Compactly Supported Multivariate Wavelet Frames Obtained by Convolution

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May 10, 2005

Abstract

In this paper, we construct pairs of compactly supported dual wavelet frames for general dilations. We obtain smooth wavelets with small supports by convolution. Our approach leads to wavelets that have a high number of vanishing moments. Moreover, the corresponding refinable function is interpolating. The wavelets for the quincunx dilation matrix in our examples additionally have very high symmetries.

AMS subject classification: 65T60, 42C40, 42C15, 41A63

Key words: wavelet frames, vanishing moments, interpolating refinable functions, multiresolution analysis.

1 Introduction

Wavelet analysis and fast wavelet algorithms are widely used in applied mathematics. Wavelet shrinkage and thresholding is applied in image and signal analysis as well as wavelets are used for the numerical treatment of operator equations. In general, the construction of compactly supported wavelets is based on the construction of a refinable function $\phi$. That means that $\phi$ satisfies the refinement equation

$$\phi(\cdot) = \sum_{k \in \mathbb{Z}^d} a_k \phi(M \cdot -k)$$

where $M$ is an expanding integer matrix and $(a_k)_{k \in \mathbb{Z}^d}$ is a finite sequence. Then the wavelets are defined by finite linear combinations of $\phi(M \cdot -k)$ for $k \in \mathbb{Z}^d$. In many applications one is
interested in refinable functions \( \phi \) that in addition are \textbf{interpolating}. That means that \( \phi \) is continuous and satisfies
\[
\phi(k) = \delta_{0,k} \quad \text{for all } k \in \mathbb{Z}^d.
\]
Interpolating refinable functions are, for example, used in subdivision schemes for computer aided geometric design. The smoothness of the wavelets is another property that is often desired. The characterization of smoothness spaces, for instance, requires smooth wavelets. Furthermore, the smoothness leads to high approximation orders, see [12, 13, 30] for details. Thus, we are interested in the construction of smooth refinable functions so that the construction by finite linear combinations leads to wavelets that have the same smoothness as the refinable function.

Let us say a wavelet \( \psi \) has \( L \) \textbf{vanishing moments} if
\[
\int_{\mathbb{R}^d} x^\alpha \psi(x) dx = 0 \quad \text{for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| < L.
\]

It can be shown that vanishing moments influence compression rates, see for example [2, 14, 16]. Furthermore, good localization of the wavelets is very important in many applications. Therefore, we are especially interested in wavelets with small supports. Moreover, symmetric wavelets may lead to better results, as for instance in image and signal analysis. For the application of symmetric wavelets in image analysis, see for example [31].

In this paper, we construct wavelets that have all of the above mentioned properties at the same time. Thus, they can even be applied in several application areas.

Arbitrary smooth, compactly supported orthogonal wavelet bases in one dimension were constructed by I. Daubechies, see [14] for details. But the construction of smooth compactly supported orthogonal wavelets already seems to be very complicated in \( \mathbb{R}^2 \) for the quincunx dilation matrix. In [1], arbitrary smooth orthogonal wavelets for \( \mathbb{R}^2 \) have been constructed. However, the quincunx matrix is not treated there and neither it is in [22]. In [9, 28], orthogonal wavelets for the quincunx matrix are constructed, but differentiability is not achieved. The wavelets in [4, 18] do not have compact support. It should be mentioned that already in dimension one it is not possible to construct continuous orthogonal wavelets that have compact support and whose refinable function is interpolating. Moreover, there can not be constructed orthogonal wavelets with compact support that satisfy certain symmetry conditions. In recent years, smooth orthogonal multi–wavelets whose refinable function vector is interpolating have successfully been constructed in one dimension, see [26] for example.

A general construction method for arbitrary smooth orthogonal wavelets with compact support for arbitrary expanding integer matrices is still not known. There are general construction principles for biorthogonal wavelets, see [23] for example. However, to obtain smoother biorthogonal wavelets, their support increases dramatically. This seems to be problematic in many applications. One can overcome these difficulties with the concept of wavelet frames. By skipping the geometrical orthogonality or biorthogonality conditions, we obtain more freedom for the construction of the wavelets. In contrast to bases, frames do not necessarily lead to unique expansions. On the one hand, these redundancies increase the amount of data for expansions, but on the other hand, for instance, the redundancy can be used for denoising strategies or pattern recognition.
In the last years, there have been several publications for the construction of symmetric tight wavelet frames in one dimension, see for example [7, 16]. In [19, 32], multivariate tight wavelet frames are constructed by convolving a given tight wavelet frame. The number of vanishing moments does not increase for all wavelets by this approach. Therefore, to obtain a tight wavelet frame with a high number of vanishing moments, one has to start with a tight wavelet frame that already has this property. In [20], it has been shown that one can obtain arbitrary smooth tight wavelet frames in any dimension. Starting with one dimensional wavelet frames, multivariate wavelet frames are obtained by tensor products and a linear transform. This approach leads to tight wavelet frames with a high number of vanishing moments. Unfortunately, one obtains separable wavelets for the box-spline dilation matrix. In case of the quincunx dilation matrix, high smoothness leads to a large number of wavelets or to large supports. This approach does not lead to symmetric wavelets in general. In [15], one-dimensional pairs of compactly supported dual wavelet frames have been constructed. The wavelets have a high number of vanishing moments and they are symmetric.

In this paper, we construct multivariate pairs of dual wavelet frames. We start with symbols that satisfy the necessary conditions for the construction of biorthogonal wavelet bases. It is worth mentioning that we do not assume any smoothness of the corresponding starting wavelets. They can be tempered distributions that are not contained in $L_2(\mathbb{R}^d)$. Therefore, we are able to start with filters that have small supports. Finally, we obtain a pair of compactly supported dual wavelet frames by convolution. The corresponding refinable function is interpolating and we achieve high regularity as well as a high number of vanishing moments for all wavelets with small supports. The number of wavelets is $m^2 - 1$ where $m$ is the modulus of the determinant of the dilation matrix. We are very close to tight wavelet frames because primal and dual functions coincide. The only difference is the labelling of primal and dual wavelets. In our examples, the whole construction is applied to the case of the quincunx dilation matrix. We additionally obtain very symmetric wavelets. As far as we know, no existing approach leads to all these properties.

The outline of this paper is as follows: in Section 2, we introduce wavelet frames and the multiresolution analysis. The mixed extension principle (MEP) is presented for the construction of pairs of dual wavelet frames, see [33]. In Section 3, we state our construction principle that is based on the MEP. There we multiply a given family of starting symbols by itself. Moreover, we present a construction method for these starting symbols, see [23], and show that the corresponding wavelets have a high number of vanishing moments. In Section 4, we apply our construction by using the starting symbols in Section 3. Finally, we give two concrete examples for the quincunx dilation matrix.

2 General Setting

In this section, we introduce wavelet frames, the multiresolution analysis and the refinement equation. Furthermore, we present the MEP in [33], that we apply for the construction of pairs of dual wavelet frames in the following sections.
2.1 Wavelet Frames

A wavelet basis consists of a finite number of functions whose dilations and translations form a basis of $L_2(\mathbb{R}^d)$. Let us specify: an integer matrix $M$ is called expanding, if all its eigenvalues are larger than one in modulus. Let $M$ be an expanding integer matrix throughout the paper. We call it the dilation matrix and set $m := |\det(M)|$. Let $\Gamma_M$ be a complete set of representatives of $M^{t-1}\mathbb{Z}^d/\mathbb{Z}^d$ with $0\in\Gamma_M$ and $\Gamma_M^*$ of $\mathbb{Z}^d/M\mathbb{Z}^d$, respectively, with $0\in\Gamma_M^*$.

For a function $\psi \in L_2(\mathbb{R}^d)$ and $j \in \mathbb{Z}, k \in \mathbb{Z}^d$ we set

$$\psi_{j,k}(\cdot) := m^j \psi(M^j \cdot -k) .$$

Moreover, we say that $\{\psi^1, \ldots, \psi^r\} \subset L_2(\mathbb{R}^d)$ generates a wavelet basis if

$$\{\psi_{j,k}^\mu | 1 \leq \mu \leq r, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$$

is a basis of $L_2(\mathbb{R}^d)$. The concept of orthogonal or biorthogonal wavelet bases seems to be very restrictiv for the construction. Thus, in the following, we weaken the classical basis concept to obtain more freedom in the construction methods.

Let $I$ be a countable index set. A set $\{f_i | i \in I\}$ in a Hilbert space $H$ is called a frame in $H$ if there exist two positive constants $A$ and $B$ so that

$$A\|f\|_H^2 \leq \sum_{i \in I} |\langle f, f_i \rangle_H|^2 \leq B\|f\|_H^2$$

for all $f \in H$.

If in addition $A = B = 1$, we call $\{f_i | i \in I\}$ a (normalized) tight frame. In this case, every $f \in H$ has the expansion

$$f = \sum_{i \in I} \langle f, f_i \rangle_H f_i .$$

To obtain expansions for the case $A \neq B$, we need the concept of dual frames. The two sets $\{f_i | i \in I\}$ and $\{\tilde{f}_i | i \in I\}$ in $H$ are called a pair of dual frames if both are frames in $H$ and the expansion

$$f = \sum_{i \in I} \langle f, \tilde{f}_i \rangle_H \tilde{f}_i$$

is valid for every $f \in H$.

At this point, let $\psi^\mu, \tilde{\psi}^\mu \in L_2(\mathbb{R}^d)$ for $1 \leq \mu \leq r$. We say that

$$\{(\psi^\mu, \tilde{\psi}^\mu) | 1 \leq \mu \leq r\}$$

generates a pair of dual wavelet frames if $\{\psi_{j,k}^\mu | 1 \leq \mu \leq r, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ and $\{\tilde{\psi}_{j,k}^\mu | 1 \leq \mu \leq r, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ are a pair of dual frames in the Hilbert space $L_2(\mathbb{R}^d)$. In this case by applying (2), one has the wavelet expansion

$$f = \sum_{\mu=1}^r \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\psi}_{j,k}^\mu \rangle_{L_2(\mathbb{R}^d)} \psi_{j,k}^\mu$$

for all $f \in L_2(\mathbb{R}^d)$. We call $\{\psi^\mu | 1 \leq \mu \leq r\}$ the set of primal and $\{\tilde{\psi}^\mu | 1 \leq \mu \leq r\}$ the set of dual wavelets. Notation (3) is used because the labelling of primal and dual wavelets is very important in our construction methods in the following sections.
2.2 Multiresolution Analysis and the Mixed Extension Principle

At this point, we introduce the concept of multiresolution analysis, followed by a presentation of the MEP for the construction of wavelet frames.

An increasing sequence \((V_j)_{j \in \mathbb{Z}}\) of closed subspaces in \(L_2(\mathbb{R}^d)\) is called a multiresolution analysis if the following holds:

(i) \(f(\cdot) \in V_j\) if and only if \(f(M^{-j}\cdot) \in V_0\).

(ii) \(\bigcap_{j \in \mathbb{Z}} V_j = \{0\}\).

(iii) \(\bigcup_{j \in \mathbb{Z}} V_j = L_2(\mathbb{R}^d)\).

(iv) There exists a function \(\phi \in V_0\) so that \(V_0\) is the closed linear span of its \(\mathbb{Z}^d\) translates.

For a given multiresolution analysis we say that the function \(\phi\) in (iv) generates it. By (i) and (iv) the sequence \((V_j)_{j \in \mathbb{Z}}\) is determined by \(\phi\).

We will use the multiresolution analysis for the construction of wavelet frames. For \(j \in \mathbb{Z}\) and \(1 \leq \mu \leq r\), our aim is to find a subspace \(W_0^\mu\) in \(L_2(\mathbb{R}^d)\), that is the closed linear span of the \(\mathbb{Z}^d\) translates of a function \(\psi^\mu\), so that

\[
V_1 = V_0 + \sum_{\mu=1}^r W_0^\mu
\]

is valid and that \(\{\psi_{j,k}^\mu \mid 1 \leq \mu \leq r, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}\) is a wavelet frame. Furthermore, defining the space \(W_j^\mu\) by

\[
f(\cdot) \in W_j^\mu \quad \text{if and only if} \quad f(M^{-j}\cdot) \in W_0^\mu
\]

for \(j \in \mathbb{Z}\) and \(1 \leq \mu \leq r\) leads to the not necessarily direct decompositions

\[
V_{j+1} = V_j + \sum_{\mu=1}^r W_j^\mu.
\]

Let us have a closer look at the construction of a multiresolution analysis. Very often, one requires that the generator \(\phi\) has stable integer shifts. That means that there are positive constants \(A\) and \(B\) so that

\[
A\|c\|_{\ell_2(\mathbb{Z}^d)} \leq \| \sum_{k \in \mathbb{Z}^d} c_k \phi(\cdot - k) \|_{L_2(\mathbb{R}^d)} \leq B\|c\|_{\ell_2(\mathbb{Z}^d)} \quad \text{for all} \quad c = (c_k)_{k \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d).
\]

By using \(\phi \in V_0 \subset V_1\), the stability and (i), one can show that there exists a sequence \((a_k)_{k \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d)\) so that

\[
\phi(\cdot) = \sum_{k \in \mathbb{Z}^d} a_k \phi(M \cdot -k).
\]
Eq. (4) is called \textbf{refinement equation}. We will use the refinement equation as a starting point for the construction of a multiresolution analysis and wavelet frames. Some preparations are necessary. We apply the Fourier transform of a function $f \in L_1(\mathbb{R}^d)$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-\xi(x)}dx \quad \text{for} \quad \xi \in \mathbb{R}^d$$

where

$$e_w(\cdot) := e^{2\pi i \langle w | \cdot \rangle} \quad \text{for} \quad w \in \mathbb{R}^d.$$

Let us assume that $(a_k)_{k \in \mathbb{Z}^d}$ is a finite sequence. By applying the Fourier transform to both sides of the refinement equation (4), one obtains

$$\hat{\phi}(\cdot) = a(M^{-1} \cdot) \hat{\phi}(M^{-1} \cdot)$$

where

$$a := \frac{1}{m} \sum_{k \in \mathbb{Z}^d} a_k e^{-k}.$$

We call the trigonometric polynomial $a$ a \textbf{symbol} and the finite sequence $(a_k)_{k \in \mathbb{Z}^d}$ the \textbf{mask} or the \textbf{filter}. Let $a$ be a symbol with $a(0) = 1$ and let $\hat{\phi}$ be continuous in $0$ with $\hat{\phi}(0) = 1$. The iteration of (5) leads to

$$\hat{\phi}(\cdot) = \prod_{k=1}^{\infty} a(M^{-k} \cdot).$$

It can be shown that the infinite product converges uniformly on compact sets. Therefore, $\hat{\phi}$ is holomorphic. Furthermore, it has been shown that the function $\hat{\phi}$ given by (6) is polynomially bounded. Thus, by applying the Paley Wiener Theorem, $\phi$ is a tempered distribution with compact support, see [14] for details.

In the following, we want to use the refinement equation (4) also for tempered distributions. Therefore, we give a short overview of operations on tempered distributions, see [34] for details.

Let us denote the space of all Schwartz functions by $\mathcal{S}(\mathbb{R}^d)$. For a tempered distribution $g$ and a Schwartz function $\eta$, we denote

$$\langle g | \eta \rangle := g(\eta).$$

For $k \in \mathbb{R}^d$, the translation $g(\cdot - k)$ is defined by

$$\langle g(\cdot - k) | \eta \rangle := \langle g | \eta(\cdot + k) \rangle \quad \text{for all} \quad \eta \in \mathcal{S}(\mathbb{R}^d)$$

and the dilation $g(S^{-1} \cdot)$ for $S \in \mathbb{R}^{d \times d}$ by

$$\langle g(S^{-1} \cdot) | \eta \rangle := |\det(S)| \langle g | \eta(S \cdot) \rangle \quad \text{for all} \quad \eta \in \mathcal{S}(\mathbb{R}^d).$$

The Fourier transform $\hat{g}$ is applied by

$$\langle \hat{g} | \eta \rangle := \langle g | \hat{\eta} \rangle \quad \text{for all} \quad \eta \in \mathcal{S}(\mathbb{R}^d).$$
For a function or a distribution $h$, we denote $h^-(\cdot) := h(-\cdot)$. The convolution of a Schwartz function $f$ with the tempered distribution $g$ is defined by

$$\langle f * g | \eta \rangle := \langle g | \eta * f^\gamma \rangle$$

for all $\eta \in \mathcal{S}(\mathbb{R}^d)$.

Let $h$ be a tempered distribution so that $\eta * h \in \mathcal{S}(\mathbb{R}^d)$ for all $\eta \in \mathcal{S}(\mathbb{R}^d)$. Then, we can define $h * g$ by

$$\langle h * g | \eta \rangle := \langle g | \eta * h^\gamma \rangle$$

for all $\eta \in \mathcal{S}(\mathbb{R}^d)$.

Now, for a symbol $a$, we can read the refinement equation (4) more general in terms of tempered distributions. We call a tempered distribution $\phi$ refinable with respect to the trigonometric polynomial $a$ if it satisfies the refinement equation with corresponding mask $(a_k)_{k \in \mathbb{Z}^d}$. Let $a$ be a symbol with $a(0) = 1$. It has been shown in [5] that there exists a unique tempered distribution $\phi$ with compact support and $\hat{\phi}(0) = 1$ that is refinable with respect to $a$. It is given by (6). In the sequel, the term refinable distribution (or refinable function in case $\phi \in L^2(\mathbb{R}^d)$) corresponding to a symbol $a$ with $a(0) = 1$ always refers to this unique solution in (6).

**Remark 2.1.** Throughout the paper, symbols are trigonometric polynomials. Therefore, their masks are finite sequences. Thus, refinable distributions always have compact support in this paper. It should be mentioned that the term symbol is often used in a more general way in literature.

Let us have a closer look at interpolating refinable functions. We say a symbol $a$ satisfies the **interpolation condition** if

$$\sum_{\gamma \in \Gamma_M} a(\cdot + \gamma) = 1. \quad (7)$$

The following theorem has been shown in [29].

**Theorem 2.2.** A continuous refinable function $\phi$ corresponding to the symbol $a$ with $a(0) = 1$ is interpolating if and only if both $\phi$ has stable integer shifts and $a$ satisfies the interpolation condition.

Moreover, the stability of $\phi$ can be checked by the zeros of the symbol $a$, see [8] for details.

Let us mention a neccessary property for smooth refinable functions. We say the refinable function $\phi$ satisfies the **Strang Fix conditions** of order $L$ if

$$\partial^\alpha \hat{\phi}(k) = 0 \quad \text{for all} \quad k \in \mathbb{Z}^d \setminus \{0\} \quad \text{and} \quad \alpha \in \mathbb{N}^d, |\alpha| < L.$$ 

With increasing smoothness, the refinable functions $\phi$ satisfies a higher order of Strang Fix conditions. On the other hand, the order of the Strang Fix conditions is connected to the reproduction of polynomials of the corresponding shift invariant space. Thus, it influences approximation rates, see [24] for details.

Now, let us come to the construction of wavelet frames. Given symbols $a^0, b^0, \ldots, a^r, b^r$, we say that $\{(a^\mu, b^\mu) \mid 0 \leq \mu \leq r\}$ satisfies the **perfect reconstruction conditions** (PR conditions) if

$$\sum_{\mu=0}^{r} a^\mu(\cdot) b^\mu(\cdot + \gamma) = \delta_{0,\gamma} \quad \text{for all} \quad \gamma \in \Gamma_M. \quad (8)$$
Furthermore, we say that \( \{(a^\mu, b^\mu) \mid 0 \leq \mu \leq r\} \) satisfies condition (I) if the following conditions hold:

(a) \( a^0(0) = b^0(0) = 1 \).
(b) \( a^\mu(0) = b^\mu(0) = 0 \) for all \( 1 \leq \mu \leq r \).
(c) \( \{(a^\mu, b^\mu) \mid 0 \leq \mu \leq r\} \) satisfies the PR conditions.

The next theorem has been shown in [33]. With the results in [16] and [20], we can state it in the following way.

**Theorem 2.3 (MEP).** Let the symbol family \( \{(a^\mu, b^\mu) \mid 0 \leq \mu \leq r\} \) satisfy condition (I) and let \( \phi \) be refinable with respect to \( a^0 \) and \( \tilde{\phi} \) be refinable with respect to \( b^0 \). In addition, let \( \phi \) and \( \tilde{\phi} \) be contained in \( L^2(\mathbb{R}^d) \). We define for \( \mu = 1, \ldots, r \)

\[
\psi^\mu(\cdot) := \sum_{k \in \mathbb{Z}^d} a^\mu_k \phi(M \cdot -k) \quad \text{and} \quad \tilde{\psi}^\mu(\cdot) := \sum_{k \in \mathbb{Z}^d} b^\mu_k \tilde{\phi}(M \cdot -k) \quad (9)
\]

Then, the set

\( \{(\psi^\mu, \tilde{\psi}^\mu) \mid 1 \leq \mu \leq r\} \)

generates a pair of dual wavelet frames.

In Theorem 2.3, we call \( \phi \) the **primal** and \( \tilde{\phi} \) the **dual** refinable function. By applying the Fourier transform to (9), we obtain the equivalent equations

\[
\hat{\psi}^\mu(\cdot) = a^\mu(M^{t-1} \cdot) \hat{\phi}(M^{t-1} \cdot) \quad \text{and} \quad \hat{\tilde{\psi}}^\mu(\cdot) = b^\mu(M^{t-1} \cdot) \hat{\tilde{\phi}}(M^{t-1} \cdot). \quad (10)
\]

Under the notation and assumptions of Theorem 2.3, the refinable functions \( \phi \) and \( \tilde{\phi} \) generate two multiresolution analyses \( (V_j)_{j \in \mathbb{Z}} \) and \( (\tilde{V}_j)_{j \in \mathbb{Z}} \), respectively, see [6] for details. Furthermore, we define for \( 1 \leq \mu \leq r \) and \( j \in \mathbb{Z} \)

\[
W^\mu_j := \text{span}\{\psi^\mu_{j,k} \mid k \in \mathbb{Z}^d\} \quad \text{and} \quad \tilde{W}^\mu_j := \text{span}\{\tilde{\psi}^\mu_{j,k} \mid k \in \mathbb{Z}^d\}. \quad (11)
\]

By applying the PR conditions, one obtains

\[
V_{j+1} = V_j + \sum_{\mu=1}^r W^\mu_j \quad \text{and} \quad \tilde{V}_{j+1} = \tilde{V}_j + \sum_{\mu=1}^r \tilde{W}^\mu_j,
\]

see [6] for details. In our setting, these decompositions are not necessarily direct.

### 3 Compactly Supported Wavelet Frames

First, we present our method that leads to a pair of dual wavelet frames. We multiply a given family of starting symbols by itself. Next, we describe the construction in [23] that can be applied for the construction of our starting symbols. Furthermore, for a given symbol that satisfies the interpolation condition (7), a characterization of all dual symbols is presented. Following, we show that the starting wavelet symbols that can be constructed by the method in [23] have a high number of vanishing moments.
3.1 Pairs of Dual Wavelet Frames

Our construction of pairs of dual wavelet frames is based on the multiplication of a given family of symbols by itself. Thus, this approach leads to wavelets that are convolutions of tempered distributions. First, we need the following lemma.

**Lemma 3.1.** Let the symbol family \( \{(a^\mu, b^\mu) \mid 0 \leq \mu \leq r\} \) satisfy the PR conditions (8) and let another symbol family \( \{(c^\nu, d^\nu) \mid 0 \leq \nu \leq s\} \) satisfy
\[
\sum_{\nu=0}^{s} c^\nu(\cdot) d^\nu(\cdot) = 1 .
\] (12)

(a) The family
\[
\{(a^\mu c^\nu, d^\nu b^\mu) \mid 0 \leq \mu \leq r \text{ and } 0 \leq \nu \leq s\}
\]
also satisfies the PR conditions.

(b) For \( 0 \leq \nu \leq s \) and \( 1 \leq \mu \leq r \), we define
\[
\tilde{a}^\nu(\cdot) := a^0(\cdot) c^\nu(M^\gamma \cdot) \quad b^\nu(\cdot) := b^m(M^\gamma \cdot) =: \tilde{b}^\nu(\cdot).
\]

The symbol family
\[
\{\tilde{(a^\mu, b^\mu)} \mid 0 \leq \mu \leq r + s\}
\]
satisfies the PR conditions.

**Proof.** (a) Let \( \gamma \in \Gamma_M \). By applying the PR conditions (8) and Eq. (12), we obtain
\[
\sum_{\mu, \nu} a^\mu(\cdot) c^\nu(\cdot) d^\nu(\cdot + \gamma) = \sum_{\nu} c^\nu(\cdot) d^\nu(\cdot + \gamma) \sum_{\mu} a^\mu(\cdot) b^\nu(\cdot + \gamma)
\]
\[=
\sum_{\nu} c^\nu(\cdot) d^\nu(\cdot + \gamma) \delta_{0, \gamma}
\]
\[=
\delta_{0, \gamma}.
\]

(b) For \( \gamma \in \Gamma_M \), we have \( d^\nu(M^\gamma \cdot + \gamma) = d^\nu(M^\gamma \cdot) \) because \( d^\nu \) is \( \mathbb{Z}^d \) periodic and \( M^\gamma \) is contained in \( \mathbb{Z}^d \). This leads to
\[
\sum_{\mu=0}^{r+s} \tilde{a}^\mu(\cdot) \tilde{b}^\mu(\cdot + \gamma) = \sum_{\nu=0}^{s} a^0(\cdot) c^\nu(M^\gamma \cdot) b^\nu(\cdot + \gamma) d^\nu(M^\gamma \cdot + \gamma) + \sum_{\mu=1}^{r} a^\mu(\cdot) b^\nu(\cdot + \gamma)
\]
\[=
\sum_{\nu=0}^{s} a^0(\cdot) c^\nu(M^\gamma \cdot) d^\nu(M^\gamma \cdot) + \sum_{\mu=1}^{r} a^\mu(\cdot) b^\nu(\cdot + \gamma)
\]
By applying (12) and the PR conditions, we obtain
\[
\sum_{\mu=0}^{r+s} \tilde{a}^\mu(\cdot) \tilde{b}^\mu(\cdot + \gamma) = a^0(\cdot) b^0(\cdot + \gamma) + \sum_{\mu=1}^{r} a^\mu(\cdot) b^\mu(\cdot + \gamma)
\]
\[=
\delta_{0, \gamma} .
\]
Now, let us state our theorem for the construction of pairs of dual wavelet frames that is based on the MEP 2.3.

**Theorem 3.2.** Let the symbol family \( \{(a^\mu, b^\mu) \mid 0 \leq \mu \leq r\} \) satisfy condition (I) and let \( \phi \ast \tilde{\phi} \) be contained in \( L_2(\mathbb{R}^d) \).

(a) The family \( \{(a^\mu b^\nu, a^\nu b^\mu) \mid 0 \leq \mu, \nu \leq r\} \) satisfies the PR conditions.

(b) With the notation in (9) and \( \psi^0 = \phi \) and \( \tilde{\psi}^0 = \tilde{\phi} \), the wavelet family
\[
\{(\psi^\mu \ast \tilde{\psi}^\nu, \psi^\nu \ast \tilde{\psi}^\mu) \mid 0 \leq \mu, \nu \leq r, \; \mu + \nu > 0\} \tag{13}
\]
generates a pair of dual wavelet frames.

(c) For \( 0 \leq \mu, \nu \leq r \), we define
\[
W^{\mu,\nu}_j := \text{span} \left\{ (\psi^\mu \ast \tilde{\psi}^\nu)_{j,k} \mid k \in \mathbb{Z}^d \right\},
\]
\[
\tilde{W}^{\mu,\nu}_j := \text{span} \left\{ (\psi^\nu \ast \tilde{\psi}^\mu)_{j,k} \mid k \in \mathbb{Z}^d \right\}
\]
and we obtain
\[
W^{\mu,\nu}_j = \tilde{W}^{\nu,\mu}_j
\]
for all \( 0 \leq \mu, \nu \leq r \) and \( j \in \mathbb{Z} \). Furthermore, \( \phi \ast \tilde{\phi} \) generates the multiresolution analysis \( (W^{0,0}_j)_{j \in \mathbb{Z}} = (\tilde{W}^{0,0}_j)_{j \in \mathbb{Z}} \) and we have the following decompositions for \( j \in \mathbb{Z} \)
\[
W^{0,0}_{j+1} = \sum_{0 \leq \mu, \nu \leq r} W^{\mu,\nu}_j.
\]

It is worth mentioning that we do not have to assume \( \phi, \tilde{\phi} \in L_2(\mathbb{R}^d) \) in Theorem 3.2. Therefore, we can start with small masks \( (a^\mu_k)_{k \in \mathbb{Z}^d} \) and \( (b^\mu_k)_{k \in \mathbb{Z}^d} \). The constructed primal and dual functions coincide. They are only differently labelled. Thus, we are very close to the case of tight wavelet frames. Furthermore, the number of vanishing moments of the starting symbols is preserved.

Let us prove Theorem 3.2

**Proof.** (a) This part follows by applying part (a) of Lemma 3.1 with \( s := r \) and
\[
(c', d') := (b', a') \quad \text{for all } 0 \leq \nu \leq r.
\]

(b) By applying (10), we obtain
\[
\left( \tilde{\psi}^\mu \tilde{\psi}^\nu \right)(\cdot) = a^\mu (M^{t-1}) b^\nu (M^{t-1}) \tilde{\phi}(M^{t-1}) \tilde{\phi}(M^{t-1}) \cdot \tag{14}
\]
Therefore, \( \phi \ast \tilde{\phi} \) is the primal and dual refinable function with corresponding symbol \( a^0 b^0 \).

We have \( (a^\mu b^\nu)(0) = 1 \) and \( (a^\mu b^\nu)(0) = 0 \) for all \( 0 \leq \mu, \nu \leq r \) with \( \mu + \nu > 0 \). Further, we have assumed that \( \phi \ast \tilde{\phi} \in L_2(\mathbb{R}^d) \). With (a) and (14), part (b) follows by applying Theorem 2.3.
(c) The refinable function $\phi \ast \tilde{\phi}$ has compact support and we know $(\hat{\phi} \hat{\tilde{\phi}})(0) = 1$. In [3], it has been shown that such a refinable function generates a multiresolution analysis. The PR conditions imply the decompositions, see [6] for details.

Let us have a look at the constructed wavelet spaces. The following theorem shows us the structure of $W^{\mu,\nu}_j$. We also apply notation (1) for tempered distributions.

**Theorem 3.3.** Under the notation of Theorem 3.2, let the symbol family $\{(a^\mu, b^\mu) \mid 0 \leq \mu \leq r\}$ satisfy condition (I) and let $\phi \ast \tilde{\phi}$ be contained in $L_2(\mathbb{R}^d)$.

(a) Let us suppose that $\phi$ is contained in $L_2(\mathbb{R}^d)$ and that $\tilde{\phi}$ is bounded. For $0 \leq \mu \leq r$ and $j \in \mathbb{Z}$, we have

$$W^{\mu}_j \ast \tilde{\psi}^\nu_{j,0} = W^{\mu,\nu}_j.$$

(b) Suppose that $\hat{\phi}$ and $\hat{\tilde{\phi}}$ have no real $\mathbb{Z}^d$ periodical zeros and $r = m - 1$. Then, the functions $\phi \ast \tilde{\phi}$ and $\psi^{\mu} \ast \tilde{\psi}^\mu$ for $1 \leq \mu \leq m - 1$ have stable integer shifts. If, in addition, $\phi \ast \tilde{\phi}$ is continuous and $b^0$ is real–valued, $\phi \ast \tilde{\phi}$ is interpolating.

We need the following lemma in order to prove Theorem 3.3.

**Lemma 3.4.** Let $g$ be a tempered distribution and $f \in L_2(\mathbb{R}^d)$. Then, we have

$$f_{j,k} \ast g_{j,0} = m^{-\frac{j}{2}}(f \ast g)_{j,k} \quad \text{for all } j \in \mathbb{Z}, \ k \in \mathbb{Z}^d. \quad (15)$$

We will also apply the following characterization of functions with stable integer shifts in [25].

**Theorem 3.5.** A compactly supported function $\phi \in L_2(\mathbb{R}^d)$ has stable integer shifts if and only if $\hat{\phi}$ has no real $\mathbb{Z}^d$ periodical zeros.

Now, let us prove Theorem 3.3.

**Proof of Theorem 3.3.** (a) The function $\tilde{\phi}$ is bounded. Therefore, it can be verified that the map

$$\cdot \ast \tilde{\psi}^\nu_{j,0} : f \mapsto f \ast \tilde{\psi}^\nu_{j,0}$$

is a bounded linear map from $L_2(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$. This implies $W^{\nu}_j \ast \tilde{\psi}^\nu_{j,0} \subset L_2(\mathbb{R}^d)$. By applying Lemma 3.4, it follows that

$$\text{span}\{(\psi^{\mu} \ast \tilde{\psi}^\nu)_{j,k} \mid k \in \mathbb{Z}^d\} \subset W^{\mu}_j \ast \tilde{\psi}^\nu_{j,0}.$$

This leads to $W^{\mu,\nu}_j \subset W^{\nu}_j \ast \tilde{\psi}^\nu_{j,0}$.

Now, let $f \in W^{\mu}_j \ast \tilde{\psi}^\nu_{j,0}$. Thus, there exists $g \in W^{\mu}_j$ so that $f = g \ast \tilde{\psi}^\nu_{j,0}$. Moreover, there are $g_n \in \text{span}\{\psi^{\mu}_{j,k} \mid k \in \mathbb{Z}^d\}$ converging to $g$ in $L_2(\mathbb{R}^d)$. By the continuity of the mapping $\cdot \ast \tilde{\psi}^\nu_{j,0}$, the functions $g_n \ast \tilde{\psi}^\nu_{j,0}$ converge to $f = g \ast \tilde{\psi}^\nu_{j,0}$. By applying Lemma 3.4, we have $g_n \ast \tilde{\psi}^\nu_{j,0} \in W^{\mu,\nu}_j$. Now, $f$ is contained in $W^{\mu,\nu}_j$ because the space $W^{\mu,\nu}_j$ is closed. Finally, we have also shown $W^{\nu}_j \ast \tilde{\psi}^\nu_{j,0} \subset W^{\mu,\nu}_j$. 


(b) It follows by Theorem 3.5 that \( \phi * \tilde{\phi} \) has stable integer shifts.

Let \( \{ \gamma_0, \ldots, \gamma_{m-1} \} = \Gamma_M \) with \( \gamma_0 = 0 \). We define matrices for the symbols \( a^\mu \) and \( b^\mu \) by

\[
a := (a^\mu(\cdot + \gamma_\nu))_{\nu=0,\ldots,m-1} \quad \text{and} \quad b := (b^\mu(\cdot + \gamma_\nu))_{\nu=0,\ldots,r}.
\]

The PR conditions are equivalent to \( ab^t = E \). For \( r = m - 1 \) the matrices \( a \) and \( b \) are quadratic. Therefore, \( b^t \) is the inverse of \( a \) and we obtain \( a^t b = E \). This leads to

\[
\sum_{\gamma \in \Gamma_M} a^\mu(\cdot + \gamma)b^\mu(\cdot + \gamma) = 1 \quad \text{for} \quad \mu = 0, \ldots, m - 1.
\] (16)

Let \( 1 \leq \mu \leq m - 1 \). Again, by applying Theorem 3.5, the function \( \psi^\mu * \tilde{\psi}^\mu \) has stable integer shifts if and only if \( \hat{\psi}^\mu \tilde{\psi}^\mu \) has no real \( \mathbb{Z}^d \) periodical zeros. We assume that \( \xi \in \mathbb{R}^d \) is a \( \mathbb{Z}^d \) periodical zero of \( \hat{\psi}^\mu \tilde{\psi}^\mu \). Eq. (16) shows that there exists \( \gamma \in \Gamma_M \) with

\[
(a^\mu b^\mu)(M^{t-1}\xi + \gamma) \neq 0.
\]

For all \( k \in \mathbb{Z}^d \), Eq. (10) yields

\[
0 = (\hat{\psi}^\mu \tilde{\psi}^\mu)(\xi + M^t\gamma + M^tk)
= (a^\mu b^\mu)(M^{t-1}\xi + \gamma) \left( \hat{\phi} \tilde{\phi} \right)(M^{t-1}\xi + \gamma + k).
\]

It follows that \( \hat{\phi} \tilde{\phi} \) has a \( \mathbb{Z}^d \) periodical zero in \( M^{t-1}\xi + \gamma \). This is a contradiction to the stability of \( \phi * \tilde{\phi} \). Therefore, we have shown that \( \psi^\mu * \tilde{\psi}^\mu \) has stable integer shifts.

If \( b^0 \) is real–valued, (16) implies that \( a^0b^0 \) satisfies the interpolation condition (7). Thus, the rest of part (b) follows by Theorem 2.2.

Now, let us prove Lemma 3.4.

**Proof of Lemma 3.4.** It can be verified that for all Schwartz functions \( \eta \) the function \( f * \eta \) is again a Schwartz function. Therefore, \( f * g \) is a tempered distribution. Let \( \eta \) be an arbitrary Schwartz function. It can be shown that \( \langle g_{j,k} | \eta \rangle = \langle g | \eta_{-j,-M^j-M^j} \rangle \) is valid. We obtain

\[
\langle f_{j,k} * g_{j,0} | \eta \rangle = \langle g_{j,0} | \eta * (f_{j,k})^- \rangle
= \left\langle g_{j,0} \right| \int \eta(x)(f_{j,k})^-(-x)dx \right\rangle
= m^2 \left\langle g_{j,0} \right| \int \eta(x)f(-M^j \cdot - M^j x - k)dx \right\rangle
= \left\langle g \right| \int \eta(x)f(- \cdot + M^j x - k)dx \right\rangle.
\]

12
On the other hand, we have
\[
\langle (f \ast g)_{j,k} | \eta \rangle &= \langle f \ast g | \eta_{-j,-M^{-1}k} \rangle \\
&= m^{-\frac{3}{2}} \left\langle g \int \eta(M^{-j}x + M^{-j}k)f(- \cdot + x)dx \right\rangle \\
&= m^{-\frac{3}{2}} \left\langle g \int \eta(x)f(- \cdot + M^2x - k)dx \right\rangle.
\]
Thus, we have shown (15). \qed

The construction by Theorem 3.2 leads to \((r + 1)^2 - 1\) wavelets. For increasing \(r\), this might be a problem for numerical applications. In the following theorem, we present a construction method that leads to fewer wavelets. It is a generalization of the construction method for tight wavelet frames in [19, 32].

**Theorem 3.6.** Let the symbol family \(\{(a^\mu, b^\nu) \mid 0 \leq \mu \leq r\}\) satisfy condition (I) and let \(\phi \ast \tilde{\phi}\) be contained in \(L_2(\mathbb{R}^d)\). With the notation in (9) and \(\psi^0 = \phi\) and \(\tilde{\psi}^0 = \tilde{\phi}\), we define
\[
\Psi^0 := \frac{1}{m} \psi^0 \ast \tilde{\psi}^0(M^{-1} \cdot), \quad \frac{1}{m} \psi^0 \ast (\psi^0(M^{-1} \cdot)) =: \tilde{\Psi}^0,
\]
\[
\ldots
\]
\[
\Psi^r := \frac{1}{m} \psi^0 \ast (\tilde{\psi}^r(M^{-1} \cdot)), \quad \frac{1}{m} \tilde{\psi}^0 \ast (\psi^r(M^{-1} \cdot)) =: \tilde{\Psi}^r,
\]
\[
\Psi^{r+1} := \frac{1}{m} \psi^1 \ast \psi^0, \quad \frac{1}{m} \tilde{\psi}^1 \ast \psi^0 =: \tilde{\Psi}^{r+1},
\]
\[
\ldots
\]
\[
\Psi^{2r} := \frac{1}{m} \psi^r \ast \tilde{\psi}^0, \quad \frac{1}{m} \tilde{\psi}^r \ast \psi^0 =: \tilde{\Psi}^{2r}.
\]
Then,
\[
\{(\Psi^\mu, \tilde{\Psi}^\mu) \mid 1 \leq \mu \leq 2r\}
\]
generates a pair of dual wavelet frames.

**Proof.** For \(0 \leq \nu \leq r\) and \(1 \leq \mu \leq r\), let us define
\[
\tilde{a}^\nu(\cdot) := a^\nu(\cdot)b^\nu(M^1 \cdot) \quad b^\nu(\cdot)a^\nu(M^1 \cdot) =: \tilde{b}^\nu(\cdot)
\]
\[
\tilde{a}^{\nu+\mu} := a^\mu \quad b^\nu =: \tilde{b}^{\nu+\mu}.
\]
By applying part (b) of Lemma 3.1 with \(s := r\) and
\[
(c^\nu, d^\nu) := (b^\nu, a^\nu) \quad \text{for all } 0 \leq \nu \leq r,
\]
we obtain that
\[
\{(\tilde{a}^\mu, \tilde{b}^\mu) \mid 0 \leq \mu \leq 2r\}
\]
satisfies the PR conditions. It can be shown that \(\Psi^0\) and \(\tilde{\Psi}^0\) are refinable with respect to \(a^0\) and \(\tilde{b}^0\). Applying the refinement equation (4) for \(\phi\) and \(\tilde{\phi}\), the assumption \(\phi \ast \tilde{\phi} \in L_2(\mathbb{R}^d)\) implies that also \(\Psi^0\) and \(\tilde{\Psi}^0\) are contained in \(L_2(\mathbb{R}^d)\). Furthermore, it can be verified that \(\Psi^\mu\) and \(\tilde{\Psi}^\mu\) are primal and dual wavelets corresponding to the wavelet symbols \(\tilde{a}^\mu\) and \(\tilde{b}^\mu\). It can easily be shown that the symbol family \(\{(\tilde{a}^\mu, \tilde{b}^\mu) \mid 0 \leq \mu \leq 2r\}\) satisfies the condition (I). Thus, the assertion follows by the MEP 2.3. \qed
It should be mentioned that $\Psi^0$ as well as $\tilde{\Psi}^0$ generates a multiresolution analysis. Furthermore, we also obtain the decompositions (11).

Theorem 3.6 leads to $2^r$ wavelets. Our construction principle in Theorem 3.2 yields $(r + 1)^2 - 1$ wavelets. However, we pay for this smaller number of wavelets with the loss of stability. It can be shown that $\Psi^0$ and $\tilde{\Psi}^0$ do not have stable integer shifts and, in general, we have $\Psi^0 \neq \tilde{\Psi}^0$. Therefore, $\Psi^0$ and $\tilde{\Psi}^0$ are not interpolating. The instability may also lead to problems in numerical applications. Thus, we will only apply Theorem 3.2 to construct concrete examples of wavelet frames in Section 4 but not Theorem 3.6.

### 3.2 The Construction of Starting Symbols

By applying Theorem 3.2 with $r = m - 1$ and under the assumptions of part (b) in Theorem 3.3, we obtain an interpolating refinable function. Furthermore, the choice $r = m - 1$ minimizes the number of the wavelets for a given dilation matrix. Therefore, in this section, we describe the construction of starting symbols $\{ (a_\mu, b_\mu) \mid 0 \leq \mu \leq m - 1 \}$. It is based on [23].

We need some notations and auxiliary results. For a given symbol $a$, we call

$$ A_{\gamma^*} := \sum_{k \in \mathbb{Z}^d} a_{Mk + \gamma^*} e_{-k} $$

the **subsymbol** of $a$ for $\gamma^* \in \Gamma^*_M$. It is well known by character sums that, for $k \in \mathbb{Z}^d$, we have

$$ \sum_{\gamma \in \Gamma_M} e_k(\gamma) = \begin{cases} m, & \text{if } k \in M\mathbb{Z}^d, \\ 0, & \text{otherwise} \end{cases} \quad (17) $$

Applying (17), one obtains the following relations

$$ A_{\gamma^*}(M^t\cdot) = \sum_{\gamma \in \Gamma_M} e_{\gamma^*}(\cdot + \gamma) a(\cdot + \gamma) \quad (18) $$

and

$$ a(\cdot) = \frac{1}{m} \sum_{\gamma^* \in \Gamma^*_M} A_{\gamma^*}(M^t\cdot) e_{-\gamma^*}(\cdot) . \quad (19) $$

For a family of symbols $\{(a^\mu, b^\mu) \mid 0 \leq \mu \leq m - 1\}$, let $A^*_{\gamma^*}$ and $B^*_{\gamma^*}$ be the subsymbols of $a^\mu$ and $b^\mu$, respectively. Let $\{\gamma^*_0, \ldots, \gamma^*_m\} = \Gamma^*_M$ with $\gamma^*_0 = 0$. The matrices

$$ A = (A^\mu_{\gamma^*_\nu})_{\mu=0,\ldots,m-1, \nu=0,\ldots,m-1} \quad \text{and} \quad B = (B^\mu_{\gamma^*_\nu})_{\mu=0,\ldots,m-1, \nu=0,\ldots,m-1} $$

are called the primal and dual *polyphase* matrices for the symbol family $\{(a^\mu, b^\mu) \mid 0 \leq \mu \leq m - 1\}$. By applying (17), it can be verified that the symbol family $\{(a^\mu, b^\mu) \mid 0 \leq \mu \leq m - 1\}$ satisfies the PR conditions (8) if and only if the corresponding polyphase matrices satisfy

$$ AB^T = mE_{m-1} \quad , \quad (20) $$

14
see [23] for details. For \( r = m - 1 \), the PR conditions are equivalent to (16). Therefore, \( b^0 \) has to satisfy
\[
\sum_{\gamma \in \Gamma_M} a^0(\cdot + \gamma)b^0(\cdot + \gamma) = 1.
\]
We call such a symbol \( b^0 \) dual to \( a^0 \). Thus, \( b^0 \) is dual to \( a^0 \) if and only if \( a^0b^0 \) satisfies the interpolation condition (7). For a symbol \( a^0 \) that satisfies the interpolation condition, an explicit construction method for \( b^0 \) is given in [17, 23]. Moreover, in the next section, we give a complete characterization. Therefore, we focus on symbols \( a^0 \) that satisfy the interpolation condition.

Let us assume we are given a symbol \( a^0 \) that satisfies the interpolation condition (7) and a dual symbol \( b^0 \) with corresponding subsymbols \( A^0_{\gamma_1} \) and \( B^0_{\gamma_1} \), respectively. In this case, a construction of the whole family of starting symbols \( \{(a^\mu, b^\mu) \mid 0 \leq \mu \leq m - 1\} \) is known. By applying the algorithms in [23], we obtain the following primal and dual polyphase matrices:

\[
A = \begin{pmatrix}
1 & A^0_{\gamma_1} & A^0_{\gamma_2} & \ldots & A^0_{\gamma_{a-1}} \\
-B^0_{\gamma_1} & m - B^0_{\gamma_2}A^0_{\gamma_1} & -B^0_{\gamma_2}A^0_{\gamma_2} & \ldots & -B^0_{\gamma_{a-1}}A^0_{\gamma_{a-1}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-B^0_{\gamma_{a-1}} & -B^0_{\gamma_{a-1}}A^0_{\gamma_1} & \ldots & m - B^0_{\gamma_{a-1}}A^0_{\gamma_{a-1}}
\end{pmatrix}
\]

(21)

and

\[
B = \begin{pmatrix}
B^0_{\gamma_1} & B^0_{\gamma_2} & \ldots & B^0_{\gamma_{a-1}} \\
-A^0_{\gamma_1} & E_{a-1} \\
\vdots & \vdots & \ddots & \ddots \\
-A^0_{\gamma_{a-1}} & \end{pmatrix}
\]

(22)

It can easily be verified that \( AB^0 = mE_{m-1} \). Now, the symbol family \( \{(a^\mu, b^\mu) \mid 0 \leq \mu \leq m - 1\} \) can be reconstructed by (19).

It should be mentioned that the polyphase matrices in [27], which are constructed by means of the lifting scheme, are very similar to (21) and (22).

### 3.3 A Characterization of Dual Symbols

For the construction of the starting symbols in Section 3.2, we have to find a symbol \( a^0 \) that satisfies the interpolation condition (7) and a symbol \( b^0 \) that is dual to \( a^0 \). The next theorem gives a characterization of all such dual symbols \( b^0 \). For preparation, let us say a symbol \( a \) satisfies the Strang Fix conditions of order \( L \) if
\[
\partial^\alpha a(\gamma) = 0 \quad \text{for all } \gamma \in \Gamma_M \setminus \{0\} \text{ and } \alpha \in \mathbb{N}^d, |\alpha| < L.
\]

If the symbol \( a \) with \( a(0) = 1 \) satisfies the Strang Fix conditions of order \( L \), then the corresponding refinable function also satisfies them, see [24] for details.
Theorem 3.7. Let $a$ and $c$ be symbols satisfying the interpolation condition (7) and the Strang Fix conditions of order $L$.

(a) The symbol
\[ b(\cdot) = c(\cdot) + 1 - \sum_{\gamma \in \Gamma} (c\overline{\alpha})(\cdot + \gamma) \] (23)
is dual to $a$ and satisfies the Strang Fix conditions of order $L$.

(b) Every symbol $b$, that is dual to $a$ and satisfies the Strang Fix conditions of order $L$, is of the form (23).

It should be mentioned that for $m = 2$ and $c = \pi$ the mask of $b$ in (23) coincides with the smallest nontrivial dual masks constructed in [17, 23].

To prove Theorem 3.7, we need some auxiliary results. Let $A_{\gamma^*}$ and $B_{\gamma^*}$ be the subsymbols of $a$ and $b$, respectively. By applying (17), (18) and (19), it can be verified that $b$ is dual to $a$ if and only if
\[ \sum_{\gamma^* \in \Gamma_M} A_{\gamma^*} B_{\gamma^*} = m. \] (24)

It was shown in [11] that a symbol $a$ satisfies the interpolation condition (7) if and only if $A_0 = 1$. Thus, the duality condition (24) reduces to
\[ B_0 = m - \sum_{\gamma^* \in \Gamma_M \setminus \{0\}} B_{\gamma^*} \overline{A_{\gamma^*}}. \] (25)

Our characterization is based on the next theorem that has been shown in [24]. In the following, we denote $\Pi_L$ for the space of all polynomials of total degree equal or less than $L$.

Theorem 3.8 (Sum Rules). A symbol $a$ satisfies the Strang Fix conditions of order $L$ if and only if
\[ \sum_{k \in \mathbb{Z}^d} a_{MK}(MK) = \sum_{k \in \mathbb{Z}^d} a_{MK+\gamma^*}(MK + \gamma^*) \quad \text{for all} \ p \in \Pi_{L-1} \text{ and } \gamma^* \in \Gamma_M^*. \] (26)

The conditions (26) are called sum rules. In the following, we restate these sum rules in terms of subsymbols and incorporate the conditions for interpolation and duality.

We define the semi-convolution of a trigonometric polynomial $Q = \sum_{k \in \mathbb{Z}^d} f_k e^{-k}$ with an arbitrary sequence $(h_k)_{k \in \mathbb{Z}^d}$ by
\[ Q \ast (h_k)_{k \in \mathbb{Z}^d} := (f_k)_{k \in \mathbb{Z}^d} \ast (h_k)_{k \in \mathbb{Z}^d}. \]

For a polynomial $p$ and $\gamma^* \in \mathbb{Z}^d$, we set
\[ p_{\gamma^*} := (p(k + M^{-1} \gamma^*))_{k \in \mathbb{Z}^d}. \] (27)

The following corollary shows that the Strang Fix conditions of a symbol can be expressed in terms of convolutions of the corresponding subsymbols with polynomial sequences.
Corollary 3.9. A symbol $a$ satisfies the Strang Fix conditions of order $L$ if and only if the corresponding subsymbols satisfy

$$A_{\gamma^*} p_0 = A_0 * p_{\gamma^*} \quad \text{for all } p \in \Pi_{L-1} \text{ and } \gamma^* \in \Gamma^*_M.$$  \hfill (28)

Proof. 1. Let $a$ satisfy the Strang Fix conditions of order $L$ and let $p \in \Pi_{L-1}$. Furthermore, let $l \in \mathbb{Z}^d$ be given. With the notation

$$q^l(\cdot) := p(l - M^{-1} \cdot + M^{-1} \gamma^*) \in \Pi_{L-1},$$

we obtain

$$(A_{\gamma^*} * p_0)(l) = \sum_{k \in \mathbb{Z}^d} a_{Mk+\gamma^*} p(l - k)$$

$$= \sum_{k \in \mathbb{Z}^d} a_{Mk+\gamma^*} q^l(Mk + \gamma^*).$$

By applying (26), it follows that

$$(A_{\gamma^*} * p_0)(l) = \sum_{k \in \mathbb{Z}^d} a_{Mk} q^l(Mk)$$

$$= \sum_{k \in \mathbb{Z}^d} a_{Mk}(l - k + M^{-1} \gamma^*)$$

$$= (A_0 * p_{\gamma^*})(l).$$

2. Let us assume that (28) is valid and let $p \in \Pi_{L-1}$ be given. We define

$$q(\cdot) := p(-M \cdot + \gamma^*) \in \Pi_{L-1}.$$ 

Thus, we obtain

$$\sum_{k \in \mathbb{Z}^d} a_{Mk} p(Mk) = \sum_{k \in \mathbb{Z}^d} a_{Mk} q(-k + M^{-1} \gamma^*)$$

$$= (A_0 * q_{\gamma^*})(0).$$

By applying (28), it follows

$$\sum_{k \in \mathbb{Z}^d} a_{Mk} p(Mk) = (A_{\gamma^*} * q_0)(0)$$

$$= \sum_{k \in \mathbb{Z}^d} a_{Mk+\gamma^*} q(-k)$$

$$= \sum_{k \in \mathbb{Z}^d} a_{Mk+\gamma^*} p(Mk + \gamma^*).$$

Now, Theorem 3.8 implies the assertion. \hfill \Box
In the next lemma, we show the consequences of Corollary 3.9 for symbols that satisfy the interpolation condition and their duals.

**Corollary 3.10.** Let be a symbol that satisfies the interpolation condition (7).

(a) The symbol satisfies the Strang Fix conditions of order if and only if

\[ A_{\gamma^*} \ast p_0 = p_{\gamma^*} \quad \text{for all } p \in \Pi_{L-1} \text{ and } \gamma^* \in \Gamma_M. \]  

(b) Let satisfy the Strang Fix conditions of order and let the symbol be dual to . Then, satisfies the Strang Fix conditions of order if and only if

\[ B_{\gamma^*} \ast p_0 = p_{\gamma^*} \quad \text{for all } p \in \Pi_{L-1} \text{ and } \gamma^* \in \Gamma_M. \]

**Proof.** (a) The subsymbol of a symbol that satisfies the interpolation condition, is given by \( A_0 = 1 \). Thus, (29) follows from Corollary 3.9.

(b) Let \( p \in \Pi_{L-1} \) and \( \gamma^* \in \Gamma_M \). It has been shown in [27] that

\[ A_{\gamma^*} \ast p_0 = p_{\gamma^*} \quad \text{if and only if } \quad A_{\gamma^*} \ast p_0 = p_{\gamma^*}. \]  

Let (28) be valid. By multiplying two trigonometric polynomials \( Q = \sum_{k \in \mathbb{Z}^d} f_k e_k \) and \( \tilde{Q} = \sum_{k \in \mathbb{Z}^d} \tilde{f}_k e_k \) we obtain again a trigonometric polynomial \( Q \tilde{Q} \) whose coefficient sequence is given by \( (f_k)_{k \in \mathbb{Z}^d} \ast (\tilde{f}_k)_{k \in \mathbb{Z}^d} \). The classical convolution is associative. Thus, for the semi–convolution, it follows that

\[ (B_{\gamma^*} A_{\gamma^*} \ast p_{\gamma^*}) \ast p_{\gamma^*} = (B_{\gamma^*} A_{\gamma^*}) \ast (p_{\gamma^*} \ast p_{\gamma^*}) . \]

By applying (25), we obtain

\[ B_0 \ast p_{\gamma^*} = m \ast p_{\gamma^*} - \sum_{\gamma^* \in \Gamma_M \setminus \{0\}} (B_{\gamma^*} A_{\gamma^*} \ast p_{\gamma^*}) \ast p_{\gamma^*} . \]

Moreover, (29), (30) and Corollary 3.9 lead to

\[ B_0 \ast p_{\gamma^*} = mp_{\gamma^*} - \sum_{\gamma^* \in \Gamma_M \setminus \{0\}} B_{\gamma^*} \ast p_{\gamma^* - \gamma^*} \]

\[ = mp_{\gamma^*} - \sum_{\gamma^* \in \Gamma_M \setminus \{0\}} B_0 \ast p_{\gamma^*} \]

\[ = mp_{\gamma^*} - (m - 1) B_0 \ast p_{\gamma^*} . \]

It follows that \( B_0 \ast p_{\gamma^*} = p_{\gamma^*} \). By applying Corollary 3.9, we have shown the equivalence.

Now, we have collected all ingredients for the proof of Theorem 3.7.
Proof of Theorem 3.7. (a) Let $A_{\gamma^*}$ and $C_{\gamma^*}$ be the subsymbols of $a$ and $c$, respectively. We define a symbol $d$ with corresponding subsymbols $D_{\gamma^*}$ by

$$D_0 := m - \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} A_{\gamma^*}C_{\gamma^*} \quad \text{and} \quad D_{\gamma^*} := C_{\gamma^*} \quad \text{for} \quad \gamma^* \in \Gamma^*_M \setminus \{0\}.$$  

By (24), the symbol $d$ is dual to $a$. Now, Corollary 3.10 yields that $d$ satisfies the Strang Fix conditions of order $L$. Next, we will show that $d = b$. Therefore, we reconstruct $d$ from its subsymbols by using (19)

$$d(\xi) = \frac{1}{m} \left( m - \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} A_{\gamma^*}(M^t\xi)C_{\gamma^*}(M^t\xi) + \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} C_{\gamma^*}(M^t\xi)e^{-\gamma^*}(\xi) \right).$$

We apply (18) and $C_0 = 1$:

$$d(\xi) = 1 - \frac{1}{m} \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} \sum_{\gamma, \overline{\gamma} \in \Gamma_M} e_{-\gamma^*}(\xi + \overline{\gamma})a(\xi + \overline{\gamma})e_{\gamma^*}(\xi + \gamma)c(\xi + \gamma) + c(\xi) - \frac{1}{m}$$

$$= 1 + c(\xi) - \frac{1}{m} \sum_{\gamma, \overline{\gamma} \in \Gamma_M} c(\xi + \gamma)a(\xi + \overline{\gamma}) \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} e_{\gamma^*}(\gamma - \overline{\gamma}).$$

Eq. (17) yields

$$\sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} e_{\gamma^*}(\gamma - \overline{\gamma}) = m\delta_{\gamma, \overline{\gamma}} - 1.$$ 

Therefore, it follows that

$$d(\xi) = 1 + c(\xi) - \frac{1}{m} \sum_{\gamma, \overline{\gamma} \in \Gamma_M} c(\xi + \gamma)a(\xi + \overline{\gamma})(m\delta_{\gamma, \overline{\gamma}} - 1)$$

$$= 1 + c(\xi) - \frac{1}{m} \sum_{\gamma \in \Gamma_M} c(\xi + \gamma)a(\xi + \gamma) + \frac{1}{m} \sum_{\gamma, \overline{\gamma} \in \Gamma_M} c(\xi + \gamma)a(\xi + \overline{\gamma}).$$

The interpolation condition (7) for the symbols $c$ and $\overline{\sigma}$ yields

$$\sum_{\gamma, \overline{\gamma} \in \Gamma_M} c(\xi + \gamma)a(\xi + \overline{\gamma}) = 1.$$ 

Thus, we obtain

$$d(\xi) = 1 + c(\xi) - \sum_{\gamma \in \Gamma_M} c(\xi + \gamma)a(\xi + \gamma).$$

(b) For the subsymbols $C_{\gamma^*}$ of a symbol $c$, we set

$$C_0 := 1 \quad \text{and} \quad C_{\gamma^*} := B_{\gamma^*} \quad \text{for} \quad \gamma^* \in \Gamma^*_M \setminus \{0\}.$$ 

Therefore, $c$ satisfies the interpolation condition. By applying Corollary 3.10, it also satisfies the Strang Fix conditions of order $L$. Following the lines of part (a), we obtain the representation (23).
3.4 Vanishing Moments of the Starting Symbols

The construction methods in Theorem 3.2 and Theorem 3.6 preserve the number of vanishing moments but do not increase them. Therefore, we need starting symbols that already have a high number of vanishing moments.

We say that a symbol \( c \) has \( L \) vanishing moments if

\[
\partial^\alpha c(0) = 0 \quad \text{for all } \alpha \in \mathbb{N}^d, |\alpha| < L.
\]

Let the wavelet \( \psi^\mu \) be constructed by the MEP. By applying (10), vanishing moments of the symbol \( a^\mu \) imply vanishing moments of the corresponding wavelet.

In this section, we will prove the following theorem.

**Theorem 3.11.** Let the symbol family \( \{(a^\mu, b^\mu) \mid 0 \leq \mu \leq m - 1\} \) satisfy the PR conditions (8) and let \( a^0 \) and \( b^0 \) satisfy the Strang Fix conditions of order \( L \). Then, all symbols \( a^\mu \) and \( b^\mu \) for \( \mu = 1, \ldots, m - 1 \) have \( L \) vanishing moments.

In order to prove Theorem 3.11, we need some auxiliary results. It can be shown that a symbol \( c \) has \( L \) vanishing moments if and only if

\[
\sum_{k \in \mathbb{Z}^d} c(k)p(k) = 0 \quad \text{for all } p \in \Pi_{L-1}. \tag{31}
\]

This enables us to state the following lemma.

**Lemma 3.12.** Under the notation of (27), the symbol \( a \) has \( L \) vanishing moments if and only if

\[
\sum_{\gamma^* \in \Gamma_M^*} A_{\gamma^*} \ast p_{-\gamma^*} = 0 \quad \text{for all } p \in \Pi_{L-1}. \tag{32}
\]

**Proof.** Let (32) be valid and let \( p \in \Pi_{L-1} \) be given. We define \( q(\cdot) := p(-M\cdot) \in \Pi_{L-1} \) and obtain

\[
\sum_{k \in \mathbb{Z}^d} a_k p(k) = \sum_{\gamma^* \in \Gamma_M} \sum_{k \in \mathbb{Z}^d} a_{Mk+\gamma^*} p(Mk + \gamma^*)
\]

\[
= \sum_{\gamma^* \in \Gamma_M} \sum_{k \in \mathbb{Z}^d} a_{Mk+\gamma^*} q(-k - M^{-1}\gamma^*)
\]

\[
= \sum_{\gamma^* \in \Gamma_M} \sum_{k \in \mathbb{Z}^d} a_{Mk+\gamma^*} q(-\gamma^*)(-k)
\]

\[
= \sum_{\gamma^* \in \Gamma_M^*} (A_{\gamma^*} \ast p_{-\gamma^*})(0).
\]

Using (32) yields

\[
\sum_{k \in \mathbb{Z}^d} a_k p(k) = 0.
\]
It follows by (31) that \( a \) has \( L \) vanishing moments.

Now, we assume that the symbol \( a \) has \( L \) vanishing moments. We will show that (32) is valid. Let \( p \in \Pi_{L-1} \) and \( l \in \mathbb{Z}^d \) be given. Next, we define \( q'(\cdot):= p(l - M^{-1} \cdot) \in \Pi_{L-1} \). This yields

\[
\sum_{\gamma^* \in \Gamma^*_{M}} (A_{\gamma^*} \ast p_{-\gamma^*})(l) = \sum_{\gamma^* \in \Gamma^*_{M}} \sum_{k \in \mathbb{Z}^d} a_{Mk + \gamma^*} p_{-\gamma^*}(l - k)
\]
\[
= \sum_{\gamma^* \in \Gamma^*_{M}} \sum_{k \in \mathbb{Z}^d} a_{Mk + \gamma^*} p(l - k - M^{-1} \gamma^*)
\]
\[
= \sum_{\gamma^* \in \Gamma^*_{M}} \sum_{k \in \mathbb{Z}^d} a_{Mk + \gamma^*} q'(Mk + \gamma^*)
\]
\[
= \sum_{k \in \mathbb{Z}^d} a_k q'(k).
\]

By applying (31), we obtain

\[
\sum_{\gamma^* \in \Gamma^*_{M}} (A_{\gamma^*} \ast p_{-\gamma^*})(l) = 0.
\]

At this point, we can prove Theorem 3.11.

**Proof of Theorem 3.11.** It follows by (20) that for \( 0 \leq \mu, \nu \leq m - 1 \) the corresponding subsymbols satisfy

\[
\sum_{\gamma^* \in \Gamma^*_{M}} A_{\gamma^*}^\mu B_{\gamma^*}^{\nu} = m \delta_{\mu, \nu}.
\]

(33)

Applying (33) for \( 1 \leq \mu \leq m - 1 \) yields

\[
0 = \sum_{\gamma^* \in \Gamma^*_{M}} \left( A_{\gamma^*}^\mu B_{\gamma^*}^{\nu} \right) \ast p_0.
\]

By Corollary 3.9, we obtain

\[
0 = \sum_{\gamma^* \in \Gamma^*_{M}} \left( A_{\gamma^*}^\mu B_{\gamma^*}^{\nu} \right) \ast p_{-\gamma^*}
\]
\[
= \sum_{\gamma^* \in \Gamma^*_{M}} \left( B_{\gamma^*}^{\nu} A_{\gamma^*}^\mu \right) \ast p_{-\gamma^*}
\]
\[
= B_{0}^{\nu} \ast \sum_{\gamma^* \in \Gamma^*_{M}} A_{\gamma^*}^\mu \ast p_{-\gamma^*}.
\]

Thus,

\[
0 = \left( A_{0}^\mu B_{0}^{\nu} \right) \ast \sum_{\gamma^* \in \Gamma^*_{M}} A_{\gamma^*}^\mu \ast p_{-\gamma^*}.
\]

(34)
is valid. It can be easily verified that $\sum_{\gamma^* \in \Gamma^*_M} A_{\gamma^*}^\mu \ast p_{-\gamma^*}$ is a polynomial sequence and that its corresponding polynomial $q$ is contained in $\Pi_{L-1}$. Applying (27) yields
\[
q_0 = \sum_{\gamma^* \in \Gamma^*_M} A_{\gamma^*}^\mu \ast p_{-\gamma^*}
\]
and (24) leads to
\[
0 = \left( m - \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} A_{\gamma^*}^0 \overline{B}_{\gamma^*}^0 \right) \ast q_0.
\]
By applying Corollary (3.9), it can be shown that $\overline{B}_{\gamma^*}^0 \ast q_0 = \overline{B}_{0}^0 \ast q_{-\gamma^*}$. Thus, we have
\[
0 = mq_0 - \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} \left( A_{\gamma^*}^0 \overline{B}_{\gamma^*}^0 \right) \ast q_{-\gamma^*},
\]
\[
= mq_0 - \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} \left( \overline{B}_{0}^0 A_{\gamma^*}^0 \right) \ast q_{-\gamma^*},
\]
\[
= mq_0 - \sum_{\gamma^* \in \Gamma^*_M \setminus \{0\}} \left( \overline{B}_{0}^0 A_{\gamma^*}^0 \right) \ast q_0,
\]
\[
= mq_0 - (m-1) \left( A_{0}^0 \overline{B}_{0}^0 \right) \ast q_0.
\]
Due to (34), it holds that
\[
0 = q_0.
\]
Thus, by applying Lemma 3.12, we obtain that $a^\mu$ has $L$ vanishing moments. The vanishing moments of the symbols $b^\mu$ can be shown in the same way. \(\square\)

4 Explicit Constructions

We start our construction with a symbol $a^0$ that satisfies the interpolation condition (7) and the Strang Fix conditions of order $L$. The dual symbol $b^0$ can be constructed by applying Theorem 3.7. It also satisfies the Strang Fix conditions of order $L$. By applying (21) and (22), we obtain a family of starting symbols that satisfies the PR conditions. Theorem 3.11 ensures that the starting wavelet symbols have $L$ vanishing moments. Finally, we apply Theorem 3.2 with these starting symbols. Thus, we have $r = m - 1$ and we obtain a pair of dual wavelet frames. All $m^2 - 1$ wavelets have at least $L$ vanishing moments. To obtain pairs of dual wavelet frames which can be used in applications, the number of wavelets should be small. Therefore, we use a dilation matrix with a small determinant.

Let $M$ be a dilation matrix with $m = 2$ and let us be given a symbol $a^0$ that satisfies the interpolation condition and the Strang Fix conditions of order $L$. We choose $A = a^0$ in Theorem 3.7. For simplicity, we denote $\Gamma_M = \{0, \gamma\}$ and $\Gamma^*_M = \{0, \gamma^*\}$. Thus, we obtain
\[
b^0(\cdot) = a^0(\cdot) + 1 - |a^0(\cdot)|^2 - |a^0(\cdot + \gamma)|^2.
\]
The symbol \( b^0 \) is dual to \( a^0 \) and satisfies the Strang Fix conditions of order \( L \). Moreover, we define for \( t \in \mathbb{R} \)

\[
a^1_t(\cdot) := \frac{1}{t} b^0(\cdot + \gamma) e_{-\gamma}(\cdot) \quad \text{and} \quad b^1_t(\cdot) := t a^0(\cdot + \gamma) e_{-\gamma}(\cdot). \tag{36}
\]

It can be shown that the primal and dual polyphase matrices (21) and (22) correspond to the symbol family

\[
\{ (a^0, b^0), (a^1_t, b^1_t) \}
\]

with \( t = 1 \). Therefore, it satisfies the PR conditions. With this result, it can easily be shown that (37) also satisfies the PR conditions for arbitary \( t \in \mathbb{R} \). Since \( a^0 \) and \( b^0 \) satisfy the Strang Fix conditions of order \( L \), the symbols \( a^1_t \) and \( b^1_t \) have \( L \) vanishing moments.

Now, we apply Theorem 3.2 with the starting symbols (37). To obtain a tight wavelet frame, the functions \( \phi, \psi \) and \( \tilde{\phi}, \tilde{\psi} \) have to coincide. We observed that the choice \( t = 2 \) reduces the \( L_2(\mathbb{R}^d) \) distance of \( \phi \ast \tilde{\psi} \) and \( \psi \ast \tilde{\phi} \) in the following examples. The distance can be estimated by the following lemma.

**Lemma 4.1.** Under the notation of Theorem 3.2, let \( \{ (a^\mu, b^\mu) \mid 0 \leq \mu \leq r \} \) be a symbol family with \( 0 \leq a^0 b^0 \leq 1 \) and \( \phi \ast \tilde{\phi} \in L_2(\mathbb{R}^d) \). Furthermore, let \( (\phi \ast \tilde{\phi}) \in L_1(\mathbb{R}^d) \). For \( 0 \leq \mu, \mu', \nu, \nu' \leq r \), we obtain

\[
\| \psi^\mu \ast \tilde{\psi}^\nu - \psi^\mu' \ast \tilde{\psi}^\nu' \|_{L_2(\mathbb{R}^d)} \leq \sqrt{1 + \frac{1}{m} (\phi \ast \tilde{\phi})(0) \| (a^\mu_k)_{k \in \mathbb{Z}^d} \ast (b^\nu_k)_{k \in \mathbb{Z}^d} - (a^\mu_k)_{k \in \mathbb{Z}^d} \ast (b^\nu_k)_{k \in \mathbb{Z}^d} \|_{L^1(\mathbb{Z}^d)}}. \tag{38}
\]

Proof. To simplify the notation, we set \( \Phi := \phi \ast \tilde{\phi} \). Furthermore, let us define

\[
(c^\mu_k)_{k \in \mathbb{Z}^d} := (a^\mu_k)_{k \in \mathbb{Z}^d} \ast (b^\nu_k)_{k \in \mathbb{Z}^d},
\]

By \( 0 \leq a^0 b^0 \leq 1 \), we obtain \( 0 \leq \hat{\Phi} \leq 1 \). This implies

\[
\| \Phi \|_{L_2(\mathbb{R}^d)} = \| \hat{\Phi} \|_{L_2(\mathbb{R}^d)} \leq \left( \int_{\mathbb{R}^d} \hat{\Phi}(\xi) d\xi \right)^{\frac{1}{2}} = \sqrt{\Phi(0)}. \tag{38}
\]

We have the following estimates:

\[
\| \psi^\mu \ast \tilde{\psi}^\nu - \psi^\mu' \ast \tilde{\psi}^\nu' \|_{L_2(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \left| \sum_{k \in \mathbb{Z}^d} (c^\mu_k - c^\mu_k') \Phi(Mx - k) \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_{\mathbb{R}^d} \left( \sum_{k \in \mathbb{Z}^d} |c^\mu_k - c^\mu_k'| \Phi(Mx - k) \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_{k, l \in \mathbb{Z}^d} |c^\mu_k - c^\mu_k'| |c^\nu_l - c^\nu_l'| \left( \int_{\mathbb{R}^d} \Phi(Mx - k) \Phi(Mx - l) dx \right)^2 \right)^{\frac{1}{2}}.
\]
By applying the Cauchy–Schwartz inequality, we have
\[
\|\psi^\mu * ˆ{\psi}^\nu - \psi^\mu * ˆ{\psi}^\nu'\|_{L^2(\mathbb{R}^d)} \leq \left( \sum_{k,l \in \mathbb{Z}^d} |c^\mu_{k} - c'^\mu_{k}'| |c^\nu_{k} - c'^\nu_{k}'| \frac{1}{m} \|\Phi\|^2_{L^2(\mathbb{R}^d)} \right)^{\frac{1}{2}}.
\]

\[
= \frac{1}{\sqrt{m}} \|\Phi\|_{L^2(\mathbb{R}^d)} \|(c^\mu_{k})_{k \in \mathbb{Z}^d} - (c'^\mu_{k}')_{k \in \mathbb{Z}^d}\|_{l^1(\mathbb{Z}^d)}.
\]

Now, the assertion follows by (38).

With the notation of Theorem 3.2, the functions \( \phi \) and \( \tilde{\phi} \) are refinable with respect to \( a^0 \) and \( b^0 \). We start with a smooth refinable function \( \phi \). High regularity for the constructed pair of dual wavelet frames can be achieved by applying Theorem 3.2 although we convolute \( \phi \) with a tempered distribution. Starting with a real–valued symbol \( a^0 \) that satisfies the interpolation condition (7), Eq. (35) yields
\[
b^0 = a^0(3 - 2a^0).
\]

Therefore, we obtain \( \tilde{\phi} = \phi * \eta \) with a tempered distribution \( \eta \) that is refinable with respect to \( 3 - 2a^0 \). This leads to \( \phi * \tilde{\phi} = \phi * \phi * \eta \). By the convolution product \( \phi * \tilde{\phi} \), we increase the smoothness and by the convolution with the tempered distribution \( \eta \), we do not decrease the regularity too much.

At this point, we are able to apply Theorem 3.2 to construct concrete examples of smooth pairs of compactly supported dual wavelet frames. The constructed wavelets have a high number of vanishing moments and the corresponding refinable function is even interpolating. Moreover, the wavelets as well as the refinable function satisfy several symmetry conditions.

We choose the quincunx dilation matrix
\[
M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

which satisfies \( m = 2 \). In both of the following examples, we start with a symbol \( a^0 \) that satisfies the interpolation condition. Next, we define the starting symbol \( b^0 \) by (35). By denoting \( \gamma^t = (1,0)^t \), the wavelet symbols \( a^1 = a_i^1 \) and \( b^1 = b_i^1 \) are defined by (36) for \( t = 2 \). With these starting symbols, we apply Theorem 3.2. The smoothness of the refinable function in the following examples was estimated with an implementation of the Villemoes algorithm, cf. [35, 36].

**Example 4.2.** As starting symbol \( a^0 \), we choose the Laplace symbol
\[
a^0 = \frac{1}{2} \left( 1 + \frac{1}{4} e^{-(1,0)^t} + \frac{1}{4} e^{(1,0)^t} + \frac{1}{4} e^{-(0,1)^t} + \frac{1}{4} e^{(0,1)^t} \right).
\]

It has been shown in [9] that \( a^0 \) satisfies the Strang Fix conditions of order 2 and the interpolating condition. The corresponding refinable function \( \phi \) is interpolating and contained in the Hölder class \( C^\alpha(\mathbb{R}^2) \) for \( \alpha = 0.6 \). The refinable distribution \( \tilde{\phi} \) is not contained in \( L_2(\mathbb{R}^2) \),
but the convolution product $\phi \ast \tilde{\phi}$ is contained in the Hölder class $C^\alpha(\mathbb{R}^2)$ for $\alpha = 1.3$. It is interpolating and satisfies the Strang Fix conditions of order 4. By applying (13), we obtain wavelets $\phi \ast \tilde{\psi}$ and $\psi \ast \tilde{\phi}$ that have 2 vanishing moments. The wavelet $\psi \ast \tilde{\psi}$ has 4 vanishing moments. Due to the symmetry properties of the corresponding mask, the refinable function is symmetric to the four lines $x_1 = 0$, $x_2 = 0$, $x_1 = x_2$ and $x_1 = -x_2$, see [21] for details. The wavelets satisfy the same symmetry conditions modulo translation. It can be shown that the assumptions of Lemma 4.1 are satisfied. Therefore, we obtain

$$\|\phi \ast \tilde{\psi} - \psi \ast \tilde{\phi}\|_{L_2(\mathbb{R}^2)} \leq 0.083.$$  

See Figure 1 for the corresponding masks and Figure 2 for the plotted functions.

In the second example, we start with a symbol $a^0$ that leads to a smoother refinable function and satisfies a higher order of the Strang Fix conditions. Therefore, the smoothness and the number of vanishing moments of the constructed pair of dual wavelet frames increase.

**Example 4.3.** Let the symbol $a^0$ be given by the mask in Figure 3. It satisfies the interpolation condition. In [10], it has been shown that the corresponding refinable function $\phi$ is interpolating. The Strang Fix conditions of order 4 are satisfied and $\phi$ is contained in the Hölder class $C^\alpha(\mathbb{R}^2)$ for $\alpha = 1.5$. Similar as in Example 4.2, the refinable distribution $\tilde{\phi}$ is not contained in $L_2(\mathbb{R}^2)$ but $\phi \ast \tilde{\phi}$ is very smooth. The convolution product is contained in the Hölder class $C^\alpha(\mathbb{R}^2)$ for $\alpha = 2.9$. Furthermore, it is interpolating and satisfies the Strang Fix conditions of order 8. The corresponding wavelets $\phi \ast \tilde{\psi}$ and $\psi \ast \tilde{\phi}$ have 4 vanishing moments. The wavelet $\psi \ast \tilde{\psi}$ has
Figure 2: Refinable function and wavelets in Example 4.2
8 vanishing moments. Like in Example 4.2, the refinable function \( \phi \ast \tilde{\phi} \) is symmetric to the four lines \( x_1 = 0 \), \( x_2 = 0 \), \( x_1 = x_2 \) and \( x_1 = -x_2 \) and all wavelets possess similar symmetry properties. For \( s \geq 0 \), we define
\[
\diamondsuit_s := \{ k \in \mathbb{Z}^2 \mid |k_1| + |k_2| \leq s \}.
\]
The masks of \( a^0 b^0 \), \( a^0 b^1 \), \( a^1 b^0 \) and \( a^1 b^1 \) are supported in \( \diamondsuit_9 \), \( \diamondsuit_6 + (1,0)^t \), \( \diamondsuit_{12} + (1,0)^t \) and \( \diamondsuit_9 + (2,0)^t \), respectively. Again, it can be verified that the assumptions of Lemma 4.1 are satisfied and we obtain
\[
\| \phi \ast \tilde{\psi} - \psi \ast \tilde{\phi} \|_{L_2(\mathbb{R}^2)} \leq 0.111.
\]
We have plotted the corresponding functions in Figure 4.

Both examples show that we can construct pairs of compactly supported dual wavelet frames with high smoothness. All wavelets have a high number vanishing moments and the corresponding refinable function is interpolating. The wavelets as well as the refinable function are very symmetric. Furthermore, primal and dual wavelets only differ by their labelling. As far as we know, no existing approach leads to all these properties.

Acknowledgement: The author would like to thank Stephan Dahlke and Karsten Koch for their helpful comments and accurate review.

References


Figure 4: Refinable function and wavelets in Example 4.3

28


