+2m}(K)) \cap L_2((0,T), B^m_{2,\infty}(K)) \hookrightarrow L_2((0,T), B^\gamma_{\tau,\infty}(K)). \end{aligned}$$

Concerning the restriction on τ, Theorem 4.3 with $\tau_0 = 2$ yields

$$\frac{1}{2} < \frac{1}{\tau} < \frac{1}{\tau^*} = \frac{\gamma}{3} + \frac{1}{2}.$$

As for γ, we have $a \in [-m, m]$ from Theorem 3.6, which together with the lower bound on a from Theorem 4.3 yields

$$m = \min(m, a + 2m) > \frac{\delta}{d}\gamma = \frac{\gamma}{3}, \quad \text{i.e.,} \quad \gamma < 3m.$$

□

Remark 4.6 The above theorem relies on the fact that problem (3.1) has a generalized solution $u \in W_2^{m,1}(K_T) = L_2((0,T), W_2^m(K)) \cap W_2^1((0,T), L_2(K)) \hookrightarrow L_2((0,T), W_2^m(K))$, cf. Proposition 3.1. We strongly believe that (in good agreement with the elliptic case) this result could be improved by studying the regularity of (3.1) in fractional Sobolev spaces $W_p^s(K)$, $s \geq 0$, cf. (2.2) and the explanations given. In this case (assuming that the generalized solution of (3.1) satisfies $u \in L_2((0,T), W_p^s(K))$ for some $s > 0$) under the assumptions of Theorem 4.5, using Theorem 3.6 and Theorem 4.3 (with $k = 0$), we would obtain

$$\varphi u \in L_2((0,T), \mathcal{K}^{\gamma}_{2,a+2m}(K)) \cap L_2((0,T), W_2^s(K)) \hookrightarrow L_2((0,T), B^{\gamma}_{\tau,\infty}(K)),$$

where again $\frac{1}{2} < \frac{1}{\tau} < \frac{\gamma}{3} + \frac{1}{2}$ but the restriction on γ now reads as

$$\gamma < 3\min(s, a + 2m) \qquad \text{for some} \qquad a \in [-m, m].$$

For general Lipschitz domains and $m = 1$ we expect that the solution to (3.1) is contained in $W_p^s(K)$ for all $s < \frac{3}{2}$ (as was shown in the elliptic case for the Poisson equation in [34]), which then leads to $\gamma < \frac{9}{2}$ (if we choose $a > 0$). For convex domains it probably even holds that $s = 2$ (for the heat equation this was already proven in [55]), which in turn would yield $\gamma < 6$ (choosing $a > 0$). However, to establish these kind of regularity results is clearly beyond the scope of this paper and will be the topic of further studies.

Example 4.7 As a parabolic model case for (3.1) we consider the heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{on } K_T, \\ u\big|_{t=0} &= 0 \quad \text{on } K, \end{aligned}$$

where $f \in L_2(K_T)$. In Theorem 4.5, we have parameters $m = 1$, $a' = a + 2m \in [1, 3]$, and $s = m = 1$. This yields the maximal Besov regularity

$$\gamma < 3\min(s, a') = 3 \tag{4.6}$$

for the solution φu of (3.1). If we additonally assume that our polyhedral cone K_0 is convex, i.e., $\theta_k \in (0, \pi)$ for all $k = 1, \ldots, n$, we can do even better. In this case now $s = 2$, cf. [55, Thm. 6.2], thus, according to what was said in Remark 4.6, the upper bound for the maximal Besov regularity is

$$\gamma < 3\min(s, a') = 3 \cdot 2 = 6.$$

4.2 Besov regularity of nonlinear parabolic PDEs

4.2.1 The fundamental problem

We modify (3.1) and now consider for some $\varepsilon > 0$, $M \in \mathbb{N}$, the following nonlinear parabolic problem

$$\left\{ \begin{aligned} \frac{\partial}{\partial t}u + (-1)^m L(x,t;D_x)u + \varepsilon u^M &= f \quad && \text{in } K_T, \\ \frac{\partial^{k-1}u}{\partial \nu^{k-1}}\Big|_{\Gamma_{j,T}} &= 0, && k = 1, \ldots, m, \; j = 1, \ldots, n, \\ u\Big|_{t=0} &= 0 && \text{in } K. \end{aligned} \right\} \tag{4.7}$$

We are interested in the Besov regularity of solutions u of problem (4.7). Our strategy to reach this goal is as follows. We show that the regularity estimates in Kondratiev spaces as stated in Theorem 3.6 carry over to (4.7), provided that ε is sufficiently small. Then, we proceed as in the proof of Theorem 4.5, i.e., we once again use embedding results of Kondratiev spaces into Besov spaces. To establish Kondratiev regularity we interpret (4.7) as a fix point problem in the following way. Let D and S be Banach-spaces (note that the letters D and S stand for different spaces here, which we will specify in each of the theorems below) and let $\tilde{L}^{-1}: D \to S$ be the linear operator defined in (3.12). Equation (4.7) is equivalent to

$$\tilde{L}u = f - \varepsilon u^M =: Nu,$$

where $N : S \to D$ is a nonlinear operator. If we can show that N maps S into D, then a solution to (4.7) is a fixed point of the problem

$$(\tilde{L}^{-1} \circ N)u = u.$$

Our aim is to apply Banach's fixed point theorem, which will also guarantee uniqueness of the solution if we can show that $T := (\tilde{L}^{-1} \circ N) : S_0 \to S_0$ is a contraction mapping, i.e.,

$$\|T(x) - T(y)|S\| \le q\|x - y|S\| \quad \text{for all} \quad x, y \in S_0, \quad q \in [0, 1),$$

where the corresponding metric space $S_0 \subset S$ is a small closed ball with center $\tilde{L}^{-1}f$ (the solution of the corresponding linear problem) and suitably chosen radius $r > 0$ in S.

4.2.2 Nonlinear regularity results

Theorem 4.8 (Nonlinear Kondratiev regularity) *Let $\tilde{L}$ and N be as described above. Assume the assumptions of Theorem 3.8 are satisfied and, additionally, we have $a > \frac{1}{2}$. Put $D := W_2^{l+1}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K))$ and $S := W_2^{l+1}((0,T), \mathcal{K}_{2,a'}^{\gamma}(K)) \cap W_2^{l+2}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K))$ with $a' = a + 2m$, and suppose that f satisfies*

$$f \in D, \qquad \partial_{t^k} f(x, 0) = 0, \quad k = 0, \ldots, l.$$

Put $\eta := \|f|D\|$ and $r_0 > 1$. Moreover, we choose $\varepsilon > 0$ so small that

$$\eta\|\tilde{L}\|^{-\frac{M}{M-1}} \le \left(\frac{1}{c\varepsilon M}\right)^{\frac{1}{M-1}} (r_0 - 1)^{\frac{1}{M-1}} \left(\frac{1}{r_0}\right)^{\frac{M}{M-1}}, \qquad \text{if} \quad r_0\|\tilde{L}\|^{-1}\eta > 1,$$

and

$$\|\tilde{L}\|^{-1} < \frac{r_0 - 1}{r_0} \left(\frac{1}{c\varepsilon M}\right), \qquad \text{if} \quad r_0\|\tilde{L}\|^{-1}\eta < 1,$$

where $c > 0$ denotes the constant in (4.16) resulting from our estimates below. Then there exists a unique solution $u \in S_0 \subset S$ of problem (4.7), where S_0 denotes a small ball around $\tilde{L}^{-1}f$ (the solution of the corresponding linear problem) with radius $r = (r_0 - 1)\eta\|\tilde{L}\|^{-1}$.

Proof : Let u be the solution of the linear problem $\tilde{L}u = f$. Theorem 3.8 shows that

$$\tilde{L}^{-1} : W_2^{l+1}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K)) \to W_2^{l+1}((0,T), \mathcal{K}_{2,a'}^{\gamma}(K)) \cap W_2^{l+2}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K)).$$

is a bounded operator. Formula (4.13) proved below implies that $u^M \in W_2^{l+1}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K))$, therefore

$$\begin{aligned} Nu &= f - \varepsilon u^M \in W_2^{l+1}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K)), \\ Nu\big|_{t=0} &= f\big|_{t=0} + \varepsilon u^M\big|_{t=0} = 0, \end{aligned}$$

which establishes the desired mapping properties of the nonlinear part N. Hence, we can apply Theorem 3.8 now with right hand side Nu. Since

$$(\tilde{L}^{-1} \circ N)(v) - (\tilde{L}^{-1} \circ N)(u) = \tilde{L}^{-1}(f - \varepsilon v^M) - \tilde{L}^{-1}(f - \varepsilon u^M) = \varepsilon \tilde{L}^{-1}(u^M - v^M)$$

one sees that $\tilde{L}^{-1} \circ N$ is a contraction if, and only, if

$$\varepsilon\|\tilde{L}^{-1}(u^M - v^M)\|_S \le q\|u - v\|_S \quad \text{for some } q < 1, \tag{4.8}$$

where $u, v \in S_0$ (meaning $u, v \in B_r(\tilde{L}^{-1}f)$ in S). We analyse the resulting condition with the help of the formula $u^M - v^M = (u-v)\sum_{j=0}^{M-1} u^j v^{M-1-j}$. This together with Theorem 3.8 gives

$$\begin{aligned}
&\|\tilde{L}^{-1}(u^M - v^M)|S\| \\
&\le \|\tilde{L}\|^{-1}\|(u^M - v^M)|D\| \\
&= \|\tilde{L}\|^{-1}\|(u^M - v^M)|W^{l+1}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))\| \\
&= \|\tilde{L}\|^{-1}\left\|(u-v)\sum_{j=0}^{M-1} u^j v^{M-1-j}|W^{l+1}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))\right\| \\
&\sim \|\tilde{L}\|^{-1}\sum_{k=0}^{l+1}\left\|\frac{\partial}{\partial t^k}\left[(u-v)\sum_{j=0}^{M-1} u^j v^{M-1-j}\right]|L_2((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))\right\|
\end{aligned} \tag{4.9}$$

Concerning the derivatives, we use Leibniz's formula twice and we see that

$$\begin{aligned}
&\frac{\partial}{\partial t^k}(u^M - v^M n) \\
&= \frac{\partial}{\partial t^k}\left[(u-v)\sum_{j=0}^{M-1} u^j v^{M-1-j}\right] \\
&= \sum_{l=0}^{k}\binom{k}{l}\frac{\partial}{\partial t^l}(u-v)\cdot\frac{\partial}{\partial t^{k-l}}\left(\sum_{j=0}^{M-1} u^j v^{M-1-j}\right) \\
&= \sum_{l=0}^{k}\binom{k}{l}\frac{\partial}{\partial t^l}(u-v)\cdot\left[\left(\sum_{j=0}^{M-1}\sum_{r=0}^{k-l}\binom{k-l}{r}\frac{\partial}{\partial t^r}u^j\cdot\frac{\partial}{\partial t^{k-l-r}}v^{M-1-j}\right)\right].
\end{aligned} \tag{4.10}$$

In order to estimate the terms $\frac{\partial}{\partial t^r}u^j$ and $\frac{\partial}{\partial t^{k-l-r}}v^{M-1-j}$ we apply Faà di Bruno's formula

$$\frac{\partial}{\partial t^r}(f\circ g) = \sum \frac{r!}{k_1!\dots k_r!}\left(\frac{\partial}{\partial t^{k_1+\dots+k_r}}f\circ g\right)\prod_{m=1}^{r}\left(\frac{\frac{\partial}{\partial t^m}g}{m!}\right)^{k_m}, \tag{4.11}$$

where the sum runs over all r-tuples of nonnegative integers $(k_1, \dots, k_r)$ satisfying

$$1\cdot k_1 + 2\cdot k_2 + \dots + r\cdot k_r = r. \tag{4.12}$$

We apply the formula to $g = u$ and $f(x) = x^j$ and make use of the embeddings (2.10) and the pointwise multiplier results from Theorem 2.8. This yields

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t^r}u^j\ |\mathcal{K}^{\gamma-2m}_{2,a}(K)\right\| \\
&\le c_{r,j}\left\|\sum_{k_1+\dots+k_r\le j} u^{j-(k_1+\dots+k_r)}\prod_{m=1}^{r}\left|\frac{\partial}{\partial t^m}u\right|^{k_m}|\mathcal{K}^{\gamma-2m}_{2,a}(K)\right\| \\
&\lesssim \sum_{k_1+\dots+k_r\le j}\left\|u|\mathcal{K}^{\gamma}_{2,a'}(K)\right\|^{j-(k_1+\dots+k_r)} \\
&\qquad \prod_{m=1}^{r-1}\left\|\frac{\partial}{\partial t^m}u|\mathcal{K}^{\gamma}_{2,a'}(K)\right\|^{k_m}\left\|\frac{\partial}{\partial t^r}u|\mathcal{K}^{\gamma-2m}_{2,a}(K)\right\|^{k_r}.
\end{aligned} \tag{4.13}$$

We proceed similarly for $\frac{\partial}{\partial t^{k-l-r}} v^{M-1-j}$. In particular, from (4.12) we see that $k_{l+1} \leq 1$ for $r = l+1$, therefore the highest derivative $u^{(l+1)}$ appears at most once. Now (4.10) together with (4.13) inserted in (4.9) give

$$
\begin{aligned}
&\|\tilde{L}^{-1}(u^M - v^M)|S\| \\
&\quad \lesssim \|\tilde{L}\|^{-1} \sum_{k=0}^{l+1} \left(\int_0^T \left\| \frac{\partial}{\partial t^k} \left[(u-v) \sum_{j=0}^{M-1} u^j v^{M-1-j} \right] |\mathcal{K}^{\gamma-2m}_{2,a}(K) \right\|^2 \mathrm{d}t \right)^{1/2} \\
&\quad \lesssim \|\tilde{L}\|^{-1} \sum_{k=0}^{l+1} \left(\int_0^T \left\| \frac{\partial}{\partial t^k} (u-v) |\mathcal{K}^{\gamma}_{2,a'}(K) \right\|^2 \cdot \sum_{j=0}^{M-1} \sum_{\substack{k_1+\ldots+k_{l+1} \leq j, \\ k_1+2k_2 \ldots +(l+1)k_{l+1} \leq l+1}} \max_{w \in \{u,v\}} \right. \\
&\qquad \left. \left\| w|\mathcal{K}^{\gamma}_{2,a'}(K) \right\|^{2(j-(k_1+\ldots+k_{l+1}))} \prod_{m=1}^{l} \left\| \frac{\partial}{\partial t^m} w |\mathcal{K}^{\gamma}_{2,a'}(K) \right\|^{2k_m} \left\| \frac{\partial}{\partial t^{l+1}} w |\mathcal{K}^{\gamma-2m}_{2,a}(K) \right\|^{2k_{l+1}} \mathrm{d}t \right)^{1/2} \\
&\quad \lesssim \|\tilde{L}\|^{-1} M \|(u-v)|W^{l+1}((0,T), \mathcal{K}^{\gamma}_{2,a'}(K))\| \\
&\qquad \cdot \max_{w \in \{u,v\}} \max_{m=0,\ldots,l} \left(\left\| \partial_{t^m} w | L_\infty((0,T), \mathcal{K}^{\gamma}_{2,a'}(K)) \right\|, \ \left\| \partial_{t^{l+1}} w | L_\infty((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K)) \right\|, \ 1 \right)^{M-1}.
\end{aligned} \tag{4.14}
$$

From Theorem 2.3 (Sobolev embedding) we conclude that

$$
\begin{aligned}
u, v \in S &= W^{l+1}_2((0,T), \mathcal{K}^{\gamma}_{2,a'}(K)) \cap W^{l+2}_2((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K)) \\
&\hookrightarrow \mathcal{C}^{l,\frac{1}{2}}((0,T), \mathcal{K}^{\gamma}_{2,a'}(K)) \cap \mathcal{C}^{l+1,\frac{1}{2}}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K)) \\
&\hookrightarrow C^{l}((0,T), \mathcal{K}^{\gamma}_{2,a'}(K)) \cap C^{l+1}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K)),
\end{aligned} \tag{4.15}
$$

hence, the term involving the maxima, $\max_{w\in\{u,v\}} \max_{m=0,\ldots,l}(\ldots)^{M-1}$ in (4.14), is bounded by $\max(r + \|\tilde{L}^{-1} f|S\|, 1)^{M-1}$. Moreover, since u and v are taken from $B_r(\tilde{L}^{-1} f)$ in $S = W^{l+1}_2((0,T), \mathcal{K}^{\gamma}_{2,a'}(K)) \cap W^{l+2}((0,T), \mathcal{K}^{\gamma-2m}_{2,a}(K))$, we obtain from (4.14),

$$
\begin{aligned}
\|\tilde{L}^{-1}(u^M - v^M)|S\| &\leq c_0 \|\tilde{L}\|^{-1} M \max(r + \|\tilde{L}^{-1} f|S\|, 1)^{M-1} \|u - v|S\| \\
&\leq c \|\tilde{L}\|^{-1} M \max(r + \|\tilde{L}\|^{-1} \cdot \|f|D\|, 1)^{M-1} \|u - v|S\| \\
&= c \|\tilde{L}\|^{-1} M \max(r + \|\tilde{L}\|^{-1} \eta, 1)^{M-1} \|u - v|S\|,
\end{aligned} \tag{4.16}
$$

where we put $\eta := \|f|D\|$ in the last line, c_0 denotes the constant resulting from (4.13) and (4.14) and $c = c_0 c_1$ with c_1 being the constant from the estimates in Theorem 3.8. For $\tilde{L}^{-1} \circ N$ to be a contraction, we therefore require

$$
c\varepsilon \|\tilde{L}\|^{-1} M \max(r + \|\tilde{L}\|^{-1} \eta, 1)^{M-1} < 1,
$$

cf. (4.8). In case of $\max(r + \|\tilde{L}\|^{-1}\eta, 1) = 1$ this leads to

$$
\|\tilde{L}\|^{-1} < \frac{1}{c\varepsilon M}. \tag{4.17}
$$

On the other hand, if $\max(r + \|\tilde{L}\|^{-1}\eta, 1) = r + \|\tilde{L}\|^{-1}\eta$, we choose $r = (r_0 - 1)\eta \|\tilde{L}\|^{-1}$, which gives rise to the condition

$$
c\varepsilon \|\tilde{L}\|^{-1} M (r_0 \|\tilde{L}\|^{-1} \eta)^{M-1} < 1, \quad \text{i.e.,} \quad \eta \|\tilde{L}\|^{-\frac{M}{M-1}} < \left(\frac{1}{c\varepsilon M} \right)^{\frac{1}{M-1}} \frac{1}{r_0}. \tag{4.18}
$$

The next step is to show that $(\tilde{L}^{-1} \circ N)(B_r(\tilde{L}^{-1} f)) \subset B_r(\tilde{L}^{-1} f)$ in S. Since $(\tilde{L}^{-1} \circ N)(0) = \tilde{L}^{-1}(f - \varepsilon 0^M) = \tilde{L}^{-1} f$, we only need to apply the above estimate (4.16) with $v = 0$. This gives

$$
\begin{aligned}
\varepsilon \|\tilde{L}^{-1} u^M |S\| &\leq c\varepsilon \|\tilde{L}\|^{-1} M \max(r + \|\tilde{L}\|^{-1}\eta, 1)^{M-1} (r + \|\tilde{L}\|^{-1}\eta) \\
&\overset{!}{\leq} r = (r_0 - 1)\eta \|\tilde{L}\|^{-1},
\end{aligned}
$$

which, in case that $\max(r + \|\tilde{L}\|^{-1}\eta, 1) = 1$, leads to

$$\|\tilde{L}\|^{-1} < \frac{r_0 - 1}{r_0}\left(\frac{1}{c\varepsilon M}\right), \tag{4.19}$$

whereas for $\max(r + \|\tilde{L}\|^{-1}\eta, 1) = r + \|\tilde{L}\|^{-1}\eta$ we get

$$\eta\|\tilde{L}\|^{-\frac{M}{M-1}} \leq \left(\frac{1}{c\varepsilon M}\right)^{\frac{1}{M-1}} (r_0 - 1)^{\frac{1}{M-1}} \left(\frac{1}{r_0}\right)^{\frac{M}{M-1}}. \tag{4.20}$$

We see that condition (4.19) implies (4.17). Furthermore, since

$$(r_0 - 1)^{\frac{1}{M-1}} \left(\frac{1}{r_0}\right)^{\frac{M}{M-1}} = (r_0 - 1)^{\frac{1}{M-1}} \left(\frac{1}{r_0}\right)^{1+\frac{1}{M-1}} = \frac{1}{r_0}\left(\frac{r_0 - 1}{r_0}\right)^{\frac{M}{M-1}} < \frac{1}{r_0},$$

also condition (4.20) implies (4.18). Thus, by applying Banach's fixed point theorem in a sufficiently small ball around the solution of the corresponding linear problem, we obtain a unique solution of problem (4.7). □

Remark 4.9 The restriction $a > \frac{1}{2}$ in Theorem 4.8 comes from the pointwise multiplication results, cf. Theorem 2.8, that we applied. In particular, it is required that $2 = p > \frac{d}{2} = \frac{3}{2}$ (which is clearly true), as well as

$$a > \frac{d}{p} - 1 = \frac{3}{2} - 1 = \frac{1}{2}.$$

Together with the restriction $a \in [0, m]$ from Theorem 3.8 this gives $a \in \left[\frac{1}{2}, m\right]$. We also refer to Remark 3.7 for further information. For the heat equation it follows from the calculations there, that the restriction $a \geq 0$ is satisfied if the underlying polyhedral cone of our problem is convex, i.e., we need $\theta_k < \pi$ for all $k = 1, \ldots, n$.

Theorem 4.8 tells us that the nonlinear problem (4.7) has a unique solution in the Kondratiev spaces. In order to obtain some results concerning the Besov regularity we wish to apply Theorem 4.3. Therefore, we also have to estimate the regularity of the nonlinear solution in the Sobolev scale $W_2^s(K) \hookrightarrow B_{2,\infty}^s(K)$, which is done in the following theorem.

Theorem 4.10 (Nonlinear Sobolev regularity) *Let $\tilde{L}$ and N be as described above. Let the assumptions of Theorem 3.8 be satisfied and, additionally, $m \geq 2$. Furthermore, put $D := W_2^{l+1}((0,T), L_2(K))$ and $S := W_2^{l+1}((0,T), W_2^m(K)) \cap W_2^{l+2}((0,T), L_2(K))$. Suppose that f satisfies*

$$f \in D, \qquad \partial_{t^k} f(x, 0) = 0, \quad k = 0, \ldots, l.$$

Put $\eta := \|f|D\|$ and $r_0 > 1$. Moreover, we choose $\varepsilon > 0$ so small that

$$\eta\|\tilde{L}\|^{-\frac{M}{M-1}} \leq \left(\frac{1}{c\varepsilon M}\right)^{\frac{1}{M-1}} (r_0 - 1)^{\frac{1}{M-1}} \left(\frac{1}{r_0}\right)^{\frac{M}{M-1}}, \qquad \text{if} \quad r_0\|\tilde{L}\|^{-1}\eta > 1,$$

and

$$\|\tilde{L}\|^{-1} < \frac{r_0 - 1}{r_0}\left(\frac{1}{c\varepsilon M}\right), \qquad \text{if} \quad r_0\|\tilde{L}\|^{-1}\eta < 1,$$

where $c > 0$ denotes the constant in (4.27) resulting from our estimates below. Then there exists a unique solution $u \in S_0 \subset S$ of problem (4.7), where S_0 denotes a small ball around $\tilde{L}^{-1}f$ (the solution of the corresponding linear problem) with radius $r = (r_0 - 1)\eta\|\tilde{L}\|^{-1}$.

Proof :
Let u be the solution of the linear problem $\tilde{L}u = f$. By [43, Lem. 3.1] we know that

$$\tilde{L}^{-1} : D \to S \tag{4.21}$$

is a bounded operator. Formula (4.24) proved below implies that $u^M \in W_2^{l+1}((0,T), L_2(K))$, therefore

$$\begin{aligned} Nu &= f - \varepsilon u^M \in W_2^{l+1}((0,T), L_2(K)), \\ Nu\big|_{t=0} &= f\big|_{t=0} + \varepsilon u^n\big|_{t=0} = 0, \end{aligned}$$

which establishes the desired mapping properties of the nonlinear part N. Hence, we can apply [43, Lem. 3.1] now with right hand side Nu. We proceed as in Theorem 4.8. $\tilde{L}^{-1} \circ N$ is a contraction if, and only, if

$$\varepsilon \|\tilde{L}^{-1}(u^M - v^M)\|_S \le q\|u - v\|_S \quad \text{for some } q < 1,$$

where $u, v \in S_0$ meaning $u, v \in B_r(\tilde{L}^{-1} f)$ in S. Again $u^M - v^M = (u - v)\sum_{j=0}^{M-1} u^j v^{M-1-j}$ together with (4.21) leads to the estimate

$$\begin{aligned} &\|\tilde{L}^{-1}(u^M - v^M)|S\| \\ &\le \|\tilde{L}\|^{-1}\|(u^M - v^M)|D\| \\ &= \|\tilde{L}\|^{-1}\|(u^M - v^M)|W^{l+1}((0,T), L_2(K))\| \\ &= \|\tilde{L}\|^{-1}\left\|(u-v)\sum_{j=0}^{M-1} u^j v^{M-1-j}|W^{l+1}((0,T), L_2(K))\right\| \\ &\sim \|\tilde{L}\|^{-1}\sum_{k=0}^{l+1}\left\|\frac{\partial}{\partial t^k}\left[(u-v)\sum_{j=0}^{M-1} u^j v^{M-1-j}\right]|L_2((0,T), L_2(K))\right\| \\ &= \|\tilde{L}\|^{-1}\sum_{k=0}^{l+1}\left\|\sum_{l=0}^{k}\binom{k}{l}\frac{\partial}{\partial t^l}(u-v)\cdot\right. \\ &\qquad \left.\left[\left(\sum_{j=0}^{M-1}\sum_{r=0}^{k-l}\binom{k-l}{r}\frac{\partial}{\partial t^r}u^j \cdot \frac{\partial}{\partial t^{k-l-r}}v^{M-1-j}\right)\right]|L_2(K_T)\right\| \\ &\lesssim \|\tilde{L}\|^{-1}\sum_{k=0}^{l+1}\left\|\sum_{l=0}^{k}\frac{\partial}{\partial t^l}(u-v)\cdot\right. \\ &\qquad \left.\left[\left(\sum_{j=0}^{M-1}\sum_{r=0}^{k-l}\frac{\partial}{\partial t^r}u^j \cdot \frac{\partial}{\partial t^{k-l-r}}v^{M-1-j}\right)\right]|L_2(K_T)\right\|, \end{aligned} \tag{4.22}$$

where we used Leibniz's formula twice as in (4.10) in the second by last line. Again Faà di Bruno's formula, cf. (4.11), is applied in order to estimate the derivatives in (4.22). We use a special case of the multiplier result from [48, Sect. 4.6.1, Thm. 1(i)], which states that for parameters $s_1 > s_2$, $s_1 + s_2 > n \max\left(0, \frac{2}{p} - 1\right)$, $s_2 > \frac{d}{p}$, and $q \ge \max(q_1, q_2)$, we have

$$\|uv|F^{s_1}_{p,q_1}\| \lesssim \|u|F^{s_2}_{p,q_2}\| \cdot \|v|F^{s_1}_{p,q_1}\|,$$

where $F^s_{p,q}$ denote the Triebel-Lizorkin spaces closely linked with the Besov spaces by interchanging the order in which the ℓ_q- and L_p-Norms are taken, cf. [48] and the references given there. In particular, choosing $s_1 = 0$, $s_2 = m \ge 2$, $d = 3$, $q_1 = q_2 = p = 2$ and using the coincidences $F^0_{2,2} = L_2$ and $F^m_{2,2} = W^m_2$, we obtain

$$\|uv|L_2\| \le \|u|W_2^m\| \cdot \|v|L_2\|. \tag{4.23}$$

This is exactly the point where our assumption $m \ge 2$ comes into play, since $s_2 = m > \frac{d}{p} = \frac{3}{2}$ is needed.

With this we obtain

$$\begin{aligned}
&\left\| \frac{\partial}{\partial t^r} u^j | L_2(K) \right\| \\
&\leq c_{r,j} \left\| \sum_{k_1+\ldots+k_r \leq j} u^{j-(k_1+\ldots+k_r)} \prod_{m=1}^{r} \left| \frac{\partial}{\partial t^m} u \right|^{k_m} | L_2(K) \right\| \\
&\lesssim \sum_{k_1+\ldots+k_r \leq j} \|u|W_2^m(K)\|^{j-(k_1+\ldots+k_r)} \prod_{m=1}^{r-1} \left\| \frac{\partial}{\partial t^m} u | W_2^m(K) \right\|^{k_m} \left\| \frac{\partial}{\partial t^r} u | L_2(K) \right\|^{k_r} .
\end{aligned} \tag{4.24}$$

Similar for $\frac{\partial}{\partial t^{k-l-r}} v^{M-1-j}$. As before, from (4.12) we observe $k_{l+1} \leq 1$ for $r = l+1$, therefore the highest derivative $u^{(l+1)}$ appears at most once. Now (4.24) inserted in (4.22) gives

$$\begin{aligned}
&\|\tilde{L}^{-1}(u^M - v^M)|S\| \\
&\lesssim \quad \|\tilde{L}\|^{-1} \sum_{k=0}^{l+1} \Bigg(\int_0^T \left\| \frac{\partial}{\partial t^k}(u-v)|W_2^m(K) \right\|^2 \cdot \sum_{j=0}^{M-1} \sum_{\substack{k_1+\ldots+k_{l+1} \leq j, \\ k_1+2k_2+\ldots+(l+1)k_{l+1} \leq l+1}} \max_{w \in \{u,v\}} \\
&\qquad \|w|W_2^m(K)\|^{2(j-(k_1+\ldots+k_{l+1}))} \prod_{m=1}^{l} \left\| \frac{\partial}{\partial t^m} w | W_2^m(K) \right\|^{2k_m} \left\| \frac{\partial}{\partial t^{l+1}} w | L_2(K) \right\|^{2k_{l+1}} \mathrm{d}t \Bigg)^{1/2} \\
&\lesssim \quad \|\tilde{L}\|^{-1} M \|(u-v)|W^{l+1}((0,T), W_2^m(K))\|^2 \\
&\qquad \max_{w \in \{u,v\}} \max_{m=0,\ldots,l} \left(\|\partial_{t^m} w | L_\infty((0,T), W_2^m(K))\|, \ \|\partial_{t^{l+1}} w | L_\infty((0,T), L_2(K))\|, \ 1 \right)^{M-1}
\end{aligned} \tag{4.25}$$

From Theorem 2.3 we see that

$$\begin{aligned}
u, v \in S \quad &= \quad W_2^{l+1}((0,T), W_p^m(K)) \cap W_2^{l+2}((0,T), L_2(K)) \\
&\hookrightarrow \quad \mathcal{C}^{l,\frac{1}{2}}((0,T), W_2^m(K)) \cap \mathcal{C}^{l+1,\frac{1}{2}}((0,T), L_2(K)) \\
&\hookrightarrow \quad C^l((0,T), W_2^m(K)) \cap C^{l+1}((0,T), L_2(K)),
\end{aligned} \tag{4.26}$$

hence the term $\max_{w\in\{u,v\}} \max_{m=0,\ldots,l}(\ldots)^{M-1}$ in (4.25) is bounded. Moreover, since u and v are taken from $B_r(\tilde{L}^{-1}f)$ in $S = W_2^{l+1}((0,T), W_2^m(K)) \cap W^{l+2}((0,T), L_2(K))$, as in (4.16) we obtain from (4.25) and (4.26),

$$\|\tilde{L}^{-1}(u^M - v^M)|S\| \quad \leq \quad c\|\tilde{L}\|^{-1} M \max(r + \|\tilde{L}\|^{-1}\eta, 1)^{M-1} \cdot \|u - v|S\|, \tag{4.27}$$

where we put $\eta := \|f|D\|$ and c denotes the constant arising from our estimates (4.25) and (4.26) above. We now proceed as in Theorem 4.8. In particular, for $\tilde{L}^{-1} \circ N$ to be a contraction and for $(\tilde{L}^{-1} \circ N)(B_r(\tilde{L}^{-1}f)) \subset B_r(\tilde{L}^{-1}f)$ in S we end up with the two conditions (4.18) and (4.20), respectively. The proof is complete. □

Remark 4.11 Note that the restriction $m \geq 2$ in Theorem 4.10 comes from the fact that we require $m > \frac{d}{p} = \frac{3}{2}$ in (4.23). This assumption can probably be weakened, since we expect for the solution to satisfy $u \in L_2((0,T), W_2^s(K))$ for all $s < \frac{3}{2}$, see also Remark 4.5 and the explanations given there.

From Theorems 4.8 and 4.10 we obtain the following result concerning Besov regularity of our nonlinear parabolic problem (4.7).

Theorem 4.12 (Nonlinear Besov regularity) *Let $\tilde{L}$ and N be as described above and let the assumptions of Theorem 3.8 be satisfied. Additionally, we assume $m \geq 2$ and $a \in \left[\frac{1}{2}, m\right]$. Furthermore, put $D := W_2^{l+1}((0,T), L_2(K)) \cap W_2^{l+1}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K))$ and $S := W_2^{l+1}((0,T), W_2^m(K)) \cap W_2^{l+2}((0,T), L_2(K)) \cap W_2^{l+1}((0,T), \mathcal{K}_{2,a'}^{\gamma}(K)) \cap W_2^{l+2}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K))$. Suppose that f satisfies*

$$f \in D, \qquad \partial_{t^k} f(x,0) = 0, \quad \text{and} \quad k = 0, \ldots, l.$$

Put $\eta := \|f|D\|$ and $r_0 > 1$. Moreover, we choose $\varepsilon > 0$ so small that

$$\eta \|\tilde{L}\|^{-\frac{M}{M-1}} \leq \left(\frac{1}{\varepsilon M}\right)^{\frac{1}{M-1}} (r_0 - 1)^{\frac{1}{M-1}} \left(\frac{1}{r_0}\right)^{\frac{M}{M-1}}, \qquad \text{if} \quad r_0 \|\tilde{L}\|^{-1}\eta > 1, \tag{4.28}$$

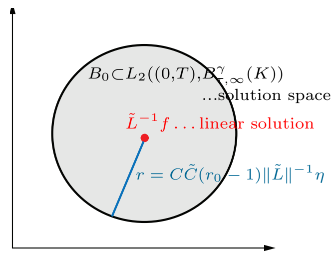
and

$$\|\tilde{L}\|^{-1} < \frac{r_0 - 1}{r_0} \left(\frac{1}{\varepsilon M}\right), \qquad \text{if} \quad r_0 \|\tilde{L}\|^{-1}\eta < 1. \tag{4.29}$$

Let φ denote the cut-off function as described in (4.4). Then there exists a solution u to (4.7), whose truncated version φu satisfies $\varphi u \in B_0 \subset B$,

$$B := L_2((0,T), B_{\tau,\infty}^{\gamma}(K)) \quad \text{for all} \quad 0 < \gamma < 3m,$$

$\frac{1}{2} < \frac{1}{\tau} < \frac{\gamma}{3} + \frac{1}{2}$, and B_0 denotes a small ball around $\tilde{L}^{-1} f$ (the solution of the corresponding linear problem) with radius $r = C\tilde{C}(r_0 - 1)\eta\|\tilde{L}\|^{-1}$.



Proof : This is a consequence of the regularity results in Kondratiev and Sobolev spaces from Theorems 4.8 and 4.10, respectively. To be more precise, Theorem 4.8 establishes the existence of a fixed point u_1 in $S_0^1 \subset S^1 := W_2^{l+1}((0,T), \mathcal{K}_{2,a'}^{\gamma}(K)) \cap W_2^{l+2}((0,T), \mathcal{K}_{2,a}^{\gamma-2m}(K))$ whereas from Theorem 4.10 we have a fixed point u_2 in $S_0^2 \subset S^2 = W_2^{l+1}((0,T), W_2^m(K)) \cap W_2^{l+2}((0,T), L_2(K))$. By Theorem A.2 the fixed points coincide, i.e., $u = u_1 = u_2$, on some subset $S_0 \subset S_0^1 \cap S_0^2$ of S. This together with the embedding results for Besov spaces from Theorem 4.3 (choosing $k = 0$) completes the proof, in particular, we calculate for the solution

$$\begin{aligned} &\|\varphi u - \varphi \tilde{L}^{-1} f | L_2((0,T), B_{\tau,\infty}^{\gamma}(K))\| \\ &\qquad \leq C \|\varphi u - \varphi \tilde{L}^{-1} f | L_2((0,T), \mathcal{K}_{2,a'}^{\gamma}(K))\| \\ &\qquad \leq C\tilde{C} \|u - \tilde{L}^{-1} f | L_2((0,T), \mathcal{K}_{2,a'}^{\gamma}(K))\| \\ &\qquad \leq C\tilde{C}(r_0 - 1)\eta \|\tilde{L}\|^{-1}, \end{aligned} \tag{4.30}$$

where in the second step we used that $\varphi \in C_0^{\infty}(K)$ is a multiplier for Kondratiev spaces, cf. (2.11). Furthermore, it can be seen from (4.30) that new constants C and $\tilde{C}$ appear when considering the radius r around the linear solution where the problem can be solved compared to Theorem 4.8. □

Remark 4.13 A few words concerning the parameters appearing in Theorem 4.12 seem to be in order. Usually, the operator norm $\|\tilde{L}\|^{-1}$ as well as ε are fixed; but we can change η and r_0 according to our needs. From this we see that by choosing η small enough the conditions (4.28) and (4.29) can always be satisfied. Moreover, one can see easily that the smaller the nonlinear perturbation $\varepsilon > 0$ is, the larger we can choose the radius r of the ball B_0 where the solution to the nonlinear problem is unique.

4.3 Space-time adaptivity and Hölder-Besov regularity

So far we have not exploited the fact that Theorems 3.6 and 3.8 not only provide regularity results of the solutions u of (3.1) but also of their partial derivatives $\partial_{t^k} u$. In this section we will use this fact together with the Sobolev embedding theorem 2.3 in order to obtain some mixed Hölder-Besov regularity results

on the whole space-time cylinder K_T.
For parabolic SPDEs, results in this direction have been obtained in [9]. However, for SPDEs, the time regularity is limited in nature. This is caused by the nonsmooth character of the driving processes. Typically, Hölder regularity $\mathcal{C}^{0,\beta}$ can be obtained, but not more. In contrast to this, it is well-known that deterministic parabolic PDEs are smoothing in time. Therefore, we expect that in the deterministic case considered here, higher regularity results in time can be obtained compared to the probabilistic setting.

Theorem 4.14 (Hölder-Besov regularity) *Let γ, l be nonnegative integers, $\gamma \geq 2m$, $a \in \mathbb{R}$, and $a \in [-m, m]$. Assume that the Assumptions 3.2 hold and the right hand side f of (3.1) satisfies*

(i) $f \in W_2^{l+1}((0,T), L_2(K)) \cap W_2^{l+1}((0,T), \mathcal{K}_{2,a}^{\gamma-2m,2}(K))$,

(ii) $\partial_{t^k} f(x,0) = 0, \quad k = 0, 1, \ldots, l.$

Let φ denote the cutoff function from (4.4). Then for the solution u of problem (3.1), we have

$$\varphi u \in \mathcal{C}^{l,\frac{1}{2}}((0,T), B_{\tau,\infty}^{\gamma}(K)) \quad \text{for all} \quad 0 < \gamma < 3m, \quad \frac{1}{2} < \frac{1}{\tau} < \frac{\gamma}{3} + \frac{1}{2}.$$

In particular, we have the a priori estimate

$$\|\varphi u | \mathcal{C}^{l,\frac{1}{2}}((0,T), B_{\tau,\infty}^{\gamma}(K))\| \leq C \sum_{k=0}^{l} \|\partial_{t^k} f | \mathcal{K}_{2,a}^{\gamma-2m,2}(K_T)\| + \sum_{k=0}^{l+1} \|\partial_{t^k} f | L_2(K_T)\|,$$

where C is a constant independent of u and f.

Proof : Theorems 3.6, 3.8, and Proposition 3.1 show together with Theorems 4.3 and 2.3, that under the given assumptions on the initial data f, we have

$$\begin{aligned} \varphi u &\in W_2^{l+1}((0,T), \mathcal{K}_{2,a'}^{\gamma}(K)) \cap W_2^{l+1}((0,T), W_2^m(K)) \\ &\hookrightarrow W_2^{l+1}((0,T), \mathcal{K}_{2,a'}^{\gamma}(K) \cap W_2^m(K)) \\ &\hookrightarrow \mathcal{C}^{l,\frac{1}{2}}((0,T), \mathcal{K}_{2,a'}^{\gamma}(K) \cap W_2^m(K)), \\ &\hookrightarrow \mathcal{C}^{l,\frac{1}{2}}((0,T), B_{\tau,\infty}^{\gamma}(K)), \end{aligned}$$

which completes the proof. □

5 Parabolic Besov regularity on general Lipschitz domains

We turn our attention towards Besov regularity results for parabolic PDEs on general Lipschitz domains using regularity results in weighted Sobolev spaces from [36]. It is important to note that for general Lipschitz domains the definition of the Kondratiev spaces is different when compared to polyhedral domains, since in this case the whole boundary $\partial\mathcal{O}$ coincides with the singular set. Therefore, the singularities induced by the boundary have a much stronger influence. As a consequence, the regularity results for polyhedral cones are much stronger compared to the Lipschitz case, see Remark 5.7 below.
Surprisingly, it turns out that the spatial regularity results in the deterministic case are more or less the same as for the case of SPDEs that was already studied in [8] based on [35, 37].
However, for the time regularity we nevertheless expect a significant difference, cf. Subsection 4.3. Moreover, the reader should observe that our results stated in Section 4 are more general in the sense that there differential operators of arbitrary order are considered, whereas the analysis in this section is restricted to second order operators.
Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and put $\varrho(x) = \operatorname{dist}(x, \partial\mathcal{O})$. We consider the following class of parabolic equations

$$\left\{ \begin{array}{lcll} \frac{\partial}{\partial t} u &=& \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} u + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} u + cu + f & \text{on } \mathcal{O}_T, \\ u(0,\cdot) &=& u_0 & \text{on } \mathcal{O}, \end{array} \right\} \tag{5.1}$$

where the coefficients are assumed to satisfy the assumptions listed below. We need some more notation. Put $\rho(x,y)=\rho(x)\wedge\rho(y)$. For $\alpha\in\mathbb{R}$, $\delta\in(0,1]$, and $m\in\mathbb{N}_0$, we set

$$\begin{aligned}
[f]_m^{(\alpha)} &:= \sup_{x\in\mathcal{O}} \rho^{m+a}(x)|D^m f(x)|,\\
[f]_{m+\delta}^{(\alpha)} &:= \sup_{\substack{x,y\in\mathcal{O}\\ |\beta|=m}} \rho^{m+\alpha}(x,y)\frac{|D^\beta f(x)-D^\beta f(y)|}{|x-y|^\delta},\\
|f|_m^{(\alpha)} &:= \sum_{l=0}^m [f]_l^{(\alpha)} \qquad \text{and} \qquad |f|_{m+\delta}^{(\alpha)} := |f|_m^{(\alpha)} + [f]_{m+\delta}^{(\alpha)},
\end{aligned}$$

whenever it makes sense. Furthermore, fix a constant $\varepsilon>0$. Then for $\gamma\geq 0$ we define

$$\gamma_+ = \begin{cases} \gamma, & \text{if } \gamma\in\mathbb{N},\\ \gamma+\varepsilon_0, & \text{otherwise.}\end{cases}$$

Assumption 5.1 (Assumptions on the coefficients)

(i) Parabolicity: There are constants $\delta_0, K\in(0,\infty)$ such that for all $\lambda\in\mathbb{R}^d$

$$\delta_0|\lambda|^2 \leq a_{ij}(t,x)\lambda_i\lambda_j \leq K|\lambda|^2$$

(ii) The behaviour of the coefficients b_i and c can be controlled near the boundary of $\mathcal{O}$:

$$\lim_{\substack{\varrho(x)\to 0,\\ x\in\mathcal{O}}} \sup_t \left(\varrho(x)|b_i(t,x)| + \varrho^2(x)|c(t,x)|\right) = 0$$

(iii) The coefficients $a_{ij}(t,\cdot)$ are uniformly continuous in x, i.e., for any $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$ such that

$$|a_{ij}(t,\cdot)-a_{ij}(t,y)|<\varepsilon$$

for all $x,y\in\mathcal{O}$ with $|x-y|<\delta$

(iv) For any $t>0$

$$|a_{ij}(t,\cdot)|_{|\gamma|+}^{(0)} + |\varrho(x)b_i(t,\cdot)|_{|\gamma|+}^{(0)} + \left|\varrho^2(x)c(t,\cdot)\right|_{|\gamma|+}^{(0)} \leq K.$$

We define Kondratiev spaces $\mathcal{K}_{p,a}^m(\mathcal{O})$ on bounded Lipschitz domains similar to (2.4), i.e.,

$$\|u|\mathcal{K}_{p,a}^m(\mathcal{O})\| := \left(\sum_{|\alpha|\leq m}\int_{\mathcal{O}} |\varrho(x)|^{p(|\alpha|-a)}|D_x^\alpha u(x)|^p \mathrm{d}x\right)^{1/p} < \infty, \tag{5.2}$$

where $a\in\mathbb{R}$, $1<p<\infty$, $m\in\mathbb{N}_0$, $\alpha\in\mathbb{N}_0^n$, and the weight function $\varrho:\mathcal{O}\to[0,1]$ now stands for the smooth distance to the singular set of $\mathcal{O}$. We generalize the above Kondratiev spaces with the help of complex interpolation, cf. [42], to non-integer values $m\geq 0$ as follows. If $0<\eta<1$, $m_1,m_2\in\mathbb{N}_0$, $p_0,p_1\in(1,\infty)$, and $a_0,a_1\in\mathbb{R}$, put

$$\mathcal{K}_{p,a}^m(\mathcal{O}) := \left[\mathcal{K}_{p_0,a_0}^{m_0}(\mathcal{O}), \mathcal{K}_{p_1,a_1}^{m_1}(\mathcal{O})\right]_\eta, \qquad \text{where} \quad m=(1-\eta)m_0+\eta m_1,$$

$\frac{1}{p}=\frac{1-\eta}{p_0}+\frac{\eta}{p_1}$, and $a=(1-\eta)a_0+\eta a_1$.

The following theorem was proven in [36, Th. 2.5]. For convenience of the reader we adapt the notation from the paper to our needs.

Theorem 5.2 (Kondratiev regularity) *Let $p \in [2,\infty)$, $\gamma \in [0,\infty)$ and Assumption 5.1 be satisfied. Then there exists $\beta_0 = \beta_0(p,d,\mathcal{O}) > 0$ such that for*

$$a \in \left(\frac{2-p-\beta_0}{p}, \frac{2-p+\beta_0}{p} \right),$$

any $f \in L_p([0,T], \mathcal{K}^{\gamma}_{p,a-1}(\mathcal{O}))$ and initial data $u_0 \in \mathcal{K}^{\gamma+2-\frac{2}{p}}_{p,a+\frac{p-2}{p}}(\mathcal{O})$, equation (5.1) admits a unique solution $u \in L_p([0,T], \mathcal{K}^{\gamma+2}_{p,a+1}(\mathcal{O}))$ with $\partial_t u \in L_p([0,T], \mathcal{K}^{\gamma}_{p,a-1}(\mathcal{O}))$. In particular, we have

$$\sum_{k=0}^{1} \left\| \partial_{t^k} u | L_p([0,T], \mathcal{K}^{\gamma+2-2k}_{p,a+1-2k}(\mathcal{O})) \right\| \\ \leq C \left(\| f | L_p([0,T], \mathcal{K}^{\gamma}_{p,a-1}(\mathcal{O})) \| + \left\| u_0 | \mathcal{K}^{\gamma+2-\frac{2}{p}}_{p,a+\frac{p-2}{p}}(\mathcal{O}) \right\| \right),$$

where $C = C(d,p,\gamma,\theta,\delta_0,K,T,\mathcal{O})$.

Remark 5.3 In [38] similar results for C^1 domains were established. In particular, the condition on a above in this case has to be replaced by

$$a \in \left(\frac{1}{p} - 1, \frac{1}{p} \right).$$

We rewrite [8, Th. 5.1] an obtain the following embedding of weighted Sobolev spaces into the scale of Besov spaces.

Theorem 5.4 *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Fix $\gamma \in (0,\infty)$, $p \in [2,\infty)$, and $a \in \mathbb{R}$. Then*

$$L_p([0,T], \mathcal{K}^{\gamma}_{p,a}(\mathcal{O})) \hookrightarrow L_p([0,T], B^{\alpha}_{\tau,\tau}(\mathcal{O})),$$

for all α and τ with

$$\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p} \quad \text{and} \quad 0 < \alpha < \min \left\{ \gamma, a\frac{d}{d-1} \right\}.$$

Remark 5.5 In contrast to Theorems 4.1 and 4.3 the result in Theorem 5.4 is weaker, since here we have the restriction $\alpha < a\frac{d}{d-1}$. On the other hand, in the embedding above no knowledge about the Sobolev regularity (or regularity in the spaces $B^s_{p,\infty}$) is needed.

Using this, we get the following Besov regularity for the solutions of (5.1).

Theorem 5.6 (Besov regularity) *Let $p \in [2,\infty)$, $\gamma \in [0,\infty)$ and Assumption 5.1 be satisfied. Then there exists $\beta_0 = \beta_0(p,d,\mathcal{O}) > 0$ such that for*

$$a \in \left(\frac{2-p-\beta_0}{p}, \frac{2-p+\beta_0}{p} \right),$$

any $f \in L_p([0,T], \mathcal{K}^{\gamma}_{p,a-1}(\mathcal{O}))$ and initial data $u_0 \in \mathcal{K}^{\gamma+2-\frac{2}{p}}_{p,a+\frac{p-2}{p}}(\mathcal{O})$, equation (5.1) admits a unique solution

$$u \in L_p([0,T], B^{\alpha}_{\tau,\tau}(\mathcal{O})), \qquad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \qquad 0 < \alpha < \min \left\{ \gamma + 2, (a+1)\frac{d}{d-1} \right\}.$$

In particular, we have

$$\left\| u | L_p([0,T], B^{\alpha}_{\tau,\tau}(\mathcal{O})) \right\| \leq C \left(\| f | L_p([0,T], \mathcal{K}^{\gamma}_{p,a-1}(\mathcal{O})) \| + \left\| u_0 | \mathcal{K}^{\gamma+2-\frac{2}{p}}_{p,a+\frac{p-2}{p}}(\mathcal{O}) \right\| \right),$$

where $C = C(d,p,\gamma,\theta,\delta_0,K,T,\mathcal{O})$.

Remark 5.7 We calculate for $a+1$ appearing in the upper bound for α that

- $a+1 \in \left(\frac{2-\beta_0}{p}, \frac{2+\beta_0}{p}\right)$ for Lipschitz domains $\mathcal{O}$
- $a+1 \in \left(\frac{1}{p}, 1+\frac{1}{p}\right)$ for C^1 domains $\mathcal{O}$

In particular, for $p=2$ and $d=3$ as an upper bound for α we get

$$\alpha < \left\{ \begin{matrix} \min\left\{\gamma+2, \left(1+\frac{\beta_0}{2}\right)\frac{3}{2}\right\} & \text{for Lipschitz domains} \\ \min\left\{\gamma+2, \frac{9}{4}\right\} & \text{for } C^1 \text{ domains} \end{matrix} \right\} < \frac{9}{4},$$

whereas Theorem 4.5 yields $\alpha < 3$ (since we have $m=1$ in (5.1)).

Example 5.8 (Heat equation) In [36, L. 3.10] it is shown that the heat equation

$$u_t = \Delta u \quad \text{on } \mathcal{O} \times [0,T], \qquad u(\cdot,0)=0,$$

has a solution $u \in L_p([0,T], \mathcal{K}^2_{p,\frac{2}{p}}(\mathcal{O}))$. Using Theorem 5.6 we obtain for the Besov regularity of the solution (now $\gamma >> 0$) that

$$u \in L_p\left([0,T], B^\alpha_{\tau,\tau}(\mathcal{O})\right) \quad \text{with} \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p} \quad \text{and} \quad 0 < \alpha < \left(1+\frac{2}{p}\right)\frac{d}{d-1}.$$

For $p=2$ and $d=3$ this even yields $\alpha < 3$. However, comparing this with our considerations in Example 4.7, we see that for polyhedral cones $K \subset \mathbb{R}^3$ our results concerning Besov regularity are better than what can be expected for the heat equation on arbitrary Lipschitz domains. We also refer to [4] in this context, where the investigations (subject to some restrictions) lead to $\alpha < \frac{3}{2}s$.

Remark 5.9 Since β_0 depends also on p in general Theorem 5.2 does not hold for all $p > 4$. A deterministic counterexample is discussed after [8, Th. 3.13], where the heat equation is considered. Thus, one should distinguish between $p \in [2,\infty)$ and $p \in [2, p_0)$ as was done in [8, Th. 3.13, Th. 5.2] also in the deterministic case.

6 Hyperbolic Besov regularity

Our special Lipschitz domains Ω from Definition 2.2 that we deal with in this section are not bounded polyhedral domains as considered in Theorems 4.1, 4.3. However, regarding embeddings of the Kondratiev spaces into the scale of Besov spaces, modifying the arguments from [33, Sect. 5, Thm. 3], we show that the results can be generalized to our context.

Theorem 6.1 *Let $\Omega \subset \mathbb{R}^d$ be a special Lipschitz domain from Definition* 2.2. *Then we have a continuous embedding*

$$\mathcal{K}^m_{p,a}(\Omega) \cap B^s_{p,p}(\Omega) \hookrightarrow B^r_{\tau,\tau}(\Omega), \qquad \frac{1}{\tau} = \frac{r}{d} + \frac{1}{p}, \qquad 1 < p < \infty, \tag{6.1}$$

for all $0 \le r < \min(m, \frac{sd}{d-1})$ and $m > a > \frac{\delta}{d}r$, where $\delta = d-2 = \dim(l_0)$.

Proof : Since for $r=0$ the result is clear, we assume in the sequel that $r > 0$ and $0 < \tau < p$. The proof is based on the wavelet characterization of Besov spaces presented in Subsection 2.4. Theorem 2.12 implies that it is enough to show

$$\left(\sum_{(I,\psi)\in\Lambda} |I|^{\left(\frac{1}{p}-\frac{1}{2}\right)\tau} |\langle \tilde{u}, \psi_I\rangle|^\tau \right)^{1/\tau} \le c \max\{\|u|\mathcal{K}^m_{p,a}(\Omega)\|, \|u|B^s_{p,p}(\Omega)\|\}.$$

Step 1: We explain why the first term in (2.24) can be incorporated in the estimates that follow in Step 2. Since our domain Ω is Lipschitz, we can extend every $u \in B^s_{p,p}(\Omega)$ to some function $\tilde{u} = Eu \in B^s_{p,p}(\mathbb{R}^d)$. Then the first term reads as

$$\sum_{k\in\mathbb{Z}^d} \langle \tilde{u}, \phi(\cdot - k)\rangle \phi(\cdot - k).$$

Since ϕ shares the same smoothness and support properties as the wavelets ψ_I for $|I| = 1$ (note that below the vanishing moments of ψ_I only become relevant for $|I| < 1$), the coefficients $\langle \tilde{u}, \phi(\cdot - k)\rangle$ can be treated exactly like any of the coefficients $\langle \tilde{u}, \psi_I\rangle$ in Step 2.

Step 2: For our analysis we shall split the index set Λ as follows. For $j \in \mathbb{N}_0$ the refinement level j is denoted by

$$\Lambda_j := \{(I, \Psi) \in \Lambda : \ |I| = 2^{-jd}\}.$$

Furthermore, for $k \in \mathbb{N}_0$ put

$$\Lambda_{j,k} := \{(I, \psi) \in \Lambda_j : \ k2^{-j} \le \rho_I(x) < (k+1)2^{-j}\},$$

where $\rho_I = \inf_{x\in Q(I)} \rho(x)$. In particular, we have $\Lambda_j = \bigcup_{k=0}^{\infty} \Lambda_{j,k}$ and $\Lambda = \bigcup_{j=0}^{\infty} \Lambda_j$.
We consider first the situation when $\rho_I > 0$ corresponding to $k \ge 1$ and therefore put $\Lambda_j^0 = \bigcup_{k\ge 1} \Lambda_{k,j}$. Moreover, we require $Q(I) \subset \Omega$. Recall Whitney's estimate regarding approximation with polynomials, cf. [26, Sect. 6.1], which states that for every I there exists a polyomial P_I of degree less than m, such that

$$\|\tilde{u} - P_I|L_p(Q(I))\| \le c_0|Q(I)|^{m/d}|\tilde{u}|_{W^m_p(Q(I))} \le c_1|I|^{m/d}|\tilde{u}|_{W^m_p(Q(I))}$$

for some constant c_1 independent of I and u, where

$$|u|_{W^m_p(Q(I))} := \left(\int_{Q(I)} |\nabla^m u(x)|^p \mathrm{d}x\right)^{1/p}.$$

Note that ψ_I satisfies moment conditions of order up to m, i.e., it is orthogonal to any polynomial of degree up to $m-1$. Thus, using Hölder's inequality with $p > 1$ we estimate

$$\begin{aligned}
|\langle \tilde{u}, \psi_I\rangle| &= |\langle \tilde{u} - P_I, \psi_I\rangle| \le \|\tilde{u} - P_I|L_p(Q(I))\| \cdot \|\psi_I|L_{p'}(Q(I))\| \\
&\le c_1|I|^{m/d}|\tilde{u}|_{W^m_p(Q(I))}|I|^{\frac{1}{2}-\frac{1}{p}} \\
&\le c_1|I|^{\frac{m}{d}+\frac{1}{2}-\frac{1}{p}}\rho_I^{a-m}\left(\sum_{|\alpha|=m}\int_{Q(I)} |\rho(x)|^{m-a}\partial^\alpha\tilde{u}(x)|^p \mathrm{d}x\right)^{1/p} \\
&=: c_1|I|^{\frac{m}{d}+\frac{1}{2}-\frac{1}{p}}\rho_I^{a-m}\mu_I.
\end{aligned}$$

On the refinement level j, using Hölder's inequality with $\frac{p}{\tau} > 1$, we find

$$\begin{aligned}
&\sum_{(I,\psi)\in\Lambda_j^0} |I|^{\left(\frac{1}{p}-\frac{1}{2}\right)\tau}|\langle \tilde{u}, \psi_I\rangle|^\tau \\
&\qquad \le \sum_{(I,\psi)\in\Lambda_j^0} \left(|I|^{\frac{m}{d}}\rho_I^{a-m}\mu_I\right)^\tau \\
&\qquad \le c_1\left(\sum_{(I,\psi)\in\Lambda_j^0}\left(|I|^{\frac{m}{d}\tau}\rho_I^{(a-m)\tau}\right)^{\frac{p}{p-\tau}}\right)^{\frac{p-\tau}{p}}\left(\sum_{(I,\psi)\in\Lambda_j^0}\mu_I^p\right)^{\tau/p}.
\end{aligned}$$

For the second factor we observe that there is a controlled overlap between the cubes $Q(I)$, meaning each

$x \in \Omega$ is contained in a finite number of cubes inepentent of x, such that we get

$$\begin{aligned}\left(\sum_{(I,\psi)\in\Lambda_j^0} \mu_I^p\right)^{1/p} &= \left(\sum_{(I,\psi)\in\Lambda_j^0}\sum_{|\alpha|=m}\int_{Q(I)} |\rho(x)^{m-a}\partial^\alpha \tilde{u}(x)|^p \mathrm{d}x\right)^{1/p} \\ &\leq c_2\left(\sum_{|\alpha|=m}\int_\Omega |\rho(x)^{m-a}\partial^\alpha \tilde{u}(x)|^p \mathrm{d}x\right)^{1/p} \leq c_2\|u|\mathcal{K}^m_{p,a}(\Omega)\|.\end{aligned}$$

For the first factor by choice or ρ we always have $\rho_I \leq 1$, hence the index k is at most 2^j for the sets $\Lambda_{j,k}$ to be non-empty. The number of elements in $\Lambda_{j,k}$ is bounded by $k^{d-1-\delta}2^{j\delta}$. With this we find

$$\begin{aligned}&\left(\sum_{(I,\psi)\in\Lambda_j^0} \left(|I|^{\frac{m}{d}\tau}\rho_I^{(a-m)\tau}\right)^{\frac{p}{p-\tau}}\right)^{\frac{p-\tau}{p}} \\ &\qquad\leq \left(\sum_{(I,\psi)\in\Lambda_j^0} \left(2^{-jm\tau}(k2^{-j})^{(a-m)\tau}\right)^{\frac{p}{p-\tau}}\right)^{\frac{p-\tau}{p}} \\ &\qquad\leq \left(\sum_{k=1}^{2^j}\sum_{(I,\psi)\in\Lambda_{j,k}} \left(2^{-ja\tau}k^{(a-m)\tau}\right)^{\frac{p}{p-\tau}}\right)^{\frac{p-\tau}{p}} \\ &\qquad\leq \left(c_3 2^{-ja\frac{p\tau}{p-\tau}}\sum_{k=1}^{2^j} k^{(a-m)\frac{p\tau}{p-\tau}}k^{d-1-\delta}2^{j\delta}\right)^{\frac{p-\tau}{p}} \\ &\qquad= c_4 2^{-ja\tau}2^{j\delta\frac{p-\tau}{p}}\left(\sum_{k=1}^{2^j} k^{(a-m)\frac{p\tau}{p-\tau}+d-1-\delta}\right)^{\frac{p-\tau}{p}}.\end{aligned}$$

Looking at the value of the exponent in the last sum we see that

$$(a-m)\frac{p\tau}{p-\tau} + d - 1 - \delta > -1 \qquad \Longleftrightarrow \qquad a - m + r\frac{d-\delta}{d} > 0,$$

which leads to

$$\begin{aligned}&\left(\sum_{(I,\psi)\in\Lambda_j^0} \left(|I|^{\frac{m}{d}\tau}\rho_I^{(a-m)\tau}\right)^{\frac{p}{p-\tau}}\right)^{\frac{p-\tau}{p}} \\ &\qquad\leq c_4 2^{-ja\tau}2^{j\delta\frac{p-\tau}{p}}\begin{cases} 2^{j\left((a-m)\tau+(d-\delta)\frac{p-\tau}{p}\right)}, & a-m+r\frac{d-\delta}{d} > 0, \\ (j+1)^{\frac{p-\tau}{p}}, & a-m+r\frac{d-\delta}{d} = 0, \\ 1, & a-m+r\frac{d-\delta}{d} < 0.\end{cases}\end{aligned}$$

<u>*Step 3:*</u> We now put $\Lambda^0 := \bigcup_{j\geq 0}\Lambda_j^0$. Summing the first line of the last estimate over all j, we obtain

$$\begin{aligned}&\sum_{(I,\psi)\in\Lambda^0} |I|^{\left(\frac{1}{p}-\frac{1}{2}\right)\tau}|\langle \tilde{u},\psi_I\rangle|^\tau \\ &\qquad\leq c_4\sum_{j=0}^\infty 2^{-j(m\tau - d\frac{p-\tau}{p})}\|u|\mathcal{K}^m_{p,a}(\Omega)\|^\tau \lesssim \|u|\mathcal{K}^m_{p,a}(\Omega)\|^\tau < \infty,\end{aligned}$$

if the geometric series converges, which happens if

$$m\tau > d\frac{p-\tau}{p} \quad \Longleftrightarrow \quad m > d\frac{r}{d} \quad \Longleftrightarrow \quad m > r.$$

Similarly, in the second case we see that

$$\sum_{(I,\psi)\in\Lambda^0} |I|^{\left(\frac{1}{p}-\frac{1}{2}\right)\tau} |\langle \tilde{u}, \psi_I\rangle|^\tau \\ \leq c_4 \sum_{j=0}^{\infty} 2^{-j(a\tau - \delta\frac{p-\tau}{p})} (j+1)^{\frac{p-\tau}{p}} \|u|\mathcal{K}^m_{p,a}(\Omega)\|^\tau \lesssim \|u|\mathcal{K}^m_{p,a}(\Omega)\|^\tau < \infty,$$

where the series converges if

$$a\tau > \delta\frac{p-\tau}{p}, \quad \text{i.e.,} \quad a > \delta\frac{r}{d}, \quad \text{i.e.,} \quad m > r\frac{d-\delta}{d} + \frac{\delta}{d}r = r, \quad \text{i.e.,} \quad m > r,$$

which is the same condition as before. Finally, in the third case we find

$$\sum_{(I,\psi)\in\Lambda^0} |I|^{\left(\frac{1}{p}-\frac{1}{2}\right)\tau} |\langle \tilde{u}, \psi_I\rangle|^\tau \\ \leq c_4 \sum_{j=0}^{\infty} 2^{-j(a\tau - \delta\frac{p-\tau}{p})} \|u|\mathcal{K}^m_{p,a}(\Omega)\|^\tau \lesssim \|u|\mathcal{K}^m_{p,a}(\Omega)\|^\tau < \infty,$$

whenever

$$a\tau > \delta\frac{p-\tau}{p} \quad \Longleftrightarrow \quad a > \delta\frac{r}{d}$$

as in the second case above.

Step 4: We need to consider the sets $\Lambda_{j,0}$, i.e., the wavelets close to l_0. Here, we shall make use of the assumption $\tilde{u} \in B^s_{p,p}(\mathbb{R}^d)$. Since the number of elements in $\Lambda_{j,0}$ is bounded from above by $c_7 2^{j\delta}$ we estimate using Hölder's inequality with $\frac{p}{\tau} > 1$ and obtain

$$\sum_{(I,\psi)\in\Lambda_{j,0}} |I|^{\left(\frac{1}{p}-\frac{1}{2}\right)\tau} |\langle \tilde{u}, \psi_I\rangle|^\tau \\ \leq c_7^{\frac{p-\tau}{p}} 2^{j\delta\frac{p-\tau}{p}} 2^{-jd\left(\frac{1}{p}-\frac{1}{2}\right)\tau} \left(\sum_{(I,\psi)\in\Lambda_{j,0}} |\langle \tilde{u}, \psi_I\rangle|^p\right)^{\tau/p} \\ = c_7^{\frac{p-\tau}{p}} 2^{j\delta\frac{p-\tau}{p}} 2^{-js\tau} \left(\sum_{(I,\psi)\in\Lambda_{j,0}} 2^{j\left(s+\frac{d}{2}-\frac{d}{p}\right)p} |\langle \tilde{u}, \psi_I\rangle|\right)^{\tau/p}.$$

Summing up over j and once more using Hölder's inequality with $\frac{p}{\tau} > 1$ gives

$$\begin{aligned} &\sum_{j=0}^{\infty} \sum_{(I,\psi)\in\Lambda_{j,0}} |I|^{\left(\frac{1}{p}-\frac{1}{2}\right)\tau} |\langle \tilde{u}, \psi_I\rangle|^\tau \\ \leq\ & c_7^{\frac{p-\tau}{p}} \sum_{j=0}^{\infty} 2^{j\delta\frac{p-\tau}{p}} 2^{-js\tau} \left(\sum_{(I,\psi)\in\Lambda_{j,0}} 2^{j\left(s+\frac{d}{2}-\frac{d}{p}\right)p} |\langle \tilde{u}, \psi_I\rangle|\right)^{\tau/p} \\ \leq\ & c_7^{\frac{p-\tau}{p}} \left(\sum_{j=0}^{\infty} 2^{j\delta} 2^{-js\tau\frac{p}{p-\tau}}\right)^{\frac{p-\tau}{p}} \cdot \left(\sum_{j=0}^{\infty} \sum_{(I,\psi)\in\Lambda_{j,0}} 2^{j\left(s+\frac{d}{2}-\frac{d}{p}\right)p} |\langle \tilde{u}, \psi_I\rangle|\right)^{\tau/p} \\ \lesssim\ & \|\tilde{u}|B^s_{p,p}(\mathbb{R}^d)\|^\tau \lesssim \|u|B^s_{p,p}(\Omega)\|^\tau, \end{aligned}$$

under the condition

$$\delta < \frac{sp\tau}{p-\tau} \quad \Longleftrightarrow \quad \frac{s}{\delta} > \frac{1}{\tau} - \frac{1}{p} = \frac{r}{d} \quad \Longleftrightarrow \quad r < \frac{sd}{\delta}.$$

Step 5: Finally, we need to consider those ψ_I whose support intersect $\partial\Omega$. In this case we can estimate similar as in Step 4 with δ replaced by $d-1$. This results in the condition

$$\sum_{(I,\psi)\in\Lambda:\ \mathrm{supp}\ \psi_I\cap\partial\Omega\neq\emptyset} |I|^{\left(\frac{1}{p}-\frac{1}{2}\right)\tau} |\langle \tilde{u}, \psi_I\rangle|^\tau \lesssim \|\tilde{u}|B^s_{p,p}(\mathbb{R}^d)\|^\tau \lesssim \|u|B^s_{p,p}(\Omega)\|^\tau$$

if $r < \frac{sd}{d-1}$. Altogether, we have proved

$$\|u|B^r_{\tau,\tau}(\Omega)\| \leq \|\tilde{u}|B^r_{\tau,\tau}(\mathbb{R}^d)\| \lesssim \|u|B^s_{p,p}(\Omega)\| + \|u|\mathcal{K}^m_{p,a}(\Omega)\|,$$

with constants independent of u. □

As an immediate consequence of Theorem 6.1 and the definition of corresponding function spaces on Ω_T we have the following generalized embedding result.

Theorem 6.2 *Let Ω_T be defined as above. Then for $1 < p < \infty$ and $0 < q \leq \infty$, we have*

$$L_q((0,T), \mathcal{K}^m_{p,a}(\Omega)) \cap L_q((0,T), B^s_{p,p}(\Omega)) \hookrightarrow L_q\left((0,T), B^r_{\tau,\tau}(\Omega)\right), \quad \frac{1}{\tau} = \frac{r}{d} + \frac{1}{p},$$

or all $0 \leq r < \min(m, \frac{sd}{d-1})$ and $m > a > \frac{\delta}{d} r$, where $\delta = d-2 = \dim(l_0)$.

Now Lemma 3.9 together with Theorem 6.2 give the following result concerning the Besov regularity of the solution to (3.13).

Theorem 6.3 (Hyperbolic Besov regularity) *Let $\Omega \subset \mathbb{R}^d$ be a special Lipschitz domain from Definition 2.2. Furthermore, let $m \in \mathbb{N}$, $m \geq 2$, $m-1 \leq a < \min(m, \frac{\pi}{\omega} - 1)$, and assume f satisfies*

(i) $\partial_{t^j} f \in L_\infty((0,T), \mathcal{K}^{m-j}_{2,a-j}(\Omega))$, $0 \leq j \leq m-1$,

(ii) $\partial_{t^j} f(x,0) = 0$, $0 \leq j \leq m-2$.

Then for the generalized solution u of problem (3.13) we have

$$u \in L_\infty((0,T), B^r_{\tau,\tau}(\Omega)), \qquad \frac{1}{\tau} = \frac{r}{d} + \frac{1}{p}, \qquad 0 \leq r < \min\left\{m, \frac{d}{d-1}\right\}.$$

In particular, the following a priori estimate holds

$$\|u|L_\infty((0,T), B^r_{\tau,\tau}(\Omega))\| \lesssim \sum_{j=0}^{k-1} \|\partial_{t^j} f|L_\infty((0,T), \mathcal{K}^{m-j}_{2,a}(\Omega))\|.$$

Proof: According to Lemma 3.9 we know $u \in L_\infty((0,T), W^1_2(\Omega)) \cap L_\infty((0,T), \mathcal{K}^m_{2,a}(\Omega))$ for $m-1 \leq a < \min(m, \frac{\pi}{\omega} - 1)$. Now using Theorem 6.2 with $s = 1$ and $\max(m-1, \frac{\delta}{d} r) < a < \min(m, \frac{\pi}{\omega} - 1)$ yields the desired embedding result. As for the restriction on a we further observe that

$$\frac{\delta}{d} r < \frac{d-2}{d} \min\left(m, \frac{d}{d-1}\right) = \frac{d-2}{d} \cdot \frac{d}{d-1} \leq 1,$$

using $m \geq 2$ and $d \geq 3$. Since $m-1 \geq 1 > \frac{\delta}{d} r$ we can use equality in the lower bound. Thus, the restriction on a reads as $m-1 \leq a < \min(m, \frac{\pi}{\omega} - 1)$. □

Remark 6.4 There are more results in [44] compared to what we used in this section. In particular, in [44, Thm. 3.5] weighted Sobolev regularity of nonlinear hyperbolic problems was investigated. Therefore, using a fixed point theorem as in subsection 4.2 it should be possible to study Besov regularity of nonlinear hyperbolic problems as well. But this is out of our scope for now and will possibly be treated in a forthcoming paper.

7 Relations to Adaptive Algorithms

In Section 1 we already sketched why we expect that the results proved in this paper will have some impact concerning the theoretical foundation of adaptive algorithms. In this section, we want to explain these relationships in more detail.
Let us start with adaptive wavelet algorithms as, e.g., discussed in [11, 49]. Let $\Psi = \{\psi_I : (I, \psi) \in \Lambda\}$ be a wavelet system with sufficiently high differentiability and vanishing moments, such that all relevant (unweighted) Sobolev and Besov spaces can be characterized in terms of expansion coefficients w.r.t. Ψ, see again Subsection 2.4. Then, the best thing we can expect from an adaptive numerical algorithm based on this wavelet basis is that it realizes the convergence order of best N-term wavelet approximation schemes. In this sense, best N-term wavelet approximation serves as the benchmark for the performance of adaptive algorithms. Let $\mathcal{O} \subset \mathbb{R}^d$ denote some bounded Lipschitz domain. The error of best N-term approximation is defined by

$$\sigma_N\big(u; L_p(\mathcal{O})\big) = \inf_{\Gamma\subset\Lambda:\#\Gamma\le N} \inf_{c_\gamma} \Bigg\| u - \sum_{\gamma=(I,\psi)\in\Gamma} c_\gamma \psi_I \Bigg| L_p(\mathcal{O}) \Bigg\| , \tag{7.1}$$

i.e., as the name suggests we consider the best approximation by linear combinations of the basis functions consisting of at most N terms. Of course, in the context of the numerical approximation of the solutions to operator equations, such an approximation scheme would never be implementable because this would require the knowledge of all wavelet coefficients, i.e., of the solution itself. Nevertheless, it has been shown that the recently developed adaptive wavelet algorithms indeed asyptotically realize the same order of approximation [11, 49]! To quantify the approximation rate, we introduce the approximation classes $\mathcal{A}^\alpha_q(X)$, $\alpha > 0$, $0 < q \le \infty$ by requiring

$$\|u|\mathcal{A}^\alpha_q(L_p(\mathcal{O}))\| = \left(\sum_{N=0}^{\infty} \Big((n+1)^\alpha \sigma_N\big(u; L_p(\mathcal{O})\big)\Big)^q \frac{1}{N+1} \right)^{1/q} < \infty . \tag{7.2}$$

Then a fundamental result of DeVore, Jawerth, and Popov [27] states that

$$\mathcal{A}^{m/d}_\tau(L_p(\mathbb{R}^d)) = B^m_{\tau,\tau}(\mathbb{R}^d) , \qquad \frac{1}{\tau} = \frac{m}{d} + \frac{1}{p} .$$

Consequently, the optimal approximation rate that can be achieved by adaptive wavelet schemes depends on the Besov smoothness of the unknown solution in the adaptivity scale (1.4). In contrast to this, the convergence order of classical nonadaptive (uniform) schemes depends on the Sobolev smoothness of the solution, see, e.g., [17, 32] for details. The results presented in this paper imply that for each $t \in (0, T)$ the Besov regularity of the unknown solutions of the problems studied here is much higher than the Sobolev regularity, which justifies the use of spatial adaptive wavelet algorithms. This corresponds to the classical time-marching schemes such as the Rothe method. We refer, e.g., to the monographs [41, 51] for a detailed discussion. Of course, it would be tempting to employ adaptive wavelet strategies in the whole space-time cylinder. First results in this direction have been reported in [50]. To justify also these schemes, Besov regularity in the whole space-time cylinder has to be established. This case will be studied in a forthcoming paper.
Quite recently, it has also turned out that the basic relationships outlined above also carry over to discretization schemes based on finite element schemes [29]. The starting point is an initial triangulation $\mathcal{T}_0$ of the polyhedral domain $D \subset \mathbb{R}^d$. Furthermore, $\mathbb{T}$ denotes the family of all conforming, shape-regular partitions $\mathcal{T}$ of D obtained from $\mathcal{T}_0$ by refinement using bisection rules. Moreover, $V_\mathcal{T}$ denotes the finite element space of continuous piecewise polynomials of degree at most r, i.e.,

$$V_\mathcal{T} = \big\{ v \in C(\overline{D}) : v|_T \in \mathcal{P}_r \text{ for all } T \in \mathcal{T} \big\} .$$

Then the counterpart to the quantity $\sigma_N(u)$ is given by

$$\sigma^{FE}_N\big(u; L_p(D)\big) = \min_{\substack{\mathcal{T}\in\mathbb{T}:\\ \#\mathcal{T}-\#\mathcal{T}_0\le N}} \inf_{v\in V_\mathcal{T}} \|u - v\|_{L_p(D)}, \quad 0 < p < \infty .$$

Then [29, Thm. 2.2] gives direct estimates,

$$\sigma_N^{FE}\big(u; L_p(D)\big) \leq C\, N^{-s/d} \|f|B^s_{\tau,\tau}(D)\|\,.$$

Therefore, the results presented in this paper also justify the use of adaptive time-marching schemes based on finite elements.

A Supplementary results

We provide some auxiliary information on results used throughout the paper.

A.1 Embeddings of generalized Hölder spaces

Remark A.1 Let Y be some (quasi-)Banach space such that $X \hookrightarrow Y$. Then it follows that

$$C^k(I,X) \hookrightarrow C^k(I,Y) \qquad \text{and} \qquad \mathcal{C}^{k,\alpha}(I,X) \hookrightarrow \mathcal{C}^{k,\alpha}(I,Y).$$

This is an immediate consequence of the definition of the spaces. Let $u : I \to X \in C^k(I,X)$ with Taylor expansion

$$u(t+h) = u(t) + u'(t)h + \frac{1}{2}u''(t)h^2 + \ldots + \frac{1}{k!}u^{(k)}(t)h^k + r_k(t,h).$$

Then also $u : I \to Y$ and we have

- $u^{(j)}(t)$ depends continuously on t for all $j = 0, \ldots, k$, since

$$|t - t_0| < \delta \quad \Longrightarrow \quad \|u^{(j)}(t) - u^{(j)}(t_0)|Y\| \leq \|u^{(j)}(t) - u^{(j)}(t_0)|X\| < \varepsilon,$$

- $\displaystyle\lim_{|h|\to 0} \frac{\|r_k(t,h)|Y\|}{|h|^k} \leq \lim_{|h|\to 0} \frac{\|r_k(t,h)|X\|}{|h|^k} = 0,$

from which we deduce that $u \in C^k(I,Y)$. Moreover, concerning the generalized Hölder spaces we now observe that

$$\begin{aligned}\|u|\mathcal{C}^{k,\alpha}(I,Y)\| &= \|u|C^k(I,Y)\| + \sup_{\substack{t,s\in I\\ t\neq s}} \frac{\|u(t)-u(s)|Y\|}{|t-s|^\alpha} \\ &\leq \|u|C^k(I,X)\| + \sup_{\substack{t,s\in I\\ t\neq s}} \frac{\|u(t)-u(s)|X\|}{|t-s|^\alpha} = \|u|\mathcal{C}^{k,\alpha}(I,X)\|,\end{aligned}$$

which completes the proof.

A.2 Fixed point theorems in intersections of Banach spaces

Theorem A.2 *Let V, W be Banach spaces satisfying $V \cap W \neq \emptyset$. Furthermore, let $V_0 := B_{r_V}(x_0, V) \subset V$ and $W_0 := B_{r_W}(x_0, W) \subset W$ be small balls with center x_0 and suitably chosen radii r_V and r_W in V and W, respectively. Assume that $T : V_0 \to V_0$ is a contraction mapping, i.e.,*

$$\|T(x) - T(y)|V\| \leq q_{V_0}\|x - y|V\| \quad \textit{for all} \quad x, y \in V_0, \quad q_{V_0} \in [0,1),$$

and the same holds with V and V_0 replaced by W and W_0. Then it follows that $T : S_0 \to S_0$, where $S_0 = B_{r_{V\cap W}}(x_0, V\cap W) \subset V_0 \cap W_0$ with $r_{V\cap W} := \min(r_V, r_W)$, is also a contraction mapping, i.e.,

$$\|T(x) - T(y)|V\cap W\| \leq q_{S_0}\|x - y|V\cap W\| \quad \textit{for all} \quad x, y \in S_0, \quad q_{S_0} \in [0,1).$$

P r o o f : Let $x \in B_{r_{V\cap W}}(x_0)$. Since

$$T(B_r(x_0,U)) \subset B_r(x_0,U) \quad \text{for all} \quad r \le r_U, \quad U \in \{V,W\},$$

we see that

$$\begin{aligned} \|T(x)-T(x_0)|V\cap W\| &= \max\left(\|T(x)-T(x_0)|V\|, \|T(x)-T(x_0)|W\|\right) \\ &\le \max(r_{V\cap W}, r_{V\cap W}) = r_{V\cap W}, \end{aligned}$$

thus, we see that $T(S_0)\subset S_0$. The contraction property follows from

$$\begin{aligned} \|T(x)-T(y)|V\cap W\| &= \max\left(\|T(x)-T(y)|V\|, \|T(x)-T(y)|W\|\right) \\ &\le \max\left(q_{V_0}\|x-y|V\|, q_{W_0}\|x-y|W\|\right) \\ &\le \max\left(q_{V_0}, q_{W_0}\right)\max\left(\|x-y|V\|, \|x-y|W\|\right) \\ &= q_{S_0}\|x-y|V\cap W\|, \end{aligned}$$

for some $q_{S_0} := \max\left(q_{V_0}, q_{W_0}\right) < 1$, which completes the proof. □

A.3 Generalized Sobolev embedding

Proof of Theorem 2.3: Note that the theorem of Meyers-Serrin extends to the spaces $W_p^m(I,X)$, cf. [40, Th. 4.11]. Hence, $C^\infty(I,X)$ is dense in $W_p^m(I,X)$. It is also shown in [40, Prop. 4.3] that in this case weak derivatives coincide with normal derivatives. Since $k=m-1$, using [7, Thm. 1.4.35] together with Bochner's Theorem and Hölder inequality gives for $u\in C^\infty(I,X)$,

$$\begin{aligned} \left\| u^{(k)}(t+h)-u^{(k)}(t)\right| X\| &= \left\|\int_t^{t+h} u^{(k+1)}(s)\mathrm{d}s|X\right\| \\ &\le \int_t^{t+h}\left\|u^{(m)}|X\right\|\mathrm{d}s \\ &\le h^{1/p'}\left(\int_t^{t+h}\left\|u^{(m)}|X\right\|^p\mathrm{d}s\right)^{1/p} \\ &\le h^\alpha\,\|u|\,W_p^m(I,X)\|. \end{aligned}$$

Hence, we see that id is a linear and bounded operator from the dense subset $C^\infty(I,X)$ of $W_p^m(I,X)$ into the Banach-space $\mathcal{C}^{k,\alpha}(I,X)$ admitting an extension $\widetilde{\mathrm{id}}$ onto $W_p^m(I,X)$ with equal norm. This completes the proof.

A.4 Different scales of weighted Sobolev spaces

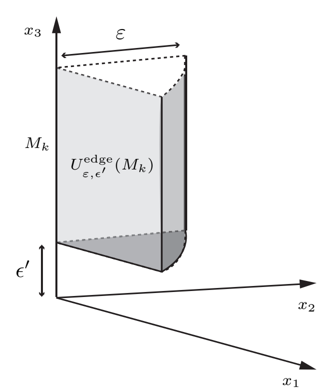
Proof of Lemma 2.5: We decompose the cone $K=K^{\text{reg}}\cup K^{\text{sing}}$ into a regular part $K^{\text{reg}}=K\setminus U_\varepsilon(S)$ not containing the vertex and the edges and a singular part $U_\varepsilon(S)$ corresponding to the union of different neighbourhoods of edges, vertices and vertex-edges as follows. This idea is taken from [5].

We start by considering the singular parts corresponding to edges. W.l.o.g. assume that $M_k=\{x=(x_1,x_2,x_3):\ x_1=x_2=0,\ 0\le x_3<\infty\}$. We define a neighborhood of the edge M_k in the following way

$$\begin{aligned} &U^{\text{edge}}_{\varepsilon,\epsilon'}(M_k) \\ &=\{x\in K: 0<r=\text{dist}(x,M_k)<\varepsilon,\ \epsilon'<x_3<\infty\}, \end{aligned}$$

which can be written in cylindrical coordinates as

$$U^{\text{edge}}_{\varepsilon,\epsilon'}(M_k)=\{(r,\theta,x_3):\ 0<r<\varepsilon,\ 0<\theta<\theta_k,\ \epsilon'<x_3<\infty\}.$$

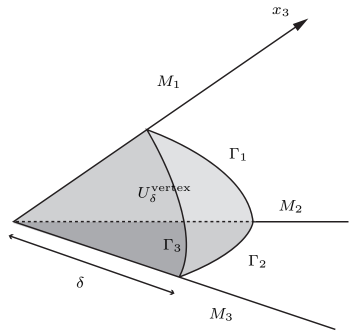


Here ε, ϵ' are chosen small enough such that $U^{\text{edge}}_{\varepsilon,\epsilon'}(M_k)$ has an empty intersection with all faces $\overline{\Gamma}_j$ non-adjacent to M_k (and this is true for all neighbourhoods of edges $U^{\text{edge}}_{\varepsilon,\epsilon'}(M_k)$, $k = 1,\ldots,n$). Moreover, put $U^{\text{edge}}_{\varepsilon,\epsilon'} = \bigcup_{k=1}^n U^{\text{edge}}_{\varepsilon,\epsilon'}(M_k)$.

Next we denote by U^{vertex}_δ a neighbourhood of the vertex $x = 0$,

$$U^{\text{vertex}}_\delta = \{x \in K : \ 0 < \rho_0(x) < \delta\}$$

and assume that $\delta < 1$ is small enough such that $U^{\text{vertex}}_\delta \cap \Omega = \emptyset$.

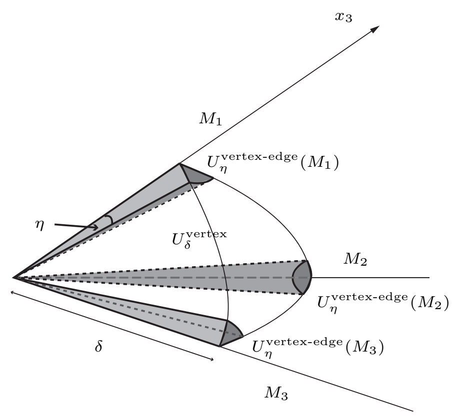


Moreover, by $U^{\text{vertex-edge}}_\eta(M_k)$ we denote the subset of U^{vertex}_δ, which belongs to a neighbourhood of the edge M_k, in particular,

$$U^{\text{vertex-edge}}_\eta(M_k) = \{x \in U^{\text{vertex}}_\delta : \ 0 < \varphi_k(x) < \eta\},$$

where $\varphi_k(x)$ is the angle between the edge M_k and the radial from 0 to the point x, assuming that η again is small enough such that $\overline{U}^{\text{vertex-edge}}_\eta(M_k) \cap \overline{U}^{\text{vertex-edge}}_\eta(M_j) = \{0\}$ for all $k \neq j$. We put

$$U^{\text{vertex-edge}}_\eta = \bigcup_{k=1}^n U^{\text{vertex-edge}}_\eta(M_k).$$



Choosing $\varepsilon < \min(\epsilon', \delta, \eta)$, we put

$$U_\varepsilon(S) := U^{\text{edge}}_{\varepsilon,\epsilon'} \cup \left(U^{\text{vertex}}_\delta \setminus U^{\text{vertex-edge}}_\eta\right) \cup U^{\text{vertex-edge}}_\eta.$$

This defines an ε-neighborhood of the singular set S in the sense that $\text{dist}(x, S) \geq \varepsilon$ for $x \in K^{\text{reg}}$. Now we prove (2.8) by distinguishing different cases when $x \in K$ belongs to the above neighbourhoods.

Case I: First assume that $x \in K^{\text{reg}}$. By the above definitions we have $r_k(x) = \sin\varphi_k(x) \cdot \rho_0(x)$ with $0 < c_\eta \leq \sin\varphi_k(x) \leq 1$, since $\varphi_k(x) \in [\eta, \pi - \eta]$ for $x \in K^{\text{reg}}$. This implies $\rho_0(x) \sim r_k(x)$ for all $k = 1,\ldots,n$ and we get

$$\rho_0(x) \prod_{k=1}^n \frac{r_k(x)}{\rho_0(x)} \sim \rho_0(x) \sim \min_{k=1,\ldots,n} r_k(x) \sim \varrho(x), \qquad \text{for all } x \in K^{\text{reg}}.$$

Case II: Let $x \in U^{\text{edge}}_{\varepsilon,\epsilon'}$, in particular, let x be in the neighourhood of the edge M_l, i.e., $x \in U^{\text{edge}}_{\varepsilon,\epsilon'}(M_l)$ for some $l \in \{1,\ldots,n\}$. Then $\varrho(x) = r_l(x)$ and $\rho_0(x) \sim r_k(x)$ for $k \neq l$ as in Case I. This yields

$$\varrho(x) \sim r_l(x) \sim r_l(x) \prod_{\substack{k=1,\\ k\neq l}}^n \frac{r_k(x)}{\rho_0(x)} \sim \rho_0(x) \prod_{k=1}^n \frac{r_k(x)}{\rho_0(x)}, \qquad \text{for all } x \in U^{\text{edge}}_{\varepsilon,\epsilon'}(M_l),$$

$l = 1,\ldots,n$, where the constants only depend on ε and the cone K.

Case III: Let $x \in U^{\text{vertex}}_\delta \setminus U^{\text{vertex-edge}}_\eta$. Then $r_k \sim \rho_0(x)$ for all $k = 1,\ldots,n$ as in Case I and $\varrho(x) \sim \rho_0(x)$, whenever $x \in U^{\text{vertex}}_\delta \setminus U^{\text{vertex-edge}}_\eta$. The estimate follows via

$$\varrho(x) \sim \rho_0(x) \sim \rho_0(x) \prod_{k=1}^n \frac{r_k(x)}{\rho_0(x)}, \qquad \text{for all } x \in U^{\text{vertex}}_\delta \setminus U^{\text{vertex-edge}}_\eta.$$

Case IV: Let $x \in U_\eta^{\text{vertex-edge}}$, w.l.o.g. assume $x \in U_\eta^{\text{vertex-edge}}(M_l)$ for some $l \in \{1, \dots, n\}$. Then for all $k \neq l$ we have $r_k(x) \sim \rho_0(x)$ as in Case I. Moreover, $r_l(x) = \sin\varphi_l(x)\rho_0(x) \leq c_\eta \rho_0(x)$ if $x \in U_\eta^{\text{vertex-edge}}$, thus $\varrho(x) \sim r_l(x)$. The result now follows from

$$\varrho(x) = r_l(x) \sim r_l(x) \prod_{\substack{k=1,\\ k\neq l}}^{n} \frac{r_k(x)}{\rho_0(x)} \sim \rho_0(x) \prod_{k=1}^{n} \frac{r_k(x)}{\rho_0(x)}.$$

This completes the proof.

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Stephan Dahlke
Philipps-Universität Marburg
FB12 Mathematik und Informatik
Hans-Meerwein Straße, Lahnberge
35032 Marburg
Germany
E-mail: dahlke@mathematik.uni-marburg.de
Web: http://www.mathematik.uni-marburg.de/∼dahlke/

Cornelia Schneider
Friedrich-Alexander-Universität Erlangen-Nürnberg
Department Mathematik, AM3
Cauerstr. 11
91058 Erlangen
Germany
E-mail: schneider@math.fau.de
Web: http://www.math.fau.de/∼schneider/